An Example of $\mathbf{\Pi_3^0}$-complete Infinitary Rational Relation *

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Abstract

We give in this paper an example of infinitary rational relation, accepted by a 2-tape Büchi automaton, which is $\mathbf{\Pi_3^0}$-complete in the Borel hierarchy. Moreover the example of infinitary rational relation given in this paper has a very simple structure and can be easily described by its sections.

Keywords: infinitary rational relations; topological properties; Borel hierarchy; $\mathbf{\Pi_3^0}$-complete set.

1 Introduction

Acceptance of infinite words by finite automata was firstly considered in the sixties by Büchi in order to study decidability of the monadic second order theory of one successor over the integers [Büc62]. Then the so called ω-regular languages have been intensively studied and many applications have been found, see [Tho90, Sta97, PP02] for many results and references.
Since then many extensions of ω-regular languages have been investigated as the classes of ω-languages accepted by pushdown automata, Petri nets, Turing machines, see [Tho90, EH93, Sta97] for a survey of this work.

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On the other side rational relations on finite words were studied in the sixties and played a fundamental role in the study of families of context free languages [Ber79]. Investigations on their extension to rational relations on infinite words were carried out or mentioned in the books [BT70, LS77]. Gire and Nivat studied infinitary rational relations in [Gir81, GN84]. Infinitary rational relations are subsets of $\Sigma_1^\omega \times \Sigma_2^\omega \times \ldots \times \Sigma_n^\omega$, where n is an integer ≥ 2 and $\Sigma_1, \Sigma_2, \ldots \Sigma_n$ are finite alphabets, which are accepted by n-tape finite Büchi automata with n asynchronous reading heads. So the class of infinitary rational relations extends both the class of finitary rational relations **and** the class of ω-regular languages.
They have been much studied, in particular in connection with the rational functions they may define, see for example [CG99, BCPS00, Sim92, Sta97, Pri00] for many results and references.
Notice that a rational relation $R \subseteq \Sigma_1^\omega \times \Sigma_2^\omega \times \ldots \times \Sigma_n^\omega$ may be seen as an ω-language over the product alphabet $\Sigma_1 \times \Sigma_2 \times \ldots \times \Sigma_n$.
A way to study the complexity of languages of infinite words accepted by finite machines is to study their topological complexity and firstly to locate them with regard to the Borel and the projective hierarchies. This work is analysed for example in [Sta86, Tho90, EH93, LT94, Sta97]. It is well known that every ω-language accepted by a Turing machine with a Büchi or Muller acceptance condition is an analytic set and that ω-regular languages are boolean combinations of $\mathbf{\Pi}_2^0$-sets hence $\mathbf{\Delta}_3^0$-sets, [Sta97, PP02].
The question of the topological complexity of relations on infinite words also naturally arises and was asked by Simonnet in [Sim92]. It was also posed in a more general form by Lescow and Thomas in [LT94] (for infinite labelled partial orders) and in [Tho89] where Thomas suggested to study reducibility notions and associated completeness results.
Every infinitary rational relation is an analytic set. We showed in [Fin03a] that there exist some infinitary rational relations which are analytic but non Borel sets. Considering Borel infinitary rational relations we prove in this paper that there exist some infinitary rational relations, accepted by 2-tape Büchi automata, which are $\mathbf{\Pi}_3^0$-complete.
Examples of $\mathbf{\Sigma}_3^0$-complete and $\mathbf{\Pi}_3^0$-complete infinitary rational relations have already been given in the conference paper [Fin03c]. But the proof of the existence of $\mathbf{\Pi}_3^0$-complete infinitary rational relations was only sketched and we used a coding of ω^2-words by pairs of infinite words. We use in this paper a different coding of ω^2-words. This way we get some infinitary rational relations which have a very simple structure and can be easily described by their sections.
The result given in this paper has two interests: 1) It gives a complete proof of a result of [Fin03c]. 2) Some new ideas are here introduced with

a new coding of ω^2-words. Some of these new ideas led us further to the proof of very surprising results, answering the long standing open questions of the topological complexity of context free ω-languages and of infinitary rational relations. In particular infinitary rational relations have the same topological complexity as ω-languages accepted by Büchi Turing machines [Fin05, Fin06]and for every recursive ordinal α there exist some $\mathbf{\Pi}^0_\alpha$-complete and some $\mathbf{\Sigma}^0_\alpha$-complete infinitary rational relations.
The result presented in this paper is still interesting, although the result of the paper [Fin06] is stronger; we use here a coding of ω^2-words while in [Fin05, Fin06] we used a simulation of Turing machines and the examples of infinitary rational relations we obtain are different.
The result of this paper may also be compared with examples of $\mathbf{\Sigma}^0_3$-complete ω-languages accepted by deterministic pushdown automata with the acceptance condition: "some stack content appears infinitely often during an infinite run", given by Cachat, Duparc, and Thomas in [CDT02] or with examples of $\mathbf{\Sigma}^0_\mathbf{n}$-complete and $\mathbf{\Pi}^0_\mathbf{n}$-complete ω-languages, $n \geq 1$, accepted by non-deterministic pushdown automata with Büchi acceptance condition given in [Fin01].
The paper is organized as follows. In section 2 we introduce the notion of infinitary rational relations. In section 3 we recall definitions of Borel sets, and we prove our main result in section 4.

2 Infinitary rational relations

Let Σ be a finite alphabet whose elements are called letters. A non-empty finite word over Σ is a finite sequence of letters: $x = a_1 a_2 \ldots a_n$ where for all integers $i \in [1; n]$ $a_i \in \Sigma$. We shall denote $x(i) = a_i$ the i^{th} letter of x and $x[i] = x(1) \ldots x(i)$ for $i \leq n$. The length of x is $|x| = n$. The empty word will be denoted by λ and has 0 letter. Its length is 0. The set of finite words over Σ is denoted $\Sigma^\star$. A (finitary) language L over Σ is a subset of $\Sigma^\star$. The usual concatenation product of u and v will be denoted by $u.v$ or just uv. For $V \subseteq \Sigma^\star$, we denote $V^\star = \{v_1 \ldots v_n \mid \ \forall i \in [1; n] \quad v_i \in V \ \} \cup \{\lambda\}$.

The first infinite ordinal is ω. An ω-word over Σ is an ω -sequence $a_1 a_2 \ldots a_n \ldots$, where for all integers $i \geq 1 \quad a_i \in \Sigma$. When σ is an ω-word over Σ, we write $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)\ldots$ and $\sigma[n] = \sigma(1)\sigma(2)\ldots\sigma(n)$ the finite word of length n, prefix of σ. The set of ω-words over the alphabet Σ is denoted by Σ^ω. An ω-language over an alphabet Σ is a subset of Σ^ω. For $V \subseteq \Sigma^\star$, $V^\omega = \{\sigma = u_1 \ldots u_n \ldots \in \Sigma^\omega \mid \forall i \geq 1 \ \ u_i \in V\}$ is the ω-power of V. The concatenation product is extended to the product of a finite word u and an

ω-word v: the infinite word $u.v$ is then the ω-word such that: $(u.v)(k) = u(k)$ if $k \leq |u|$, and $(u.v)(k) = v(k - |u|)$ if $k > |u|$.

If A is a subset of B we shall denote $A^{-} = B - A$ the complement of A (in B).

We assume the reader to be familiar with the theory of formal languages and of ω-regular languages. We recall that ω-regular languages form the class of ω-languages accepted by finite automata with a Büchi acceptance condition and this class is the omega Kleene closure of the class of regular finitary languages.

We are going now to introduce the notion of infinitary rational relation $R \subseteq \Sigma_1^\omega \times \Sigma_2^\omega$ via acceptance by 2-tape Büchi automata.

Definition 2.1 *A 2-tape Büchi automaton is a 7-tuple $\mathcal{T} = (K, \Sigma_1, \Sigma_2, \Delta, q_0, F)$, where K is a finite set of states, Σ_1, Σ_2, are finite alphabets, Δ is a finite subset of $K \times \Sigma_1^\star \times \Sigma_2^\star \times K$ called the set of transitions, q_0 is the initial state, and $F \subseteq K$ is the set of accepting states.*
A computation $\mathcal{C}$ of the 2-tape Büchi automaton $\mathcal{T}$ over the pair $(u, v) \in \Sigma_1^\omega \times \Sigma_2^\omega$ is an infinite sequence of transitions

$$(q_0, u_1, v_1, q_1), (q_1, u_2, v_2, q_2), \ldots (q_{i-1}, u_i, v_i, q_i), (q_i, u_{i+1}, v_{i+1}, q_{i+1}), \ldots$$

such that: $u = u_1.u_2.u_3 \ldots$ and $v = v_1.v_2.v_3 \ldots$.
The computation is said to be successful iff there exists an accepting state $q_f \in F$ and infinitely many integers $i \geq 0$ such that $q_i = q_f$.
The infinitary rational relation $R(\mathcal{T}) \subseteq \Sigma_1^\omega \times \Sigma_2^\omega$ accepted by the 2-tape Büchi automaton $\mathcal{T}$ is the set of pairs $(u, v) \in \Sigma_1^\omega \times \Sigma_2^\omega$ such that there is some successful computation $\mathcal{C}$ of $\mathcal{T}$ over (u, v).
The set of infinitary rational relations accepted by 2-tape Büchi automata will be denoted RAT_2.

As noticed in the introduction an infinitary rational relation $R \subseteq \Sigma_1^\omega \times \Sigma_2^\omega$ may be considered as an ω-language over the product alphabet $\Sigma_1 \times \Sigma_2$. We shall use this fact to investigate the topological complexity of infinitary rational relations.

3 Borel sets

We assume the reader to be familiar with basic notions of topology which may be found in [Mos80, Kec95, LT94, Sta97, PP02].

For a finite alphabet X we shall consider X^ω as a topological space with the Cantor topology. The open sets of X^ω are the sets in the form $W.X^\omega$, where $W \subseteq X^\star$. A set $L \subseteq X^\omega$ is a closed set iff its complement $X^\omega - L$ is an open set.

Define now the next classes of the Hierarchy of Borel sets of finite ranks:

Definition 3.1 *The classes $\mathbf{\Sigma^0_n}$ and $\mathbf{\Pi^0_n}$ of the Borel Hierarchy on the topological space X^ω are defined as follows:*
$\mathbf{\Sigma^0_1}$ is the class of open sets of X^ω.
$\mathbf{\Pi^0_1}$ is the class of closed sets of X^ω.
And for any integer $n \geq 1$:
$\mathbf{\Sigma^0_{n+1}}$ is the class of countable unions of $\mathbf{\Pi^0_n}$-subsets of X^ω.
$\mathbf{\Pi^0_{n+1}}$ is the class of countable intersections of $\mathbf{\Sigma^0_n}$-subsets of X^ω.

The Borel Hierarchy is also defined for transfinite levels, but we shall not need them in the present study. There are also some subsets of X^ω which are not Borel. In particular the class of Borel subsets of X^ω is strictly included into the class $\mathbf{\Sigma^1_1}$ of analytic sets which are obtained by projection of Borel sets, see for example [Sta97, LT94, PP02, Kec95] for more details.

Recall also the notion of completeness with regard to reduction by continuous functions. For an integer $n \geq 1$, a set $F \subseteq X^\omega$ is said to be a $\mathbf{\Sigma^0_n}$ (respectively, $\mathbf{\Pi^0_n}$, $\mathbf{\Sigma^1_1}$)-complete set iff for any set $E \subseteq Y^\omega$ (with Y a finite alphabet): $E \in \mathbf{\Sigma^0_n}$ (respectively, $E \in \mathbf{\Pi^0_n}$, $E \in \mathbf{\Sigma^1_1}$) iff there exists a continuous function $f : Y^\omega \rightarrow X^\omega$ such that $E = f^{-1}(F)$.
A $\mathbf{\Sigma^0_n}$ (respectively, $\mathbf{\Pi^0_n}$, $\mathbf{\Sigma^1_1}$)-complete set is a $\mathbf{\Sigma^0_n}$ (respectively, $\mathbf{\Pi^0_n}$, $\mathbf{\Sigma^1_1}$)-set which is in some sense a set of the highest topological complexity among the $\mathbf{\Sigma^0_n}$ (respectively, $\mathbf{\Pi^0_n}$, $\mathbf{\Sigma^1_1}$)-sets. $\mathbf{\Sigma^0_n}$ (respectively, $\mathbf{\Pi^0_n}$)-complete sets, with n an integer ≥ 1, are thoroughly characterized in [Sta86].

Example 3.2 *Let $\Sigma = \{0, 1\}$ and $\mathcal{A} = (0^\star.1)^\omega \subseteq \Sigma^\omega$. $\mathcal{A}$ is the set of ω-words over the alphabet Σ with infinitely many occurrences of the letter 1. It is well known that $\mathcal{A}$ is a $\mathbf{\Pi^0_2}$-complete set and its complement $\mathcal{A}^-$ is a $\mathbf{\Sigma^0_2}$-complete set: it is the set of ω-words over $\{0, 1\}$ having only a finite number of occurrences of letter 1.*

4 $\mathbf{\Pi^0_3}$-complete infinitary rational relations

We had got in [Fin03c] some $\mathbf{\Pi^0_3}$-complete infinitary rational relations. We used a coding of ω^2-words over a finite alphabet Σ by pairs of ω-words over

$\Sigma \cup \{A\}$ where A is an additionnal letter not in Σ.
We shall modify the previous proof (only sketched in [Fin03c]) by coding an ω^2-word over a finite alphabet Σ by a **single** ω-word over $\Sigma \cup \{A\}$. This way we can get some $\mathbf{\Pi^0_3}$-complete infinitary rational relation having some extra property.

Theorem 4.1 *Let $\Gamma = \{0, 1, A\}$ be an alphabet having three letters, and α be the ω-word over the alphabet Γ which is defined by:*

$$\alpha = A.0.A.0^2.A.0^3.A.0^4.A.0^5.A \ldots A.0^n.A.0^{n+1}.A \ldots$$

Then there exists an infinitary rational relation $R \subseteq \Gamma^\omega \times \Gamma^\omega$ such that:
$R_\alpha = \{\sigma \in \Gamma^\omega \mid (\sigma, \alpha) \in R\}$ is a $\mathbf{\Pi^0_3}$ − complete subset of Γ^ω, and for all $u \in \Gamma^\omega - \{\alpha\}$ $\quad R_u = \{\sigma \in \Gamma^\omega \mid (\sigma, u) \in R\} = \Gamma^\omega$. Moreover R is a $\mathbf{\Pi^0_3}$-complete subset of $\Gamma^\omega \times \Gamma^\omega$.

Proof. We shall use a well known example of $\mathbf{\Pi^0_3}$-complete set which is a subset of the topological space Σ^{ω^2}.

The set Σ^{ω^2} is the set of ω^2-words over the finite alphabet Σ. It may also be viewed as the set of (infinite) $(\omega \times \omega)$-matrices whose coefficients are letters of Σ. If $x \in \Sigma^{\omega^2}$ we shall write $x = (x(m, n))_{m \geq 1, n \geq 1}$. The infinite word $x(m, 1)x(m, 2) \ldots x(m, n) \ldots$ will be called the m^{th} column of the ω^2-word x and the infinite word $x(1, n)x(2, n) \ldots x(m, n) \ldots$ will be called the n^{th} row of the ω^2-word x. Thus an element of Σ^{ω^2} is completely determined by the (infinite) set of its columns or of its rows.
The set Σ^{ω^2} is usually equipped with the product topology of the dicrete topology on Σ (for which every subset of Σ is an open set), see [Kec95] [PP02]. This topology may be defined by the following distance d. Let x and y be two ω^2-words in Σ^{ω^2} such that $x \neq y$, then

$$d(x, y) = \frac{1}{2^n} \qquad \text{where}$$

$$n = min\{p \geq 1 \mid \exists (i, j)\ x(i, j) \neq y(i, j) \text{ and } i + j = p\}$$

Then the topological space Σ^{ω^2} is homeomorphic to the above defined topological space Σ^ω. The Borel hierarchy and the projective hierarchy on Σ^{ω^2} are defined from open sets in the same manner as in the case of the topological space Σ^ω. The notion of $\mathbf{\Sigma^0_n}$ (respectively $\mathbf{\Pi^0_n}$)-complete sets are also defined in a similar way.

Let now

$$P = \{x \in \{0,1\}^{\omega^2} \mid \forall m \exists^{<\infty} n \ x(m,n) = 1\}$$

where $\exists^{<\infty}$ means "there exist only finitely many",
P is the set of ω^2-words having all their columns in the $\mathbf{\Sigma^0_2}$-complete subset $\mathcal{A}^-$ of $\{0,1\}^\omega$ where $\mathcal{A}$ is the $\mathbf{\Pi^0_2}$-complete ω-regular language given in Example 3.2.

Recall the following classical result, [Kec95, p. 179]:

Lemma 4.2 *The set P is a $\mathbf{\Pi^0_3}$-complete subset of* $\{0,1\}^{\omega^2}$.

Proof. Let $\mathcal{B}_m = \{x \in \Sigma^{\omega^2} \mid x(m,1)x(m,2)\ldots x(m,n)\ldots \in \mathcal{A}^-\}$ be the set of ω^2-words over $\Sigma = \{0,1\}$ having their m^{th} column in the $\mathbf{\Sigma^0_2}$-complete set $\mathcal{A}^-$. In order to prove that, for every integer $m \geq 1$, the set $\mathcal{B}_m$ is a $\mathbf{\Sigma^0_2}$-subset of Σ^{ω^2}, consider the function $i_m : \Sigma^{\omega^2} \rightarrow \Sigma^\omega$ defined by $i_m(x) = x(m,1)x(m,2)\ldots x(m,n)\ldots$ for every $x \in \Sigma^{\omega^2}$. The function i_m is continuous and $i_m^{-1}(\mathcal{A}^-) = \mathcal{B}_m$ holds. Therefore $\mathcal{B}_m$ is a $\mathbf{\Sigma^0_2}$-subset of Σ^{ω^2} because the class $\mathbf{\Sigma^0_2}$ is closed under inverse images by continuous functions.

Thus the set

$$P = \bigcap_{m \geq 1} \mathcal{B}_m$$

of ω^2-words over Σ having all their columns in $\mathcal{A}^-$ is a countable intersection of $\mathbf{\Sigma^0_2}$-sets so it is a $\mathbf{\Pi^0_3}$-set.

It remains to show that P is $\mathbf{\Pi^0_3}$**-complete**. Let then L be a $\mathbf{\Pi^0_3}$-subset of Σ^ω. We know that $L = \cap_{i \in \mathbb{N}^\star} A_i$ for some $\mathbf{\Sigma^0_2}$-subsets A_i, $i \geq 1$, of Σ^ω. But $\mathcal{A}^-$ is $\mathbf{\Sigma^0_2}$-complete therefore, for each integer $i \geq 1$, there is some continuous function $f_i : \Sigma^\omega \rightarrow \Sigma^\omega$ such that $f_i^{-1}(\mathcal{A}^-) = A_i$.

Let now f be the function from Σ^ω into Σ^{ω^2} which is defined by $f(x)(m,n) = f_m(x)(n)$. The function f is continuous because each function f_i is continuous.

For $x \in \Sigma^\omega$ $f(x) \in P$ iff the ω^2-word $f(x)$ has all its columns in the ω-language $\mathcal{A}^-$, i.e. iff for all integers $m \geq 1$

$$f_m(x) = f_m(x)(1)f_m(x)(2)\ldots f_m(x)(n)\ldots \in \mathcal{A}^-$$

iff $\forall m \geq 1 \ \ x \in A_m$. Thus $f(x) \in P$ iff $x \in L = \cap_{m \geq 1} A_m$ so $L = f^{-1}(P)$.

We have then proved that all $\mathbf{\Pi^0_3}$-subsets of Σ^ω are inverse images by continuous functions of the $\mathbf{\Pi^0_3}$-set P therefore P is a $\mathbf{\Pi^0_3}$-complete set. $\square$

In order to use this example we shall firstly define a coding of ω^2-words over Σ by ω-words over the alphabet $(\Sigma \cup \{A\})$ where A is a new letter not in Σ.

Let us call, for $x \in \Sigma^{\omega^2}$ and p an integer ≥ 1:

$$T^x_{p+1} = \{x(p,1), x(p-1,2), \ldots, x(2,p-1), x(1,p)\}$$

the set of elements $x(m,n)$ with $m+n = p+1$ and

$$U^x_{p+1} = x(p,1).x(p-1,2)\ldots x(2,p-1).x(1,p)$$

the sequence formed by the concatenation of elements $x(m,n)$ of T^x_{p+1} for increasing values of n.

We shall code an ω^2-word $x \in \Sigma^{\omega^2}$ by the ω-word $h(x)$ defined by

$$h(x) = A.U^x_2.A.U^x_3.A.U^x_4.A.U^x_5.A.U^x_6.A\ldots A.U^x_n.A.U^x_{n+1}.A\ldots$$

Let then h be the mapping from Σ^{ω^2} into $(\Sigma \cup \{A\})^\omega$ such that, for every ω^2-word x over the alphabet Σ, $h(x)$ is the code of the ω^2-word x as defined above. It is easy to see, from the definition of h and of the order of the enumeration of letters $x(m,n)$ in $h(x)$ (they are enumerated for increasing values of $m+n$), that h is a continuous function from Σ^{ω^2} into $(\Sigma \cup \{A\})^\omega$.

Remark that the above coding of ω^2-words resembles the use of the Cantor pairing function as it was used to construct the complete sets P_i and S_i in [SW78] (see also [Sta86] or [Sta97, section 3.4]).

Lemma 4.3 *Let Σ be a finite alphabet. If $L \subseteq \Sigma^{\omega^2}$ is $\mathbf{\Pi^0_3}$-complete then*

$$h(L) \cup h(\Sigma^{\omega^2})^-$$

is a $\mathbf{\Pi^0_3}$-complete subset of $(\Sigma \cup \{A\})^\omega$.

Proof. The topological space Σ^{ω^2} is compact thus its image by the continuous function h is also a compact subset of the topological space $(\Sigma \cup \{A\})^\omega$. The set $h(\Sigma^{\omega^2})$ is compact hence it is a closed subset of $(\Sigma \cup \{A\})^\omega$ and its complement

$$(h(\Sigma^{\omega^2}))^- = (\Sigma \cup \{A\})^\omega - h(\Sigma^{\omega^2})$$

is an open (i.e. a $\mathbf{\Sigma^0_1}$) subset of $(\Sigma \cup \{A\})^\omega$.

On the other hand the function h is also injective thus it is a bijection from Σ^{ω^2} onto $h(\Sigma^{\omega^2})$. But a continuous bijection between two compact sets is an homeomorphism therefore h induces an homeomorphism between Σ^{ω^2} and $h(\Sigma^{\omega^2})$. By hypothesis L is a $\mathbf{\Pi^0_3}$-subset of Σ^{ω^2} thus $h(L)$ is a $\mathbf{\Pi^0_3}$-subset of $h(\Sigma^{\omega^2})$ (where Borel sets of the topological space $h(\Sigma^{\omega^2})$ are defined from open sets as in the cases of the topological spaces Σ^ω or Σ^{ω^2}).

The topological space $h(\Sigma^{\omega^2})$ is a topological subspace of $(\Sigma \cup \{A\})^\omega$ and its topology is induced by the topology on $(\Sigma \cup \{A\})^\omega$: open sets of $h(\Sigma^{\omega^2})$ are traces on $h(\Sigma^{\omega^2})$ of open sets of $(\Sigma \cup \{A\})^\omega$ and the same result holds for closed sets. Then one can easily show by induction that for every integer $n \geq 1$, $\mathbf{\Pi^0_n}$-subsets (resp. $\mathbf{\Sigma^0_n}$-subsets) of $h(\Sigma^{\omega^2})$ are traces on $h(\Sigma^{\omega^2})$ of $\mathbf{\Pi^0_n}$-subsets (resp. $\mathbf{\Sigma^0_n}$-subsets) of $(\Sigma \cup \{A\})^\omega$, i.e. are intersections with $h(\Sigma^{\omega^2})$ of $\mathbf{\Pi^0_n}$-subsets (resp. $\mathbf{\Sigma^0_n}$-subsets) of $(\Sigma \cup \{A\})^\omega$.

But $h(L)$ is a $\mathbf{\Pi^0_3}$-subset of $h(\Sigma^{\omega^2})$ hence there exists a $\mathbf{\Pi^0_3}$-subset T of $(\Sigma \cup \{A\})^\omega$ such that $h(L) = T \cap h(\Sigma^{\omega^2})$. But $h(\Sigma^{\omega^2})$ is a closed i.e. $\mathbf{\Pi^0_1}$-subset (hence also a $\mathbf{\Pi^0_3}$-subset) of $(\Sigma \cup \{A\})^\omega$ and the class of $\mathbf{\Pi^0_3}$-subsets of $(\Sigma \cup \{A\})^\omega$ is closed under finite intersection thus $h(L)$ is a $\mathbf{\Pi^0_3}$-subset of $(\Sigma \cup \{A\})^\omega$.

Now $h(L) \cup (h(\Sigma^{\omega^2}))^-$ is the union of a $\mathbf{\Pi^0_3}$-subset and of a $\mathbf{\Sigma^0_1}$-subset of $(\Sigma \cup \{A\})^\omega$ therefore it is a $\mathbf{\Pi^0_3}$-subset of $(\Sigma \cup \{A\})^\omega$ because the class of $\mathbf{\Pi^0_3}$-subsets of $(\Sigma \cup \{A\})^\omega$ is closed under finite union.

In order to prove that $h(L) \cup (h(\Sigma^{\omega^2}))^-$ is **$\mathbf{\Pi^0_3}$-complete** it suffices to remark that

$$L = h^{-1}[h(L) \cup (h(\Sigma^{\omega^2}))^-]$$

This implies that $h(L) \cup (h(\Sigma^{\omega^2}))^-$ is $\mathbf{\Pi^0_3}$-complete because L is assumed to be $\mathbf{\Pi^0_3}$-complete. □

Lemma 4.4 *Let* $P = \{x \in \{0,1\}^{\omega^2} \mid \forall m \exists^{<\infty} n \ x(m,n) = 1\}$ *and* $\Sigma = \{0,1\}$. *Then*

$$\mathcal{P} = h(P) \cup (h(\Sigma^{\omega^2}))^-$$

is a $\mathbf{\Pi^0_3}$*-complete subset of* $(\Sigma \cup \{A\})^\omega$.

Proof. It follows directly from the two preceding Lemmas. □

Let now $\Sigma = \{0, 1\}$ and let α be the ω-word over the alphabet $\Sigma \cup \{A\}$ which is defined by:

$$\alpha = A.0.A.0^2.A.0^3.A.0^4.A.0^5.A \ldots A.0^n.A.0^{n+1}.A \ldots$$

We can now state the following Lemma.

Lemma 4.5 *Let $\Sigma = \{0, 1\}$ and α be the ω-word over $\Sigma \cup \{A\}$ defined as above. Then there exists an infinitary rational relation $R_1 \subseteq (\Sigma \cup \{A\})^\omega \times (\Sigma \cup \{A\})^\omega$ such that:*

$$\forall x \in \Sigma^{\omega^2} \quad (x \in P) \text{ iff } ((h(x), \alpha) \in R_1)$$

Proof. We define now the relation R_1. A pair $y = (y_1, y_2)$ of ω-words over the alphabet $\Sigma \cup \{A\}$ is in R_1 if and only if it is in the form

$y_1 = U_k.u_1.v_1.A.u_2.v_2.A.u_3.v_3.A \ldots A.u_n.v_n.A. \ldots$
$y_2 = V_k.w_1.z_1.A.w_2.z_2.A.w_3.z_3.A \ldots A.w_n.z_n.A \ldots$

where k is an integer ≥ 1, $U_k, V_k \in (\Sigma^\star.A)^k$, and, for all integers $i \geq 1$,

$$v_i, w_i, z_i \in 0^\star \text{ and } u_i \in \Sigma^\star \text{ and}$$

$$|w_i| = |v_i| \quad \text{ and } \quad [\ |u_{i+1}| = |z_i| + 1 \text{ or } |u_{i+1}| = |z_i|\ ]$$

and there exist infinitely many integers i such that $|u_{i+1}| = |z_i|$.

We prove first that the relation R_1 satisfies:

$$\forall x \in \Sigma^{\omega^2} \quad (x \in P) \text{ iff } ((h(x), \alpha) \in R_1)$$

Assume that for some $x \in \Sigma^{\omega^2}$ $(h(x), \alpha) \in R_1$. Then $(h(x), \alpha)$ may be written in the above form (y_1, y_2) with
$y_1 = U_k.u_1.v_1.A.u_2.v_2.A.u_3.v_3.A \ldots A.u_n.v_n.A. \ldots$
$y_2 = V_k.w_1.z_1.A.w_2.z_2.A.w_3.z_3.A \ldots A.w_n.z_n.A \ldots$

$y_1 = h(x)$ implies that for all integers $n \geq 1$ $\quad U^x_{k+n} = u_n.v_n$ thus $|u_n.v_n| = k + n - 1$.
$y_2 = \alpha$ implies that for all integers $n \geq 1$ $\quad w_n.z_n = 0^{k+n-1}$ thus $|w_n.z_n| = k + n - 1$.

So $|u_n.v_n| = |w_n.z_n|$ but by hypothesis $|w_n| = |v_n|$ therefore $|u_n| = |z_n|$.

Moreover $|u_{n+1}| = |z_n| + 1$ or $|u_{n+1}| = |z_n|$.
If $|u_{n+1}| = |z_n| + 1$ then $|u_{n+1}| = |u_n| + 1$ and $|v_{n+1}| = |v_n|$ because $|u_{n+1}| + |v_{n+1}| = |u_n| + |v_n| + 1$.
If $|u_{n+1}| = |z_n|$ then $|u_{n+1}| = |u_n|$ and $|v_{n+1}| = |v_n| + 1$ because $|u_{n+1}| + |v_{n+1}| = |u_n| + |v_n| + 1$.

This proves that the sequence $(|v_n|)_{n\geq 1}$ is increasing because for all integers $n \geq 1 \quad |v_{n+1}| = |v_n|$ or $|v_{n+1}| = |v_n| + 1$. Moreover by definition of R_1 we know that there exist infinitely many integers $n \geq 1$ such that $|u_{n+1}| = |z_n|$ hence also $|v_{n+1}| = |v_n| + 1$. Thus

$$\lim_{n\to+\infty} |v_n| = +\infty$$

Let now K be an integer ≥ 1 and let us prove that the K first columns of the ω^2-word x have only finitely many occurrences of the letter 1.

$\lim_{n\to+\infty} |v_n| = +\infty$ thus there exists an integer $N \geq 1$ such that $\forall n \geq N$ $|v_n| \geq K$.

Consider now, for $n \geq N$,

$$U^x_{k+n} = u_n.v_n = x(k+n-1, 1).x(k+n-2, 2) \ldots x(2, k+n-2).x(1, k+n-1)$$

We know that $v_n \in 0^\star$ thus

$$x(|v_n|, k+n-|v_n|) = x(|v_n|-1, k+n+1-|v_n|) = \ldots = x(2, k+n-2) = x(1, k+n-1) = 0$$

and in particular

$$x(K, k+n-K) = x(K-1, k+n+1-K) = \ldots = x(2, k+n-2) = x(1, k+n-1) = 0$$

because $|v_n| \geq K$.

These equalities hold for all integers $n \geq N$ and this proves that the K first columns of the ω^2-word x have only finitely many occurrences of the letter 1.
But this is true for all integers $K \geq 1$ so **all columns** of x have a finite number of occurrences of the letter 1 and $x \in P$.

Conversely it is easy to see that for each $x \in P$ the pair $(h(x), \alpha)$ may be written in the above form $(y_1, y_2) \in R_1$.

It remains only to prove that the above defined relation R_1 is an infinitary rational relation. It is easy to see that the following 2-tape Büchi automaton $\mathcal{T}$ accepts the infinitary rational relation R_1.

$\mathcal{T} = (K, \Gamma, \Gamma, \delta, q_0, F)$, where $K = \{q_0, q_1, q_2, q_3, q_4, q_5\}$ is a finite set of states, $\Gamma = \Sigma \cup \{A\} = \{0, 1, A\}$, with $\Sigma = \{0, 1\}$, q_0 is the initial state, and $F = \{q_4\}$ is the set of final states. Moreover $\delta \subseteq K \times \Gamma^\star \times \Gamma^\star \times K$ is the finite set of transitions, containing the following transitions:

(q_0, a, λ, q_0), for all $a \in \Sigma$,
(q_0, λ, a, q_0), for all $a \in \Sigma$,
(q_0, A, A, q_0),
(q_0, A, A, q_1),
(q_1, a, λ, q_1), for all $a \in \Sigma$,
$(q_1, \lambda, \lambda, q_2)$,
$(q_2, a, 0, q_2)$, for all $a \in \Sigma$,
(q_2, A, λ, q_3),
$(q_3, a, 0, q_3)$, for all $a \in \Sigma$,
$(q_3, \lambda, \lambda, q_4)$
(q_3, a, λ, q_5), for all $a \in \Sigma$,
(q_4, λ, A, q_2),
(q_5, λ, A, q_2).

Remark 4.6 *Using classical constructions from automata theory, we could have avoided the set of transitions to contain some transitions in the form* $(q_i, \lambda, \lambda, q_j)$*, like* $(q_1, \lambda, \lambda, q_2)$ *or* $(q_3, \lambda, \lambda, q_4)$*.*

Lemma 4.7 *The set*

$$R_2 = (\Sigma \cup \{A\})^\omega \times (\Sigma \cup \{A\})^\omega - (h(\Sigma^{\omega^2}) \times \{\alpha\})$$

is an infinitary rational relation.

Proof. By definition of the mapping h, we know that a pair of ω-words over the alphabet $(\Sigma \cup \{A\})$ is in $h(\Sigma^{\omega^2}) \times \{\alpha\}$ iff it is in the form (σ_1, σ_2), where
$\sigma_1 = A.u_1.A.u_2.A.u_3.A.u_4.A \ldots .A.u_n.A.u_{n+1}.A \ldots$
$\sigma_2 = \alpha = A.0.A.0^2.A.0^3.A.0^4.A \ldots A.0^n.A.0^{n+1}.A \ldots$

where for all integers $i \geq 1$, $u_i \in \Sigma^\star$ and $|u_i| = i$.

So it is easy to see that $(\Sigma \cup \{A\})^\omega \times (\Sigma \cup \{A\})^\omega - (h(\Sigma^{\omega^2}) \times \{\alpha\})$ is the union of the sets $\mathcal{C}_j$ where:

- $\mathcal{C}_1 = \{(\sigma_1, \sigma_2) \ / \ \sigma_1, \sigma_2 \in (\Sigma \cup \{A\})^\omega$ and $(\sigma_1 \in \mathcal{B}$ or $\sigma_2 \in \mathcal{B})\}$
where $\mathcal{B}$ is the set of ω-words over $(\Sigma \cup \{A\})$ having only a finite number of letters A.

- $\mathcal{C}_2$ is formed by pairs (σ_1, σ_2) where
σ_1 or σ_2 has not any initial segment in $A.\Sigma.A.\Sigma^2.A$.

- $\mathcal{C}_3$ is formed by pairs (σ_1, σ_2) where
$\sigma_2 \notin \{0, A\}^\omega$.

- $\mathcal{C}_4$ is formed by pairs (σ_1, σ_2) where
$\sigma_1 = A.w_1.A.w_2.A.w_3.A.w_4.A \ldots A.w_n.A.u.A.z_1$
$\sigma_2 = A.w'_1.A.w'_2.A.w'_3.A.w'_4.A \ldots A.w'_n.A.v.A.z_2$

  where n is an integer ≥ 1, for all $i \leq n$ $\ w_i, w'_i \in \Sigma^\star$, $z_1, z_2 \in (\Sigma \cup \{A\})^\omega$ and
$$u, v \in \Sigma^\star \text{ and } |v| \neq |u|$$

- $\mathcal{C}_5$ is formed by pairs (σ_1, σ_2) where
$\sigma_1 = A.w_1.A.w_2.A.w_3.A.w_4 \ldots A.w_n.A.w_{n+1}.A.v.A.z_1$
$\sigma_2 = A.w'_1.A.w'_2.A.w'_3.A.w'_4 \ldots A.w'_n.A.u.A.z_2$

  where n is an integer ≥ 1, for all $i \leq n$ $\ w_i, w'_i \in \Sigma^\star$, $w_{n+1} \in \Sigma^\star$, $z_1, z_2 \in (\Sigma \cup \{A\})^\omega$ and
$$u, v \in \Sigma^\star \text{ and } |v| \neq |u| + 1$$

Each set $\mathcal{C}_j$, $1 \leq j \leq 5$, is easily seen to be an infinitary rational relation $\subseteq (\Sigma \cup \{A\})^\omega \times (\Sigma \cup \{A\})^\omega$ (the detailed proof is left to the reader). The class RAT_2 is closed under finite union thus
$$R_2 = (\Sigma \cup \{A\})^\omega \times (\Sigma \cup \{A\})^\omega - (h(\Sigma^{\omega^2}) \times \{\alpha\}) = \bigcup_{1 \leq j \leq 5} \mathcal{C}_j$$
is an infinitary rational relation. □

Return now to the proof of Theorem 4.1. Let
$$R = R_1 \cup R_2 \subseteq \Gamma^\omega \times \Gamma^\omega$$
The class RAT_2 is closed under finite union therefore R is an infinitary rational relation.

Lemma 4.5 and the definition of R_2 imply that $R_\alpha = \{\sigma \in \Gamma^\omega \mid (\sigma, \alpha) \in R\}$ is equal to the set $\mathcal{P} = h(P) \cup (h(\Sigma^{\omega^2}))^-$ which is a $\mathbf{\Pi^0_3}$-complete subset of $(\Sigma \cup \{A\})^\omega$ by Lemma 4.4.

Moreover, for all $u \in \Gamma^\omega - \{\alpha\}$, $R_u = \{\sigma \in \Gamma^\omega \mid (\sigma, u) \in R\} = \Gamma^\omega$ holds by definition of R_2.

In order to prove that R is a $\mathbf{\Pi^0_3}$-set remark first that R may be written as the union:

$$R = \mathcal{P} \times \{\alpha\} \bigcup \Gamma^\omega \times (\Gamma^\omega - \{\alpha\})$$

We already know that $\mathcal{P}$ is a $\mathbf{\Pi^0_3}$-complete subset of $(\Sigma \cup \{A\})^\omega$. Then it is easy to show that $\mathcal{P} \times \{\alpha\}$ is also a $\mathbf{\Pi^0_3}$-subset of $(\Sigma \cup \{A\})^\omega \times (\Sigma \cup \{A\})^\omega$. On the other side it is easy to see that $\Gamma^\omega \times (\Gamma^\omega - \{\alpha\})$ is an open subset of $\Gamma^\omega \times \Gamma^\omega$. Thus R is a $\mathbf{\Pi^0_3}$-set because the Borel class $\mathbf{\Pi^0_3}$ is closed under finite union.

Moreover let $g : \Sigma^{\omega^2} \to (\Sigma \cup \{A\})^\omega \times (\Sigma \cup \{A\})^\omega$ be the function defined by:

$$\forall x \in \Sigma^{\omega^2} \qquad g(x) = (h(x), \alpha)$$

It is easy to see that g is continuous because h is continuous. By construction it turns out that for all ω^2-words $x \in \Sigma^{\omega^2}$ $(x \in P)$ iff $(g(x) \in R)$. This means that $g^{-1}(R) = P$. This implies that R is $\mathbf{\Pi^0_3}$-complete because P is $\mathbf{\Pi^0_3}$-complete. $\square$

Remark 4.8 *The structure of the $\mathbf{\Pi^0_3}$-complete infinitary rational relation R we have just got is very different from the structure of a previous example given in [Fin03c]. It can be described very simply by the sections R_u, $u \in \Gamma^\omega$. All sections but one are equal to Γ^ω, so they have the lowest topological complexity and exactly one section is a $\mathbf{\Pi^0_3}$-complete subset of Γ^ω.*

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