Nonequilibrium fluctuations in a resistor

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In small systems where relevant energies are comparable to thermal agitation, fluctuations are of the order of average values. In systems in thermodynamical equilibrium, the variance of these fluctuations can be related to the dissipation constant in the system, exploiting the Fluctuation-Dissipation Theorem (FDT) [1, 2, 3]. In non-equilibrium steady systems, Fluctuations Theorems (FT) [4, 5, 6, 7] additionally describe symmetry properties of the probability density functions (PDFs) of the fluctuations of injected and dissipated energies. We experimentally probe a model system: an electrical dipole driven out of equilibrium by a small constant current I, and show that FT are experimentally accessible and valid. Furthermore, we stress that FT can be used to measure the dissipated power $\bar{\mathcal{P}} = RI^2$ in the system by just studying the PDFs symmetries.

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Introduction

By studying the fluctuations of an extensive quantity such as the voltage across an electric dipole [1, 2], Fluctuation-Dissipation Theorem [3] allows one to measure at equilibrium the parameter quantifying the dissipative part of the system. The first relation of this type was given for Brownian motion in 1905 [8, 9], relating the mean squared displacement of a particle in a thermal viscous bath with the viscosity and the temperature, via a microscopic expression of the diffusion coefficient of the particle. Since Johnson [1] and Nyquist [2], FDT is also known to apply in electrical circuits, relating equilibrium fluctuations of voltage U across a dipole with the resistive part of this dipole.

In the last decade, a new type of relations, Fluctuation Theorems (FT) [4, 5], appeared in nonequilibrium statistical physics that relate the asymmetry of fluctuations of energies (or powers) with the dissipated power required to maintain the nonequilibrium steady state of the system, which is a measure of the distance from equilibrium. Using a powerful analogy with a forced Langevin equation [6], a precise formulation was recently given for electrical circuits [7]. While FDT is derived for equilibrium systems and can still be used close to equilibrium where generalizations are inferred, FT require the system to be out of equilibrium, but at an arbitrary distance of it.

We drive an electrical dipole out of equilibrium by injecting in it a small current and we use this system as a model nonequilibrium system. By looking at the injected power, and the dissipated heat in the system, we checked Fluctuation Theorems. We show that they are as a predictive tool as the FDT of Nyquist, extending it away from equilibrium.

Our system is an electrical dipole constituted of a resistance R in parallel with a capacitor C (Fig. 1). We drive it out of equilibrium by making a constant current I flow in it. The injected power is typically of some $k_{\rm B}T$ per second, of the same order as in biophysics or nanoscale physics experiments. This represent the fundamental case of a system in contact with two different (electrons-) reservoirs, one of the simplest and most fundamental problems of nonequilibrium physics [10]. Nyquist FDT as well as FT are verified experimentally.

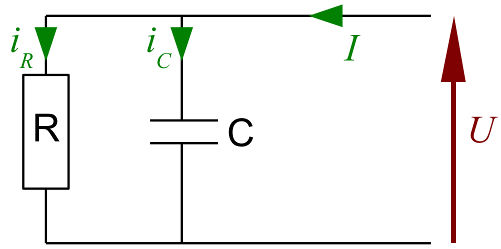

FIG. 1: Model circuit : an electrical dipole is composed of a resistive part R and a capacitive part C. Due to thermal fluctuations of charges positions, a fluctuating voltage U is observed. We drive the system away from equilibrium by imposing a constant flux of electrons, via a constant current I.

Experimental setup

The circuit we use is composed of a resistor in parallel with a capacitor, as depicted on Fig.1. The resistance is a standard metallic one of nominal value $R = 9.52$ MΩ. In parallel, we have an equivalent capacitor of value $C = 280$ pF. This accounts for the capacitance of the all set of coaxial connectors and cables that we used. The time constant of the circuit is $\tau_0 \equiv RC = 2.67$ms. Using a 50 GΩ resistance, we inject in the circuit a constant current I ranging from 0 to 6×10^{-13} A. This current corresponds to an injected power I^2R ranging from 0 to $1000 k_{\rm B}T$/s, where k_B is the Boltzmann constant and T is the absolute temperature. Experiments are conducted at room temperature $T = 300$ K. The typical values of the injected energy for, e.g., $I = 1.4 \times 10^{-13}$ A and $\tau = 10\tau_0$ are of order of a few hundreds of $k_{\rm B}T$, which is small enough to ensure that the resistance is not heating;

expected changes of temperature are estimated to be less than 10^{-14} K over a one hour experiment. Moreover, the resistance is thermostated: all the heat dissipated by Joule effect is absorbed by the thermal bath.

The fluctuating voltage U across the dipole is measured with a resolution of 10^{-11} V sampled at 819.2 Hz. This is achieved by first amplifying the signal by 10^4, using a FET amplifier, with a 4 GΩ input impedance, a voltage noise level of 5 nV/$\sqrt{\text{Hz}}$ and a current noise of 10^{-15} A/$\sqrt{\text{Hz}}$. The signal is then digitized with a 24-bits data acquisition card at frequency 8192 Hz, and then decimating after averaging ten consecutive points.

Fluctuation-Dissipation theorem

If one considers a electrical dipole, as a pure resistance R in parallel with a capacitor C, as modeled as on Fig. 1 the complex impedance of the dipole reads $Z(f) = 1/(1/R + i2\pi RCf)$ where $i^2 = -1$ and f is the frequency. The effective dissipative part is the real part of Z. It was experimentally observed by Johnson [1], and then demonstrated by Nyquist [2] that the potential difference U across the dipole fluctuates with a stationary power spectral density $S(f)df$ such that

$$S(f)df = 4k_{\mathrm{B}}T\Re(Z)df\,, \tag{1}$$

In average, no current is flowing in the circuit, and the mean value of U is zero. Integrating over all positive frequencies, one gets the variance of U:

$$\langle U^2\rangle = \frac{k_{\mathrm{B}}TR}{\tau_0} = \frac{k_{\mathrm{B}}T}{C}\,. \tag{2}$$

Eqs. (1) and (2) are the expressions of the Fluctuation-Dissipation theorem (FDT) for electrical circuits.

The exact value of our capacitance was determined by fitting the power density spectrum of equilibrium fluctuations (at imposed $I = 0$ A) by a Lorentzian low-pass transfer function (eq.(1)), as illustrated on Fig. 2. Application of FDT leads with a very good accuracy to the determination of R, in perfect accordance with the measured nominal value (Fig.2). When $I = 1.4\times10^{-13}$ A, we found the very same power spectral density for U, and performing the same treatments gave the same estimates of R and C. We therefore conclude that FDT is still holding in our system driven out of equilibrium.

Fluctuation theorems

The power injected in the circuit is $\mathcal{P}_{\mathrm{in}} = UI$ but power is dissipated in the resistive part only, so the dissipated power is $\mathcal{P}_{\mathrm{diss}} = Ui_{\mathrm{R}}$ where i_{R} is the current flowing in the resistor (Fig. 1). As already noted [11], in average, one expects $\langle\mathcal{P}_{\mathrm{in}}\rangle = \langle\mathcal{P}_{\mathrm{diss}}\rangle \equiv \bar{\mathcal{P}}$, where the brackets stand for time average over sufficiently long times compared to τ_0. This is very well checked in our experiment. $\mathcal{P}_{\mathrm{in}}$ and $\mathcal{P}_{\mathrm{diss}}$ fluctuate in time because U itself is fluctuating. If one assumes that fluctuations of U have a Gaussian distribution, which is the case at equilibrium when $I = 0$, then $\mathcal{P}_{\mathrm{in}}$ has also a Gaussian distribution,

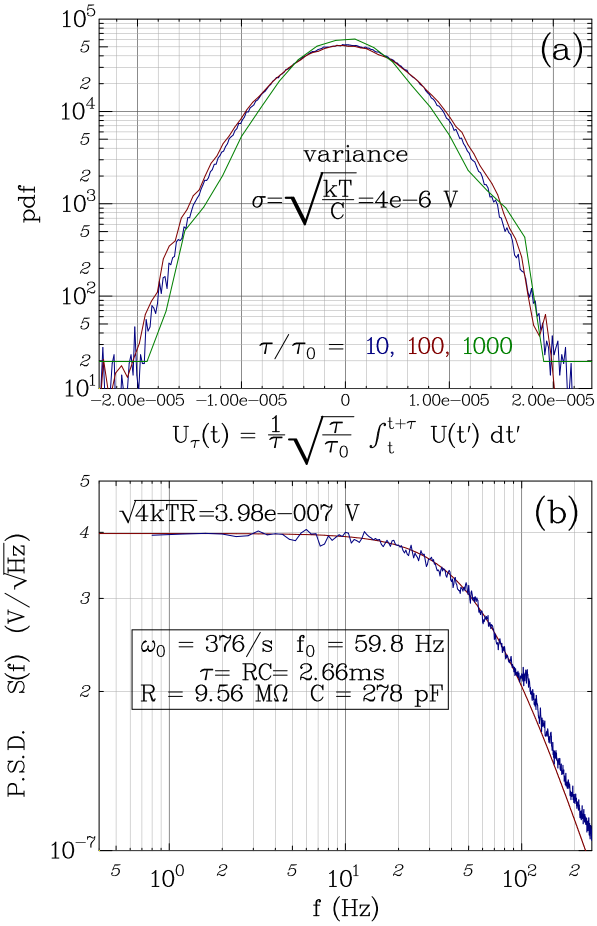

FIG. 2: For $I = 0$ A, Johnson-Nyquist noise has a Gaussian distribution. Relation (2) is verified. The noise is white up to the cutoff frequency f_0 of the RC dipole. We use FDT to extract from the energy of the noise the value of the resistive, dissipative part of the circuit R: for low frequency, the noise level spectral density is constant, equal to $\sqrt{4k_BTR}$ in a band $[f;\ f+df]$. A Lorentzian fit of the spectrum additionally gives $\tau_0 = RC$.

because I is constant. On the contrary, i_R fluctuates, as we see from Kirchoff's laws:

$$I = i_R + C\frac{\mathrm{d}U}{\mathrm{d}t}\,, \qquad \text{so} \qquad \mathcal{P}_{\mathrm{in}} = \mathcal{P}_{\mathrm{diss}} + \frac{1}{2}C\frac{\mathrm{d}U^2}{\mathrm{d}t}\,, \tag{3}$$

and therefore, the probability distribution of $\mathcal{P}_{\mathrm{diss}}$ is not Gaussian [7]. It is worth noting that for large current I, some orders of magnitude larger than the one we use, $\mathcal{P}_{\mathrm{in}}$ and $\mathcal{P}_{\mathrm{diss}}$ will be much larger than the conservative part $\frac{C}{2}\frac{\mathrm{d}U^2}{\mathrm{d}t}$ and therefore the probability distributions of both the injected and dissipated power will be Gaussian, as it is usually expected in macroscopic systems.

We call $\langle g\rangle_\tau(t) = \frac{1}{\tau}\int_t^{t+\tau} g(t')dt'$ the time-averaged value of a function g over a time τ.

Reasoning with energies instead of powers, we define $W_\tau(t) = \tau\langle\mathcal{P}_{\mathrm{in}}\rangle_\tau(t)$, the energy injected in the circuit

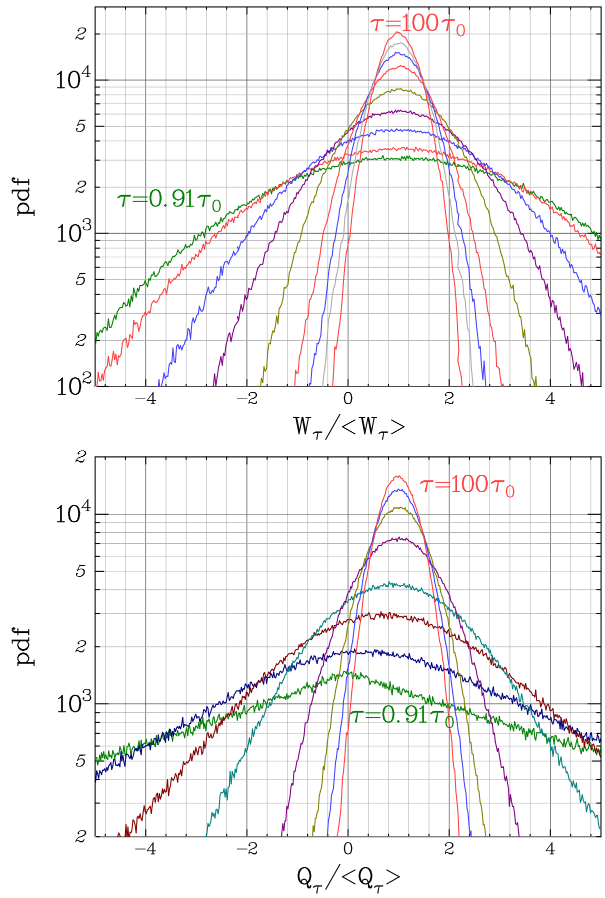

FIG. 3: Histograms of W_τ and Q_τ when a current $I = 1.4 \times 10^{-13}$A is flowing in the dipole ($\bar{\mathcal{P}} = RI^2 = 45 k_\mathrm{B}T/\mathrm{s}$).

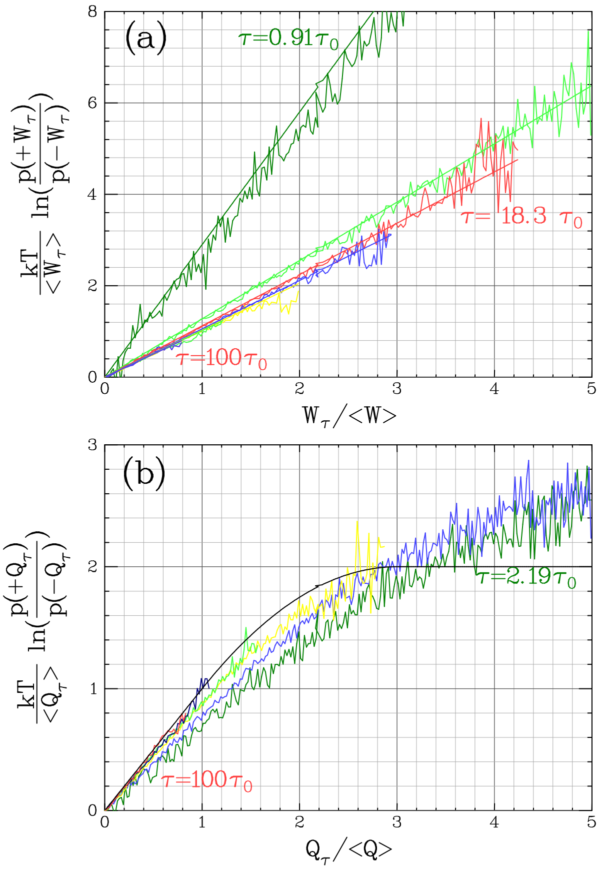

FIG. 4: Normalized symmetry functions S_W and S_Q for W_τ and Q_τ when a current $I = 1.4 \times 10^{-13}$A is flowing in the dipole. For any τ, S_W is a linear function of $a = W_\tau/\langle W_\tau\rangle$. For $\tau \to \infty$, $S_W(a)$ tends to have a slope of 1, whereas $S_Q(a)$ tends to a limit function (black curve) which is a straight line of slope 1 for $a < 1$ only.

during time τ, analogous to the work performed on the system (positive when received by the system). In the same way, we write $Q_\tau(t) = \tau\langle\mathcal{P}_\mathrm{diss}\rangle_\tau(t)$, the energy dissipated by Joule effect during time τ, analogous to the heat given by the system (positive when given by the system). We used values of τ spanning from τ_0 up to hundreds of τ_0.

From the experimentally measured $U(t)$ for a given value of I, we compute $\mathcal{P}_\mathrm{in}$ and $\mathcal{P}_\mathrm{diss}$. We then build the probability density functions of cumulated variables W_τ and Q_τ using 10^6 points; their typical distributions are plotted on Fig. 3. As noticed above, fluctuations of W_τ are Gaussian for any τ; on the contrary, heat fluctuations are not Gaussian for small values of τ, but exponential: large fluctuations of heat Q_τ are more likely to occur than large fluctuations of work W_τ.

We then look at the symmetry function:

$$S_E(\tau, a) = \ln \frac{p(E_\tau = a)}{p(E_\tau = -a)} ,$$

where E_τ stands for either W_τ or Q_τ. If the Fluctuation Theorem for energy E_τ holds, then one should have, for large enough τ, the relationship

$$S_E(\tau, a) = \frac{\tau \bar{\mathcal{P}}}{k_\mathrm{B}T} f_E(\tau, a) , \qquad (4)$$

where for the work $\lim_{\tau\to\infty} f_W(\tau, a) = a$. In contrast, for the heat and $\tau \to \infty$, the asymptotic values of $f_Q(\tau, a)$ are $f_Q^\infty(a) = a$ for $a \le 1$, $f_Q^\infty(a) = 2$ for $a \ge 3$, and there is a continuous parabolic connection for $1 \le a \le 3$ that has a continuous derivative [6, 7]. From histograms of Fig. 3, we deduce the symmetry functions $S_W(\tau, a)$ and $S_Q(\tau, a)$ (Fig. 4).

Work fluctuations First, for any given τ we checked that the symmetry function $S_W(\tau, a)$ is linear in a (Fig. 4a). We measured the corresponding proportionality coefficient $\sigma_W(\tau)$ such that $f_W(\tau, a) = \sigma_W(\tau)a$. This coefficient $\sigma_W(\tau)$ tends to 1 when τ is increased (see Fig. 5).

Heat fluctuations We found that $S_Q(\tau, a)$ is linear in a only for $a < 1$, as expected [6, 7]. Again, as $\tau \to \infty$,

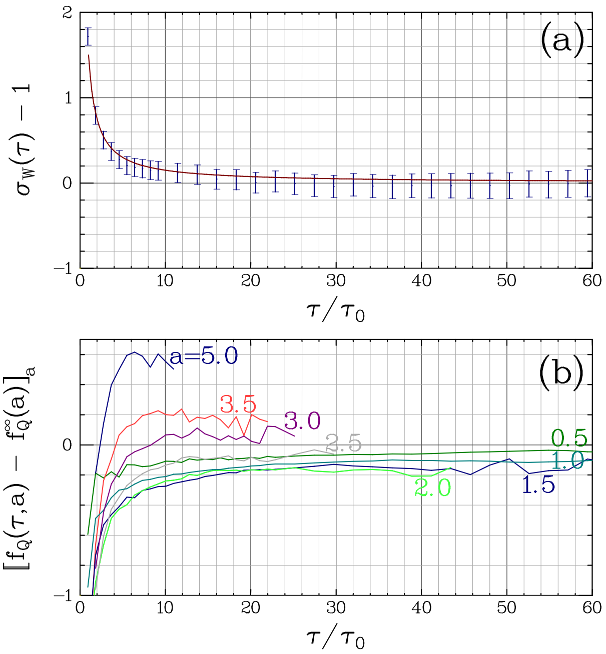

FIG. 5: (a): dependence on τ of the slope of S_W in Fig. 4. Slope is converging from above. (b): distance between $S_Q(\tau)$ for finite τ and theoretical prediction of S_Q for infinite τ, for several values of $a = Q/\langle Q \rangle$. for $a < 3$, convergence is from below, whereas it is from above for $a > 3$.

the limit slope of the symmetry function is 1 whereas for $a > 3$, $S_Q(\tau, a)$ tends to two.

Asymptotic symmetry functions and convergence In [6, 7], expressions for the convergence towards the asymptotic limits $f_W^\infty(a)$ and $f_Q^\infty(a)$ are given in terms of τ. We can check these predictions with our data. The convergence for the work W_τ is very well reproduced by these predictions, as can be seen on Fig. 4a: continuous straight lines are theoretical predictions for small values of τ, using eqs. (9) and (10) from [7], with no adjustable parameters. On Fig. 5a the slope $\sigma_W(\tau)$ of the experimental symmetry function is plotted. The continuous line is the prediction of [7], which perfectly agrees with our data.

For the heat, we distinguish two regimes for the convergence towards the asymptotic symmetry function. For $a = Q_\tau/\langle Q \rangle < 3$, we find that when τ is increased, symmetry functions are converging to the asymptotic function from below (Fig. 4), which is the opposite of what is observed for the work. On the contrary, for $a > 3$, convergence to the asymptotic function is from above, thus enhancing the peculiarity of the point $a = 3$. On Fig. 5, we have plotted the evolution of $f_Q(\tau, a)$ versus τ for several fixed values of a. The convergence from above for $a \geq 3$ and from below for $a < 3$ is clear. For increasing values of τ, only smaller and smaller values of a are accessible because of the averaging process. Therefore the accessible values of a are quickly lower than 1, and only the linear part of $S_Q(\tau, a)$ can be experimentally tested. Nevertheless, for intermediate time scales τ we see in Fig. 4 that the data converge towards the theoretical asymptotic nonlinear symmetry function [6, 7] (smooth curve in Fig. 4b).

We observed that the convergence reproduced on Fig. 5 depends on the injected current I, as pointed out in [7]. Other experiments with a larger current ($\bar{\mathcal{P}} = 186 k_{\rm B}T/{\rm s}$) give a faster convergence; corresponding results will be reported elsewhere.

Conclusions

We have shown experimentally that the asymmetry of the probability distribution functions of work and heat in a simple electrical circuit driven out of equilibrium by a fixed current I, is linked to the averaged dissipated power in the system. The recently proposed Fluctuation Theorems for first order Langevin systems are then experimentally confirmed. Exploiting formula (4), FT can be used to measure an unknown averaged dissipated power $\bar{\mathcal{P}} = \lim_{\tau\to\infty} \frac{Q_\tau}{\tau}$ by using only the symmetries of the fluctuations, *i.e.* computing S_W or S_Q and measuring their asymptotic slope.

We operated with energies of order of $k_B T$ in order to have strong fluctuations compared to the averaged values. It is worth noting that as the driving current is increased up to macroscopic values, the fluctuations become more and more negligible, therefore Fluctuation Theorems become harder and harder to use, and therefore less relevant.

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