Examples of Groups that are Measure Equivalent to the Free Group

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Abstract

Measure Equivalence (ME) is the measure theoretic counterpart of quasi-isometry. This field grew considerably during the last years, developing tools to distinguish between different ME classes of countable groups. On the other hand, contructions of ME equivalent groups are very rare. We present a new method, based on a notion of measurable free-factor, and we apply it to exhibit a new family of groups that are measure equivalent to the free group. We also present a quite extensive survey on results about Measure Equivalence for countable groups.

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1 Introduction

During the last years, much progress has been made in the classification of countable discrete groups up to Measure Equivalence (ME). All of them describe criteria ensuring that certain groups don't belong to the same equivalence class (see for instance the work of R. Zimmer [Zim84, Zim91], S. Adams [Ada90, Ada94, Ada95], Adams-Spatzier [AS90], A. Furman [Fur99a, Fur99b], D. Gaboriau [Gab00, Gab02], Monod-Shalom [MS02],... We also would like to draw the attention of the reader to the very nice paper [HK05] of Hjorth-Kechris where similar results are developed.

This notion (ME), introduced by M. Gromov [Gro93] is a measure theoretic analogue of Quasi-Isometry (QI) (see sect. 2). But, it takes its roots in the pioneering work of H. Dye [Dye59, Dye63], and even, somehow, back to Murray-von Neumann [MvN36].

The simplest instances of ME groups are *commensurable groups* or more generally *commensurable up to finite kernel*[1], and *groups that are lattices* (=discrete, finite covolume subgroup) *in the same locally compact second countable group*. Recall that cocompact lattices are QI.

The first non-elementary iso-ME-class result is due to Dye [Dye59, Dye63] who puts together many amenable groups in the same ME-class, for example, all the infinite groups with polynomial growth. A series of improvements led to Ornstein-Weiss' theorem [OW80] asserting that, in fact, *all infinite amenable groups are ME to each other*. On the other hand, since amenability is a ME invariant, the ME class of $\mathbb{Z}$ consists precisely in all the infinite amenable groups [Fur99a].

Next, to build further ME groups, one can elaborate on these constructions via elementary procedures: direct products and free products (see Section 2 for precise statements). And these are essentially the only known methods.

The family of groups ME to a free group[2] contains for instance the finite and amenable groups, as well as the fundamental groups $\pi_1(\Sigma_g)$ of the compact orientable surfaces of genus ≥ 2 (lattices in $\mathrm{SL}(2,\mathbb{R})$); and it is stable

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[1] in the sense of [Har00, IV.B.27]: **commensurability up to finite kernel** is the equivalence relation on the set of countable groups generated by the equivalence of two terms of an exact sequence $1 \to A \to B \to C \to 1$ as soon as the third term is a finite group.

[2] For consistency, we are led to include the free groups $\mathbf{F}_\infty$ on countably many elements and $\mathbf{F}_0$ on 0 element (i.e. the trivial group $\{1\}$).

under taking free products (see Section 2.2, Property $\mathbf{P}_{\mathrm{ME}}\mathbf{7}$) and subgroups (see Section 2.2, Property $\mathbf{P}_{\mathrm{ME}}\mathbf{9}$). For instance, the following groups are ME to $\mathbf{F}_2$: $\mathbf{F}_2 * \pi_1(\Sigma_g)$, $\mathbb{Z}^2 * \mathbb{Z}^2$, $\mathbb{Q} * \mathbb{Z}/3\mathbb{Z}$, the triangular groups $T_{a,b,c}$ of isometries of the hyperbolic plane (with fundamental domain a triangle of angles $\frac{\pi}{a}, \frac{\pi}{b}, \frac{\pi}{c}$, $2 \leq a \leq b \leq c$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$), $(\mathrm{SL}(2,\mathbb{Z}) \times \mathbb{Z}/5\mathbb{Z}) * \mathbb{Q}^2 * A$, where A is any amenable group.

Following a theorem of G. Hjorth (see [Hjo02] or also [KM04, Th. 28.2, p. 98]), being ME to a free group is equivalent with the following elementarily equivalent conditions (sect. 2.2, $\mathbf{P}_{\mathrm{ME}}\mathbf{8}$):
– being treeable in the sense of Peres-Pemantle [PP00]
– admitting a treeable p.m.p.[3] free[4] action in the sense of [Ada90, Gab00]
– having ergodic dimension 0 (for finite groups) or 1 in the sense of [Gab02]

On the other hand, the free groups (and thus the family of groups ME to a free group) split into four different ME-classes: the class of $\mathbf{F}_0$ (finite groups), that of $\mathbf{F}_1 = \mathbb{Z}$ (amenable groups), that of $\mathbf{F}_p$, $2 \leq p < \infty$, (all the $\mathbf{F}_p$ are commensurable) and that of $\mathbf{F}_\infty$. The last two classes are distinguished by their ℓ^2-Betti numbers – $\beta_1 \in (0,\infty)$, resp. $\beta_1 = \infty$ – [Gab02] (or also by their cost – $\mathcal{C} \in (1,\infty)$, resp. $\mathcal{C} = \infty$ – [Gab00]) (see Section 2).

However, the classification of groups ME to a free group seems, nowadays, completely out of reach[5]. In fact, the only groups whose ME-class is classified are finite groups, amenable groups and lattices in simple connected Lie groups with finite center and real rank ≥ 2 [Fur99a] (see Property $\mathbf{P}_{\mathrm{ME}}\mathbf{19}$).

The contribution of this paper consists in a new construction of ME groups, leading to the exhibition of new groups that are ME to free groups $\mathbf{F}_2$ and $\mathbf{F}_\infty$.

Let Γ be a countable group and Λ a subgroup. We denote by $\overset{n}{\underset{\Lambda}{*}}\, \Gamma$ the **iterated amalgamated free product** $\Gamma \underset{\Lambda}{*} \Gamma \underset{\Lambda}{*} \cdots \underset{\Lambda}{*} \Gamma$ of n copies of Γ above the corresponding n copies of Λ, where the injection morphisms are just the identity.

Consider the free group $\mathbf{F}_{2p} = \langle a_1, \cdots, a_p, b_1, \cdots, b_p \rangle$ and its cyclic subgroup C generated by the product of commutators: $\kappa := \prod_{i=1}^{i=p} [a_i, b_i]$.

Theorem 1.1 (Cor. 3.7) *For each $n \in \mathbb{N}$, the iterated amalgamated free product $\overset{n}{\underset{C}{*}}\, \mathbf{F}_{2p}$ is measure equivalent to the free group $\mathbf{F}_2$.*

This iterated amalgamated free product $\overset{n}{\underset{C}{*}}\, \mathbf{F}_{2p}$ is the fundamental group $\pi_1(\Sigma)$ of a "branched surface" (figure 1). Our examples for various n are not obviously lattices in the same l.c.s.c. group (i.e. the question asked to several

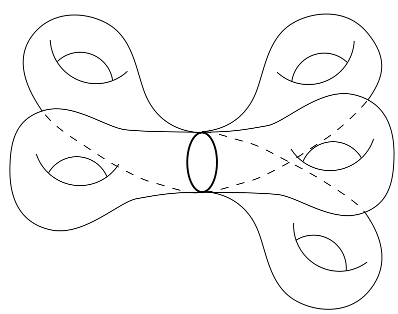

Figure 1: Branched surface

specialists remains open). However, they are certainly not cocompact lattices in the same l.c.s.c. group nor commensurable up to finite kernel since they are not quasi-isometric:

[3] p.m.p.: probability measure preserving, on a standard Borel probability space.
[4] In this measure theoretic context, free means "essentially free", i.e. up to removing a set of measure zero.
[5] To compare with, recall that a group quasi-isometric to a free group is virtually a free group [Sta68].

Proposition 1.2 *The number n of factors $\mathbf{F}_{2p}$ in the iterated amalgamated free product $\overset{n}{\underset{C}{*}}\mathbf{F}_{2p}$ is a quasi-isometry invariant.*

PROOF: The boundary of the group is disconnected by pairs of points into either 1, 2 or n connected components. ■

Theorem 1.1 extends to an infinite amalgamated free product: $\overset{\infty}{\underset{C}{*}}\mathbf{F}_{2p}$ is measure equivalent to the free group $\mathbf{F}_\infty$ (Cor. 3.7).

Corollary 1.3 *If r is any non trivial element of $\mathbb{Z}$, then the amalgamated free product $\Gamma_r = \mathbb{Z}\underset{r=\kappa}{*}\mathbf{F}_{2p}$ is measure equivalent to the free group $\mathbf{F}_2$.*

PROOF: Consider the natural homomorphism $\mathbb{Z}\underset{r=\kappa}{*}\mathbf{F}_{2p}\to\mathbb{Z}/r\mathbb{Z}$ that kills $\mathbf{F}_{2p}$. Its kernel has index r in Γ_r and is isomorphic with $\overset{r}{\underset{C}{*}}\mathbf{F}_{2p}$. ■

More generally, one gets:

Theorem 1.4 *(Cor. 3.18) Let G be any countable group, H an infinite cyclic subgroup and C the cyclic subgroup generated by κ in $\mathbf{F}_{2p}$. Then $G\underset{H=C}{*}\mathbf{F}_{2p}$ is ME to $G*\mathbf{F}_{2p-1}$. In particular, if G is ME to $\mathbf{F}_2$, then $G\underset{H=C}{*}\mathbf{F}_{2p}$ is ME to $\mathbf{F}_2$.*

By contrast, an amalgamated free product $\mathbb{Z}\underset{\mathbb{Z}}{*}\mathbb{Z}$ is measure equivalent to a free group if and only if the common $\mathbb{Z}$ coincides with one the components or injects as a subgroup of index 2 in both (see $\mathbf{P}_{\text{ME}}\mathbf{15}^{**}$). Also, if instead of C one considers $2C$, generated by κ^2, and if H is contained as a subgroup of index at least three in a greater abelian group H' of G, then $G\underset{H=2C}{*}\mathbf{F}_{2p}$ cannot be ME to a free group (by $\mathbf{P}_{\text{ME}}\mathbf{9}$ and $\mathbf{P}_{\text{ME}}\mathbf{15}^{**}$): it contains the nonamenable subgroup $H'\underset{H=2C}{*}C$ with $\beta_1=0$.

To obtain these results we show that the subgroup C happens to appear as a free factor of the group $\mathbf{F}_{2p}$ in a measure theoretic sense (see Def. 3.1). Our proof makes use of standard percolation techniques developed in [LP05].

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2 Generalities about Measure Equivalence

Definition 2.1 ([Gro93, 0.5.E]) *Two countable groups Γ_1 and Γ_2 are* ***Measure Equivalent (ME)*** *if there exist commuting, measure preserving, free actions of Γ_1 and Γ_2 on some Lebesgue measure space (Ω,m) such that the action of each of the groups admits a* finite measure *fundamental domain (D_i for Γ_i).*

In this case we say that Γ_1 is ME to Γ_2 with **index** $\iota=[\Gamma_1:\Gamma_2]_\Omega := m(D_2)/m(D_1)$, and we denote the existence of such a **coupling** Ω by

$$\Gamma_1 \underset{\iota}{\overset{\text{ME}}{\sim}} \Gamma_2 \quad\text{or simply}\quad \Gamma_1 \overset{\text{ME}}{\sim} \Gamma_2$$

when the particular value of the index is irrelevant. Measure Equivalence is an equivalence relation on the set of countable groups. The grounds are established by A. Furman in [Fur99a, Sect. 2].

The notion of Measure Equivalence has been introduced by M. Gromov as the *measure theoretic* analogue of the notion of Quasi-Isometry, which is more *topological*, as testified by the following criterion:

Criterion for quasi-isometry ([Gro93, 0.2.$\mathbf{C}_2'$]) Two finitely generated groups Γ_1 and Γ_2 are *quasi-isometric (QI)* iff there exist commuting, continuous actions of Γ_1 and Γ_2 on some locally compact space M, such that the action of each of the groups is properly discontinuous and has a *compact* fundamental domain.

We anthologize basic and less basic properties of ME, with references, or proofs when necessary.

2.1 Basic Properties of ME

$\mathbf{P}_{\mathrm{ME}}\mathbf{1}$ The ME class of the trivial group $\{1\}$ consists in all finite groups.

$\mathbf{P}_{\mathrm{ME}}\mathbf{2}$ Groups that are commensurable up to finite kernel are ME.

Commensurability up to finite kernel (footnote 1) corresponds exactly to ME with a countable atomic space Ω.

$\mathbf{P}_{\mathrm{ME}}\mathbf{3}$ Lattices in the same locally compact second countable group are ME.

Actions of the two lattices by left multiplication and right multiplication by the inverse on the group deliver the desired ME coupling.

$\mathbf{P}_{\mathrm{ME}}\mathbf{4}$ (Direct Product) $\Gamma_1 \overset{\mathrm{ME}}{\sim} \Lambda_1$ and $\Gamma_2 \overset{\mathrm{ME}}{\sim} \Lambda_2 \Rightarrow \Gamma_1 \times \Gamma_2 \overset{\mathrm{ME}}{\sim} \Lambda_1 \times \Lambda_2$

This is quite obvious, by taking the product actions on the product space. The converse fails in general. For instance by Ornstein-Weiss' theorem recalled in the introduction (see also $\mathbf{P}_{\mathrm{ME}}\mathbf{10}$), all infinite amenable groups are ME so that[6] $\mathbf{F}_2 \times \mathbb{Z} \overset{\mathrm{ME}}{\sim} \mathbf{F}_2 \times (\mathbb{Z} \times \mathbb{Z}) = (\mathbf{F}_2 \times \mathbb{Z}) \times \mathbb{Z}$. but $\mathbf{F}_2 \overset{\mathrm{ME}}{\not\sim} \mathbf{F}_2 \times \mathbb{Z}$ (see [Ada94, Th. 6.1] or $\mathbf{P}_{\mathrm{ME}}\mathbf{15}$**).

However, it becomes true when restricted to a quite large class of groups (see $\mathbf{P}_{\mathrm{ME}}\mathbf{20}$).

Recall that two p.m.p. actions α_i of Γ_i on (X_i, μ_i) $(i = 1, 2)$ are **stably orbit equivalent (SOE)** if there are complete sections $A_i \subset X_i$ such that the orbit equivalence relations $(\mathcal{R}_{\alpha_i(\Gamma_i)}, \mu_i)$ restricted to A_i are isomorphic via a measure scaling isomorphism $f : A_1 \to A_2$ $(f_*\mu_{1|A_1} = \lambda\, \mu_{2|A_2})$. The **compression constant** is defined as $\iota := \mu_2(A_2)/\mu_1(A_1) = 1/\lambda$. The connection with ME is very useful and is proved in [Fur99b, Lem. 3.2, Th. 3.3], where it is credited to Zimmer and Gromov. An easy additional argument gives the freeness condition (see [Gab00b, Th. 2.3]). **Orbit Equivalence (OE)** corresponds to the case where $A_i = X_i$ (and thus $\iota = 1$) and where the fundamental domains D_1, D_2 of the measure equivalence coupling may be taken to coincide.

$\mathbf{P}_{\mathrm{ME}}\mathbf{5}$ (ME $\Leftrightarrow$ SOE) – Two groups Γ_1 and Γ_2 are ME (with index ι) if and only if they admit stably orbit equivalent **(SOE)** free p.m.p. actions (with compression constant ι).
– They are ME with a common fundamental domain if and only if they admit orbit equivalent **(OE)** free p.m.p. actions. We shall denote this last situation by

$$\Gamma_1 \overset{\mathrm{OE}}{\sim} \Gamma_2$$

– If the only possible value of the index $\iota = [\Gamma_1 : \Gamma_2]$ is 1, then $\Gamma_1 \overset{\mathrm{OE}}{\sim} \Gamma_2$.

For its conciseness, we present a proof of this result.

Proof: ME $\Rightarrow$ SOE. Consider an action of $\Gamma_1 \times \Gamma_2$ on (Ω, m) witnessing $\Gamma_1 \underset{\iota}{\overset{\mathrm{ME}}{\sim}} \Gamma_2$. The existence of fundamental domains of finite measures D_i ensures the existence of a Γ_1-equivariant Borel map onto a finite measure Borel space $\pi_1 : \Gamma_1 \times \Gamma_2 \curvearrowright (\Omega, m) \to \Gamma_1 \curvearrowright X_1 := \Omega/\Gamma_2 \simeq D_2$ and similarly for π_2 by exchanging the subscripts 1 and 2. Any two points y, y' of Ω are $\Gamma_1 \times \Gamma_2$-equivalent iff their images under π_i are Γ_i-equivalent $(i = 1, 2)$. Since π_1, π_2 have countable fibers, one can find a non-null $Y \subset \Omega$ on which both π_1 and π_2 are one-to-one. The natural isomorphism f between $\pi_1(Y)$ and $\pi_2(Y)$ preserves the restriction of the equivalence relations. Normalizing the induced measure on X_i to a probability measure leads to identify index and compression constant. The freeness is obtained by just considering any free p.m.p. action of Γ_1 on some space (Z, ν) (with trivial action of Γ_2) and replacing the action on Ω by the diagonal action on the product measure space $\Omega \times Z$. In case of coincidence, $Y := D_1 = D_2 \simeq X_1 \simeq X_2$ delivers OE actions.

[6] More generally $(\Gamma \times A) \overset{\mathrm{ME}}{\sim} (\Gamma \times A) \times B$, for any group Γ, and any infinite amenable groups A and B.

PROOF: SOE $\Rightarrow$ ME. Let Γ_i act on (X_i, μ_i) and assume $X_1 \supset Y_1 \simeq Y_2 \subset X_2$ witnesses the SOE. Rescale the measures μ_1, μ_2 so that Y_1 and Y_2 get the same measure and so that the glueing $X := (X_1 \coprod X_2)/(Y_1 = Y_2)$ of X_1, X_2 along Y_1, Y_2 becomes a probability space. Define on X the equivalence relation $\mathcal{R}$ extending $\mathcal{R}_{\Gamma_i}$ on X_i. Consider the coordinate projections π_1 and π_2 from $\mathcal{R}$ to X. The Borel subset $\Omega \subset \mathcal{R} \subset X \times X$ defined as the preimage by $\pi_1 \times \pi_2$ of $X_1 \times X_2$ gives the desired ME coupling: Γ_i acts on the i-th coordinate (and trivially on the other one). Any Borel section of π_1 gives a Borel fundamental domain for Γ_2 and any Borel section of π_2 gives a Borel fundamental domain for Γ_1. Their measures, for the natural measure on $\mathcal{R}$ (see [FM77]), are compared via the rescaling to the compression constant. In case of OE, $\Omega = \mathcal{R}$ and the diagonal $\Delta \subset \mathcal{R}$ is a fundamental domain for both groups. Observe that ergodic SOE actions on X_i are in fact OE if and only if $\iota = 1$. The condition, when $\iota = 1$ is the only possible index, follows by considering an ergodic component. ■

2.2 Free Products and Amalgamations

Stability under taking free products requires the additional assumption that not only the groups are ME but also that they admit a coupling with a common fundamental domain. That such a result may be true is suggested in [MS02, Rem. 2.27]. We prove it below (after Lemma 2.6) as well as some generalizations.

$\mathbf{P}_{\text{ME}}\mathbf{6}$ (Free Product) $\Gamma_1 \overset{\text{OE}}{\sim} \Lambda_1$ and $\Gamma_2 \overset{\text{OE}}{\sim} \Lambda_2 \Rightarrow \Gamma_1 * \Gamma_2 \overset{\text{OE}}{\sim} \Lambda_1 * \Lambda_2$

And more generally:

$\mathbf{P}_{\text{ME}}\mathbf{6}^*$ (Infinite Free Product) $\Gamma_j \overset{\text{OE}}{\sim} \Lambda_j$ for all $j \in \mathbb{N} \Rightarrow \underset{j\in\mathbb{N}}{*}\Gamma_j \overset{\text{OE}}{\sim} \underset{j\in\mathbb{N}}{*}\Lambda_j$

The hypothesis $\overset{\text{OE}}{\sim}$ cannot be replaced by $\overset{\text{ME}}{\sim}$ since the following three groups belong to three different ME-classes[7]: $\{1\} * \mathbb{Z}/2\mathbb{Z}$ (finite) (see $\mathbf{P}_{\text{ME}}\mathbf{1}$), $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ (amenable) (see $\mathbf{P}_{\text{ME}}\mathbf{10}$) and $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ (nonamenable). Similarly, free groups of different ranks are not OE (for a proof, we draw the attention of the reader to the simplified preliminary version [Gab98] of [Gab00]), and the groups $\mathbf{F}_p * (\mathbf{F}_2 \times \mathbf{F}_2)$ all belong to different ME-classes for different p's: their ℓ^2 Betti numbers[8] do not agree with $\mathbf{P}_{\text{ME}}\mathbf{15}$. However, this hypothesis may be relaxed when working with groups ME to free groups:

$\mathbf{P}_{\text{ME}}\mathbf{7}$ If each Γ_j is ME to some free group, then $\underset{j\in\mathbb{N}}{*}\Gamma_j$ is ME to a free group.

In particular,

$\mathbf{P}_{\text{ME}}\mathbf{7}^*$ $\Gamma_1 \overset{\text{ME}}{\sim} \mathbf{F}_p$ and $\Gamma_2 \overset{\text{ME}}{\sim} \mathbf{F}_q$, for $p, q \in \{1, 2, \infty\} \Rightarrow \Gamma_1 * \Gamma_2 \overset{\text{ME}}{\sim} \mathbf{F}_{p+q}$

For instance, non trivial free products of finitely many infinite amenable groups are ME to $\mathbf{F}_2$ (see [Gab00, p. 145]). Observe that free products of finitely many finite groups are virtually free. [9]

We make an intensive use of countable measured equivalence relations $\mathcal{R}$ on standard probability measure spaces (see [FM77] and the reminders of [Gab00, Gab02]) : **they all are assumed to preserve the measure**.

A countable group Γ is **treeable** in the sense of Peres-Pemantle [PP00] if the set of trees with vertex set Γ supports a Γ-invariant probability measure. The group Γ is **not anti-treeable** in the sense of [Gab00] if it admits a treeable free action α, i.e. an action whose associated equivalence relation $\mathcal{R}_\alpha$ is generated by a treeing. Recall that a **graphing** of $\mathcal{R}$ is a countable family $\Phi = (\varphi_j)_{j\in J}$ of partially defined isomorphisms $\varphi_j : A_j \to B_j$ between Borel subsets of X such that $\mathcal{R}$ is the smallest equivalence relation satisfying for all

[7] The same phenomenon occurs in the analogous setting of geometric group theory and for similar reasons: Bi-Lipschitz equivalence of groups passes to free products, but not just QI of them. However, the (counter-)example of $\{1\} * \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ is essentially the only one and is satisfactorily worked around by a result of Papasoglu-Whyte: *If two families of non trivial ($\neq \{1\}$) finitely generated groups $(\Gamma_i)_{i=1,\cdots,s}$ and $(\Lambda_j)_{j=1,\cdots,t}$ $s, t \geq 2$ define the same sets of quasi-isometry types (without multiplicity), then the free products $\Gamma = \Gamma_1 * \Gamma_2 * \cdots \Gamma_s$ and $\Lambda = \Lambda_1 * \Lambda_2 * \cdots * \Lambda_t$ are quasi-isometric unless Γ or Λ is $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ [PW02, Th. 02].*

[8] The sequence of ℓ^2 Betti numbers of $\mathbf{F}_p * (\mathbf{F}_2 \times \mathbf{F}_2)$ is $(\beta_i(\mathbf{F}_p * (\mathbf{F}_2 \times \mathbf{F}_2)))_{i\in\mathbb{N}} = (0, p, 1, 0, 0, \cdots, 0, \cdots)$. Incidentally, no one of QI or ME classification is finer than the other one, as illustrated by this family of groups (that they are QI is due to K. Whyte, see footnote 7).

[9] It follows from $\mathbf{P}_{\text{ME}}\mathbf{7}$ that all the groups Γ_S that appear in [HK05, Th. 1] are ME when S is finite (resp. infinite).

$j \in J$ for all $x \in A_j$, $x \sim \varphi_j(x)$. It equips each orbit with a graph structure. When these graphs are trees, the graphing is called a **treeing** [Ada90] and the equivalence relation **treeable**.

The two following notions taken from [Gab02], which we mention just for completeness (see also $\mathbf{P}_{\text{ME}}\mathbf{16}$), won't be seriously used in the remainder of the paper. Equivalence relations, as groupoids, admit discrete actions on fields of simplicial complexes. The **ergodic dimension** is the smallest possible dimension of such a field of contractible simplicial complexes. For the **approximate ergodic dimension**, one considers increasing exhaustions of $\mathcal{R}$ by measurable subrelations and the smallest possible ergodic dimensions of these approximations.

$\mathbf{P}_{\text{ME}}\mathbf{8}$ The following conditions on a countable group Γ are equivalent:
-i- being treeable in the sense of Peres-Pemantle
-ii- admitting a treeable p.m.p. free action in the sense of [Gab00]
-iii- having ergodic dimension 0 (for finite groups) or 1
-iv- being ME to a free group

The equivalence of the first three conditions is elementary, while the connection with the fourth one is a result of G. Hjorth: see [Hjo02] or also [KM04, Th. 28.2, Th. 28.5]. In order to apply [KM04, Th. 28.2, Th. 28.5], observe that after considering an ergodic component, the cost can be arranged, by SOE, to be an integer.

We introduce some terminology. Let $\mathcal{R}$ and $\mathcal{S}$ be two countable measured equivalence relations on the standard probability measure spaces (X, μ) and (Y, ν) respectively. A measurable map $p : X \to Y$ is a **morphism** from $\mathcal{R}$ to $\mathcal{S}$ if almost every $\mathcal{R}$-equivalent points x, x' of X have $\mathcal{S}$-equivalent images $p(x), p(x')$ and if the pushforward measure $p_*\mu$ is equivalent to ν. Say that such a morphism is **locally one-to-one** if, for almost every $x \in X$, p induces a one-to-one map from the $\mathcal{R}$-class of x to the $\mathcal{S}$-class of $p(x)$.

Example 2.2 *If $\mathcal{R}_\alpha$ and $\mathcal{S}_\beta$ are the orbit equivalence relations of two actions α, β of a countable group Γ on X (resp. Y), then any Γ-equivariant measurable map $p : X \to Y$ is a morphism from $\mathcal{R}_\alpha$ to $\mathcal{S}_\beta$. If the action β is moreover free, then p is locally one-to-one.*

Lemma 2.3 *If $p : X \to Y$ is a locally one-to-one morphism from $\mathcal{R}$ to $\mathcal{S}$ and if $\mathcal{S}$ is the orbit equivalence relation of an action β of a countable group Γ, then there exists a unique action α of Γ on X inducing the equivalence relation $\mathcal{R}$ and for which p is Γ-equivariant. If β is moreover free, then α is free.* ■

And more generally, for graphings (see [Gab00]):

Lemma 2.4 *If $p : X \to Y$ is a locally one-to-one morphism from $\mathcal{R}$ to $\mathcal{S}$ and if $\mathcal{S}$ is generated by a graphing $\Phi = (\varphi_1, \varphi_2, \cdots, \varphi_j, \cdots)$, then there exists a unique graphing $\tilde{\Phi} = (\tilde{\varphi}_1, \tilde{\varphi}_2, \cdots, \tilde{\varphi}_j, \cdots)$ generating $\mathcal{R}$ and for which p is equivariant (i.e. $\forall \varphi_j$, $x \in \mathrm{Dom}(\tilde{\varphi}_j)$ iff $p(x) \in \mathrm{Dom}(\varphi_j)$ and in this case $p\tilde{\varphi}_j(x) = \varphi_j(p(x))$). In particular, if $\mathcal{S}$ is treeable, then $\mathcal{R}$ is treeable.* ■

We have introduced in [Gab00, Sect. IV.B] the notions of an equivalence relation $\mathcal{R}$ that splits as a **free product** of two (or more) subrelations $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$ and more generally of an equivalence relation $\mathcal{R}$ that splits as a **free product with amalgamation** of two subrelations over a third one $\mathcal{R} = \mathcal{R}_1 \underset{\mathcal{R}_3}{*} \mathcal{R}_2$:

– $\mathcal{R}$ is *generated* by $\mathcal{R}_1$ and $\mathcal{R}_2$ (i.e. $\mathcal{R}$ is the smallest equivalence relation containing $\mathcal{R}_1$ and $\mathcal{R}_2$ as subrelations)

– for (almost) every $2p$-tuple $(x_j)_{j \in \mathbb{Z}/2p\mathbb{Z}}$ that are successively $\mathcal{R}_1$ and $\mathcal{R}_2$-equivalent (i.e. $x_{2i-1} \overset{\mathcal{R}_1}{\sim} x_{2i} \overset{\mathcal{R}_2}{\sim} x_{2i+1}$ $\forall i$), there exists two consecutive ones that are $\mathcal{R}_3$-equivalent (i.e. there exists a j such that $x_j \overset{\mathcal{R}_3}{\sim} x_{j+1}$).

Free products correspond to triviality of $\mathcal{R}_3$, i.e. the second condition becomes: there exists a j such that $x_j = x_{j+1}$.

Standard examples are of course given by free actions of groups that split in a similar way.

Lemma 2.5 *Assume that $\mathcal{R}$ splits as a free product $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$ and that $\mathcal{R}_i$ is produced by a free action of Γ_i. Then the induced action of the free product $\Gamma_1 * \Gamma_2$ produces $\mathcal{R}$ and is free.* ■

Lemma 2.6 *If $p : X \to Y$ is a locally one-to-one morphism from $\mathcal{R}$ to $\mathcal{S}$ and if $\mathcal{S}$ splits as a free product with amalgamation $\mathcal{S} = \mathcal{S}_1 \underset{\mathcal{S}_3}{*} \mathcal{S}_2$, then $\mathcal{R}$ admits a corresponding splitting $\mathcal{R} = \mathcal{R}_1 \underset{\mathcal{R}_3}{*} \mathcal{R}_2$ for which p is a locally one-to-one morphism from $\mathcal{R}_i$ to $\mathcal{S}_i$, for $i = 1, 2, 3$.* ■

Proof of $\mathbf{P}_{\text{ME}}\mathbf{6}^*$ (the proof of $\mathbf{P}_{\text{ME}}\mathbf{6}$ follows by letting $\Gamma_j = \Lambda_j = \{1\}$ for $j \geq 3$): Let $\mathcal{S}_j$, $(j \in \mathbb{N})$, be measured equivalence relations on (Y_j, ν_j), given by orbit equivalent free actions of Γ_j and Λ_j. Take any free p.m.p. action of $*_{j\in\mathbb{N}}\Gamma_j$ on a standard probability measure space (Z, ν) and consider the diagonal action on the product measure space $(X, \mu) := (Z \times \prod_{j\in\mathbb{N}} Y_j, \nu \times \prod_{j\in\mathbb{N}} \nu_j)$ (i.e. $\gamma.x = \gamma.(z, y_1, y_2, \cdots, y_j, \cdots) = (\gamma.z, \gamma.y_1, \gamma.y_2, \cdots, \gamma.y_j, \cdots)$, where $*_{j\in\mathbb{N}}\Gamma_j$ acts on Y_i via the natural homomorphism $*_{j\in\mathbb{N}}\Gamma_j \to \Gamma_i$). This action is free and p.m.p. Denote by $\mathcal{R}$ the equivalence relation it defines on X and by $\mathcal{R}_j$ the equivalence sub-relations defined by the restriction of this action to Γ_j. The measurable map $p_i : X = Z \times \prod_{j\in\mathbb{N}} Y_j \to Y_i$ is Γ_i-equivariant and thus is a locally one-to-one morphism from $\mathcal{R}_i$ to $\mathcal{S}_i$ (ex. 2.2). Lemma 2.3 provides a free action of Λ_i on X producing $\mathcal{R}_i$; and thus an action of the free product $*_{j\in\mathbb{N}}\Lambda_j$ giving $\mathcal{R}$. The free product structure $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2 * \cdots * \mathcal{R}_j * \cdots$ of the equivalence relation, given by the free product structure of the group $*_{j\in\mathbb{N}}\Gamma_j$, (see [Gab00, Déf. IV.9]) ensures that the action of the free product $*_{j\in\mathbb{N}}\Lambda_j$ is free (see Lemma 2.5). We have constructed orbit equivalent free actions of $*_{j\in\mathbb{N}}\Gamma_j$ and $*_{j\in\mathbb{N}}\Lambda_j$. ■

Proof of $\mathbf{P}_{\text{ME}}\mathbf{7}$: While following the lines of the preceding proof, forget about the groups $\Lambda_j = \mathbf{F}_{p_j}$ and just retain that $\mathcal{S}_j$, generated by a free action of Γ_j, is treeable. The treeings of $\mathcal{R}_j$ $(j \in \mathbb{N})$, delivered by lemma 2.4, together define a graphing of $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2 * \cdots * \mathcal{R}_j * \cdots$ that happen to be a treeing due to the free product structure of $\mathcal{R}$. Then apply $\mathbf{P}_{\text{ME}}\mathbf{8}$. ■

$\mathbf{P}_{\text{ME}}\mathbf{9}$ The family of groups ME to some free group is stable under taking subgroups.

This follows from [Gab00, Th. 5] and $\mathbf{P}_{\text{ME}}\mathbf{8}$.

2.3 Some Invariants

$\mathbf{P}_{\text{ME}}\mathbf{10}$ (Amenability) The ME class of $\mathbb{Z}$ consists in all infinite amenable groups. Moreover, any ergodic action of any two infinite amenable groups are OE and also SOE for any compression constant ι [OW80].

This property is sometimes interpreted as a kind of *elasticity*. This in wide contrast with the *rigidity* phenomenon described by Zimmer and Furman [Zim84, Fur99a, Fur99b] for lattice in higher rank semi-simple Lie groups see also $\mathbf{P}_{\text{ME}}\mathbf{19}$.

Recall that Kazhdan property T is often considered as opposite to amenability. More recently, the general opinion switched, after M. Gromov to oppose Kazhdan property T with the greater class of groups with Haagerup property (= a-T-menability, see [CCJJV01]).

$\mathbf{P}_{\text{ME}}\mathbf{11}$ (Kazhdan Property T) Kazhdan property T is a ME invariant [Fur99a].

Lattices of $\mathrm{Sp}(n, 1)$ are not ME to lattices of $\mathrm{SU}(p, 1)$ or $\mathrm{SO}(p, 1)$. We already knew from [AS90] that groups with property T are not ME to groups that split non trivially as free product with amalgamation or HNN-extension.

$\mathbf{P}_{\text{ME}}\mathbf{12}$ (Cowling-Haagerup constant) Cowling-Haagerup constant $\Lambda(\Gamma)$ is a ME invariant [CZ89, Jol01].

The Cowling-Haagerup constant $\Lambda(\Gamma)$ of the group Γ is the infimum of the constants C such that the Fourier algebra $\mathcal{A}(\Gamma)$ admits an approximate unit bounded by C in the multipliers norm. Jolissaint in [Jol01] extends the above result of Cowling-Zimmer [CZ89] from OE to ME. Lattices in various $\mathrm{Sp}(n, 1)$ are not ME ($\Lambda(\Gamma) = 2n-1$). Recall that $\Lambda(\Gamma) = 1$ implies that Γ has Haagerup property, but that it is unknown whether the converse holds (Cowling conjecture).

$\mathbf{P}_{\text{ME}}\mathbf{13}$ (a-T-menability) Haagerup property is a ME invariant

This result was obtained independently by several people, including Jolissaint[Jol01], Popa [Pop01] and Shalom (personal communication).

Two kinds of numerical invariants of countable groups are studied in [Gab00, Gab02] in connection with ME: Cost $\mathcal{C}(\Gamma) \in [0, \infty]$ and ℓ^2 Betti numbers $\beta_0(\Gamma), \beta_1(\Gamma), \cdots, \beta_i(\Gamma), \cdots$.

$\mathbf{P}_{\text{ME}}\mathbf{14}$ (Cost) Having cost $= 1$ (resp. $= \infty$) is a ME invariant [Gab00].

For all the groups for which the computation has been carried out until now, $\mathcal{C}(\Gamma) = \beta_1(\Gamma) - \beta_0(\Gamma) + 1$.

$\mathbf{P}_{\mathrm{ME}}$15 (ℓ^2 Betti Numbers) ME groups Γ and Λ have proportional ℓ^2 Betti numbers: there is $\lambda > 0$ such that for every $n \in \mathbb{N}$, $\beta_n(\Gamma) = \lambda\beta_n(\Lambda)$ [Gab02].

This gives[10] the splitting of groups ME to free groups into four ME-classes according to whether $\beta_0 \neq 0, \beta_1 = 0$ (finite groups) $\beta_0 = 0,\ \beta_1 = 0$ (amenable ones), $\beta_1 \in (0, \infty)$ (class of $\mathbf{F}_p$, $1 < p < \infty$) or $\beta_1 = \infty$ (class of $\mathbf{F}_\infty$). Moreover (see **$\mathbf{P}_{\mathrm{ME}}$15*** below), if Γ is ME to a free group and $\beta_1(\Gamma) = p - 1$ is an integer, then $\Gamma \overset{\mathrm{OE}}{\sim} \mathbf{F}_p$.
Also, $\mathbf{F}_{p_1} \times \mathbf{F}_{p_2} \times \cdots \times \mathbf{F}_{p_j} \overset{\mathrm{ME}}{\sim} \mathbf{F}_{q_1} \times \mathbf{F}_{q_2} \times \cdots \times \mathbf{F}_{q_k}$ iff $j = k$ (with $\infty > p_i, q_i \geq 2$)[11], and in this case they are commensurable.
In the following three claims, the only non zero ℓ^2 Betti number is in the middle dimension of the associated symmetric space:
If two lattices of $\mathrm{Sp}(n, 1)$ and $\mathrm{Sp}(p, 1)$ are ME, then $p = n$ (see also **$\mathbf{P}_{\mathrm{ME}}$12**).
If two lattices of $\mathrm{SU}(n, 1)$ and $\mathrm{SU}(p, 1)$ are ME, then $p = n$.
If two lattices of $\mathrm{SO}(2n, 1)$ and $\mathrm{SO}(2p, 1)$ are ME, then $p = n$.
If it would happen that lattices in $\mathrm{SU}(n, 1)$ and $\mathrm{SO}(p, 1)$ are ME, then $p = 2n$, but we suspect that lattices of $\mathrm{SU}(n, 1)$ and $\mathrm{SO}(2n, 1)$ are never ME.

Thanks to the great flexibility in constructing groups with prescribed ℓ^2 Betti numbers (see [CG86]), it follows from **$\mathbf{P}_{\mathrm{ME}}$15** that there are uncountably many different ME classes.

$\mathbf{P}_{\mathrm{ME}}$15* If one of the $\beta_n(\Gamma)$ is different from $0, \infty$, and if $\Gamma \overset{\mathrm{ME}}{\sim} \Lambda$, with index $\iota = [\Gamma : \Lambda]$, then $\iota = \frac{1}{\lambda}$ is imposed. In this case, $\Gamma \overset{\mathrm{OE}}{\sim} \Lambda$ iff $\lambda = 1$ (see **$\mathbf{P}_{\mathrm{ME}}$5**).

$\mathbf{P}_{\mathrm{ME}}$15** An infinite group with $\beta_1 = 0$ is ME to a free group if and only if it is amenable.

An amalgamated free product $\mathbb{Z} \underset{\mathbb{Z}}{*} \mathbb{Z}$ is ME to a free group if and only if it is amenable, i.e. the common $\mathbb{Z}$ coincides with one the components or injects as a subgroup of index 2 in both.

Remark 2.7 *If $\Gamma = \Gamma_1 *_D \Gamma_2$ is ME to a free group and nonamenable, with D infinite, then $\beta_1(\Gamma_1) + \beta_1(\Gamma_2) - \beta_1(D) > 0$. To see it, first observe that $\beta_1(\Gamma) > 0$ and $\beta_2(\Gamma) = 0$. The Mayer-Vietoris' exact sequence then gives the estimate. For instance, amalgamated free products $\mathbf{F}_2 *_{\mathbf{F}_3} \mathbf{F}_2$ are not ME to $\mathbf{F}_2$.*

$\mathbf{P}_{\mathrm{ME}}$16 Ergodic dimension and approximate ergodic dimension are ME invariants (see [Gab02, Sect. 5.3 and Prop. 6.5]).

Both invariants are useful when all the ℓ^2 Betti numbers are 0. For instance, very little is known about the ME classification of lattices in $\mathrm{SO}(m, 1)$ for m odd. However, a reasoning on ergodic dimension shows that: If two lattices in $\mathrm{SO}(2n + 1, 1)$ and $\mathrm{SO}(2p + 1, 1)$ are ME, with $p \leq n$, then $n \leq 2p$ [Gab02, Cor. 6.9]. Similarly, such a group as the following Γ has approximate dimension j, and thus $\Gamma = \mathbf{F}_{p_1} \times \mathbf{F}_{p_2} \times \cdots \times \mathbf{F}_{p_j} \times \mathbb{Z} \overset{\mathrm{ME}}{\sim} \mathbf{F}_{q_1} \times \mathbf{F}_{q_2} \times \cdots \times \mathbf{F}_{q_k} \times \mathbb{Z}$ iff $j = k$ (all the $p_i, q_i \geq 2$). In this case they are commensurable.

$\mathbf{P}_{\mathrm{ME}}$17 The set $I_{\mathrm{ME}}(\Gamma)$ of all indices of ME couplings of Γ with itself [Gab00b, Sect. 2.2] is a ME invariant.

Since non-ergodic ME couplings are allowed, $I_{\mathrm{ME}}(\Gamma)$ is a convex subset of $\mathbb{R}_+^*$. If one of the ℓ^2 Betti numbers $\beta_n(\Gamma) \neq 0, \infty$, then $I_{\mathrm{ME}}(\Gamma) = \{1\}$. When Γ is the direct product of an infinite amenable group with any group then $I_{\mathrm{ME}}(\Gamma) = \mathbb{R}_+^*$.

Question 2.8 *Are there groups Γ such that the set of all indices of **ergodic** ME couplings of Γ with itself is discrete $\neq \{1\}$?*

[10] For $p \neq 0$, $\beta_1(\mathbf{F}_p) = p - 1$ and all the other β_i are zero
[11] The only non zero ℓ^2 Betti number is the j-th, resp. the k-th.

Recall that the **fundamental group** of a p.m.p. ergodic Γ-action α on (X,μ) is the subgroup of $\mathbb{R}_+^*$ generated by the set of measures $\mu(A)$ for those $A \subset X$ such that the restriction of $\mathcal{R}_\alpha$ to A is isomorphic with $\mathcal{R}_\alpha$ (equivalently the set of compression constants of α with itself). To compare with, I_{ME} is related to the bigger set of measures $\mu(A)$ for those $A \subset X$ such that the restriction of $\mathcal{R}_\alpha$ to A may be produced by *some* free action of Γ.

In [MS02], Monod-Shalom introduce the class $\mathcal{C}_{\mathrm{reg}}$ of all groups Γ such that $\mathrm{H}_b^2(\Gamma, \ell^2(\Gamma))$ is non-zero and the larger class $\mathcal{C}$ of all groups Γ admitting a mixing unitary representation π such that $\mathrm{H}_b^2(\Gamma,\pi)$ is non-zero. Non amenable free products and non-elementary subgroups of hyperbolic groups all belong to the class $\mathcal{C}_{\mathrm{reg}} \subset \mathcal{C}$ [MS02, MMS04]. A group in the class $\mathcal{C}$ has finite center and is not a direct product of two infinite groups. Also, being in the class $\mathcal{C}$ passes to normal subgroups.

$\mathbf{P}_{\mathrm{ME}}$18 Belonging to the class $\mathcal{C}_{\mathrm{reg}}$ (resp. $\mathcal{C}$) is a ME invariant [MS02].

For instance, a normal subgroup Λ of a group Γ ME to a non elementary hyperbolic group has finite center and is not a direct product of two infinite groups. This statement for $\Lambda = \Gamma$ were obtained by S. Adams [Ada94, Ada95]. Lattices in $\mathrm{SO}(2n,1)$ are not ME to lattices in $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \cdots \times \mathrm{SL}(2,\mathbb{R})$ (follows from [Ada94]). Observe that hyperbolicity itself is not preserved since $\mathbb{Z}^2 * \mathbb{Z}^2 \overset{\mathrm{OE}}{\sim} \mathbf{F}_2$.

2.4 Some "Rigidity" Results

The following result is a spectacular achievement of the rigidity phenomena attached to higher rank lattices, after Margulis' super-rigidity and Zimmer's cocycle super-rigidity [Zim84]. In fact, Zimmer obtained the similar result with the additional assumption that the mysterious group admits a linear representation with infinite image [Zim84, Zim91].

$\mathbf{P}_{\mathrm{ME}}$19 (ME Rigidity – Higher Rank Lattices) Any countable group which is ME to a lattice in a connected simple Lie group G with finite center and real rank ≥ 2, is commensurable up to finite kernel with a lattice in G [Fur99a].

The prototype of such Lie groups are $G = \mathrm{SL}(n,\mathbb{R})$ or $\mathrm{SO}(p,q)$, for $n-1, p, q \geq 2$.

Direct products sometimes appear as analogue to higher rank lattices. Here, bounded cohomology and belonging to the class $\mathcal{C}$ (see **$\mathbf{P}_{\mathrm{ME}}$18**) rigidifies the situation:

$\mathbf{P}_{\mathrm{ME}}$20 (ME Rigidity – Products) If $\Gamma_1 \times \cdots \times \Gamma_n \overset{\mathrm{ME}}{\sim} \Lambda_1 \times \cdots \times \Lambda_p$ are ME products of (non-trivial) torsion-free countable groups with $n \leq p$, where all the Γ_i's are in the class $\mathcal{C}$, then $n = p$ and after permutation of the indices $\Gamma_i \overset{\mathrm{ME}}{\sim} \Lambda_i$ for all i [MS02, Th. 1.14].

For direct products of free groups, with $p_i, q_i \geq 2$ the value $p_i, q_i = \infty$ being now allowed, $\mathbf{F}_{p_1} \times \mathbf{F}_{p_2} \times \cdots \times \mathbf{F}_{p_j} \overset{\mathrm{ME}}{\sim} \mathbf{F}_{q_1} \times \mathbf{F}_{q_2} \times \cdots \times \mathbf{F}_{q_k}$ iff $j = k$ and the number of times ∞ occurs is the same; and in this case they are commensurable (compare with **$\mathbf{P}_{\mathrm{ME}}$15** which gives only $j = k$).

$\mathbf{P}_{\mathrm{ME}}$21 (ME Rigidity – Quotients by Radicals) Let $M \triangleleft \Gamma$, $N \triangleleft \Lambda$ be amenable normal subgroups of Γ, Λ such that Γ/M and Λ/N are in the class $\mathcal{C}$ and torsion-free. If $\Gamma \overset{\mathrm{ME}}{\sim} \Lambda$, then $\Gamma/M \overset{\mathrm{ME}}{\sim} \Lambda/N$ [MS02, Th. 1.15].

2.5 Application

Consider Lie groups of the form $\prod_{i\in I} \mathrm{Sp}(m_i,1) \times \prod_{j\in J} \mathrm{SU}(n_j,1) \times \prod_{k\in K} \mathrm{SO}(p_k,1)$, where I, J, K are finite sets and $m_i, n_j, p_k \geq 2$. Let Γ and Γ' be lattices of such Lie groups G and G'. Assume they are ME. Each of them is ME to a product of torsion free cocompact (thus hyperbolic) lattices of the factors $\prod_{t\in I,J,K} \Gamma_t \overset{\mathrm{ME}}{\sim} \Gamma \overset{\mathrm{ME}}{\sim} \Gamma' \overset{\mathrm{ME}}{\sim} \prod_{t'\in I',J',K'} \Gamma_{t'}$. By **$\mathbf{P}_{\mathrm{ME}}$20**, the pieces correspond under ME, after reordering. It follows that the number of factors coincide. Thanks to property T (**$\mathbf{P}_{\mathrm{ME}}$11**), the pieces in the $\mathrm{Sp}(\cdot,1)$'s are ME. Moreover, **$\mathbf{P}_{\mathrm{ME}}$12** or the examination of the ℓ^2 Betti numbers (**$\mathbf{P}_{\mathrm{ME}}$15**) ensures that the sets $\{m_i : i \in I\}$ and $\{m_{i'} : i' \in I'\}$ are the same. Similarly, the number of odd p_k, $p_{k'}$ are the same and the set of n_j and even p_k on the one hand, coincide with the set $n_{j'}$ and even $p_{k'}$ on the other hand. For instance if K, K' are empty, then $\Gamma \overset{\mathrm{ME}}{\sim} \Gamma'$ implies that G, G' are isomorphic. And similarly if J, J' are empty and all the p_k are even.

3 A Construction of Measure Equivalent Groups

3.1 Measure Free-Factor

We first introduce a measure theoretic notion analogue to free factors in group theory.

Definition 3.1 *A subgroup $\Lambda < \Gamma$ is called a* ***measure free-factor of*** Γ *if* Γ *admits a free p.m.p. action* α *such that* $\mathcal{R}_{\alpha(\Lambda)}$ *is freely supplemented in* $\mathcal{R}_{\alpha(\Gamma)}$*; i.e. there exists a subrelation* $\mathcal{S} < \mathcal{R}$ *such that* $\mathcal{R} = \mathcal{R}_{\alpha(\Lambda)} * \mathcal{S}$.

This notion is clearly invariant under the automorphism group of Γ.
Example: If Λ is a free-factor of Γ, i.e. if Γ decomposes as a free product $\Gamma = \Lambda * \Lambda'$, then Λ is a measure free-factor. This definition, motivated by Theorem 3.6, of course challenges to exhibit non-trivial examples. This is done in the following:

Theorem 3.2 *The cyclic subgroup C generated by the product of commutators $\kappa := \prod_{i=1}^{i=p} [a_i, b_i]$ is a measure free-factor in the free group $\mathbf{F}_{2p} = \langle a_1, \cdots, a_p, b_1, \cdots, b_p\rangle$.*

Since it vanishes in the abelianization, C is not a free factor in the usual sense.

Proof of Th. 3.2: This result is obtained by exhibiting a free p.m.p. action σ of $\mathbf{F}_{2p}$ on an (X, μ) whose equivalence relation is also generated by a **treeing** $\Phi = (\varphi_\gamma)_{\gamma\in\{\kappa,a_1,\cdots,a_p,b_1,\cdots,b_p\}}$ made of an automorphism $\varphi_\kappa = \sigma(\kappa)$, defined on the whole of X, and of partially defined isomorphims φ_γ that are restrictions of the generators γ.

It follows from the classification of surfaces that the free group $\mathbf{F}_{2p}$ is isomorphic with the fundamental group of the oriented surface of genus p with one boundary component (supporting the base point), via an isomorphism sending the product of commutators κ to the boundary curve and the generators to simple closed curves that are disjoint up to the base point.

The Cayley graph $\mathcal{G}$ and the universal cover of the Cayley complex associated to the (non-free) presentation $\langle \kappa, a_1, \cdots, a_p, b_1, \cdots, b_p \mid \kappa := \prod_{i=1}^{i=p}[a_i, b_i]\rangle$ are thus planar, with one orbit of bounded 2-cells, and boundary edges labelled κ. Its dual graph $\mathcal{G}^*$ is a regular tree of valency $2p$ which does not cross the edges labelled κ. It is equipped with a natural vertex-transitive and free action of $\mathbf{F}_{2p}$ which thus may be seen as a "standard" Cayley tree for the free group $\mathbf{F}_{2p}$. Denote by E^* its edge set and by $\mathcal{F}^*$ the subset of $\{0, 1\}^{E^*}$, corresponding to the (characteristic functions of the) forests, whose connected components are infinite trees with one end. The subset $\mathcal{F}^*$ is equipped with a natural $\mathbf{F}_{2p}$-action.

Proposition 3.3 *The subset $\mathcal{F}^*$ supports an $\mathbf{F}_{2p}$-invariant probability measure ν.*

This is an immediate consequence of [Häg98, Cor 2.6 or Th. 4.2]. For completeness, we will give an elementary proof below; now we continue with the proof of Th. 3.2.

Denote the edge set of $\mathcal{G}$ by E. A forest $f^* \in \mathcal{F}^*$ defines a subgraph $H(f^*) \in \{0, 1\}^E$ by removing the edges of E that f^* crosses. The end-points of an edge removed (thanks to an edge $e^* \in f^*$) are connected in $H(f^*)$ by the finite planar path surrounding the finite bush of $f^* \setminus \{e^*\}$. It follows that $H(f^*)$ is a connected subgraph of $\mathcal{G}$ containing all the edge labelled κ (since the dual $\mathcal{G}^*$ does not cross the edges of $\mathcal{G}$ labelled κ). By planarity and since all the connected components of f^* are infinite, $H(f^*)$ has no cycle: it is a tree.

The equivariant map $H : \mathcal{F}^* \to \{0, 1\}^E$ thus pushes the measure ν to an $\mathbf{F}_{2p}$-invariant probability measure μ, supported on the set of (connected) subtrees of $\mathcal{G}$, containing all the edges labelled κ.

Making the action free if necessary (by considering the diagonal action on $(X, \mu) = (X', \mu') \times (\{0, 1\}^E, \mu)$ for some free p.m.p. $\mathbf{F}_{2p}$-space (X', μ'), see [Gab04, 1.3.c]), and according, for example, to [Gab04, 1.3.f] this is equivalent with the existence of the required graphing on (X, μ). More precisely, the natural $\mathbf{F}_{2p}$-equivariant map $\pi : X \to \{0, 1\}^E$ forgetting the first coordinate sends $x \in X$ to the subset $\pi(x) \subset E$ of the edge set of $\mathcal{G}$. For $\gamma \in \{\kappa, a_1, \cdots, a_p, b_1, \cdots, b_p\}$ define the Borel subset A_γ of those $x \in X$ for which the edge $[id, \gamma]$ of $\mathcal{G}$ belongs to $\pi(x)$ and consider the partially defined isomorphism $\varphi_\gamma := \gamma^{-1}_{|A_\gamma}$. The graphing $\Phi = (\varphi_\kappa, \varphi_{a_1}, \cdots, \varphi_{a_p}, \varphi_{b_1}, \cdots, \varphi_{b_p})$ matches the conditions: The graph associated with the orbit of x is isomorphic with the subgraph defined by $\pi(x)$, and thus it is a tree for μ-almost all $x \in X$ (Φ is a treeing), Φ generates the same equivalence relation as the $\mathbf{F}_2$-action since the graph $\pi(x)$ is connected, and $A_\kappa = X$ (up to a μ-null set), i.e. φ_κ is defined almost everywhere. ∎

Proof of Prop. 3.3: Let (Y, m) be a Borel standard probability space with a partition $Y = \coprod_{i\in\mathbb{N}}(U_i \coprod V_i)$, where $m(U_i) = m(V_i) = \frac{1}{2^{i+1}}$. Let $g_1, \cdots, g_{2p}$ be a family of p.m.p. automorphisms of (Y, m) such that $g_1(U_i) = U_{i+1} \cup V_{i+1}$ and $g_2(V_i) = U_{i+1} \cup V_{i+1}$. They define a p.m.p. (non-free) action α of $\mathbf{F}_{2p} = \langle \gamma_1, \cdots, \gamma_{2p}\rangle$ on Y. Denote $A_1 = \coprod_{i\in\mathbb{N}} U_i$ and $A_2 = \coprod_{i\in\mathbb{N}} V_i$. The restrictions of the automorphisms $\varphi_1 = g_{1|A_1}$ and $\varphi_2 = g_{2|A_2}$ define a graphing, which is a treeing whose associated graphs are all one-ended trees, and which generates a subrelation of $\mathcal{R}_\alpha$. If e_i^* denotes the edge $[id, \gamma_i^{-1} id]$ in the Cayley graph of $\mathbf{F}_{2p}$ associated with the generating family (γ_i), then the conditions $\pi(y)(e_1^*) = 1$ iff $y \in A_1$, $\pi(y)(e_2^*) = 1$ iff $y \in A_2$, and $\pi(y)(e_i^*) = 0$ for the other i's, extend by $\mathbf{F}_{2p}$-equivariance to a map $Y \to \mathcal{F}^*$. Pushing forward the invariant measure m delivers an instance of the required measure ν. ■

Proposition 3.4 *If $\Lambda < \Gamma$ is a measure free-factor of Γ and Γ' is any countable group, then Λ as well as $\Lambda * \Gamma'$ is a measure free-factor of $\Gamma * \Gamma'$.*

Proof: Consider an action of Γ, that witnesses the measure free-factor condition. Look at it as an action of $\Gamma * \Gamma'$ via the obvious map $\Gamma * \Gamma' \to \Gamma$, and consider the direct product action α with any free p.m.p. action of $\Gamma * \Gamma'$. The lifting properties of locally one-to-one morphisms (see Lemma 2.6) applied to the restriction of α to Γ show that α matches the required condition. ■

Corollary 3.5 *Let Σ be an oriented surface with $r > 0$ boundary components. Let $\pi_1(\Sigma, *) \simeq \mathbf{F}_q$ be its fundamental group and $\Lambda \simeq \mathbf{F}_r$ the subgroup generated by the boundary components. Then Λ is a measure free-factor of $\mathbf{F}_q$.*

Proof: $\pi_1(\Sigma, *)$ admits a free generating set $a_i, b_i, \kappa_1, \kappa_2, \cdots \kappa_{r-1}$ for which $\prod_{i=1}^p [a_i, b_i], \kappa_1, \kappa_2, \cdots \kappa_{r-1}$ freely generates Λ. In fact, the product of the boundary components $\kappa_1, \kappa_2, \cdots \kappa_r$ equals $\prod_{i=1}^p [a_i, b_i]$. ■

3.2 Application

Theorem 3.6 *If Γ is ME to the free group $\mathbf{F}_p$ ($p = 2$ or $p = \infty$) and admits a measure free-factor subgroup Λ, then the iterated amalgamated free product $\overset{n}{\underset{\Lambda}{*}} \Gamma = \Gamma \underset{\Lambda}{*} \Gamma \underset{\Lambda}{*} \cdots \underset{\Lambda}{*} \Gamma$ is ME to $\mathbf{F}_p$, and the infinite iterated amalgamated free product $\overset{\infty}{\underset{\Lambda}{*}} \Gamma = \Gamma \underset{\Lambda}{*} \Gamma \underset{\Lambda}{*} \cdots \underset{\Lambda}{*} \Gamma \underset{\Lambda}{*} \cdots$ is ME to $\mathbf{F}_\infty$. Moreover, Λ remains a measure free-factor in the resulting group.*

Applied with Th. 3.2, this gives Th. 1.1 of the introduction:

Corollary 3.7 *For each $n \in \mathbb{N}$, the iterated amalgamated free product $\overset{n}{\underset{C}{*}} \mathbf{F}_{2p}$ is measure equivalent to the free group $\mathbf{F}_2$. The infinite amalgamated free product: $\overset{\infty}{\underset{C}{*}} \mathbf{F}_{2p}$ is measure equivalent to the free group $\mathbf{F}_\infty$. Moreover, C is a measure free-factor in the resulting group.*

Proof of Th. 3.6: The diagonal action β on the product measure space X of the two actions witnessing each condition on Γ, besides being free and p.m.p.,both is treeable (Lemma 2.4) and admits a splitting as $\mathcal{S} = \mathcal{S}_{\beta(\Lambda)} * \mathcal{S}'$ (the splitting condition lifts, see Lemma 2.6). It thus admits a treeing that splits into two pieces $\Phi = \Phi_\Lambda \vee \Phi'$ where Φ_Λ generates $\mathcal{S}_{\beta(\Lambda)}$ and Φ' generates $\mathcal{S}'$ ($\mathcal{S}'$ is treeable by [Gab00, Th. 5]). Consider

(1) the natural surjective homomorphism $\pi : \overset{n}{\underset{\Lambda}{*}} \Gamma \to \Gamma$, defined by the identity on each copy of Γ,

(2) the (non free) p.m.p. action of $\overset{n}{\underset{\Lambda}{*}} \Gamma$ on X via $\beta \circ \pi$,

(3) any free p.m.p. action of $\overset{n}{\underset{\Lambda}{*}} \Gamma$ on a standard Borel space Y,

(4) the diagonal $\overset{n}{\underset{\Lambda}{*}} \Gamma$-action α^n on the product measure space $Z := X \times Y$ (it is p.m.p. and free),

(5) the $\overset{n}{\underset{\Lambda}{*}} \Gamma$-equivariant projection to the first coordinate $\Pi : X \times Y \to X$.

For each $i = 1, \cdots, n$, denote by $\mathcal{R}_i := \mathcal{R}_{\alpha^n(\Gamma_i)}$ the equivalence relation generated by the restriction of the action α^n to the i-th copy Γ_i of Γ (resp. $\mathcal{R}_\Lambda := \mathcal{R}_{\alpha^n(\Lambda)}$ for the restriction to Λ). The morphism Π is locally one-to-one when restricted to $\mathcal{R}_i$ (Ex. 2.2). Denote by $\tilde{\Phi}_i = \tilde{\Phi}_{i\,\Lambda} \vee \tilde{\Phi}'_i$ the graphing of $\mathcal{R}_i$ given from Φ by the lifting lemma 2.4. More concretely: Up to subdividing the domains, each generator $\varphi : A_\varphi \to B_\varphi$ of the treeing Φ is the restriction of the β-action of one element γ_φ of Γ. Each σ_j of the n homomorphic sections $\sigma_j : \Gamma \to \underset{\Lambda}{\overset{n}{*}} \Gamma$ of π delivers a graphing on Z of the following form: $\tilde{\Phi}_j = \tilde{\Phi}_{j\Lambda} \vee \tilde{\Phi}'_j$, where each element $\tilde{\varphi} : \tilde{A_\varphi} \to \tilde{B_\varphi}$ is defined from the element $\varphi A_\varphi \to B_\varphi$ of Φ as the restriction of the α^n-action of $\sigma_j(\gamma_\varphi)$ to $\tilde{A_\varphi} := \Pi^{-1}(A_\varphi)$.

By the coincidence of the copies of Λ, it follows that all the $\tilde{\Phi}_{i\,\Lambda}$ coincide, so that $\tilde{\Phi} = \tilde{\Phi}_{1\,\Lambda} \vee \tilde{\Phi}'_1 \vee \tilde{\Phi}'_2 \cdots \vee \tilde{\Phi}'_n$ still generates $\mathcal{R}_{\alpha^n}$. Since it lifts a treeing, each $\tilde{\Phi}'_i$ (as well as $\tilde{\Phi}_{1\,\Lambda}$) is itself a treeing. The structure of free product with amalgamation of $\mathcal{R} = \mathcal{R}_1 \underset{\mathcal{R}_\Lambda}{*} \mathcal{R}_2 \underset{\mathcal{R}_\Lambda}{*} \cdots \underset{\mathcal{R}_\Lambda}{*} \mathcal{R}_n$ ensures that $\tilde{\Phi}$ is globally a treeing: $\underset{\Lambda}{\overset{n}{*}} \Gamma$ admits a treeable free action, and $\mathcal{R}_\Lambda$ is a free-factor. A similar reasoning give the similar result for $\underset{\Lambda}{\overset{\infty}{*}} \Gamma$. These groups are consequently ME to a free group by $\mathbf{P}_{\mathrm{ME}}\mathbf{8}$. The precise ME-class is determined by the first ℓ^2 Betti number (by $\mathbf{P}_{\mathrm{ME}}\mathbf{15}$). ■

Remark 3.8 *The treeing Φ produced in the proof of theorem 3.2 cannot be a quasi-isometry, i.e. there is some generator $\gamma \in \{a_1, \cdots, a_p, b_1, \cdots, b_p\}$ for which the length of the Φ-words sending x to $\gamma(x)$ is not bounded a.s. For otherwise, the graphing produced in the proof of theorem 3.6 for $n = 2$ would equip each equivalence class with a graph structure (tree) quasi-isometric to the group itself $\mathbf{F}_{2p} *_C \mathbf{F}_{2p}$ (the fundamental group of a closed compact surface, without boundary). It follows from the particular form of Φ that the surface group $\langle a_1, \cdots, a_p, b_1, \cdots, b_p\rangle *_{C=C'} \langle a'_1, \cdots, a'_p, b'_1, \cdots, b'_p\rangle$ has a treeing made of C and restrictions of the "natural" generators a_i, b_i, a'_i, b'_i.*

Question 3.9 *What are all the measure free-factors of the free group $\mathbf{F}_2$?*

Observe that in a group Γ ME to a free group, a cyclic subgroup D strictly contained in a greater cyclic subgroup E cannot be a measure free-factor by Th. 3.6, since iterated amalgamated free product $\Gamma \underset{D}{*} \Gamma \underset{D}{*} \Gamma$ is not ME to a free group: it contains the nonamenable subgroup $E \underset{D}{*} E \underset{D}{*} E$ for which $\beta_1 = 0$, then use $\mathbf{P}_{\mathrm{ME}}\mathbf{15}^{**}$ and $\mathbf{P}_{\mathrm{ME}}\mathbf{9}$.

Question 3.10 *It is known that if an amalgamated free product $\mathbf{F}_p *_{\mathbb{Z}} \mathbf{F}_q$ happens to be a free group, then $\mathbb{Z}$ is a free factor in one of $\mathbf{F}_p$ or $\mathbf{F}_q$ (see [BF94, Ex. 4.2]). Is it true that similarly if $\mathbf{F}_p *_{\mathbb{Z}} \mathbf{F}_q \overset{\mathrm{ME}}{\sim} \mathbf{F}_2$ then $\mathbb{Z}$ is a measure free-factor in $\mathbf{F}_p$ or $\mathbf{F}_q$?*

3.3 Orbit Equivalence of Pairs

We will now make use of the following refined notion of Orbit Equivalence.

Definition 3.11 *Consider for $i = 1, 2$ a group Γ_i and a subgroup Γ_i^0. An **orbit equivalence of the pairs** $(\Gamma_1^0 < \Gamma_1)$ and $(\Gamma_2^0 < \Gamma_2)$ is the data of two p.m.p. actions α_1 and α_2 of Γ_1 and Γ_2 on the probability measure standard Borel space (X, μ), that generate the same equivalence relation ($\mathcal{R}_{\alpha_1(\Gamma_1)} = \mathcal{R}_{\alpha_2(\Gamma_2)}$) and such that the restrictions of α_1 to Γ_1^0 and of α_2 to Γ_2^0 also define a common sub-relation: ($\mathcal{R}_{\alpha_1(\Gamma_1^0)} = \mathcal{R}_{\alpha_2(\Gamma_2^0)}$). The existence of such an equivalence of pairs is denoted by:*

$$(\Gamma_1^0 < \Gamma_1) \overset{\mathrm{OE}}{\sim} (\Gamma_2^0 < \Gamma_2)$$

Definition 3.12 *It is a **strong orbit equivalence of the pairs** if moreover the sub-groups Γ_1^0 and Γ_2^0 are isomorphic, via an isomorphism $\phi : \Gamma_1^0 \to \Gamma_2^0$ that turns the actions conjugate: $\forall \gamma \in \Gamma_1^0, \alpha_1(\gamma) = \alpha_2(\phi(\gamma))$ in* $\mathrm{Aut}(X, \mu)$. *The existence of such an equivalence of pairs is denoted by:*

$$(\Gamma_1^0 < \Gamma_1) \underset{st}{\overset{\mathrm{OE}}{\sim}} (\Gamma_2^0 < \Gamma_2)$$

Measure free-factors naturally lead to orbit equivalence of pairs:

Theorem 3.13 *Let Γ be a group ME to the free group $\mathbf{F}_2$ and let Γ_0 be a subgroup such that $\beta_1(\Gamma)-\beta_1(\Gamma_0)=q$ is an integer. If Γ_0 is a measure free-factor of Γ, then there exists a strong orbit equivalence of pairs $(\Gamma_0 < \Gamma)\underset{st}{\overset{\mathrm{OE}}{\sim}}(\Gamma_0 < \Gamma_0 * \mathbf{F}_q)$.*

Proof of Th. 3.13: Like in the proof of Th. 3.6, consider (by taking a diagonal action of two actions witnessing both properties of Γ) a free p.m.p. action β of Γ that both is treeable and admits a splitting $\mathcal{S}_\beta = \mathcal{S}_{\beta(\Gamma_0)} * \mathcal{S}'$. Up to considering the ergodic decomposition, β may be assumed to be ergodic. The subrelation $\mathcal{S}_{\beta(\Gamma_0)}$ admits a treeing Φ and $\mathcal{S}'$ admits a treeing Ψ of cost q [Gab02, Cor. 3.16, Cor. 3.23]). A recursive use of Theorem 28.3 of [KM04] (due to Hjorth) gives that $\mathcal{S}'$ may be replaced in the above decomposition by $\mathcal{S}''$ produced by a free action of the free group $\mathbf{F}_q$. The free product decomposition of $\mathcal{S}_\beta = \mathcal{S}_{\beta(\Gamma_0)} * \mathcal{S}''$ asserts that the action of Γ_0 fits well to produce the pair $\mathcal{S}_{\beta(\Gamma_0)} < \mathcal{S}_\beta$ by a free action of $\Gamma_0 * \mathbf{F}_q$. The resulting orbit equivalence of pairs is strong by construction. ■

Example 3.14 *If C is the cyclic subgroup generated by the product of commutators κ in $\mathbf{F}_{2p}$ then $(C < \mathbf{F}_{2p})\underset{st}{\overset{\mathrm{OE}}{\sim}}(\mathbb{Z} < \mathbb{Z} * \mathbf{F}_{2p-1})$.*

Example 3.15 *If Λ is the subgroup of $\mathbf{F}_q$ as in Corollary 3.5, then $(\mathbf{F}_r \simeq \Lambda < \mathbf{F}_q)\underset{st}{\overset{\mathrm{OE}}{\sim}}(\mathbf{F}_r < \mathbf{F}_r * \mathbf{F}_{q-r})$.*

Remark 3.16 *Observe that in a free product situation $(\Gamma^0 < \Gamma_1)\overset{\mathrm{OE}}{\sim}(\Gamma^0 < \Gamma^0 * \Gamma_2')$, the orbit equivalence of pairs may always be assumed strong, by changing the Γ^0-action in the free product.*

Theorem 3.17 *If $(\Gamma_1^0 < \Gamma_1)\underset{st}{\overset{\mathrm{OE}}{\sim}}(\Gamma_2^0 < \Gamma_2)$ are two pairs of groups admitting a strong orbit equivalence of pairs and G is a countable group with a subgroup Γ^0 isomorphic with Γ_1^0 and Γ_2^0, then the following pairs admit a strong orbit equivalence of pairs:*

$$(G < G \underset{\Gamma^0=\Gamma_1^0}{*} \Gamma_1)\underset{st}{\overset{\mathrm{OE}}{\sim}}(G < G \underset{\Gamma^0=\Gamma_2^0}{*} \Gamma_2).$$

In particular, if $(\Gamma^0 < \Gamma_1)\overset{\mathrm{OE}}{\sim}(\Gamma^0 < \Gamma^0 * \Gamma_2')$ and G contains a subgroup isomorphic with Γ^0, then

$$(G < G \underset{\Gamma^0}{*} \Gamma_1)\underset{st}{\overset{\mathrm{OE}}{\sim}}(G < G * \Gamma_2').$$

Corollary 3.18 *Let G be any countable group and H an infinite cyclic subgroup. Let C be the cyclic subgroup generated by the product of commutators κ in $\mathbf{F}_{2p}$. Then $G \underset{H=C}{*} \mathbf{F}_{2p} \overset{\mathrm{OE}}{\sim} G * \mathbf{F}_{2p-1}$. In particular, if G is ME to $\mathbf{F}_2$, then $G \underset{H=C}{*} \mathbf{F}_{2p} \overset{\mathrm{ME}}{\sim} \mathbf{F}_2$.*

Proof of Th. 3.17: Let β_1 and β_2 be p.m.p. actions on (Y,ν) defining a strong OE of the pairs given by the assumption. Consider a free p.m.p. action $\tilde\alpha_1$ of $G \underset{\Gamma^0=\Gamma_1^0}{*} \Gamma_1$ on some (X,μ), with a Γ_1-equivariant map $p: Y \to X$ (the co-induced action from Γ_1 to $G \underset{\Gamma^0=\Gamma_1^0}{*} \Gamma_1$ satisfies this property, see section 3.4 below). Call α_1 the restriction of $\tilde\alpha_1$ to Γ_1. The locally one-to-one morphism p from $\mathcal{R}_{\alpha_1(\Gamma_1)}$ to $\mathcal{S}_{\beta_1(\Gamma_1)} = \mathcal{S}_{\beta_2(\Gamma_2)}$ allows to lift (cf. Lemma 2.3) the action β_2 to an action α_2 of Γ_2 on Y, which is orbit equivalent with α_1. By uniqueness in Lemma 2.3, strongness also lifts, i.e. α_1 and α_2 coincide on Γ_1^0 and Γ_2^0, via the given isomorphism between these groups[12]. Now, using $\tilde\alpha_1$, one extends the Γ_2^0-restriction of α_2 to a free action of G, so as to produce an action $\tilde\alpha_2$ of $G \underset{\Gamma^0=\Gamma_2^0}{*} \Gamma_2$. Given the amalgamated free product structure of $\mathcal{R}_{\tilde\alpha_2} = \mathcal{R}_{\tilde\alpha_2(G)} \underset{\mathcal{R}_{\tilde\alpha_2(\Gamma^0=\Gamma_1^0)}}{*} \mathcal{R}_{\tilde\alpha_2(\Gamma_1)}$, and since $\tilde\alpha_1$ and $\tilde\alpha_2$ produce the same equivalence relation when restricted, on the one hand to G, on the second hand to Γ_1^0 and Γ_2^0, and on the third hand (!) to Γ_1 and Γ_2, it follows that the action $\tilde\alpha_2$ is free, and forms with $\tilde\alpha_1$ the required strong orbit equivalence of pairs. For the "in particular" part, just observe with remark 3.16 that $(G < G \underset{\Gamma^0}{*} \Gamma_1)\underset{st}{\overset{\mathrm{OE}}{\sim}}(G < G \underset{\Gamma^0}{*} \Gamma^0 * \Gamma_2') = (G < G * \Gamma_2')$. ■

[12] Thus, α_1 and α_2 also form a strong orbit equivalence for the pairs $(\Gamma_1^0 < \Gamma_1)$ and $(\Gamma_2^0 < \Gamma_2)$; with an additional potentiality.

Question 3.19 *A limit group is a finitely generated group Γ that is ω-residually free, i.e. for every finite subset $K \subset \Gamma$ there exists a homomorphism $\Gamma \to F$ to a free group, that is injective on K. It is a natural question to ask whether limit groups are ME to a free group.*

3.4 Co-induced action

Let A be a countable group and B a subgroup. The co-induction is a canonical way, from an action β of the "small" group B on Y, to produce an action α of the overgroup $A > B$ on a space X together with a surjective B-equivariant map $X \to Y$. Let λ and ρ be the left A-actions by left multiplication and right multiplication by the inverse on A: $\lambda(g) : h \mapsto gh$ and $\rho(g) : h \mapsto hg^{-1}$.
The space $X := \mathrm{coInd}_B^A Y$ is the set of (B, ρ, β)-equivariant maps from A to Y:

$$\mathrm{coInd}_B^A Y := \{\Psi : A \to Y \mid \Psi(\rho(b)a) = \beta(b)(\Psi(a)); \forall b \in B, a \in A\}.$$

The **co-induced action** $\alpha := \mathrm{coInd}_B^A \beta$ is the action of A on $\mathrm{coInd}_B^A Y$ defined from the A-action λ on the source:

$$\mathrm{coInd}_B^A \beta(a) : \Psi(.) \mapsto \Psi(\lambda(a)(.)), \quad \forall a \in A$$

The map $\mathrm{coInd}_B^A Y \to Y; \Psi \mapsto \Psi(1)$, where 1 is the identity element of A, is B-equivariant.

A section $s : A/B \to A$ of the canonical map being chosen, the A-space $\mathrm{coInd}_B^A Y$ naturally identifies with $Y^{A/B} = \prod_{h \in A/B} Y_h$, equipped with an action σ of A which permutes the coordinates Y_h and then permutes the points in each coordinate Y_h, via the β action of an element of B. More precisely, for all $a \in A$, $f \in Y^{A/B}$, $h \in A/B$, the action gives $[\sigma(a)(f)]_h = \beta(b^{-1})(f_{h'})$, where $b \in B$ and $h' \in A/B$ are defined by the equation $s(h')b = a^{-1}s(h)$.

It follows that if β preserves a probability measure ν on Y, then σ preserves the product probability measure on $Y^{A/B}$, and in view of the description of σ, the corresponding (A, α)-invariant measure μ on X is independent of the choice of the section s. If β is essentially free, then α is essentially free.

Remark: It is interesting to compare co-induction with the usual notion of induction (the A-action induced by λ on the second factor, on the quotient of $Y \times A$ by the diagonal B-action (β, ρ)). They produce respectively right-adjoint and left-adjoint functors for the restriction functor $res : \mathcal{A} \to \mathcal{B}$ between the categories $\mathcal{A}$ of spaces with an A-action and $\mathcal{B}$ of spaces with a B-action, with equivariant maps as morphisms. In our context above, induction delivers an action of A on $(Y \times A)/(B, \beta, \rho) \simeq Y \times A/B$ with a natural invariant measure which is *infinite* when B has infinite index in A.

References

[Ada90] S. Adams. Trees and amenable equivalence relations. *Ergodic Theory Dynamical Systems*, 10(1):1–14, 1990.

[Ada94] S. Adams. Indecomposability of equivalence relations generated by word hyperbolic groups. *Topology*, 33(4):785–798, 1994.

[Ada95] S. Adams. Some new rigidity results for stable orbit equivalence. *Ergodic Theory Dynam. Systems*, 15(2):209–219, 1995.

[AS90] S. Adams, R. Spatzier. Kazhdan groups, cocycles and trees. *Amer. J. Math.*, 112(2):271–287, 1990.

[BF94] M. Bestvina, M. Feighn. Outer limits. Preprint (1994).

[BLPS01] I. Benjamini, R. Lyons, Y. Peres, O. Schramm. Uniform spanning forests. *Ann. Probab.*, 29(1):1–65, 2001.

[CCJJV01] P.-A. Cherix, M. Cowling, P. Jolissaint, P. Julg, A. Valette. *Groups with the Haagerup property.* Gromov's a-T-menability. Progr. Math. 197. Birkhäuser Verlag, Basel, 2001.

[CG86] J. Cheeger, M. Gromov. L_2-cohomology and group cohomology. *Topology*, 25(2):189–215, 1986.

[CZ89] M. Cowling, R. Zimmer. Actions of lattices in $\mathrm{Sp}(1, n)$. *Ergodic Theory Dynamical Systems*, 9(2):221–237, 1989.

[Dye59] H. Dye. On groups of measure preserving transformation. I. *Amer. J. Math.*, 81:119–159, 1959.

[Dye63] H. Dye. On groups of measure preserving transformations. II. *Amer. J. Math.*, 85:551–576, 1963.

[FM77] J. Feldman, C. Moore. Ergodic equivalence relations, cohomology, and von Neumann algebras. I. *Trans. Amer. Math. Soc.*, 234(2):289–324, 1977.

[Fur99a] A. Furman. Gromov's measure equivalence and rigidity of higher rank lattices. *Ann. of Math. (2)*, 150(3):1059–1081, 1999.

[Fur99b] A. Furman. Orbit equivalence rigidity. *Ann. of Math. (2)*, 150(3):1083–1108, 1999.

[Gab98] D. Gaboriau. Mercuriale de groupes et de relations. *C. R. Acad. Sci. Paris Sér. I Math.*, 326(2):219–222, 1998.

[Gab00] D. Gaboriau. Coût des relations d'équivalence et des groupes. *Invent. Math.*, 139(1):41–98, 2000.

[Gab00b] D. Gaboriau. On Orbit Equivalence of Measure Preserving Actions. In *Rigidity in dynamics and geometry* (Cambridge, 2000), 167–186, Springer, Berlin, 2002.

[Gab02] D. Gaboriau. Invariants L^2 de relations d'équivalence et de groupes. *Publ. math. Inst. Hautes Étud. Sci.*, 95(1):93–150, 2002.

[Gab04] D. Gaboriau. Invariant Percolation and Harmonic Dirichlet Functions. Preprint 2004. arXiv math.PR/0405458. HAL, http://hal.ccsd.cnrs.fr/ccsd-00001606.

[Gro93] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[Häg98] O. Häggström Uniform and Minimal Essential Spanning Forests on Trees. *Random Structures Algorithms*, 12:27–50, 1998.

[Har00] P. de la Harpe. *Topics in geometric group theory.* University of Chicago Press, Chicago, IL, 2000.

[Hjo02] G. Hjorth. A lemma for cost attained. Preprint 2002.

[HK05] G. Hjorth, A. Kechris. Rigidity theorems for actions of product groups and countable Borel equivalence relations. To appear in *Memoirs of the Amer. Math. Soc.*, 2005.

[Jol01] P. Jolissaint. Approximation properties for Measure Equivalent groups. Preprint 2001.

[KM04] A. Kechris, B. Miller *Lectures on Orbit Equivalence*, volume 1852 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2004.

[LP05] R. Lyons, Y. Peres. *Probability on Trees and Networks.* Book in preparation. To appear. Cambridge University Press. http://mypage.iu.edu/∼rdlyons/prbtree/prbtree.html.

[MMS04] I. Mineyev, N. Monod, Y. Shalom. Ideal bicombings for hyperbolic groups and applications. *Topology*, 43(6):1319–1344, 2004.

[MS02] N. Monod, Y. Shalom. Orbit equivalence rigidity and bounded cohomology. Preprint. (2002). To appear in *Ann. of Math.*

[MvN36] F. Murray, J. von Neumann. On rings of operators. *Ann. of Math., II. Ser.*, 37:116–229, 1936.

[OW80] D. Ornstein, B. Weiss. Ergodic theory of amenable group actions. I. The Rohlin lemma. *Bull. Amer. Math. Soc. (N.S.)*, 2(1):161–164, 1980.

[Pop01] S. Popa. On a class of type II_1 factors with Betti numbers invariants. MSRI preprint no 2001-0024. Revised arXiv math.OA/0209130.

[PP00] R. Pemantle, Y. Peres. Nonamenable products are not treeable. *Israel J. Math.*, 118:147–155, 2000.

[PW02] P. Papasoglu, K. Whyte. Quasi-isometries between groups with infinitely many ends. *Comment. Math. Helv.*, 77:133–144, 2002.

[Sta68] J. Stallings. On torsion-free groups with infinitely many ends. *Ann. Math.*, 88:312-334, 1968.

[Zim84] R. J. Zimmer. *Ergodic theory and semisimple groups*. Birkhäuser Verlag, Basel, 1984.

[Zim91] R. J. Zimmer. Groups generating transversals to semisimple Lie group actions. *Israel J. Math.*, 73(2):151–159, 1991.

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