Observed and Unobserved Heterogeneity in Failure Data Analysis: A Case Study

Rezgar Zaki[1], Abbas Barabadi[1], Javad Barabadi[1], Ali Nouri Qarahasanlou*[2]

[1]Department of Engineering and Safety, The Arctic University of Norway (UiT), Tromsø, Norway
[2] Faculty of Technical & Engineering, Imam Khomeini International University, Qazvin, Iran

Abstract: The results of reliability analysis for heterogeneous data can differ substantially from those in a homogeneous case. Covariates can introduce observed and unobserved heterogeneity among data failures collected from a specific equipment in which they working at different locations under various operational and environmental conditions (e.g. operator skill, maintenance strategies, low temperature, etc). In most reliability studies observed heterogeneity due to observed covariates are discussed. However, unobserved heterogeneity for unobserved reasons, which may have a significant impact on reliability, are neglected. This can lead to erroneous model selection for the time to failure of the item, as well as wrong conclusions and decisions. There is a lack of systematic approach, to model the unobserved covariate in the area of reliability analysis. In this study, the required statistical tests and available models for observed and unobserved heterogeneity in the reliability analysis of failure data are reviewed, and then a methodology is developed to facilitate the application of these models. The methodology is based on the mixed proportional hazards model and its extension, which provides an appropriate tool for modeling observed and unobserved heterogeneity under the different types of maintenance strategies. In the second part of the study the application of the proposed methodology is shown by investigating of observed and unobserved heterogeneity in the failure data of chain part from three excavators that put into service at Golgohar Sirjan Iron Mine in Iran.

Keywords: Reliability, Observed Covariate, Unobserved Covariate, Gamma Frailty Models, Mixed Proportional Hazard

1- Introduction

In many reliability studies, data sets are assumed to be homogeneous, where the failure data are independent and identically distributed (1–3). However, in reality they are often working at different locations under various operational and environmental conditions (e.g. operator skill, maintenance strategies, low temperature, etc). (4). This may introduce heterogeneity into the data (5,6). In general, differences in failure intensity are called heterogeneities and can be due to either observed or unobserved influence risk factors, which are called covariates (7–9). Covariates describe the item's characteristics or the environment in which the item operates (10) and they may have different levels. For example, as a covariate on the reliability of a pump, vibration may be of high, low or medium levels (11). Observed covariates may have different levels and effects and they are recorded with the failure data. For example, as a observed covariate on the reliability of a pump, vibration may be of high, low or medium levels (11). They can be time-dependent or time-independent. Time-dependent observed covariates vary continuously with time. Unobserved covariates are independent variables that may have a

*- E-mail address: Alinoorimine@gmail.com (Phone: +989149405561)

significant impact on failure time of an equipment however they are not available in the failure database (12).

Unobserved covariates may lead to unobserved heterogeneity (11,13,14). For example, in a production process, some pumps may have a soft foot problem, due to a defect in the installation process. The soft foot problem will put the bearing in an over-stressed situation; this should be considered as a covariate for reliability analysis. In the case that there is no information regarding soft foot in the failure database of the bearing, an unobserved covariate should be defined, to capture the effect of soft foot on the reliability of the bearing.

In general, due to the quality of manufacturing, installation, operation and maintenance procedures, some items may become frailer, while others are more robust. In the presence of unobserved covariates, different items may have different levels of frailty. Unobserved covariates are typically unknown or not available for each item; hence, they cannot be explicitly included in the analysis. The result of our literature review revealed that, in many cases, unobserved covariates are eliminated during the failure data analysis (7,9,11,12,15). However, if unobserved covariates are neglected, the result of the reliability analysis only represents the reliability of items with an average level of frailty and not that of the individual items. High-risk items (high frailty) tend to fail earlier than low-risk items (low frailty) for unobserved reasons, and, thus, the population composition changes over time. Hence, in time, the analysis represents the item with low frailty, and the estimated reliability increases more with time than the reliability of a randomly selected item of the population (16,17).

The Cox regression model family, such as the proportional hazards model (PHM) and its extension, is the most dominant statistical approach for capturing the effect of covariates on the reliability performance of an item (4,11,15,18–22). In PHM, the hazard rate of an item is the product of a baseline hazard rate and a positive functional term that describes how the hazard rate changes as a function of covariates. However, the PHM is very sensitive to the omission of the covariates and is unable to isolate the effect of unobserved covariates (11). The frailty model introduced by Clayton (23) and Vaupel et al. (14) is used to describe the influence of unobserved covariates in a proportional hazards model. A frailty model is a random effects model for time variables, where the random effect (the frailty) has a multiplicative effect on the hazard (13,21). Gamma distribution, inverse Gaussian or exponential distribution can be used to model the frailty (4,12,13,24).

Recently, some studies in the reliability field have used the frailty model to model the effect of missing covariates on the reliability of an item (4,7,22,25). However some of them are related to the application of a frailty model in reliability engineering with a focus on maintenance purpose.. Asha (26) incorporated the frailty model into the load share systems and describe the effect of observed and unobserved covariates on the reliability analysis. Xu and Li (27) obtained stochastic properties of univariate frailty models, which are a special case of multivariate frailty models, and Misra (28) used stochastic orders to compare frailty models arising from different choices of frailty distribution. Giorgio et al. (9) applied the model to a real set of failure time data of powertrain systems mounted on 33 buses, employed on urban and suburban routes in Italy Slimacek and Lindqvist (30) have implemented frailty model and Poisson process to show unobserved covariates effect on the reliability of wind turbines. Finkelstein (29) Studied the ability to survive a single shock and the intensity of these shocks in time on system reliability. He noted that heterogeneity is a natural feature in many populations and frailty model gives an appropriate tool and flexible way to description of lifetimes.

However, these studies do not discuss how the time-dependent covariates should be handled in the frailty model. Moreover, the required statistical tests for the investigation of observed and unobserved heterogeneity among the failure data are not discussed.. To overcome these challenges, the main contribution of this paper is to present a methodology for failure data analysis in the presence of unobserved and observed covariates. The framework is based on the mixed proportional hazards model, which was originally developed by Lancaster (31) in order to determine the causes of variation among unemployed persons in the length of time they are out of work.

. This paper is organized as follows. In section 2 the basic concept is presented. Section 3 describe the proposed methodology. Thereafter, the application of the methodology will be illustrated in the reliability analysis of a mining excavator in Section 4. Finally, Section 5 provides the conclusions.

2- Basic concept

The COX regression models for reliability analysis considering the effect of covariates can be categorized into two main families: *i)* the mixed proportional hazard model family and *ii)* the proportional hazard model family. In these models, the hazard rate of an item is the product of a baseline hazard rate and a positive functional term that describes how the hazard rate changes as a function of unobserved and observed covariates. The baseline hazard rate is assumed to be identical and equal to the total hazard rate when the observed and unobserved covariates have no influence on the failure pattern (11). The family of mixed proportional hazard models is able to handle the effect of unobserved covariates.

2-1- Mixed proportional hazard model (MPHM) family

In the mixed proportional hazards model, the baseline hazard acts multiplicatively on the *i)* observed covariate function $\psi(z;\eta)$ and *ii)* a time-independent frailty function α_j. Suppose we have a fleet of *j* items, the hazard function for an item at time $t > 0$ is:

$$\lambda_j(t;z;\alpha) = \alpha_j \lambda_0(t) \psi(z;\eta) \quad (1)$$

where $\lambda_0(t)$ is an arbitrary baseline hazard rate, dependent on time alone, z is a row vector consisting of the observed covariates associated with the item, η is a column vector consisting of the regression parameters for identified observed covariates, and α_j is a time-independent frailty function for item *j* and represents the cumulative effect of one or more unobserved covariates. The baseline hazard rate $(\lambda_0(t))$ may either be left unspecified or can be modeled used a specific parametric form such as Weibull distribution or Non-Homogeneous Poisson Process (NHPP).

According to the mixed proportional hazards model, the fleet of items (the population) is represented as a mixture, in which the $\lambda_0(t)$ and $\psi(z;\eta)$ are common to all items, although each item has its own frailty. The observed and unobserved covariates can affect the hazard rate, so that the actual hazard rate $(\lambda_j(t;z;\alpha))$ is either greater (e.g. in the case of higher vibration level or poor maintenance) or smaller (e.g. better training for operators, installation of a new ventilation system) than the baseline hazard rate. Moreover, the equipment with $\alpha_j > 1$ are frailer and may have decreased time to failure. The items for whom $\alpha_j < 1$ are less frail and they tend to be more reliable.

Different functional forms of $\psi(z;\eta)$ and α_j may be used to model the observed and unobserved covariate functions. For example, the exponential form $exp(z\eta)$, the logistic form $log(1 + \exp(z\eta))$, the inverse linear form $1/_{(1+z\eta)}$, and the linear form $(1 + z\eta)$ are some of the functions used for observed covariate function (11,32,33). Moreover, gamma, inverse Gaussian and exponential distribution are used to model the frailty function (21,34,35). Generally, the exponential distribution for $\psi(z;\eta)$ and gamma distribution, with the mean equal to one and variance of θ, are the most generally used functions for observed and unobserved distribution, respectively.

In Eq. (1), the assumption is that all covariates are time-independent. However, in reality, there are many cases where the covariates are time-dependent. It is of great practical importance to decide whether covariate effects are constant over time or their effects change (36). For example, a covariate that is used to represent crack growth may change over the operational time of the item. Such a covariate is time-dependent (11); hence, it should be modeled as a time-dependent covariate by using the crack propagation geometry. The hazard rate of an item in the presence of time-dependent covariates ($z(t)$) takes the following form (12):

$$\lambda_j(t;z;z(t);\alpha) = \alpha_j.\lambda_0(t)\psi(z,z(t);\eta;\delta) \tag{2}$$

where $z(t)$ is a row vector consisting of the observed time-dependent covariates associated with the item (e.g. ambient temperature, pressure on the failure time, etc.) η and δ are the corresponding regression coefficients (i.e., the effect size) of time-independent and time-independent observed covariates. As Eq. (2) is an extension of Eq. (1) in this paper, Eq. (2) is named as extension mixed proportional hazards model (EMPHM).

Considering gamma distribution (with the mean to one and variance θ) for unobserved covariates and exponential function for observed covariates, the reliability function can be written as (12):

$$R_\theta(t;z;z(t);\alpha) = [1-\theta\ln(R_i(t;z;z(t))]^{-1/\theta} \tag{3}$$

where $R_i(t;z;z(t)$ is the item reliability function considering and considering the exist of p_1 time-independent observed covariates and p_2 time-dependent observed covariates. It can be estimated by:

$$R_i(t;z;z(t)) = \left[exp\left(-\int_0^t \lambda_0(x)exp\left[\sum_{j=1}^{p_2}\delta_{sj}z_{sj}(t)\right]dx\right)\right]^{\exp[\sum_{i=1}^{p_1}\eta_{si}z_{si}]} \tag{4}$$

If all the covariates are time-independent Eq.(4) reduce to:

$$R_i(t;z;z(t)) = R_0(t)^{\exp[\sum_{i=1}^{p_1}\eta_{si}z_{si}]} \tag{5}$$

where $R_0(t)$ is baseline reliability function, η_i represents regression parameters for i time-independent covariate (z_i). Thus, Eq. (3) can be written as:

$$R_\theta(t;z;z(t);\alpha) = \left[1-\theta\ln(R_0(t)^{\exp[\sum_{i=1}^{p_1}\eta_{si}z_{si}]}\right]^{-1/\theta} \tag{6}$$

Which is named as the mixed proportional hazard model (MPHM).

2-1-1- Shared frailty model

In some cases, a group of items share the same frailty value (37). For example, consider a company where identified excavators are utilized in two different shifts: night and day. Here, the shift can be considered a shared frailty. A shared frailty is a group-specific latent random effect that multiplies into the hazard function and will generate dependence between those items which share frailties (12). The distribution of the shared frailty is gamma, with mean 1 and variance to be estimated from the data. For data consisting of *n* groups, with the *s*th group comprised of n_s items $(s = 1,...,n)$, the shared-frailty model can be written as:

$$\lambda(t; z; z(t); \alpha) = \alpha\lambda_{0s}(t)\text{exp}\left[\sum_{i=1}^{p_1}\eta_{si}z_{si} + \sum_{j=1}^{p_2}\delta_{sj}z_{sj}(t)\right] \tag{7}$$

That is, for any member of the *i*th group, the standard hazard function is now multiplied by the shared frailty, α_s. In the case that there is no time dependent covariates Eq.(7) will reduce to:

$$\lambda(t; z; z(t); \alpha) = \alpha\lambda_{0s}(t)\text{exp}\left[\sum_{i=1}^{p_1}\eta_{si}z_{si}\right] \tag{8}$$

2-1-2- Stratification approach model

In the case of existence time-dependent covariates ($z_j(t)$), the stratification approach can be used (15). In this approach it is possible to categorize a categorical or time depended covariate with several categories into different stratum with different baseline hazards for each category. For example, in the case where we are going to model the effect of ambient temperature on the reliability of a pump which is installed outside, then the collected failure data can be categorized into four groups, based on the seasons (spring, summer, fall and winter). Figure 1 shows a graphical representation of this example, where *i* is the number of failures occurring within each stratum.

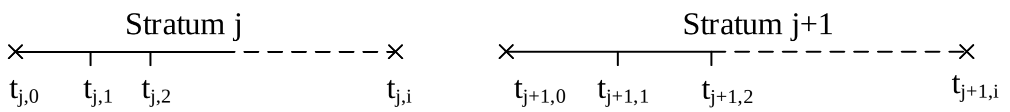


Figure 1: A graphical representation of the strata for a data set

In the stratification approach, the baseline hazard function differs for defined strata, but the regression coefficients are the same for all covariates. Hence, the hazard rate for the system in strata *r* in the presence of unobserved covariates will be:

$$\lambda_r(t; z; \alpha) = \alpha_r.\lambda_{0r}(t)\text{exp}\left[\sum_{i=1}^{p_1}\eta_i z_i\right] \tag{9}$$

where λ_{0r} is the baseline hazard for stratum *r*, and, if there is no unobserved covariate, then Eq.(9) will be reduced to:

$$\lambda_r(t; z) = \lambda_{0r}(t)\text{exp}\left[\sum_{i=1}^{p_1}\eta_i z_i\right] \tag{10}$$

2-2- Proportional hazard model family

The main assumption in the proportional hazard model family is that all influence covariates are identified and there is no omission of covariates (no unobserved covariates). In the case that of no unobserved covariates' effect on the hazard rate of the item, then, in Eq. (3) gamma distribution will be equal to 1 and the successive equation is:

$$\frac{\alpha^{\frac{1}{\theta}-1} e^{-\frac{\alpha}{\theta}}}{\Gamma\left(\frac{1}{\theta}\right) \theta^{\frac{1}{\theta}}} = 1 \tag{11}$$

and the hazard rate can be written as:

$$\lambda(t; z; z(t)) = \lambda_0(t)\exp\left[\sum_{i=1}^{p_1} \eta_i z_i + \sum_{j=1}^{p_2} \delta_j z_j(t)\right] \tag{12}$$

In the literature, this model is mainly referred to as an extension of proportional hazard model (EPHM) (15,38). Proportionality assumption implies that the effect of a covariate is independent of time and the ratio of any two hazard rates is constant with respect to time, i.e.:

$$\frac{\lambda_1(t; z_1; \alpha)}{\lambda_2(t; z_2; \alpha)} = \frac{\lambda_0(t)\exp\,(\eta_1 z_1)}{\lambda_0(t)\exp\,(\eta_2 z_2)} = \exp(\eta_1 z_1 - \eta_2 z_2) = Constant \tag{13}$$

where $z_1$and z_2 are any two different sets of time-independent observed covariates assumed to be associated with the item. If there is no time-dependent covariate, then Eq.**Error! Reference source not found.** will reduce to the PHM as follows:

$$\lambda(t; z) = \lambda_0(t)\exp\left[\sum_{i=1}^{p_1} \eta_i z_i\right] \tag{14}$$

2-3- Parameter estimation

For EMPHM, given the relationship between the hazard rate and the reliability functions, it can be shown that the conditional (item) reliability function, $R(t; z; z(t)|\alpha)$, conditional on the frailty, α, is (12):

$$R(t; z; z(t)|\alpha) = \{R(t; z; z(t))\}^{\alpha} \tag{15}$$

The unconditional (population) reliability function can then be estimated by integrating out the unobserved α. If α has probability density function *g(α)*, then the population or unconditional reliability function is given by:

$$R_\theta(t; z; z(t)) = \int_0^\infty \{R(t; z; z(t))\}^{\alpha} g(\alpha) d\alpha \tag{16}$$

where we use the subscript θ to emphasize the dependence on the frailty variance θ. The relationship between the reliability function and the hazard function still holds unconditional on α, and, thus, we can obtain the population hazard function using (12):

$$\lambda_\theta\big(t;z;z(t)\big) = -\frac{d}{dt}R_\theta(t;z;z(t))[R_\theta(t;z;z(t))]^{-1} \tag{17}$$

Having the gamma distribution with unobserved covariates (12):

$$R_\theta(t;z;z(t)) = [1-\theta ln\{R(t;z;z(t))\}]^{-1/\theta} \tag{18}$$

Having the event times (t_{0i}, t_i, d_i), for $i = 1,,n$ with the *i*th observation corresponding to the time span $(t_{0i},\ t_i]$, with either failure occurring at time t_i $(d_i = 1)$ or the failure time being right-censored at time t_i $(d_i = 0)$, the likelihood function for reliability data is given by:

$$LnL = ln\prod_{i=1}^{n}\frac{\{R_{\theta i}(t_{0i}, z_i, z_i(t)\}^{1-d_i}\{f_{\theta i}(t_i, z_i, z_i(t)\}^{d_i}}{R_{\theta i}(t_i, z_i, z_i(t))} \tag{19}$$

where, $f_{\theta i}$ is the probability density function.

In a shared frailty model for failure data of unrepairable units, that is, to the so called “classical distributions”, suppose we have data for $i = 1,...,n$ groups, with $j = 1,...,n_i$ observations per group, consisting of the trivariate response $(t_{0ij}, t_{ij}, d_{ij})$, which indicates the start time, end time, and failure/censoring for the *j*th item from the *i*th group, while the shared frailties follow a gamma distribution, L_i can be expressed compactly as (12):

$$L_i = \left[\prod_{j=1}^{n_i}\{\lambda_{ij}(t_{ij})\}^{d_{ij}}\right]\frac{\Gamma(1/_\theta + D_i)}{\Gamma(1/_\theta)}\theta^{D_i}\left\{1-\theta\sum_{j=1}^{n_i}Ln\frac{R_{ij}(t_{ij})}{R_{ij}(t_{0ij})}\right\}^{-1/_\theta + D_i} \tag{20}$$

where $D_i = \sum_{j=1}^{n_i} d_{ij}$. Given the unconditional group likelihoods, we can estimate the regression parameters and frailty variance θ, by maximizing the overall log-likelihood $LnL = \sum_{i=1}^{n}\ln L_i$. In shared-frailty Cox models, the estimation consists of two steps. In the first step, the optimization is in terms of θ alone. For fixed θ, the second step consists of fitting a standard Cox model via penalized log-likelihood, with the v_i introduced as estimable coefficients of dummy variables identifying the groups. The same approach can be used to estimate the likelihood functions for EPHM, MPHM and PHM. For more information, see (7,11,12).

In a minimally repaired system (NHPP), the times between failures (TBF) are not independent and identically distributed random variables (except for the special case of constant failure intensity, that is, of homogeneous Poisson process), and the log-likelihood function relative to "m" minimally repaired systems, whose failure intensity is given by Eq.(2), results in (12):

$$lnL = ln\left[\prod_{j=1}^{m}\int g(\alpha)\left(\prod_{i=1}^{n_j}f\big(t_{i,j}\big|t_{i-1,j};z,z(t),\alpha\big)\right)\frac{R\big(T_j;z,z(t),\alpha\big)}{R\left(t_{n_j,j};z,z(t),\alpha\right)}d\alpha\right] \tag{21}$$

where $g(\alpha)$ denotes the pdf of the frailty parameter α, n_j is the number of observed failures of the j-th system, $t_{i,j}\big(i = 1,\dots,n_j; j = 1,\dots,m\big)$ is the i-th failure time of the j-th system observed up to T_j, $t_{o,j} = 0$. Where conditional pdf of the failure time $t_{i,j}$, given the previous failure time $t_{i-1,j}$, is (12):

$$f_T\big(t_{i,j}\big|t_{i-1,j};z,z(t),\alpha\big) = \alpha\lambda_0\big(t_{i,j}\big)\psi(z,z(t);\eta;\delta)\frac{R\big(t_{i,j};z,z(t),\alpha\big)}{R\big(t_{i-1,,j};z,z(t),\alpha\big)} \tag{22}$$

The reliability function, $R(t; z; z(t)|\alpha)$, conditional on the frailty, α, is (12):

$$R(t; z; z(t)|\alpha) = \left[exp\left(-\int_0^t \lambda_0(t_{i,j})\psi(z, z(t); \eta; \delta)dx\right)\right]^{\alpha} \quad (23)$$

The log-likelihood function relative to "m" repairable systems subject to perfect repairs (renewal process), whose hazard function is given by Eq.(2), results in:

$$lnL = ln\left[\prod_{j=1}^{m}\int g(\alpha)\left(\prod_{i=1}^{n_j} f(t_{i,j}|t_{i-1,j}; z, z(t), \alpha)\right)R(X_j; z, z(X_j), \alpha)d\alpha\right] \quad (24)$$

where, here, $x_{i,j} = t_{i,j} - t_{i-1,j}$ $(i = 1, \dots, n_j; j = 1, \dots, m)$ is the i-th time between failures (TBF) of the j-the system, $X_j = T_j - t_{n_j,j}$, and the (unconditional) pdf of the TBF $x_{i,j}$ is:

$$f_X(x_{i,j}; z, z(t), \alpha) = \alpha\lambda_0(x_{ij})\psi(z, z(x_{ij}); \eta; \delta)R(x_{ij}; z, z(t), \alpha) \quad (25)$$

Only in absence of unobservable heterogeneity, the log-likelihood function in Eq.(24) reduces to the log-likelihood in Eq.(19). Indeed, in presence of unobserved heterogeneity, the same value of the frailty variable α characterizes the whole path of each repairable system, so that the whole conditional likelihood function $L_j|\alpha$ of each system "j", given α, must be multiplied by $g(\alpha)$, and hence integrated on α. In addition, it must be mentioned that the log-likelihood function in Eq. (24) becomes much more complex when the j-th system is observed starting from a generic time which is not a failure time.

3- Proposed framework

The systematic framework for reliability analysis of reliability data in the presence of observed and unobserved covariates (or heterogeneity) is described in Figure 2. This methodology is based on four important steps:

- Establishing the context and data collection
- Identifying the baseline hazard rate, based on maintenance nature
- Modeling the effect of the covariates
- Parameter estimation

As this figure shows, in the first step, the context should be established. In this step, all external and internal parameters to be taken into account when analyzing failure data and setting the scope and assumptions for the reliability analysis should be defined. External context is the external environment in which the item is going to work such as ambient temperature, pressure, humidity, etc. Internal context is the internal conditions related to the item itself and the company running and maintaining the item, including the repair and physics of failure, operator condition, maintenance crew, etc. Understanding the external and internal context is important in order to identify the observed covariates. For example, based on the physics of failure, road condition can contribute to the failure of a excavator in a mine; hence, it should be considered as a covariate in the reliability analysis of the excavator. In this step, the possible relationships between different covariates should be investigated, as well as the possible level for each of them.

In the next step, failure data and all possible observed covariates associated with each failure should be collected.

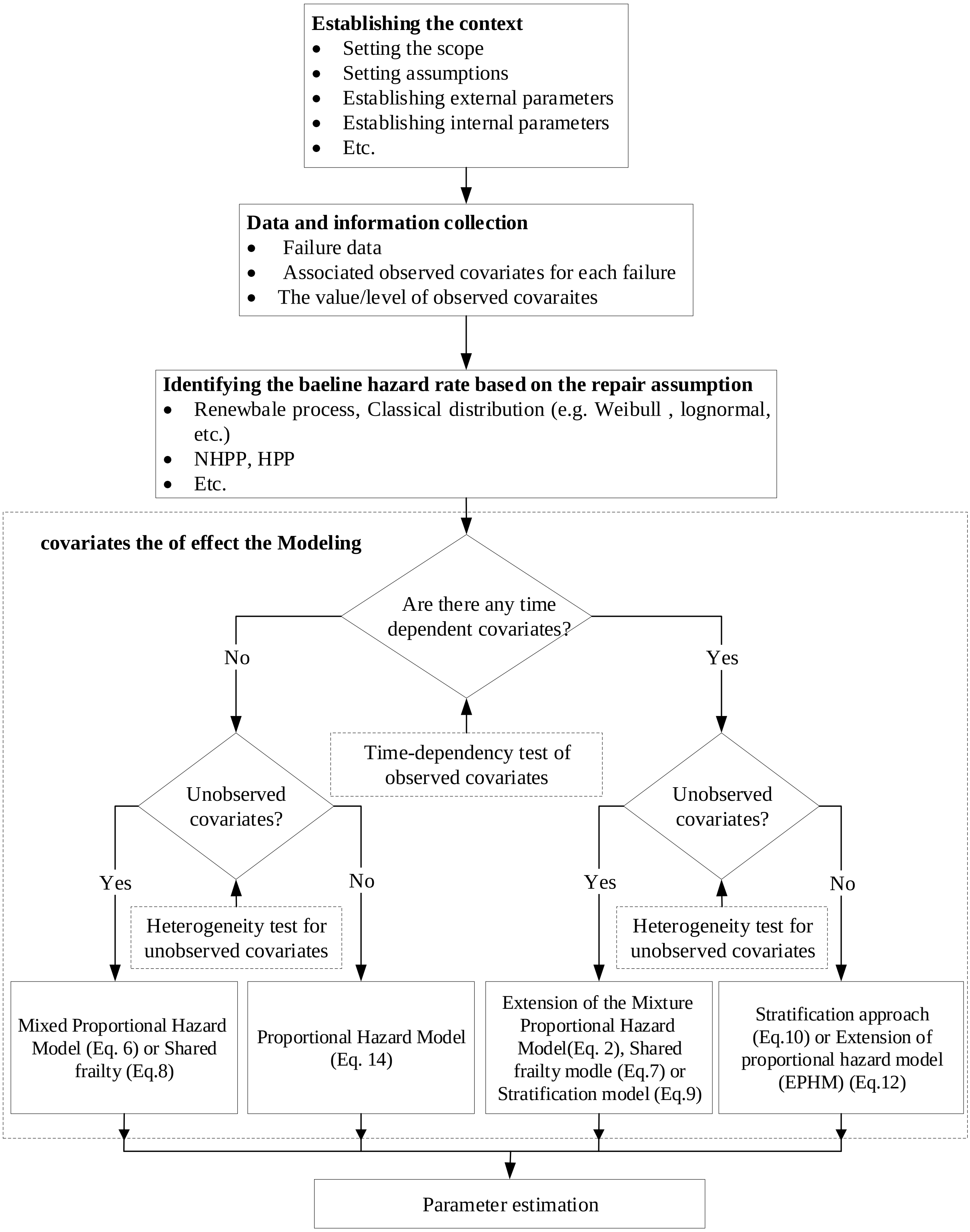


Figure 2: A framework for reliability model selection in the presence of observed and unobserved covariates

Thereafter, based on the nature of the failure data (e.g. trend behavior of the data) and the type of repair strategy, the appropriate baseline hazard should be selected for the data. For example, the common assumption for a repairable system can be i) perfect repair or good-as-new condition, ii) minimal repair or bad-as-old condition, or iii) jumps in the hazard rate after repair or different baseline hazard rate. Under the perfect repair strategy, the item is restored to as 'good-as-new' condition and the main assumption is that the hazard rate is reset to that of a new system after maintenance. If the times between failures are independent and identically distributed (iid), it can be concluded that the item went through perfect repair (21). In such cases, classical distribution, such as Weibull distribution, can be used to model the baseline hazard rate.

In the case of minimal repair (bad-as-old), an item has the same intensity function after repair as before the failure. The failure times when minimal repair is carried out can be thought of as a non-homogeneous Poisson process (NHPP). In other words, the baseline hazard rate will be modeled using a non-homogeneous Poisson model. However, it should be mentioned that, on some occasions, such as overhaul, the system may return to as 'good-as-new' condition. Under this condition, it is assumed that the NHPP is cyclic, with each cycle starting as a renewal process and, within the cycle, failure times follow the NHPP. In this case, the failure data will then be categorized by these occasions (for example, overhaul) and then a stratification approach is used to estimate the effect of each covariate, while the baseline hazard rate is modeled by NHPP model. However, when a fleet of items is analyzed, after some time and undergoing several repairs, the baseline hazard rate will change. For example, in some cases, as the number of failures increases, the average failure time decreases; hence, the baseline hazard rate will not be identical for a particular failure number. Here, the failure data can be categorized based on the failure number; it can be used to define strata, and then the stratification approach can be used to model the fleet failure data.

In general, the first step in analyzing the collected failure data of a repairable system is to check the trend of the failure data. In the case that the data shown trend the NHPP or trend renewal process (TRP) can be used to model the baseline hazard rate. However, when there is no trend in the data, classical distribution, such as Weibull distribution, can be used to model the baseline hazard rate. However, some goodness-of-fit test, such as residual test, should be used to find the best fit distribution for failure data. For more information regarding the trend test, see (7).

In the next step, the time dependency of observed covariates should be checked. Later, the failure data need to be investigated for unobserved covariates. Data sets without unobserved heterogeneity will be analyzed using the classical proportional hazards model, including the proportional hazards model (when all observed covariates are time-independent) and the extension of the proportional hazards model (in the presence of time-dependent covariates). Moreover, data sets with unobserved heterogeneity will be analyzed using the mixed proportional hazards model family.

3-1- Time dependency test of observed covariates

There are two general approaches for checking the time dependency of covariates: *i)* the graphical procedure and *ii)* the goodness-of-fit testing procedure (15). The developed graphical procedure can generally be categorized into three main groups: *i)* cumulative hazards plots, *ii)* average hazards plots and *iii)* residual plots (11). For example, in cumulative hazards plots, the data will be categorized based on the different level of the covariate that is to be checked for time dependency. Consider that a covariate can be categorized into r levels, in which the covariate is equal to z_r. Thereafter, the hazard rate can be written as:

$$\lambda_r(t;z;\alpha) = \alpha_s.\lambda_{0r}(t)\exp\left[\sum_{i=1}^{p_1}\eta_i z_i\right] \tag{26}$$

where $\eta_i z_i$ is the same as before, with $\eta_r z_r$ omitted, with $i=1,2,...p_1$ and $j\neq r$. If the PH assumption is justified, then we will end up with:

$$\lambda_{0r}(t) = C_r\lambda_0(t), and\ C_r = \alpha_s \exp\left(\eta_r z_r\right) \tag{27}$$

A similar relation can be concluded for the cumulative baseline hazard rate. Hence, if the assumption of PH is justified, then the plots of the logarithm of the estimated cumulative baseline hazard rates against time for defined categories should simply be shifted by an additive constant, η_r. In other words, they should be approximately parallel and separated, corresponding to the different values of the covariates. Departure from parallelism of the above plots for different categories may suggest that z_r is a time-dependent covariate. For a review of other graphical approaches, see (11,15,39–41).

In the same way as the cumulative baseline hazard rate, a log–log Kaplan-Meier curve over different (combinations of) categories of variables can be used to check the assumption of PH. A log–log reliability curve is simply a transformation of an estimated reliability curve that results from taking the natural log of an estimated reliability probability twice. If we use a PHM or MPHM and plot the estimated log–log reliability curves for defined categories on the same graph, the two plots would be approximately parallel (11). In the residuals plot in the first step, the residual should be estimated by using the estimated values of the cumulative hazard rate, $H_0(t_i)$, and the regression vector η as:

$$e_i = -H_0(t_i)\exp\left(\eta_r z_r\right) \tag{28}$$

If the PH assumption is justified, then the logarithm of the estimated reliability function of e_i against the residuals should lie approximately on a straight line with slope -1 (11,42). A transformed plot of the partial residual suggested by Schoenfeld can also be used as an exploratory tool to detect the time-varying effects of a covariate, even when the a priori form of time dependence is unknown (43–45). The Schoenfeld Residuals Test is analogous with testing whether the slope of the scaled residuals on time is zero or not. If the slope is not zero then the proportional hazard assumption has been violated (45). When the covariates are quantitative, using graphical approaches is challenging, as it is difficult both to define different levels for quantitative covariates and to decide whether the plots are parallel or not. In such cases, it is better to use a goodness-of-fit testing procedure such as the chi-squared goodness-of-fit test (3,46,47), the log rank test (3,46), the likelihood ratio test (3,46), score tests (46,48), the doubly cumulative hazard function (49), the Wilcoxon test (50) and generalized moments specification tests (51). For example, if the PH assumption is justified, the different two-sample tests, e.g. generalized Wilcoxon and log rank tests, should have the same results (11).

3-2- Heterogeneity test for unobserved covariates

Several statistical tests are available in the literature for identifying and quantifying the effects of unobserved heterogeneity. For example, Kimber (52) developed a Weibull-based score test for heterogeneity and then demonstrated its application in two case studies on infant nutrition. Under the assumption that the data follow a stratified proportional hazards model, where the hazard rate can be different within different strata, Gray (53) used the martingale residuals to test for variation over groups in reliability data. Commenges and Andersen (54) used marginal partial likelihood to develop a score test of homogeneity for reliability data, when the frailty model is used to model the covariates. The score test is valid for general distributions of the frailty variable, not only for the frequently used gamma distribution. In the meta-analysis, Cochran's Q test (Q test) is normally used to check the homogeneity among data sets. However, the Q test only checks the presence versus the absence of heterogeneity; it does not report on the extent of such heterogeneity. However, these statistical tests and their applications are limited, mainly due to their requirements, in terms of data and assumptions. Each test is the optimum to detect the heterogeneity of a specific form (55,56). For example, a shortcoming of the Q statistic is that it has poor power to detect true heterogeneity, among studies when the meta-analysis includes a small number of studies, and excessive power to detect negligible variability with a high number of studies. Recently, the I^2 index has been proposed to quantify the degree of heterogeneity in a meta-analysis (57). A likelihood ratio test, the Akaike information criterion (AIC) and Bayesian information criterion (BIC) are common tests for checking the hypothesis of the presence of heterogeneity against the null hypothesis of non-heterogeneity ($\hat{\theta}$ = 0). In general, the AIC performs well when heterogeneity is small, but, if heterogeneity is large, the BIC will often perform better (7,10,58). For example, in the case of Weibull distribution for the baseline hazard rate, likelihood ratio can be written as:

$$R_H = 2\left(\ln L(\hat{\lambda},\hat{\beta},\hat{\eta},\hat{\theta}) - lnL(\hat{\lambda}_0,\hat{\beta}_0,\hat{\eta}_0,0\,)\right) \tag{29}$$

Here, $\hat{\lambda}$ and $\hat{\beta}$ are estimated parameters for Weibull distribution, $\hat{\eta}$ is the regression coefficient for observed covariates and $\hat{\theta}$ can be interpreted as the degree of heterogeneity (7). These parameters can be estimated by maximizing the full likelihood function. On a 5% significance level, the null hypothesis (no heterogeneity) will be rejected if $R \geq 2.706$. Moreover, under the minimal repair strategy, a power law can be used to represent the intensity function. Under the assumption of the power law intensity function, in order to check whether a significant amount of heterogeneity among units exists, a three-step likelihood ratio test procedure can be performed (7). As the first step, the null hypothesis, say $H_0\colon \lambda_1 = \lambda_2,\ \lambda_m = \lambda_0\ ,\beta_1 = \beta_2,$ $\beta_m = \beta_0$, should be tested against the alternative hypothesis, $H_1\colon \lambda_1 \neq \lambda_2,\ \lambda_m \neq \lambda_0\ ,\beta_1 \neq \beta_2,\ \beta_m \neq \beta_0$. In the second and third steps, common λ, uncommon β and uncommon λ, common β should be carried out respectively (9).

4- Mixed proportional hazards modeling of unrepairable systems: Case study

The excavator as one must important machine in mine needs to have strong undercarriage and chain to provide excellent reliability and durability during working on rocky ground or blasted rock. Thus, the case study refers to chain failure data of three Caterpillar 390DL excavators (Figure 3) put into service in Golgohar Sirjan Iron Mine in Iran during two years.

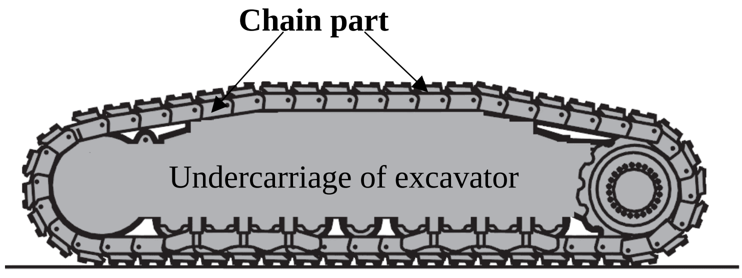


Figure 3: Undercarriage of excavator system and chain part of excavator

This mine is located in the south-west of Kerman Province, Iran and it contains six ore bodies spread over an area of 40 km^2 that named by 1, 2, 3, 4, 5 and 6.

The main design characteristics (weight, size, maximum capacity, etc.) of the Excavators are identical. The number of observed failures of the excavators, n_i, ranges from 24 to 51 for a total $n_{T=} \sum_i^4 n_i = 103$. We used the trend test and the serial correlative to check the assumption independent and identically distributed (iid) assumption in the collected data.

In this paper the serial correlation tests were performed by T-test (TSTA) and Ljung-Box-Q (LBQ) statistics. TSTA and LBQ statistics compare the values of the test statistic with the critical values 1.960 and 3.841 at 5% level of significance at log 1 for accepting the null hypothesis of no autocorrelation in the systems is verified. Trend test was performed by three analytical tests including MIL-Hdbk-189 (MIL), Laplace's and Anderson-Darling (A-D).

The results of the trend and correlation tests for each excavators are presented in Table 1. As shown, the test statistics of TSTA and LBQ confirm the acceptance of no autocorrelation of the log1.

Table 1- Trend and serial correlation tests of each system

System	**Trend Tests**				**Serial correlation tests**	
	Subject	**MIL**	**Laplace's**	**A-D**	**TSTA**	**LBQ**
Excavator 1	Test Statistic	13.74	4.64	13.29	0.86	0.82
	P-Value	0	0	0		
	DF	50				
Excavator 2	Test Statistic	18.52	4.02	9.52	1.18	1.57
	P-Value	0	0	0		
	DF	50				
Excavator 3	Test Statistic	57.04	3.24	6.93	0.55	0.32
	P-Value	0	0.001	0		
	DF	100				

Also, the null hypothesis (H_0: No Trend) was not rejected at a 5% significance level (p-value>α). Thus, we can conclude that the data were independent and identically distributed. Hence, the classical distribution can be used to model the baseline hazard rate. The Akaike information criterion (AIC) and Bayesian information criterion (BIC) can be used to find the best fit distribution for the baseline hazard rate (60). The candidate distribution with the smallest AIC and BIC value is the best fit distribution to model the baseline hazard rate.

According to the framework in Figure 2, in addition to failure data (TBF), all associated observed covariates should be collected. To this aim, the observed covariates should be identified. Table 2 shows the selected observed covariates. As the table shows, 5 observed covariates are identified which may affect the reliability of the excavators. For example, here in this analysis, the rock type is considered as an observed covariate. This is due to the fact that the current operation and the maintenance strategy differ in the two types, which may have affected the excavators' failure rate. Moreover, the training processes for operator differ in these types, which may lead to different levels of skill among operators and, consequently, different levels of stress on the excavators. This can lead to different failure rates for identified excavators in these companies. The numbers in the brackets in Table 2 are used to nominate (formulate) the covariates. For example, excavators work in three different shifts: namely, morning, afternoon, and night shifts; here, 3, 2 and 1 are used to represent these shifts, respectively.

Table 2: The identified observed covariates for the excavators

Covariate	Covariate level	Covariate	Covariate level
Working shift (z_s)	Morning shift [3]	Excavator Code (z_e)	Machine No.1 [1]
	Afternoon shift [2]		Machine No.2 [2]
	Night shift [1]		Machine No.3 [3]
Machin movement (z_m)	Little [<=3 m]	Rock type (z_r)	West [1]
	Medium [3-13 m]		Ore [2]
	Large [13 m <]	Precipitation (z_p)	Continuous covariate
		Temperature (z_t)	Continuous covariate

Table 3 shows a sample of failure data and their associated observed covariates. According to the framework in Figure 2, after collecting the data on failures and observed covariates, the time dependency of the covariate should be checked. Here, the graphical approach (a ln–ln reliability curve) is used to check the time dependency of all the covariates.

Table 3: A sample of failure data and their associated observed covariates

N. failure	TBF	Status	Covariates					
			z_s	z_t	z_p	z_r	z_m	z_e
1.00	5061.43	0.00	3.00	15.94	0.00	2.00	3.00	1.00
2.00	914.80	1.00	3.00	8.78	0.00	2.00	3.00	1.00
3.00	1770.59	1.00	3.00	11.36	0.20	2.00	3.00	1.00
4.00	16.24	1.00	3.00	13.71	0.00	2.00	1.00	1.00
5.00	6.78	1.00	3.00	13.71	0.00	2.00	1.00	1.00

Figure 4 shows the *–ln (-ln reliability)* against the *ln (analysis time)* for an observed covariates: namely, Machin movement (z_m). As this graph show, the curves are approximately parallel; hence, the assumption of proportionality is correct for the data sets, and it can be concluded that the covariates are time-independent. The *–ln (-ln reliability)* for other covariates confirms the same result.

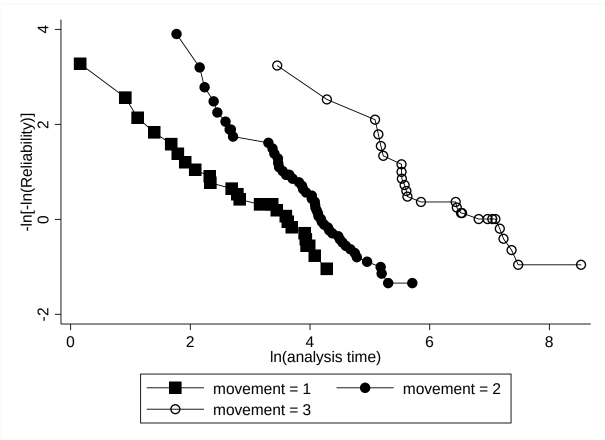


Figure 4: The log minus log graph for time between failures of the excavators, based on movement covariate

We used an analytical test to check the PH assumption in this study. The test of Harrell and Lee (1986) is a variation of a test originally proposed by Shenfield (1982) and is based on the residuals. The PH testing approach is attractive because it provides a test statistic and p-value (P(PH)) for checking the PH assumption for a given predictor of interest. Thus, a more objective decision provide by a statistical test than graphical approach. The P(PH) is used for evaluating the PH assumption for that variable. Table 4 is illustrated statistical of PH test for all covariates. The P(PH) values for α=0.05 are quite high for all variables satisfying the PH assumption.

Table 4: Analytical test approach results for PH assumption

Covariates	ρ	χ^2	Df.	P(PH)
z_s	-0.05595	0.3	1	0.5842
z_t	-0.05343	0.25	1	0.6182
z_p	0.00836	0	1	0.95
z_r	-0.10143	1.14	1	0.286
z_m	0.01796	0.03	1	0.8742
z_e	-0.00187	0	1	0.9852

In the next step of the framework, the presence of unobserved covariates (heterogeneity test) should be checked. For this, the best fit distribution for the baseline hazard rate needs to be identified. The AIC and BIC procedures are applied to select the best fit distribution for the baseline hazard rate, as well as to check the heterogeneity of data. Table 5 shows the values of the AIC and BIC for the different nominated distributions for the baseline hazard rate with the same covariates under two assumptions: *i)* with frailty and *ii)* without frailty. As the result in Table 5 shows, the Weibull MPHM is the most suitable model for the data, as it has the smallest AIC or BIC among all the models. Therefore, the model with unobserved heterogeneity can give a better estimation of reliability of the excavators.

Table 5: Goodness of fit of different reliability models

Model		Observations	d.f.	AIC	BIC	Log likelihood
With frailty	Exponential MPHM	103	8	315.1397	336.2175	-149.570
	Weibull MPHM	103	9	306.9757	330.6882	-144.488
	Gompertz MPHM	103	9	311.9349	335.6475	-146.968
Without frailty	Exponential PHM	103	7	313.3724	331.8155	-149.686
	Weibull PHM	103	8	314.8206	335.8984	-149.410
	Gompertz PHM	103	8	309.9349	331.0128	-146.968

Moreover, we used the ratio test to also check the unobserved heterogeneity in unrepairable part. Under the assumption of Weibull MPHM, while *i)* the gamma distribution represents the frailty model (with mean equal to one and variance equal to θ) and *ii)* there is no time-dependent covariates, the hazard rate can be written as:

$$\lambda(t; z; z(t); \alpha) = \frac{\alpha^{\frac{1}{\theta}-1} e^{-\frac{\alpha}{\theta}}}{\Gamma(\frac{1}{\theta})\theta^{\frac{1}{\theta}}} . (mt^{m-1}) \exp\left[\sum_{i=1}^{p_1} \eta_i z_i\right] \tag{30}$$

For this, likelihood ratio tests are performed as below:

$$R_H = 2\left(\left(\ln L(\hat{\lambda}, \hat{\beta}, \hat{\eta}, \hat{\theta}) - lnL(\hat{\lambda}_0, \hat{\beta}_0, \hat{\eta}_0, 0\,)\right)\right) = 9.84 \tag{31}$$

The p-value for R_H=9.84 will be equal to 0.001, which hints at the existence of an unobserved covariates' (unobserved heterogeneity) effect on the reliability of the excavators. Hence, the Weibull MPHM should be used to analyze the data. There is software available, which can estimate the parameters in MPHM, such as Stata, R and SAS. Table 6 and Table 7 show the results of the analysis in STATA.

Table 6: The result of covariates effect analysis under the assumption of MPHM

Covariate	Coef.	Std. Error	Z	P>|Z|	[95% Conf. Interval]	
z_s	0.094	0.287	0.330	0.744	-0.469	0.656
z_t	0.018	0.028	0.630	0.528	-0.037	0.072
z_p	-2.615	9.615	-0.270	0.786	-21.460	16.230
z_r	0.547	0.456	1.200	0.230	-0.347	1.441
z_m	-3.274	0.744	-4.400	0.000	-4.732	-1.816
z_e	0.927	0.339	2.730	0.006	0.262	1.592

Table 7: The constant value, baseline and unobserved parameters estimation of MPHM

Parameters	Coef.	Std. Error	[95% Conf. Interval]	
Constant value	-5.709	1.896	-9.425	-1.994
Baseline Weibull Ancillary parameter (m)	1.940	0.403	1.291	2.913
Variance of Gamma Distribution (θ)	1.576	0.714	0.649	3.829

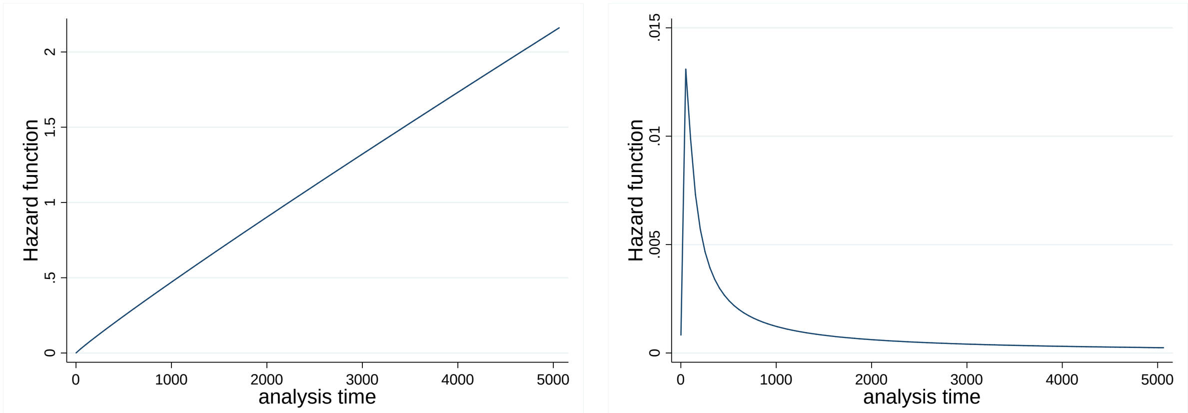
The result of the analysis in Table 7 shows that, constant value is -5.709 and Machine movement (z_m) and Excavator code (z_e) have a significant effect on the excavators' reliability. Based on Eq.18 the unconditional reliability of Weibull MPHM can be written as:

$$R_\theta(t;z) = \left[1 - \theta ln\left(\mathrm{e}^{\left(-(t^m)\left(\exp\left[\text{Constant value}+\sum_{i=1}^{p_1}\eta_i z_i\right]\right)\right)}\right)\right]^{-1/\theta} \tag{32}$$

Having the regression coefficient for covariates, the unconditional reliability of the excavators will be equal to:

$$R_\theta(t;z) = \left[1 - 1.576 ln\left(\mathrm{e}^{\left(-(t^{1.94})(\exp(-5.709-3.274\mathrm{z_m}+0.937\mathrm{z_e}))\right)}\right)\right]^{-1/1.576} \tag{33}$$

Figure 5 shows the unconditional and conditional hazard functions of the excavators. Figure 5-a, is population hazard function where the curve is unconditional on the frailty and is "averaged" over the frailty distribution and Figure 5-b, is individual hazard function that are conditional on a frailty value of one $\left(\alpha_j = 1 \text{ in Eq. } 1\right)$. As the figure 5 shows, there is a big difference between the hazard rate of the excavators' population and that of an individual excavator.



(a) (b)

Figure 5: a) The unconditional (population) hazard function of the excavators on the mean of covariate, b) the excavator conditional (individual) hazard function on the mean of covariate

In the next step, in order to compare how much bias will be associated with analysis if the effect of unobserved covariates is ignored, analysis is performed on the assumption that there are no effects of unobserved covariates. The result of analysis, using Gompertz-PHM as the selected model, is shown in Table 8 and Table 9.

Table 8: The result of covariates effect analysis under the assumption of Gompertz-PHM

Covariates	Coef.	Std. Error	z	P>|z|	[95% Conf. Interval]	
$\mathbf{z_s}$	-0.125	0.136	-0.920	0.359	-0.392	0.142
$\mathbf{z_t}$	0.009	0.014	0.620	0.536	-0.019	0.037
$\mathbf{z_p}$	0.884	4.008	0.220	0.825	-6.971	8.739
$\mathbf{z_r}$	0.269	0.252	1.070	0.285	-0.224	0.762
$\mathbf{z_m}$	-1.606	0.172	-9.320	0.000	-1.943	-1.268
$\mathbf{z_e}$	0.365	0.149	2.450	0.014	0.073	0.657

Table 9: The constant value, baseline and unobserved parameters estimation of Gompertz-PHM

Parameters	Coef.	Std. Error	[95% Conf. Interval]	
Constant value	-2.844	0.728	-4.271	-1.416
Baseline Gompertz Ancillary parameter (γ)	-0.001	0.000	-0.001	0.000

The results of the analysis under the assumption of Gompertz-PHM showed that Machine movement (z_m) and Excavator code (z_e) have a significant effect on the excavators' reliability. Having the regression coefficient for covariates, the reliability of the excavators will be equal to:

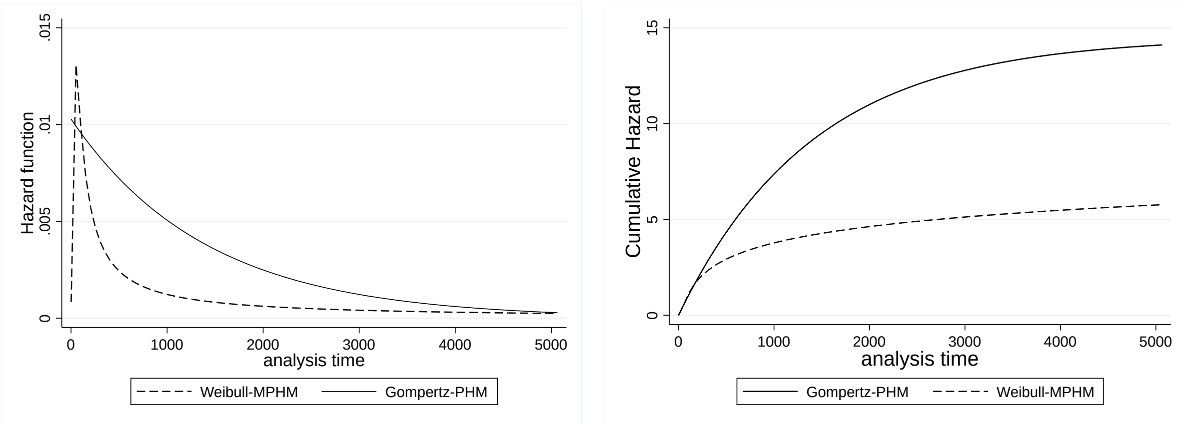
$$R(t;z) = \mathrm{e}^{\left(\left(\frac{e^{-0.001t}-1}{0.001}\right)(\exp\left(-2.884-1.606\mathrm{z_m}+0.365\mathrm{z_e}\right))\right)} \tag{34}$$

As you can see in Table 10, the regression coefficients of the observed covariates that have a significant effect on the hazard rate are different estimations in PHM than in MPHM.

Table 10: The difference between the effect of each covariate in PHM and MPHM

Covariates	Exp (coef.)		Differences	
	MPHM	PHM		
z_m	0.038	0.201	81%	overestimate
z_e	2.527	1.440	-43%	underestimate

Figure 6 compares the excavators' hazard and cumulative hazard rates in both models. As this figure shows, after approximately 300 hours we gain different hazard rates which hints that the unobserved covariates have a significant effect on the hazard rates of excavators; ignoring this factor may mislead a further decision on the operation and maintenance strategy.



(a) (b)

Figure 6: Comparison of the a) hazard function and b) cumulative hazard function under the Weibull-MPHM and Gompertz-PHM

5- Conclusion

The results of reliability analysis for heterogeneous data can differ substantially from those in a homogeneous case. In most cases, failing to account for heterogeneity would lead to significant differences in the estimation of the effects of covariates. Our recommendation is that all data sets should be checked for unobserved heterogeneity, using an appropriate statistical test. In the analysis of the data sets with observed and unobserved heterogeneity, in the first step, the time dependency test of the observed covariates needs to be performed. Thereafter, the presence of unobserved covariates should be checked, using an appropriate statistical test. Finally, considering the type of repair strategy carried out on the item, the most appropriate model among the mixed proportional hazards model family should be selected. The large variability in failure data and the differences in failure intensity of the excavators indicate heterogeneity among the collected data, which can be explained by observed and unobserved covariates. The analytical approach is used to check the trend and correlation of failure data. The result showed no trend and correlation among the data which can justify the *iid* assumption. Hence, the renewal process can represent the baseline hazard of excavators. The result of time-dependency and heterogonous tests (ratio test) indicated that all identified observed covariates are time-independent and that there is an unobserved heterogeneity among the failure data. This means that some other factors, which were not included in this study, might have an effect on the reliability of the excavators. Therefore, we need to further explore and model the effect of the unobserved factors, to enhance the accuracy of the estimation. Having these results and the developed framework (Figure 2), the mixed proportional hazards model (MPHM) are used to analyze the data. The result of analysis showed that the two of the identified observed covariates have a significant effect on the hazard rate of the excavators. Ignoring the effect of unobserved covariates, and using PHM instead of MPHM, will 43 percent underestimate the effect of Excavator type and 81 percent overestimate the effect of Machin movement. Moreover, under the assumption of PHM baseline is different when MPHM is used to model the failure data. Thus failure rate of these model completely different. Hence, for any decisions on the operation and maintenance strategy, the effect of unobserved covariates should be considered.

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