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CAUSAL PROPERTIES OF ADS-ISOMETRY GROUPS II: BTZ MULTI BLACK-HOLES

THIERRY BARBOT

Abstract. This paper is the continuation of [8]. We essentially prove that the familly of strongly causal spacetimes defined in [8] associated to generic achronal subsets in Ein_2 contains all the examples of BTZ multi black-holes. It provides new elements for the global description of these multi black-holes. We also prove that any strongly causal spacetime locally modelled on the anti-de Sitter space admits a well-defined maximal strongly causal conformal boundary locally modelled on Ein_2.

1. Introduction

In [8], we studied certain aspects of causal properties of AdS-spacetimes, i.e. lorentzian manifolds of dimension 3 with negative constant curvature; in other words, locally modeled on the anti-de Sitter space AdS. In particular, we made a detailed analysis on the causality notion in AdS, which, to be meaningfull, has to be understood as the projection of the causality relation in the universal covering $\widetilde{\mathrm{AdS}}$. The Einstein universes Ein_3 and Ein_2 play an important role in this study: the first one as containing a conformal copy of AdS, the second being the conformal boundary of AdS (see § 4 in [8]).

We also introduced the notion of *(generic) closed achronal subset* of the conformal boundary Ein_2 (§ 5 of [8]). We associate to it the invisible domains $E(\Lambda) \subset \mathrm{AdS}$ and $\Omega(\Lambda) \subset \mathrm{Ein}_2$ (§ 8). One of the main results of [8] was the following: assume that Λ is non-elementary, and that a torsion-free discrete group of isometries of AdS preserves Λ. Then, the action of Γ on $E(\Lambda)$ is free, properly discontinuous, and the quotient space $M_\Lambda(\Gamma) = \Gamma\backslash E(\Lambda)$ is strongly causal (Theorem 10.1). We observed that this construction provides, when Λ is a topological circle, the entire family of maximal globally hyperbolic AdS-spacetimes admitting a Cauchy-complete Cauchy surface (§ 11; the same result is obtained in [10] with a different proof). We also discussed in which condition a discrete group of isometries Γ is admissible, i.e. preserves a closed achronal subset as above: essentially, up to a permutation of space and time, admissible groups are precisely the positively proximal groups which are precisely the groups preserving a proper closed convex domain of $\mathbb{R}P^3$ (Proposition 10.23 of [8]). We also stressed out that there is a natural 1-1 correspondance between admissible groups and pairs of (marked) complete hyperbolic metrics on the same surface (not necesserily closed). Finally, if Γ is admissible and non-abelian, there exists an unique minimal closed generic

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achronal Γ-invariant subset $\Lambda(\Gamma)$ which is contained in every closed achronal Γ-invariant subset (Theorem 10.13, corollary 10.14 of [8]). Hence, following the classical terminology used for isometry groups of $\mathbb{H}^n$, it is natural to call $\Lambda(\Gamma)$ the *limit set* of Γ.

The present paper completes this study in the elementary case, i.e. the case where Λ is contained in the past or future of one point in Ein_2. In particular, we give the description of invisible domains from elementary generic achronal domains (§ 3).

There is another equivalent definition of invisible domains $E(\Lambda(\Gamma))$ when $\Lambda(\Gamma)$ is the limit set of Γ (since $\Lambda(\Gamma)$ is minimal, $E(\Lambda(\Gamma))$ is in some way maximal among the Γ-invariant invisible domains). For any element γ of Γ we define the *standard causal domain* $C(\gamma)$ as the set of points x in AdS which are not causally related to their image γx. The interior of the intersection of all the $C(\gamma)$ is then the standard causal domain of Γ, denoted $C(\Gamma)$. Every element γ is also the time 1 of a Killing vector field. We then define the *absolute causal domain* $D(\gamma)$ as the domain of AdS where this Killing vector field has positive norm. The interior of the intersection of all the $D(\gamma)$ is the absolute causal domain $D(\Gamma)$ of Γ. According to Theorem 8.2 and corollary 8.4, *if* Γ *is a non-cyclic admissible group, then* $D(\Gamma)$ *and* $C(\Gamma)$ *both coincide with* $E(\Lambda(\Gamma))$. A first version of this theorem, in a very particular case, appears in [3], § 7.

But, most of all, we prove that the spacetimes $M_\Lambda(\Gamma)$ form a global family of *BTZ multi black-holes,* containing and enlarging all the previous examples ([1, 2, 3, 5, 6, 11, 12]), except the non-static single BTZ black-holes themselves (see remark 10.3)

In order to develop this assertion, since we don't assume any acquaintance of the reader with this notion, we need some preliminary discussion about black-holes.

1.1. **A quick insight into Schwarzschild and Kerr black-holes.** A honest introduction to the notion of black-hole requires a minimal historical exposition. First of all, a black-hole is a lorentzian manifold (M, g) solution of the Einstein equation[1]:

$$\mathrm{Ric}_g - \frac{R}{2}g = \Lambda.g$$

where:

– Ric_g is the Ricci tensor,

– R is the scalar curvature, i.e. the trace of the Ricci tensor (relatively to the metric g),

– Λ is a prescribed real number, the *cosmological constant.*

It was soon realized by K. Schwarzschild that this equation with $\Lambda = 0$ admits the following 1-parameter family of solutions:

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 ds_0^2$$

[1]The full Einstein equation contains an additionnal term on the right side, which is a symmetric trace-free tensor describing the matter and physical forces in the spacetime. The equation stated here is an equation in the void.

In this expression, $ds_0^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the usual round metric on the sphere $\mathbb{S}^2$. The term r is $\sqrt{x^2+y^2+z^2}$, where (x, y, z, t) are coordinates of $\mathbb{R}^4$, and (θ, ϕ) are the spherical coordinates of $(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}) \in \mathbb{S}^2$.

Moreover, in some manner, the Schwarzschild metric is the "unique" solution of the Einstein equation on $\mathbb{R}^4$ with spherical symmetry, i.e. invariant by the usual $\mathrm{SO}(3,\mathbb{R})$-action (Birkhoff Theorem, see [24] for a more rigorous and detailed exposition). Hence, this metric suits perfectly the role of model of a non-rotating isolated object (for example, a star) with mass M. But these solutions contains a psychological difficulty: the radius of the object could be inferior to $r = 2M$, in which case the singularity $r = 2M$ of the metric cannot be dropped out by considering that it is hidden inside the object, where the general relativity theory does not apply anymore. It has been realized that the singularity is not a real one: it appears only in reason of the selected coordinate system. Actually, the Schwarzschild metric on $\mathbb{R}^4 \setminus \{r = 0, 2M\}$ can be isometrically embedded in another lorentzian manifold M_{KS}, diffeomorphic to $\mathbb{R}^2 \times \mathbb{S}^2$, and maximal: M_{KS} cannot be isometrically embedded in a bigger spacetime. This maximal spacetime is usually described in the following way: equip $\mathbb{R}^2 \times \mathbb{S}^2$ with coordinates (u, v, θ, ϕ) - the *Kruskal-Szekeres coordinates* - where (θ, ϕ) are still the spherical coodinates. Then, M_{KS} is the open domain $U \times \mathbb{R}^2$, where $U = \{v^2 - u^2 < 1\}$, equipped with the metric:

$$ds_{KS}^2 = (32M^3/r)e^{-r/2M}(-dv^2 + du^2) + r^2 ds_0^2$$

The term r, defined by $(r/2M - 1)e^{r/2M} = u^2 - v^2$ coincide with the r-coordinate on the Schwarzschild domain: hence, the "singularity" $r = 2M$ vanishes: it corresponds to the locus $\{u^2 = v^2\}$.

The domain $O = \{0 < u^2 - v^2 < 1\}$ is the *outer domain:* it is thought as the region where the typical (prudent) observer takes place. There are two other regions: $B^\pm = \{u^2 - v^2 < 0, \pm v > 0\}$. Moreover, M_{KS} is time-oriented so that v increases with time.

B^+ enjoys the following remarkable property: there is no future-oriented causal curve starting from a point in B^+ and reaching a point in O. Similarly, a past-oriented causal curve cannot escape from B^-. In a more "physical" language, photons cannot escape from B^+: B^+ is invisible from O. In other words, B^+ is a *black-hole.* The no-return frontier $\{v =\mid u \mid\}$ is the *(event) horizon.*

The family of Schwarzschild metrics is actually included in a more general family of solutions, the *Kerr metrics:*

$$-(1-\frac{2Mr}{\rho^2})dt^2+\frac{\rho^2}{\Delta}dr^2-\sin^2(\theta)\frac{4Mra}{\rho^2}d\phi dt+\rho^2d\theta^2+(r^2+a^2+\sin^2(\theta)\frac{2Mra}{\rho^2})\sin^2(\theta)d\phi^2$$

where:
– $\Delta = r^2 - 2Mr + a^2$, t $\rho^2 = r^2 + a^2\cos^2(\theta)$,
– M is positive (the "mass"),
– a is a real number (the "angular momenta per mass unit").

A phenomenom similar to the embedding of Schwarzschild spacetime in Kruskal-Szekeres coordinates still applies: every Kerr-spacetime can be embedded in a natural way into a maximal spacetime M^{max}_{Kerr} where the singularities appearing in the Kerr-coordinates vanish, and where some domains deserve the appelation "black-hole". But the description of this completion is much more involved: it is the entire matter of the excellent book [25], which is also one of the best references for a rigorous description of Schwarzschild black-holes.

1.2. **Towards a general definition of black-holes.** The basic facts on Schwarzschild spacetimes presented above are sufficient to provide a quite satisfactory illustration of essential notions that should appear in any mathematical definition of black-holes. Let's be more precise and express some problems arising when trying to elaborate such a general theory, i.e. in our meaning, to specify what are the spacetimes deserving the appelation of spacetime with black-hole.

(1) *Where are the typical observers?* The description of the Schwarzschild black-hole makes clear that this notion is relative to the region O where the observers are assumed to stay. A black-hole is simply a connected component of the region invisible from O. How to define the region O in M without specifying the black-hole itself?
(2) *Is some part of M missing?* The notion of black-hole is a global property, depending on the entire M. For example, if we delete from the Minkowski space some regions, we can easily produce in an artificial way regions invisible from the observers (assuming solved the first question above) which does not correspond to a physically relevant example of black-hole. Hence, we have to define a notion of "full" spacetime ensuring that some part of the invisible domain is not simply due to the absence of some relevant region.

Remark 1.1. To these two basic problems, another important requirement, traditionnally appearing in the physical litterature, should be added: the *Cosmic Censorship.* This condition admits many different formulations. It is most of the time expressed in the form "singularities of the spacetime must be hidden to the observers by the horizons". There is a geometrical way to translate this notion, interpreting singularities as final points of non-complete causal geodesics, or as causal singularities: we could define spacetimes satisfying the Cosmic Censorship if causal closed curves, if any, all belong to a black hole, and if any non-complete causal future-oriented ray in M, with starting point visible from one observer, must enter in a black-hole

There is a particular case of the Cosmic Censorship conjecture often stated as follows: "generically" (to be defined!), maximal globally hyperbolic spacetimes are maximal, i.e. cannot be embedded in a bigger spacetime, even not globally hyperbolic. But the spacetimes containing a black-hole we will consider here are not globally hyperbolic, hence this expression of Cosmic Censorship is not relevant for them.

Some answer to the first question above seems to be widely accepted: *observers lie near the conformal boundary.* Then, we can define the black-holes as the connected components of the *invisible domain,* i.e. the interior of the region of M containing the points x such that no causal future-oriented curve starting from x tends to a point in the conformal boundary of M. Of course, this solution arises immediately another question: what is the conformal boundary? This new question is not answered in full generality - for a nice recent survey on related questions see [18]. We can summarize many works by observing that this point is most of the time solved simply by prescribing from the beginning what is the conformal boundary, without making sure that the proposed boundary is maximal in any meaning[2]. However, in our special AdS context, we have a completely satisfactory and easy definition of conformal boundary of *strongly causal* AdS spacetimes (see next §).

Concerning the second question, encouraged by the examples of Kruskal-Szekeres and Kerr spacetimes, one could hope that a good answer is simply to require M to be maximal in the sense that it cannot be isometrically embedded in a bigger spacetime satisfying the Einstein equation. Unfortunately, this attempt fails for BTZ black-holes: they are not maximal in this meaning. Furthermore, such a definition of full would lead to some incoherence in the maximal Kerr spacetime. The restriction to causal spacetimes, i.e. to require that M is a causal spacetime, and that it cannot be embedded in a bigger causal spacetime does not solve the problem: once more, it does not apply to BTZ black-holes (see remark 10.7).

Our deceiving conclusion is that we still don't know how to express in a satisfying way what is a good notion of full spacetime, even in the AdS-background. As a positive element of answer, we stress out that all the examples reproduced later enjoy the following properties of "maximal" nature: every connected component of their conformal boundary is maximal globally hyperbolic, and every black-hole, i.e. every connected component of the invisible domain (but this domain will always be connected) is maximal globally hyperbolic too (see remark 10.9). Observe that these requirements (global hyperbolicity of black-holes and observer-spacetime) reflect in some way the Cosmic Censorship principle for globally hyperbolic spacetimes (remark 1.1).

1.3. **BTZ black-holes.** First of all, BTZ-multi-black-holes have dimension $2+1$, i.e. are 3-dimensional. In this low dimension the Einstein equation is remarkably simplified: the solutions have all constant sectional curvature with the same sign that the cosmological constant Λ. We only consider the case $\Lambda < 0$. Hence, up to rescaling, the spacetimes satisfying the Einstein equation are precisely the spacetimes locally modeled on AdS.

This special feature allows us to propose a correct answer to the first question in the preceding paragraph, in the $2+1$-dimensional case, i.e. with AdS-background, thanks to the natural conformal completion of AdS by Ein_2. More precisely, the spacetimes M we will consider are locally modeled in AdS - in short, they are AdS-spacetimes. Hence, their universal covering admits a development $\mathcal{D} : \widetilde{M} \to \mathrm{AdS}$. Define then the lifted conformal

[2]Anyway, for the Kruskal-Szekeres spacetimes M_{KS} and maximal Kerr-spacetimes M^{max}_{Kerr}, which are all asymptotically flat, the conformal boundary is well-defined in a fully satisfying way.

boundary Ω as the interior of the set of final extremities in $\mathrm{Ein}_2 = \partial\mathrm{AdS}$ of future-oriented causal curves with relative interior contained in $\mathcal{D}(\widetilde{M})$. We could try to define the conformal boundary of M as the quotient of Ω by the holonomy group $\rho(\Gamma)$. Unfortunately, this definition admits an uncomfortable drawback: the development $\mathcal{D}$ could be non-injective, which requires a modification of the definition above. Moreover, we have no guarantee *a priori* that the action of $\rho(\Gamma)$ on Ω is free and proper.

In § 9, we show to solve these difficulties in the context of strongly causal spacetimes.: every strongly causal AdS-spacetime admits a natural strongly causal conformal boundary (Theorem 9.5).

For that reason (besides the physical coherence of such an assumption) we restrict from now to strongly causal spacetimes. Now we can state our peculiar definition:

Definition 1.2. *An AdS-spacetime with black hole is a strongly causal AdS spacetime M such that:*

– M admits a non-empty strongly causal conformal boundary O,

– the past of O, i.e. the region of M made of initial points of future oriented causal curves ending in O,is not the entire M.

Every connected component of the interior of the complement in M of the past of O is a black-hole.

Let's now collect the examples. While considering discrete groups of isometries of AdS, we have constructed many examples of spacetimes locally modeled on AdS: the quotients $M(\Gamma) = \Gamma\backslash E(\Gamma)$ where Γ is a non-abelian torsion-free discrete admissible subgroup of Isom(AdS). *Every $M(\Gamma)$ contains a black-hole, except if it is globally hyperbolic, i.e. except if Λ is a topological circle.*

This construction applies in more cases, even if Γ is abelian, but the elementary achronal subset Λ must be selected: it must contain at least two points, and Γ must be a cyclic subgroup, generated by an isometry γ_0. More precisely:

– (the conical case) This is the case where Λ is the union of two lightlike geodesic segments $[y, x]$, $[z, x]$, each of them not reduced to single point and with a common extremity x, which is their common past. Then γ_0 must be a hyperbolic - hyperbolic element (see definition 4.2) and y, z must be attractive fixed points of γ_0.

– (the splitting case) This is the case where Λ is a pair of non-causally related points in Ein_2. As in the previous case, these points must be the attractive and repulsive fixed points of the hyperbolic-hyperbolic element γ_0.

– (the extreme case) This last case is the case where Λ is a lightlike segment, not reduced to a single point. Then, γ_0 must be a parabolic-hyperbolic element (see definition 4.2) fixing the two extremities of Λ.

In all these cases, the quotient $M_\Lambda(\Gamma) = \Gamma\backslash E(\Lambda)$ is still a spacetime with conformal boundary, and with non-empty invisible set: a black-hole.

Since it requires a basic knowledge of AdS geometry, the description of these spacetimes is postponed to § 10. We just mention in this introduction that they essentially[3] include all the previously examples named BTZ black-holes ([5, 6, 1, 2, 3, 11, 12], etc...), but contains also new examples: mainly, the case where Γ is not finitely generated, and also the conical case, which is not considered as a spacetime with black-hole in these references, probably because it is obviously non-maximal, since, in this case, $M_\Lambda(\Gamma)$ embeds isometrically into $M_{yz}(\Gamma)$.

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Notations

This paper should be readen jointly with [8] in which all basic notions and notations are introduced. Let's recall some of them, for the reader convenience:

- AdS is the anti-de Sitter space, i.e. the locus $\{Q = -1\}$ in $E \approx \mathbb{R}^{2,2}$ equipped with a quadratic form Q of signature $(2, 2)$. It is isometric to the Lie group $\mathrm{SL}(2, \mathbb{R})$ endowed with the lorentzian metric defined by the Killing form. Its universal covering is denoted by $\widetilde{\mathrm{AdS}}$ and its projection into the sphere $S(E)$ of rays in E - the Klein model - is denoted $\mathbb{ADS}$. The projective Klein model $\overline{\mathbb{ADS}}$ is the projection in the projective space $P(E)$. It is naturally identified with the Lie group $G = \mathrm{PSL}(2, \mathbb{R})$. All these lorentzian manifolds are oriented and chronologically oriented.
- A spacetime is a lorentzian manifold which is oriented and chronologically oriented. An AdS - spacetime is a spacetime locally modelled on the anti-de Sitter space AdS.
- Affine domains in AdS, $\widetilde{\mathrm{AdS}}$, $\mathbb{ADS}$ are lifts of affine domains in $\overline{\mathbb{ADS}}$, i.e. intersection in $P(E)$ between $\overline{\mathbb{ADS}}$ and affine domains U of $P(E)$ such that the intersection between the projective plane ∂U and $\overline{\mathbb{ADS}}$ is a spacelike surface; it is then an isometric copy of the hyperbolic plane $\mathbb{H}^2$.
- The isometry group of $\overline{\mathbb{ADS}}$ is $G \times G$. The isometry group of $\widetilde{\mathrm{AdS}}$ is isomorphic to $\widetilde{G} \times \widetilde{G}$ quotiented by the cyclic diagonal group generated by (δ, δ), where δ is a generator of the center of $\widetilde{G}$, the universal covering of G.
- There is a conformal embedding of AdS into the Einstein universe Ein_3, which is $\mathbb{S}^2 \times \mathbb{S}^1$ equipped with (the conformal class of) the lorentzian metric $ds^2 - dt^2$, where ds^2 is the round metric on the 2-sphere and dt^2 the usual metric on $\mathbb{S}^1 \approx \mathbb{R}/2\pi\mathbb{Z}$. The universal covering of Ein_3 is denoted by $\widehat{\mathrm{Ein}}_3$. There is a conformal embeding of $\widetilde{\mathrm{AdS}}$ into $\widehat{\mathrm{Ein}}_3$. The boundary of the image of this embedding is

[3] Except the particular the case of non-static single BTZ black-holes, see remark 10.3.

the conformal boundary $\partial\widetilde{\mathrm{AdS}} \approx \widehat{\mathrm{Ein}}_2$. The projection $\overline{\mathrm{Ein}}_2$ of Ein_2 in $P(E)$ is the conformal boundary of $\overline{\mathbb{ADS}}$.

- $\overline{\mathrm{Ein}}_2$ is bifoliated by two transverse foliations $\overline{\mathcal{G}}_L$, $\overline{\mathcal{G}}_R$. Every leaf of $\overline{\mathcal{G}}_L$ (the left foliation) or of $\overline{\mathcal{G}}_R$ (the right foliation) is a lightlike geodesic, canonically isomorphic to the real projective line $\mathbb{R}P^1$. Every leaf of $\overline{\mathcal{G}}_L$ intersects every leaf of $\overline{\mathcal{G}}_R$ at one and only one point. Finally, the leaf space of the left (resp. right) foliation is naturally isomorphic to the real projective line. It is denoted by $\mathbb{R}P^1_L$ (resp. $\mathbb{R}P^1_R$).
- For all causality notions, we refer to [9]. In [8] we also present the causality notions pertinent for our purpose. An achronal subset of $\widehat{\mathrm{Ein}}_2$ is a subset $\widetilde{\Lambda}$ such that for any pair (x, y) of distinct elements of $\widetilde{\Lambda}$, x and y are not causally related in $\widehat{\mathrm{Ein}}_2$. An achronal subset $\widetilde{\Lambda}$ is pure lightlike if it contains two opposite elements x, $\delta(x)$; if not, it is generic. A generic achronal subset is elementary if it is contained in an union $l \cup l'$, where l, l' are lightlike geodesic segments in $\widehat{\mathrm{Ein}}_2$ (see § 8.7 in [8]). All these notions project on similar notions in Ein_2. See [8], § 5 for more details.
- The invisible domain of a generic closed achronal subset $\widetilde{\Lambda}$ in $\widehat{\mathrm{Ein}}_2$ is the set of points in $\widehat{\mathrm{Ein}}_2$ which are not causally related to any element of $\widetilde{\Lambda}$. It is denoted by $\Omega(\widetilde{\Lambda})$. It is the region between the graphs $\widetilde{\Lambda}^+$ and $\widetilde{\Lambda}^-$ of two 1-Lipschitz functions from $\mathbb{S}^1$ into $\mathbb{R}$ (see § 8.6 in [8]). The invisible domain $E(\widetilde{\Lambda})$ in $\widetilde{\mathrm{AdS}}$ is the set of points in $\widetilde{\mathrm{AdS}} \subset \widehat{\mathrm{Ein}}_3$ which are not causally related to any element of $\widetilde{\Lambda}$ in $\widehat{\mathrm{Ein}}_3$.

2. Strongly causal spacetimes

A lorentzian manifold M is *strongly causal* if for every point x in M every neighborhood of x contains an open neighborhood U which is *causally convex,* i.e. such that any causal curve in M joining two points in U is actually contained in U (see §2.3 in [8]). An isometric action of a group Γ on a strongly causal lorentzian manifold M is strongly causal if any point x of M admits arbitrarily small neighborhoods U such that for every non trivial element γ of Γ no point of U is causally related to a point of γU. Clearly, if Γ is a group of isometries of a strongly causal lorentzian manifold M acting freely and properly discontinuously, the quotient manifold $\Gamma\backslash M$ is strongly causal if and only if the action of Γ is strongly causal.

In this paper we need to prove that certain isometric actions are strongly causal. In this § we provide two lemmas useful for this purpose.

In the first lemma we consider a strongly causal lorentzian manifold M admitting a Killing vector field X. We assume that X is everywhere spacelike. Let ϕ^t be the flow generated by X and assume that the $\mathbb{R}$-action defined by ϕ^t is free and properly discontinuous. Let Q^ϕ be the orbit space of ϕ^t and $\pi : M \to Q^\phi$ the projection map. It is easy to show that π is a locally trivial fibration with fibers homeomorphic to $\mathbb{R}$. For every tangent vector v of Q^ϕ at a point y let x be any element of $\pi^{-1}(y)$ and let w be the unique tangent vector to M at x orthogonal to $X(x)$ and such that $d\pi_x(w) = v$.

Since ϕ^t is isometric the norm of w does not depend on the choice of x: we define the norm $\mid v \mid$ as this norm $\mid w \mid$. This procedure defines a lorentzian metric on Q^ϕ.

Lemma 2.1. *If the lorentzian manifold Q^ϕ is strongly causal then the action of ϕ^1 on M is strongly causal.*

Proof Let x be any element of M and U any neighborhood of x. Let $y = \pi(x)$. Since Q^ϕ is assumed strongly causal, $U' = \pi(U)$ contains a causally convex open neighborhood V' of y. Define $V = \pi^{-1}(V')$: this open subset is diffeomorphic to $V' \times \mathbb{R}$, such that the restriction of ϕ^t to V corresponds to the translation on the $\mathbb{R}$-factor. We also take the convention that the $\mathbb{R}$-coordinate of x is zero: $x \approx (y, 0)$. Shrinking V' if necessary, we can assume that the spacelike norm of X on V is uniformly bounded by some constant C, and moreover that any causal curve in V' has lorentzian length less than $1/2C$. It then follows that the $\mathbb{R}$-coordinate of any causal curve in V cannot vary more than $1/2$. Let W be a causally convex open neighborhood of x contained in $V' \times] - 1/4, +1/4[\subset V$. We claim that for every non zero integer n non point of $\phi^n W$ is causally related to a point of W. Indeed: assume by contradiction that there is a causal curve $c : I \to M$ joining an element (z, t) to an element $(z', s + n)$ with z, z' in V' and s, t in $] - 1/4, +1/4[$. The projection $\pi \circ c$ is a causal curve in Q joining two $\pi(z)$, $\pi(z')$. Since V' is causally convex $\pi \circ c$ is contained in V', i.e. c is contained in V. Hence the $\mathbb{R}$-coordinate cannot vary along c more than $1/2$. We obtain a contradiction since $s + n - t$ has absolute value bigger than $1/2$. ∎

Remark 2.2. Lemma 2.1 remains true when the action of ϕ^t is not free but periodic. In this case the group generated by ϕ^1 is finite. Details are left to the reader.

We will apply lemma 2.1 in the case where M has dimension 3 and is simply connected. In this situation the map π is a fibration and the homotopy sequence of this fibration implies that Q^ϕ is a 2-dimensional manifold which is simply connected. Hence a very nice complement is:

Lemma 2.3. *Any simply connected 2-dimensional lorentzian manifold is strongly causal.*

Proof Any simply connected surface is diffeomorphic to the sphere or the plane $\mathbb{R}^2$. Since the sphere does not admit any lorentzian metric we just have to consider the case of the plane $\mathbb{R}^2$. Any reader acquainted with dynamical systems will recognize that the lemma follows quite easily from the Poincaré-Bendixon Theorem applied to the lightlike foliations: every leaf of one of these foliations is a closed embedding of the real line in $\mathbb{R}^2$ and for any point x in $\mathbb{R}^2$ the future (resp. the past) of x is the domain bounded by $r_1 \cup r_2$ where r_1, r_2 are the future oriented (resp. past oriented) lighlike geodesic rays starting from x. Details are left to these readers.

For other readers, we refer to Theorem 3.43 of [9] where a slightly better statement is proved: lorentzian metrics on the 2-plane are *stably* causal, which is stronger than being strongly causal. ∎

3. Description of the elementary generic achronal subsets of $\widehat{\mathrm{Ein}}_2$

In this section, we complete the descriptions of invisible domains $E(\widetilde{\Lambda})$ and $\Omega(\widetilde{\Lambda})$ by considering the elementary cases. For the reader convenience, we start with the splitting case, which is already described in [8], § 8.8.

3.1. **The splitting case.** It is the case $\widetilde{\Lambda} = \{x, y\}$, where x, y are two non-causally related points in $\widehat{\mathrm{Ein}}_2$. Then, $\{x, y\}$ is a gap pair and there are two associated ordered gap pairs that we denote respectively by (x, y) and (y, x) (see definition 8.24 in [8]). $\widetilde{\Lambda}^+$ is an union $\mathcal{T}^+_{xy} \cup \mathcal{T}^+_{yx}$ of two future oriented lightlike segments with extremities x, y that we call *upper tents*. Such an upper tent is the union of two lightlike segments, one starting from x, the other from y, and stopping at their first intersection point, that we call the *upper corner.*

Similarly, $\widetilde{\Lambda}^-$ is an union $\mathcal{T}^-_{xy} \cup \mathcal{T}^-_{yx}$ of two *lower tents* admitting a similar description, but where the lightlike segments starting from x, y are now past oriented (see Figure 1, Figure 2), and sharing a common extremity: the *lower corner.*

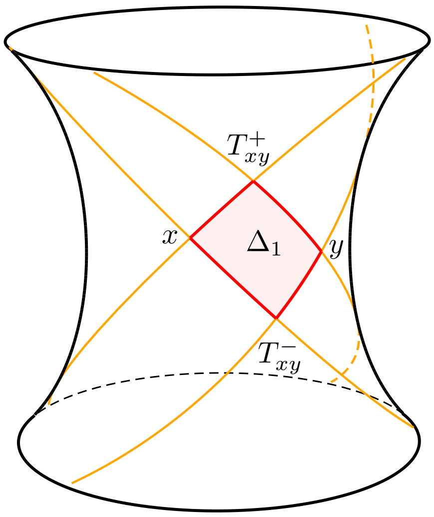

Figure 1. *Upper and lower tents*

The invisible domain $\Omega(\widetilde{\Lambda})$ from $\widetilde{\Lambda}$ in $\widehat{\mathrm{Ein}}_2$ is the union of two diamond-shape regions $\widetilde{\Delta}_1$, $\widetilde{\Delta}_2$. The boundary of $\widetilde{\Delta}_1$ is the union $\mathcal{T}^+_{xy} \cup \mathcal{T}^-_{xy}$, and the boundary of $\widetilde{\Delta}_2$ is $\mathcal{T}^+_{yx} \cup \mathcal{T}^-_{yx}$. We project all the picture in some affine region $V \approx \mathbb{R}^3$ of $S(E)$ such that:

- $V \cap \mathbb{ADS}$ is the interior of the hyperboloid: $\{x^2 + y^2 < 1 + z^2\}$,
- $\Lambda = \{(1, 0, 0), (-1, 0, 0)\}$.

Then, $E(\Lambda)$ is region $\{-1 < x < 1\} \cap \mathbb{ADS}$. One of the diamond-shape region $\widetilde{\Delta}_i$ projects to $\Delta_1 = \{-1 < x < 1, y > 0, x^2 + y^2 = 1 + z^2\}$, the other projects to $\Delta_2 = \{-1 < x < 1, y < 0, x^2 + y^2 = 1 + z^2\}$. The past of Δ_1 in $E(\Lambda)$ is $P_1 = \{(x, y, z) \in E(\Lambda)/z < y\}$. and the future of Δ_1 in $E(\Lambda)$ is $F_1 = \{(x, y, z) \in E(\Lambda)/z > -y\}$. We have of course a similar description for the future F_2 and the past P_2 of Δ_2 in $E(\Lambda)$. Observe:

– the intersections $F_1 \cap F_2$ and $P_1 \cap P_2$ are disjoint. They are tetraedra in $S(E)$: $F_1 \cap F_2$ is the interior of the convex hull of Λ^+, and $P_1 \cap P_2$ is the interior of the convex hull of Λ^-.

– the intersection $F_1 \cap P_1$ (resp. $F_2 \cap P_2$) is the intersection between $\mathbb{ADS}$ and the interior of a tetraedron in $S(E)$: the convex hull of Δ_1 (resp. Δ_2).

Definition 3.1. *$E^+(\Lambda) = F_1 \cap F_2$ is the future globally hyperbolic convex core; $E^-(\Lambda) = P_1 \cap P_2$ is the past globally convex core.*

This terminology is justified by the following (easy) fact: $F_1 \cap F_2$ (resp. $P_1 \cap P_2$) is the invisible domain $E(\Lambda^+)$ (resp. $E(\Lambda^-)$). Hence, they are indeed globally hyperbolic.

The intersection between the closure of $E(\Lambda)$ in $S(E)$ and the boundary $\mathcal{Q}$ of $\mathbb{ADS}$ is the union of the closures of the diamond-shape regions. Hence, $\Delta_{1,2}$ can be thought as the conformal boundaries at infinity of $E(\Lambda)$. Starting from any point in $E(\Lambda)$, to Δ_i we have to enter in $F_i \cap P_i$, hence we can adopt the following definition:

Definition 3.2. *$F_1 \cap P_1$ is an end of $E(\Lambda)$.*

Finally:

Definition 3.3. *The future horizon is the past boundary of $F_1 \cap F_2$; the past horizon is the future boundary of $P_1 \cap P_2$.*

Proposition 3.4. *$E(\Lambda)$ is the disjoint union of the future and past globally hyperbolic cores $E^{\pm}(\Lambda)$, of the two ends, and of the past and future horizons.* ∎

Remark 3.5. In the conventions of [5, 6, 12], the globally hyperbolic convex cores $F_1 \cap F_2$ and $P_1 \cap P_2$ are *regions of type II,* also called *intermediate regions.* The ends $F_1 \cap P_1$ and $F_2 \cap P_2$ are *outer regions,* or *regions of type I.*

3.2. **The extreme case.** The extreme case is harder to picture out since $\Omega(\widetilde{\Lambda})$ and $E(\widetilde{\Lambda})$ are *not* contained in an affine domain (see figure 3). Assume that y is in the future of x. Observe that $\widetilde{\Lambda}^{\pm}$ are then pure lightlike. Hence, $E(\widetilde{\Lambda}^{\pm})$ are empty. The region $\Omega(x, y)$ is a "diamond" in Ein_2 (we call it an *extreme diamond*) admitting as boundary four lightlike segments: the segments $[y, \delta(x)]$, $[x, \delta(x)]$, $[\delta^{-1}(y), x]$, and $[y, \delta^{-1}(y)]$.

A carefull analysis shows that $E(x, y)$ is precisely the intersection between the past and the future of $\Omega(x, y)$.

Keeping x fixed, and considering a sequence y_n converging to y, with y_n non-causally related to x, then one of the diamond-shape region Δ_i^n of the associated $\Omega(x, y_n)$ - let's say, Δ_2^n - vanishes. The other converges to the entire region $\Omega(x, y)$. The various parts of the domains $E(x, y_n)$, namely

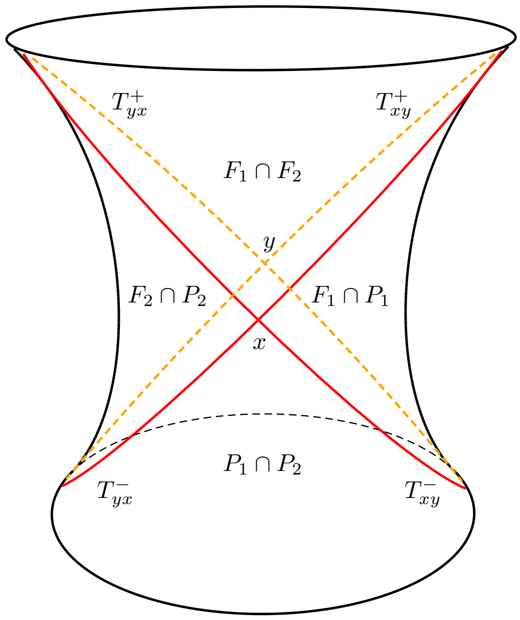

Figure 2. *The splitting case. The domain $E(\Lambda)$ is between the hyperplanes $x^\perp$ and $y^\perp$. These hyperplanes, tangent to the hyperboloid, are not drawn, except their intersections with the hyperboloid, which are the upper and lower tents $\mathcal{T}^\pm_{xy}$, $\mathcal{T}^\pm_{yx}$.*

the globally hyperbolic convex cores $E^\pm(x, y_n)$ and the ends, vanish, except one of the end, which becomes closer and closer to the entire $E(x, y)$.

3.3. **The conical case.** In the conical case, Λ is an upper or lower tent. By symetry, we can consider only the upper case: $\Lambda = \mathcal{T}^+_{xy} = [x, z] \cup [z, y]$. Then, $\Lambda^+ = \Lambda \cup \mathcal{T}^+_{yx}$, and Λ^- is the pure lightlike subset $\Lambda \cup \mathcal{T}^-_{yx}$. In the notations of § 3.1, Λ^+ is the future boundary of the diamond Δ_1, and $\Omega(\Lambda)$ is the diamond Δ_2. Then, in some affine domain, $E(\Lambda) = \{(x, y, z)/ -1 < x < 1, z > y\}$. It can be described as the future in V of Δ_2. It is also the intersection between the past of x, the past of y, and the complement of the past of z. Finally, $E(\Lambda)$ is the union of $F_1 \cap F_2$, $F_2 \cap P_2$ and the component of the past horizon of $E(x, y)$ separating these two regions (see figure 2).

4. Causal domains of isometries of AdS

4.1. **The isometry group.** We recall some facts established in [8], § 9, concerning isometries preserving generic achronal subsets of Ein_2. We use the identification $\overline{\mathbb{ADS}} \approx G = \mathrm{PSL}(2, \mathbb{R})$ (cf. § Notations). Then $\widetilde{\mathrm{AdS}}$ can be identified with the universal covering $\widetilde{G} = \widetilde{\mathrm{SL}}(2, \mathbb{R})$. Denote by $\bar{p} : \widetilde{G} \to G$ the covering map, and Z the kernel of $\bar{p}$: Z is cyclic, it is the center of $\widetilde{G}$. Let δ be a generator of Z: we select it in the future of the neutral element *id*.

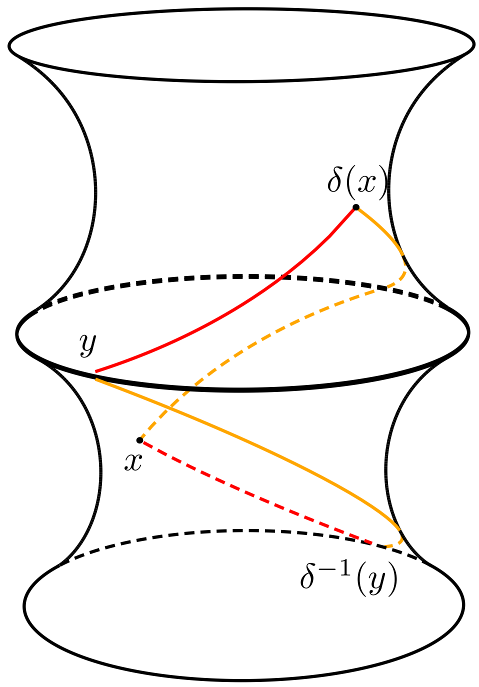

FIGURE 3. *The extreme case. The point $\delta(x)$ is in the front of the upper hyperboloid. The point x is in the rear of the lower hyperboloid. The domain $E(\Lambda)$ is the domain between the hyperplanes $\delta(x)^\perp$ and $\delta^{-1}(y)^\perp$.*

$\widetilde{G}\times\widetilde{G}$ acts by left and right translations on $\widetilde{G}$. This action is not faithfull: the elements acting trivially are precisely the elements in $\mathcal{Z}$, the image of Z by the diagonal embedding. The isometry group $\widehat{\mathrm{SO}}_0(2,2)$ is then identified with $(\widetilde{G}\times\widetilde{G})_{/\mathcal{Z}}$.

Let $\mathcal{G}$ be the Lie algebra $\mathrm{sl}(2,\mathbb{R})$ of G: the Lie algebra of $(\widetilde{G}\times\widetilde{G})_{/\mathcal{Z}}$ is $\mathcal{G}\times\mathcal{G}$. We assume the reader familiar with the notion of elliptic, parabolic, hyperbolic elements of $\mathrm{PSL}(2,\mathbb{R})$. Observe that hyperbolic (resp. parabolic) elements of $\mathrm{PSL}(2,\mathbb{R})$ are the exponentials $\exp(A)$ of *hyperbolic* (resp. *parabolic, elliptic*) elements of $\mathcal{G}=\mathrm{sl}(2,\mathbb{R})$, i.e. such that $\det(A)<0$ (resp. $\det(A)=0$, $\det(A)>0$).

Definition 4.1. *An element of $\widetilde{G}$ is* hyperbolic *(resp. parabolic, elliptic) if it is the exponential of a hyperbolic (resp. parabolic, elliptic) element of $\mathcal{G}$.*

Definition 4.2. *An element $\gamma=(\gamma_L,\gamma_R)$ of $\widetilde{G}\times\widetilde{G}$ is synchronised if, up to a permutation of left and right components, it has one of the following form:*

- *(hyperbolic translation): γ_L is trivial and γ_R is hyperbolic,*
- *(parabolic translation): γ_L is trivial and γ_R is parabolic,*

- *(hyperbolic - hyperbolic)* γ_L *and* γ_R *are both non-trivial and hyperbolic,*
- *(parabolic - hyperbolic)* γ_L *is parabolic and* γ_R *is hyperbolic,*
- *(parabolic - parabolic)* γ_L *and* γ_R *are both non-trivial and parabolic,*
- *(elliptic)* γ_L *and* γ_R *are elliptic elements conjugate in* $\widetilde{G}$.

An element γ *of* $(\widetilde{G} \times \widetilde{G})_{/\mathcal{Z}}$ *is synchronized if it is represented by a synchronized element of* $\widetilde{G} \times \widetilde{G}$.

Lemma 4.3 (Lemma 9.6 in [8])**.** *An isometry* γ *is synchronized if and only if there is an affine domain* U *in* $\widetilde{AdS}$ *such that* $\gamma^n(U) \cap U \neq \emptyset$ *for every* n *in* $\mathbb{Z}$. ∎

Observe:

Lemma 4.4 (Lemma 5.6 of [8])**.** *Every generic closed achronal subset* Λ *of* $\widehat{Ein}_n$ *is contained in a de Sitter domain.* ∎

Hence:

Corollary 4.5. *Any isometry preserving a generic closed achronal subset of* $\widehat{Ein}_2$ *is synchronized.* ∎

4.2. **Causal open subsets.** Let $\gamma = (\gamma_L, \gamma_R) = (\exp(X_L), \exp(X_R))$ be a synchronised element of $\widetilde{G} \times \widetilde{G}$.

Definition 4.6. *The standard causal subset of* γ*, denoted by* $C(\gamma)$*, is the set of points* x *of* $\widetilde{AdS}$ *for which* γx *is not causally related to* x.

Observe that $C(\gamma) = C(\gamma^{-1})$, and $C(\gamma)$ is γ-invariant. The inclusions $C(\gamma^k) \subset C(\gamma)$ follows.

Definition 4.7. *The convex causal subset of* γ*, denoted by* $C_\infty(\gamma)$*, is the set of points of* $\widetilde{AdS}$ *admitting in their future no* γ*-iterates of themselves.*

Clearly, $C_\infty(\gamma)$ is the decreasing intersection of all $C(\gamma^n)$ when n describes all $\mathbb{Z}$. It is γ-invariant.

At first glance, it seems natural to consider $C_\infty(\gamma)$ as the prefered γ-invariant subset such that the quotient is *causal,* i.e. does not admit closed causal curves (see [9], page 7). Actually, there exists a bigger subset with the same property, which, in some way, is a maximal open subset with this property. The construction goes as follows: γ is the time one map of the flow $\gamma^t = (\exp(tX_L), \exp(tX_R))$ induced by some Killing vector field X_γ of $\widetilde{\mathrm{AdS}}$.

Definition 4.8. *The absolute causal subset of* γ*, denoted by* $D(\gamma)$*, is the open subset of* $\widetilde{AdS}$ *where* X_γ *is spacelike.*

Lemma 4.9. *The open domain* $D(\gamma)$ *is the union of all the* $C(\gamma^{\frac{1}{n}})$. ∎

Clearly, since X_γ is lightlike on the boundary of $D(\gamma)$:

Lemma 4.10. *Let* $U \subset \widetilde{AdS}$ *be a* γ^t*-invariant subset containing* $D(\gamma)$*. If the quotient of* U *by* γ *is causal, then* $U = D(\gamma)$. ∎

Proposition 4.11. *If γ is not a pair (γ_L, γ_R) of elliptic elements with irrationnal rotation angle, the quotient space of $D(\gamma)$ by γ is a strongly causal spacetime.*

Proof

Denote:

$$R_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Delta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$H = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Up to conjugacy in $\widetilde{G} \times \widetilde{G}$, inversion of time orientation and permutation of the left-right components, we have 7 cases to consider:

(1) $(X_L, X_R) = (\lambda R_0, \lambda R_0)$ $(\lambda > 0)$,
(2) $(X_L, X_R) = (\lambda \Delta, 0)$ $(\lambda > 0)$,
(3) $(X_L, X_R) = (H, 0)$,
(4) $(X_L, X_R) = (H, -H)$,
(5) $(X_L, X_R) = (H, H)$,
(6) $(X_L, X_R) = (\lambda \Delta, \mu \Delta)$ $(0 < \lambda \leq \mu)$,
(7) $(X_L, X_R) = (\lambda \Delta, H)$ $(\lambda > 0)$.

According to lemmas 2.1, 2.3 the proposition is proved as soon as we check in every case that every connected component of $D(\gamma)$ is simply connected (when not empty).

For every $\tilde{g}$ in $\widetilde{\mathrm{AdS}} \approx \widetilde{G}$, the norm of $X_\gamma(\tilde{g})$ is $-det$ of $X_L - gX_Rg^{-1} = X_L - Ad(g)X_R$ (where $g = p(\tilde{g})$). It follows easily that in case (3) and (4), $D(\gamma)$ is actually empty.

4.2-a. *Case* (1)*: conjugacy by an elliptic element.* In this case $D(\gamma)$ is $\widetilde{G} \setminus \mathbf{R}$, where $\mathbf{R} = \{\exp(tR_0)\}$, i.e. the complement of the set of fixed points of γ. The quotient of $D(\gamma)$ by the flow γ^t is simply connected: apply remark 2.2.

4.2-b. *Case* (2)*: translation by a hyperbolic element.* In this case the action of γ is free and properly discontinuous since it is an action by left translation. $D(\gamma)$ is the entire $\widetilde{G}$: it is homeomorphic to $\mathbb{R}^3$ hence simply connected.

4.2-c. *Case* (5)*: conjugacy by a parabolic element.* In this case (5) $D(\gamma)$ is $p^{-1}(U)$, where $U \subset \mathrm{SL}(2, \mathbb{R})$ is the set of matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad ad - bc = 1, c \neq 0$$

For g in U the iterate $\exp(nH)g\exp(-nH)$ is the matrix:

$$\begin{pmatrix} a + nc & -n^2c + n(d-a) + b \\ c & -nc + d \end{pmatrix}$$

Since $c \neq 0$ it follows easily that the action on $D(\gamma)$ is free and properly discontinuous. Every connected component of $D(\gamma)$ is simply connected.

4.2-d. *Case* (6)*: the hyperbolic-hyperbolic case.* A straightforward calculus shows that in this case $\tilde{g}$ belongs to $D(\gamma)$ if and only if $bc < \frac{(\lambda-\mu)^2}{4\lambda\mu}$, where $g = p(\tilde{g})$ is:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad ad - bc = 1$$

The projection in G of the γ^n-iterate of $\tilde{g}$ is:

$$\begin{pmatrix} a\exp(n(\lambda-\mu)) & b\exp(n(\lambda+\mu)) \\ c\exp(-n(\lambda+\mu)) & d\exp(n(\mu-\lambda)) \end{pmatrix}$$

If $\lambda \neq \mu$ then the action of γ on the entire $\widetilde{\mathrm{AdS}}$ is free and properly discontinuous (see for example [27]). The strong causality of the action on $D(\gamma)$ is once more a corollary of lemmas 2.1, 2.3. When $\lambda = \mu$, the projection of $D(\gamma)$ is $\{bc < 0\}$: it is easy to see that the action on it is free and properly discontinuous and we conclude thanks to lemmas 2.1, 2.3, observing that $D(\gamma)$ is simply connected.

4.2-e. *Case* (7)*: the hyperbolic-parabolic case.* This last case is completely similar to the previous one. The action of γ on $\widetilde{\mathrm{AdS}}$ is free and properly discontinuous (see [27]), and $D(\gamma)$ is defined by:

$$-2ac < \lambda$$

Details are left to the reader.

∎

Remark 4.12. In case (6), if $\lambda \neq \mu$, the domain $D(\gamma)$ is not contained in an affine domain. This is an union of elementary domain of invisibility $\{bc < 0\}$, connected by domains $\{0 \leq bc < \frac{(\lambda-\mu)^2}{4\lambda\mu}\}$. The reader can find complementary descriptions in [12], or [6]. In the terminology of these papers, $\{bc < 0\}$ is the union of regions of type I and II, and $\{0 \leq bc < \frac{(\lambda-\mu)^2}{4\lambda\mu}\}$ are the regions of type III: the "inner regions". Compare in particular our proof of proposition 4.11 with § 3.2.5 of [6].

Remark 4.13. The region $D(\gamma)$ in the case (7) is particularly difficult to draw. The best way to catch a picture of it is to consider this case as a limit of case (6): for every $\epsilon > 0$ define $\gamma_\epsilon = (\gamma_L, \gamma_R^\epsilon)$ where:

$$\gamma_R^\epsilon = \begin{pmatrix} \exp(\epsilon) & \frac{\sinh(\epsilon)}{\epsilon} \\ 0 & \exp(-\epsilon) \end{pmatrix}$$

γ_ϵ is the exponential of $(\lambda\Delta, \epsilon\Delta + H)$. At the limit $\epsilon \to 0$, γ_ϵ tends to γ. Then $D(\gamma)$ is the limit of the domains $D(\gamma_\epsilon)$. Recall also § 3.2.

Remark 4.14. When $\gamma_L = \gamma_R$ are elliptic elements with rationnal angle the quotient of $\widetilde{\mathrm{AdS}}$ by γ is a singular spacetime with orbifold type. More precisely, the timelike line of γ-fixed points induces in the quotient a singular line, which can be considered as the trajectory of a massive particle.

This point of view can be extended to the irrationnal angle case without difficulty, but we don't want to enter here in this discussion. See for example [14, 21].

5. Actions on invisible domains from elementary achronal subsets

According to Theorem 10.1 in [8]:

Theorem 5.1. *Let $\widetilde{\Lambda}$ be a nonelementary generic achronal subset, preserved by a torsionfree discrete group $\Gamma \subset SO_0(2,2)$. Then the action of Γ on $\Omega(\widetilde{\Lambda})$ and $E(\widetilde{\Lambda})$ is free, properly discontinuous and the quotient spacetime $M_{\widetilde{\Lambda}}(\Gamma)$ is strongly causal.* ∎

This statement does not hold when $\widetilde{\Lambda}$ is elementary. As it will appear from our study, the philosophy which should be retained is that in the elementary cases the invariant achronal subset $\widetilde{\Lambda}$, even if Γ-invariant is not sufficient to reveal the causal properties of Γ: some points are missing (see § 7).

5.1. **The extreme case.** Assume that $\widetilde{\Lambda}$ is extreme, i.e. is a lightlike segment $[x, y]$. We can assume that the lightlike geodesic l containing $[x, y]$ is a leaf of the left foliation, i.e. an element of $\mathbb{R}P^1_L$. Let l_x, l_y be the right leaves, i.e. the elements of $\mathbb{R}P^1_R$, containing respectively x, y. Then, l is a fixed point of (the projection in G) of the left component γ_L of every element of Γ and l_x, l_y are fixed points of γ_R. It follows easily that γ is synchronized, that the right component γ_R is trivial or hyperbolic, and that the left component is nonelliptic (maybe trivial). In other words, with the notations involved in the proof of proposition 4.11, $\gamma_L = \exp(\lambda\Delta + \eta H)$ and $\gamma_R = \exp(\mu\Delta)$.

The first commutator group $[\Gamma, \Gamma]$ is a group of left translations. Since Γ is discrete, the same is true for $[\Gamma, \Gamma]$. Assume that $[\Gamma, \Gamma]$ is not trivial. Then it is a cyclic group $\mathrm{Aff}(\mathbb{R})$ of affine transformations of the line. A homothety of $\mathbb{R}$ cannot be a commutator of elements of $\mathrm{Aff}(\mathbb{R})$. Hence, in the last case above, $[\Gamma, \Gamma] \subset \widetilde{G}_L$ is a cyclic group of parabolic elements preserved by conjugacies by left components of elements of Γ. It follows that these left components are necessarely parabolic, i.e. translations of $\mathbb{R}$. Hence, left components of elements of $[\Gamma, \Gamma]$ are trivial: contradiction.

Therefore, Γ is an abelian discrete subgroup of A_{hyp}, A_{ext} where:
– $A_{hyp} = \{(\exp(\lambda\Delta), \exp(\mu\Delta)) \ \ (\lambda, \mu \in \mathbb{R})\}$,
– $A_{ext} = \{(\exp(\lambda H), \exp(\mu\Delta)) \ \ (\lambda, \mu \in \mathbb{R})\}$.

5.1-a. *The mixed case $\Gamma \subset A_{ext}$.* In this case the action is free since a parabolic element can be conjugate in $\widetilde{G}$ to a hyperbolic one only if they are both trivial.

Claim: the action of Γ on $E(\widetilde{\Lambda})$ is properly discontinuous.

Let's prove now the properness: assume by contradiction the existence of a compact K in $E(\widetilde{\Lambda})$ and a sequence $\gamma_n = (\exp(\lambda_n H), \exp(\mu_n\Delta))$ of elements of Γ such that every $\gamma_n K \cap K$ is not empty. Let $\|$ be the operator norm on $\mathrm{gl}(E)$. Up to a subsequence, $\gamma_n/\|\gamma_n\|$ converges to an element $\bar{\gamma}$ of the unit ball of $\mathrm{gl}(E)$. Since Γ is discrete, and since all the γ_n have determinant one, the norms $\|\gamma_n\|$ tends to $+\infty$.

If the λ_n are unbounded, up to a subsequence, we can assume that they tend to $+\infty$. Then, the kernel of $\bar{\gamma}$ is a hyperplane, and its image is a line. More precisely, the image is the line spanned by one of the extremities of $[x, y]$, let's say x; and the kernel is the Q-orthogonal $y^\perp$. The compact K

is disjoint from $y^\perp$: it follows that in $P(E) \setminus y^\perp$, the sequence γ_n converges uniformly on K towards the constant map x. This is a contradiction, since x does not belong to K.

If the λ_n are bounded, the image and the kernel of $\bar{\gamma}$ are Q-isotropic 2-planes (one of them is the 2-plane spanned by x and y): their projection in $S(E)$ is disjoint from $\mathbb{ADS}$. But the iterates $\gamma_n K$ accumulates on the projection of the image of $\bar{\gamma}$: we obtain a contradiction as above. The claim is proved.

Moreover, according to proposition 4.11, case (7), this action is strongly causal except if some right component $\exp(\mu\Delta)$ is trivial (case (3) of proposition 4.11): in this last case, the quotient is foliated by closed lightlike geodesics, which are orbits of some 1-parameter subgroup of A_{ext}.

5.1-b. *The hyperbolic case* $\Gamma \subset A_{hyp}$. There is a particular situation: the subcase $\Gamma \subset \widetilde{G}_L$. Then, Γ is cyclic. It follows from proposition 4.11, case (2), that the action on $E(\widetilde{\Lambda})$ is free, properly discontinuous and strongly causal. The same conclusion holds if $\Gamma \subset \widetilde{G}_R$.

Hence, assume that Γ is not contained in $\widetilde{G}_R$ or $\widetilde{G}_L$. The group A_{hyp} admits 4 fixed points in $\overline{\mathrm{Ein}}_2$, including the projections of x, y. We can then define two additionnal A_{hyp}-fixed points x', y' uniquely defined by the requirement that $\{x, x'\}$ and $\{y, y'\}$ are strictly achronal.

Many subcases appear, with different behavior. For example, the action of Γ on $E(\widetilde{\Lambda})$ may be free and properly discontinuous (for example, if Γ is cyclic, spanned by an element for which $\lambda > \mu$). But the action may also be non proper (the cyclic case, with $\lambda = \mu$). Anyway, this action is never causal. Indeed, if γ is an element of $\Gamma \setminus (\widetilde{G}_L \cup \widetilde{G}_R)$, $E(\widetilde{\Lambda})$ is γ^t-invariant, but is not contained in the absolute causal domain $D(\gamma)$. Then, the γ^t-orbit of a point x in $E(\widetilde{\Lambda}) \setminus D(\gamma)$ is a timelike curve containing x and γx.

5.2. **The splitting case.** We consider the splitting case $\widetilde{\Lambda} = \{x, y\}$, with x, y not causally related. Then, the leaves of $\widehat{\mathcal{G}}_R$ through x, y are two distinct fixed points in $\mathbb{R}P^1_R$. The right component of any element of Γ is therefore trivial or hyperbolic. A similar argument shows that the left components are trivial or hyperbolic too. Hence, in the notations of the previous §, we have $\Gamma \subset A_{hyp}$.

Observe that the segment $]x, y[$ is contained in $E(\widetilde{\Lambda})$. Hence, if Γ is not cyclic, its action on $E(\widetilde{\Lambda})$ cannot be properly discontinuous.

Assume that Γ is cyclic, spanned by $\gamma = (\exp(\lambda\Delta), \exp(\mu\Delta))$. If λ or μ is zero the action is free, properly discontinuous and causal.

If λ and μ are both nonzero, it follows from case (6) of proposition 4.11 that *the action of* Γ *on* $E(\widetilde{\Lambda})$ *is free, properly discontinuous and strongly causal if and only if* x, y *are attractive or repulsive fixed points of* γ.

5.3. **The conical case.** We assume here that $\widetilde{\Lambda}$ is conical, i.e. the union of two non-trivial lightlike segments $I_1 = [x_1, x]$ and $I_2 = [x, x_2]$. Then, $\{x_1, x_2\}$ is strictly achronal, $E(\widetilde{\Lambda})$ is contained in $E(x_1, x_2)$, and Γ preserves $E(x_1, x_2)$. As in the previous § we still have $\Gamma \subset A_{hyp}$.

Recall the description of $E(\widetilde{\Lambda})$ (§ 3.3): it is the union of a outer region $P_2 \cap F_2$, an intermediate region $F_2 \cap F_1$, and their common horizon boundary. It follows that $E(\widetilde{\Lambda})$ is A_{hyp}-invariant, and that all the A_{hyp}-orbits inside $E(\widetilde{\Lambda})$ are 2-dimensional. Hence, the action of A_{hyp} on $E(\widetilde{\Lambda})$ is free and properly discontinuous: the same is true for the action of Γ.

If Γ is contained in $\widetilde{G}_L$ or $\widetilde{G}_R$, then its action on $E(\widetilde{\Lambda})$ is strongly causal. If not, the statement in the previous case still holds: the action of Γ on $E(\widetilde{\Lambda})$ is free, properly discontinuous and strongly causal if and only if Γ is cyclic and x, y are attractive or repulsive fixed points of every non-trivial element of Γ.

6. Existence of invariant achronal subsets

Recall the following definition (definition 10.6 in [8]):

Definition 6.1. *Let $\rho_L : \Gamma \to G$ and $\rho_R : \Gamma \to G$ two morphisms. The representation $\rho = (\rho_L, \rho_R)$ is admissible if and only if it is faithfull, has discrete image and lifts to some representation $\tilde{\rho} : \Gamma \to (\widetilde{G} \times \widetilde{G})_{/\mathcal{Z}}$ preserving a generic closed achronal subset of $\widehat{Ein_2}$ containing at least two points.*

A ρ-admissible closed subset for an admissible representation ρ is the projection in $\overline{Ein_2}$ of $\tilde{\rho}$-invariant generic closed achronal subset of $\widehat{Ein_2}$ containing at least two points.

In [8], we claimed (theorem 10.7):

Theorem 6.2. *Let Γ be a torsionfree group, and $\rho : \Gamma \to G \times G$ a faithfull representation. Then, ρ is admissible if and only if one the following occurs:*

(1) The abelian case: $\rho(\Gamma)$ *is a discrete subgroup of A_{hyp}, A_{ext} or A_{par} where (see the notations in § 5.1 where the first two groups are already defined):*
  - $A_{hyp} = \{(\exp(\lambda\Delta), \exp(\mu\Delta))/\lambda, \mu \in \mathbb{R}\}$,
  - $A_{ext} = \{(\exp(\lambda\Delta), \exp(\eta H))/\lambda, \eta \in \mathbb{R}\}$,
  - $A_{par} = \{(\exp(\lambda H), \exp(\lambda H))/\lambda \in \mathbb{R}\}$.

(2) The non-abelian case: *The left and right morphisms ρ_L, ρ_R are faithfull with discrete image and the marked surfaces $\Sigma_L = \rho_L(\Gamma)\backslash\mathbb{H}^2$, $\Sigma_R = \rho_R(\Gamma)\backslash\mathbb{H}^2$ are homeomorphic, i.e. there is a Γ-equivariant homeomorphism $f : \mathbb{H}^2 \to \mathbb{H}^2$ satisfying:*

$$\forall \gamma \in \Gamma,\ f \circ \rho_L(\gamma) = \rho_R(\gamma) \circ f$$

Remark 6.3. Parabolic $\rho_L(\gamma)$ correspond to punctures in Σ_L. Hence, if γ is a homotopy class corresponding to a loop which is not homotopic to an isolated end of Σ_R, $\rho_L(\gamma)$ is necessarely hyperbolic.

Actually, we only proved in [8] the non-abelian case. Here, we justify the abelian case.

When Γ is cyclic, a case by case study is needed, but which follows almost immediatly from the study in the proof of Proposition 4.11. The situation can be summarized as follows:

Proposition 6.4. *When Γ is cyclic, then the representation ρ is admissible if and only if either the left or right component of $\rho(\gamma)$ is hyperbolic and* *the other component non-elliptic, or if $\rho_L(\gamma)$, $\rho_R(\gamma)$ are parabolic elements conjugate one to the other in G.* ∎

Assume now that Γ is abelian but not cyclic. It follows from the cyclic case that ρ is admissible if and only if $\rho(\Gamma)$ is contained in (a conjugate of G of) A_{hyp}, A_{ext} or A_{par}. The validity of Theorem 6.2 in the abelian case follows.

7. Minimal invariant achronal subsets

Let $\rho : \Gamma \to G \times G$ be an admissible representation.

Definition 7.1. *$\overline{\Lambda}(\rho)$ is the closure of the set of attractive fixed points in $P(E)$.*

Since attractive fixed points in $P(E)$ of elements of G belong to $\overline{\mathrm{Ein}}_2$, $\overline{\Lambda}(\rho)$ is contained in $\overline{\mathrm{Ein}}_2$. According to § 10.5 of [8]:

Theorem 7.2. *Let Γ be a non-abelian torsion-free group, and $\rho : \Gamma \to G \times G$. Then, every $\rho(\Gamma)$-invariant closed subset of $P(E)$ contains $\overline{\Lambda}(\rho)$.*

Corollary 7.3. *Let (Γ, ρ) be pair satisfying the hypothesis of Theorem 7.2. Then, $\overline{\Lambda}(\rho)$ is a $\rho(\Gamma)$-invariant generic nonelementary achronal subset of $\overline{Ein}_2$. Furthermore, for every $\rho(\Gamma)$-invariant closed achronal subset Λ in Ein_2, the invisibility domain $E(\Lambda)$ projects injectively in $\overline{\mathbb{ADS}}$ inside $E(\overline{\Lambda}(\rho))$.* ∎

Remark 7.4. We will need the following remark: still assuming that Γ is not abelian, the minimal closed achronal subset $\overline{\Lambda}(\rho)$ is the projection in $P(E)$ of $\Lambda(\rho) \cup -\Lambda(\rho)$, where $\Lambda(\rho)$ is a closed achronal subset of Ein_2, and $-\Lambda(\rho)$ is the image of $\Lambda(\rho)$ by the antipody in $S(E)$. The action of $\rho(\Gamma)$ on $\Lambda(\rho)$ is minimal, hence $\Lambda(\rho)$ and $-\Lambda(\rho)$ are exactly the minimal components of the action of $\rho(\Gamma)$ on $S(E)$ (see Lemma 10.21 in [8]).

When Γ is abelian, Theorem 7.2 and Corollary 7.3 do not hold. However, in this case, still assuming that $\rho : \Gamma \to G$ is admissible:

– *The extreme case:* let l be the unique A_{ext}-invariant left leaf, and r_1, r_2 be the two A_{ext} right invariant right leaves. Let $\overline{\Lambda}_{ext}$ be the set of A_{ext}-fixed points: $\overline{\Lambda}_{ext} = \{l \cap r_1, l \cap r_2\}$. It is easy to show that if $\rho(\Gamma) \subset A_{ext}$, any generic $\rho(\Gamma)$-invariant closed achronal subset containing at least two points must contain $\overline{\Lambda}_{ext}$. This formulation is an extension - more accurately, a limit case - of corollary 7.3, even if elements of $\rho(\Gamma)$ do not admit attractive fixed points in $P(E)$.

– *The parabolic case:* if $\rho(\Gamma) \subset A_{par}$, any $\rho(\Gamma)$-invariant closed subset contains the unique fixed point of A_{par}. Hence, for any $\rho(\Gamma)$-invariant closed achronal subset Λ, we have: $E(\Lambda) \subset D(\rho(\Gamma))$. This is in some way a limit case of the previous one.

– *The hyperbolic-hyperbolic case:* the group A_{hyp} admits 4 fixed points in $P(E)$. If $\rho(\Gamma)$ is a lattice of A_{hyp}, it is easy to see that every A_{hyp}-fixed point is an attractive fixed point in $P(E)$ of some $\rho(\gamma)$. Anyway, corollary 7.3 is false in this situation. There is however a convenient statement: any nonelementary $\rho(\Gamma)$-invariant generic closed achronal subset of $P(E)$ must contain the 4 fixed points of A_{hyp}.

8. Coincidence of standard and absolute chronological domains

In § 4.2, we have associated to any synchronized element γ of $\widetilde{G} \times \widetilde{G}$ two open domains:

– the convex causal domain $C_\infty(\gamma)$,

– the absolute causal subset $D(\gamma)$.

In all cases, we have $C_\infty(\gamma) \subset D(\gamma)$. Actually, the identity $C_\infty(\gamma) = D(\gamma)$ holds if and only if γ is a pair (γ_L, γ_R) where γ_L, γ_R are conjugate in $\widetilde{G}$.

These definitions easily extend to (lifted) admissible representations $\widetilde{\rho} : \Gamma \to \widetilde{G} \times \widetilde{G}$:

Definition 8.1. *The convex causal domain $C_\infty(\widetilde{\rho})$ is the interior of the intersection $\bigcap_{\gamma\in\Gamma} C_\infty(\widetilde{\rho}(\gamma))$. The absolute causal domain $D(\widetilde{\rho})$ is the interior of the intersection $\bigcap_{\gamma\in\Gamma} D(\widetilde{\rho}(\gamma))$.*

The inclusions $C_\infty(\gamma) \subset D(\gamma)$ implies $C_\infty(\widetilde{\rho}) \subset D(\widetilde{\rho})$. Conversely:

Theorem 8.2. *If Γ is non cyclic, then the convex causal domains and absolute causal domains coincide.*

The rest of this section is devoted to the proof of theorem 8.2.

8.1. **The flat case.** In the flat case, i.e., the case where $\widetilde{\rho}(\Gamma)$ preserves a point in $\widetilde{\mathrm{AdS}}$, the proof of Theorem 8.2 is obvious. Indeed, after conjugacy, we can assume in this case that the left and right representations ρ_L, ρ_R coincide. Then, for every γ in Γ, the identity $C_\infty(\rho(\gamma)) = D(\gamma)$ holds.

8.2. **The abelian case.** If Γ is abelian, since it is assumed non-cyclic, $\rho(\Gamma)$ is a lattice in A_{hyp} or A_{ext}. We only consider the first case, the other can be obtained in a similar way (or as a limit case).

There are two morphisms $\alpha, \beta : \Gamma \to \mathbb{R}$ such that, for every γ in Γ:

$$\rho_L(\gamma) = \begin{pmatrix} \exp(\alpha(\gamma)) & 0 \\ 0 & \exp(-\alpha(\gamma)) \end{pmatrix}, \quad \rho_R(\gamma) = \begin{pmatrix} \exp(\beta(\gamma)) & 0 \\ 0 & \exp(-\beta(\gamma)) \end{pmatrix}$$

Recall that the projections in $\mathbb{ADS} \approx \mathrm{SL}(2,\mathbb{R})$ are:

$$C_\infty(\rho(\gamma)) = \{bc < 0\}, \quad D(\rho(\gamma)) = \{bc < \sinh^2(\alpha(\gamma) - \beta(\gamma))\}$$

Theorem 8.2 follows then from the fact that, since $\rho(\Gamma)$ is a lattice in A_{hyp}, $\mid \alpha(\gamma) - \beta(\gamma) \mid$ admits arbitrarly small value.

8.3. **The proper case.** Assume that ρ is strongly irreducible. The representations ρ_L and ρ_R are faithfull, with discrete image, semi-conjugate one to the other, but not conjugate in G, and Γ is not abelian.

Let $\Lambda(\rho)$ be one of the two minimal closed subsets of $\mathcal{D} \subset S(E)$ such that the closure in $S(E)$ of the set of attractive fixed points is $\Lambda(\rho) \cup -\Lambda(\rho)$ (see remark 7.4). It projects injectively on $\overline{\Lambda}(\rho)$, the closure in $P(E)$ of the set of attractive fixed points of elements of $\rho(\Gamma)$.

Let $E(\rho)$ be the invisible domain of $\Lambda(\rho)$: this is the intersection between $\mathbb{ADS}$ and the intersection of all $\{x/\langle x \mid p\rangle < 0\}$, where p describe $\Lambda(\rho)$. Let $\overline{E}(\rho)$ be the projection of $E(\rho)$ in $\overline{\mathbb{ADS}}$. It can be defined in the following way:

$$\overline{E}(\rho) = \{[x] \in \overline{\mathbb{ADS}}/\forall p, q \in \Lambda(\rho), \langle x \mid p\rangle\langle x \mid q\rangle > 0\}$$

Indeed, although $\langle x \mid p\rangle$, $\langle x \mid q\rangle$ are not individually well defined for $[x]$ in $P(E)$, their product have a well-defined sign.

For any subset J of $\overline{\Lambda}(\rho) \approx \Lambda(\rho)$, we can define $\overline{E}(J)$ as the interior of the set: $\{[x] \in \overline{\mathbb{ADS}}/\forall [p], [q] \in J, \langle x \mid p\rangle\langle x \mid q\rangle > 0\}$.

Lemma 8.3. *If J is $\rho(\Gamma)$-invariant and non-empty, then $\overline{E}(J) = \overline{E}(\rho)$.*

Proof The inclusion $\overline{E}(\rho) \subset \overline{E}(J)$ is obvious (observing that $\overline{E}(\rho)$ is open). The reverse inclusion follows from the fact that $\rho(\Gamma)$-invariant subsets of $\overline{\Lambda}(\rho)$ are dense, that $\overline{E}(J)$ is open if J is closed, and that if $\bar{J}$ is the closure of J, $\overline{E}(\bar{J}) = \overline{E}(J)$. ∎

Corollary 8.4. *Every connected component of $C_\infty(\widetilde{\rho})$ projects injectively in $P(E)$ on $\overline{E}(\rho)$.*

Proof Let γ be a non-trivial element of Γ. We define $J(\gamma) \subset \overline{\mathcal{Q}}$ in the following way:

– if $\rho(\gamma)$ is hyperbolic - hyperbolic: $J(\gamma) = \{p^+(\gamma), p^-(\gamma)\}$, where $p^+(\gamma)$ is the attractive fixed point of $\rho(\gamma)$, and $p^-(\gamma)$ is the repulsive fixed point.

– if $\rho(\gamma)$ is hyperbolic - parabolic: $J(\gamma) = \{p(\gamma), q(\gamma)\}$, where $p(\gamma)$, $q(\gamma)$ are the two $\rho(\gamma)$ fixed points,

– if $\rho(\gamma)$ is parabolic - parabolic: $J(\gamma) = \{p(\gamma)\}$, where $p(\gamma)$ is the unique fixed point.

Then, the study in the proof of Proposition 4.11 shows that, in every case, every connected component of $C_\infty(\rho(\gamma))$ projects injectively on $\overline{C}_\infty(\rho(\gamma)) = \overline{E}(J(\gamma))$. On the other hand, if $\rho_L(\gamma)$ (or $\rho_R(\gamma)$) is parabolic, every closed subset of $\mathbb{R}P^1_L$ (or $\mathbb{R}P^1_R$) which is $\rho_L(\gamma)$-invariant (or $\rho_R(\gamma)$-invariant) contains the unique $\rho_L(\gamma)$-fixed point (resp. the $\rho_R(\gamma)$-fixed point). Hence, we have the inclusion $\overline{J}(\gamma) \subset \overline{\Lambda}(\rho)$. In other words, $\overline{C}_\infty(\rho(\gamma))$ is contained in $\overline{E}(\rho)$. The corollary 8.4 follows then from lemma 8.3. ∎

Corollary 8.5. *Let γ_1 be an element of Γ such that $\rho_L(\gamma_1)$ and $\rho_R(\gamma_1)$ are both hyperbolic. Then: $C_\infty(\widetilde{\rho}) = \bigcap_{\gamma\in\Gamma} C_\infty(\widetilde{\rho}(\gamma\gamma_1\gamma^{-1}))$.*

Proof Corollary of lemma 8.3 and corollary 8.4, since $\bigcap_{\gamma\in\Gamma} \overline{C}_\infty(\rho(\gamma\gamma_1\gamma^{-1}))$ is equal to $\overline{E}(J)$, where J is the Γ-orbit of the attractive fixed points of $\rho(\gamma_1)$ and $\rho(\gamma_1^{-1})$. ∎

Proof of 8.2 According to § 8.1 and 8.2, we just have to consider the case where $\rho(\Gamma)$ is non-abelian and does not preserve a point in AdS. The surfaces $\Sigma_L = \rho_L(\Gamma)\backslash\mathbb{H}^2$ and $\Sigma_R = \rho_R(\Gamma)\backslash\mathbb{H}^2$ are homeomorphic: let Σ be any surface homeomorphic to Σ_R, Σ_L.

Let c_1 be a closed loop in Σ which is not freely homotopic to an isolated end of Σ. It represents a conjugacy class $[\gamma_1]$ in Γ. According to remark 6.3, every $\rho_{L,R}(\gamma_1)$ is hyperbolic. After conjugacy, we can assume:

$$\rho_L(\gamma_1) = \begin{pmatrix} \exp(\lambda) & 0 \\ 0 & \exp(-\lambda) \end{pmatrix}, \quad \rho_R(\gamma_1) = \begin{pmatrix} \exp(\mu) & 0 \\ 0 & \exp(-\mu) \end{pmatrix}$$

with $\lambda \geq \mu > 0$.

Since Γ is non-abelian, the Euler characteristic of Σ is negative. Hence, there is a closed loop c_2 in Σ, not freely homotopic to an end of Σ, and such that every loop freely homotopic to c_1 intersects every loop freely homotopic to c_2.

Let γ_2 be any element of Γ corresponding to the free homotopy class of c_2. We express the coefficients of $\rho_{L,R}(\gamma_2) = \exp(A_{L,R})$:

$$A_L = \begin{pmatrix} \alpha_L & \beta_L \\ \nu_L & -\alpha_L \end{pmatrix}, \quad A_R = \begin{pmatrix} \alpha_R & \beta_R \\ \nu_R & -\alpha_R \end{pmatrix}$$

The fixed points in $\mathbb{R}P^1_L$ of $\rho_L(\gamma_1)$ are 0 and ∞. Hence, the connected components of the complement in $\mathbb{R}P^1_L$ of these fixed points are $]-\infty, 0[$ and $]0, +\infty[$.

The fixed points in $\mathbb{R}P^1_L$ of A_L are $\alpha_L \frac{1 \pm \sqrt{1+\beta_L\nu_L/\alpha_L^2}}{\nu_L}$. Replacing γ_2 by its inverse if necessary, and since the intersection number between c_1 and c_2 is not trivial, we can assume that the attractive $\rho_L(\gamma_2)$-fixed point belongs to $]0, +\infty[$, and that the repulsive fixed point belongs to $]-\infty, 0[$. In other words, we can assume that the products $\beta_L\nu_L$ and $\alpha_L\nu_L$ are positive. Hence, after conjugacy by a diagonal matrix, we can assume $\nu_L = \beta_L \neq 0$.

Then, since ρ_L and ρ_R are semi-conjugate, the right components satisfy the same properties: we can assume $\beta_R = \nu_R \neq 0$.

Assume now by contradiction that the inclusion $C_\infty(\widetilde{\rho}) \subset D(\widetilde{\rho})$ is strict. Then, there is an element $\tilde{x}$ of $D(\widetilde{\rho})$ which is in the boundary of $C_\infty(\widetilde{\rho})$. Let $\widetilde{U}$ be an open neighborhood of $\tilde{x}$ in $D(\widetilde{\rho})$. Let U be the projection of $\widetilde{U}$ in $P(E)$. Then, according to corollary 8.5, there is an element γ of Γ, a fixed point x_1 of $\rho(\gamma_1)$, and an element p of U such that $\langle \rho(\gamma)x_1 \mid p\rangle = 0$. After conjugacy of γ_1 by γ, we can assume that γ is trivial. Moreover, we can also assume without loss of generality that x_1 is the attractive fixed point of $\rho(\gamma_1)$. Then, the equation $\langle x_1 \mid p\rangle = 0$ means $c = 0$ where p is expressed by the matrix:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Observe that we can also assume, after a slight modification of $p \approx g$ if necessary, $b \neq 0$.

We consider the elements $\gamma_n = \gamma_1^n \gamma_2 \gamma_1^{-n}$ of Γ. We have:

$$\rho_L(\gamma_n) = \exp(n\lambda\Delta)\exp(A_L)\exp(-n\lambda\Delta)$$
$$\rho_R(\gamma_n) = \exp(n\mu\Delta)\exp(A_R)\exp(-n\mu\Delta)$$

Hence, the norm at g of the Killing vector field generating γ_n is the opposite of the determinant of X_n, with:

$$X_n = \exp(n\lambda\Delta)\exp(A_L)\exp(-n\lambda\Delta)g - g\exp(n\mu\Delta)\exp(A_R)\exp(-n\mu\Delta)$$

After computation, we see that X_n is the matrix:

$$\begin{pmatrix} a(\alpha_L - \alpha_R) - b\beta_R\exp(-2n\mu) & b(\alpha_L + \alpha_R) + d\beta_L\exp(2n\lambda) - a\beta_R\exp(2n\mu) \\ a\beta_L\exp(-2n\lambda) - d\beta_R\exp(-2n\mu) & b\beta_L\exp(-2n\lambda) - d(\alpha_L - \alpha_R) \end{pmatrix}$$

We distinguish two subcases:

8.3-a. *The case* $\lambda > \mu$*:* Observe that b and d are nonzero. If $\lambda > \mu$, the leading term of $-\det(X_n)$ for $n \to +\infty$ is:

$$-d^2\beta_L\beta_R\exp(2n(\lambda-\mu))$$

On the other hand, the leading term for $n \to -\infty$ is $b^2\beta_L\beta_R\exp(-2n(\lambda+\mu))$. But, since g correspond to an element of $D(\widetilde{\rho})$, all the $-\det(X_n)$ are positive. Hence, the product $\beta_L\beta_R$ must be positive and negative. Contradiction. ∎

8.3-b. *The case* $\lambda = \mu$*:* More precisely, the remaining case is the case where $\lambda = \mu$ for any choice of pairs γ_1, γ_2 as above, such that the corresponding homotopy classes have non-trivial intersection number. It is equivalent to the fact that $\mathrm{Tr}(\rho_L(\gamma_1)) = \mathrm{Tr}(\rho_R(\gamma_1))$ for every γ_1 in Γ which is not freely homotopic to a loop around an isolated end of Σ.

Select such a pair (γ_1, γ_2) of Γ satisfying the following additionnal property: the product $\gamma_3 = \gamma_1\gamma_2$ is not freely homotopic to an isolated end of Σ (we leave to the reader the proof of the fact that such a pair exists). Let Γ_1 be the group generated by γ_1, γ_2. Hence, we can assume the identity $\mathrm{Tr}(\rho_L(\gamma_i)) = \mathrm{Tr}(\rho_R(\gamma_i))$ for $i = 1, 2, 3$. By Fricke-Klein Theorem ([16, 17, 19]) these equalities imply that the restrictions of $\rho_L$and ρ_R to Γ_1 are representations conjugated in $\mathrm{SL}(2, \mathbb{R})$.

Therefore, after conjugacy,we can assume: $\rho_L(\gamma_i) = \rho_R(\gamma_i)$ $(i = 1, 2)$. In other words, $\lambda = \mu$, $\alpha_L = \alpha_R$, $\beta_L = \beta_R$. A straightforward computation shows that the leading term of $-\det(X_n)$ for $n \to +\infty$ is $(a\beta_L - d\beta_R)(d\beta_L - a\beta_R) = -(a-d)^2\beta_L^2$. We obtain a contradiction since this term should be nonnegative, whereas $d = 1/a \neq a$. ∎

9. Conformal boundaries of strongly causal spacetimes

As we have seen in the introduction, the notion of black-hole is related to the notion of conformal boundary.

Definition 9.1 (Compare with § 4.2 in [15])**.** *An AdS-spacetime with boundary is a triple $(M, O, \mathcal{M})$ where $\mathcal{M}$ is a manifold with boundary O and interior M which is (AdS, Ein_2)-modeled, i.e.:*

– there exist a morphism (the holonomy representation) $\rho = (\rho_L, \rho_R) : \Gamma \to G \times G$, where Γ is the fundamental group of $\mathcal{M}$,

– there exist a ρ-equivariant local homeomorphism (the developing map) $\mathcal{D} : \widetilde{\mathcal{M}} \to \overline{Ein}_3$, where $\widetilde{\mathcal{M}}$ is the universal covering of $\overline{M}$,

– the image by $\mathcal{D}$ of $\widetilde{M}$, the interior of $\widetilde{\mathcal{M}}$, is contained in $\overline{\mathbb{ADS}}$,

– the image by $\mathcal{D}$ of the boundary $\widetilde{O}$ of $\widetilde{\mathcal{M}}$ is contained in $\overline{Ein}_2$.

Observe that if $(M, O, \mathcal{M})$ is AdS-spacetime with boundary, M inherits a well-defined AdS-structure, and O a Ein_2-structure.

Our aim is to attach to any AdS-spacetime M a AdS-spacetime with boundary $(M, O, \mathcal{M})$. This procedure should be canonical.

Definition 9.2. *Let $(M, O, \mathcal{M})$, $(M', O', \mathcal{M}')$ be two AdS-spacetime with boundary. A morphism between them is a local homeomorphism from $\mathcal{M}$ into $\mathcal{M}'$ inducing a AdS-morphism $M \to M'$.*

Such a morphism lifts to a map $\tilde{F} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}'$ such that $\mathcal{D}' \circ \tilde{F} = g \circ \mathcal{D}$ for some isometry g of $\overline{\mathbb{ADS}}$. In particular, it induces a Ein_2-morphism $O \to O'$. Such a morphism if an *isomorphism* if it is moreover a homeomorphism.

Definition 9.3. *An AdS-spacetime with boundary $(M, O, \mathcal{M})$ is an universal conformal completion of M if for any AdS-spacetime with boundary $(M, O', \mathcal{M}')$ there exist an injective morphism $(M, O', \mathcal{M}') \to (M, O, \mathcal{M})$.*

It should be clear to the reader that if a AdS-spacetime M admits an universal conformal completion, then this completion is unique up to isomorphism. In this case, the boundary is denoted by O_M, and called the *natural conformal boundary of M.*

C. Frances proved that *complete* AdS spacetimes, i.e. quotients of the entire AdS by discrete torsion-free subgroups, admits an universal completion ([15], Theorem 1). But our spacetimes are *never* complete, and we will see that some of them do not admit universal conformal completion as defined above (see remark 9.9). However, we can prove the existence of such universal completions if we restrict to the strongly causal category:

Observe first that causal curves in an AdS-spacetime with boundary is a well defined notion since they are well defined in $\overline{\mathbb{ADS}} \cup \overline{\mathrm{Ein}_2}$. We can therefore define the causality relation in such a manifold with boundary, and in particular the strong causality property (see § 2).

Definition 9.4. *An AdS-spacetime with boundary $(M, O, \mathcal{M})$ is an universal strongly causal conformal completion of M if it is strongly causal and for any strongly causal AdS-spacetime with boundary $(M, O', \mathcal{M}')$ there exist an injective morphism $(M, O', \mathcal{M}') \to (M, O, \mathcal{M})$.*

Obviously, an AdS-spacetime can admits a conformal strongly causal completion only if it is already strongly causal. Conversely:

Theorem 9.5. *Every strongly causal AdS-spacetime admits an universal strongly causal conformal completion.*

In the proof we will need the following lemmas, valid for any local homeomorphism $\varphi : X \to Y$ between manifolds (for proofs, see for example [7], § 2.1):

Lemma 9.6 (Lemme des assiettes emboîtées)**.** *Let U, U' be two open domains in X such that the restrictions of φ on U, U' are injective. Assume that $U \cap U'$ is not empty, and that $\varphi(U')$ contains $\varphi(U)$. Then, U' contains U.* ∎

Lemma 9.7 (Fermeture des assiettes)**.** *Assume that φ is injective on some open domain U in X, and that the image $V = \varphi(U)$ is locally connected in Y, i.e. that every point y in the closure of V admits arbitrarly small neighborhood W such that $V \cap W$ is connected. Then, the restriction of φ to the closure of U in X is injective.* ∎

Proof of 9.5

Step 1: the construction of the AdS-spacetime with conformal boundary $(\widetilde{M}, \widetilde{O}, \widetilde{\mathcal{M}})$

We need to start with a definition: an *end* in $\widetilde{M}$ is an open domain U in $\widetilde{M}$ such that the restriction of $\mathcal{D}$ to U is injective, with image an end V in $\overline{\mathbb{ADS}}$ (see definition 3.2, Figure 2). The proof relies on the geometric understanding of ends, hence, we insist on their description: an end is the intersection $F \cap P$, where P is the past of an element a of $\overline{\mathrm{Ein}}_2$, F the future of an element b of $\overline{\mathrm{Ein}}_2$, and such that a, b are strictly causally in a de Sitter domain containing V. The interior in $\overline{\mathrm{Ein}}_2$ of the intersection between $\overline{\mathrm{Ein}}_2$ and the closure of V is a diamond-shape region, denoted by ∂U. The end itself is the intersection between $\overline{\mathbb{ADS}}$ and the convex hull in $P(E)$ of the boundary of ∂U.

Let $\overline{U}$ be the union $U \cap \partial U$. Observe that for any end U the triple $(U, \partial U, \overline{U})$ is an AdS-spacetime with boundary. A *marked* end is a pair (U, x), where U is an end in $\widetilde{M}$, and x an element of $\partial U \subset \overline{\mathrm{Ein}}_2$. Let $\mathcal{E}$ be the set of marked ends in $\widetilde{M}$. On $\mathcal{E}$, let $\sim$ be the equivalence relation identifying two marked ends (U, x), (U', x) if there is a third marked end (U'', x) with $U'' \subset U \cap U'$. Let Υ be the quotient space of $\sim$. Let Ξ be the union $\Upsilon \cup \widetilde{M}$. For any end U, let $\partial\widehat{U} \subset \Upsilon$ be the set $\{[U', y] \in \Upsilon / U' \subset U\}$, and let $\widehat{U}$ be the union in Ξ of $\partial\widehat{U}$ with U. The $\partial\widehat{U}$ form the basis of a topology on Υ, and the $\widehat{U}$ form, with the open subset of $\widetilde{M}$, the basis of a topology in Ξ. It should be clear to the reader that in the special case where $\widetilde{M}$ is the end V, all this process gives as final output topogical spaces Υ, Ξ respectively homeomorphic to ∂V, $V \cup \partial V$, where ∂V is the interior of the intersection between $\overline{\mathrm{Ein}}_2$ and the closure of V.

The inclusion $\widetilde{M} \subset \Xi$ is clearly a homeomorphism onto its image, which is dense in Ξ. Similarly, any marked end (U, x), the open domain $\widehat{U}$ is a neighborhood of $[U, x]$ in Ξ, which is homeomorphic to $V \cup \partial V$, the conformal completion in $\overline{\mathrm{Ein}}_3$ of the end $V = \mathcal{D}(U)$ in $\overline{\mathbb{ADS}}$. It follows that Ξ is a manifold, with chards the $\widehat{U}$ and the chards of $\widetilde{M}$. Indeed, the only remaining point to check (with the second countability that we leave to the reader) is the Hausdorff property: let x_1, x_2 be two elements of Ξ such that every neighborhood of x_1 intersects every neighborhood of x_2. Then, clearly, if x_1 belongs to $\widetilde{M}$, the same is true for x_2, and $x_1 = x_2$. If not, x_1 and x_2 belong to Υ: $x_1 = [U_1, x'_1]$, $x_2 = [U_2, x'_2]$ with $(U_i, x_i) \in \mathcal{E}$. By hypothesis, the neighborhoods $\widehat{U}_1$ and $\widehat{U}_2$ must overlap. If $U_1 \cap U_2 \subset \widetilde{M}$ is empty, then some $[U_3, x_3]$ must belong to $\widehat{U}_1 \cap \widehat{U}_2$. Then, points in U_3 correspond to points in $\widehat{U}_3$ which are in $U_1 \cap U_2$: contradiction. Hence, $U_1 \cap U_2$ must intersect for any choice of the marked ends (U_i, x_i). Fix one choice (U_i^0, x_i^0) of these ends, and consider for every $i = 1, 2$ smaller ends $U_i \subset U_i^0$. More precisely, fix U_1, with $[U_1^0, x_1] = [U_1, x_1]$ and $U_1 \subset U_1^0$. Then, if U_2 is sufficiently small, its image $\mathcal{D}(U_2)$, which intersect $\mathcal{D}(U_1)$, is contained in $\mathcal{D}(U_1^0)$. According to Lemma 9.6, U_2 is then contained in U_1^0. But, since x_1, x_2 are not separated one from the other, they must have the same image under $\mathcal{D}$. Applying Lemma 9.7, we obtain $x = y$, i.e. Ξ is Hausdorff.

The combination of the developing map $\mathcal{D} : \widetilde{M} \to \overline{\mathbb{ADS}}$ with the inclusions $\overline{U} \subset \overline{\mathrm{Ein}_3}$ induce a well-defined continuous map $\mathcal{D} : \Xi \to \overline{\mathrm{Ein}_3}$. Moreover, the restriction of $\mathcal{D}$ to (the projection of) any closed end $\overline{U}$ is injective: $\mathcal{D}$ is a local homeomorphism.

Finally, the action of Γ on $\widetilde{M}$ extends naturally on Ξ: for any γ in Γ, define $\gamma[x \in \overline{U}]$ as being $[\rho(\gamma)x \in \rho(\gamma)\overline{U}]$.

Observe that this action is continuous and preserves $\Upsilon = \Xi \setminus \widetilde{M}$. Moreover, the map $\mathcal{D} : \Xi \to \overline{\mathrm{Ein}_3}$ is equivariant for this action.

We can now define $\widetilde{\mathcal{M}}$: this is the set of Γ-causally wandering points in Ξ, i.e. the set of elements x of Ξ admitting neighborhood W such that for every non-trivial element γ of Γ no element of W is causally related in Ξ to an element of γW. Observe that Ξ is open, Γ-invariant, and that it contains $\widetilde{M}$ since the action of Γ on $\widetilde{M}$ is strongly causal. $\widetilde{O}$ is the complement $\widetilde{\mathcal{M}} \setminus \widetilde{M}$ (hence, the set of Γ-non causally wandering points).

Step 2: the action of Γ *on* $\widetilde{\mathcal{M}}$ *is free, proper and strongly causal.*

Observe that the action is free, since the fixed points are not wandering. Assume that the action of Γ on $\widetilde{\mathcal{M}}$ is not proper.

Then, there are sequences $(x_n)_{(n\in\mathbb{N})}$, $(y_n)_{(n\in\mathbb{N})}$ in $\widetilde{\mathcal{M}}$ and a sequence γ_n in Γ such that:

– $\gamma_n x_n = y_n$,
– $x_n \to x \in \widetilde{\mathcal{M}}$,
– $y_n \to y \in \widetilde{\mathcal{M}}$,
– y is not in the Γ-orbit of x.

Let $\bar{x} = \mathcal{D}(x)$, $\bar{y} = \mathcal{D}(y)$, $\bar{x}_n = \mathcal{D}(x_n)$, $\bar{y}_n = \mathcal{D}(y_n)$. We also decompose the $g_n = \rho(\gamma_n)$ along their left and right components: $g_n = (g_L^n, g_R^n)$.

Since the action of Γ on $\widetilde{M}$ is proper (it is the group of covering automorphisms), x or y must belong to $\widetilde{O}$; let's say x. Then, x belongs to a diamond shape region ∂U, where U is an end in $\widetilde{M}$. Since x is Γ-wandering, we can choose U so that $\gamma\overline{U} \cap \overline{U} = \emptyset$ for every non-trivial γ in Γ. Moreover, since $\overline{U}$ is open in $\widetilde{\mathcal{M}}$, we can assume that the x_n all belongs to U.

Define $U_n = \gamma_n U$, $\Delta = \partial U$, $\Delta_n = \gamma_n \Delta$, and $\overline{\Delta}_n = \mathcal{D}(\Delta_n)$, $\overline{U}_n = \mathcal{D}(U_n)$. The image $\overline{\Delta} = \mathcal{D}(\Delta)$ is a diamond-shape region. Since the U_n are necessarily disjoint we can assume that none of them contain y.

Claim: y belongs to $\widetilde{O}$.

Assume not. There is a small neighborhood W of y in $\widetilde{M}$ such that the restriction of $\mathcal{D}$ is injective with image a small ellipsoid $\overline{W}$ in $\overline{\mathbb{ADS}}$. For n sufficiently big y_n belongs to W, hence the intersection $\mathcal{I}_n = W \cap U_n$ is not empty. On the other hand, $\overline{U}_n$ is an end: it is a connected component of the intersection between $\overline{\mathbb{ADS}}$ and the complement of two hyperplanes in $P(E)$. It follows that the intersection $\overline{\mathcal{I}}_n$ between the ellipsoid $\overline{W}$ and $\overline{U}_n$ is convex: it is the trace in an ellipsoid of a half-space or a quarter of space. Moreover, the restriction of $\mathcal{D}$ to $\mathcal{I}_n$ and W is injective. Hence, according to Lemma 9.6, the image of $\mathcal{I}_n$ by $\mathcal{D}$ is the entire $\overline{\mathcal{I}}_n$. In other words, the screen $\overline{W}$ reflects faithfully how the ends U_n intersect W. But it is geometrically clear that $\overline{U}_n$ cannot accumulate to $\bar{y}$ if they are disjoint one to the other: visualize by considering an ellipsoid W' in the Minkowski space conformally equivalent to $\overline{W}$; then, in this conformal chard, the $\overline{U}_n$ are intersections between the past of lightlike plane and the future of another lightlike plane. It leads to a contradiction: the claim is proved.

Replace the ellipsoid W in the proof above by a neighborhood $\overline{U}'$ with $U' \in \mathcal{E}$ and such that y belongs to $\partial U'$. We can assume that all the y_n belong to U'. The argument above, based on lemma 9.6, shows that the image by $\mathcal{D}$ of the intersection $U_n \cap U'$ is also the entire $\overline{U}' \cap \overline{U}_n$. Applying once more this lemma to the closure, we obtain that $\mathcal{D}$ projects faithfully the intersections between Δ' and Δ_n over the entire $\overline{\Delta}_n \cap \overline{\Delta}'$. But these intersections are even simpler to visualize than intersection between an ellipsoid and ends: indeed, through the identification $\overline{\mathbb{ADS}} \approx \mathbb{R}P^1_L \times \mathbb{R}P^1_R$, a diamond-shape region corresponds to the product of two open intervals $I_L \times I_R$. Denote by I^0_L, I^0_R the open intervals in the projective line such that $I^0_L \times I^0_R = \partial\overline{\Delta}'$: $\bar{y}$ correspond to a pair $(y_L, y_R) \in I^0_L \times I^0_R$. For every integer n let $I^n_L \subset I^0_L$ and $I^n_L \subset I^0_R$ be the intervals such that $\overline{\Delta}_n \cap \overline{\Delta}' = I^n_L \times I^n_R$.

Assume that for some integers $n \neq m$ the intersection $I^n_L \cap I^m_L$ is not empty. Then there is lightlike segment in $\overline{\Delta}'$ with one extremity in $\overline{\Delta}_n$ and the other in $\overline{\Delta}_m$. It follows that there is a causal curve joining an element of $U_n \cap U'$ to an element of $U_m \cap U'$. Now, for the first time, we use the fact that M is strongly causal, i.e. that the action of Γ on $\widetilde{M}$ is strongly causal: it means that the open domain U can be selected so that for every non trivial element of Γ no element of γU can be causally related to an element of U. Apply this remark to $\gamma_n^{-1}\gamma_m$: we obtain a contradiction.

Hence, for every $n \neq m$ we have $I^n_L \cap I^m_L = \emptyset$. Similarly $I^n_R \cap I^m_R = \emptyset$. But since the y_n converge to $y = (y_L, y_R)$ it follows that $(I^n_L)_{(n \in \mathbb{N})}$ (resp. $(I^n_R)(n \in \mathbb{N})$) is a sequence of intervals smaller and smaller converging uniformly to $\{y_L\}$ (resp. $\{y_R\}$). Hence for big n we have $U_n \subset U'$. We obtain a contradiction since U' can be chosen so that $\gamma U' \cap U' = \emptyset$ for every non trivial γ.

This final contradiction achieves the proof of step 2. Hence, the quotient $\mathcal{M} = \Gamma \backslash \widetilde{\mathcal{M}}$ is a manifold, with boundary $O = \Gamma \backslash \widetilde{O}$.

Step 3: $(M, O, \mathcal{M})$ is an universal conformal completion.

Let $(M, O', \mathcal{M}')$ be another conformal completion of M. Let $\pi' : \widetilde{\mathcal{M}}' \to \mathcal{M}'$ be the universal covering of $\mathcal{M}$. The interior of $\widetilde{\mathcal{M}}'$ is simply-connected, hence it can be identified with the universal covering of M. Hence M, $\mathcal{M}'$ (and $\mathcal{M}$) have the same fundamental group Γ. Let $\mathcal{D}' : \widetilde{\mathcal{M}}' \to \overline{\mathrm{Ein}}_3$ and $\rho : \Gamma \to G \times G$ be the developing map and the holonomy representation of $(M, O, \mathcal{M})$. The developing map for M is then the restriction to $\widetilde{M}$ of $\mathcal{D}'$, and ρ is the holonomy representation of the AdS-spacetime M.

The main observation is the following: *every point x in $\widetilde{O}'$ admits a neighborhood U' in $\widetilde{\mathcal{M}}'$ such that the restriction of $\mathcal{D}'$ to U' is injective and the image $\mathcal{D}'(U \cap \widetilde{M})$ is an end of AdS.* Hence, $(U \cap \widetilde{M}, \mathcal{D}'(x))$ is a marked end of $\widetilde{M}$. It defines a map $\widetilde{O} \to \Upsilon$. With the identity map $\widetilde{M} \to \widetilde{M}$, and after composition in the quotient space, we obtain a map $\widetilde{F} : \widetilde{\mathcal{M}}' \to \Xi$. The proof that $\widetilde{F}$ is a homeomorphism onto its image is straightforward and left to the reader.

The main point is to show that $\widetilde{F}$ takes value in the domain of Γ-wandering points $\widetilde{\mathcal{M}}$. Assume by contradiction the existence of a point x in $\widetilde{\mathcal{M}'}$ such that $x' = \widetilde{F}(x)$ is not a Γ-wandering point. Since Γ acts properly on $\widetilde{\mathcal{M}'}$, there is a neighborhood W of x such that $\gamma W \cap W = \emptyset$ for every non-trivial γ. Moreover, since it is not Γ-wandering, x' does not belong to $\widetilde{M}$. Hence, x belongs to $\widetilde{O'}$: we can choose W so that $W \cap \widetilde{M}$ is an end of $\widetilde{M}$. Then, by hypothesis, there is a non-trivial γ such that $\gamma\overline{W} \cap \overline{W}$ is not empty since $\overline{W}$ is a neighborhood of x' in $\widetilde{\mathcal{M}}$. But completed ends overlapping in $\widetilde{\Xi}$ must also overlap in the interior of $\widetilde{\Xi}$, i.e. in $\widetilde{M}$. Hence, this overlapping exists also $\widetilde{\mathcal{M}'}$. Contradiction.

Hence, $\widetilde{F}$ takes value in $\widetilde{\mathcal{M}}$. Since the construction is Γ-equivariant, $\widetilde{F}$ induces the required morphism $(M, \mathcal{M}', O') \to (M, O, \mathcal{M})$. ∎

Remark 9.8. Consider the quotient of $\widetilde{\mathrm{AdS}}$ by a cyclic group Γ generated by a hyperbolic left translation $\gamma = (\gamma_L, id)$. Then the set of fixed points of γ is the union of two left lightlike leaves l_1, l_2. The action of Γ on $\widetilde{\mathrm{AdS}} \cup \widehat{\mathrm{Ein}}_2 \setminus (l_1 \cup l_2)$ is free and properly discontinuous, hence the quotient space is an universal conformal boundary for $\Gamma\backslash\widetilde{\mathrm{AdS}}$. But since every Γ-orbit of elements of $\widehat{\mathrm{Ein}}_2$ is contained in a right lightlike leaf, it follows that the strongly causal conformal boundary of $\Gamma\backslash\widetilde{\mathrm{AdS}}$ is empty.

Remark 9.9. Restricting to the strongly causal category is essential to ensure the uniqueness of the maximal conformal extension. Indeed: let $\gamma = (\exp(\lambda\Delta), \exp(\mu\Delta))$ be a hyperbolic-hyperbolic element of $\widetilde{G} \times \widetilde{G}$ with $\lambda \neq \mu$. The action of γ on $\widetilde{\mathrm{AdS}}$ is free and properly discontinuous. In $\overline{\mathrm{Ein}}_2$ t here are two γ-invariant left leaves l_1, l_2, and two γ-invariant right leaves r_1, r_2.

The preimage $\tilde{l}_i$, $\tilde{r}_i$ in $\widehat{\mathrm{Ein}}_2$ of l_i, r_i are lightlike geodesics. It is easy to see that the action of Γ on $\widetilde{\mathrm{AdS}} \cup \widehat{\mathrm{Ein}}_2 \setminus (l_1 \cup l_2)$ is free and properly discontinuous, and the same is true for the action on $\widetilde{\mathrm{AdS}} \cup \widehat{\mathrm{Ein}}_2 \setminus (r_1 \cup r_2)$. Each of these extensions provides a conformal completion of $\Gamma\backslash\widetilde{\mathrm{AdS}}$. But it can be proved that each of this extension is maximal, i.e. does not embeds in a larger conformal completion of $\Gamma\backslash\widetilde{\mathrm{AdS}}$. Hence $\Gamma\backslash\widetilde{\mathrm{AdS}}$ does not admit an universal conformal completion.

A similar situation appears in the so-called "Taub - NUT" examples, see [20, 23].

10. BTZ black holes and multi black-holes

In this section, we consider various pairs (ρ, Λ), where $\rho = (\rho_L, \rho_R) : \Gamma \to G \times G$ is an admissible representation and Λ a ρ-admissible closed subset of $\overline{\mathrm{Ein}}_2$ (see definition 6.1). We prove in the selected cases that the spacetimes $M_\Lambda(\Gamma)$ are AdS-spacetimes with black-holes in the meaning of definition 1.2.

Remark 10.1. We don't pretend to study all the possibilities: for example, for non-abelian Λ, we only consider the case $\Lambda = \Lambda(\Gamma)$, whereas the case $\Lambda \neq \Lambda(\Gamma)$ give other examples of spacetimes with black-holes. Moreover, we don't consider the case where Γ is trivial, which would lead, for the Λ selected below, to AdS-spacetimes with black-hole too!

A justification for this omission is that all these spacetimes are not maximal for (too much) obvious reasons, because of the embedding $M_\Lambda(\Gamma) \subset M_{\widetilde{\Lambda(\Gamma)}}(\Gamma)$ - especially in the case $\Gamma = id$, where the whole spacetime embeds in $\widetilde{\mathrm{AdS}}$.

10.1. **The conical black-holes.** This is the case where Λ is conical (see § 3.3). More precisely, we have to consider the case where Λ is a upper lower tent $\mathcal{T}^+_{xy} = [x,z] \cup [z,y]$ (if it is a lower tent, then the spacetime is full, but without black-hole). The domain $E(\Lambda)$ has been described in § 3.3. From this description (see also the Figure 2), it appears clearly that the conformal boundary $\widetilde{O}$ of $E(\Lambda)$ is the diamond-shape region $\Delta_2 = \Omega(\Lambda)$, and the full completion of $E(\Lambda)$ is $\mathcal{M}(\Lambda) = E(\Lambda) \cup \Delta_2$. The region $F_1 \cap F_2$ is the region invisible from $\widetilde{O}$. We can understand a part of the terminology reported in remark 3.5: the outer region $F_2 \cap P_2$ is the region visible by observers in O.

According to § 5.3 the action of Γ on $E(\Lambda)$ is free, properly discontinuous and strongly causal if and only if Γ is a cyclic group generated by an element $\gamma = (\gamma_L, \gamma_R) = (\exp(u\Delta), \exp(v\Delta))$. We can assume without loss of generality $v \geq u \geq 0$. It appears then clearly that the action of Γ on $\widetilde{O} = \Delta_2$ is free and proper. Moreover, this action is strongly causal if $u = 0$. Hence, its quotient O is the natural conformal boundary of $M_\Lambda(\Gamma)$ and it is also the strongly causal conformal boundary if $u \neq 0$ - if $u = 0$, the strongly causal conformal boundary is empty (see remark 9.8). Obviously, the invisible domain in $M_\Lambda(\Gamma)$ from O is the quotient of the "intermediate region" $F_1 \cap F_2$.

It follows that $M_\Lambda(\Gamma)$ is an AdS-spacetime with black-hole in the meaning of definition 1.2 if and only if $u \neq 0$.

Remark 10.2. The topology is very simple: M_Λ is homeomorphic to the product of the annulus by $\mathbb{R}$. The same is true for the outer region and the black-hole, which are separated by a lightlike annulus, the horizon.

10.2. **Splitting black holes:** Here, we consider the case where Λ is splitting, i.e. two non-causally related points (x, y). This case is fully detailed and described in § 3.1. Figure 2 is still useful. The discussion above remains essentially the same, but now the conformal boundary of $E(\Lambda)$ has two connected components: Δ_1 and Δ_2. Hence, the invisible domain from their union is still $F_1 \cap P_1$.

According to § 5.2, in order to act properly on $E(\Lambda)$, the group Γ must be as in the conical case: generated by a hyperbolic translation, or a hyperbolic-hyperbolic element. Hence, the conformal completion of the quotient is the quotient of these two diamond-shapes regions. The strongly causal conformal boundary is $\Delta_1 \cup \Delta_2$, except if γ is an hyperbolic translation, in which case the strongly causal boundary is empty.

When γ is hyperbolic-hyperbolic, $M_\Lambda(\Gamma)$ contains a black-hole - the quotient of $F_1 \cap F_2$ - isometric to the black-hole of the conical case.

The topological description is the same than in the conical case. But the horizon is not C^2, and the visible domain is not globally hyperbolic.

Remark 10.3. Something similar to what was discussed in remark 10.1 appears: the conical spacetime $M_\Lambda(\Gamma)$ embeds isometrically in $M_{xy}(\Gamma)$: hence, it is not maximal. This is a good reason for considering that conical case does not contain a black-hole, as in the "classical" litterature. But observe that $M_{xy}(\Gamma)$ itself is not maximal too: when $u < v$, $M_{xy}(\Gamma)$ embeds in the spacetime $M^D(\Gamma) = \Gamma\backslash D(\Gamma)$, where $D(\Gamma)$ is the absolute causality domain (see definition 4.8, proposition 4.11, case (6)). Observe that $D(\Gamma)$ is an open domain in $\widetilde{\mathrm{AdS}}$ not contained in a affine domain. Actually, the proof of case (6) of Proposition 4.11 shows that $D(\Gamma)$ contains all the preimage in $\widetilde{\mathrm{AdS}}$ of E_{xy}, which are connected one to the other by regions (the "inner regions" with the conventions in [5, 6, 12]) which project in AdS $\approx \mathrm{SL}(2,\mathbb{R})$ to the domain $0 < bc < \frac{(\exp(u)-\exp(v))^2}{4\exp(u+v)}$. Observe that these new regions has empty conformal boundary: their closure intersect Ein_2 only along the union of two lightlike geodesics. Hence, the conformal boundary of $D(\Gamma)$ is the preimage in $\widehat{\mathrm{Ein}}_2$ of all the preimage of $\Delta_1 \cup \Delta_2$.

In other words, let A is an affine domain containing $E(\Lambda)$, and let A_i be the infinite family of affine domains in $\widetilde{\mathrm{AdS}}$ such that $\widetilde{\mathrm{AdS}}$ is the union of the $\overline{A}_i$ (see § 3.5 in [8]). Every $\overline{A}_i$ contains a copy E_i of $E(\Lambda)$. Moreover, A can be selected so that the conformal boundary of $\overline{A}_i$ contains the conformal boundary $\Delta_1^i \cup \Delta_2^i$ of E_i. The inner regions connect every $\overline{A}_i$ to the following $\overline{A}_{i+1}$, offering the way to some causal curves to pass from $\overline{A}_i$ to $\overline{A}_{i+1}$. It is easy to show that, thanks to these connecting inner regions, $D(\Gamma)$ is entirely visible from its conformal boundary: it has no hole.

However, the quotient $M^D(\Gamma)$ is considered in the litterature devoted to BTZ black-holes (including [5]) has the typical spacetime containing a *single non-static ($u \neq v$) BTZ-black-hole.* It means that the point of view to adopt is to pay attention to simple blocks $\overline{A}_i \cap D(\Gamma)$ individually, to consider the observers only in the boundary components of *one* of them, and to consider other blocks as being other parts of the universe which can be reached only by going through the horizon of the black-hole. Adopting this point of view, we observe that $M_{xy}(\Gamma)$ is therefore enough to give a picture of the considered black-hole.

Remark 10.4. This causal description is actually very similar to the description of black-holes in the maximal Kerr spacetime M_{Kerr}^{max} (see the Introduction).

10.3. **The extreme black hole.** The case where Λ is extreme is described in § 3.2. In this case, the invisible domain is not contained in an affine domain, hence we need two successive domains, and Λ must be considered as a closed subset of $\widehat{\mathrm{Ein}}_2$ (see Figure 3). In this case, the conformal boundary is the extreme diamond $\Omega(x, y)$, and the entire $E(\Lambda)$ is visible from the boundary: there is no black-hole!

According to § 5.1, the group Γ must be contained in A_{hyp} or A_{ext}, and since we want the action of Γ on $E(\Lambda)$ to be causal, the case $\Gamma \subset A_{hyp}$ must be excluded (§ 5.1-b). Hence, the action of Γ on $E(\Lambda)$ is free, properly discontinuous and strongly causal, except if it contains a parabolic translation (§ 5.1-a).

Anyway, as we have seen, $E(\Lambda)$ does not contain any black-hole. But we can use the same trick as for single non-static BTZ black-holes (remark 10.3): consider the absolute causal domain $D(\Gamma)$. Observe that $D(\Gamma) = E(\Lambda)$ if

Γ is not cyclic (Theorem 8.2). Hence, this trick will apply only for cyclic subgroups of A_{ext}. Moreover, according to Proposition 4.11, case (3), the absolute causal domain of parabolic translations is trivial: we must exclude them. Finally, elements of Γ are not hyperbolic translations since $\Gamma \subset A_{hyp}$ is excluded. Hence, Γ must be generated by a parabolic-hyperbolic element. According to Proposition 4.11, case (7), the action of Γ on $D(\Gamma)$ is free, properly discontinuous and proper.

We can reproduce nearly the same comment than in remark 10.3: $D(\Gamma)$ has to be understood as a δ-invariant subset of $\widetilde{\mathrm{AdS}}$, and the quotient space $M^D(\Gamma)$ is an union of "local universes". There is a small difference: the conformal boundary of every simple block is now connected (the other connect component "vanished"), and there is no intermediate region: the black-holes correspond to inner regions.

10.4. **Multi-black holes.** The last case we consider is the non-elementary case: Γ is non-abelian, and $\Lambda = \Lambda(\Gamma)$ is not elementary. The key point is to use Proposition 8.50 of [8]: $E(\Lambda)$ *is the union of the past and future globally hyperbolic cores, with the closed ends* $\Omega(I)$ *associated to gaps* I *of* Λ. Observe that the gaps in this case are not extreme (remark 8.25 of [8]), hence, the diamonds $\Omega(I)$ are not extreme. Moreover, since in this case the left and right morphisms are both faithfull, the stabilizer of $\Omega(I)$ is generated by a hyperbolic - hyperbolic element.

It follows clearly that the connected components of the conformal boundary of $E(\Lambda)$ are precisely the diamonds $\Omega(I)$: the only way to approach this conformal boundary is to enter in a closed end. Observe that the closed end associated to a gap I is isometric to the outer region of a conical spacetime. Moreover, if Λ is a topological circle, then the conformal boundary is empty: there is no observor, no black-hole. Finally, the invisible domain from $\Omega(\Lambda)$ is precisely the future globally hyperbolic core $E(\Lambda^+)$.

Now, according to Theorem 5.1, and since every element of Γ is hyperbolic - hyperbolic, the action of Γ on $\Omega(\Lambda) \cup E(\Lambda)$ is free, properly discontinuous and strongly causal (observe that it is true even if $\Lambda \neq \Lambda(\Gamma)$). The quotient spacetime $M_\Lambda(\Gamma)$ is then a AdS-spacetime with one black-hole: the (globally hyperbolic) quotient of $E(\Lambda^+)$ by Γ.

Remark 10.5. There is an obvious 1-1 correspondance between the connected components of the conformal boundary of $M_\Lambda(\Gamma)$ and Γ-orbits of gaps, i.e. non-cuspidal boundary components of the surfaces $\Sigma_L \approx \Sigma_R$ (see remark 6.3).

Remark 10.6. According to Theorem 8.2, the trick used in remark 10.3 in order to enlarge the spacetime by considering the absolute causal domain $D(\Gamma)$ gives nothing new: the quotient is the same spacetime.

Remark 10.7. Anyway, even if the "obvious" trick above does not work, many spacetimes $M_\Lambda(\Gamma)$ are not maximal AdS-spacetimes. Indeed, add to $E(\Lambda)$ a very small end U, such that the intersection ∂U of the closure of U with $\overline{\mathrm{Ein}_2} \approx \mathbb{R}P^1_L \times \mathbb{R}P^1_L$ is a small rectangle $I_L \times I_L$ around a point x in the boundary of $\Omega(I)$ (for some gap I). Don't take as point x one of the corner points of $\Omega(I)$. Then, if for some element γ of Γ the intersection $\partial U \cap \gamma\partial U$

is not empty, then $\gamma\Omega(I) \cap \Omega(I)$ is not empty. Then, γ must belong to the cyclic subgroup Γ_I of elements preserving the gap I (see the proof of Theorem 5.1). Thus, U can be chosen so that the intersection $\gamma U \cap U$ never happens. Then, the union of $E(\Lambda)$ with all the $\gamma U \ \ (\gamma \in \Gamma)$ is a Γ-invariant connected spacetime E' on which Γ acts freely and properly discontinuously. Its quotient is a bigger spacetime M' in which $M_\Lambda(\Gamma)$ embeds isometrically.

Remark 10.8. Consider once more the spacetime M' constructed in the remark above. Let's prove that M' is strongly causal. There is no loss of generality in assuming that the point x has been selected so that its (local) future does not meet $E(\Lambda) \cup \Omega(\Lambda)$. Then, no future oriented causal curve starting from a point in $E' \setminus E(\Lambda)$ can enter into $E(\Lambda)$: we say that $E(\Lambda)$ is *past-convex* in E'.

Let y be an element of E'. If y belongs to $E(\Lambda)$, since $E(\Lambda)$ is causally convex and strongly causal, there is a neighborhood V of y which is not causally related to any non-trivial Γ-iterate of itself. Assume now that y belongs to the end U but not to $E(\Lambda)$. Observe that by construction U meets $E(\Lambda)$ only in the closed end $E(I)$ associated to I. It follows that small neighborhoods of y are contained in $U \setminus E(\Lambda)$. Any causal curve c joining a point near y to a point near γy with $\gamma \neq id$ must enter in $E(\Lambda)$. Since $E(\Lambda)$ is past-convex in E' the causal curve c must be past-oriented. But the same argument applied near γy shows that c must be future oriented.

This contradiction implies that the action of Γ on E' is strongly causal. Therefore, M' is strongly causal.

It is quite clear that the conformal boundary of M' is the quotient by Γ of the union of $\Omega(\Lambda)$ with the disjoint union of all the $\gamma\partial U$. Moreover, by causal convexity of $E(\Lambda) \cup \Omega(\Lambda)$, the strongly causal conformal boundary of M' contains as an open subset the conformal boundary of $M_\Lambda(\Gamma)$. We claim that the conformal boundary of $M_\Lambda(\Gamma)$ is actually a connected component of the strongly causal conformal boundary of M'. Indeed, for every y in ∂U and in the boundary of $\Omega(I)$ in the Einstein space, for any neighborhood V of y in ∂U and for any non-trivial element γ of Γ_I there are elements of $V \cap \Omega(I)$ causally related to elements of $\gamma V \cap \partial U$.

It follows quite easily that the strongly causal conformal boundary of M' is the quotient by Γ of the union of $\Omega(\Lambda)$ with the disjoint union of all $\gamma(\partial U \setminus \overline{\Omega}(I))$, where γ describes all Γ, and $\overline{\Omega}(I)$ is the closure in the Einstein space of $\Omega(I)$.

Points in $E' \setminus E(\Lambda)$, i.e. elements of $\gamma U \ \ (\gamma \in \Gamma)$ are in the past of the strongly causal boundary of E'. Hence, the black-hole in M' is contained in the black-hole of $M_\Lambda(\Gamma)$, in particular, smaller. Moreover, if the point x has been selected in the past of $\Omega(I)$ (the proof of strong causality still applies, just replace past-convex above by future-convex) then the black-hole in M' is equal to the black-hole in $M_\Lambda(\Gamma)$: the part of the conformal boundary we added didn't reveal any new point. But if the point x has been selected in the future $\Omega(I)$ there is no general answer: the black-hole in M' might be strictly smaller than the black-hole in $M_\Lambda(\Gamma)$.

Remark 10.9. The example described in remarks 10.7, 10.8 shows that spacetimes with BTZ multi black-holes are far from being maximal as spacetimes, even in the strongly causal category. But observe that these examples become forbidden if in the definition of spacetimes with black-holes we impose the following additionnal requirement: *the strongly causal conformal boundary must be globally hyperbolic.*

10.5. **Kerr-like coordinates.** Contrary to the habits in the BTZ litterature, we end by the presentation of the Kerr-like expression of the BTZ metric. This expression concerns the metric on the "outer regions", i.e. the ends. Hence we just have to consider the splitting and extreme cases.

10.5-a. *The Kerr-like metric on the outer region of a splitting spacetime.* In this case, Γ is generated by γ, where, up to conjugacy:

$$\gamma = (\gamma_L, \gamma_R) = (\exp(u\Delta), \exp(v\Delta)), \quad v \geq u > 0$$

The elements x, y of Λ must be the attractive and repulsive fixed points of γ.

Define $r_\pm$ ($r_+ > r_- \geq 0$) by:

$$u = \pi(r_+ - r_-) \quad v = \pi(r_+ + r_-)$$

Then:

$$\gamma_L g \gamma_R^{-1} = \begin{pmatrix} a\exp(-2\pi r_-) & b\exp(2\pi r_+) \\ c\exp(-2\pi r_+) & d\exp(2\pi r_-) \end{pmatrix}$$

Let (x_1, x_2, y_1, y_2) be coordinates of E for which $Q = dx_1^2 + dx_2^2 - dy_1^2 - dy_2^2$. Consider the identification:

$$\begin{array}{rcl} \mathrm{AdS} & \to & \mathrm{SL}(2,\mathbb{R}) \\ (x_1, x_2, y_1, y_2) & \mapsto & \begin{pmatrix} y_1 + x_1 & x_2 + y_2 \\ x_2 - y_2 & y_1 - x_1 \end{pmatrix} \end{array}$$

Then, in the coordinates (x_1, y_1, x_2, y_2), the matrix in $\mathrm{SO}(2,2)$ corresponding to γ is:

$$\begin{pmatrix} \cosh(2\pi r_-) & -\sinh(2\pi r_-) & 0 & 0 \\ -\sinh(2\pi r_-) & \cosh(2\pi r_-) & 0 & 0 \\ 0 & 0 & \cosh(2\pi r_+) & \sinh(2\pi r_+) \\ 0 & 0 & \sinh(2\pi r_+) & \cosh(2\pi r_+) \end{pmatrix}$$

The attractive and repulsive fixed points have coordinates $(0, \pm 1, 0, 1)$. Thus, the outer region is $\{\mid y_1 \mid < x_1, \mid x_2 \mid < y_2\}$. A natural associated coordinate system on this domain is:

$$\begin{aligned} x_1 &= \rho_1 \cosh(T) \\ y_1 &= \rho_1 \sinh(T) \\ x_2 &= \rho_2 \sinh(\phi) \\ y_2 &= \rho_2 \cosh(\phi) \end{aligned}$$

with $\rho_1^2 = \rho_2^2 - 1$. The AdS-metric $dx_1^2 + dx_2^2 - dy_1^2 - dy_1^2$ in the coordinates $(T, \phi, \rho = \rho_2)$ is:

$$\frac{1}{\rho^2-1}d\rho^2 + \rho^2 d\phi^2 - (\rho^2-1)dT^2$$

Observe that ϕ, T may have any real value, and that ρ takes value in $]1,+\infty[$.

The action of γ in the coordinates $(\phi, T, \rho_1, \rho_2)$ is simply the translation by $2\pi r_+$ on ϕ and by $-2\pi r_-$ on T. Hence, it preserves the function $t = \frac{r_-\phi + r_+ T}{r_+^2 - r_-^2}$, and adds to $\varphi = \frac{r_+\phi + r_- T}{r_+^2 - r_-^2}$ the term 2π. Therefore, we introduce the coordinates (t, φ) instead of (T, ϕ): φ is considered as a polar coordinate, the action by γ being represented by $\varphi \to \varphi + 2\pi$, the other coordinates (t, ρ) remaining unchanged. Actually, replace ρ by the coordinate r defined by: $\rho^2 = \frac{r^2 - r_-^2}{r_+^2 - r_-^2}$. Then, the AdS-metric becomes:

$$-N(r)dt^2 + N(r)^{-1}dr^2 + r^2(d\varphi + \frac{J}{2r^2}dt)^2$$

where:

$$\begin{aligned} J &= -2r_- r_+ \\ N(r) &= \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2} \end{aligned}$$

The coordinates (φ, t, r) are the *Kerr-like coordinates.* Considering φ as defined modulo 2π, it provides an expression of the outer region of $M_{xy}(\Gamma)$. The analogy with the Kerr metric is striking if we observe $N(r) = r^2 - M + (\frac{J}{2r})^2$, where $M = r_+^2 + r_-^2$.

Remark 10.10. In the Kerr-like coordinates, the level set of the time function t are *not* homogeneous. The stabilizer of the outer region is A_{hyp}, the orbits of which in the outer region are timelike. Observe that this action of A_{hyp} is an action by translations on the coordinates (t, φ). Actually, the (scalar) curvature of the level sets of t is not constant (except if $r_- = 0$). What is the specific geometric feature of these level sets?

A remarkable fact is that the level sets $\{t = t_0\}$ are *maximal,* i.e. have zero mean curvature. A quick way to perform the computation is as follows: parametrize the level set $S_0 = \{t = t_0\}$ by parameters η, φ, where $\rho = \cosh(\eta)$:

$$\begin{aligned} x_1 &= \sinh(\eta)\cosh(T) \\ y_1 &= \sinh(\eta)\sinh(T) \\ x_2 &= \cosh(\eta)\sinh(\phi) \\ y_2 &= \cosh(\eta)\cosh(\phi) \end{aligned}$$

with $T = r_+ t_0 - r_-\varphi$, $\phi = r_+\varphi - r_- t_0$. Let p be a point of S_0 of coordinates (η, φ). Identify the tangent space to AdS at p with the Q-orthogonal $p^\perp$ in E. Then, the tangent vectors of S_0 at p are generated by:

$$\partial_\eta = \begin{pmatrix} \cosh(\eta)\cosh(T) \\ \cosh(\eta)\sinh(T) \\ \sinh(\eta)\sinh(\phi) \\ \sinh(\eta)\cosh(\phi) \end{pmatrix}, \quad \partial_\varphi = \begin{pmatrix} -r_-\sinh(\eta)\sinh(T) \\ -r_-\sinh(\eta)\cosh(T) \\ r_+\cosh(\eta)\cosh(\phi) \\ r_+\cosh(\eta)\sinh(\phi) \end{pmatrix}$$

The future oriented normal n_0 to S_0 at p is the following vector n, divided by the square root of the opposite of its norm, which is $r_-^2\sinh(\eta)^2 - r_+^2\cosh(\eta)^2 = -r^2$:

$$n = \begin{pmatrix} r_+\cosh(\eta)\sinh(T) \\ r_+\cosh(\eta)\cosh(T) \\ -r_-\sinh(\eta)\cosh(\phi) \\ -r_-\sinh(\eta)\sinh(\phi) \end{pmatrix}$$

Now, the second fundamental form of S_0 on a tangent vector field X at p is obtained by computing $\langle n_0 \mid \nabla_X X\rangle$. But the Levi-Civita connection of AdS is just the orthogonal projection on TAdS of the usual flat connection of E. Hence, the (extrinsic) curvature of the curves $\{\varphi = Cte\}$ and $\{\eta = Cte\}$ are the Q-scalar products with n_0 of the following vectors:

$$\partial_{\eta\eta} = \begin{pmatrix} \sinh(\eta)\cosh(T) \\ \sinh(\eta)\sinh(T) \\ \cosh(\eta)\sinh(\phi) \\ \cosh(\eta)\cosh(\phi) \end{pmatrix} = p, \quad \partial_{\varphi\varphi} = \begin{pmatrix} r_-^2\sinh(\eta)\cosh(T) \\ r_-^2\sinh(\eta)\sinh(T) \\ r_+^2\cosh(\eta)\sinh(\phi) \\ r_+^2\cosh(\eta)\cosh(\phi) \end{pmatrix}$$

These scalar products are null. Since the curves $\{\eta = Cte\}$ and $\{\varphi = Cte\}$ are everywhere orthogonal, the mean curvature of S_0 at p is thus 0 (but these curves do not define the principal directions if $r_- \neq 0$). Observe that $\{\varphi = Cte\}$ is actually a spacelike geodesic in AdS.

Moreover, $\langle n \mid \partial_{\eta\varphi}\rangle = r_- r_+$. It follows that the second fundamental form is $\mathrm{II} = r^{-1}r_-r_+ d\eta d\varphi$. The pair $(\partial_\eta, r^{-1}\partial_\varphi)$ is an orthonormal basis of the tangent space at p. Hence, the Gauss curvature of S_0 is:

$$\frac{r_-^2 r_+^2}{4r^4}$$

10.5-b. *The Kerr-like metric on the outer region of an extreme spacetime.* Observe that the Kerr-like metric remains meaningfull when $r_- = r_+$, even if the coordinate transformations considered in the previous § are not anymore valid. Let $E'(r_+)$ be (simply connected) lorentzian manifold consisting of $\mathbb{R}^2\times]r_+, +\infty[$ with coordinates (t, φ, ρ), equipped with the metric:

$$-N(r)dt^2 + N(r)^{-1}dr^2 + r^2(d\varphi + \frac{J}{2r^2}dt)^2$$

where:

$$\begin{aligned} J &= -2r_+^2 \\ N(r) &= \frac{(r^2 - r_+^2)^2}{r^2} = r^2 - 2r_+^2 + (\frac{J}{2r})^2 \end{aligned}$$

$M'(r_+)$ is the quotient of $E'(r_+)$ by the translation $\varphi \to \varphi + 2\pi$, the other coordinates remaining the same. We want to prove that $E'(r_+)$ is isometric to the outer region of an extreme black-hole as defined in § 10.3.

The sectional curvature of the Kerr-like metric for $r_- = r_+ - \epsilon$ is -1 for any ϵ, it then remains true at the limit $\epsilon = 0$. Hence, $E'(r_+)$ and $M'(r_+)$ are locally AdS. Observe also that ∂_φ and ∂_t generates a rank 2 abelian Lie algebra of Killing vector fields. This is the pull-back by the developing map of an abelian Lie algebra $\mathcal{A}(r_+)$ of Killing vector fields on AdS. This Lie algebra is of course a limit of algebras $\mathcal{A}(r_+, r_+ - \epsilon)$ which are all Lie algebras of subgroups conjugate to A_{hyp}. It follows easily that $\mathcal{A}(r_+)$ is either conjugate to the Lie algebra of A_{hyp}, or to the Lie algebra of A_{ext}. A quick calculus shows that $\partial_\varphi + \partial_t$ is an everywhere lightlike Killing vector field: it excludes the hyperbolic-hyperbolic case A_{hyp}, hence, up to conjugacy, the isometry group generated by ∂_t and ∂_φ is A_{ext}.

Parametrize a line $\{t = Cte, \rho = Cte\}$ by φ, and compute $\langle \partial_{\varphi\varphi} \mid p \rangle = -r^2$ (once more, we can compute for $r_- = r_+ - \epsilon$). Hence, the orbits of the translations on the φ-coordinate are not geodesic. It follows that they are not hyperbolic translations: they are hyperbolic-parabolic. In other words, the monodromy of the translation $\varphi \to \varphi + 2\pi$ is conjugate to $\gamma' = (\exp(2\pi r_0 \Delta), \exp(H))$, for some $r_0 > 0$. Actually, $r_0 = r_+$: indeed, for $r_- = r_+ - \epsilon$, the left component of the monodromy of $\varphi \to \varphi + 2\pi$ is conjugate to $\exp(2\pi r_+)$, it remains true at the limit $\epsilon = 0$ by continuity of the monodromy under deformations of the AdS-structure. Hence, $\gamma' = \gamma$.

We observe now that ∂_φ is spacelike on $E'(r_+)$. Hence, the image of the developing map of $E'(r_+)$ is contained in $D(\gamma)$. The curve $c(r) = (t = 0, \varphi = 0, r)$ is a geodesic (for example, the study in the preceding section shows that it is true in the case $r_- < r_+$, and the case $r_- = r_+$ is a limit case). Moreover, the length between two points $c(r)$, $c(r')$ is $\frac{1}{2}\log(\frac{r'^2 - r_+^2}{r^2 - r_+^2})$: c is a complete spacelike geodesic. According to the description of absolute causal domains of hyperbolic-parabolic elements (case (7) of Proposition 4.11), the complete spacelike geodesics entirely contained in $D(\gamma)$ must actually lie in $C(\gamma)$, which is the outer region E of the extreme black-hole associated to γ. Moreover, every A_{ext}-orbit in $E'(r_+)$ intersects c. Hence, the developing image of $E'(r_+)$ is contained in E. The action of A_{ext} on E is free: it follows that the restriction of the developing map $\mathcal{D}$ to any A_{ext}-orbit in $E'(r_+)$ is a homeomorphism onto an entire A_{ext}-orbit in E. Finally, the restriction of c is injective, with image a complete spacelike geodesic $\bar{c}$ in E. Every A_{ext}-orbit in E intersect $\bar{c}$. It follows that $\mathcal{D}$ is an isometry between $E'(r_+)$ and E: the coordinates (t, φ, r) parametrizes the entire outer region E.

Remark 10.11. The limit "$r = r_+$" is a lightlike segment τ in ∂AdS. It follows that the time levels $\{t = Cte\}$ are closed in AdS. Their closure in AdS $\cup$ ∂AdS intersects ∂AdS along an achronal topological circle which contains the lightlike segment τ, and is spacelike outside τ. Compare this situation with the example described page 45 in [22].

Of course, these level sets have zero mean curvature since it is true in the non-extreme case $r_- < r_+$. Furthermore, they have Gauss curvature $r_+^4/4r^4$.

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