Actions of noncompact semisimple groups on Lorentz manifolds

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ACTIONS OF NONCOMPACT SEMISIMPLE GROUPS ON LORENTZ MANIFOLDS

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Abstract. The above title is the same, but with "semisimple" instead of "simple," as that of a notice by N. Kowalsky. There, she announced many theorems on the subject of actions of simple Lie groups preserving a Lorentz structure. Unfortunately, she published proofs for essentially only half of the announced results before her premature death. Here, using a different, geometric approach, we generalize her results to the semisimple case, and give proofs of all her announced results.

1. Introduction

Isometric actions on Lorentz manifolds were first investigated in the compact case. The natural question was then: how can a compact Lorentz manifold have a noncompact isometry group? There is a strong evidence that such a question is in fact "decidable" for a wide class of geometric structures (see, for instance [DAG], [Zi1]).

1.1. **Framework.** One new aspect of N. Kowalsky's work was to deal with actions of groups on *noncompact* Lorentz manifolds. Obviously, nothing can be said about such actions without compensating for noncompactness with a dynamical counterpart ensuring some kind of reccurrence. A natural and rather weak condition used by Kowalsky is nonproperness of the action.

Let us here appreciate consideration of the noncompact case, at least from a physical point of view, according to which compact spacetimes have little interest. Having a nonproper isometry group is a manifestation of a non-Riemannian character of the geometry of spacetime. It is in such spaces that one can observe "dilation of length" and "contraction of time." It is surely interesting to try to classify spacetimes with nonproper isometry groups. This job, however, does not seem to be easy. Some extra hypotheses are therefore in order. N. Kowalsky restricts herself to actions of *simple Lie groups.*

Date: January 13, 2006.

1.2. **Kowalsky's main theorem.** The de Sitter and anti-de Sitter spaces, dS_n and AdS_{n+1}, respectively, are the homogeneous Lorentz spaces $O(1,n)/O(1,n-1)$ and $O(2,n)/O(1,n)$. Geometrically, they are the universal Lorentz spaces of constant positive and negative curvature, respectively. A striking fact proved by Kowalsky is that, *at a group level*, they are the only Lorentz nonproper G-spaces, with G simple:

Theorem 1.1 (Kowalsky [K1] 5.1)**.** *Let G be a simple Lie group with finite center acting isometrically and nonproperly on a connected Lorentz manifold. Then G is locally isomorphic to either $O(1,n), n \geq 2$, or $O(2,n), n \geq 3$.*

Remark 1.2. *The groups $O(1,1)$ and $O(2,2)$ are not simple.*

1.3. **Geometry of semisimple isometric actions.** Once the acting group is known, the problem arises to understand the geometry of the Lorentz space, or at least that of orbits. Here, one hopes the space looks like de Sitter or anti-de Sitter space, depending on whether G is locally $O(1,n)$ or $O(2,n)$. N. Kowalsky announced results to this effect in [K2] and wrote proofs for the $O(1,n)$ case in her thesis [K3]. We will recall their statements below in §1.7. Unfortunately, she prematurely died, before publishing proofs.

1.4. **The technique.** When a Lie group G acts on M preserving a pseudo-Riemannian metric g, one can consider a Gauss map from M to $S^2(\mathfrak{g})$, the space of quadratic forms on the Lie algebra $\mathfrak{g}$ of G. When $S^2(\mathfrak{g})$ is endowed with the natural adjoint G-action, the Gauss map is equivariant, and the G-space $S^2(\mathfrak{g})$ reflects the dynamics on M. It is via this map that the nonproperness condition is translated as a geometric condition on the induced metrics on orbits. This idea, due to N. Kowalsky, has become a basic tool in similar questions on the subject, e.g. Adams-Stuck [AS1], [AS2], and Bader-Nevo [BN] (Remark here that variants of the Gauss map, with other natural spaces instead of $S^2(\mathfrak{g})$ were used by other authors, e.g. Margulis, Zimmer). However, this is the starting point; further work in the proof is algebraic Lie theoretic. The proof of Theorem 1.1 is a real algebraic "tour de force."

1.5. **Other works.** Another proof of Theorem 1.1 was proposed by S. Adams [A3]; his methods are largely algebraic. In other directions, S. Adams investigated Lorentz isometric actions, for some more general groups, sometimes with a stronger dynamical condition ([A1], [A2]).

Concerning Kowalsky's unpublished proofs, we notice a contribution by D. Witte Morris [W]. He considers the homogeneous case. More precisely, he takes G to be $O(1,n)$, or $O(2,n)$, the isometry group of de Sitter or anti-de Sitter space, respectively, and considers a nonproper Lorentz homogeneous space G/H—that is, H is noncompact, and the G-action on G/H preserves some Lorentz metric. He proves that G/H is de Sitter or anti-de Sitter space—that is, $H = O(1,n-1)$ or $O(1,n)$. Witte's proof is quite algebraic.

1.6. **On the present contribution.** Our investigation here highly relies on the approach of [ADZ], although the two articles can be read completely independently. From [ADZ], we will use the result recalled below as Theorem 2.2. In light of this result, on the structure of nonproper orbits of Lorentz type, the present paper addresses the case in which the acting group has a nonproper degenerate orbit.

Before stating our results, let us give some motivations and emphasize new features:

• *Completing Kowalsky's work:* One major goal is to prove the announced results of N. Kowalsky.

• *Geometric approach:* The approach here is different from that of Kowalsky (as well as from others', for instance Adams'). Together with [ADZ], we get a geometric proof of the main results, in particular, of Theorem 1.1, where one sees the global structure of proofs.

• *From simple to semisimple:* More important, we generalize results to the semisimple case, assuming there are no local $SL(2, \mathbf{R})$-factors. A semisimple Lie group is essentially a product of simple Lie groups, but, in general, a nonproper action of a product does not derive from a nonproper action of one factor. However, in the Lorentz setting, we conclude that is the case—that is, the semisimple case reduces to the simple one. This is really an important fact, since it leads one to hope to reduce the remaining work to the case in which the group is solvable. Of course, the reason for this is the Levi decomposition of Lie groups, which says that a Lie group is essentially a semidirect product of a semisimple and a solvable group.

1.7. **Kowalsky's legacy.** In [K2], the following theorems are stated. For Theorem 1.3 below, see also [K3] 6.2. The manifold M and group G are assumed connected throughout.

Theorem 1.3. *Let G be locally isomorphic to $O(1, n), n \geq 3$, and suppose that G acts on a manifold M preserving a Lorentz metric. Then all noncompact stabilizers have a Lie algebra isomorphic to either $\mathfrak{o}(1, n)$, $\mathfrak{o}(1, n-1)$, or $\mathfrak{o}(n-1) \ltimes \mathbf{R}^{n-1}$.*

Theorem 1.4. *Let G be locally isomorphic to $O(2, n), n \geq 6$, with G having finite center. Suppose that G acts non-trivially on a manifold M preserving a Lorentz metric. Then all noncompact stabilizers have a Lie algebra isomorphic to $\mathfrak{o}(1, n)$.*

Theorem 1.5. *Let G and M be as in Theorem 1.4 above, and assume there is a point of M with noncompact stabilizer. Then the universal cover $\widetilde{M}$ is Lorentz isometric to a warped product $L \times_w \widetilde{AdS}_{n+1}$, where $\widetilde{AdS}_{n+1}$ is the simply connected $(n+1)$-dimensional Lorentz space of constant curvature -1, and L is a Riemannian manifold. Further, the induced action of the* *universal cover* $\widetilde{G}$ *on* $\widetilde{M}$ *is via the canonical* $\widetilde{G}$*-action on* $\widetilde{AdS}_{n+1}$ *and the trivial action on* L.

See Section 2 below for the definition of warped product.

1.8. **Results.** As said above in §1.6, we provide here proofs of all previous statements of Kowalsky, together with some improvements.

A submanifold N in a Lorentz manifold is *degenerate* if $T_xN^\perp \cap T_xN \neq \mathbf{0}$. In Minkowski space $\mathbf{R}^{1,n}$, the simple subgroup $O(1,n) \subset Isom(\mathbf{R}^{1,n})$ has two degenerate orbits. Together with the origin $\mathbf{0}$, these form the *light cone*, the set of all isotropic vectors in $\mathbf{R}^{1,n}$. The stabilizer in $O(1,n)$ of a point in either component is isomorphic to $O(n-1) \ltimes \mathbf{R}^{n-1}$, where the action of $O(n-1)$ on $\mathbf{R}^{n-1}$ is the usual representation.

In the degenerate case, we have the following theorem, which says that a degenerate orbit for a simple group acting nonproperly is isometric to the Minkowski light cone, up to finite cover. Together with Theorem 1.5 of [ADZ], which classifies nonproper orbits of Lorentz type, it implies Theorems 1.3 and 1.4 of Kowalsky above.

Theorem 1.6. *Suppose* G *is a connected group with finite center locally isomorphic to* $O(1,n)$ *or* $O(2,n)$ *for* $n \geq 3$. *If* G *acts isometrically on a Lorentz manifold and has a degenerate orbit with noncompact stabilizer* $G(x)$, *then* $\mathfrak{g} \cong \mathfrak{o}(1,n)$, *and* $\mathfrak{g}(x) \cong \mathfrak{o}(n-1) \ltimes \mathbf{R}^{n-1}$.

The following result implies Theorem 1.5 above.

Theorem 1.7. *If* G, *a group with finite center locally isomorphic to* $O(2,n)$, $n \geq 3$, *acts isometrically on a Lorentz manifold* M, *with some noncompact stabilizer, then, up to finite covers,* M *is equivariantly isometric to a warped product* $L \times_w AdS_{n+1}$ *of a Riemannian manifold* L *with the anti-de Sitter space* AdS_{n+1}.

We extend the above results to semisimple groups. Note that the noncompact stabilizer assumption is weakened to nonproperness of the action.

Theorem 1.8. *Let* G *be a semisimple group with finite center and no local factor locally isomorphic to* $SL(2,\mathbf{R})$, *acting isometrically and nonproperly on a Lorentz manifold* M. *Then*

(1) *G has a local factor G_1 locally isomorphic to $O(1,n)$ or $O(2,n)$*
(2) *There exists a Lorentz manifold S, isometric, up to finite cover, to dS_n or AdS_{n+1}, depending whether G_1 is locally isomorphic to $O(1,n)$ or $O(2,n)$, and an open subset of M, in which each G_1-orbit is homothetic to S.*
(3) *Any such orbit as above has a neighborhood isometric to a warped product $L \times_w S$, for L a Riemannian manifold.*

As a fusion, we can give the following "full" theorem:

Theorem 1.9. *Let G be a semisimple Lie group with finite center and no local factor locally isomorphic to $SL(2, \mathbf{R})$, acting nonproperly and isometrically on a Lorentz manifold (M, g). Then, G has a simple local factor G_1 that acts nonproperly. All the other local factors act properly. There are two possibilities for G_1:*

(1) *G_1 locally isomorphic to $O(2, n)$. In this case, there is a Lorentz manifold S isometric, up to finite cover, to AdS_{n+1}, such that all G_1-orbits are homothetic to S. In fact, up to a finite cover, M is a warped product $L \times_w AdS_{n+1}$.*
(2) *G_1 locally isomorphic to $O(1, n)$. There is a Lorentz manifold S isometric, up to finite cover, to dS_n, such that, on an open nonempty subset U of M, all G_1-orbits are homothetic to S.*
    - *Any point of U has a neighborhood isometric to a warped product $L \times_w S$.*
    - *Orbits on the boundary of U are isometric, up to finite cover, to the light cone of the Minkowski space $\mathbf{R}^{1,n}$.*

Remark 1.10. One can prove that in the last case, the other orbits are homothetic to the (Riemannian) hyperbolic space $\mathbf{H}^n = O(1, n)/O(n)$, and that near these orbits, there is a similar warped product.

2. Background: warped product near Lorentz orbits

Definition 2.1. *For two pseudo-Riemannian manifolds (L, λ) and (S, σ), a* warped product *$L \times_w S$ is given by a positive function w on L : the metric at (l, s) is $\lambda_l + w(l)\sigma_s$. The factor S is called the* normal factor*.*

2.1. Results of [ADZ].

We will make use of the following theorem:

Theorem 2.2. *([ADZ] 1.5) Let G be a connected semisimple Lie group acting isometrically on a Lorentz manifold (M, g) of dimension ≥ 3. Suppose that no local factor of G is locally isomorphic to $SL(2, \mathbf{R})$ and that there exists an orbit O of Lorentz type with noncompact isotropy.*

Then, up to a finite cover, G factors $G = G_2 \times G_1$, where:

(1) *G_1 possesses an orbit O_1 which is a Lorentz space of constant, nonvanishing curvature, and G_1 equals $Isom^0(O_1)$.*
(2) *There is a G-invariant neighborhood U of O_1 which is a warped product $L \times_w O_1$.*
(3) *The factor O_1 corresponds to G_1-orbits, and G_2 acts along the L-factor.*

3. Properties of the isotropy representation

Here we collect some algebraic facts about the structure of nonproper degenerate orbits. Suppose that x is a point of M with degenerate G-orbit. Denote this orbit by O. Let k be the dimension of M, and assume that G is semisimple.

Fix an isometric isomorphism of T_xM with $\mathbf{R}^{1,k-1}$, determining an isomorphism $O(T_xM) \cong O(1,k-1)$. Let V be the image of T_xO under this isomorphism. Let $\Phi : G(x) \to O(1,k-1)$ be the resulting isotropy representation. Because G acts properly and freely on the bundle of Lorentz frames of M, the isotropy representation is an injective, proper map. The invariant subspace V is degenerate, so $\Phi(G(x))$ leaves invariant the line $V^\perp \cap V$. The stabilizer of an isotropic line is conjugate in $O(1,k-1)$ to the minimal parabolic

$$P = (K \times A) \ltimes U$$

where $U \cong \mathbf{R}^{k-2}$ is unipotent, $A \cong \mathbf{R}^*$, and $K \cong O(k-2)$ with the conjugation action of $K \times A$ on U equivalent to the standard conformal representation of $O(k-2) \times \mathbf{R}^*$ on $\mathbf{R}^{n-2}$. Denote by $\mathfrak{p}$ the Lie algebra of P, and by $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{u}$, the subalgebras corresponding to K, A and U.

Lemma 3.1. *Let Y be an element of $\mathfrak{p}$ with eigenvalue $\lambda > 0$ on V. Then $ad_{\mathfrak{p}}(Y)$ has no negative eigenvalue. If Y has eigenvalue $\lambda < 0$ on V, then $ad_{\mathfrak{p}}(Y)$ has no positive eigenvalue.*

Proof. Assume $\lambda > 0$; the case $\lambda < 0$ is analogous. There is a decomposition

$$\varphi(Y) = Y_1 + Y_2 + Y_3 \qquad Y_1 \in \mathfrak{a},\ Y_2 \in \mathfrak{k},\ Y_3 \in \mathfrak{u}$$

It is easy to compute that because $\varphi(Y)$ has eigenvalue $\lambda > 0$ on V, the adjoint of Y_1 has eigenvalue λ with eigenspace $\mathfrak{u}$; the only other eigenvalue for $\mathrm{ad}(Y_1)$ is 0. Note that $[\mathfrak{p},\mathfrak{p}] = \mathfrak{k} \ltimes \mathfrak{u}$. Suppose there were $W \in \mathfrak{p}$ for which $[\varphi(Y), W] = \alpha W$ with $\alpha < 0$. There is a decomposition $W = W_2 + W_3$, with $W_2 \in \mathfrak{k}$ and $W_3 \in \mathfrak{u}$. Then

$$[Y_2, W_2] + [Y_1 + Y_2, W_3] + [Y_3, W_2] = \alpha(W_2 + W_3)$$

Considering $\mathfrak{k}$-components gives $\alpha W_2 = [Y_2, W_2]$. Because $\mathrm{ad}(Y_2)$ is skew-symmetric, $W_2 = \mathbf{0}$. Then $W \in \mathfrak{u}$, and $[Y_1 + Y_2, W] = \alpha W$. Now $\mathrm{ad}(Y_1)$ and $\mathrm{ad}(Y_2)$ are simultaneously diagonalizable, but $\mathrm{ad}(Y_1)$ has only nonnegative eigenvalues, and $\mathrm{ad}(Y_2)$ has only purely imaginary eigenvalues. Therefore, no such W can exist. $\square$

Note that $\mathfrak{g}/\mathfrak{g}(x)$ can be identified with T_xO by the map

$$Y \mapsto \overline{Y}(x) = \left.\frac{\partial}{\partial t}\right|_0 e^{tY} x$$

For $g \in G(x)$, differentiating the relation $ge^{tY}x = (ge^{tY}g^{-1})x$ gives

$$D_x g(\overline{Y}(x)) = \overline{\mathrm{Ad}g(Y)}(x)$$

In other words, Φ restricted to V is equivalent to the representation $\overline{\mathrm{Ad}}$ of $G(x)$ on $\mathfrak{g}/\mathfrak{g}(x)$ arising from the adjoint representation. Let $\varphi : \mathfrak{g}(x) \to \mathfrak{o}(1, k-1)$ be the Lie algebra representation tangent to Φ and $\overline{\mathrm{ad}}$ be the representation tangent to $\overline{\mathrm{Ad}}$.

An element Y of $\mathfrak{g}$ is called nilpotent if $\mathrm{ad}(Y)$ is nilpotent. An element Y is semisimple if $\mathrm{ad}(Y)$ is diagonalizable over $\mathbf{C}$, and Y is $\mathbf{R}$-split if $\mathrm{ad}(Y)$ is diagonalizable over $\mathbf{R}$.

Lemma 3.2. *The stabilizer subalgebra $\mathfrak{g}(x) \subset \mathfrak{g}$ has the following properties:*

(1) *For all $Y \in \mathfrak{g}(x)$, the endomorphism $\overline{\mathrm{ad}}(Y)$ has no real nonzero eigenvalues. In fact, the same is true for φ, so $\varphi(\mathfrak{g}(x))$ is conjugate to a subalgebra of $\mathfrak{k} \ltimes \mathfrak{u}$.*
(2) *The stabilizer subalgebra $\mathfrak{g}(x)$ contains no element $\mathbf{R}$-split in $\mathfrak{g}$.*
(3) *There exists a subalgebra $\mathfrak{s}(x)$ in which $\mathfrak{g}(x)$ has codimension one such that $[\mathfrak{s}(x), \mathfrak{s}(x)] \subset \mathfrak{g}(x)$, and the representation of $\mathfrak{g}(x)$ on $\mathfrak{g}/\mathfrak{s}(x)$ is skew-symmetric—that is, every endomorphism is diagonalizable with purely imaginary eigenvalues.*

Proof.

(1) Suppose that $\overline{\mathrm{ad}}(Y)$ has eigenvalue $\lambda > 0$. Then λ is also an eigenvalue of $\varphi(Y)$ on V. Since $\varphi(Y)$ is skew-symmetric on $V/(V \cap V^{\perp})$, the generalized eigenspace for λ is one-dimensional and equals $V^{\perp} \cap V$. The trace of $\varphi(Y)$ on V is λ, so the trace of $\overline{\mathrm{ad}}(Y)$ on $\mathfrak{g}/\mathfrak{g}(x)$ is λ.

The trace of $\mathrm{ad}(\varphi(Y))$ on $\varphi(\mathfrak{g}(x))$ is nonnegative, because $\mathrm{ad}(\varphi(Y))$ has no negative eigenvalues on $\mathfrak{p}$ by Lemma 3.1. Then the trace of $\mathrm{ad}(Y)$ on $\mathfrak{g}(x)$ is also nonnegative. Finally, the trace of $\mathrm{ad}(Y)$ on $\mathfrak{g}$ is positive, contradicting the unimodularity of $\mathfrak{g}$. If $\lambda < 0$, the same argument shows that the trace of $\mathrm{ad}(Y)$ on $\mathfrak{g}$ is negative. Therefore, no $\overline{\mathrm{ad}}(Y)$ has any nonzero real eigenvalues, and no $\varphi(Y)$ has any nonzero real eigenvalues on V.

If $\varphi(Y)$ has a nonzero real eigenvalue on $\mathbf{R}^{1,k-1}$, then an eigenvector must be isotropic. It either lies in V or is not orthogonal to $V^{\perp} \cap V$. In either case, $\varphi(Y)$ has a nonzero real eigenvalue on V, a contradiction.

(2) If an $\mathbf{R}$-split element $H \in \mathfrak{g}(x)$, then by (1), all root vectors on which $\mathrm{ad}(H)$ is nontrivial must project to $\mathbf{0}$ in $\mathfrak{g}/\mathfrak{g}(x)$. In this case, $\mathfrak{g}(x)$

contains a subalgebra isomorphic to $\mathfrak{sl}_2$. Applying the monomorphism φ would yield a subalgebra isomorphic to $\mathfrak{sl}_2$ in $\mathfrak{p}$, which is impossible.

(3) Take any Z' spanning $V^\perp \cap V$, so $\varphi(\mathfrak{g}(x))(Z') = \mathbf{0}$ by (1). Take the corresponding vector in $\mathfrak{g}/\mathfrak{g}(x)$, and let Z be any lift to $\mathfrak{g}$. Then $\mathrm{ad}(\mathfrak{g}(x))(Z) \subseteq \mathfrak{g}(x)$, so $\mathfrak{s}(x) = \mathbf{R}Z + \mathfrak{g}(x)$ is the desired subalgebra.

From the equivalence of $\varphi|_V$ with $\overline{\mathrm{ad}}$, the representations $V/(V \cap V^\perp)$ and $\mathfrak{g}/\mathfrak{s}(x)$ are equivalent. The former is skew-symmetric.

□

4. Root spaces in isotropy subalgebra

By a nonproper orbit we will mean one with noncompact isotropy. Theorem 1.8 of [ADZ] asserts the existence of a nonproper orbit under our assumptions. The proof was easily deduced from the following result of [K1]:

Proposition 4.1. *If the G-action is nonproper, then there is $x \in M$, and an $\mathbf{R}$-split element H of $\mathfrak{g}$ such that the negative root space*

$$\Sigma_{\alpha(H)<0}\mathfrak{g}_\alpha$$

is isotropic at x.

The subalgebra $\mathfrak{s}(x)$ of the previous section is exactly the subspace of $Y \in \mathfrak{g}$ such that $\overline{Y}(x) \in T_xM$ is isotropic.

Fact 4.2 of [ADZ] is that

$$\Sigma_{\alpha(H)<0}\mathfrak{g}_\alpha \cap \mathfrak{g}(x) \neq \mathbf{0}$$

If this intersection were $\mathbf{0}$, then the subalgebra $\oplus_{\alpha(H)\geq 0}\mathfrak{g}_\alpha$ would have codimension one in $\mathfrak{g}$. Such a subalgebra could only exist if $\mathfrak{sl}_2$ were a factor of $\mathfrak{g}$, but our hypotheses exclude this.

Denote by $\mathfrak{a}$ and Δ the Cartan subalgebra and root system, respectively, of Proposition 4.1. The remainder of this section is devoted to showing the following proposition.

Proposition 4.2. *There exist $J \in \mathfrak{a}$ and $S \subset \Delta$ such that*

(1) $\mathfrak{s}(x) = \mathbf{R}J + \mathfrak{g}(x)$
(2) $\alpha(J) < 0$ *for all* $\alpha \in S$
(3) $\Sigma_{\alpha\in S}\mathfrak{g}_\alpha \subseteq \mathfrak{g}(x)$
(4) $dim(\Sigma_{\alpha\in S}\mathfrak{g}_\alpha) \geq 2$

Proof. Let

$$\mathbf{0} \neq X \in \Sigma_{\alpha(H)<0}\mathfrak{g}_\alpha \cap \mathfrak{g}(x)$$

There exist $J \in \mathfrak{a}$ and a nilpotent Y in $\mathfrak{g}$ such that $[J, X] = -2X$ and $[X, Y] = J$ (see [S] 2.4.B). The operator $\overline{\mathrm{ad}}(X)$ is nilpotent; on the other hand, by Lemma 3.2 (3), $\overline{\mathrm{ad}}(X)$ is skew-symmetric on $\mathfrak{g}/\mathfrak{s}(x)$, so J must belong to $\mathfrak{s}(x)$.

Note $J \notin \mathfrak{g}(x)$ by Lemma 3.2 (2). Therefore $\mathfrak{s}(x) = \mathbf{R}J + \mathfrak{g}(x)$, proving (1).

Let S be the set of $\alpha \in \Delta$ such that $\alpha(H) < 0$ and $\alpha(J) < 0$, so (2) is obviously satisfied. Note that S is not empty since $[J, X] = -2X$.

For $\alpha \in S$, we have $\mathfrak{g}_\alpha \subset \mathfrak{s}(x)$ and

$$\mathfrak{g}_\alpha = [J, \mathfrak{g}_\alpha] \subset [\mathfrak{s}(x), \mathfrak{s}(x)] \subset \mathfrak{g}(x)$$

by Lemma 3.2 (3), showing statement (3) above.

Now, replacing X by a nonzero element of some $\mathfrak{g}_\alpha$, $\alpha \in S$, we may assume that $-J$ is a *basic translation*—that is, there exists $c_\alpha < 0$ such that

$$\alpha(J) = -2 \qquad \text{and} \qquad \alpha(Z) = c_\alpha \kappa(J, Z)$$

for any $Z \in \mathfrak{a}$, where κ denotes the Killing form. In this case, we have that for any root β, the reflection

$$\sigma_\alpha(\beta) = \beta + \beta(J)\alpha$$

is again a root (see [S] II.5.A).

Now, to show (4) it suffices to show that $\dim(\mathfrak{g}_\alpha) \geq 2$ or that there exists some $\gamma \neq \alpha$ also in S.

Suppose $\dim(\mathfrak{g}_\alpha) = 1$. The assumption that $\mathfrak{g}$ has no $\mathfrak{sl}_2(\mathbf{R})$-factor implies that there exists some nonzero root $\delta \neq \alpha$ such that $\delta(J) \neq 0$. We may assume $\delta(J) < 0$. If $\delta(H) < 0$, then $\delta \in S$, as desired. So suppose that $\delta(H) \geq 0$. Now let

$$\gamma = -\sigma_\alpha(\delta) = -\delta - \delta(J)\alpha$$

Then

$$\gamma(J) = -\delta(J) - \delta(J)\alpha(J) = -\delta(J)(1-2) = \delta(J) < 0$$

and

$$\gamma(H) = -\delta(H) - \delta(J)\alpha(H) < 0$$

so $\gamma \in S$. □

5. Proof of Theorem 1.8

Reduction of the proof. It was proved in [ADZ] (Theorem 1.8) that G has an orbit with a noncompact stabilizer. It was also proved (Theorem 1.5) that if such an orbit is Lorentzian, then the situation is exactly as described in Theorem 1.8. It remains to consider the case where this orbit is degenerate. The proof would be finished using the following proposition which states that a nonproper degenerate orbit hides a nonproper Lorentzian one. ♢

Proposition 5.1. *Let O be a degenerate G-orbit with noncompact isotropy. Then, G has (near O) Lorentzian orbits with noncompact isotropy.*

The remainder of the section is devoted to the proof of this proposition.

5.1. The asymptotic geodesic hypersurface $\mathcal{F}_x$.

Fact 5.2. *For $x \in O$, let $\mathbf{R}n_x$ be the null direction in T_xO. Then, the orthogonal $n_x^\perp$ is tangent to a lightlike geodesic hypersurface $\mathcal{F}_x$ (defined in a neighborhood of x).*

Proof. Let X be a nilpotent element of $\mathfrak{g}(x)$ given by Proposition 4.2, and consider the isometry $f = e^{tX}$, for some $t \neq 0$. Let k be the dimension of M, and g the Lorentz metric. The graph $Graph(f) \subset M \times M$ is an isotropic totally geodesic k-dimensional submanifold of $M \times M$, when equipped with the metric $g \oplus (-g)$. The graphs $Graph(f^m)$ converge to E', a k-dimensional isotropic totally geodesic submanifold, which is no longer a graph, since f^m is divergent. Its intersection with $\{x\} \times M$ is nontrivial, but has at most dimension 1, because it is isotropic, and M is Lorentzian. Therefore, the projection $\mathcal{F}_x$ of E is a totally geodesic hypersurface in $M \times \{x\}$, which is easily seen to be lightlike and satisfies the required conditions.

□

Fact 5.3. *The hypersurface $\mathcal{F}_x$ carries a 1-dimensional foliation $\mathcal{C}$ such that:*

1. *Any isotropic curve in $\mathcal{F}_x$ is tangent to a leaf of $\mathcal{C}$.*
2. *Each leaf of $\mathcal{C}$ is an isotropic geodesic.*
3. *The (local) quotient space $\mathcal{F}_x/\mathcal{C}$ inherits a Riemannian metric, preserved by the elements of G preserving $\mathcal{F}_x$.*

Proof. At any point y of a degenerate hypersurface $\mathcal{F}$, there exists a unique tangent isotropic direction C_y, determining a characteristic 1-dimensional foliation $\mathcal{C}$ of $\mathcal{F}$, proving (1). Since $\mathcal{F}$ is totally geodesic, (2) follows. For (3), it is known (see, for instance, [Ze1]) that $\mathcal{C}$ is transversally Riemannian if and only if $\mathcal{F}$ is totally geodesic. Here, transversally Riemannian means that the flow along any parameterization of $\mathcal{C}$ preserves the induced degenerate metric, or equivalently, that the degenerate metric is projectable as a Riemannian metric on the (local) quotient space $\mathcal{F}_x/\mathcal{C}$. □

Fact 5.4. *The subalgebra $\mathfrak{s}(x)$ preserves the isotropic geodesic $\mathcal{C}_x$.*

Proof. Indeed, any $Y \in \mathfrak{s}(x)$ has $\overline{Y}(x)$ isotropic, and hence the whole Y-orbit of x is isotropic. But, as stated above, isotropic curves of $\mathcal{F}_x$ are contained in leaves of $\mathcal{C}$—that is, all Y-orbits through x are contained in $\mathcal{C}_x$. The image of $\mathcal{C}_x$ by any element of the one-parameter group generated by Y is an isotropic geodesic tangent to $\mathcal{C}_x$ at some point, thus equals $\mathcal{C}_x$. □

Fact 5.5. *The tangent space of the (local) quotient space $\mathcal{F}_x/\mathcal{C}$ at the point corresponding to $\mathcal{C}_x$ is identified with $n_x^{\perp}/\mathbf{R}n_x$. The action of $\Sigma_{\alpha \in S}\mathfrak{g}_\alpha$ on it is trivial.*

Proof. Note that the subspace $\mathbf{R}J + \Sigma_{\alpha \in S}\mathfrak{g}_\alpha$ as in Proposition 4.2 is in fact a subalgebra of $\mathfrak{s}(x)$. We have a representation ρ of $\mathbf{R}J + \Sigma_{\alpha \in S}\mathfrak{g}_\alpha$ into the orthogonal algebra of $n_x^{\perp}/\mathbf{R}n_x$, which is endowed with a positive definite inner product. But in such an orthogonal algebra, an equality $[\rho(J), \rho(Y)] = \lambda\rho(Y)$, becomes trivial—that is $\rho(Y) = 0$ (since $\lambda \neq 0$); □

Corollary 5.6. *$\Sigma_{\alpha \in S}\mathfrak{g}_\alpha$ acts trivially on the (local) quotient space $\mathcal{F}_x/\mathcal{C}$. That is, $\Sigma_{\alpha \in S}\mathfrak{g}_\alpha$ preserves individually each leaf of $\mathcal{C}$.*

Proof. The action of $\Sigma_{\alpha \in S}\mathfrak{g}_\alpha$ on $\mathcal{F}_x/\mathcal{C}$ is trivial, since it is a Riemannian action with a fixed a point and a trivial derivative at it. □

Corollary 5.7. *Any point of $\mathcal{F}_x$ has a noncompact isotropy algebra.*

Proof. Indeed, $\Sigma_{\alpha \in S}\mathfrak{g}_\alpha$ has dimension ≥ 2 and has orbits of dimension 1. Therefore, stabilizers are nontrivial. They are not compact since all elements of $\Sigma_{\alpha \in S}\mathfrak{g}_\alpha$ are nilpotent. □

Fact 5.8. *Let Γ be the set of fixed points of $\Sigma_{\alpha \in S}\mathfrak{g}_\alpha$ in $\mathcal{F}_x$. Then, Γ has an empty interior (in $\mathcal{F}_x$). In particular the orbit of any point of $\mathcal{F}_x - \Gamma$, under $\Sigma_{\alpha \in S}\mathfrak{g}_\alpha$, contains its $\mathcal{C}$-leaf.*

Proof. No element of $\Sigma_{\alpha \in S}\mathfrak{g}_\alpha$ can fix points of an open subset of $\mathcal{F}_x$. Indeed, in general, a Lorentz transformation fixing one point and acting trivially on a tangent lightlike hyperplane at that point has a trivial derivative, and is therefore trivial. □

Corollary 5.9. *No point of $\mathcal{F}_x$ has a spacelike G-orbit.*

Proof. If a point $y \in \mathcal{F}_x$ has a spacelike orbit, then all orbits of points in a neighborhood of y are spacelike, as well. However, any neighborhood of y meets $\mathcal{F}_x - \Gamma$; orbits of points in here cannot be spacelike, because they contain at least one isotropic geodesic. □

5.2. **End of the proof of Proposition 5.1.**

Fact 5.10. *The degenerate orbit O cannot be contained in $\mathcal{F}_x$.*

Proof. Suppose $O \subseteq \mathcal{F}_x$. Then the group G preserves $\mathcal{F}_x$. From Corollary 5.6, the action of G on the quotient space $Q = \mathcal{F}_x/\mathcal{C}$ is not faithful. More precisely, any factor $\mathfrak{b}$ of $\mathfrak{g}$ which contains an element like J (in Proposition

4.2) must act trivially on Q. However, orbits of $\mathfrak{b}$ cannot be 1-dimensional, since $\mathfrak{g}$ has no $\mathfrak{sl}(2,\mathbf{R})$-factor. Therefore, $\mathfrak{b}$ acts trivially on $\mathcal{F}_x$. As in the proof of Fact 5.8, this implies $\mathfrak{b}$ acts trivially on M. □

Now, from Corollary 5.7, the proof of Proposition 5.1 would be finished once one proves that there is a point of $\mathcal{F}_x$ with a Lorentz orbit. It suffices to show existence of nondegenerate orbits, since from Corollary 5.9, points of $\mathcal{F}_x$ cannot have spacelike orbits. Assume, for a contradiction, that all G-orbits of points of $\mathcal{F}_x$ are degenerate. For any $y \in \mathcal{F}_x - \Gamma$, the orbit $G.y$ contains the isotropic geodesic $\mathcal{C}_y$ by Fact 5.8; therefore, $T_y(G.y) \subset n_y^{\perp} = T_y(\mathcal{F}_x)$. There is an open $U \subset \mathcal{F}_x - \Gamma$ on which the tangent spaces to G-orbits form an integrable distribution. The leaves are again G-orbits. Then $\mathcal{F}_x$ contains a degenerate nonproper G-orbit, contradicting the previous fact. ♢

Corollary 5.11. *(from proof) There is a local factor G_1 of G for which the G_1-orbit of x is a point or degenerate with noncompact stabilizer. In other words, if G has a nonproper orbit that is either a point or degenerate, then a subgroup G_1 locally isomorphic to $O(1,n)$ or $O(2,n)$ has an orbit with the same properties.*

Proof. We have seen that some nilpotent elements stabilizing a point in the degenerate orbit O stabilize points with Lorentz orbits. But, from Theorem 2.2, a Lorentz orbit can be nonproper only if there is a local factor G_1 acting nonproperly; moreover, any nilpotent elements stabilizing a point in such an orbit must belong to G_1. We infer from this that G_1 acts nonproperly on the degenerate orbit O. Because G_1 is simple and the stablizer $G_1(x)$ is noncompact, the orbit $G_1.x$ must be degenerate or one point. □

6. Proof of Theorem 1.6

6.1. Excluding $\mathfrak{o}(2,n)$.

Let $\alpha, J, \mathfrak{a}, \Delta$, and X be as above.

Now suppose that $\mathfrak{g} \cong \mathfrak{o}(2,n)$. Let β and γ be distinct positive roots, each with $(n-2)$-dimensional root spaces. The other positive roots are $\beta-\gamma$ and $\beta+\gamma$, with one-dimensional root spaces.

First suppose $\alpha = \beta$ and let $X \in \mathfrak{g}_\beta \subset \mathfrak{g}(x)$. Let L span $\mathfrak{g}_{-\beta-\gamma}$. The order of nilpotence of $\mathrm{ad}(X)$ on L is three:

$$\begin{aligned}
[X, L] &= W \qquad \text{where } \mathbf{0} \neq W \in \mathfrak{g}_{-\gamma} \\
[X, W] &= S \qquad \text{where } \mathbf{0} \neq S \in \mathfrak{g}_{\beta-\gamma} \\
[X, S] &= \mathbf{0}
\end{aligned}$$

The nilpotent subalgebra generated by X and L cannot be contained in $\mathfrak{s}(x)$ because $[\mathfrak{g}(x), \mathfrak{s}(x)] \subseteq \mathfrak{g}(x)$, and any nilpotent subalgebra of $\mathfrak{g}(x)$ is abelian.

Let $\overline{L}$ be the image of L modulo $\mathfrak{s}(x)$. Because the adjoint of X on $\mathfrak{g}/\mathfrak{s}(x)$ is skew-symmetric, W must be in $\mathfrak{s}(x)$. Because X and W generate a non-abelian nilpotent algebra, $W \notin \mathfrak{g}(x)$. Then $cW - J \in \mathfrak{g}(x)$ for some $c \in \mathbf{R}$. But $\overline{L}$ would be an eigenvector with real non-zero eigenvalue for $\overline{\mathrm{ad}}(cW - J)$, contradicting 3.2 (1).

Therefore, X cannot be in $\mathfrak{g}_\beta$. The same argument shows X cannot be in $\mathfrak{g}_\gamma$. In fact, $\mathfrak{g}(x) \cap \mathfrak{g}_\omega$ must be $\mathbf{0}$ for $\omega = \pm\beta, \pm\gamma$.

Next suppose $\alpha = \beta - \gamma$ and $X \in \mathfrak{g}_{\beta-\gamma} \subset \mathfrak{g}(x)$. The bracket $[X, \mathfrak{g}_\gamma] = \mathfrak{g}_\beta$, so the skew-symmetry condition 3.2 (3) forces $\mathfrak{g}_\beta \subset \mathfrak{s}(x)$. From $[X, \mathfrak{g}_{-\beta}] = \mathfrak{g}_{-\gamma}$, also $\mathfrak{g}_{-\gamma} \subset \mathfrak{s}(x)$. There is some nonzero $B + C \in \mathfrak{g}(x)$, where $B \in \mathfrak{g}_\beta$ and $C \in \mathfrak{g}_{-\gamma}$. Since neither $\mathfrak{g}_\beta$ nor $\mathfrak{g}_{-\gamma}$ can intersect $\mathfrak{g}(x)$, both B and C must be nonzero. Then the action of $\mathrm{ad}(B+C)$ on $L \in \mathfrak{g}_{-\beta-\gamma}$ contradicts properties of the isotropy representation, as above.

Last, suppose $X \in \mathfrak{g}_{\beta+\gamma}$. A similar contradiction results from $[X, \mathfrak{g}_{-\beta}] = \mathfrak{g}_\gamma$ and $[X, \mathfrak{g}_{-\gamma}] = \mathfrak{g}_\beta$.

6.2. **Full stabilizer.** Now $\mathfrak{g}$ must be $\mathfrak{o}(1,n)$. Let α and J be as above. Let $\mathfrak{m}$ be the centralizer of J in $\mathfrak{g}$; it is isomorphic to $\mathfrak{o}(n-1)$.

Suppose $Y \in \mathfrak{s}(x) \cap \mathfrak{g}_{-\alpha}$. By Lemma 3.2 (3),

$$J \in [Y, \mathfrak{g}_\alpha] \subset [\mathfrak{s}(x), \mathfrak{s}(x)] \subset \mathfrak{g}(x)$$

But this contradicts Lemma 3.2 (2). Therefore, $\mathfrak{s}(x) \cap \mathfrak{g}_{-\alpha} = \mathbf{0}$.

On the other hand, since $\overline{\mathrm{ad}}(X)$ is nilpotent for all $X \in \mathfrak{g}_\alpha$, Lemma 3.2 (3) forces

$$\mathfrak{m} \subset [\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha] \subset \overline{\mathrm{ad}}(\mathfrak{g}_\alpha)(\mathfrak{g}) \subset \mathfrak{s}(x)$$

Since $\mathfrak{g} = \mathfrak{g}_{-\alpha} + \mathfrak{m} + \mathbf{R}J + \mathfrak{g}_\alpha$, the algebra $\mathfrak{s}(x)$ is exactly $\mathfrak{m} + \mathbf{R}J + \mathfrak{g}_\alpha$. Suppose there were $X = cJ + M \in (\mathbf{R}J + \mathfrak{m}) \cap \mathfrak{g}(x)$ for some nonzero $c \in \mathbf{R}$. The subspace $\mathfrak{g}_{-\alpha}$ is $\mathrm{ad}(X)$-invariant and maps onto $\mathfrak{g}/\mathfrak{s}(x)$. But $\mathrm{ad}(X)$ is clearly not skew-symmetric here, contradicting lemma 3.2 (3). Therefore, $\mathfrak{g}(x)$ is exactly $\mathfrak{m}+\mathfrak{g}_\alpha$, which is isomorphic as a Lie algebra to $\mathfrak{o}(n-1) \ltimes \mathbf{R}^{n-1}$.

7. Proofs of Theorems 1.7 and 1.9

7.1. **Proof of Theorem 1.7.** Suppose G is locally isomorphic to $O(2,n)$, $n \geq 3$, with finite center, and G acts isometrically on M. We know that a neighborhood of some G-orbit is a warped product of the form $L \times_w S$, where S is isometric to AdS_{n+1} up to finite cover. The set of orbits having a neighborhood isometric to $L \times_w S$, for some Riemannian manifold L and $w : L \to \mathbf{R}^+$, is open. Let us prove that this set is also closed, and thus equals the whole of M. A limit O of a sequence O_k of such orbits is an orbit O of dimension $\leq n+1$. Let $x_k \in O_k$ be a sequence converging to $x \in O$. The stabilizers of the x_k are all conjugate to the stabilizer of a point in S—that is, to some H locally isomorphic to $O(1,n)$. A limit of such conjugates is a noncompact subgroup of G. The stabilizer of x contains such a limit and is therefore noncompact; in particular, O cannot be spacelike. From Theorem 1.6, O cannot be degenerate; hence, it is Lorentzian. From Theorem 2.2, a neighborhood of this orbit is isometric to a warped product as above. Therefore, any orbit of M has a neighborhood isometric to $L \times_w S$, for some L and w.

From this, one sees in particular that the G-action determines a foliation $\mathcal{O}$. This foliation is locally trivial in the sense that every leaf has a neighborhood that is a product of leaves with a transverse submanifold. In addition, $\mathcal{O}$ admits an orthogonal foliation $\mathcal{O}^{\perp}$. We will use these two properties, together with the G-action, to show that the pair of foliations $\mathcal{O}$ and $\mathcal{O}^{\perp}$ arise from a global warped product of the form $L \times_w AdS_{n+1}$ on a finite cover of M.

Choose a point $x_0 \in M$. Let O_0 and $O_0^{\perp}$ be the leaves of x_0 in the foliations $\mathcal{O}$ and $\mathcal{O}^{\perp}$, respectively. Let H_0 be the stabilizer of x_0. Note that $O_0^{\perp}$ is a component of the fixed set $Fix(H_0)$. The full $Fix(H_0)$ is $N(H_0).O_0^{\perp}$, where $N(H_0)$ is the normalizer of H_0 in G. It is well known that $N(H_0)/H_0$ is finite; then since $O_0^{\perp}$ is H_0-invariant, $Fix(H_0)$ has finitely-many components, each isometric to $O_0^{\perp}$.

Let i and $i^{\perp}$ denote the respective inclusions of O_0 and $Fix(H_0)$ in M. Let G act on $Fix(H_0) \times O_0$ by $g.(x,y) = (x, g.y)$. Define a mapping $\phi : Fix(H_0) \times O_0 \to M$, by $\phi(x,y) = g.(i^{\perp}(x))$, where $g \in G$ is any transformation sending the base point x_0 to y. One sees that ϕ is well-defined, and it is in fact the G-equivariant extension of the inclusions.

Next, ϕ is a covering map. Clearly ϕ is a local diffeomorphism. It is also easy to see that ϕ is surjective: the orbit of any $y \in M$ is isometric to S. Let H_y be its stabilizer. There is some $g \in G$ conjugating H_y to H_0. Then $g.y \in Fix(H_0)$, and $y = \phi(g.y, g^{-1}.x_0)$. Finally, ϕ is everywhere N-to-1, where $N = |N(H_0)/H_0|$, because every G-orbit in M is isometric to S. An N-to-1 surjective local diffeomorphism is a covering map.

The submanifold $Fix(H_0)$ is Riemannian, and ϕ is clearly a local isometry. Since S and AdS_{n+1} have a common finite cover, M is isometric, up to finite cover, to a warped product $L \times_w AdS_{n+1}$, where $L \cong Fix(H_0)$. $\diamondsuit$

7.2. **Proof of Theorem 1.9.** In fact, all statements but the last part of point (2) follow immediately from Theorems 1.8 and 1.7. Suppose G_1 is locally isomorphic to $O(1,n)$, and G_1-orbits are isometric to S, where S has a common finite cover with dS_n. It remains to show that limits of orbits isometric to S are fixed points or isometric to the Minkowski light cone, up to finite cover (or, of course, to S, itself, if the limit is nonsingular). If

x_n have orbits isometric to S and converge to x, then, up to choosing a subsequence, the stabilizers of x_n in G_1 converge to a subgroup $H \subset G_1$, contained in the stabilizer of x; in particular, x has a noncompact stabilizer. If the orbit is of Lorentz type, then from Theorem 2.2, it must be isometric to S. If the orbit is degenerate, then we apply Theorem 1.6 to deduce that this orbit is a point or is isometric to the Minkowski light cone, up to finite cover. $\diamondsuit$

Acknowledgements: The first two authors would like to thank the École Normale Supérieure de Lyon for their hospitality.

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