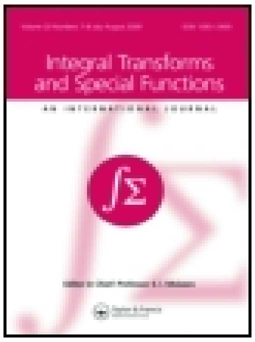

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Indefinite integrals involving the exponential integral function

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ABSTRACT

The exponential integral function Ei(x) is given as an indefinite integral of an elementary expression. This allows a second-order linear differential equation for the function to be constructed, which is of conventional form. A limitless number of differential equations can be derived from the original by elementary transformations, and many integrals are given by applying the method of fragments to some of these transformed equations. Results are presented here both for simple transformations and other transformations obtained by solving simple Riccati equations. Some of the Integrals are presented combine Ei(x) with Bessel functions, modified Bessel functions and Whittaker functions. All results have been checked by differentiation using Mathematica.

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1. Introduction

The exponential integral function is defined by the integral [1,2]

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{\mathrm{e}^t}{t}\,\mathrm{d}t \tag{1.1}$$

for which

$$\mathrm{Ei}'(x) = \frac{\mathrm{e}^x}{x}. \tag{1.2}$$

Any arbitrary function $y(x)$ trivially obeys a differential equation of the form

$$\bar{y}''(x) - \left(\frac{y''(x)}{y'(x)}\right)\bar{y}'(x) = 0 \tag{1.3}$$

which has the general solution

$$\bar{y}(x) = C_1 + C_2 y(x).$$

From Equations (1.1)–(1.3) the function $y(x) = \mathrm{Ei}(x)$ obeys the differential equation

$$\bar{y}''(x) + \left(\frac{1}{x} - 1\right)\bar{y}'(x) = 0 \tag{1.4}$$

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which is of conventional form. This paper applies the method of fragments introduced in [3,4] to derive indefinite integrals involving $\mathrm{Ei}(x)$. As Equation (1.4) by itself provides only a limited number of fragments, various transformations will be applied to this equation, which provide in principle an unlimited number of cases, but only a limited number of these are interesting.

1.1. The method of fragments

The general second-order linear homogeneous differential equation is

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \tag{1.5}$$

and in [3,4] the integration formula

$$\int f(x)(h''(x) + p(x)h'(x) + q(x)h(x))y(x)\,\mathrm{d}x = f(x)(h'(x)y(x) - h(x)y'(x)) \tag{1.6}$$

was derived, where $y(x)$ is any solution of Equation (1.1) and $h(x)$ is an arbitrary twice differentiable complex- valued function of x. The function $f(x)$ in Equation (1.2) is the reciprocal of the Wronskian for Equation (1.1) and is also the integrating factor for the two leftmost terms of this equation. It also appears in the Lagrangian form [3] of Equation (1.1) and is given by

$$f(x) = \exp\left(\int p(x)\,\mathrm{d}x\right). \tag{1.7}$$

Suitable choices of the arbitrary function $h(x)$ in Equation (1.6) give integrals involving $y(x)$. A useful technique for obtaining interesting integrals is to take $h(x)$ to be a solution of a fragment of Equation (1.5), where a fragment is defined [3] as the differential equation with one or more terms deleted or modified, for example

$$h''(x) + q(x)h(x) = 0.$$

For an equation of the form (1.3), the factor $f(x)$ in Equation (1.7) is given as

$$\bar{f}(x) = \exp\left(-\int \frac{y''(x)}{y'(x)}\,\mathrm{d}x\right)$$

and as any constant multiplicative factor in the definition of $f(x)$ would cancel in Equation (1.2), we can always take

$$\bar{f}(x) = \frac{1}{y'(x)}$$

and for Equation (1.4) this gives

$$\bar{f}(x) = x\,\mathrm{e}^{-x}.$$

Equation (1.4) is simple with few terms, and the small number of obvious fragments are

$$h''(x) = 0 \Rightarrow h(x) = x$$

$$h''(x) + \frac{1}{x}h'(x) = 0 \Rightarrow h(x) = \ln(x)$$
$$h''(x) - h'(x) = 0 \Rightarrow h(x) = e^x.$$

Substituting these results into Equation (1.6) gives the three integrals

$$\int (1-x)\,e^{-x}\,\mathrm{Ei}(x)\,dx = x(e^{-x}\,\mathrm{Ei}(x) - 1) \tag{1.8}$$
$$\int e^{-x}\,\mathrm{Ei}(x)\,dx = \ln(x) - e^{-x}\,\mathrm{Ei}(x)$$
$$\int \mathrm{Ei}(x)\,dx = x\,\mathrm{Ei}(x) - e^x \tag{1.9}$$

and subtracting Equation (1.9) from Equation (1.8) gives the additional integral

$$\int xe^{-x}\,\mathrm{Ei}(x)\,dx = x + \ln(x) - (1+x)\,e^{-x}\,\mathrm{Ei}(x).$$

All of these cases are known integrals [2].

1.2. Transformations of the differential equation

Equation (1.4) is of the baseline form

$$\bar{y}''(x) + \bar{p}(x)\bar{y}'(x) = 0$$

with

$$\bar{f}(x) = \exp\left(\int \bar{p}(x)\,dx\right). \tag{1.10}$$

Equation (1.10) can be transformed [3] to an equation in $y(x)$ by the substitution $\bar{y}(x) = g(x)y(x)$, which gives

$$y''(x) + \left(2\frac{g'(x)}{g(x)} + \bar{p}(x)\right)y'(x) + \left(\frac{g''(x)}{g(x)} + \bar{p}(x)\frac{g'(x)}{g(x)}\right)y(x) = 0. \tag{1.11}$$

In [3] there was a typographical error equivalent to stating $y(x) = g(x)\bar{y}(x)$, but all the related formulas were given correctly. Defining

$$p(x) = 2\frac{g'(x)}{g(x)} + \bar{p}(x)$$

then

$$g(x) = \sqrt{\frac{f(x)}{\bar{f}(x)}}$$

$$y(x) = \sqrt{\frac{\bar{f}(x)}{f(x)}}\bar{y}(x)$$

and Equation (1.11) can be expressed as [3]

$$y''(x) + p(x)y'(x) + [\tfrac{1}{2}(p(x) - \bar{p}(x))' + \tfrac{1}{4}(p^2(x) - \bar{p}^2(x))]y(x) = 0. \tag{1.12}$$

The general solution of Equation (1.12) is given in terms of the general solution of Equation (1.4) by [3]

$$y(x) = \sqrt{\frac{\bar{f}(x)}{f(x)}}\bar{y}(x).$$

For Equation (1.4) for $\mathrm{Ei}(x)$, the transformed Equation (1.12) becomes

$$y''(x) + p(x)y'(x) + \left[\frac{p'(x)}{2} + \frac{p^2(x)}{4} - \frac{1}{4} + \frac{1}{2x} + \frac{1}{4x^2}\right]y(x) = 0. \tag{1.13}$$

Employing Equation (1.13) in Equation (1.6) gives a large number of integrals involving $\bar{y}(x) = C_1 + C_2\,\mathrm{Ei}(x)$, as $p(x)$ is arbitrary, but only sample results can be given here. As in [3,4], interesting integrals can be obtained by applying the method of fragments to various forms of Equation (1.13). Equation (1.13) can be given concrete form either by directly specifying the function $p(x)$ or by solving a Riccati equation for $p(x)$ such that the term in square brackets takes some desired simpler form. Sample cases obtained by specifying $p(x)$ are examined in Section 2 below, and some cases where $p(x)$ is the solution of a Riccati equation are examined in Section 3. All results presented have been checked using Mathematica [5].

2. Integrals from specifying $p(x)$ directly

Choosing $p(x) = 0$ in Equation (1.13) gives the differential equation

$$y''(x) + \left(-\frac{1}{4} + \frac{1}{2x} + \frac{1}{4x^2}\right)y(x) = 0 \tag{2.1}$$

with the general solution

$$y(x) = \sqrt{x}\,\mathrm{e}^{-\frac{1}{2}x}(C_1 + C_2\,\mathrm{Ei}(x)) \tag{2.2}$$

and with the first derivative

$$y'(x) = \frac{1}{\sqrt{x}}\left(\frac{1-x}{2}\,\mathrm{e}^{-\frac{1}{2}x}\,(C_1 + C_2\,\mathrm{Ei}(x)) + C_2\,\mathrm{e}^{\frac{1}{2}x}\right).$$

Equation (2.1) is a special case of the general Whittaker equation

$$y''(x) + \left(-\frac{1}{4} + \frac{\lambda}{x} + \frac{\frac{1}{4} - \mu^2}{x^2}\right)y(x) = 0 \tag{2.3}$$

which has the general solution

$$y(x) = \hat{W}_{\lambda,\mu}(x) \equiv C_3 M_{\lambda,\mu}(x) + C_4 W_{\lambda,\mu}(x).$$

It might be expected that the solution given by Equation (2.2) could also be expressed in terms of the two Whittaker functions $M_{\frac{1}{2},0}(x)$ and $W_{\frac{1}{2},0}(x)$, but this is not the case. An equation for $M_{\lambda,\mu}(x)$ in terms of $W_{-\lambda,\mu}(x)$ and $W_{\lambda,\mu}(x)$ is given in [6] as

$$\begin{aligned} M_{\lambda,\mu}(x) = &\frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\lambda+\frac{1}{2}\right)}\, \mathrm{e}^{\mathrm{i}\pi\lambda} W_{-\lambda,\mu}\left(\mathrm{e}^{\mathrm{i}\pi} x\right) \\ &+ \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu+\lambda+\frac{1}{2}\right)} \exp\left[\mathrm{i}\pi\left(\lambda-\mu-\frac{1}{2}\right)\right] W_{\lambda,\mu}(x) \end{aligned}$$

and for $\lambda = \frac{1}{2}$ and $\mu = 0$ this equation reduces to $M_{\frac{1}{2},0}(x) = W_{\frac{1}{2},0}(x)$, so that Equation (2.1) is a degenerate case of the Whittaker equation. The relation [6]

$$M_{n+\mu+\frac{1}{2},\mu}(x) = \frac{x^{\frac{1}{2}-\mu}\mathrm{e}^{\frac{1}{x}x}}{(2\mu+1)(2\mu+2)\cdots(2\mu+n)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^{n+2\mu}\mathrm{e}^{-x})$$

reduces for $n = 0$ and $\mu = 0$ to

$$M_{\frac{1}{2},0}(x) = \sqrt{x}\mathrm{e}^{-\frac{x}{2}}.$$

Hence for this degenerate case $M_{\frac{1}{2},0}(x)$ and $W_{\frac{1}{2},0}(x)$ both reduce to the elementary function in Equation (2.2) and the extra solution of the degenerate equation is $y(x) = \sqrt{x}\mathrm{e}^{-\frac{1}{2}x}\,\mathrm{Ei}(x)$.

Taking the differential equation in Equation (1.6) to be Equation (2.1) and specifying $h(x)$ to be any solution of the general Whittaker Equation (2.3) gives the integral

$$\begin{aligned} &\int \frac{\mathrm{e}^{-\frac{x}{2}}}{\sqrt{x}}\left(\frac{1}{2}-\lambda+\frac{\mu^2}{x}\right)\hat{W}_{\lambda,\mu}(x)(C_1 + C_2\,\mathrm{Ei}(x))\,\mathrm{d}x \\ &\quad = \hat{W}'_{\lambda,\mu}(x)\sqrt{x}\,\mathrm{e}^{-\frac{1}{2}x}(C_1 + C_2\,\mathrm{Ei}(x)) - \hat{W}_{\lambda,\mu}(x)(\sqrt{x}\,\mathrm{e}^{-\frac{1}{2}x}(C_1 + C_2\,\mathrm{Ei}(x)))'. \end{aligned}$$

For $C_2 = 0$ this integral reduces to

$$\begin{aligned} &\int \frac{\mathrm{e}^{-\frac{x}{2}}}{\sqrt{x}}\left(\frac{1}{2}-\lambda+\frac{\mu^2}{x}\right)\hat{W}_{\lambda,\mu}(x)\,\mathrm{d}x \\ &\quad = \sqrt{x}\,\mathrm{e}^{-\frac{x}{2}}\left[\hat{W}'_{\lambda,\mu}(x) + \frac{1}{2}\left(1-\frac{1}{x}\right)\hat{W}_{\lambda,\mu}(x)\right] \end{aligned}$$

and for $C_1 = 0$ it reduces to

$$\begin{aligned} &\int \frac{\mathrm{e}^{-\frac{x}{2}}}{\sqrt{x}}\left(\frac{1}{2}-\lambda+\frac{\mu^2}{x}\right)\hat{W}_{\lambda,\mu}(x)\,\mathrm{Ei}(x)\,\mathrm{d}x \\ &\quad = \sqrt{x}\,\mathrm{e}^{-\frac{1}{2}x}\left[\hat{W}'_{\lambda,\mu}(x)\,\mathrm{Ei}(x) + \hat{W}_{\lambda,\mu}(x)\left(\frac{1}{2}\left(1-\frac{1}{x}\right)\mathrm{Ei}(x) - \frac{\mathrm{e}^x}{x}\right)\right]. \end{aligned}$$

Further simplifications of these integrals are obtained for $\lambda = \frac{1}{2}$ or $\mu = 0$ and $\hat{W}_{\lambda,\mu}(x) = M_\lambda^\mu(x)$ or $W_\lambda^\mu(x)$.

Choosing $h(x)$ to be a solution of the equation

$$h''(x) + \left(-\frac{1}{4} + \frac{1}{4x^2}\right) h(x) = 0 \tag{2.4}$$

gives $h(x)$ in terms of a general modified Bessel function of order zero as

$$h(x) = \sqrt{x}\hat{K}_0\left(\frac{1}{2}x\right).$$

where

$$\hat{K}_0\left(\frac{1}{2}x\right) \equiv C_3 I_0\left(\frac{1}{2}x\right) + C_4 K_0\left(\frac{1}{2}x\right).$$

The modified Bessel function $I_0(\frac{1}{2}x)$ and the MacDonald function $K_0(\frac{1}{2}x)$ both obey Equation (2.4), but they have different formulas for their derivatives [7], with

$$I_0'\left(\frac{1}{2}x\right) = \frac{1}{2}I_1\left(\frac{1}{2}x\right)$$

$$K_0'\left(\frac{1}{2}x\right) = -\frac{1}{2}K_1\left(\frac{1}{2}x\right)$$

and when deriving explicit integration formulas it is simpler to treat the two cases separately. We have the alternative formulas

$$h(x) = \sqrt{x}I_0\left(\frac{1}{2}x\right); \quad h'(x) = \frac{1}{2\sqrt{x}}\left(I_0\left(\frac{1}{2}x\right) + xI_1\left(\frac{1}{2}x\right)\right)$$

$$h(x) = \sqrt{x}K_0\left(\frac{1}{2}x\right); \quad h'(x) = \frac{1}{2\sqrt{x}}\left(K_0\left(\frac{1}{2}x\right) - xK_1\left(\frac{1}{2}x\right)\right)$$

Substituting substituting these choices into Equation (1.6) and simplifying gives the integrals

$$\begin{aligned}&\int I_0\left(\frac{1}{2}x\right) \mathrm{e}^{-\frac{1}{2}x}\left(C_1 + C_2 \operatorname{Ei}(x)\right) \mathrm{d}x \\ &\quad = x\mathrm{e}^{-\frac{1}{2}x}\left(I_0\left(\frac{1}{2}x\right) + I_1\left(\frac{1}{2}x\right)\right)\left(C_1 + C_2 \operatorname{Ei}(x)\right) - 2C_2\,\mathrm{e}^{\frac{1}{2}x} I_0\left(\frac{1}{2}x\right)\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\int K_0\left(\frac{1}{2}x\right) \mathrm{e}^{-\frac{1}{2}x}\left(C_1 + C_2 \operatorname{Ei}(x)\right) \mathrm{d}x \\ &\quad = x\,\mathrm{e}^{-\frac{1}{2}x}\left(K_0\left(\frac{1}{2}x\right) - K_1\left(\frac{1}{2}x\right)\right)\left(C_1 + C_2 \operatorname{Ei}(x)\right) - 2C_2\,\mathrm{e}^{\frac{1}{2}x} K_0\left(\frac{1}{2}x\right)\end{aligned} \tag{2.6}$$

For $C_2 = 0$ Equations (2.5)–(2.6) reduce to

$$\int \mathrm{e}^{-x} I_0(x)\,\mathrm{d}x = x\,\mathrm{e}^{-x}(I_0(x) + I_1(x))$$

$$\int \mathrm{e}^{-x} K_0(x)\,\mathrm{d}x = x\,\mathrm{e}^{-x}\left(K_0(x) - K_1(x)\right)$$

which are known integrals [2]. For $C_1 = 0$, Equations (2.5)–(2.6) reduce to

$$\int \mathrm{e}^{-x} I_0(x)\,\mathrm{Ei}(2x)\,\mathrm{d}x = x\,\mathrm{e}^{-x}(I_0(x) + I_1(x))\,\mathrm{Ei}(2x) - \mathrm{e}^{x} I_0(x) \tag{2.7}$$

$$\int \mathrm{e}^{-x} K_0(x)\,\mathrm{Ei}(2x)\,\mathrm{d}x = x\,\mathrm{e}^{-x}(K_0(x) - K_1(x))\,\mathrm{Ei}(2x) - \mathrm{e}^{x} K_0(x) \tag{2.8}$$

and Equations (2.7)–(2.8) appear to be new.

Choosing $h(x)$ to be a solution of the fragment

$$h''(x) + \left(\frac{1}{2x} + \frac{1}{4x^2}\right) h(x) = 0$$

gives

$$h(x) = \sqrt{x} Z_0(\sqrt{2x}); \quad h'(x) = \frac{Z_0(\sqrt{2x}) - \sqrt{2x} Z_1(\sqrt{2x})}{2\sqrt{x}} \tag{2.9}$$

where $Z_0(x) \equiv C_3 J_0(x) + C_4 Y_0(x)$ is the general cylinder function of order zero. Substituting Equations (2.9) into Equation (1.6) gives the integral

$$\begin{aligned} &\int x\,\mathrm{e}^{-\frac{1}{2}x} Z_0(\sqrt{2x})(C_1 + C_2\,\mathrm{Ei}(x))\,\mathrm{d}x \\ &= 2\sqrt{2x} Z_1(\sqrt{2x})\,\mathrm{e}^{-\frac{1}{2}x}(C_1 + C_2\,\mathrm{Ei}(x)) \\ &\quad - Z_0(\sqrt{2x})\left(2x\,\mathrm{e}^{-\frac{1}{2}x}(C_1 + C_2\,\mathrm{Ei}(x)) - 4C_2\,\mathrm{e}^{\frac{1}{2}x}\right). \end{aligned} \tag{2.10}$$

For $C_2 = 0$ Equation (2.10) reduces to

$$\int x\,\mathrm{e}^{-\frac{1}{2}x} Z_0(\sqrt{2x})\,\mathrm{d}x = 2\,\mathrm{e}^{-\frac{1}{2}x}(\sqrt{2x} Z_1(\sqrt{2x}) - x Z_0(\sqrt{2x}) t)$$

and for $C_1 = 0$ this equation reduces to

$$\begin{aligned} &\int x\,\mathrm{e}^{-\frac{1}{2}x} Z_0(\sqrt{2x})\,\mathrm{Ei}(x)\,\mathrm{d}x \\ &= 2(\sqrt{2x} Z_1(\sqrt{2x})\,\mathrm{e}^{-\frac{1}{2}x}\,\mathrm{Ei}(x) - Z_0(\sqrt{2x})(x\,\mathrm{e}^{-\frac{1}{2}x}\,\mathrm{Ei}(x) - 2\mathrm{e}^{\frac{1}{2}x})). \end{aligned}$$

Choosing $p(x) = 1/x$ gives the differential equation

$$y''(x) + \frac{1}{x} y'(x) + \left(\frac{1}{2x} - \frac{1}{4}\right) y(x) = 0 \tag{2.11}$$

with $f(x) = x$ and the general solution is

$$y(x) = \mathrm{e}^{-\frac{1}{2}x}(C_1 + C_2\,\mathrm{Ei}(x)).$$

Only the special case $y(x) = \mathrm{e}^{-\frac{1}{2}x}\,\mathrm{Ei}(x)$ will be considered here, for which

$$y'(x) = \frac{\mathrm{e}^{\frac{1}{2}x}}{x} - \frac{\mathrm{e}^{-\frac{1}{2}x}}{2}\,\mathrm{Ei}(x).$$

Some fragments of Equation (2.11) are

$$h''(x) + \frac{1}{x} h'(x) = 0 \Rightarrow h(x) = C_3 + C_4 \ln(x)$$

$$h''(x) + \frac{1}{x}h'(x) - \frac{1}{4}h(x) = 0 \Rightarrow h(x) = \hat{K}_0\left(\frac{x}{2}\right)$$
$$\frac{1}{x}h'(x) + \left(\frac{1}{2x} - \frac{1}{4}\right)h(x) = 0 \Rightarrow h(x) = \mathrm{e}^{\frac{1}{8}x(x-4)}$$

and these give the respective integrals below for the special case $y(x) = \mathrm{Ei}(x)$.

$$\int \left(1 - \frac{x}{2}\right)\ln(x)\,\mathrm{e}^{-\frac{1}{2}x}\,\mathrm{Ei}(x)\,\mathrm{d}x = 2\mathrm{e}^{-\frac{1}{2}x}\,\mathrm{Ei}(x) + \ln(x)(x\,\mathrm{e}^{-\frac{1}{2}x}\,\mathrm{Ei}(x) - 2\mathrm{e}^{\frac{1}{2}x})$$

for $h(x) = \ln(x)$.

$$\int I_0\left(\frac{x}{2}\right)\mathrm{e}^{-\frac{1}{2}x}\,\mathrm{Ei}(x)\,\mathrm{d}x = x\left(I_0\left(\frac{x}{2}\right) + I_1\left(\frac{1}{2}x\right)\right)\mathrm{e}^{-\frac{1}{2}x}\,\mathrm{Ei}(x) - 2I_0\left(\frac{x}{2}\right)\mathrm{e}^{\frac{1}{2}x}$$
$$\int K_0\left(\frac{x}{2}\right)\mathrm{e}^{-\frac{1}{2}x}\,\mathrm{Ei}(x)\,\mathrm{d}x = x\left(K_0\left(\frac{x}{2}\right) - K_1\left(\frac{1}{2}x\right)\right)\mathrm{e}^{-\frac{1}{2}x}\,\mathrm{Ei}(x) - 2K_0\left(\frac{x}{2}\right)\mathrm{e}^{\frac{1}{2}x}$$
$$\int x(x^2 - 4x + 8)\,\mathrm{e}^{\frac{1}{8}x^2 - x}\,\mathrm{Ei}(x)\,\mathrm{d}x = 4x^2\,\mathrm{e}^{\frac{1}{8}x^2 - x}\,\mathrm{Ei}(x) - 16\mathrm{e}^{\frac{1}{8}x^2}.$$

Choosing $p(x) = -1$ gives the differential equation

$$y''(x) - y'(x) + \frac{1 + 2x}{4x^2}y(x) = 0$$

for which

$$f(x) = \mathrm{e}^{-x}$$
$$y(x) = C_1\sqrt{x} + C_2\sqrt{x}\,\mathrm{Ei}(x)$$
$$y'(x) = \frac{C_1 + C_2\,\mathrm{Ei}(x)}{2\sqrt{x}} + C_2\frac{\mathrm{e}^x}{\sqrt{x}}.$$

For $y(x) = \sqrt{x}\,\mathrm{Ei}(x)$ the fragments

$$h''(x) = 0 \Rightarrow h(x) = C_1 + C_2x$$
$$h''(x) - h'(x) = 0 \Rightarrow h(x) = C_1 + C_2\mathrm{e}^x$$
$$-h'(x) + \frac{1 + 2x}{4x^2}h(x) = 0 \Rightarrow h(x) = \sqrt{x}\mathrm{e}^{-\frac{1}{4x}}$$

give the integrals

$$\int \frac{1 + 2x}{x^{\frac{3}{2}}}\mathrm{e}^{-x}\,\mathrm{Ei}(x)\,\mathrm{d}x = -\frac{2\mathrm{e}^{-x}\,\mathrm{Ei}(x)}{\sqrt{x}} - \frac{4}{\sqrt{x}}$$
$$\int \frac{1 - 2x}{\sqrt{x}}\mathrm{e}^{-x}\,\mathrm{Ei}(x)\,\mathrm{d}x = 2\sqrt{x}(\mathrm{e}^{-x}\,\mathrm{Ei}(x) - 2)$$
$$\int \frac{1 + 2x}{x^{\frac{3}{2}}}\,\mathrm{Ei}(x)\,\mathrm{d}x = \frac{(4x - 2)\,\mathrm{Ei}(x) - 4\mathrm{e}^x}{\sqrt{x}}$$
$$\int \frac{4x^2 + 4x - 1}{x^3}\,\mathrm{e}^{-\frac{1}{4x} - x}\,\mathrm{Ei}(x)\,\mathrm{d}x = 4\mathrm{e}^{-\frac{1}{4x}}\left(4 - \frac{1}{x}\mathrm{e}^{-x}\,\mathrm{Ei}(x)\right)$$

3. Equations from solutions of Riccati equations

The expression in square brackets in Equation (1.13) is

$$\frac{p'(x)}{2}+\frac{p^2(x)}{4}-\frac{1}{4}+\frac{1}{2x}+\frac{1}{4x^2}$$

and this can be simplified by setting various combinations of the terms equal to zero and solving the resulting Riccati equations for $p(x)$. One case is

$$\frac{p'(x)}{2}+\frac{p^2(x)}{4}-\frac{1}{4}=0$$

which is separable in $p(x)$, such that

$$\int \frac{\mathrm{d}p}{1-p^2}=\frac{1}{2}\int \mathrm{d}x.$$

This equation gives

$$\operatorname{arctanh}(p(x))=\frac{x}{2}+c \tag{3.1}$$

where c is an arbitrary constant. Setting $c=0$ and taking the tanh of both sides of Equation (3.1) gives the solution

$$p(x)=\frac{\mathrm{e}^x-1}{\mathrm{e}^x+1} \tag{3.2}$$

Equation (3.2) gives the differential equation

$$y''(x)+\frac{\mathrm{e}^x-1}{\mathrm{e}^x+1}y'(x)+\frac{1+2x}{4x^2}y(x)=0 \tag{3.3}$$

with

$$f(x)=\exp\left(\int \frac{\mathrm{e}^x-1}{\mathrm{e}^x+1}\,\mathrm{d}x\right)=\mathrm{e}^{-x}(\mathrm{e}^x+1)^2 \tag{3.4}$$

and solution

$$y(x)=\frac{\sqrt{x}(C_1+C_2\,\mathrm{Ei}(x))}{\mathrm{e}^x+1}. \tag{3.5}$$

The most interesting case of this solution is

$$y(x)=\frac{\sqrt{x}\,\mathrm{Ei}(x)}{\mathrm{e}^x+1} \tag{3.6}$$

with

$$y'(x)=\frac{(\mathrm{e}^x+1-2x\mathrm{e}^x)\,\mathrm{Ei}(x)}{2\sqrt{x}(\mathrm{e}^x+1)^2}+\frac{\mathrm{e}^x}{\sqrt{x}(\mathrm{e}^x+1)}.$$

Employing Equations (3.3)–(3.6) with $h(x)=1$ in Equation (1.6) gives, after some simplification, the integral

$$\int \frac{(1+2x)(1+\mathrm{e}^{-x})}{x^{\frac{3}{2}}}\,\mathrm{Ei}(x)\,\mathrm{d}x=-2\frac{(1-2x+\mathrm{e}^{-x})\,\mathrm{Ei}(x)+2(\mathrm{e}^x+1)}{\sqrt{x}}.$$

A simpler integral can be obtained by taking $h(x)$ to be a solution of the fragment

$$h''(x) + \frac{e^x - 1}{e^x + 1} h'(x) = 0$$

for which

$$h(x) = \frac{1}{e^x + 1}; \quad h'(x) = -\frac{e^x}{(e^x + 1)^2}.$$

Employing these results in Equation (1.6) gives the integral

$$\int \frac{1 + 2x}{x^{\frac{3}{2}}} e^{-x} \operatorname{Ei}(x) \, dx = -\frac{2}{\sqrt{x}} (e^{-x} \operatorname{Ei}(x) + 2).$$

Another separable case is

$$\frac{p'(x)}{2} + \frac{p^2(x)}{4} = 0$$

which has a solution

$$p(x) = \frac{2}{x}$$

and this gives the differential equation

$$y''(x) + \frac{2}{x} y'(x) + \left(-\frac{1}{4} + \frac{1}{2x} + \frac{1}{4x^2} \right) y(x) = 0 \tag{3.7}$$

with the solution

$$y(x) = \frac{e^{-\frac{1}{2}x}}{\sqrt{x}} (C_1 + C_2 \operatorname{Ei}(x))$$

and

$$f(x) = x^2.$$

For $C_1 = 0$ and $C_2 = 1$ then

$$y(x) = \frac{e^{-\frac{1}{2}x}}{\sqrt{x}} \operatorname{Ei}(x)$$

and

$$y'(x) = \frac{e^{\frac{1}{2}x}}{x^{\frac{3}{2}}} - e^{-\frac{1}{2}x} \frac{(1 + x) \operatorname{Ei}(x)}{2x^{\frac{3}{2}}}.$$

The fragment

$$h''(x) + \left(-\frac{1}{4} + \frac{1}{2x} + \frac{1}{4x^2} \right) h(x) = 0$$

of Equation (3.7) is identical to Equation (2.1) and has the same solution given by Equation (2.2). For $C_1 = 0$ and $C_2 = 1$ then

$$h(x) = e^{-\frac{1}{2}x} \sqrt{x} \operatorname{Ei}(x)$$

$$h'(x) = \mathrm{e}^{-\frac{1}{2}x}\frac{(1-x)\,\mathrm{Ei}(x)}{2\sqrt{x}} + \frac{\mathrm{e}^{\frac{1}{2}x}}{\sqrt{x}}$$

and substituting these results in the integral formula (1.6) allows an integral in $\mathrm{Ei}^2(x)$ to be derived. Employing these formulas in Equation (1.6) gives initially

$$\int (\mathrm{e}^{-x}(1-x)\,\mathrm{Ei}(x) + 2)\,\mathrm{Ei}(x)\,\mathrm{d}x = x\,\mathrm{e}^{-x}\,\mathrm{Ei}^2(x)$$

which can be simplified using Equation (1.8) to give

$$\int \mathrm{e}^{-x}(1-x)\,\mathrm{Ei}^2(x)\,\mathrm{d}x = x\,\mathrm{e}^{-x}\,\mathrm{Ei}^2(x) - 2x\,\mathrm{Ei}(x) + 2\mathrm{e}^x. \tag{3.1}$$

3.1. Non separable cases of the Riccati equation

A Riccati equation of the form

$$\frac{1}{2}\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{1}{2}p^2 + Q(x) = 0$$

can be solved by the substitution $p(x) = 2u(x)$ which gives

$$u'(x) + u^2(x) + Q(x) = 0.$$

Euler [8] showed that solutions of the Riccati equation

$$u'(x) + u^2(x) + P(x)u(x) + Q(x) = 0$$

are given by

$$u(x) = \frac{z'(x)}{z(x)} \tag{3.8}$$

where $z(x)$ obeys the linear equation

$$z''(x) + P(x)z'(x) + Q(x)z(x) = 0.$$

The form of Equation (3.10) means that $f(x)$ is trivially given by

$$f(x) = z^2(x)$$

Hence solutions of the equation

$$\frac{1}{2}p'(x) + \frac{p^2(x)}{4} + \frac{1}{2x} = 0 \tag{3.9}$$

are given by solutions of the equation

$$z''(x) + \frac{1}{2x}z(x) = 0$$

which has the general solution

$$z(x) = \sqrt{2x}Z_1(\sqrt{2x})$$

where $Z_1(x) \equiv C_3J_1(x) + C_4Y_1(x)$ is the general cylinder function of order one, and

$$z'(x) = Z_0(\sqrt{2x}).$$

Hence a solution of Equation (3.10) is given by

$$p(x) = \frac{2Z_0(\sqrt{2x})}{\sqrt{2x}Z_1(x)}$$

and a transformed form of Equation (1.13) is

$$y''(x) + \frac{2Z_0(\sqrt{2x})}{\sqrt{2x}Z_1(\sqrt{2x})}y'(x) + \frac{1-x^2}{4x^2}y(x) = 0. \tag{3.10}$$

for which

$$f(x) = xZ_1^2(\sqrt{2x}).$$

Equation (3.10) has the general solution

$$y(x) = \frac{\mathrm{e}^{-\frac{1}{2}x}(C_1 + C_2\,\mathrm{Ei}(x))}{Z_1(\sqrt{2x})}$$

with

$$y'(x) = ((1-x)Z_1(\sqrt{2x}) - \sqrt{2x}Z_0(\sqrt{2x}))\frac{\mathrm{e}^{-\frac{1}{2}x}(C_1 + C_2\,\mathrm{Ei}(x))}{2xZ_1(\sqrt{2x})^2} + \frac{C_2\,\mathrm{e}^{\frac{1}{2}x}}{xZ_1(\sqrt{2x})}.$$

For the differential equation (3.8), taking $h(x) = 1$ in the integration formula (1.6) gives the integral:

$$\int \frac{1-x^2}{x}Z_1(\sqrt{2x})\,\mathrm{e}^{-\frac{1}{2}x}(C_1 + C_2\,\mathrm{Ei}(x))\,\mathrm{d}x$$
$$= 2((x-1)Z_1(\sqrt{2x}) + \sqrt{2x}Z_0(\sqrt{2x}))\,\mathrm{e}^{-\frac{1}{2}x}(C_1 + C_2\,\mathrm{Ei}(x)) - 4C_2Z_1(\sqrt{2x})\,\mathrm{e}^{\frac{1}{2}x}.$$

For $C_2 = 0$ this equation reduces to

$$\int \frac{1-x^2}{x}Z_1(\sqrt{2x})\,\mathrm{e}^{-\frac{1}{2}x} = 2((x-1)Z_1(\sqrt{2x}) + \sqrt{2x}Z_0(\sqrt{2x}))\,\mathrm{e}^{-\frac{1}{2}x}$$

and for $C_1 = 0$ it reduces to

$$\int \frac{1-x^2}{x}Z_1(\sqrt{2x})\,\mathrm{e}^{-\frac{1}{2}x}\,\mathrm{Ei}(x)\,\mathrm{d}x$$
$$= 2((x-1)Z_1(\sqrt{2x}) + \sqrt{2x}Z_0(\sqrt{2x}))\,\mathrm{e}^{-\frac{1}{2}x}\,\mathrm{Ei}(x) - 4Z_1(\sqrt{2x})\,\mathrm{e}^{\frac{1}{2}x}.$$

Both of these integrals appear to be new.

The equation

$$\frac{1}{2}\frac{\mathrm{d}p}{\mathrm{d}x}+\frac{p^2}{4}+\left(-\frac{1}{4}+\frac{1}{4x^2}\right)=0$$

has the solutions

$$p(x)=\frac{1}{x}+\frac{I_1\left(\frac{1}{2}x\right)}{I_0\left(\frac{1}{2}x\right)}\Rightarrow f(x)=xI_0^2(\frac{1}{2}x)$$

$$p(x)=\frac{1}{x}-\frac{K_1\left(\frac{1}{2}x\right)}{K_0\left(\frac{1}{2}x\right)}\Rightarrow f(x)=xK_0^2(\frac{1}{2}x).$$

These equations give the respective differential equations, solutions and derivatives:

$$y''(x)+\left(\frac{1}{x}+\frac{I_1\left(\frac{1}{2}x\right)}{I_0\left(\frac{1}{2}x\right)}\right)y'(x)+\frac{1}{2x}y(x)=0$$

$$y(x)=\frac{\mathrm{e}^{-\frac{1}{2}x}(C_1+C_2\,\mathrm{Ei}(x))}{I_0\left(\frac{1}{2}x\right)}$$

$$y'(x)=C_2\frac{\mathrm{e}^{\frac{1}{2}x}}{xI_0\left(\frac{1}{2}x\right)}-\mathrm{e}^{-\frac{1}{2}x}\frac{I_0\left(\frac{1}{2}x\right)+I_1\left(\frac{1}{2}x\right)}{2I_0\left(\frac{1}{2}x\right)^2}(C_1+C_2\,\mathrm{Ei}(x)) \tag{3.11}$$

and

$$y''(x)+\left(\frac{1}{x}-\frac{K_1\left(\frac{1}{2}x\right)}{K_0\left(\frac{1}{2}x\right)}\right)y'(x)+\frac{1}{2x}y(x)=0$$

$$y(x)=\frac{\mathrm{e}^{-\frac{1}{2}x}\,(C_1+C_2\,\mathrm{Ei}(x))}{K_0\left(\frac{1}{2}x\right)}. \tag{3.12}$$

The fragment

$$h''(x)+\frac{1}{x}h'(x)+\frac{1}{2x}h=0$$

of both Equation (3.11) and (3.12) has the exact solution and derivative

$$h(x)=Z_0(\sqrt{2x});\quad h'(x)=-\frac{Z_1(\sqrt{2x})}{\sqrt{2x}}.$$

Employing these results in the integration formula (1.6) gives the integrals

$$\begin{aligned}&\int\sqrt{2x}Z_1(\sqrt{2x})I_1\left(\frac{x}{2}\right)\mathrm{e}^{-\frac{1}{2}x}(C_1+C_2\,\mathrm{Ei}(x))\mathrm{d}x\\&=\sqrt{2x}Z_1(\sqrt{2x})I_0\left(\frac{x}{2}\right)\mathrm{e}^{-\frac{1}{2}x}(C_1+C_2\,\mathrm{Ei}(x))\\&\quad+Z_0(\sqrt{2x})\left(2C_2\,\mathrm{e}^{\frac{1}{2}x}I_0\left(\frac{x}{2}\right)-x\,\mathrm{e}^{-\frac{1}{2}x}\left(I_0\left(\frac{1}{2}x\right)+I_1\left(\frac{1}{2}x\right)\right)(C_1+C_2\,\mathrm{Ei}(x))\right)\end{aligned}$$

$$\int \sqrt{2x}Z_1(\sqrt{2x})K_1\left(\frac{x}{2}\right)\mathrm{e}^{-\frac{1}{2}x}(C_1+C_2\,\mathrm{Ei}(x))\mathrm{d}x$$
$$=-\sqrt{2x}Z_1(\sqrt{2x})K_0\left(\frac{x}{2}\right)\mathrm{e}^{-\frac{1}{2}x}(C_1+C_2\,\mathrm{Ei}(x))$$
$$+Z_0(\sqrt{2x})\left(x\,\mathrm{e}^{-\frac{1}{2}x}\left(K_0\left(\frac{1}{2}x\right)-K_1\left(\frac{1}{2}x\right)\right)(C_1+C_2\,\mathrm{Ei}(x))-2C_2\,\mathrm{e}^{\frac{1}{2}x}K_0\left(\frac{x}{2}\right)\right).$$

The four main special cases of these integrals are

$$\int \sqrt{2x}Z_1(\sqrt{2x})I_1\left(\frac{x}{2}\right)\mathrm{e}^{-\frac{1}{2}x}\,\mathrm{Ei}(x)\,\mathrm{d}x=\sqrt{2x}Z_1(\sqrt{2x})I_0\left(\frac{x}{2}\right)\mathrm{e}^{-\frac{1}{2}x}\,\mathrm{Ei}(x)$$
$$+Z_0(\sqrt{2x})\left(2\mathrm{e}^{\frac{1}{2}x}I_0\left(\frac{x}{2}\right)-x\,\mathrm{e}^{-\frac{1}{2}x}\left(I_0\left(\frac{1}{2}x\right)+I_1\left(\frac{1}{2}x\right)\right)\mathrm{Ei}(x)\right)$$
$$\int \sqrt{2x}Z_1(\sqrt{2x})I_1\left(\frac{x}{2}\right)\mathrm{e}^{-\frac{1}{2}x}\,\mathrm{d}x=\sqrt{2x}Z_1(\sqrt{2x})I_0\left(\frac{x}{2}\right)\mathrm{e}^{-\frac{1}{2}x}$$
$$-xZ_0(\sqrt{2x})\,\mathrm{e}^{-\frac{1}{2}x}\left(I_0\left(\frac{1}{2}x\right)+I_1\left(\frac{1}{2}x\right)\right)$$
$$\int \sqrt{2x}Z_1(\sqrt{2x})K_1\left(\frac{x}{2}\right)\mathrm{e}^{-\frac{1}{2}x}\,\mathrm{Ei}(x)\,\mathrm{d}x=-\sqrt{2x}Z_1(\sqrt{2x})K_0\left(\frac{x}{2}\right)\mathrm{e}^{-\frac{1}{2}x}\,\mathrm{Ei}(x)$$
$$+Z_0(\sqrt{2x})\left(x\mathrm{e}^{-\frac{1}{2}x}\left(K_0\left(\frac{1}{2}x\right)-K_1\left(\frac{1}{2}x\right)\right)\mathrm{Ei}(x)-2\mathrm{e}^{\frac{1}{2}x}K_0\left(\frac{x}{2}\right)\right).$$
$$\int \sqrt{2x}Z_1(\sqrt{2x})K_1\left(\frac{x}{2}\right)\mathrm{e}^{-\frac{1}{2}x}\,\mathrm{d}x=-\sqrt{2x}Z_1(\sqrt{2x})K_0\left(\frac{x}{2}\right)\mathrm{e}^{-\frac{1}{2}x}$$
$$+x\mathrm{e}^{-\frac{1}{2}x}Z_0(\sqrt{2x})\left(K_0\left(\frac{1}{2}x\right)-K_1\left(\frac{1}{2}x\right)\right).$$

All of these integrals appear to be new.

For the simpler where case $h(x)=1$ we obtain the integrals

$$\int I_0\left(\frac{x}{2}\right)\mathrm{e}^{-\frac{1}{2}x}\,(C_1+C_2\mathrm{Ei}(x))\,\mathrm{d}x$$
$$=x\,\mathrm{e}^{-\frac{1}{2}x}\left(I_0\left(\frac{1}{2}x\right)+I_1\left(\frac{1}{2}x\right)\right)(C_1+C_2\,\mathrm{Ei}(x))-2C_2\,\mathrm{e}^{\frac{1}{2}x}I_0\left(\frac{1}{2}x\right)$$
$$\int K_0\left(\frac{x}{2}\right)\mathrm{e}^{-\frac{1}{2}x}\,(C_1+C_2\,\mathrm{Ei}(x))\,\mathrm{d}x$$
$$=x\,\mathrm{e}^{-\frac{1}{2}x}\left(K_0\left(\frac{1}{2}x\right)-K_1\left(\frac{1}{2}x\right)\right)(C_1+C_2\,\mathrm{Ei}(x))-2C_2\,\mathrm{e}^{\frac{1}{2}x}K_0\left(\frac{1}{2}x\right)$$

and these have the special cases

$$\int I_0\left(\frac{x}{2}\right)\mathrm{e}^{-\frac{1}{2}x}\,\mathrm{Ei}(x)\,\mathrm{d}x$$
$$=x\,\mathrm{e}^{-\frac{1}{2}x}\left(I_0\left(\frac{1}{2}x\right)+I_1\left(\frac{1}{2}x\right)\right)\mathrm{Ei}(x)-2\mathrm{e}^{\frac{1}{2}x}I_0\left(\frac{1}{2}x\right)\tag{3.13}$$

$$\int I_0\left(\frac{x}{2}\right) \mathrm{e}^{-\frac{1}{2}x}\,\mathrm{d}x = x\,\mathrm{e}^{-\frac{1}{2}x}\left(I_0\left(\frac{1}{2}x\right) + I_1\left(\frac{1}{2}x\right)\right) \tag{3.14}$$

$$\int K_0\left(\frac{x}{2}\right) \mathrm{e}^{-\frac{1}{2}x}\,\mathrm{Ei}(x)\,\mathrm{d}x$$

$$= x\,\mathrm{e}^{-\frac{1}{2}x}\left(K_0\left(\frac{1}{2}x\right) - K_1\left(\frac{1}{2}x\right)\right)\mathrm{Ei}(x) - 2\mathrm{e}^{\frac{1}{2}x}K_0\left(\frac{1}{2}x\right) \tag{3.15}$$

$$\int K_0\left(\frac{x}{2}\right) \mathrm{e}^{-\frac{1}{2}x}\,\mathrm{d}x = x\,\mathrm{e}^{-\frac{1}{2}x}\left(K_0\left(\frac{1}{2}x\right) - K_1\left(\frac{1}{2}x\right)\right). \tag{3.16}$$

The integrals (3.13) and (3.15) appear to be new, but the integrals (3.14) and (3.16) are given in [2].

Disclosure statement

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