Dynamics of nonautonomous eco-pidemiological models

Versão final após defesa

Lopo Ferreira de Jesus

Tese para obtenção do Grau de Doutor em
Matemática e Aplicações
(3º ciclo de estudos)

Orientador: Prof. Doutor César Augusto Teixeira Marques da Silva
Co-orientador: Prof. Doutor Helder Soares Vilarinho

Júri:
Profª. Doutora Maria Carlota da Rocha Xavier Rebelo Gonçalves
Prof. Doutor Jorge das Neves Duarte
Prof. Doutor José Joaquim Martins Oliveira
Prof. Doutor José Carlos Matos Duque

5 de julho de 2021

Dedicatory

To Naomi, Neri and Lopo Júnior

To Loide

Acknowledgments

First of all i want to thank God for giving me the strength to do this arduous task.

I want to thank Professor César Silva for his orientation, availability and support even in those moments he was not suppose to. He was crucial in the most difficult times and i will never forget how important he was to inspire me to persecute my goals.

A special thank you to Professor Helder Vilarinho that promptly attended the challenge to work with us in this difficult task.

The Beira Interior University for all conditions provided which allowed me to develop the present work, as well as the people from the Mathematics Department, particularly, i want to thank Professor José Duque for his support in some simulations that we had to make and also Filipa Raposo for her kindness and the way she promptly attended some of the issues that i went thought.

Thank you too, to Professor Joaquim Mateus, Professor at Instituto Politécnico of Guarda for his support in what concerns to some simulations.

I want to thank my parents for the teachings they gave me that made the person i am today; Thank you too, to my sons Naomi, Nerida and Lopo Júnior, for their patience in my less good moments; last, but not least, i want to thank my dearest wife, Loide, for her love, patience, dedication and understanding even when i was tired or absent. Without God, this work would never be possible.

Resumo

Nesta tese consideramos um modelo eco-epidemiológico geral que inclui uma grande variedade de modelos eco-epidemiológicos presentes na literatura. Assumimos que os parâmetros dependem do tempo e consideramos funções gerais para a predação de presas infectadas e não infectadas e também para a dinâmica vital de presas não infectadas e da população de predadores. Estudamos estes modelos em quatro cenários: não-autónomo geral, periódico, discreto e aleatório. Nos casos não-autónomo geral e discreto analisamos a persistência forte e extinção da doença, no caso periódico estudamos as condições para a existência de uma órbita periódica endémica e, finalmente, no caso aleatório estudamos a existência de atratores globais aleatórios.

Palavras-chave

Modelos eco-epidemiológicos; não-autónomo; periódico; discreto; aleatório; persistência e extinção; atractor global.

Resumo alargado

Nesta tese, consideramos um modelo geral que inclui uma grande família de modelos eco-epidemiológicos propostos na literatura. Assumimos que os parâmetros são dependentes do tempo e consideramos funções gerais para a predação de presas infectadas e não infectadas e também para a dinâmica vital de presas não infectadas e da população de predadores. Especificamente, consideramos o seguinte modelo eco-epidemiológico

$$\begin{cases} S' = G(t,S) - a(t)f(S,I,P)P - \beta(t)SI \\ I' = \beta(t)SI - \eta(t)g(S,I,P)I - c(t)I \\ P' = H(t,P) + \gamma(t)a(t)f(S,I,P)P + \theta(t)\eta(t)g(S,I,P)I \end{cases}, \qquad (0.0.1)$$

onde S, I e P correspondem, respectivamente, à presa suscetível, à presa infectada e ao predador, $\beta(t)$ é a taxa de incidência da doença, $\eta(t)$ é a taxa de predação de presas infectadas, $c(t)$ é a taxa de mortalidade na classe dos infectados, $\gamma(t)$ é a taxa de conversão de presas suscetíveis em predadores (transferência de biomassa), $\theta(t)$ é a taxa de conversão de presas infectadas em predadores, $G(t,S)$ e $H(t,P)$ representam a dinâmica vital das populações de presas suscetíveis e predadores, respectivamente, $a(t)f(S,I,P)$ é a predação de presas suscetíveis e $\eta(t)g(S,I,P)$ representa a predação de presas infectadas. Supõe-se que apenas as presas suscetíveis são capazes de se reproduzir, ou seja, a presa infectada é removida por morte (incluindo morte natural e relacionada à doença) ou por predação antes de ter a possibilidade de se reproduzir.

No Capítulo 1 consideramos o modelo não-autónomo (0.0.1) com $H(t,x) = h(t,x)x$, para alguma função $h : (\mathbb{R}_0^+)^2 \to \mathbb{R}$. Assumimos que as funções $f(S,I,P)$, $g(S,I,P)$ e $G(t,S)$ são localmente Lipschitz e não negativas e que $H(t,P)$ é localmente Lipschitz. O objetivo deste capítulo é discutir a persistência forte e extinção

de presas infectadas I sob alguns pressupostos adequados. Os resultados sobre persistência forte e extinção de presas infectadas são baseados na análise de sistemas relacionados com a dinâmica do modelo na ausência de presas infectadas. Aplicamos os resultados a modelos eco-epidemiológicos construídos a partir de modelos predador-presa existentes na literatura. Por fim, foram feitas simulações numéricas para ilustrar os resultados.

No Capítulo 2 abordamos a existência de órbitas periódicas para duas famílias distintas de modelos. Para a primeira, obtida do modelo geral (0.0.1) fazendo $G(t, S) = \Lambda(t) - \mu(t)S$ e $H(t, P) = (r(t) - b(t)P)P$, usamos o conhecido teorema de continuação de Mawhin para provar a existência de uma órbita periódica endêmica. Para a segunda, obtida de (0.0.1) fazendo $G(t, S) = \Lambda(t) - \mu(t)S$ e $H(t, P) = \Upsilon(t) - \zeta(t)P$, apresentamos um resultado semelhante ao anterior usando uma estratégia recente que se baseia na unicidade das órbitas periódicas no espaço livre de doenças.

No Capítulo 3 consideramos um modelo discreto não-autónomo correspondente ao modelo (0.0.1), com $G(t, S) = \Lambda(t) - \mu(t)S$ e $H(t, P) = (r(t) - b(t)P)P$, obtido pela aplicação do método de discretização de Micken ao modelo de tempo contínuo. O objetivo neste capítulo é discutir a persistência forte e extinção de presas infectadas I para o modelo discreto obtido.

No capítulo 4, consideramos perturbações aleatórias do modelo (0.0.1) com $G(t, S) = \Lambda(t) - \mu S(t)$ e $H(t, P) = -\delta_1 P(t) - \delta_2 P(t)^2$. Para isso, introduzimos perturbações aleatórias na taxa de natalidade (Λ) usando uma variável aleatória (*ruído real*), considerando todos os outros parâmetros como constantes e positivos. Demonstramos a existência de um atractor global aleatório, a persistência de presas suscetíveis e fornecemos condições para a extinção simultânea de predadores e presas infectadas. Também discutimos as dinâmicas do modelo SI aleatório e do modelo predador-presa aleatório. Obtemos para esses casos um atractor global aleatório, discutimos a prevalência de presas suscetíveis e fornecemos condições para a extinção de predadores ou presas infectadas.

O trabalho apresentado no Capítulo 2 foi publicado no artigo [**55**]. Os conteúdos dos Capítulos 1, 3 e 4 integram as pré-publicações [**54**], [**56**] e [**57**], respetivamente.

Abstract

We consider a general eco-epidemiological model which includes a large variety of eco-epidemiological models available in the literature. We assume that the parameters are time dependent and we consider general functions for the predation on infected and uninfected prey and also for the vital dynamics of uninfected prey and predator populations. We studied this model in four scenarios: non-autonomous, periodic, discrete and random. In the non-autonomous and discrete case we discussed the uniform strong persistence and extinction of the disease, in the periodic case, we studied the existence of an endemic periodic orbit, and finally, in the random case we studied the existence of random global attractors.

Keywords

Eco-epidemiological; non-autonomous; periodic; discrete; random; persistence and extinction; global attractor.

Contents

List of Figures

Introduction

The Lotka-Volterra models were considered independently by Alfred Lotka and Vito Volterra in 1925. Ecological models including more general models describing predator-prey interaction have been a major subject in mathematical biology. On the other hand, mathematical epidemiological models also have a long history that goes back to the construction of the first epidemiological model by Kermack and Mckendrick in 1926, having published together a series of papers containing contributions to the mathematical theory of epidemics [**58, 59, 60**].

Eco-epidemiological models are ecological models that include infected compartments. In many situations, these models describe more accurately the real ecological system than models where the disease is not taken into account. Thus the description of the dynamics of eco-epidemiological systems is a subject that have been receiving increasing attention by the researchers interested in mathematical biology. In particular, the inclusion of a disease in the preys or in the predators have impact on the population size of the predator-prey community [**43, 97, 45, 103, 19, 47**].

The first mathematical studies concerning eco-epidemiological models were considered in the late 1980's. To the best of our knowledge, Hadeler and Freedman, in 1989, were the first authors to consider an eco-epidemiological model in [**43**]. In that paper they construct a predator-prey model where both predator and prey are subjected to parasitism. They showed that, in the case where the uninfected predator cannot survive only on uninfected prey, the parasitization could lead to persistence of the predator provided a certain threshold of transmission is surpassed, i.e., if the product of rates of conversion of uninfected populations into infected populations is less than some threshold then the uninfected stable attractor (with respect to the uninfected system, stationary point or periodic orbit) is also stable against parasitic infection. On the other hand, if that product passes through the referred threshold then the uninfected attractor loses its stability and a stable infected attractor appear. In the following year, Chattopadhyay and Arino [**18**] considered a model

where the disease affects only the prey. Under some assumptions on the intrinsic growth rate of the susceptible population, the authors reduced their three dimensional system to a two dimensional system and were able to describe the persistence and extinction of the infected population in terms of the environmental carrying capacity. They also observed that, when the maximal renewal rate of the infected population is less than its natural mortality rate then both populations go to extinction and pointed out that there is an exchange of stability through simple bifurcation at the crossing point of trivial equilibrium to the boundary equilibrium, as well as at the crossing point of boundary equilibrium to the positive equilibrium. Moreover, they observed that for certain parametric conditions an Hopf-type bifurcation occurs for the strictly positive equilibrium. Additionally, considering a Holling-type II predator functional response, they found that the bifurcation branches are supercritical in some parametric region space, confirming the local asymptotic stability of the bifurcation orbit. Finally, using a Poincaré map, they observed that the analysis for the reduced system is valid for the original system. Later on, in 1994, Ezio Venturino [**97**] studied separately the effect of disease in the prey and in the predator, considering in each of the cases Holling-type I as well as standard functional responses to model the incidence of the disease. For the case of Holling-type I and disease in the prey, he showed that there exist seven equilibria: the origin, the neutral equilibrium, the boundary equilibrium, the positive equilibrium and three other equilibrium points. He found that the origin cannot be stable since one of its eigenvalues is positive, that the neutral equilibrium is stable under a suitable condition, that the boundary equilibrium is stable under some conditions and unstable otherwise, and that the positive equilibrium is the only nontrivial equilibrium point which is always unstable. For the stardard functional response, there are four equilibria: the origin, the neutral equilibrium, the boundary equilibrium and the positive equilibrium. He proved that origin is a saddle, the neutral equilibrium is locally asymptotically stable, the boundary and the positive equilibria are always unstable. For the case of standard functional response in the predator, we also have four equilibria: the origin, the neutral, the boundary and the positive equilibrium. He showed that origin is again a saddle since it has one eigenvalue which is positive, the neutral equilibrium is locally stable under some suitable condition, the

boundary equilibrium is always unstable. Finally, for the Holling-type I functional response and disease in the predator case, we have again four equilibria: the origin, the neutral, the boundary and the positive equilibrium. He established that the origin and the neutral equilibrium are unstable and that the boundary equilibrium is stable for some suitable condition. Considering again that the disease only affects the prey, in [103], Xiao and Chen studied an eco-epidemiological model focusing on the permanence and stability of equilibria and shown that, for some parameter values, the positive equilibrium becomes unstable, allowing a periodic solution to appear by Hopf bifurcation.

Different dynamical aspects of eco-epidemiological models in the autonomous setting have been extensively studied in several contexts. Concerning eco-epidemiological models with disease in the prey, there is an extensive literature focusing on persistence and extinction, local and global stability, existence of limit cycles and bifurcations. In [45] Haque and Chattopadhyay modified an existent eco-epidemiological model by replacing the Holling-type I incidence rates by some non-linear incidence rates. Their numerical simulations allowed them to conclude that, with this change of incidence rates, they can control the limit cycle oscillations around the positive interior equilibrium that was observed in the original model and, moreover, they determined the value for which a Hopf-bifurcation occurs for the positive interior equilibrium. Later, in [19], Samrat and Chattopadhyay consider a model with Holling-type I incidence rates and observe that the force of infection and the predation rate play important roles in maintaining the stability around the positive steady state. In [93] Upadhyay, Bairagi, Kundu and Chattopadhyay observed, in several numerical simulations, the existence of chaotic behaviour when some key parameters attain their critical values and in [94] Upadhyay and Roy considered a modified version of the model in the cited work by assuming that the predators consume the prey species according to a modified Holling-type II functional response. They studied the stability of the equilibria and, taking the death rate of predator and the growth rate of susceptible prey population as bifurcation parameters, they demonstrated the existence of backward Hopf-bifurcations and saw, in several computational experiments, that the system exhibits deterministic chaos when some control parameters attain some critical values. Still concerning bifurcation

analysis, it should be referred that in [67] Liu undertook a throughout study of bifurcations, especially near the boundary equilibrium. Several other works study, both analytically and numerically, the bifurcations of eco-epidemiological systems, including [95], where the authors considered distinct functional responses of predator for susceptible and infected preys, [87], where the authors considered Holling type II functional responses both for susceptible and infected preys, and [29] where a ratio-dependent functional response (a type of functional response that assumes that the prey eaten per unit time is a function of the ratio of prey to predator) is considered.

An analysis of stability of equilibria and bifurcations of an eco-epidemiological model was undertaken by Chakraborty, Das, Haldar and Kar in [17] for a model constructed from an epidemiological model considered by Bhattacharyya and Mukhopadhyay in [73] and including different nonlinear functional responses for susceptible and infected preys. Another very interesting work devoted to analysis of stability of equilibria of an eco-epidemiological model can be found in a paper by Sasmal and Chattopadhyay [86] where the authors compare the model with and without Allee effect and discuss the way the Allee effect affect the population dynamics of both the prey and the predator and also the conditions that allow the susceptible prey, infected prey and predator to coexist. In [81] the authors studied another model where Allee effect in the predator is considered.

There is still an active line of research aimed at the study of autonomous eco-epidemiological models as we can see by the recent papers [42], where a model involving switching (changing of preference of the predator from susceptible to infected prey) is considered, [89] where a model with Holling type II functional response functions and a type of mortality rate for the predator that the authors call hyperbolic mortality is analytically studied, [71] where the fear of predators among prey population is taken into account by assuming that an increase in the predator population leads to lowering of prey growth rate, [88] where a model is proposed that assumes, contrarily to the usual, that the infected prey is able to reproduce, and [100] where the local and global stability properties, existence of Hopf bifurcation, and permanence of the infected prey is studied for a model that considers the effect of the amount of prey refuge.

All the model referred above are autonomous models. However, to make models more realistic, it is important, in many situations, to consider time varying parameters. For instance, it is well known in epidemiology that incidence rates are seldom subject to periodic seasonal fluctuations.

In the context of eco-epidemiological models, several nonautonomous systems have been studied in the literature. In [75] Niu, Zhang and Teng consider a class of general non-autonomous eco-epidemiological models with disease in the prey, containing the periodic case as a very particular situation, and obtained threshold conditions for the extinction and persistence of the infected preys. For the same model, in [90], Silva established the existence of an endemic periodic orbit. The models in [75] and [90] assume that there is no predation on uninfected preys. It is quite true, from the biological point of view, that infected individuals can be less active and caught more easily; another possibility is that the infection modifies the behavior of the preys in such a way that they start living in parts of the habitat which are more accessible to the predators. This motivates the simplified assumption that there is no predation on uninfected prey. Nevertheless, in real life this is not always the way things happen: predators are often able to catch prey, even if they are not carriers of a certain infectious disease. In [41] Ghosh, Sardar, Biswas, Samanta and Chattopadhyay studied a model with predation of uninfected preys. Their model assumes that the uninfected prey population grows according to a logistic law and, in the absence of a specific prey population considered, the predator also follows some logistic law that depends on the availability of alternative preys. Moreover, they assume that the transmission of the disease is given by a bilinear contact rate and the predation of both uninfected and infected preys is given by some kind of Holling-type II functional response, even though in those functions the denominators depend on the uninfected prey or the infected prey instead of depending on the total prey population. The authors provided in that paper a condition for uniform persistence of the disease and conditions for the global asymptotic stability of the positive periodic orbit they found. Based on the model in [75, 90], in [68] Lu, Wang and Liu proposed a family of models that include predation on uninfected preys described by a bilinear functional response and obtained threshold conditions for the extinction and persistence of the infected prey. In their model, the authors

assume that in the absence of disease, the growth rate of the prey population is given by a linear equation, that the predator species vanishes in the absence of the considered prey and that both the transmission of the disease and the predation on infected and uninfected preys is given by Holling-type I functional responses. The extinction result in this work is given by some condition that depends on a bound for the size of an absorbing region. The model studied in [**35**] has a very different structure from the models above: it includes the possibility of recovery and a modified Leslie-Gower functional response is used to describe the dynamics between migratory preys and their predators.

Several works are already available in the literature that deal with eco-epidemiological models with disease in the predator [**92, 48, 76, 102, 9**]. Additionally, there are also works devoted to the study of eco-epidemiological models with delay [**83, 69, 74**] as well as models for which optimal control theory is used [**2, 36, 3, 84**].

In the context of periodic models, there is already a theory developed to obtain the basic reproductive number, $\mathcal{R}_0$. In fact, in [**5**], Bacaër and Guernaoui introduced $\mathcal{R}_0$ for periodic epidemiological models, and later on, in [**98**], the definition of $\mathcal{R}_0$ was adapted by Wang and Zhao to the study of periodic patchy models. In the recent article [**38**] Garrione and Rebelo adapted the theory in [**98**] to study persistence and extinction of the predator in general periodic predator-prey models.

In all non-autonomous works cited above, the functional response of the predator to prey is given by some particular function. Also the vital dynamics of predator and prey is usually assumed to follow some particular law. In this thesis we consider a general model that includes a large family of the eco-epidemiological models proposed in the literature. We assume that the parameters are time-dependent and we consider general functions for the predation on infected and uninfected prey and also for the vital dynamics of uninfected prey and predator populations. Namely, we consider the following eco-epidemiological model:

$$
\begin{cases}
S' = G(t, S) - a(t)f(S, I, P)P - \beta(t)SI \\
I' = \beta(t)SI - \eta(t)g(S, I, P)I - c(t)I \\
P' = H(t, P) + \gamma(t)a(t)f(S, I, P)P + \theta(t)\eta(t)g(S, I, P)I
\end{cases}
, \qquad (0.0.2)
$$

where S, I and P correspond, respectively, to the susceptible prey, infected prey and predator, $\beta(t)$ is the incidence rate of the disease, $\eta(t)$ is the predation rate of infected prey, $c(t)$ is the death rate in the infective class, $\gamma(t)$ is the rate converting susceptible prey into predator (biomass transfer), $\theta(t)$ is the rate of converting infected prey into predator, $G(t, S)$ and $H(t, P)$ represent the vital dynamics of the susceptible prey and predator populations, respectively, $a(t)f(S, I, P)$ is the predation of susceptible prey and $\eta(t)g(S, I, P)$ represent the predation of infected prey. It is assumed that only susceptible preys S are capable of reproducing, i.e., the infected prey is removed by death (including natural and disease-related death) or by predation before having the possibility of reproducing.

In Chapter 1 we consider a non-autonomous model where $H(t, x) = h(t, x)x$ for some function $h : (\mathbb{R}_0^+)^2 \to \mathbb{R}$. We assume that the functions $f(S, I, P)$ and $g(S, I, P)$ are locally Lipschitz and nonnegative, and $G(t, S)$ and $H(t, P)$ are locally Lipschitz. The objective of this chapter is to discuss the uniform strong persistence and extinction of the infectives I under some suitable assumptions. Our results on strong persistence and extinction of the infected prey are based on the analysis of systems related to the dynamics of the model in the absence of infected prey. We apply our results to eco-epidemiological models built from predator-prey models existent in the literature. Simulations are made to illustrate the results.

We address in Chapter 2 the existence of periodic orbits for two distinct families of models. For the first one, obtained from the general model (0.0.2) by making $G(t, S) = \Lambda(t) - \mu(t)S$ and $H(t, P) = (r(t) - b(t)P)P$, we used the well known Mawhin's continuation theorem to prove the existence of an endemic periodic orbit. For the second one, obtained from (0.0.2) by making $G(t, S) = \Lambda(t) - \mu(t)S$ and $H(t, P) = \Upsilon(t) - \zeta(t)P$, we obtain a sharp result using a recent strategy that relies on the uniqueness of periodic orbits in the disease-free space.

All the works above are concerned with continuous time models given by systems of differential equations. On the other hand, it is important to consider discrete time models. In fact, a discretization of continuous time models is fundamental to obtain approximations of the solutions of nonlinear systems of differential equations, for instance as in a numerical approach. Additionally, in some situations, at least for

epidemiological model, it has been argued that discrete time models are better to approximate the disease dynamics as they permit arbitrary time-step units [**109, 34**].

The chosen process of discretization can lead to models with a very distinct structure. In [**52**] the authors use a piecewise constant argument method that lead to a model whose right hand side includes an exponential of a function of the several compartments corresponding to the vector field of the original system of differential equations. Other discrete models of this type can be found in [**51, 31**]. In [**52**], the authors studied the stability of equilibrium solutions analitically and then, numerically, they showed that there appears a series of distinct dynamical behaviors (for example, varying the death rate of the infected prey and maintaining the initial value, there appear chaos, Hopf bifurcation, local stability, flip bifurcation, and chaos). In [**51**] the authors studied an efficient method for analyzing the global asymptotic stability for general three dimensional discrete systems based on a piecewise constant argument method. For the discrete time model obtained in [**31**] the authors studied the local asymptotic stability of equilibria, and also used explicit Hopf bifurcation and period-doubling bifurcation criteria to discuss emergence of both type of bifurcations at positive steady-states of their model.

In [**10**], the authors use the standard Euler forward scheme as well as Mickens nonstandard finite difference scheme to obtain two autonomous discrete models. Also in [**31**] Euler's forward scheme is used to obtain a discrete model from a corresponding continuous time model. In [**10**] the authors showed that the solution of non-standard finite difference scheme of the system remains positive for all positive initial values. Fixed points and their local stability properties are shown to be identical to the corresponding notions in the continuous time model, indicating its dynamic consistency. On the other hand, they showed that the dynamics of the Euler model depend on the step size and therefore we have dynamical inconsistency. Solutions in this method may be negative and allow numerical instabilities, leading to chaos. For the model obtained by Euler's forward scheme in [**31**] it is concluded that the positive equilibrium is locally asymptotically stable, and they implemented an explicit criteria for Hopf and flip bifurcations to investigate parametric conditions for existence of both type of bifurcations at positive steady-states.

We note that all the discrete models described above are autonomous models. In contrast, in Chapter 3, we consider a non-autonomous discrete model corresponding to the discrete counterpart of the model (0.0.2) with $G(t, S) = \Lambda(t) - \mu(t)S$ and $H(t, P) = (r(t) - b(t)P)P$, obtained by applying Mickens nonstandard finite difference scheme to the continuous time model. The objective in this chapter is to discuss the uniform strong persistence and extinction of the infectives I for the obtained discrete model.

Finally, in Chapter 4 we consider random perturbations of a general eco-epidemiological model. In the nondeterministic situation there are two main approaches to incorporate randomness by considering stochastic and random perturbations, which, roughly speaking, can be expressed throughout stochastic and random differential equations. For an exposition on this subjects, their comparison and application in this biological context we refer to [**1, 12, 13, 14, 15, 46**] and references therein, where perturbations of epidemiological models where taken into account both in the presence of white noise (via stochastic differential equations) or real noise (through pathwise random differential equations). One of the main concerns on these works is to understand the dynamics of the perturbed model by the presence of a (random) attractor for the system. On the other hand, in [**104, 44, 105, 7**] random perturbations of eco-epidemiological models were considered always made by the insertion of white noise in a deterministic model. In this former references, the authors aimed to prove the stochastic stability and long time behaviour around equilibrium of deterministic model.

In this chapter, we consider random perturbations of the model (0.0.2) with $G(t, S) = \Lambda(t) - \mu S(t)$ e $H(t, P) = -\delta_1 P(t) - \delta_2 P(t)^2$. For this, we introduce random perturbations in the birth rate Λ using a random variable (*real noise*), considering all other parameters to be constant and positive. We prove the existence of a global random attractor, the persistence of susceptible preys and provide conditions for the simultaneous extinction of infective and predators. We also discuss the dynamics of the corresponding random SI model and random predator-prey model. We obtain for this cases a global random attractor, prove the prevalence of susceptible preys and provide conditions for the extinctions of infective or predators.

The results in Chapter 2 were published in [**55**]. The contents of Chapters 1, 3 and 4 are included in preprints [**54**], [**56**] and [**57**], respectively.

CHAPTER 1

Non-Autonomous Eco-Epidemiological Model

In this chapter we consider a generalization of the non-autonomous model in [**75, 68**] by adding general functions corresponding to predation on infected and uninfected preys as well as general functions associated to the vital dynamics of the susceptible prey and predator populations. We obtain persistence and extinction results for the infected prey based on some assumptions on systems related to the dynamics in the absence of infected preys. We apply our results to eco-epidemiological models constructed from several predator-prey models existent in the literature. Some illustrative simulation is undertaken.

The objective of this chapter is to discuss the uniform strong persistence and extinction of the infectives I in system (0.0.2) under some hypothesis to be detailed later. Recall that the infectives are *uniformly strong persistent* in system (0.0.2) if there exist $0 < m_1 < m_2$ such that for every solution $(S(t), I(t), P(t))$ of (0.0.2) with positive initial conditions $S(t_0), I(t_0), P(t_0) > 0$, we have

$$m_1 < \liminf_{t \to \infty} I(t) \leqslant \limsup_{t \to \infty} I(t) < m_2,$$

and we say that the infectives I go to *extinction* in system (0.0.2) if

$$\lim_{t \to \infty} I(t) = 0,$$

for all solutions of (0.0.2) with positive initial conditions.

For biological reasons we will only consider for system (0.0.2) solutions with initial conditions in the set $(\mathbb{R}^+)^3$.

Our approach is very different to the one in [**75**] and [**68**]. In fact, we want to discuss the extinction and strong persistence of the infectives in system (0.0.2), having as departure point some prescribed behavior of the uninfected subsystem corresponding to the dynamics of preys and predators in the absence of disease. We will assume in the major part of this work, more specifically in section 1.1,

that we have global asymptotic stability of solutions of some special bi-dimensional systems related to the predator-prey dynamics in the uninfected subspace (the model obtained by letting $I = 0$ in the first and third equations in (0.0.2)) and that we will refer to as *uninfected subsystem* (see condition N6) in section 1.1). Thus, to apply our results to specific situations in the literature, one must first verify that the underlying uninfected subsystem satisfies our assumptions or, looking at our results differently, we can construct an eco-epidemiological model from a previously studied predator-prey model (the uninfected subsystem) that satisfies our assumptions. We believe that this approach is interesting since it highlights the relation of the dynamics of the eco-epidemiological model with the behavior of the predator-prey model used in its construction.

We note that, similarly to the thresholds obtained in [**68**], our thresholds for extinction and uniform strong persistence are not sharp. In spite of this, unlike the conditions for extinction and strong persistence in [**68**], that rely on parameters that can not, in principle, be computed explicitly (note that conditions (22) and (43) in [**68**] depend on q_1), our thresholds can be directly obtained from the parameters and the limit behavior of the predator-(uninfected) prey subsystem.

To illustrate our findings, in section 1.3 several predator-prey models available in the literature, satisfying our assumptions, are considered and thresholds conditions for the corresponding eco-epidemiological model automatically obtained from our results: in our Example 1, we consider the situation where $f \equiv 0$ in system (0.0.2), corresponding to a generalized version of the situation studied in [**75**]; in Example 2, we obtain a particular form for the threshold conditions in the context of periodic models and particularize our result for a model constructed from the predator-prey model in [**40**]; in Example 3, we start with an uninfected subsystem with Gause-type interaction (a predator-prey model with Holling type II functional response of predator to prey, logistic growth of prey in the absence of predators and exponential extinction of predator in the absence of prey) and, using [**62**], obtain the corresponding results for extinction in the eco-epidemiological model; in Example 4, we consider the eco-epidemiological model obtained from an uninfected subsystem with ratio-dependent functional response of predator to prey, a type interaction considered as an attempt to overcome some know biological paradoxes observed in models

with Gause-type interaction and again obtain the corresponding results for the eco-epidemiological model, based on the discussion of ratio-dependent predator-prey systems in [**50**]; finally, in Examples 5 and 6, we consider eco-epidemiological models, based on the discussion of the corresponding predator-prey models in [**80, 91**] where the uninfected subsystem has some specific type of non-autonomy in the prey equation (Example 5) or the predator equation (Example 6). For all these examples we present some simulation that corroborate our conclusions.

For Examples 3 to 6 we provide computational experiments that suggests that we have persistence of infected preys when the threshold is given by some function depending on solutions of the uninfected subsystem instead of solutions of the subsystems considered in our theoretical results.

1.1. Asymptotically stable behavior in uninfected subspace

In this chapter we consider the model (0.0.2) with $H(t, x) = h(t, x)x$ for some function $h : (\mathbb{R}_0^+)^2 \to \mathbb{R}$:

$$
\begin{cases}
S' = G(t, S) - a(t)f(S, I, P)P - \beta(t)SI \\
I' = \beta(t)SI - \eta(t)g(S, I, P)I - c(t)I \\
P' = h(t, P)P + \gamma(t)a(t)f(S, I, P)P + \theta(t)\eta(t)g(S, I, P)I
\end{cases}
. \qquad (1.1.1)
$$

We will assume the following hypothesis concerning the parameter functions and the functions f, g, G and h:

N1) The real valued functions a, β, η, c, γ and θ are bounded, nonnegative and continuous;

N2) The real valued functions f, g, G and $H(t, x) = h(t, x)x$ are locally Lipschitz, functions f and g are nonnegative and $f(0, 0, z) = 0$, for every $z \geqslant 0$. For fixed $x, z \geqslant 0$, functions $y \mapsto f(x, y, z)$ and $y \mapsto g(x, y, z)$ are nonincreasing. For fixed $y, z \geqslant 0$, function $x \mapsto g(x, y, z)$ is nonincreasing; for fixed $x, y \geqslant 0$, function $z \mapsto f(x, y, z)$ is nonincreasing and function $z \mapsto g(x, y, z)$ is nondecreasing;

Our next assumption relates to the ω-limit of solutions of (1.1.1) and is usually fulfilled by mathematical models in epidemiology.

N3) Each solution of (1.1.1) with positive initial condition is bounded and there is a bounded region $\mathcal{R}$ that contains the ω-limit of all solutions of (1.1.1) with positive initial conditions.

Notice in particular that condition N3) implies that there is $L > 0$ such that

$$\limsup_{t \to +\infty} (S(t) + I(t) + P(t)) < L,$$

for all solutions $(S(t), I(t), P(t))$ of (1.1.1) with positive initial conditions.

To proceed, we need to consider two auxiliary equations and one auxiliary system. First, we consider the equation

$$s' = G(t, s), \tag{1.1.2}$$

that corresponds to the dynamics of uninfected preys in the absence of infected preys and predators (the first equation in system (1.1.1) with $I = 0$, $S = s$ and $P = 0$). We assume the following properties for the solutions of (1.1.2):

N4) Each solution $s(t)$ of (1.1.2) with positive initial condition is bounded, bounded away from zero, and globally attractive on $]0, +\infty[$, that is $|s(t) - v(t)| \to 0$ as $t \to +\infty$ for each solution $v(t)$ of (1.1.2) with positive initial condition.

The second auxiliary equation we consider is the equation

$$y' = h(t, y)y, \tag{1.1.3}$$

that corresponds to the dynamics of predators in the absence of the considered preys (the third equation in system (1.1.1) with $I = 0$, $S = 0$ and $P = y$). We need the following property for the solutions of (1.1.3):

N5) Each fixed solution $y(t)$ of (1.1.3) with positive initial condition is bounded and globally attractive on $[0, +\infty)$.

Finally, we need the following auxiliary system

$$\begin{cases} x' = G(t, x) - a(t)f(x, 0, z)z \\ z' = h(t, z)z + \gamma(t)a(t)f(x, 0, z)z \end{cases} \tag{1.1.4}$$

that describes the behavior of preys and predators in the absence of infected preys (the first and third equations of system (1.1.1) with $I = 0$, $S = x$ and $P = z$). We

will refer to system (1.1.4) as the *uninfected subsystem*. We assume that we are able to construct families of auxiliary subsystems:

$$\begin{cases} x' = G_{1,\varepsilon}(t,x) - a(t)f(x,0,0)\hat{z}_\varepsilon(t) - v(\varepsilon)\rho(t)x \\ z' = h_{1,\varepsilon}(t,z)z + \gamma(t)a(t)f(x,v(\varepsilon)\rho^u,z)z \end{cases} \tag{1.1.5}$$

where $(\hat{x}_\varepsilon(t),\hat{z}_\varepsilon(t))$ is a solution of

$$\begin{cases} x' = G_{2,\varepsilon}(t,x) \\ z' = h_{2,\varepsilon}(t,z)z + \gamma(t)a(t)f(x,0,z)z + v(\varepsilon)\rho(t)g(x,0,z) \end{cases} \tag{1.1.6}$$

satisfying the following assumptions.

N6) The following holds for systems (1.1.5) and (1.1.6):

 N6.1) for sufficiently small $\varepsilon > 0$, the functions $G_{i,\varepsilon}$ and $h_{i,\varepsilon}$, $i = 1,2$, are continuous, the functionals $\varepsilon \mapsto G_{i,\varepsilon}$ and $\varepsilon \mapsto h_{i,\varepsilon}$, $i = 1,2$, are continuous, $G_{1,0} = G_{2,0} = G$, $h_{1,0} = h_{2,0} = h$,

$$G_{1,\varepsilon}(t,x) \leqslant G(t,x) \leqslant G_{2,\varepsilon}(t,x)$$

and

$$h_{1,\varepsilon}(t,x) \leqslant h(t,x) \leqslant h_{2,\varepsilon}(t,x);$$

 N6.2) the real valued function $v : [0,+\infty[\to \mathbb{R}$ verifies $v(\varepsilon) > 0$ for $\varepsilon \in]0,+\infty[$, $v(0) = 0$ and is differentiable near $\varepsilon = 0$ with

$$A < v'(\varepsilon) < B,$$

for some $A,B > 0$ and sufficiently small $\varepsilon \geqslant 0$;

 N6.3) the function ρ is continuous and there are constants ρ^u,ρ^ℓ such that, for all $t \geqslant 0$,

$$0 < \rho^\ell \leqslant \rho(t) \leqslant \rho^u;$$

 N6.4) there is a family of nonnegative solutions, $\{(x_{1,\varepsilon}^*(t),\, z_{1,\varepsilon}^*(t))\}$ of system (1.1.5), one solution for each $\varepsilon \geqslant 0$ sufficiently small, depending on a solution $(x_{2,\varepsilon}^*(t), z_{2,\varepsilon}^*(t))$ of system (1.1.6), such that each solution in the family is globally asymptotically stable in a set containing the set $\{(x,z) \in (\mathbb{R}_0^+)^2 :$

$x > 0 \ \wedge \ z > 0\}$ and the function

$$\varepsilon \mapsto (x_{1,\varepsilon}^*(t), z_{1,\varepsilon}^*(t)) \quad \text{is continuous;}$$

N6.5) the family of nonnegative solutions $\{(x_{2,\varepsilon}^*(t), z_{2,\varepsilon}^*(t))\}$ of system (1.1.6), one solution for each $\varepsilon \geqslant 0$ sufficiently small, is such that each solution in the family is globally asymptotically stable in a set containing the set $\{(x, z) \in (\mathbb{R}_0^+)^2 : x > 0 \ \wedge \ z > 0\}$ and the function

$$\varepsilon \mapsto (x_{2,\varepsilon}^*(t), z_{2,\varepsilon}^*(t)) \quad \text{is continuous.}$$

We write $x_{1,0}^* = x_1^*$, $x_{2,0}^* = x_2^*$, $z_{1,0}^* = z_1^*$ and $z_{2,0}^* = z_2^*$ for the components of the solutions in N6.4) and N6.5) corresponding to $\varepsilon = 0$. For the continuity of the functionals in N6.1), N6.4) and N6.5) we consider the usual supremum norm, $\|\cdot\|_0$. Also notice that, by N3) the solutions are bounded. Note that we only aim to control two suitable families of perturbations of the uninfected subsystem, so that condition N6) is sufficiently flexible to adapt to a wide range of uninfected subsystems associated to the eco-epidemiological models.

We emphasize that our setting includes several of the most common functional response functions: $f(S, I, P) = kS$ and $g(S, I, P) = kP$ (Holling-type I), $f(S, I, P) = kS/(1+m(S+I))$ and $g(S, I, P) = kP/(1+m(S+I))$ (Holling-type II), $f(S, I, P) = kS^\alpha/(1+m(S+I)^\alpha)$ and $g(S, I, P) = kP^\alpha/(1+m(S+I)^\alpha)$ with $\alpha > 0$ (Holling-type III), $f(S, I, P) = kS/(a + b(S + I) + cP)$ and $g(S, I, P) = kP/(a + b(S + I) + cP)$ (Beddington-De Angelis), $f(S, I, P) = kS/(a + b(S + I) + cP + d(S + I)P)$ and $g(S, I, P) = kP/(a + b(S + I) + cP + d(S + I)P)$ (Crowley-Martin). In the functions above $k, m, a, b, c, d > 0$. Notice that while g can be of the form $g(S, I, P) = kP/(a + b(S + I) + c(S + I)^2)$ (Holling-type IV), this type of functional response corresponds in the case of f to take $f(S, I, P) = kS/(a+b(S+I)+c(S+I)^2)$. In this case $x \mapsto kx/(a + b(x + y) + c(x + y)^2)$ is not nondecreasing when $c \neq 0$ and cannot take f such that the corresponding functional response function is Holling-type IV.

1.2. Extinction and uniform strong persistence of infectives

In this section we will establish our results on the extinction and uniform strong persistence of the infective prey in system (1.1.1). Given a function f we will use

the notations $f^\ell = \inf_{t \geqslant 0} f(t)$, $f^u = \sup_{t \geqslant 0} f(t)$ and, for a ω-periodic function f we use the notation $\bar{f} = (1/\omega) \int_0^\omega f(s)\, ds$.

We define

$$\mathcal{R}^\ell(\lambda) = \liminf_{t \to +\infty} \int_t^{t+\lambda} \beta(s)x_1^*(s) - \eta(s)g(x_1^*(s), 0, z_2^*(s)) - c(s)\, ds \qquad (1.2.1)$$

where we still denote by $x_1^*(t)$ and $z_2^*(t)$ the components of solutions in systems (1.1.5) and (1.1.6), with $\varepsilon = 0$, and

$$\mathcal{R}^u(\lambda) = \limsup_{t \to +\infty} \int_t^{t+\lambda} \beta(s)s^*(s) - \eta(s)g(s^*(s), 0, y^*(s)) - c(s)\, ds. \qquad (1.2.2)$$

where $s^*(t)$ and $y^*(t)$ are particular solutions, respectively, of (1.1.2) and (1.1.3) with positive initial conditions.

As we will see in the following, using the global attractivity of solutions of (1.1.2) and (1.1.3) in $]0, +\infty[$ and the global attractivity of solutions given at N6.4) and N6.5) we can easily conclude that (1.2.1) is independent of the particular solutions considered in N6.4) and N6.5). Similarly, it is easy to conclude that (1.2.2) is independent of the particular solutions of (1.1.2) and (1.1.3) with positive initial conditions considered.

PROPOSITION 1.1. *The numbers* (1.2.1) *and* (1.2.2) *are independent, respectively, of the particular solutions considered in N6.4) and N6.5) and of the particular solutions of* (1.1.2) *and* (1.1.3) *with positive initial conditions chosen.*

PROOF. Let $(x_1^*(t), z_1^*(t))$, $(x_2^*(t), z_2^*(t))$ and $(\bar{x}_1^*(t), \bar{z}_1^*(t))$, $(\bar{x}_2^*(t), \bar{z}_2^*(t))$ be two distinct pairs of nonnegative solutions of (1.1.5) and (1.1.6) as in N6.4) and N6.5). Let $\delta > 0$. By N6), for $t \geq T_\delta$ sufficiently large, we have

$$x_1^*(t) - \delta \leqslant \bar{x}_1^*(t) \leqslant x_1^*(t) + \delta \quad \text{and} \quad z_2^*(t) - \delta \leqslant \bar{z}_2^*(t) \leqslant z_2^*(t) + \delta.$$

Addicionally, by N1) and N2) we have, for every $t \geq T_\delta$,

$$\left| \int_t^{t+\lambda} \beta(s)x_1^*(s) - \eta(s)g(x_1^*(s), 0, z_2^*(s)) - c(s)\, ds - \int_t^{t+\lambda} \beta(s)\bar{x}_1^*(s) - \eta(s)g(\bar{x}_1^*(s), 0, \bar{z}_2^*(s)) - c(s)\, ds \right|$$

$$\leqslant \int_t^{t+\lambda} \beta(s)\, |x_1^*(s) - \bar{x}_1^*(s)| + \eta(s)\, |g(x_1^*(s), 0, z_2^*(s)) - g(\bar{x}_1^*(s), 0, \bar{z}_2^*(s))|\, ds$$

$$\leqslant \lambda\beta^u\delta + 2\lambda\eta^u\varphi(\delta),$$

with $\varphi(\delta) \to 0$ as $\delta \to 0$. We conclude that, for every $\delta > 0$,

$$\liminf_{t\to+\infty} \int_t^{t+\lambda} \beta(s)x_1^*(s) - \eta(s)g(x_1^*(s), 0, z_2^*(s)) - c(s)\,ds - \lambda\beta^u\delta - 2\lambda\eta^u\varphi(\delta)$$

$$\leqslant \liminf_{t\to+\infty} \int_t^{t+\lambda} \beta(s)\bar{x}_1^*(s) - \eta(s)g(\bar{x}_1^*(s), 0, \bar{z}_2^*(s)) - c(s)\,ds$$

$$\leqslant \liminf_{t\to+\infty} \int_t^{t+\lambda} \beta(s)x_1^*(s) - \eta(s)g(x_1^*(s), 0, z_2^*(s)) - c(s)\,ds + \lambda\beta^u\delta + 2\lambda\eta^u\varphi(\delta),$$

Thus $\mathcal{R}^\ell(\lambda)$ is independent of the chosen solution. Taking, respectively, $\limsup$, $s^*(t)$ and $y^*(t)$ instead of $\liminf$, $x_1^*(t)$ and $z_2^*(t)$ and using the same reasoning we can prove that $\mathcal{R}^u(\lambda)$ is also independent of the particular solutions chosen. The result follows. $\qquad\square$

THEOREM 1.1. *Assume that conditions N1) to N5) hold. Assume further that either $G(t,S) = \Lambda(t) - \mu(t)S$ and $g(S+I, 0, P) \leqslant g(S, I, P)$ or g does not depend on I. If there is $\lambda > 0$ such that $\mathcal{R}^u(\lambda) < 0$, then the infectives in system (1.1.1) go to extinction.*

PROOF. Assume that there is $\lambda > 0$ such that $\mathcal{R}^u(\lambda) < 0$ and let $s^*(t)$ and $y^*(t)$ be particular solutions, respectively, of (1.1.2) and (1.1.3) with positive initial conditions. Since functions β and η are bounded, there are $\kappa > 0$, $t_0 > 0$ and $\varepsilon_0 > 0$ such that, for $t \geqslant t_0$ and $\delta \in\,]0, \varepsilon_0]$, we have

$$\int_t^{t+\lambda} \beta(s)(s^*(s) + \delta) - \eta(s)g(s^*(s) + \delta, 0, y^*(s) - \delta) - c(s)\,ds \leqslant -\kappa < 0. \quad (1.2.3)$$

Let $(S(t), I(t), P(t))$ be a solution of (1.1.1) with positive initial conditions. We will prove first that

$$\liminf_{t\to+\infty} I(t) = 0. \tag{1.2.4}$$

Assume that (1.2.4) does not hold. Then, there is $\varepsilon > 0$ such that $I(t) > \varepsilon$ for all sufficiently large t. By the first equation of (1.1.1) we have

$$S' \leqslant G(t, S) \tag{1.2.5}$$

and thus $S(t) \leqslant s(t)$, where $s(t)$ is the solution of (1.1.2) with $s(t_0) = S(t_0)$. By condition N4), given $\varepsilon \in\,]0, \varepsilon_0]$, we have $S(t) \leqslant s^*(t) + \varepsilon$, for all sufficiently large t.

By the third equation of (1.1.1), we have

$$P' \geqslant h(t, P)P \tag{1.2.6}$$

and thus $P(t) \geqslant y(t)$, where $y(t)$ is the solution of (1.1.3) with $y(t_0) = P(t_0)$. By condition N5), given $\varepsilon \in]0, \varepsilon_0]$, we have $P(t) \geqslant y^*(t) - \varepsilon$, for all sufficiently large t.

When $G(t, S) = \Lambda(t) - \mu(t)S$,

$$(S + I)' \leq \Lambda(t) - \mu(t)S - c(t)I \leq \Lambda(t) - \mu(t)(S + I),$$

and consequently, for sufficiently large t

$$S(t) + I(t) \leq s^*(t) + \varepsilon.$$

Under this assumption on G, by the second equation of (1.1.1), since we assumed that $g(S + I, 0, P) \leqslant g(S, I, P)$, we have

$$I' \leqslant [\beta(t)(s^*(t) + \varepsilon) - \eta(t)g(s^*(t) + \varepsilon, 0, y^*(t) - \varepsilon) - c(t)]I,$$

for all sufficiently large t. Notice that, for a general G, if g does not depend on I we have $g(S, I, P) \geq g(s^*(t) + \varepsilon, 0, y^*(t) - \varepsilon)$ and we still obtain the inequality above. Denoting by $\lfloor \alpha \rfloor$ the integer part of α and integrating the previous equation, we get

$$I(t) \leqslant I(t_0) \exp\left\{ \int_{t_0}^{t} \beta(r)(s^*(r) + \varepsilon) - \eta(r)g(s^*(r) + \varepsilon, 0, y^*(r) - \varepsilon) - c(r)\, dr \right\}$$

$$\leqslant I(t_0)\, e^{\lambda(\beta^u(s^*)^u + \varepsilon\beta^u)}$$

$$\times \exp\left\{ \int_{t_0}^{t_0 + \lfloor \frac{t - t_0}{\lambda} \rfloor \lambda} \beta(r)(s^*(r) + \varepsilon) - \eta(r)g(s^*(r) + \varepsilon, 0, y^*(r) - \varepsilon) - c(r)\, dr \right\}$$

$$\leqslant I(t_0)\, e^{-\lfloor (t - t_0)/\lambda \rfloor \kappa}\, e^{\lambda(\beta^u(s^*)^u + \varepsilon\beta^u)},$$

for all sufficiently large t. Since $\lfloor (t - t_0)/\lambda \rfloor \kappa \to +\infty$ as $t \to +\infty$, we get a contradiction to the hypothesis that there is $\varepsilon > 0$ such that $I(t) > \varepsilon$ for sufficiently large t. We conclude that (1.2.4) holds.

Let $\varepsilon > 0$. Next we will prove that for sufficiently large t

$$I(t) \leqslant \varepsilon\, e^{h\lambda}, \tag{1.2.7}$$

where

$$h = \sup_{t \geqslant 0} |\beta(t)(s^*(t) + \varepsilon_0) - \eta(t)g(s^*(t) + \varepsilon_0, 0, y^*(t) - \varepsilon_0) - c(t)|.$$

By (1.2.4), there exists $t_1 \geqslant t_0$ such that $I(t_1) < \varepsilon$.

Assume, by contradiction that (1.2.7) does not hold. Then, there is $t_2 > t_1$ such that $I(t_2) > \varepsilon\, e^{h\lambda}$. Since $I(t_1) < \varepsilon$, there is $t_3 \in\,]t_1, t_2[$ such that $I(t_3) = \varepsilon$ and $I(t) > \varepsilon$, for all $t \in\,]t_3, t_2[$. Integrating we get, by (1.2.3),

$$\varepsilon\, e^{h\lambda} < I(t_2)$$

$$\leqslant I(t_3) \exp\left\{ \int_{t_3}^{t_2} \beta(r)(s^*(r) + \varepsilon) - \eta(r)g(s^*(r) + \varepsilon, 0, y^*(r) - \varepsilon) - c(r)\, dr \right\}$$

$$\leqslant \varepsilon \exp\left\{ \int_{t_3 + \lfloor (t_2 - t_3)/\lambda \rfloor \lambda}^{t_2} \beta(r)(s^*(r) + \varepsilon_0) - \eta(r)g(s^*(r) + \varepsilon_0, 0, y^*(r) - \varepsilon_0) - c(r)\, dr \right\}$$

$$\leqslant \varepsilon\, e^{h\lambda},$$

witch is a contradiction. Thus, we conclude that (1.2.7) holds and, since $\varepsilon \in\,]0, \varepsilon_0]$ is arbitrary, we conclude that $I(t) \to 0$ as $t \to 0$, as claimed. $\qquad\square$

THEOREM 1.2. *Assume that conditions N1) to N3) and N6) hold. If there is $\lambda > 0$ such that $\mathcal{R}^\ell(\lambda) > 0$, then the infectives in system (1.1.1) are uniformly strong persistent.*

PROOF. Assume that there is $\lambda > 0$ such that $\mathcal{R}^\ell(\lambda) > 0$ and let us fix particular families of solutions of systems (1.1.5) and (1.1.6), respectively $(x_{1,\varepsilon}^*(t), z_{1,\varepsilon}^*(t))$ and $(x_{2,\varepsilon}^*(t), z_{2,\varepsilon}^*(t))$, with positive initial conditions and satisfying N6.4) and N6.5). Then, we can choose $t_0 > 0$, $\kappa > 0$ and $\varepsilon_0 > 0$ such that, for $t \geqslant t_0$ and $\delta \in [0, \varepsilon_0]$ we have

$$\int_t^{t+\lambda} \beta(s)(x_1^*(s) - \delta) - \eta(s)g(x_1^*(s) - \delta, \delta, z_2^*(s) + \delta) - c(s)\, ds \geqslant \kappa > 0. \quad (1.2.8)$$

Let $(S(t), I(t), P(t))$ be a solution of (1.1.1) with positive initial conditions. We will prove first that there is $\varepsilon > 0$ such that

$$\limsup_{t \to +\infty} I(t) \geqslant \frac{v(\varepsilon)\rho^\ell}{(1 + \beta^u)(1 + \theta^u \eta^u)} > 0. \quad (1.2.9)$$

Assume that for all sufficiently small $\varepsilon > 0$

$$\limsup_{t \to +\infty} I(t) < \frac{v(\varepsilon)\rho^\ell}{(1 + \beta^u)(1 + \theta^u \eta^u)}.$$

Then, we conclude that there is $t_1 > t_0$, such that

$$I(t) < \frac{v(\varepsilon)\rho^\ell}{(1 + \beta^u)(1 + \theta^u \eta^u)} < v(\varepsilon)\rho(t), \quad (1.2.10)$$

for each $t \geqslant t_1$. By the first and third equations of (1.1.1) and the inequalities in N6.1) we have

$$
\begin{cases}
S' \leqslant G_{2,\varepsilon}(t, S) \\
P' \leqslant h_{2,\varepsilon}(t, P)P + \gamma(t)a(t)f(S, 0, P)P + v(\varepsilon)\rho(t)\theta(t)\eta(t)g(S, 0, P)
\end{cases}
.
$$

Let $(\hat{x}_\varepsilon(t), \hat{z}_\varepsilon(t))$ be the solution of

$$
\begin{cases}
x' = G_{2,\varepsilon}(t, x) \\
z' = h_{2,\varepsilon}(t, z)z + \gamma(t)a(t)f(x, 0, z)z + v(\varepsilon)\rho(t)\theta(t)\eta(t)g(x, 0, z)
\end{cases}
$$

with $\hat{x}_\varepsilon(t_1) = S(t_1)$ and $\hat{z}_\varepsilon(t_1) = P(t_1)$. We have $S(t) \leqslant \hat{x}_\varepsilon(t)$ and $P(t) \leqslant \hat{z}_\varepsilon(t)$ for $t \geqslant t_1$. By the global stability assumption in N6.5), we have

$$
\left| x_{2,\varepsilon}^*(t) - \hat{x}_\varepsilon(t) \right| \to 0 \quad \text{and} \quad \left| z_{2,\varepsilon}^*(t) - \hat{z}_\varepsilon(t) \right| \to 0, \quad \text{as } t \to +\infty
$$

and, by continuity, again according to N6.5), we have for sufficiently large t

$$
\begin{aligned}
\left| x_2^*(t) - \hat{x}_\varepsilon(t) \right| &\leqslant \left| x_2^*(t) - x_{2,\varepsilon}^*(t) \right| + \left| x_{2,\varepsilon}^*(t) - \hat{x}_\varepsilon(t) \right| \\
&\leqslant \| x_2^* - x_{2,\varepsilon}^* \|_0 + \left| x_{2,\varepsilon}^*(t) - \hat{x}_\varepsilon(t) \right| \\
&\leqslant \varphi_1(\varepsilon),
\end{aligned}
$$

and

$$
\begin{aligned}
\left| z_2^*(t) - \hat{z}_\varepsilon(t) \right| &\leqslant \left| z_2^*(t) - z_{2,\varepsilon}^*(t) \right| + \left| z_{2,\varepsilon}^*(t) - \hat{z}_\varepsilon(t) \right| \\
&\leqslant \| z_2^* - z_{2,\varepsilon}^* \|_0 + \left| z_{2,\varepsilon}^*(t) - \hat{z}_\varepsilon(t) \right| \\
&\leqslant \varphi_2(\varepsilon),
\end{aligned}
$$

with $\varphi_1(\varepsilon), \varphi_2(\varepsilon) \to 0$ as $\varepsilon \to 0$. In particular, for sufficiently large t,

$$
S(t) \leqslant \hat{x}_\varepsilon(t) \leqslant \varphi_1(\varepsilon) + x_2^*(t) \quad \text{and} \quad P(t) \leqslant \hat{z}_\varepsilon(t) \leqslant \varphi_2(\varepsilon) + z_2^*(t). \tag{1.2.11}
$$

On the other hand, by (1.2.10) and by the first and the third equations of (1.1.1), we have

$$
\begin{cases}
S' \geqslant G_{1,\varepsilon}(t, S) - a(t)f(S, 0, 0)\hat{z}_\varepsilon(t) - v(\varepsilon)\rho(t)S \\
P' \geqslant h_{1,\varepsilon}(t, P)P + \gamma(t)a(t)f(S, v(\varepsilon)\rho^u, P)P
\end{cases}
$$

Letting $(\tilde{x}_\varepsilon(t), \tilde{z}_\varepsilon(t))$ be the solution of

$$\begin{cases} x' = G_{1,\varepsilon}(t,x) - a(t)f(x,0,0)\hat{z}_\varepsilon(t) - v(\varepsilon)\rho(t)x \\ z' = h_{1,\varepsilon}(t,z)z + \gamma(t)a(t)f(x,v(\varepsilon)\rho^u,z)z \end{cases}$$

with $\tilde{x}_\varepsilon(t_1) = S(t_1)$ and $\tilde{z}_\varepsilon(t_1) = P(t_1)$, we have $S(t) \geqslant \tilde{x}_\varepsilon(t)$ and $P(t) \geqslant \tilde{z}_\varepsilon(t)$, for all $t \geqslant t_1$. By the global stability assumption in N6.4), we have

$$\left| x^*_{1,\varepsilon}(t) - \tilde{x}_\varepsilon(t) \right| \to 0 \quad \text{and} \quad \left| z^*_{1,\varepsilon}(t) - \tilde{z}_\varepsilon(t) \right| \to 0, \quad \text{as } t \to +\infty.$$

and, by the continuity property in N6.4), for sufficiently large t, we have

$$\left| x^*_1(t) - \tilde{x}_\varepsilon(t) \right| \leqslant \left| x^*_1(t) - x^*_{1,\varepsilon}(t) \right| + \left| x^*_{1,\varepsilon}(t) - \tilde{x}_\varepsilon(t) \right|$$

$$\leqslant \| x^*_1 - x^*_{1,\varepsilon} \|_0 + \left| x^*_{1,\varepsilon}(t) - \tilde{x}_\varepsilon(t) \right|$$

$$\leqslant \psi_1(\varepsilon),$$

and

$$\left| z^*_1(t) - \tilde{z}_\varepsilon(t) \right| \leqslant \left| z^*_1(t) - z^*_{1,\varepsilon}(t) \right| + \left| z^*_{1,\varepsilon}(t) - \tilde{z}_\varepsilon(t) \right|$$

$$\leqslant \| z^*_1 - z^*_{1,\varepsilon} \|_0 + \left| z^*_{1,\varepsilon}(t) - \tilde{z}_\varepsilon(t) \right|$$

$$\leqslant \psi_2(\varepsilon),$$

with $\psi_1(\varepsilon), \psi_2(\varepsilon) \to 0$ as $\varepsilon \to 0$. In particular, for sufficiently large t,

$$S(t) \geqslant \tilde{x}_\varepsilon(t) \geqslant x^*_1(t) - \psi_1(\varepsilon) \quad \text{and} \quad P(t) \geqslant \tilde{z}_\varepsilon(t) \geqslant z^*_1(t) - \psi_2(\varepsilon). \qquad (1.2.12)$$

By the second equation in (1.1.1), (1.2.8), (1.2.11) and (1.2.12) we get, for $t \geqslant t_1$,

$$\int_t^{t+\lambda} \beta(s)S(s) - \eta(s)g(S(s), I(s), P(s)) - c(s)\, ds$$

$$\geqslant \int_t^{t+\lambda} \beta(s)(x^*_1(s) - \psi_1(\varepsilon)) - \eta(s)g(x^*_1(s) - \psi_1(\varepsilon), 0, z^*_2(s) + \varphi_2(\varepsilon)) - c(s)\, ds \geqslant \kappa.$$

Thus, choosing $\varepsilon > 0$ such that $\max\{\varphi_2(\varepsilon), \psi_1(\varepsilon), v(\varepsilon)\rho^u\} < \varepsilon_0$, we have

$$I(t) = I(t_1) \exp\left\{\int_{t_1}^t \beta(s)S(s) - \eta(s)g(S(s), I(s), P(s)) - c(s)\,ds\right\}$$

$$\geqslant I(t_1) \exp\left\{\int_{t_1}^t \beta(s)(x_1^*(s) - \psi_1(\varepsilon))ds\right\}$$

$$\times \exp\left\{\int_{t_1}^t -\eta(s)g(x_1^*(s) - \psi_1(\varepsilon), 0, z_2^*(s) + \varphi_2(\varepsilon)) - c(s)\,ds\right\}$$

$$\geqslant I(t_1)\,e^{-\lambda(\beta^u \psi_1(\varepsilon) + \eta^u g((x_1^*)^u - \psi_1(\varepsilon), 0, (z_2^*)^\ell + \varphi_2(\varepsilon)) + c^u)}$$

$$\times \exp\left\{\int_{t_1}^{t_1 + \lfloor(t-t_1)/\lambda\rfloor\lambda} \beta(s)(x_1^*(s) - \psi_1(\varepsilon))ds\right\}$$

$$\times \exp\left\{\int_{t_1}^{t_1 + \lfloor(t-t_1)/\lambda\rfloor\lambda} -\eta(s)g(x_1^*(s) - \psi_1(\varepsilon), 0, z_2^*(s) + \varphi_2(\varepsilon)) - c(s)\,ds\right\}$$

$$\geqslant I(t_1)\,e^{\lfloor(t-t_1)/\lambda\rfloor\kappa}\,e^{-\lambda(\beta^u \psi_1(\varepsilon) + \eta^u g((x_1^*)^u - \psi_1(\varepsilon), 0, (x_1^*)^\ell + \varphi_2(\varepsilon)) + c^u)},$$

a contradiction to the fact that, according to N3), $I(t)$ is bounded. We conclude that (1.2.9) holds.

Next we will prove that there is $m_1 > 0$ such that for any solution $(S(t), I(t), P(t))$ with positive initial condition,

$$\liminf_{t \to +\infty} I(t) > m_1. \tag{1.2.13}$$

Assume that (1.2.13) does not hold. Then, given $\varepsilon \in]0, \varepsilon_0[$, there exists a sequence of initial values $(x_n)_{n \in \mathbb{N}}$, with $x_n = (S_n, I_n, P_n)$ and $S_n > 0$, $I_n > 0$ and $P_n > 0$ such that

$$\liminf_{t \to +\infty} I(t, x_n) < \frac{\rho^u v(\varepsilon/n^2)}{(1 + \theta^u \eta^u)(1 + \beta^u)}, \tag{1.2.14}$$

where $(S(t, x_n), I(t, x_n), P(t, x_n))$ denotes the solution of (1.1.1) with initial conditions $S(0) = S_n$, $I(0) = I_n$, and $P(0) = P_n$. By (1.2.9), given $n \in \mathbb{N}$, there are two sequences $(t_{n,k})_{k \in \mathbb{N}}$ and $(s_{n,k})_{k \in \mathbb{N}}$ with

$$s_{n,1} < t_{n,1} < s_{n,2} < t_{n,2} < \cdots < s_{n,k} < t_{n,k} < \cdots$$

and $\lim_{k \to +\infty} s_{n,k} = +\infty$, such that

$$I(s_{n,k}, x_n) = \frac{\rho^\ell v(\varepsilon/n)}{(1 + \theta^u \eta^u)(1 + \beta^u)}, \quad I(t_{n,k}, x_n) = \frac{\rho^u v(\varepsilon/n^2)}{(1 + \theta^u \eta^u)(1 + \beta^u)} \tag{1.2.15}$$

and, for all $t \in]s_{n,k}, t_{n,k}[$,

$$\frac{\rho^u v(\varepsilon/n^2)}{(1 + \theta^u \eta^u)(1 + \beta^u)} < I(t, x_n) < \frac{\rho^\ell v(\varepsilon/n)}{(1 + \theta^u \eta^u)(1 + \beta^u)}. \tag{1.2.16}$$

By the second equation in (1.1.1) and N3), for sufficiently large t, we have

$$I'(t, x_n) = [\beta(t)S(t, x_n) - \eta(t)g(S(t, x_n), I(t, x_n), P(t, x_n)) - c(t)]\, I(t, x_n)$$

$$\geqslant -(\eta^u g(0, 0, L) + c^u)I(t, x_n).$$

Therefore we obtain

$$\int_{s_{n,k}}^{t_{n,k}} \frac{I'(r, x_n)}{I(r, x_n)} \, dr \geqslant -(\eta^u g(0, 0, L) + c^u)(t_{n,k} - s_{n,k})$$

and thus $I(t_{n,k}, x_n) \geqslant I(s_{n,k}, x_n)\, \mathrm{e}^{-(\eta^u g(0,0,L)+c^u)(t_{n,k}-s_{n,k})}$. By (1.2.15), and N6.3) we get

$$\frac{\rho^u v(\varepsilon/n^2)}{\rho^\ell v(\varepsilon/n)} \geqslant \frac{\rho(t_{n,k})v(\varepsilon/n^2)}{\rho(s_{n,k})v(\varepsilon/n)} \geqslant \mathrm{e}^{-(\eta^u g(0,0,L)+c^u)(t_{n,k}-s_{n,k})}$$

and therefore we have

$$t_{n,k} - s_{n,k} \geqslant \frac{\log(\rho^\ell/\rho^u) + \log(v(\varepsilon/n)/v(\varepsilon/n^2))}{\eta^u g(0, 0, L) + c^u} \to +\infty \tag{1.2.17}$$

as $n \to +\infty$, since, by N6.2) we have

$$\lim_{n \to +\infty} \frac{v(\varepsilon/n)}{v(\varepsilon/n^2)} = \lim_{n \to +\infty} \frac{n\, v'(\varepsilon/n)}{2\, v'(\varepsilon/n^2)} \geqslant \lim_{n \to +\infty} \frac{An}{2B} = +\infty.$$

By the first and third equations of (1.1.1) and (1.2.16), we have, for $t \in \,]s_{n,k}, t_{n,k}[$,

$$\begin{cases} S' \leqslant G_{2,\varepsilon}(t, S(t, x_n)) \\ P' \leqslant h_{2,\varepsilon}(t, P(t, x_n))P(t, x_n) + \gamma(t)a(t)f(S(t, x_n), 0, P(t, x_n))P(t, x_n) \\ \qquad + \rho(t)v(\varepsilon/n)\theta(t)\eta(t)g(S(t, x_n), 0, P(t, x_n)) \end{cases} .$$

Letting $(\hat{x}_{n,k}(t), \hat{z}_{n,k}(t))$ be the solution of

$$\begin{cases} x' = G_{2,\varepsilon}(t, x) \\ z' = h_{2,\varepsilon}(t, z)z + \gamma(t)a(t)f(x, 0, z)z + \rho(t)v(\varepsilon/n)\theta(t)\eta(t)g(x, 0, z) \end{cases}$$

with $\hat{x}_{n,k}(s_{n,k}) = S(s_{n,k})$ and $\hat{z}_{n,k}(s_{n,k}) = P(s_{n,k})$. We conclude that $S(t, x_n) \leqslant \hat{x}_{n,k}(t)$ and $P(t, x_n) \leqslant \hat{z}_{n,k}(t)$, for each $t \in \,]s_{n,k}, t_{n,k}[$. By N6.5), given $\delta > 0$, we have

$$\left| x_{2,\varepsilon/n}^*(t) - \hat{x}_{n,k}(t) \right| < \delta/2 \quad \text{and} \quad \left| z_{2,\varepsilon/n}^*(t) - \hat{z}_{n,k}(t) \right| < \delta/2,$$

for all sufficiently large k (that depends on n). By continuity, for sufficiently large n and all sufficiently large $k \geqslant K(n)$, we have

$$|x_2^*(t) - \hat{x}_{n,k}(t)| \leqslant \left| x_2^*(t) - x_{2,\varepsilon/n}^*(t) \right| + \left| x_{2,\varepsilon/n}^*(t) - \hat{x}_{n,k}(t) \right| \leqslant \delta.$$

and

$$\left|z_2^*(t) - \hat{z}_{n,k}(t)\right| \leqslant \left|z_2^*(t) - z_{2,\varepsilon/n}^*(t)\right| + \left|z_{2,\varepsilon/n}^*(t) - \hat{z}_{n,k}(t)\right| \leqslant \delta.$$

In particular, for sufficiently large n, all sufficiently large $k \geqslant K(n)$ and for $t \in \,]s_{n,k(n)}, t_{n,k(n)}[$, we have

$$S(t) \leqslant \hat{x}_{n,k}(t) \leqslant x_2^*(t) + \delta \quad \text{and} \quad P(t) \leqslant \hat{z}_{n,k}(t) \leqslant z_2^*(t) + \delta. \tag{1.2.18}$$

Similar computations show that, for sufficiently large n, all sufficiently large $k \geqslant K(n)$ and for $t \in \,]s_{n,k(n)}, t_{n,k(n)}[$, we obtain

$$S(t) \geqslant \tilde{x}_{n,k}(t) \geqslant x_1^*(t) - \delta \quad \text{and} \quad P(t) \geqslant \tilde{z}_{n,k}(t) \geqslant z_1^*(t) - \delta. \tag{1.2.19}$$

Notice that, for a given δ, eventually considering a larger n, we can take the same n and k in (1.2.18) and (1.2.19).

Given $l > 0$, by (1.2.17) we can choose $T > 0$ such that $t_{n,k} - s_{n,k} > l\lambda$ for all $n \geqslant T$. Therefore, by (1.2.15), (1.2.18) and (1.2.19), and by the second equation in (1.1.1), for $n \geqslant T$ and $k \geqslant K(n)$ we get

$$\frac{\rho^u v(\varepsilon/n^2)}{(1 + \theta^u \eta^u)(1 + \beta^u)}$$

$$= I(t_{n,k}, x_n)$$

$$= I(s_{n,k}, x_n) \exp\left\{\int_{s_{n,k}}^{t_{n,k}} \beta(r)S(r) - \eta(r)g(S(r), I(r), P(r)) - c(r)\, dr\right\}$$

$$\geqslant I(s_{n,k}, x_n) \times$$

$$\times \exp\left\{\kappa l + \int_{s_{n,k}+\lfloor (t_{n,k}-s_{n,k})/\lambda\rfloor}^{t_{n,k}} \beta(r)(x_1^*(r) - \delta) - \eta(r)g(x_1^*(r) - \delta, 0, z_2^*(r) + \delta) - c(r)\, dr\right\}$$

$$\geqslant \frac{\rho^\ell v(\varepsilon/n)}{(1 + \theta^u \eta^u)(1 + \beta^u)} \, e^{\kappa l - \lambda(\beta^u \delta + \eta^u(g((x_1^*)^u + \delta, 0, (z_2^*)^u - \delta) + c^u)}$$

$$> \frac{\rho^\ell v(\varepsilon/n)}{(1 + \theta^u \eta^u)(1 + \beta^u)},$$

for sufficiently large l (that requires that T is sufficiently large). We conclude that

$$\frac{\rho^u v(\varepsilon/n^2)}{\rho^\ell v(\varepsilon/n)} > 1$$

and this contradicts the fact that, by N6.2) and N6.3), we have

$$\lim_{n\to+\infty} \frac{\rho^u v(\varepsilon/n^2)}{\rho^\ell v(\varepsilon/n)} = \lim_{n\to+\infty} \frac{2\rho^u v'(\varepsilon/n^2)/n^3}{\rho^\ell v'(\varepsilon/n)/n^2} \leqslant \lim_{n\to+\infty} \frac{2\rho^u B}{n\rho^\ell A} = 0.$$

We conclude that there is $m_1 > 0$ such that $\liminf\limits_{t\to+\infty} I(t) > m_1$ and the result follows from N3). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In [**68**], the authors obtain extinction and persistence results for eco-epidemiological model with Crowley-Martin functional response. In the extinction result the authors consider auxiliary equations different from (1.1.2) and (1.1.3) using some upper bound for S and some lower bound for P related to the dimension of some positive invariant region that contains the omega limit of all solutions. We will borrow and improve the idea of that paper in our context. To this purpose, we need to consider families of auxiliary equations. We begin by noticing that, by the proof of Theorem 1.1, for that any solution $(S(t), I(t), P(t))$ of our problem with initial condition $(S(t_0), I(t_0), P(t_0)) = (S_0, I_0, P_0)$ we have $s^{1,\ell}(t) \leqslant S(t) \leqslant s^{1,u}(t)$ and $y^{1,\ell}(t) \leqslant P(t) \leqslant y^{1,u}(t)$, for all $t > 0$ sufficiently large, where $s^{1,\ell}(t) = 0$, $s^{1,u}(t)$ is the solution of (1.1.2) with initial condition $s^{1,u}(t_0) = S_0$, $y^{1,\ell}(t)$ is the solution of (1.1.3) with initial condition $y^{1,\ell}(t_0) = P_0$ and $y^{1,u}(t) = L$, where L is given in condition N3). Consider the equations:

$$s' = G(t,s) - a(t)f(s, L, y^{1,u}(t))y^{1,\ell}(t), \qquad\qquad (1.2.20)$$

and

$$s' = G(t,s) - a(t)f(s, 0, y^{1,\ell}(t))y^{1,u}(t) - \beta(t)sL, \qquad\qquad (1.2.21)$$

where $y^{1,\ell}(t)$ is a particular solution of (1.1.3). For equations (1.2.20) and (1.2.21), we assume the following:

N4') Each solution $s(t)$ of (1.2.20) (respectively (1.2.21)) with positive initial condition is bounded, bounded away from zero, and globally attractive on $]0, +\infty[$, that is $|s(t) - v(t)| \to 0$ as $t \to +\infty$ for each solution $v(t)$ of (1.2.20) (respectively (1.2.21)) with positive initial condition.

We also need to consider the equations

$$y' = h(t,y)y + \gamma(t)a(t)f(s^{2,\ell}(t), L, y)y, \qquad\qquad (1.2.22)$$

and

$$y' = h(t,y)y + \gamma(t)a(t)f(s^{2,u}(t), 0, y)y + \theta(t)\eta(t)g(0, 0, y)L, \qquad\qquad (1.2.23)$$

where $s^{2,u}(t)$ is a particular solution of (1.2.21). For the family of equations (1.2.22), we assume the following:

N5') Each fixed solution $y(t)$ of (1.2.22) (respectively (1.2.23)) with positive initial condition is bounded and globally attractive on $[0, +\infty)$.

Using the solutions of the systems above we can define the following number:

$$\mathcal{R}^{u,1}(\lambda) = \limsup_{t \to +\infty} \int_t^{t+\lambda} \beta(s)s^{\sharp}(s) - \eta(s)g(s^{\sharp}(s), 0, y^{\sharp}(s)) - c(s)\, ds. \qquad (1.2.24)$$

where $s^{\sharp}(t)$ and $y^{\sharp}(t)$ are particular solutions, respectively, of (1.2.20) and (1.2.22) with positive initial conditions. Notice that, according to our assumptions, it is easy to prove, with similar arguments to the ones in Proposition 1.1, that $\mathcal{R}^{u,1}(\lambda)$ is independent of the particular solutions considered.

THEOREM 1.3. *Assume that conditions N1) to N2), N4') and N5') hold. Assume further $x \to f(x,y,z)$ in nondecreasing and that either $G(t,S) = \Lambda(t) - \mu(t)S$ and $g(S+I,0,P) \leqslant g(S,I,P)$ or g does not depend on I. If there is $\lambda > 0$ such that $\mathcal{R}^{u,1}(\lambda) < 0$, then the infectives in system (1.1.1) go to extinction.*

PROOF. The proof consists in repeating the steps in the proof of Theorem 1.1, with the changes that we will describe below. In the first place, instead of the bounds (1.2.5) and (1.2.6), we use bounds obtained in the following way: letting $y^{1,\ell}(t)$ and $y^{2,u}(t)$ be the solutions defined above, we know that

$$S' \leqslant G(t,S) - a(t)f(S, L, y^{1,u}(t))y^{1,\ell}(t) \qquad (1.2.25)$$

and

$$S' \geqslant G(t,S) - a(t)f(S, 0, y^{1,\ell}(t))y^{1,u}(t) - \beta(t)SL. \qquad (1.2.26)$$

Thus, using the solutions $s^{2,u}(t)$ and $s^{2,\ell}(t)$ respectively of (1.2.25) and (1.2.26), we obtain

$$P' \geqslant h(t,P)P + \gamma(t)a(t)f(s^{2,\ell}(t), L, P)P \qquad (1.2.27)$$

and

$$P' \leqslant h(t,P)P + \gamma(t)a(t)f(s^{2,u}(t), 0, P)P + \theta(t)\eta(t)g(0, 0, P)L. \qquad (1.2.28)$$

The bounds in (1.2.25) and (1.2.27) allow us to conclude that, for sufficiently large $t > 0$, we have $S(t) \leqslant s^{\sharp}(t)$ and $P(t) \geqslant y^{\sharp}(t)$, where $s^{\sharp}(t)$ and $y^{\sharp}(t)$ are respectively particular solutions of (1.2.20) and (1.2.22); using the solutions $s^{\sharp}(t)$ and $y^{\sharp}(t)$ and the number $\mathcal{R}^{u,1}(\lambda)$ in (1.2.24), similar arguments to the ones in Theorem 1.1 allow us to obtain the result. □

We note that the procedure in Theorem 1.3 can be iterated to obtain new and (hopefully) better estimates of the region of extinction, as long as we can still ensure that assumptions N4') and N5') still hold for the new equations. In fact all we have to do is the following: consider equations (1.2.20) and (1.2.21) with $y^{1,\ell}(t)$ and $y^{1,u}(t)$ replaced by $y^{2,\ell}(t)$ and $y^{2,u}(t)$, the solutions of (1.2.22) and (1.2.23), and denote the solutions of those equations by $s^{3,\ell}(t)$ and $s^{3,u}(t)$; consider equations (1.2.22) and (1.2.23) with $s^{2,\ell}(t)$ and $s^{2,u}(t)$ replaced by $s^{3,\ell}(t)$ and $s^{3,u}(t)$; replace $\mathcal{R}^{u,1}(\lambda)$ by

$$\mathcal{R}^{u,2}(\lambda) = \limsup_{t \to +\infty} \int_t^{t+\lambda} \beta(s)s^{\sharp\sharp}(s) - \eta(s)g(s^{\sharp\sharp}(s), 0, y^{\sharp}(s)) - c(s)\, ds,$$

where $s^{\sharp\sharp}(t)$ and $y^{\sharp\sharp}(t)$ are particular solutions, respectively, of the new equations corresponding to (1.2.20) and (1.2.22) with positive initial conditions. With these ingredients we obtain a new theorem on extinction. As long as the assumptions corresponding to N4') and N5') still hold, we can repeat the process over and over again obtaining a sequence of theorems on extinction and (hopefully) improving the estimates at each step.

1.3. Examples

In this section we will apply Theorems 1.1 and 1.2 to some particular cases of model (1.1.1).

1.3.1. No predation on uninfected preys. In this section we will consider a family of models with no predation on uninfected preys by letting $f \equiv 0$ and $g(S, I, P) = P$. This family generalises the family of models in [75] by allowing a very general form for the vital dynamics of predators and preys. Thus, still assuming

conditions N1) to N6), we consider in this subsection the following model:

$$\begin{cases} S' = G(t, S) - \beta(t)SI \\ I' = \beta(t)SI - \eta(t)PI - c(t)I \\ P' = h(t, P)P + \theta(t)\eta(t)PI \end{cases} \quad . \tag{1.3.1}$$

In this context, (1.2.1) and (1.2.2) become

$$\mathcal{R}_{np}^{\ell}(\lambda) = \liminf_{t \to +\infty} \int_{t}^{t+\lambda} \beta(s)s^*(s) - \eta(s)y^*(s) - c(s)\,ds \tag{1.3.2}$$

and

$$\mathcal{R}_{np}^{u}(\lambda) = \limsup_{t \to +\infty} \int_{t}^{t+\lambda} \beta(s)s^*(s) - \eta(s)y^*(s) - c(s)\,ds. \tag{1.3.3}$$

where $s^*(t)$ and $y^*(t)$ are particular solutions, respectively, of (1.1.2) and (1.1.3). We obtain the corollaries of Theorems 1.1 and 1.2:

COROLLARY 1. *Assume that we have N1) to N5) and that $g(S + I, 0, P) \leqslant g(S, I, P)$. If there is $\lambda > 0$ such that $\mathcal{R}_{np}^{u}(\lambda) < 0$ then the infectives in system (1.3.1) go to extinction.*

COROLLARY 2. *Assume that we have N1) to N3) and N6). If there is $\lambda > 0$ such that $\mathcal{R}_{np}^{\ell}(\lambda) > 0$ then the infectives in system (1.3.1) are uniform strong persistent.*

As we already mentioned, model (1.3.1) includes the model discussed in [**75**] as the particular case where $G(t, S) = \Lambda(t) - \mu(t)S$ and $h(t, P) = b(t) - r(t)P$, with Λ, μ, r and b nonnegative, continuous and bounded functions satisfying:

$$\liminf_{t \to +\infty} \int_{t}^{t+\omega_1} \Lambda(s)\,ds > 0, \qquad \liminf_{t \to +\infty} \int_{t}^{t+\omega_2} \mu(s)\,ds > 0,$$

$$\liminf_{t \to +\infty} \int_{t}^{t+\omega_3} r(s)\,ds > 0 \qquad \text{and} \qquad \liminf_{t \to +\infty} \int_{t}^{t+\omega_4} b(s)\,ds > 0,$$

for some constants $w_i > 0$, $i = 1, \ldots, 4$:

$$\begin{cases} S' = \Lambda(t) - \mu(t)S - \beta(t)SI \\ I' = \beta(t)SI - \eta(t)PI - c(t)I \\ P' = (b(t) - r(t)P)P + \theta(t)\eta(t)PI \end{cases} \quad . \tag{1.3.4}$$

Note that, for the model in (1.3.4), condition N1) is assumed, condition N2) is immediate from the particular forms of the functions g and h, conditions N4) and N5) follow from Lemmas 1 and 3 in [75] and condition N6) is a consequence of the fact that, in this setting, systems (1.1.5) and (1.1.6) are uncoupled and small perturbations of each of the equations in those systems is globally asymptotically stable by Lemmas 1 and 3 in [75]. Finally, condition N3) follows from Theorem 1 in [75]. We also note that $\mathcal{R}_{np}^u(\lambda)$ and $\mathcal{R}_{np}^\ell(\lambda)$ coincide with the corresponding numbers in [75].

Another possible choice for the functions g and h is $h(t, P) = -(\delta_1(t) + \delta_2(t)P)$, with δ_1 and δ_2 continuous and nonnegative functions and $G(t, S) = k(t, S)S$ with k a continuous and bounded function satisfying the conditions: $\partial k/\partial S(t, s) < 0$, for every $t, s \geqslant 0$; $k(t, 0) > 0$ for all $t \geqslant 0$; there is $S_1(t) > 0$ such that $k(t, S_1(t)) = 0$, for every $t \geqslant 0$. This choice makes the underlying predator-uninfected prey subsystem in model (1.3.1) correspond to the model studied in section 3 of [38] with the function $f \equiv 0$. System (1.3.1) becomes in this case:

$$\begin{cases} S' = k(t, S)S - \beta(t)SI \\ I' = \beta(t)SI - \eta(t)PI - c(t)I \\ P' = -(\delta_1(t) + \delta_2(t)P)P + \theta(t)\eta(t)PI \end{cases} \qquad . \qquad (1.3.5)$$

Notice that the study of the function $k(t, S)$ in [38] allow us to conclude easily that conditions N1) to N5) are satisfied for this model. Condition N6) is a consequence of the fact that systems (1.1.5) and (1.1.6) are uncoupled and small perturbations of each of the equations in those systems is globally asymptotically stable (the global asymptotic stability of the first equation is consequence of Lemma 3.1 in [38] and the global asymptotic stability of the second equation is trivial).

To do some simulation, we consider the following particular set of parameters: $G(t, S) = (0.7 - 0.6S)S$; $\beta(t) = \beta_0(1 + 0.7\cos(2\pi t))$; $\eta(t) = 0.7(1 + 0.7\cos(\pi + 2\pi t))$; $c(t) = 0.1$; $h(t, P) = -0.2 - 0.3P$; $\theta(t) = 0.9$. We obtain the model:

$$\begin{cases} S' = (0.7 - 0.6S)S - \beta_0(1 + 0.7\cos(2\pi t))SI \\ I' = \beta_0(1 + 0.7\cos(2\pi t))SI - 0.7(1 + 0.7\cos(\pi + 2\pi t))PI - 0.1I \\ P' = -0.2 - 0.3P + 0.63(1 + 0.7\cos(\pi + 2\pi t))PI \end{cases} , \quad (1.3.6)$$

We first consider $\beta_0 = 0.075$ and obtain, $\mathcal{R}_0 \approx 0.87 < 1$ and we conclude that we have extinction of the infected prey (figure 1.1). Next we assume that $\beta_0 = 0.09$ and obtain $\mathcal{R}_0 \approx 1.05 > 1$ and we conclude that we have uniform strong persistence of the infected prey (figure 1.2). In the extinction scenario we considered the following initial conditions in time $t = 0$: $(S_0, I_0, P_0) = (2.66, 0.51, 0.9)$, $(S_0, I_0, P_0) = (1.6, 0.2, 0.3)$ and $(S_0, I_0, P_0) = (0.48, 0.7, 0.6)$. In the uniform strong persistence situation we considered the following initial conditions in time $t = 0$: $(S_0, I_0, P_0) = (0.5, 0.1, 0.4)$, $(S_0, I_0, P_0) = (0.4, 0.8, 0.7)$ and $(S_0, I_0, P_0) = (1.036, 0.387, 0.153)$.
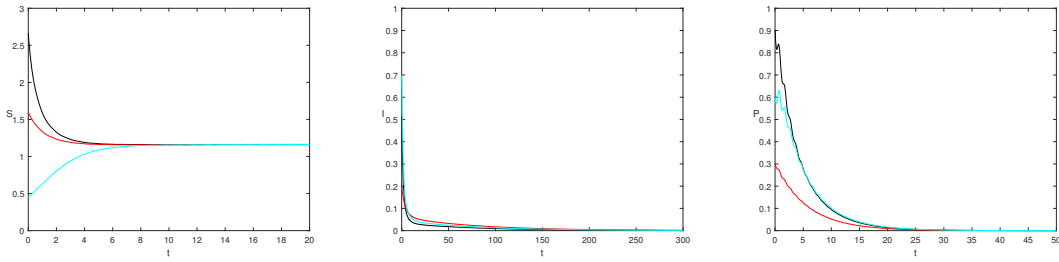
FIGURE 1.1. Extinction; no predation on uninfected prey; $\beta_0 = 0.075$.
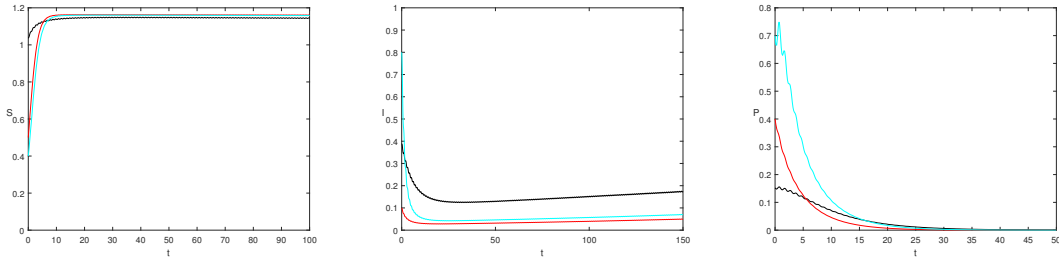
FIGURE 1.2. Persistence; no predation on uninfected prey; $\beta_0 = 0.09$.

1.3.2. Periodic coefficients. In this subsection we consider a family of models with periodic parameters and predation on uninfected preys that, in general, is not included in the general family of models considered in [**68**]. For periodic models, the thresholds become easier to deal with.

Assume that there is $\omega > 0$ such that all parameters in (1.1.1) are ω-periodic functions. In this case, (1.2.1) and (1.2.2) become, respectively,

$$\mathcal{R}^{\ell}(\omega) = \int_0^{\omega} \beta(s)x_1^*(s) - \eta(s)g(x_1^*(s), 0, z_2^*(s)) - c(s)\,ds,$$

and

$$\mathcal{R}^u(\omega) = \int_0^{\omega} \beta(s)s^*(s) - \eta(s)g(s^*(s), 0, y^*(s)) - c(s)\,ds.$$

Thus

$$\mathcal{R}^{\ell}(\omega) > 0 \quad \Leftrightarrow \quad \frac{\overline{\beta x_1^*}}{\overline{\eta g(x_1^*, 0, z_2^*) + \overline{c}}} > 1$$

and

$$\mathcal{R}^u(\omega) < 0 \quad \Leftrightarrow \quad \frac{\overline{\beta s^*}}{\overline{\eta g(s^*, 0, y^*) + \overline{c}}} < 1.$$

where $s^*(t)$ and $y^*(t)$ are particular solutions, respectively, of (1.1.2) and (1.1.3), and $x_1^*(t)$ and $z_2^*(t)$ still denote any particular solution of first and second equations in systems (1.1.5) and (1.1.6), respectively, with positive initial conditions. Define

$$\mathcal{R}_{per}^{\ell} = \frac{\overline{\beta x_1^*}}{\overline{\eta g(x_1^*, 0, z_2^*) + \overline{c}}} \quad \text{and} \quad \mathcal{R}_{per}^u = \frac{\overline{\beta s^*}}{\overline{\eta g(s^*, 0, y^*) + \overline{c}}}.$$

We obtain the following corollaries of Theorems 1.1 and 1.2:

COROLLARY 3. *Assume that we have N1) to N5) and that* $g(S + I, 0, P) \leqslant g(S, I, P)$. *If* $\mathcal{R}_{per}^u < 1$ *then the infectives in model* (1.1.1) *with periodic coefficients go to extinction.*

COROLLARY 4. *Assume that we have N1) to N3) and N6). If* $\mathcal{R}_{per}^{\ell} > 1$ *then the infectives in model* (1.1.1) *with periodic coefficients are uniform strong persistent.*

Note that the corollaries in [**75**], concerning the periodic case, are particular cases of the corollaries above. In fact, in [**75**] we have $f \equiv 0$, $G(t, S) = \Lambda(t) - \mu(t)S$ and $h(t, S) = b(t) - r(t)P$. In that case, as argued in the previous section, $(s^*(t), y^*(t))$ is a particular solution of (1.1.4), condition N1) is assumed, condition N2) is immediate, conditions N3) to N6) follow from results in [**75**]. Thus, when $f \equiv 0$, we get similar thresholds to the ones in the mentioned paper:

$$\mathcal{R}_{per}^{\ell} = \mathcal{R}_{per}^u = \frac{\overline{\beta s^*}}{\overline{\eta y^* + \overline{c}}}.$$

We will focus now on a particular models with a function G that is different from the corresponding function in [**68**]. We consider the following setting: $G(t, S) = (\Lambda - \mu S)S$; $a(t) = a$; $f(S, I, P) = S$; $g(S, I, P) = P$; $h(t, P) = b - rP$; $\gamma(t) = \gamma$. We

obtain the model:

$$
\begin{cases}
S' = (\Lambda - \mu S)S - aSP - \beta(t)SI \\
I' = \beta(t)SI - \eta(t)PI - c(t)I \\
P' = (b - rP)P + \gamma aSP + \theta(t)\eta(t)PI
\end{cases}
, \qquad (1.3.7)
$$

For this model, condition N1) is assumed, condition N2) is immediate from the particular forms of the functions g and h, conditions N4) and N5) hold for our particular functions as already discussed in section 1.3.1. In this context, an endemic equilibrium for (1.1.6) is $(\Lambda/\mu, \hat{z}_\varepsilon)$, with $\hat{z}_\varepsilon = (b\mu + a\gamma\Lambda + \varepsilon\mu)/\mu r$, and the endemic equilibrium for (1.1.5) exists if $\Lambda r > ab + a\gamma\Lambda/\mu$:

$$
\left(\frac{\hat{\Lambda}}{\mu}, \frac{b\mu + a\gamma\hat{\Lambda}}{\mu r} \right),
$$

with $\hat{\Lambda} = \Lambda - a\hat{z}_\varepsilon - \varepsilon$. These subsystems can be discussed using [40]. In fact, the global asymptotic stability result proved in section 3 of [40] implies that, if $\Lambda r > ab + a\gamma\Lambda/\mu$, condition N6) is satisfied. Finally, condition N3) is consequence of the following lemma:

LEMMA 1.1. *There is a bounded region that contains the ω-limit of all orbits of* (1.3.7).

PROOF. Let $\varepsilon > 0$. Since, by the first equation in (1.3.7), $S' \leqslant (\Lambda - \mu S)S$, we conclude that

$$
S(t) \leqslant \frac{\Lambda}{\mu} + \varepsilon, \qquad (1.3.8)
$$

for all t sufficiently large. Additionally, we get

$$
\sup_{S \in \mathbb{R}} (\Lambda - \mu S)S \leqslant \left(\Lambda - \frac{\mu\Lambda}{2\mu} \right) \frac{\Lambda}{2\mu} = \frac{\Lambda^2}{4\mu}. \qquad (1.3.9)
$$

Adding the first two equations in (1.3.7) and using (1.3.8) and (1.3.9) we have, for all t sufficiently large,

$$
\begin{aligned}
(S + I)' &= (\Lambda - \mu S)S - c(t)I \\
&\leqslant \frac{\Lambda^2}{4\mu} + c(t)S - c(t)(S + I) \\
&\leqslant \frac{\Lambda^2}{4\mu} + c^u\frac{\Lambda}{\mu} + c^u\varepsilon - c^\ell(S + I).
\end{aligned}
\qquad (1.3.10)
$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\limsup_{t\to\infty}(S+I)(t) \leqslant \frac{1}{c^\ell}\left(\frac{\Lambda^2}{4\mu} + c^u\frac{\Lambda}{\mu}\right) := A. \qquad (1.3.11)$$

Finally, by the third equation in (1.3.7) and (1.3.10), given $\varepsilon > 0$, we get

$$P' = (b - rP)P + \gamma aSP + \theta(t)\eta(t)PI$$
$$\leqslant (b + \gamma aA + \theta^u\eta^u A - rP)\,P, \qquad (1.3.12)$$

for sufficiently large t. Thus,

$$\limsup_{t\to\infty} P(t) \leqslant \frac{1}{r}\left(b + \gamma aA + \theta^u\eta^u A\right) := B.$$

Equations (1.3.10) and (1.3.12) show that the region

$$\{(S, I, P) \in \mathbb{R}^3 : 0 \leqslant S + I \leqslant A \text{ and } 0 \leqslant P \leqslant B\}$$

contains the ω-limit of any orbit. The result is proved. $\qquad\square$

To do some simulation in the this scenario, we consider the following parameters in (1.3.7): $\Lambda = 0.7$; $\mu = 0.18$; $a = 0.4$; $\beta(t) = \beta_0(1 + 0.7\cos(2\pi t))$; $\eta(t) = 0.7(1 + 0.7\cos(\pi + 2\pi t))$; $c(t) = 0.1$; $b = 0.8$; $r = 0.6$; $\theta(t) = 0.9$; $\gamma = 0.1$. We obtain the model:

$$\begin{cases} S' = (0.7 - 0.18S)S - 0.4SP - \beta_0(1 + 0.7\cos(2\pi t))SI \\ I' = \beta_0(1 + 0.7\cos(2\pi t))SI - 0.7(1 + 0.7\cos(\pi + 2\pi t))PI - 0.1I \\ P' = (0.8 - 0.6P)P + 0.04SP + 0.63(1 + 0.7\cos(\pi + 2\pi t))PI \end{cases} \qquad (1.3.13)$$

We have $\mathcal{R}_{per}^u = 3.764\beta_0$ and $\mathcal{R}_{per}^\ell = 3.705\beta_0$. Thus, if $\beta_0 < 0.266$, we have extinction of the infectives and, if $\beta_0 > 0.270$, we have persistence of the infectives. In figure 1.3, we present simulation results for $\beta_0 = 0.2$ (extinction) and in figure 1.4, we present simulation results for $\beta_0 = 1.4$ (uniform strong persistence). To obtain figures 1.3 and 1.4 where the extinction and uniform strong persistence situations were addressed, respectively, the following initial conditions, in time $t = 0$, were used $(S_0, I_0, P_0) = (0.811, 0.0624, 1.388)$ and $(S_0, I_0, P_0) = (1.388, 0.06, 1.388)$ corresponding, respectively, to a disease-free solution and an (approximately) periodic solution. The other two initial conditions are, for the

extinction case, $(S_0, I_0, P_0) = (0.6, 0.16, 0.46)$ and $(S_0, I_0, P_0) = (1.0975, 0.044, 0.76)$. For the case of uniform strong persistence we considered, in time $t = 0$, the initial conditions $(S_0, I_0, P_0) = (0.5, 0.1, 0.4)$ and $(S_0, I_0, P_0) = (0.4, 0.04, 0.7)$.
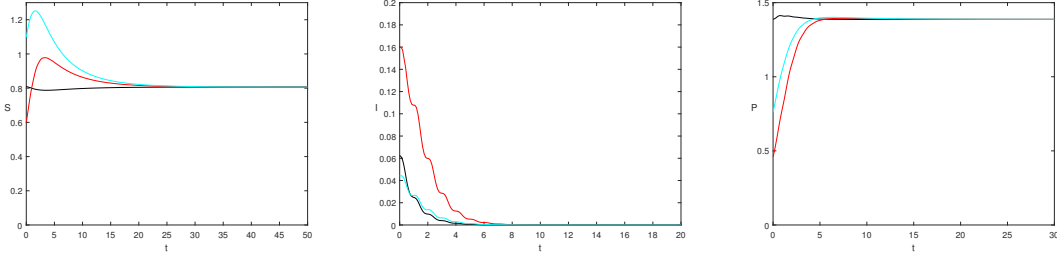
FIGURE 1.3. Extinction; periodic coefficients; $\beta_0 = 0, 2$.
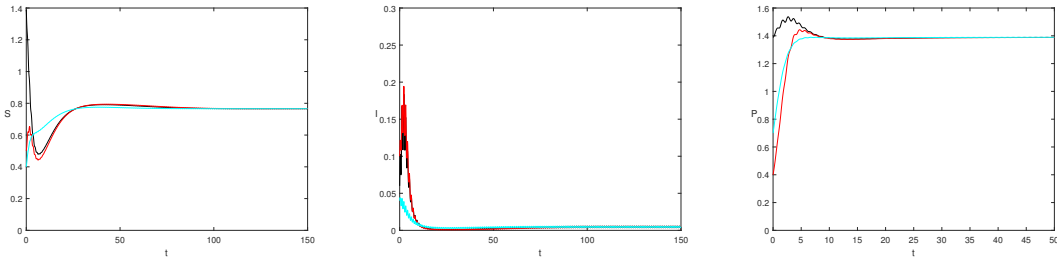
FIGURE 1.4. Persistence; periodic coefficients; $\beta_0 = 1, 4$.

A question that arises is if it is possible to obtain similar results as in Theorems 1.1 and 1.2 perturbing the uninfected subsystem (1.1.4), by considering for condition N6) the following subsystems

$$\begin{cases} x' = G_{1,\varepsilon}(t, x) - a(t)f(x, 0, z)z - v(\varepsilon)\rho(t)x \\ z' = h_{1,\varepsilon}(t, z)z + \gamma(t)a(t)f(x, v(\varepsilon)\rho^u, z)z \end{cases} \tag{1.3.14}$$

and

$$\begin{cases} x' = G_{2,\varepsilon}(t, x) - a(t)f(x, v(\varepsilon)\rho^u, z)z \\ z' = h_{2,\varepsilon}(t, z)z + \gamma(t)a(t)f(x, 0, z)z + v(\varepsilon)\rho(t)g(x, v(\varepsilon)\rho^u, z) \end{cases} , \tag{1.3.15}$$

instead (1.1.5) and (1.1.6), pespectively.

In the next examples is not straightforward to check the necessary properties of subsystems (1.1.5) and (1.1.6). To give some contributions, we check that the following examples satisty N6) if we replace (1.1.5) and (1.1.6) by (1.3.14) and (1.3.15), respectively, and we carry out some simulation that suggests we have persistence in this context, if we replace $R^\ell(\lambda)$ by

$$\widehat{\mathcal{R}}^\ell(\lambda) = \liminf_{t\to+\infty} \int_t^{t+\lambda} \beta(s)x^*(s) - \eta(s)g(x^*(s), 0, z^*(s)) - c(s)\,ds \qquad (1.3.16)$$

where $(x^*(t), z^*(t))$ is any particular solution of system (1.1.4) with positive initial conditions. It remains to answer if it is possible to obtain persistence assuming (1.3.15) and (1.3.15) for condition N6).

1.3.3. Gause-type uninfected subsystem. A model with Michaelis-Menten (or Holling type II) functional response of predator to infected prey is now considered. The uninfected model obtained is a particular case of a Gause-type model (see [**62**]). We consider the following setting: $G(t, S) = (\Lambda - \mu S)S$; $a(t) = a$; $f(S, I, P) = S/(m + S + I)$; $g(S, I, P) = P/(m + S + I)$ with $m > 0$; $h(t, P) = -dP$ and $\gamma(t) = \gamma$. We obtain the model:

$$\begin{cases} S' = (\Lambda - \mu S)S - a\frac{SP}{m+S+I} - \beta(t)SI \\ I' = \beta(t)SI - \eta(t)\frac{PI}{m+S+I} - c(t)I \\ P' = -dP + \gamma a\frac{SP}{m+S+I} + \theta(t)\eta(t)\frac{PI}{m+S+I} \end{cases}, \qquad (1.3.17)$$

where β, η, c and θ are still continuous functions. The result in [**62**] allows us to conclude that, for sufficiently small $\varepsilon > 0$, if $\gamma a \leqslant d$ or $\gamma a > d$ and $\Lambda/\mu \leqslant d(m+\varepsilon)/(\gamma a - d)$, we have that the equilibrium point $((\Lambda - \varepsilon)/\mu, 0)$ of the system

$$\begin{cases} x' = (\Lambda - \mu x)x - a\frac{xz}{m+x} - \varepsilon x \\ z' = -dz + \gamma a\frac{xz}{m+\varepsilon+x} \end{cases} \qquad (1.3.18)$$

is globally asymptotically stable in the set $\{(x, z) \in (\mathbb{R}_0^+)^2 : x > 0 \wedge z \geqslant 0\}$. Notice that system (1.3.18) corresponds, in this case, to system (1.3.14) with $v(\varepsilon) = \varepsilon$, $\rho(t) = 1$, $G_{1,\varepsilon} = (\Lambda - \mu x)x$ and $h_{1,\varepsilon} = -d$. By [**62**] we can also conclude that, for

sufficiently small $\varepsilon > 0$, If

$$\frac{d(m+\varepsilon)}{\gamma a - d} < \frac{\Lambda}{\mu} \leqslant m + \varepsilon + \frac{2d(m+\varepsilon)}{\gamma a - d},$$

the equilibrium

$$\left(\frac{d(m+\varepsilon)}{\gamma a - d}, \frac{\mu}{a} \left(\frac{\Lambda}{\mu} - \frac{d(m+\varepsilon)}{\gamma a - d} \right) \left(m + \varepsilon + \frac{d(m+\varepsilon)}{\gamma a - d} \right) \right)$$

of the system

$$\begin{cases} x' = (\Lambda - \mu x)x - a\frac{xz}{m+\varepsilon+x} \\ z' = -dz + \gamma a\frac{xz}{m+x} + \frac{\varepsilon z}{m+\varepsilon+x}, \end{cases} \tag{1.3.19}$$

is globally asymptotically stable in the set $\{(x,z) \in (\mathbb{R}_0^+)^2 : x > 0 \wedge z > 0\}$. Notice that system (1.3.19) corresponds, in this case, to system (1.3.15) with $v(\varepsilon) = \varepsilon$, $\rho(t) = 1$, $G_{2,\varepsilon} = (\Lambda - \mu x)x$ and $h_{2,\varepsilon} = -d$. For this models, the numbers in (1.3.16) and (1.2.2) become:

$$\widehat{\mathcal{R}}^\ell(\lambda) = \liminf_{t \to +\infty} \int_t^{t+\lambda} \beta(s)\frac{dm}{\gamma a - d} - \eta(s)\left(\frac{\Lambda}{\mu} - \frac{d}{\gamma a - d} \right)\left(m\frac{dm}{\gamma a - d} \right) - c(s)\,ds \tag{1.3.20}$$

and

$$\mathcal{R}^u(\lambda) = \limsup_{t \to +\infty} \int_t^{t+\lambda} \beta(s)\frac{\Lambda}{\mu} - c(s)\,ds. \tag{1.3.21}$$

When the parameters are periodic, we obtain

$$\widehat{\mathcal{R}}^\ell_{per} = \frac{dm\bar{\beta}}{\bar{\eta}\left(\frac{\Lambda}{\mu} - \frac{d}{\gamma a - d} \right)\left(m\frac{dm}{\gamma a - d} \right) + \bar{c}} \quad \text{and} \quad \mathcal{R}^u_{per} = \frac{(\Lambda/\mu)\bar{\beta}}{\bar{c}}.$$

For model (1.3.17), condition N1) is assumed, condition N2) is immediate from the particular forms of the functions G and h, condition N4) holds, as already discussed, and condition N5) is immediate. Finally, condition N3) can be obtained using similar arguments to the ones in Lemma 1.1. We have the following corollary of Theorem 1.1:

COROLLARY 5. *Assume that the parameters in model* (1.3.17) *are periodic. If* $\mathcal{R}^u_{per} < 1$ *then the infectives in model* (1.3.17) *go to extinction.*

Under the conditions above, we conclude that condition N6) holds if we replace (1.1.5) and (1.1.6) by (1.3.14) and (1.3.15), respectively. It remains to answer

that if the parameters in model (1.3.17) are periodic and $\widehat{\mathcal{R}}_{per}^{\ell} > 1$ then the infectives in model (1.3.17) are uniformly strong persistent. The following simulations suggests that this is the case.

To do some simulation, in this scenario we assumed that $G(t, S) = (0.7 - 0.6S)S$; $a = 0.978$; $\beta(t) = \beta_0(1 + 0.7\cos(2\pi t))$; $\eta(t) = 0.7(1 + 0.7\cos(\pi + 2\pi t))$; $c(t) = 0.1$; $d = 0.3$; $m = 2$; $\gamma = 0.9$; $\theta(t) = 0.9$. We obtain the model:

$$
\begin{cases}
S' = (0.7 - 0.6S)S - 0.978\,\frac{SP}{2+S+I} - \beta_0(1 + 0.7\cos(2\pi t))SI \\[2mm]
I' = \beta_0(1 + 0.7\cos(2\pi t))SI - 0.7(1 + 0.7\cos(\pi + 2\pi t))\frac{PI}{2+S+I} - 0.1I \\[2mm]
P' = -0.3P + 0.8802\,\frac{SP}{2+S+I} + 0.63(1 + 0.7\cos(\pi + 2\pi t))\frac{PI}{2+S+I}
\end{cases} \quad . \quad (1.3.22)
$$

When $\beta_0 = 0,07$ we obtain approximately $\mathcal{R}_{per}^u \approx 0,82 < 1$ and we conclude that we have extinction (figure 1.7). When $\beta_0 = 0,6$ we obtain approximately $\widehat{\mathcal{R}}_{per}^{\ell} \approx 1.2 > 1$. The simulation suggests that the infectives are uniform strong persistent in this case (figure 1.8).

In the extinction scenario we considered the following initial conditions in time $t = 0$: $(S_0, I_0, P_0) = (1.66, 0.51, 0.9)$, $(S_0, I_0, P_0) = (0.6, 0.2, 0.3)$ and $(S_0, I_0, P_0) = (2.45, 0.7, 0.6)$. In the uniform strong persistent situation we considered the initial conditions in $t = 0$: $(S_0, I_0, P_0) = (1, 0.387, 0.153)$, $(S_0, I_0, P_0) = (0.5, 0.1, 0.4)$ and $(S_0, I_0, P_0) = (0.4, 0.04, 0.7)$.

FIGURE 1.5.　　Extinction; Gause-type uninfected subsystem; $\beta_0 = 0.07$.
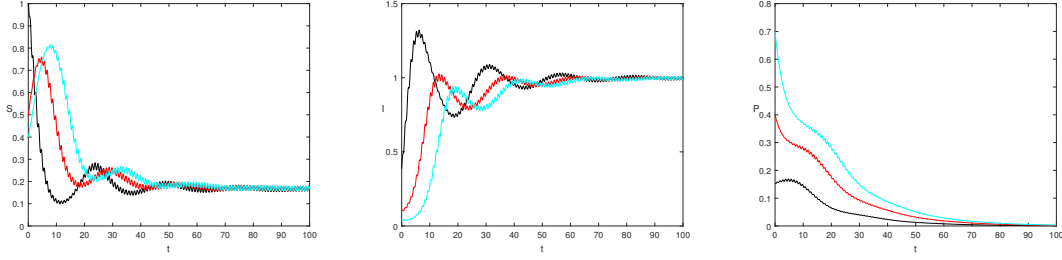
FIGURE 1.6.   Persistence; Gause-type uninfected subsystem; $\beta_0 = 0.6$.

1.3.4. Ratio-dependent uninfected subsystem. The functional response of predator to prey in the uninfected subsystem in the next example is ratio-dependent. Ratio-dependent functional responses were considered to overcome some paradoxes identified in Gause-type systems (see [**50**] and the references therein). We consider the following setting: $G(t, S) = (\Lambda - \mu S)S$; $a(t) = a$; $f(S, I, P) = S/(mP + S)$ with $m > 0$; $g(S, I, P) = P$; $h(t, P) = -d$ and $\gamma(t) = \gamma$. The following model is obtained:

$$\begin{cases} S' = (\Lambda - \mu S)S - a\frac{SP}{mP+S} - \beta(t)SI \\ I' = \beta(t)SI - \eta(t)PI - c(t)I \\ P' = -dP + \gamma a\frac{SP}{mP+S} + \theta(t)\eta(t)PI \end{cases} \qquad . \qquad (1.3.23)$$

In this context, the uninfected system was discussed in [**50**]. In that paper it was shown that, if $d \geqslant \gamma a$ and $a < m\Lambda$ we have that the equilibrium point $(\Lambda/\mu, 0)$ is globally asymptotically stable in the set $\{(x, z) \in (\mathbb{R}_0^+)^2 : x > 0 \ \wedge \ z \geqslant 0\}$ (note that this conditions lead to extinction of the predator). Under the conditions above, we conclude that condition N6) holds if we replace (1.1.5) and (1.1.6) by (1.3.14) and (1.3.15), respectively. For model (1.3.23), condition N1) is assumed, condition N2) is immediate from the particular forms of the functions f, G and h, condition N4) holds, as already discussed, and condition N5) is immediate. Finally, condition N3) can be obtained using similar arguments to the ones in Lemma 1.1.

To do some simulation, in this scenario we assumed that $G(t, S) = (0.7 - 0.6S)S$; $a = 0.4$; $\beta(t) = \beta_0(1 + 0.7\cos(2\pi t))$; $\eta(t) = 0.7(1 + 0.7\cos(\pi + 2\pi t))$; $c(t) = 0.1$;

$d = 0.4$; $m = 2$; $\gamma = 0.8$; $\theta(t) = 0.9$. We obtain the model:

$$\begin{cases} S' = (0.7 - 0.6S)S - 0.4\frac{SP}{2P+S} - \beta_0(1 + 0.7\cos(2\pi t))SI \\ I' = \beta_0(1 + 0.7\cos(2\pi t))SI - 0.7(1 + 0.7\cos(\pi + 2\pi t))PI - 0.1I \\ P' = -0.4P + 0.32\frac{SP}{2P+S} + 0.63(1 + 0.7\cos(\pi + 2\pi t))PI \end{cases} \qquad (1.3.24)$$

When $\beta_0 = 0,08$ we obtain approximately $\mathcal{R}_{per}^u \approx 0.93 < 1$ and we conclude that we have extinction (figure 1.7). When $\beta_0 = 0.25$ we obtain approximately $\widehat{\mathcal{R}}_{per}^\ell \approx 2.9 > 1$. The simulations below suggests that the infectives are uniformly strong persistent (figure 1.8).

In the extinction case we considered the following initial conditions in time $t = 0$: $(S_0, I_0, P_0) = (2.5, 0.514, 0.9)$, $(S_0, I_0, P_0) = (1.2, 0.2, 0.3)$ and $(S_0, I_0, P_0) = (0.45, 0.7, 0.6)$. In the uniform strong persistent situation we considered the initial conditions: $(S_0, I_0, P_0) = (1.0357, 0.387, 0.1525)$, $(S_0, I_0, P_0) = (0.5, 0.1, 0.4)$ and $(S_0, I_0, P_0) = (0.4, 0.04, 0.7)$, in $t = 0$.
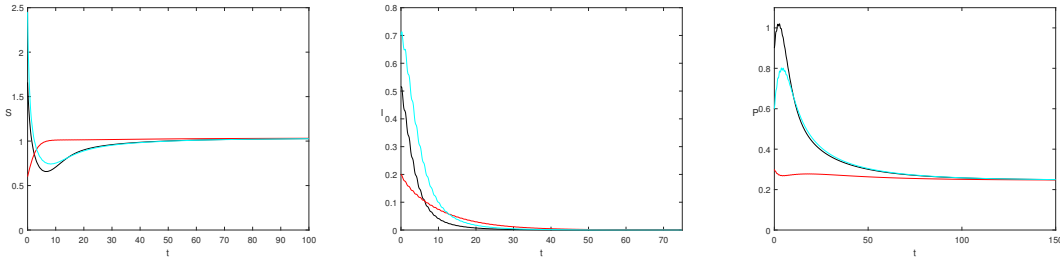

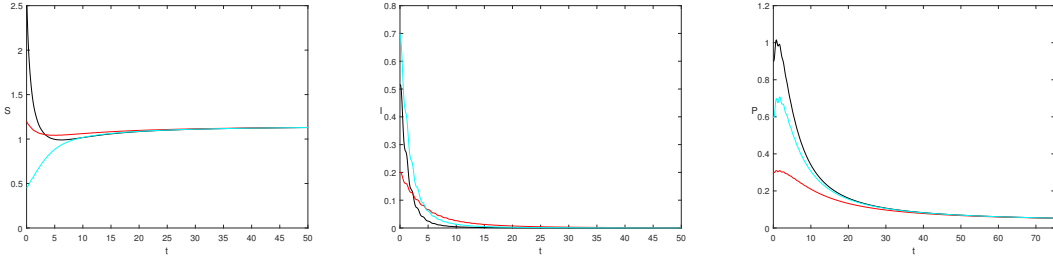FIGURE 1.7.   Extinction; ratio-dependent uninfected subsystem; $\beta_0 = 0.08$.
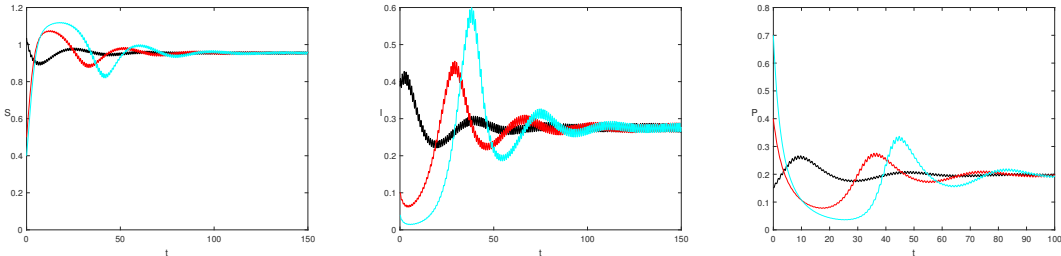

FIGURE 1.8.   Persistence; ratio-dependent uninfected subsystem; $\beta_0 = 0.25$.

1.3.5. Time-varying coefficients in the uninfected subsystem I. We now consider an example where the uninfected model is nonautonomous. For this model

we will be able to obtain explicit thresholds based on the study of the underlying susceptible prey/predator subsystem in [**91**]. Assuming that $G(t, S) = (p + qh(t) - dh(t)S)S$ with $h(t)$ continuous and satisfying $h^\ell < h(t) < h^u$ for some constants $h^\ell, h^u > 0$, $f(S, I, P) = S$, $g(S, I, P) = P$, $a(t) = b$, $h(t, P) = -q$ and $\gamma(t) = d/b$ in (1.1.1), we obtain the following particular model:

$$
\begin{cases}
S' = (p + qh(t) - dh(t)S)S - bSP - \beta(t)SI \\
I' = \beta(t)SI - \eta(t)PI - c(t)I \\
P' = -qP + dSP + \theta(t)\eta(t)PI
\end{cases}
, \qquad (1.3.25)
$$

where we continue to assume that β, η, c and θ are continuous functions.

For model (1.3.25), condition N1) is assumed, condition N2) is immediate from the particular forms of the functions G and h, condition N3) can be obtained using similar arguments to the ones in Lemma 1.1, condition N4) holds, as already discussed, and condition N5) is immediate. We will check that condition N6) holds if we replace (1.1.5) and (1.1.6) by (1.3.14) and (1.3.15), respectively.

The results in [**107**] allow us to conclude that, for sufficiently small $\varepsilon > 0$, the interior equilibrium $((q - h^u\varepsilon)/d, (p + (h^u)^2\varepsilon)/b)$ of system

$$
\begin{cases}
x' = (p + \varepsilon(h^u)^2 + (q - h^u\varepsilon)h(t) - dh(t)x)x \\
z' = -(q - h^u\varepsilon)z + dxz
\end{cases}
\qquad (1.3.26)
$$

is globally asymptotically stable in the region $\{(x, z) \in \mathbb{R}^2 : x, z > 0\}$. Notice that system (1.3.26) corresponds, in this case, to system (1.1.6) with $v(\varepsilon) = \varepsilon$, $\rho(t) = h(t)$, $G_{2,\varepsilon}(t, x) = (p + (q + h^u\varepsilon)h(t) - dh(t)x)x$ and $h_{2,\varepsilon}(t, z) = -q + (h^u - h(t))\varepsilon$.

By [**107**], we can also conclude that, for sufficiently small $\varepsilon > 0$, the interior equilibrium $((q + \varepsilon)/d, (p - 2\varepsilon h^u)/b)$ of system

$$
\begin{cases}
x' = (p - 2\varepsilon h^u + (q + \varepsilon)h(t) - dh(t)x)x - bxz \\
z' = -(q + \varepsilon)z + dxz
\end{cases}
\qquad (1.3.27)
$$

is globally asymptotically stable in the region $\{(x, z) \in \mathbb{R}^2 : x, z > 0\}$. Notice that system (1.3.27) corresponds, in this case, to system (1.3.14) with $v(\varepsilon) = \varepsilon$, $\rho(t) = h(t)$, $G_{1,\varepsilon}(t, x) = (p - 2\varepsilon h^u + qh(t) - dh(t)x)x$ and $h_{1,\varepsilon}(t, z) = -q - \varepsilon$.

Naturally, the continuity of functions

$$\varepsilon \mapsto ((q - h^u\varepsilon)/d, (p + (h^u)^2\varepsilon)/b) \quad \text{and} \quad \varepsilon \mapsto ((q + \varepsilon)/d, (p - 2\varepsilon h^u)/b)$$

is immediate and we conclude that condition N6) with the mentioned changes also holds.

For this model, the numbers (1.3.16) and (1.2.2) become:

$$\widehat{\mathcal{R}}^\ell(\lambda) = \liminf_{t \to +\infty} \int_t^{t+\lambda} \beta(s)q/d - \eta(s)p/b - c(s)\,ds$$

and

$$\mathcal{R}^u(\lambda) = \limsup_{t \to +\infty} \int_t^{t+\lambda} \beta(s)(p + qh)/d - c(s)\,ds.$$

When the parameters are periodic, we obtain

$$\widehat{\mathcal{R}}^\ell_{per} = \frac{qb\bar\beta}{d(p\bar\eta + b\bar c)} \quad \text{and} \quad \mathcal{R}^u_{per} = \frac{(p + qh)\bar\beta}{d\bar c}$$

and we have the following corollary of Theorems 1.1:

COROLLARY 6. *Assume that the parameters in model* (1.3.25) *are periodic. If* $\mathcal{R}^u_{per} < 1$ *then the infectives in model* (1.3.25) *go to extinction.*

As in the previous example, it remains to answer that if the parameters in model (1.3.25) are periodic and $\widehat{\mathcal{R}}^\ell_{per} > 1$ we have that the infectives in model (1.3.25) are uniformly strong persistent. Again, the following simulations suggests that this is the case.

To do some simulation, we consider the following setting: $p = 0.7$, $q = 0.7$, $h(t) = 0.5(1 + 0.7\cos(2\pi t))$, $d = 0.18$, $b = 0.3$, $\beta(t) = \beta_0(1 + 0.7\cos(2\pi t))$, $\eta(t) = 0.7(1 + 0.7\cos(\pi + 2\pi t))$, $c(t) = 0.1$ and $\theta(t) = 0.9$. We obtain the model:

$$\begin{cases} S' = (0.7 + (1 + 0.7\cos(2\pi t))(0.35 - 0.09S))S - \beta_0(1 + 0.7\cos(2\pi t))SI - 0.3SP \\ I' = \beta_0(1 + 0.7\cos(2\pi t))SI - 0.7(1 + 0.7\cos(\pi + 2\pi t))PI - 0.1I \\ P' = -0.7P + 0.18SP + 0.63(1 + 0.7\cos(\pi + 2\pi t))PI \end{cases}.$$

$$(1.3.28)$$

When $\beta_0 = 0.01$ we obtain $\mathcal{R}^\ell_{per} \approx 0.58 < 1$ and we conclude that the infectives go to extinction (figure 1.9). When $\beta_0 = 0,5$ we obtain $\widehat{\mathcal{R}}^\ell_{per} \approx 1.12 > 1$. Simulation results suggest that the infectives are strong persistent (figure 1.10). In the extinction and strong persistent situations we considered, in $t = 0$, respectively, the

initial condition $(S_0, I_0, P_0) = (7.16, 0.15, 4.5)$ and $(S_0, I_0, P_0) = (2.48, 0.38, 1.95)$. In both situations, we also considered the initial conditions $(S_0, I_0, P_0) = (0.5, 0.1, 0.4)$ and $(S_0, I_0, P_0) = (0.4, 0.04, 0.7)$ in $t = 0$.
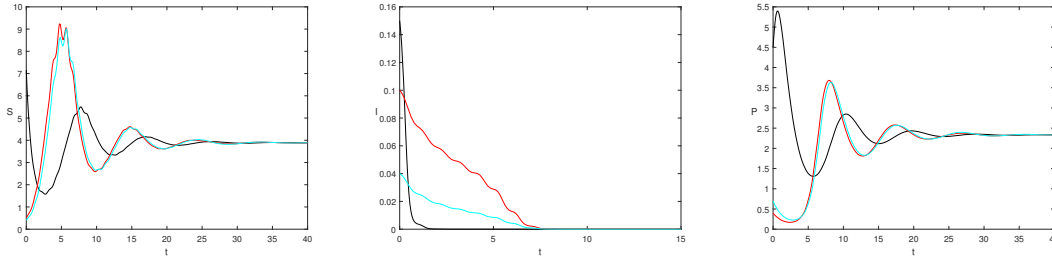
FIGURE 1.9.   Extinction; time-varying coefficients uninfected subsystem I; $\beta_0 = 0.01$.
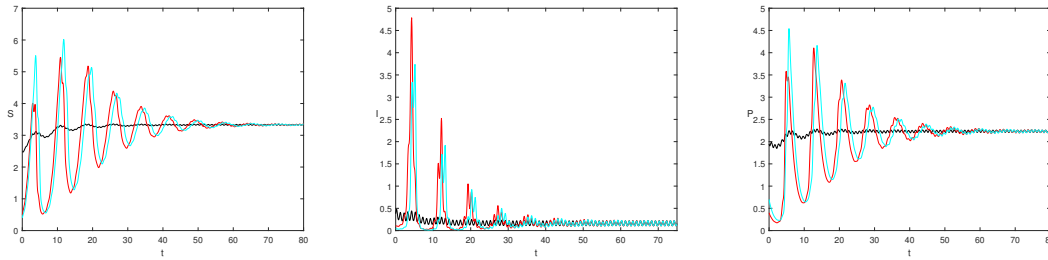
FIGURE 1.10.   Persistence; time-varying coefficients uninfected subsystem I; $\beta_0 = 0.5$.

Note that according to [**91**], if h in system (1.3.25) is replaced by a nonnegative and bounded function then, when $\varepsilon_1 = \varepsilon_2 = 0$, a sufficient condition for system (1.3.26) to be globally asymptotically stable is that $h \in \mathcal{F}_{[WIP]}$, where $F_{[WIP]}$ denotes the class of real functions defined in $[0, +\infty[$ such that

$$\sum_{n=1}^{+\infty} \int_{\tau_n}^{\sigma_n} h(t)\, dt = +\infty,$$

for every pair of sequences satisfying $\tau_n < \sigma_n < \tau_{n+1}$, $\liminf(\sigma_n - \tau_n) > 0$ and $\limsup(\tau_{n+1} - \sigma_n) > 0$. See also [**107**] for a necessary and sufficient condition. It would be interesting to have a theoretical result for this larger class of systems.

1.3.6. Time-varying coefficients in the uninfected subsystem II. Like in the previous subsection, in this we will consider as example with non-autonomous uninfected model but, unlike the uninfected model in the previous subsection, here the time-varying parameters arise in the predator equation. Assuming that $G(t, S) = q$, $f(S, I, P) = S$, $g(S, I, P) = P$, $a(t) = b$, $h(t, P) = -p + qh(t) - bh(t)P$ with $h(t)$ continuous and satisfying

$h^\ell < h(t) < h^u$ for some constants $h^\ell, h^u > 0$, and $\gamma(t) = d/b$ in (1.1.1), we obtain the following particular model:

$$\begin{cases} S' = qS - bSP - \beta(t)SI \\ I' = \beta(t)SI - \eta(t)PI - c(t)I \\ P' = (-p + qh(t) - bh(t)P)P + dSP + \theta(t)\eta(t)PI \end{cases} , \qquad (1.3.29)$$

where we continue to assume that β, η, c and θ are continuous functions.

The uninfected system associated to model (1.3.29) was considered in [80] where global asymptotic stability for the disease-free equilibrium was obtained. For the same model, in [53], under suitable conditions, the global asymptotic stability for the endemic equilibrium was also obtained. The results in [53] allow us to conclude that, for sufficiently small $\varepsilon > 0$, the interior equilibrium $((p - (h^u)^2\varepsilon)/d, (q - h^u\varepsilon)/b)$ of system

$$\begin{cases} x' = (q - h^u\varepsilon)x - dxz \\ z' = (-p + \varepsilon(h^u)^2 + (q - h^u\varepsilon)h(t) - bh(t)z)z + dxz \end{cases} \qquad (1.3.30)$$

is globally asymptotically stable in the region $\{(x, z) \in \mathbb{R}^2 : x, z > 0\}$. Notice that system (1.3.30) corresponds, in this case, to system (1.3.15) with $v(\varepsilon) = \varepsilon$, $\rho(t) = h(t)$, $G_{2,\varepsilon}(t, x) = q + (h(t) - h^u)\varepsilon$ and $h_{2,\varepsilon}(t, z) = -p + (h^u)^2\varepsilon + (q - h^u\varepsilon)h(t) - bh(t)z$.

By [80], we can also conclude that, for sufficiently small $\varepsilon > 0$, the interior equilibrium $((q - \varepsilon)/b, (p - 2\varepsilon h^u)/d)$ of system

$$\begin{cases} x' = (q - \varepsilon)x - bxz \\ z' = (-p + 2\varepsilon h^u + (q - \varepsilon)h(t) - bh(t)z)z + dxz \end{cases} \qquad (1.3.31)$$

is globally asymptotically stable in the region $\{(x, z) \in \mathbb{R}^2 : x, z > 0\}$. Notice that system (1.3.31) corresponds, in this case, to system (1.3.14) with $v(\varepsilon) = \varepsilon$, $\rho(t) = h(t)$, $G_{1,\varepsilon}(t, x) = (q - \varepsilon)x$ and $h_{1,\varepsilon}(t, z) = -p + 2\varepsilon h^u + (q - \varepsilon)h(t) - bh(t)z$.

As we can see, the continuity of functions

$$\varepsilon \mapsto ((p - (h^u)^2\varepsilon)/d, (q - h^u\varepsilon)/b) \quad \text{and} \quad \varepsilon \mapsto ((q - \varepsilon)/b, (p - 2\varepsilon h^u)/d)$$

is immediate. We conclude that condition N6) holds if we replace (1.1.5) and (1.1.6) by (1.3.14) and (1.3.15), respectively .

Let

$$\widehat{\mathcal{R}}^\ell(\lambda) = \liminf_{t \to +\infty} \int_t^{t+\lambda} \beta(s)p/d - \eta(s)q/b - c(s)\, ds$$

and

$$\mathcal{R}^u(\lambda) = \limsup_{t\to+\infty} \int_t^{t+\lambda} \beta(s)(p-qh)/d - c(s)\, ds$$

When the parameters are periodic, we obtain

$$\widehat{\mathcal{R}}_{per}^{\ell} = \frac{pb\bar{\beta}}{d(p\bar{\eta} + b\bar{c})} \quad \text{and} \quad \mathcal{R}_{per}^u = \frac{(p-qh)\bar{\beta}}{d\bar{c}}$$

and we have the following corollary of Theorem 1.1:

COROLLARY 7. *Assume that the parameters in model* (1.3.29) *are periodic. If* $\mathcal{R}_{per}^u < 1$ *then the infectives in model* (1.3.29) *go to extinction.*

It remains to answer that if the parameters in model (1.3.29) are periodic and $\widehat{\mathcal{R}}_{per}^{\ell} > 1$ we have that the infectives in model (1.3.29) are uniformly strong persistent. The following simulations suggests that this is the case.

To do some simulation, we consider the following setting: $p = 0.7$, $q = 0.9$, $h(t) = 0.5(1 + 0.7\cos(100\pi\sqrt{t}))$, $d = 0.18$, $b = 0.3$, $\beta(t) = \beta_0(1 + 0.7\cos(100\pi\sqrt{t}))$, $\eta(t) = 0.7(1 + 0.7\cos(\pi + 100\pi\sqrt{t}))$, $c(t) = 0.1$ and $\theta(t) = 0.9$. We obtain the model:

$$\begin{cases} S' = 0.9S - 0.3SP - \beta_0(1 + 0.7\cos(100\pi\sqrt{t}))SI \\ I' = \beta_0(1 + 0.7\cos(100\pi\sqrt{t}))SI - 0.7(1 + 0.7\cos(\pi + 100\pi\sqrt{t}))PI - 0.1I \\ P' = (-0.7 + (0.45 - 0.15P)(1 + 0.7\cos(100\pi\sqrt{t})))P \\ \quad + 0.63(1 + 0.7\cos(\pi + 100\pi\sqrt{t}))PI + 0.18SP \end{cases} \quad (1.3.32)$$

When $\beta_0 = 0.04$ we obtain $\mathcal{R}_{per}^u \approx 0.78 < 1$ and we conclude that the infectives go to extinction (figure 1.9). When $\beta_0 = 0,5$ we obtain $\widehat{\mathcal{R}}_{per}^{\ell} \approx 1,12 > 1$ and we conclude that the infectives are uniformly strong persistent (figure 1.10). In the extinction and strong persistent situations we considered, in $t = 0$, respectively, the initial condition $(S_0, I_0, P_0) = (3.342, 0.15, 2.23)$ and $(S_0, I_0, P_0) = (3.889, 0.15, 2.334)$ corresponding, respectively, to a disease-free solution and an (approximately) periodic solution. In both situations, we also considered, in $t = 0$, the initial conditions $(S_0, I_0, P_0) = (0.5, 0.1, 0.4)$ and $(S_0, I_0, P_0) = (0.4, 0.04, 0.7)$.
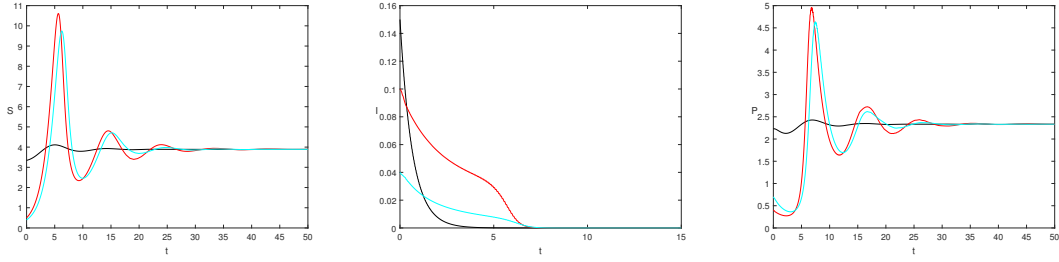
FIGURE 1.11.   Extinction; time-varying coefficients uninfected subsystem II; $\beta_0 = 0.01$.
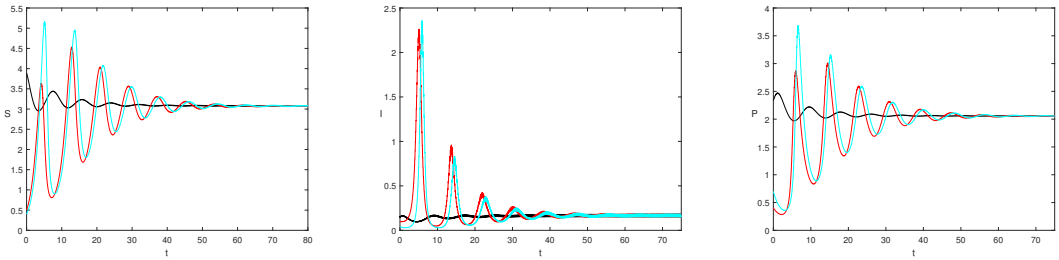
FIGURE 1.12.   Persistence; time-varying coefficients uninfected subsystem II; $\beta_0 = 0.5$.

Based on the simulation carried out, including a considerable amount of tests undertaken for the several models studied, but for parameters that fall into the regions where we are not able to decide, based on Theorems 1.1 and 1.2, if we have extinction or uniform strong persistence (regions where $\mathcal{R}^u(\lambda) > 1$ and $\mathcal{R}^\ell(\lambda) < 1$), we conjecture that for the model considered in examples 1 to 6, we have extinction when

$$\limsup_{t \to +\infty} \int_t^{t+\lambda} \beta(s)x^*(s) - \eta(t)z^*(s) - c(s)\,ds < 0,$$

where $(x^*(t), z^*(t))$ is any particular solution of system (1.1.4) with positive initial conditions.

1.4. Classical Lotka-Volterra interaction

In section 1.1 we considered that in the predator/uninfected prey subspace we had asymptotic stability. This doesn't correspond to the behavior of the classical Lotka-Volterra model, considered independently by Alfred Lotka and Vito Volterra, where the interior equilibrium was stable in the sense of Lyapunov but not asymptotically stable. It would be interesting to use a similar strategy to the one in section 1.1 to study the following eco-epidemiological model, where in the unifected subspace we have a classical Lotka-Volterra

model:

$$\begin{cases} S' = \alpha S - aSP - \beta(t)SI \\ I' = \beta(t)SI - \eta(t)PI - c(t)I \\ P' = \gamma aSP - bP + \theta(t)\eta(t)PI \end{cases} , \qquad (1.4.1)$$

where the constants α, a, γ and b are positive and the real valued functions β, η, c and θ are bounded, nonnegative and continuous.

By similarity with the previous thresholds, we define the number

$$\mathcal{R}_{LV}^{\ell} = \frac{b\beta^{\ell}}{\gamma a} - \frac{\alpha \eta^u}{a} - c^u.$$

To do some simulations we consider the following parameters: $\alpha = 0.7$, $a = 1.2$, $\beta(t) = \beta_0(1 + 0.4\cos(2\pi t))$, $\eta(t) = 0.5(1 + 0.4\cos(\pi + 2\pi t))$, $c(t) = 0.1$, $\gamma = 0.5$, $b = 0.7$, $\theta(t) = 0.9$. We obtain the model:

$$\begin{cases} S' = 0.7S - 1.2SP - \beta_0(1 + 0.4\cos(2\pi t))SI \\ I' = \beta_0(1 + 0.4\cos(2\pi t))SI - 0.5(1 + 0.4\cos(\pi + 2\pi t))PI - 0.1I \\ P' = 0.6SP + 0.45(1 + 0.4\cos(\pi + 2\pi t))PI - 0.7P \end{cases} , \qquad (1.4.2)$$

When $\beta_0 = 0.9$ we obtain $\mathcal{R}_{LV} \approx 0,13 > 0$ and the simulation suggests that the infectives are persistent (figure 1.13). Similar conclusions were obtained in different simulations.

We considered the following initial conditions in $t = 0$: $(S_0, I_0, P_0) = (0.8, 1.7, 0.7)$, $(S_0, I_0, P_0) = (0.6, 1.7, 0.5)$ and $(S_0, I_0, P_0) = (0.4, 1.3, 0.3)$ for the plots in figures 1.13 and 1.14. Additionally, in figure 1.14, we also consider orbits contained in the SI plane corresponding to following initial conditions in $t = 0$: $(S_0, I_0, P_0) = (0.0041, 0.3531, 0)$, $(S_0, I_0, P_0) = (0.0065, 1.2949, 0)$ and $(S_0, I_0, P_0) = (0.0845, 0.4234, 0)$.
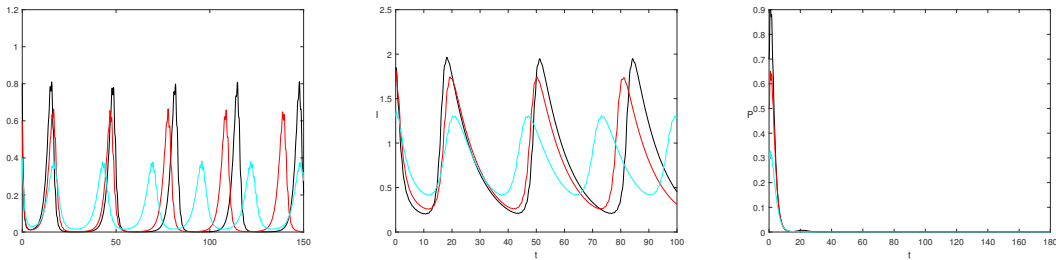
FIGURE 1.13.   Persistence classical Lotka-Volterra interaction; $\beta_0 = 0.9$.
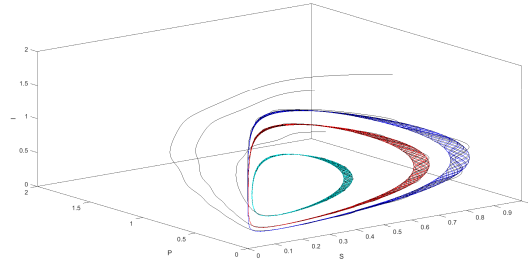
FIGURE 1.14.   Persistence classical Lotka-Volterra interaction; $\beta_0 = 0.9$.

1.5. Comments

In this chapter we studied a general non-autonomous eco-epidemiological model and obtained some results on the extinction and persistence of the infected prey I assuming that the uninfected subsystem in globally asymptotically stable: under some suitable conditions, in Theorem 1 we obtained extinction of the infected prey whenever there is $\lambda > 0$ such that $\mathcal{R}^u(\lambda) < 0$, in Theorem 2 we obtained the uniform strong persistence of the infected prey whenever there is $\lambda > 0$ such that $\mathcal{R}^\ell(\lambda) > 0$, and in Theorem 3 we improved the extinction result by considering better estimates that allowed us to consider a different condition for extinction than the one in Theorem 1, namely it was proved that the infected prey goes to extinction if there is $\lambda > 0$ such that $\mathcal{R}^{u,1}(\lambda) < 0$. Moreover, we describe an iterative process based on the ideas in the proof of Theorem 3 to obtain a sequence of conditions for extinction.

When there is no predation on uninfected preys, the integrals in (1.2.1) and (1.2.2) coincide. One of the natural questions that follows from our results is the following: under which condition we have a similar situation, even in the case of where we have predation on uninfected preys. Another related question is to explore the iterative procedure described after Theorem 1.3.

Still in this chapter we consider several examples to illustrate the applicability of the results mentioned above.

A natural question that arises is if it is possible to still obtain a result on persistence and extinction when the uninfected subsystem is a more general predator prey model.

Periodic Eco-Epidemiological Model

In this chapter we address the existence of periodic orbits for periodic eco-epidemiological system with disease in the prey for two distinct families of models obtained from the general model (0.0.2). For the first one, we use Mawhin's continuation Theorem in a wide general system that includes some models discussed in the literature, and for the second family we obtain a sharp result using a recent strategy that relies on the uniqueness of periodic orbits in the disease-free space.

In more detail, we consider in sections 2.1 and 2.2 a first family of periodic models obtained from (0.0.2) by letting $G(t,S) = \Lambda(t) - \mu(t)S$ and

$$H(t,P) = (r(t) - b(t)P)P, \tag{2.0.1}$$

and in section 2.3 a second family of periodic models obtained from (0.0.2) by letting $G(t,S) = \Lambda(t) - \mu(t)S$ and

$$H(t,P) = \Upsilon(t) - \zeta(t)P. \tag{2.0.2}$$

The first model considered generalizes the model studied in [**90**], a periodic version of the general non-autonomous model introduced in [**75**].

In the first scenario, where H is given by (2.0.1), we allow $r(t)$ to be positive or negative. When $r(t) > 0$ for all t, we obtain a model with linear vital dynamics of susceptible prey in the absence of predators and disease and with logistic vital dynamics of predators in the absence of the considered prey. This model generalizes [**75**]. When $r(t) < 0$ for all t, we obtain a model with a classical vital dynamics of the predators as in the family of Lotka-Volterra models considered in [**38**].

In the second scenario, as referred, we consider a linear vital dynamics for predators by letting H be given by (2.0.2). The model obtained has no periodic solutions on the axis, allowing us to use a different set of arguments to establish the existence of an endemic periodic orbit. Note that, when H is given by (2.0.1), there is space in our model for the possibility that predators survive in the absence of this prey. In fact, when $r(t)$ is nonnegative, the predator have a logistic behaviour. A possible biological interpretation

is that predators in this ecosystem possess different sources of food and, in the absence of the prey in this model, the behavior of the predator population is logistic. When $r(t)$ is nonpositive we obtain a usual behaviour for predators in the absence of preys. When H is given by (2.0.2) predators allways survive in the absence of the prey considered in the model and we also interpret this fact as in the corresponding situation for the first scenario.

In the first scenario, for technical reasons, we have to make the restriction $g(S, I, P) = P$, while in the second scenario we let g be a general function that satisfies some natural assumptions.

In the first situation, $r(t)$ and $b(t)$ are parameters related to the vital dynamics of the predator population that include the intra-specific competition between predators. This vital dynamics is assumed to follow a logistic law when $r(t) > 0$ for all $t \geqslant 0$ and that is similar to the vital dynamics of predator in a family of Lotka-Volterra models considered in [38] when $r(t) < 0$ for all $t \geqslant 0$. In both scenarios $\Lambda(t)$ is the recruitment rate of the prey population, $\mu(t)$ is the natural death rate of the prey population, $a(t)$ is the predation rate of susceptible prey, $\beta(t)$ is the incidence rate, $\eta(t)$ is the predation rate of infected prey, $c(t)$ is the death rate in the infective class $(c(t) \geqslant \mu(t))$, $\gamma(t)$ is the rate of converting susceptible prey into predator (biomass transfer), $\theta(t)$ is the rate of converting infected prey into predator. It is assumed that only susceptible preys S are capable of reproducing, i.e, the infected prey is removed by death (including natural and disease-related death) or by predation before having the possibility of reproducing.

2.1. Classical or logistic vital dynamics for predators

In this section we let $g(S, I, P) = P$, $G(t, S) = \Lambda(t) - \mu(t)S$ and $H(t, P) = (r(t) - b(t)P)P$, obtaining a model that generalizes the model in [90] by considering a function that corresponds to the predation of uninfected preys:

$$\begin{cases} S' = \Lambda(t) - \mu(t)S - a(t)f(S, I, P)P - \beta(t)SI \\ I' = \beta(t)SI - \eta(t)PI - c(t)I \\ P' = (r(t) - b(t)P)P + \gamma(t)a(t)f(S, I, P)P + \theta(t)\eta(t)PI \end{cases} . \qquad (2.1.1)$$

Given a ω-periodic function f we will use throughout the paper the notations $f^\ell = \inf_{t \in (0,\omega]} f(t)$, $f^u = \sup_{t \in (0,\omega]} f(t)$ and $\bar{f} = \frac{1}{\omega} \int_0^\omega f(s)\, ds$. We will assume the following structural hypothesis concerning the parameter functions and the function f appearing in our model:

P1) The real valued functions Λ, μ, a, β, η, c, γ, θ and b are periodic with period ω, nonnegative and continuous; the real valued function r is periodic with period ω and continuous and can be nonnegative or nonpositive;

P2) Function f is nonnegative and C^1;

P3) Function $x \mapsto f(x, y, z)$ is nondecreasing;

P4) Functions $z \mapsto f(x, y, z)$ and $y \mapsto f(x, y, z)$ are nonincreasing;

P5) For all (x, y, z) we have

$$f(x,y,z) + z\frac{\partial f}{\partial z}(x,y,z) > 0, \quad \overline{\eta} + \overline{a}\frac{\partial f}{\partial y}(x,y,z) > 0 \quad \text{and} \quad \overline{\theta\eta} + \overline{\gamma a}\frac{\partial f}{\partial y}(x,y,z) > 0;$$

P6) $\bar{\Lambda} > 0$, $\bar{\mu} > 0$ and $\bar{b} > 0$;

P7) There is $\alpha \geqslant 1$ and $K > 0$ such that $f(x, 0, 0) \leqslant Kx^{\alpha}$.

Note that our functional response must depend on I to be able to include functional response functions with saturation, that must depend on the total population of preys (see [**32**, **8**]). Condition P4) is included in condition N2) in chapter 1. Like in chapter 1, these conditions are satisfied by several usual functional response functions: Holling-type I, Holling-type II, Holling-type III, Beddington-De Angelis and Crowley-Martin. The assumption $f(x, y, z) + z\frac{\partial f}{\partial z}(x, y, z) > 0$ in P5) and condition P7) are also satisfied by all the functional response functions above. The other conditions in P5) are satisfied by Holling-type I functional response functions regardless of the parameter considered and by the other mentioned response functions for suitable choices of parameters.

To formulate our next assumptions we need to consider two auxiliary equations and one auxiliary system. First, for each $\lambda \in (0, 1]$, we need to consider the following equations:

$$x' = \lambda(\Lambda(t) - \mu(t)x) \tag{2.1.2}$$

and

$$z' = \lambda(r(t) - b(t)z)z. \tag{2.1.3}$$

Note that, if we identify x with the susceptible prey population, equation (2.1.2) gives the behaviour of the susceptible preys in the absence of infected preys and predator and identifying z with the predator population, equation (2.1.3) gives the behaviour of the predator in the absence of preys.

Equation (2.1.2) is a linear equation that was considered in countless papers on epidemiological models and equation (2.1.3) was already studied in [**64**]. These equations have a well known behaviour, given in the following lemmas:

LEMMA 2.1. *For each $\lambda \in (0, 1]$ there is a unique ω-periodic solution of equation (2.1.2), $x_\lambda^*(t)$, that is globally asymptotically stable in $\mathbb{R}^+$.*

LEMMA 2.2. *If the function r is nonnegative, for each $\lambda \in (0, 1]$ there is a unique ω-periodic solution of equation (2.1.3), $z_\lambda^*(t)$, that is globally asymptotically stable in $\mathbb{R}^+$. If the function r is nonpositive for each $\lambda \in (0, 1]$ the zero solution of equation (2.1.3), that we still denote by $z_\lambda^*(t)$, is globally asymptotically stable in $\mathbb{R}_0^+$.*

For each $\lambda \in (0, 1]$ and $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$ sufficiently small, we consider the family of systems:

$$\begin{cases} x' = \lambda(\Lambda(t) - \mu(t)x - a(t)f(x, 0, 0)z_{2,\varepsilon,\lambda}^*(t) - \varepsilon_1 x) \\ z' = \lambda(r(t) - b(t)z + \gamma(t)a(t)f(x, \varepsilon_2, z))z \end{cases} \quad (2.1.4)$$

where $(x_{2,\varepsilon,\lambda}^*(t), \ z_{2,\varepsilon,\lambda}^*(t))$ is a solution of

$$\begin{cases} x' = \lambda(\Lambda(t) - \mu(t)x) \\ z' = \lambda(r(t) - b(t)z + \gamma(t)a(t)f(x, 0, z) + \varepsilon)z \end{cases} \quad (2.1.5)$$

satisfying the following assumptions.

P8) The following holds for systems (2.1.4) and (2.1.5):

    P8.1) For each $\lambda \in (0, 1]$ and each $\varepsilon_1, \varepsilon_2, \varepsilon \geqslant 0$ sufficiently small, system (2.1.4) has a unique ω-periodic solution, $(x_{1,\varepsilon_1,\varepsilon_2,\varepsilon,\lambda}^*(t), z_{1,\varepsilon_1,\varepsilon_2,\varepsilon,\lambda}^*(t))$, with

$$x_{1,\varepsilon_1,\varepsilon_2,\varepsilon,\lambda}^*(t) > 0 \quad \text{and} \quad z_{1,\varepsilon_1,\varepsilon_2,\varepsilon,\lambda}^*(t)) > 0,$$

that is globally asymptotically stable in the set

$$\{(x, z) \in (\mathbb{R}_0^+)^2 : x \geqslant 0 \ \wedge \ z > 0\}.$$

We assume that $(\varepsilon_1, \varepsilon_2, \varepsilon) \mapsto (x_{1,\varepsilon_1,\varepsilon_2,\varepsilon,\lambda}^*(t), \ z_{1,\varepsilon_1,\varepsilon_2,\varepsilon,\lambda}^*(t))$ is continuous.

    P8.2) For each $\lambda \in (0, 1]$ and each $\varepsilon \geqslant 0$ sufficiently small, system (2.1.5) has a unique ω-periodic solution, $(x_{2,\varepsilon,\lambda}^*(t), z_{2,\varepsilon,\lambda}^*(t))$, with

$$x_{2,\varepsilon,\lambda}^*(t) > 0 \quad \text{and} \quad z_{2,\varepsilon,\lambda}^*(t)) > 0,$$

that is globally asymptotically stable in the set

$$\{(x, z) \in (\mathbb{R}_0^+)^2 : x \geqslant 0 \ \wedge \ z > 0\}.$$

We assume that $(\varepsilon) \mapsto (x_{2,\varepsilon,\lambda}^*(t), \ z_{2,\varepsilon,\lambda}^*(t))$ is continuous.

We write $x_{1,0,\lambda}^* = x_{1,\lambda}^*$, $x_{2,0,\lambda}^* = x_{2,\lambda}^*$, $z_{1,0,\lambda}^* = z_{1,\lambda}^*$ and $z_{2,0,\lambda}^* = z_{2,\lambda}^*$ for the components of the solutions in P8.1) and P8.2) corresponding to $\varepsilon = 0$.

Condition P8) is similar to condition N6) in chapter 1. We introduce the numbers

$$\overline{\mathcal{R}}_0 = \frac{\bar{\beta}\bar{\Lambda}/\bar{\mu}}{\bar{c} + \overline{\eta}\bar{r}/\bar{b}}, \quad \mathcal{R}_0^\lambda = \frac{\overline{\beta x_{1,\lambda}^*}}{\bar{c} + \overline{\eta z_{2,\lambda}^*}} \quad \text{and} \quad \widetilde{\mathcal{R}}_0 = \inf_{\lambda \in (0,1]} \mathcal{R}_0^\lambda. \tag{2.1.6}$$

Before presenting our main result we have to consider the averaged system corresponding to (2.1.1):

$$\begin{cases} S' = \overline{\Lambda} - \overline{\mu}S - \overline{a}f(S,I,P)P - \overline{\beta}SI \\ I' = \overline{\beta}SI - \overline{\eta}PI - \overline{c}I \\ P' = (\overline{r} - \overline{b}P)P + \overline{\gamma a}f(S,I,P)P + \overline{\theta\eta}PI \end{cases}. \tag{2.1.7}$$

The number $\overline{\mathcal{R}}_0$ is the basic reproductive number of (2.1.7) with $f \equiv 0$. We now present our main result.

THEOREM 2.1. *If $\widetilde{R}_0 > 1$, conditions P1) to P8) hold and there is a unique equilibrium of the averaged system (2.1.7) in $(\mathbb{R}^+)^3$, the interior of the first octant, then system (2.1.1) possesses an endemic periodic orbit of period ω.*

Our proof relies on an application of Mawhin's continuation theorem. We will proceed in several steps. Firstly, in subsection 2.1.2, we consider a one parameter family of systems and obtain uniform bounds for the components of any periodic solution of these systems. Next, in subsection 2.1.3 we make a suitable change of variables in our family of systems to establish the setting where we will apply Mawhin's continuation Theorem. Finally, in subsection 2.1.4, we use Mawhin's continuation Theorem to obtain our result.

2.1.1. Mawhin's continuation theorem. In the following, we state Mawhin's continuation theorem [**39**, Part IV]. Let X and Z be Banach spaces.

DEFINITION 2.1. *A linear map $\mathcal{L} : D \subseteq X \to Z$ is called a Fredholm mapping of index zero if*

1. $\dim \ker \mathcal{L} = \operatorname{codim} \operatorname{Im} \mathcal{L} \leqslant \infty$;

2. $\operatorname{Im} \mathcal{L}$ is closed in Z.

Given a Fredholm mapping of index zero $\mathcal{L} : D \subseteq X \to Z$ it is well known that there are continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that:

1. $\operatorname{Im} P = \ker \mathcal{L}$;

2. $\ker Q = \operatorname{Im} \mathcal{L} = \operatorname{Im}(I - Q)$;

3. $X = \ker \mathcal{L} \oplus \ker P$;

4. $Z = \operatorname{Im} \mathcal{L} \oplus \operatorname{Im} Q$.

It follows that $\mathcal{L}|_{D \bigcap \ker P} : (I - P)X \to \operatorname{Im} \mathcal{L}$ is invertible. We denote the inverse of that map by $\mathcal{K}$.

DEFINITION 2.2. *A continuous mapping* $\mathcal{N} : X \to Z$ *is called* $\mathcal{L}$*-compact on* $\overline{U} \subset X$, *where U is an open bounded set, if*

1. $Q\mathcal{N}(\overline{U})$ *is bounded;*

2. $\mathcal{K}(I - Q)\mathcal{N} : \overline{U} \to X$ *is compact.*

Note that, since $\operatorname{Im} Q$ is isomorphic to $\ker \mathcal{L}$, there is an isomophism $\mathcal{I} : \operatorname{Im} Q \to \ker \mathcal{L}$. We are now prepared to state the Mawhin's continuation theorem.

THEOREM 2.2 (Mawhin's continuation theorem). *Let X and Z be Banach spaces and let $U \subset X$ be an open set. Assume that $\mathcal{L} : D \subseteq X \to Z$ is a Fredholm mapping of index zero and let $\mathcal{N} : X \to Z$ be $\mathcal{L}$-compact on $\overline{U}$. Additionally, assume that*

M1) for each $\lambda \in (0, 1)$ and $x \in \partial U \cap D$ we have $\mathcal{L}x \neq \lambda \mathcal{N}x$;

M2) for each $x \in \partial U \cap \ker \mathcal{L}$ we have $Q\mathcal{N}x \neq 0$;

M3) $\deg(\mathcal{I}Q\mathcal{N}, U \cap \ker \mathcal{L}, 0) \neq 0$.

Then the operator equation $\mathcal{L}x = \mathcal{N}x$ has at least one solution in $D \cap \overline{U}$.

2.1.2. Uniform Persistence for periodic orbits. In this section, to obtain uniform bounds for the components of any periodic solution of the family of systems that we can obtain multiplying the right hand side of (2.1.1) by $\lambda \in (0, 1]$, we need to consider the auxiliary systems:

$$\begin{cases} S'_\lambda = \lambda(\Lambda(t) - \mu(t)S_\lambda - a(t)f(S_\lambda, I_\lambda, P_\lambda)P_\lambda - \beta(t)S_\lambda I_\lambda) \\ I'_\lambda = \lambda(\beta(t)S_\lambda I_\lambda - \eta(t)P_\lambda I_\lambda - c(t)I_\lambda) \\ P'_\lambda = \lambda(\gamma(t)a(t)f(S_\lambda, I_\lambda, P_\lambda)P_\lambda + \theta(t)\eta(t)P_\lambda I_\lambda + r(t)P_\lambda - b(t)P_\lambda^2) \end{cases} \quad (2.1.8)$$

We will consider separately each of the several components of any periodic orbit.

LEMMA 2.3. *Let $x^*_\lambda(t)$ be the unique solution of (2.1.2). There is $L_1 > 0$ such that, for any $\lambda \in (0, 1]$ and any periodic solution $(S_\lambda(t), I_\lambda(t), P_\lambda(t))$ of (2.1.8) with initial conditions $S_\lambda(t_0) = S_0 > 0$, $I_\lambda(t_0) = I_0 > 0$ and $P_\lambda(t_0) = P_0 > 0$, we have $S_\lambda(t) + I_\lambda(t) \leqslant x^*_\lambda(t) \leqslant \Lambda^u/\mu^\ell$ and $S_\lambda \geqslant L_1$, for all $t \in \mathbb{R}$.*

PROOF. Let $(S_\lambda(t), I_\lambda(t), P_\lambda(t))$ be some periodic solution of (2.1.8) with initial conditions $S_\lambda(t_0) = S_0 > 0$, $I_\lambda(t_0) = I_0 > 0$ and $P_\lambda(t_0) = P_0 > 0$. Since $c(t) \geqslant \mu(t)$, we have,

by the first and second equations of (2.1.8),

$$(S_\lambda + I_\lambda)' \leqslant \lambda\Lambda(t) - \lambda\mu(t)S_\lambda - \lambda c(t)I_\lambda \leqslant \lambda\Lambda(t) - \lambda\mu(t)(S_\lambda + I_\lambda).$$

Since, by Lemma 2.1, equation (2.1.2) has a unique periodic orbit, $x_\lambda^*(t)$, that is globally asymptotically stable, we conclude that $S_\lambda(t) + I_\lambda(t) \leqslant x_\lambda^*(t)$ for all $t \in \mathbb{R}$. Comparing equation (2.1.2) with equation $x' = \lambda\Lambda^u - \lambda\mu^\ell x$, we conclude that $x_\lambda^*(t) \leqslant \Lambda^u/\mu^\ell$.

Using conditions P3) and P4), by the third equation of (2.1.8), we have

$$P_\lambda' \leqslant \lambda(r(t) + \gamma(t)a(t)f(x_\lambda^*(t),0,0) + \theta(t)\eta(t)x_\lambda^*(t) - b(t)P_\lambda)P_\lambda \leqslant (\Theta^u - b^\ell P_\lambda)P_\lambda,$$

where function Θ is given by

$$\Theta(t) = \max_{t\in[0,\omega]}\{r(t),0\} + \gamma(t)a(t)f(x_\lambda^*(t),0,0) + \theta(t)\eta(t)x_\lambda^*(t).$$

Thus, comparing with equation (2.1.3) and using Lemma 2.2, we get $P_\lambda(t) \leqslant P_\lambda^*(t) \leqslant \Theta^u/b^\ell$. Using the bound obtained above, since $-\beta(t)S_\lambda(t) \geqslant -\beta(t)x_\lambda^*(t)$, we have, by conditions P3), P4) and P7),

$$\begin{aligned}
S_\lambda' &= \lambda\Lambda(t) - \lambda\mu(t)S_\lambda - \lambda a(t)f(S_\lambda, I_\lambda, P_\lambda)P_\lambda - \lambda\beta(t)S_\lambda I_\lambda \\
&\geqslant \lambda\Lambda^\ell - \left(\lambda\mu^u + \lambda a^u \frac{f(S_\lambda,0,0)}{S_\lambda}\frac{\Theta^u}{b^\ell} + \lambda\beta^u(x_\lambda^*)^u\right)S_\lambda \\
&\geqslant \lambda\Lambda^\ell - \left(\lambda\mu^u + \lambda a^u K((x_\lambda^*)^u)^{\alpha-1}\Theta^u/b^\ell + \lambda\beta^u(x_\lambda^*)^u\right)S_\lambda
\end{aligned}$$

According to computations above we have $x_\lambda^*(t) \leqslant \Lambda^u/\mu^\ell$ and thus

$$S_\lambda(t) \geqslant \frac{\lambda\Lambda^\ell}{\lambda\mu^u + \lambda a^u K(\Lambda^u/\mu^\ell)^{\alpha-1}\Theta^u/b^\ell + \lambda\beta^u\Lambda^u/\mu^\ell} =: L_1.$$

$\square$

LEMMA 2.4. *Let $z_\lambda^*(t)$ be the unique solution of (2.1.3). There is $L_2 > 0$ such that, for any $\lambda \in (0,1]$ and any periodic solution $(S_\lambda(t), I_\lambda(t), P_\lambda(t))$ of (2.1.8) with initial conditions $S_\lambda(t_0) = S_0 > 0$, $I_\lambda(t_0) = I_0 > 0$ and $P_\lambda(t_0) = P_0 > 0$, we have $z_\lambda^*(t) \leqslant P_\lambda(t) \leqslant L_2$, for all $t \in \mathbb{R}$.*

PROOF. Let $\lambda \in (0,1]$ and $(S_\lambda(t), I_\lambda(t), P_\lambda(t))$ be any periodic solution of (2.1.8) with initial conditions $S_\lambda(t_0) = S_0 > 0$, $I_\lambda(t_0) = I_0 > 0$ and $P_\lambda(t_0) = P_0 > 0$. We have

$$P_\lambda' = \lambda P_\lambda(\gamma(t)a(t)f(S_\lambda, I_\lambda, P_\lambda) + \theta(t)\eta(t)I_\lambda + r(t) - b(t)P_\lambda) \geqslant (\lambda r(t) - \lambda b(t)P_\lambda)P_\lambda.$$

Comparing the previous inequality with equation (2.1.3) and using Lemma 2.2, we get $P_\lambda(t) \geqslant z_\lambda^*(t)$.

Using the computations in proof of the previous lemma, we have $P_\lambda(t) \leqslant L_1$ and we take $L_2 = L_1$. $\qquad\square$

LEMMA 2.5. *Let $\widetilde{\mathcal{R}}_0 > 1$. There are $L_3, L_4 > 0$ such that, for any $\lambda \in (0, 1]$ and any periodic solution $(S_\lambda(t), I_\lambda(t), P_\lambda(t))$ of (2.1.8) with initial conditions $S_\lambda(t_0) = S_0 > 0$, $I_\lambda(t_0) = I_0 > 0$ and $P_\lambda(t_0) = P_0 > 0$, we have $L_3 \leqslant I_\lambda(t) \leqslant L_4$, for all $t \in \mathbb{R}$.*

PROOF. We will first prove that there is $\varepsilon_1 > 0$ such that, for any $\lambda \in (0, 1]$, we have

$$\limsup_{t \to +\infty} I_\lambda(t) \geqslant \varepsilon_1. \tag{2.1.9}$$

By contradiction, assume that (2.1.9) does not hold. Then, for any $\varepsilon > 0$, there must be $\lambda > 0$ such that $I_\lambda(t) < \varepsilon$ for all $t \in \mathbb{R}$. We have

$$\begin{cases} S'_\lambda \leqslant \lambda\Lambda(t) - \lambda\mu(t)S_\lambda \\ P'_\lambda \leqslant \lambda(\gamma(t)a(t)f(S_\lambda, 0, P_\lambda) + r(t) - b(t)P_\lambda + \varepsilon\theta^u\eta^u)P_\lambda \end{cases}.$$

We have $S_\lambda(t) \leqslant x^*_{2,\varepsilon\theta^u\eta^u,\lambda}(t)$ and $P_\lambda(t) \leqslant z^*_{2,\varepsilon\theta^u\eta^u,\lambda}(t)$, for all $t \geqslant t_1$ whenever $S_\lambda(t_1) = x^*_{2,\varepsilon\theta^u\eta^u,\lambda}(t_1)$ and $P_\lambda(t_1) = z^*_{2,\varepsilon\theta^u\eta^u,\lambda}(t_1)$. Moreover, since

$$\begin{cases} S'_\lambda \geqslant \lambda\Lambda(t) - \lambda\mu(t)S_\lambda - \lambda a(t)f(S_\lambda, 0, 0)z^*_{2,\varepsilon\theta^u\eta^u,\lambda}(t) - \varepsilon\lambda\beta^u S_\lambda \\ P'_\lambda \geqslant \lambda(\gamma(t)a(t)f(S_\lambda, \varepsilon, P_\lambda) + r(t) - b(t)P_\lambda)P_\lambda \end{cases}.$$

we have $S_\lambda(t) \geqslant x^*_{1,\varepsilon\beta^u,\varepsilon,\varepsilon\theta^u\eta^u\lambda}(t)$ and $P_\lambda(t) \geqslant z^*_{1,\varepsilon\beta^u,\varepsilon,\varepsilon\theta^u\eta^u\lambda}(t)$ for all $t \geqslant t_2 \geq t_1$, whenever $S_\lambda(t_2) = x^*_{2,\varepsilon\theta^u\eta^u,\lambda}(t_2)$ and $P_\lambda(t_2) = z^*_{2,\varepsilon\theta^u\eta^u,\lambda}(t_2)$. Thus, using condition P8), we have

$$\begin{aligned} I'_\lambda &= \lambda(\beta(t)S_\lambda - \eta(t)P_\lambda - c(t))I_\lambda \\ &\geqslant (\lambda\beta(t)x^*_{1,\varepsilon\beta^u,\varepsilon,\varepsilon\theta^u\eta^u\lambda}(t) - \lambda\eta(t)z^*_{2,\varepsilon\theta^u\eta^u,\lambda}(t) - \lambda c(t))I_\lambda \qquad (2.1.10) \\ &\geqslant (\lambda\beta(t)x^*_{1,\lambda}(t) - \lambda\eta(t)z^*_{2,\lambda}(t) - \lambda c(t) - \varphi(\varepsilon))I_\lambda, \end{aligned}$$

where φ is a nonnegative function such that $\varphi(\varepsilon) \to 0$ as $\varepsilon \to 0$ (notice that, by continuity, we can assume that φ is independent of λ and, by periodicity of the parameter functions, it is independent of t). Integrating in $[0, \omega]$ and using (2.1.10), we get

$$\begin{aligned} 0 &= \frac{1}{\omega}\left(\ln I_\lambda(\omega) - \ln I_\lambda(0)\right) = \frac{1}{\omega}\int_0^\omega I'_\lambda(s)/I_\lambda(s)\,ds \\ &\geqslant \lambda\left(\overline{\beta x^*_{1,\lambda}} - \bar{c} - \overline{\eta z^*_{2,\lambda}}\right) + \varphi(\varepsilon) = \lambda(\bar{c} + \overline{\eta z^*_{2,\lambda}})(\mathcal{R}^\lambda_0 - 1) + \varphi(\varepsilon) \end{aligned}$$

and since

$$\mathcal{R}^\lambda_0 \geqslant \inf_{\ell \in (0,1]} \mathcal{R}^\ell_0 = \widetilde{\mathcal{R}}_0 > 1,$$

we have a contradiction. We conclude that (2.1.9) holds. Next we will prove that there is $\varepsilon_2 > 0$ such that, for any $\lambda \in (0, 1]$, we have

$$\liminf_{t \to +\infty} I_\lambda(t) \geqslant \varepsilon_2. \tag{2.1.11}$$

Assuming by contradiction that (2.1.11) does not hold, we conclude that there is a sequence $(\lambda_n, I_{\lambda_n}(s_n), I_{\lambda_n}(t_n)) \subset (0, 1] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$ such that $s_n < t_n$, $t_n - s_n \leqslant \omega$,

$$I_{\lambda_n}(s_n) = 1/n, \quad I_{\lambda_n}(t_n) = \varepsilon_2/2 \quad \text{and} \quad I_{\lambda_n}(t) \in (1/n, \varepsilon_2/2), \text{ for all } t \in (s_n, t_n).$$

Since $\lambda_n \leqslant 1$, by Lemma 2.3 we have

$$I'_{\lambda_n} = (\lambda_n \beta(t) S_{\lambda_n} - \lambda_n \eta(t) P_{\lambda_n} - \lambda_n c(t)) I_{\lambda_n} \leqslant \beta^u \Lambda^u I_{\lambda_n} / \mu^\ell$$

and thus

$$\ln(\varepsilon_2 n/2) = \ln(I_{\lambda_n}(t_n)/I_{\lambda_n}(s_n)) = \int_{s_n}^{t_n} I'_{\lambda_n}(s)/I_{\lambda_n}(s) \, ds \leqslant \beta^u \Lambda^u \omega / \mu^\ell,$$

which is a contradiction since the sequence $(\ln(\varepsilon_2 n/2))_{n \in \mathbb{N}}$ goes to $+\infty$ as $n \to +\infty$, and thus is not bounded.

We conclude that there is $\varepsilon_2 > 0$ such that (2.1.11) holds. Letting $L_3 = \varepsilon_2$, we obtain $I_\lambda(t) \geqslant L_3$ for all $\lambda \in (0, 1]$.

Since $I_\lambda(t) \leqslant S_\lambda(t) + I_\lambda(t)$, by Lemma 2.3, we can take $L_4 = L_2$ and the result is established. $\qquad\square$

2.1.3. Mawhin's continuation theorem setting.

To apply Mawhin's continuation theorem to our model we make the change of variables: $S(t) = e^{u_1(t)}$, $I(t) = e^{u_2(t)}$ and $P(t) = e^{u_3(t)}$. With this change of variables, system (2.1.1) becomes

$$\begin{cases} u_1' = \Lambda(t)e^{-u_1} - a(t)f(e^{u_1}, e^{u_2}, e^{u_3})e^{u_3-u_1} - \beta(t)e^{u_2} - \mu(t) \\ u_2' = \beta(t)e^{u_1} - \eta(t)e^{u_3} - c(t) \\ u_3' = \gamma(t)a(t)f(e^{u_1}, e^{u_2}, e^{u_3}) + \theta(t)\eta(t)e^{u_2} - b(t)e^{u_3} + r(t) \end{cases} \tag{2.1.12}$$

Note that, if $(u_1^*(t), u_2^*(t), u_3^*(t))$ is an ω-periodic solution of the system (2.1.12) then $(e^{u_1(t)}, e^{u_2(t)}, e^{u_3(t)})$ is an ω-periodic solution of system (2.1.1).

To define the operators in Mawhin's theorem (see subsection 2.1.1), we need to consider the Banach spaces $(X, \|\cdot\|)$ and $(Z, \|\cdot\|)$ where X and Z are the space of ω-periodic continuous functions $u : \mathbb{R} \to \mathbb{R}^3$:

$$X = Z = \{u = (u_1, u_2, u_3) \in C(\mathbb{R}, \mathbb{R}^3) : u(t) = u(t + \omega)\}$$

and

$$\|u\| = \max_{t \in [0,\omega]} |u_1(t)| + \max_{t \in [0,\omega]} |u_2(t)| + \max_{t \in [0,\omega]} |u_3(t)|.$$

Next, we consider the linear map $\mathcal{L} : X \cap C^1(\mathbb{R}, \mathbb{R}^3) \to Z$ given by

$$\mathcal{L}u(t) = \frac{du(t)}{dt} \tag{2.1.13}$$

and the map $\mathcal{N} : X \to Z$ defined by

$$\mathcal{N}u(t) = \begin{bmatrix} \Lambda(t)e^{-u_1(t)} - a(t)f(e^{u_1}, e^{u_2}, e^{u_3})e^{u_3(t)-u_1(t)} - \beta(t)e^{u_2(t)} - \mu(t) \\ \beta(t)e^{u_1(t)} - \eta(t)e^{u_3(t)} - c(t) \\ \gamma(t)a(t)f(e^{u_1}, e^{u_2}, e^{u_3}) + \theta(t)\eta(t)e^{u_2(t)} - b(t)e^{u_3(t)} + r(t) \end{bmatrix}. \tag{2.1.14}$$

In the following lemma we show that the linear map in (2.1.13) is a Fredholm mapping of index zero

LEMMA 2.6. *The linear map $\mathcal{L}$ in (2.1.13) is a Fredholm mapping of index zero.*

PROOF. We have

$$\ker \mathcal{L} = \left\{ (u_1, u_2, u_3) \in X \cap C^1(\mathbb{R}, \mathbb{R}^3) : \frac{du_i(t)}{dt} = 0, \quad i = 1, 2, 3 \right\}$$
$$= \left\{ (u_1, u_2, u_3) \in X \cap C^1(\mathbb{R}, \mathbb{R}^3) : u_i \text{ is constant, } i = 1, 2, 3 \right\}$$

and thus $\ker \mathcal{L}$ can be identified with $\mathbb{R}^3$. Therefore $\dim \ker \mathcal{L} = 3$. On the other hand

$$\operatorname{Im} \mathcal{L} = \left\{ (z_1, z_2, z_3) \in Z : \exists\, u \in X \cap C^1(\mathbb{R}, \mathbb{R}^3) : \frac{du_i(t)}{dt} = z_i(t), \ i = 1, 2, 3 \right\}$$
$$= \left\{ (z_1, z_2, z_3) \in Z : \int_0^\omega z_i(s) \, ds = 0, \ i = 1, 2, 3 \right\}.$$

and any $z \in Z$ can be written as $z = \tilde{z} + \alpha$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ and $\tilde{z} \in \operatorname{Im} \mathcal{L}$. Thus the complementary space of $\operatorname{Im} \mathcal{L}$ consists of the constant functions. Thus, the complementary space has dimension 3 and therefore $\operatorname{codim} \operatorname{Im} \mathcal{L} = 3$.

Given any sequence (z_n) in $\operatorname{Im} \mathcal{L}$ such that

$$z_n = ((z_1)_n, (z_2)_n, (z_3)_n) \to z = (z_1, z_2, z_3),$$

we have, for $i = 1, 2, 3$ (note that $z \in Z$ since Z is a Banach space and thus it is integrable in $[0, \omega]$ since it is continuous in that interval),

$$\int_0^\omega z_i(s) \, ds = \int_0^\omega \lim_{n \to +\infty} (z_i)_n(s) \, ds = \lim_{n \to +\infty} \int_0^\omega (z_i)_n(s) \, ds = 0.$$

Thus, $z \in \operatorname{Im} \mathcal{L}$ and we conclude that $\operatorname{Im} \mathcal{L}$ is closed in Z. Thus $\mathcal{L}$ is a Fredholm mapping of index zero.   □

Consider the projectors $P : X \to X$ and $Q : Z \to Z$ given by

$$Pu(t) = \frac{1}{\omega} \int_0^\omega u(s)ds \quad \text{and} \quad Qz(t) = \frac{1}{\omega} \int_0^\omega z(s)ds.$$

Note that $\operatorname{Im} P = \ker \mathcal{L}$ and that $\ker Q = \operatorname{Im}(I - Q) = \operatorname{Im} \mathcal{L}$.

Consider the generalized inverse of $\mathcal{L}$, $\mathcal{K} : \operatorname{Im} \mathcal{L} \to D \cap \ker P$, given by

$$\mathcal{K}z(t) = \int_0^t z(s)ds - \frac{1}{\omega} \int_0^\omega \int_0^r z(s)\, ds\, dr$$

the operator $Q\mathcal{N} : X \to Z$ given by

$$Q\mathcal{N}u(t) = \begin{bmatrix} \frac{1}{\omega} \int_0^\omega \Lambda(s)e^{-u_1(s)} - a(s)f(e^{u_1(s)}, e^{u_2(s)}, e^{u_3(s)})e^{u_3(s)} - \beta(s)e^{u_2(s)}\, ds - \overline{\mu} \\[2ex] \frac{1}{\omega} \int_0^\omega \beta(s)e^{u_1(s)} - \eta(s)e^{u_3(s)}\, ds - \overline{c} \\[2ex] \frac{1}{\omega} \int_0^\omega \gamma(s)a(s)f(e^{u_1(s)}, e^{u_2(s)}, e^{u_3(s)})e^{u_3(s)} + \theta(s)\eta(s)e^{u_2(s)} - b(s)e^{u_3(s)}\, ds + \overline{r} \end{bmatrix}$$

and the mapping $\mathcal{K}(I - Q)\mathcal{N} : X \to D \cap \ker P$ given by

$$\mathcal{K}(I - Q)\mathcal{N}u(t) = B_1(t) - B_2(t) - B_3(t),$$

where

$$B_1(t) = \begin{bmatrix} \int_0^t \Lambda(s)e^{-u_1(s)} - a(s)f(e^{u_1}, e^{u_2}, e^{u_3})e^{u_3(s)} - \beta(s)e^{u_2(s)} - \mu(s)\, ds \\[2ex] \int_0^t \beta(s)e^{u_1(s)} - \eta(s)e^{u_3(s)} - c(s)\, ds \\[2ex] \int_0^t \gamma(s)a(s)f(e^{u_1}, e^{u_2}, e^{u_3})e^{u_3(s)} + \theta(s)\eta(s)e^{u_2(s)} - b(s)e^{u_3(s)}dt + r(s)\, ds \end{bmatrix},$$

$$B_2(t) = \begin{bmatrix} \frac{1}{\omega} \int_0^\omega \int_0^r \Lambda(s)e^{-u_1(s)} - a(s)f(e^{u_1}, e^{u_2}, e^{u_3})e^{u_3(s)} - \beta(s)e^{u_2(s)} - \mu(s)\, ds\, dr \\[2ex] \frac{1}{\omega} \int_0^\omega \int_0^r \beta(s)e^{u_1(s)} - \eta(s)e^{u_3(s)} - c(s)\, ds\, dr \\[2ex] \frac{1}{\omega} \int_0^\omega \int_0^r \gamma(s)a(s)f(e^{u_1}, e^{u_2}, e^{u_3})e^{u_3(s)} + \theta(s)\eta(s)e^{u_2(s)} - b(s)e^{u_3(s)} + r(s)\, ds\, dr \end{bmatrix},$$

and

$$B_3(t) = \left(\frac{t}{\omega} - \frac{1}{2} \right) \begin{bmatrix} \int_0^\omega \Lambda(s)e^{-u_1(s)} - a(s)f(e^{u_1}, e^{u_2}, e^{u_3})e^{u_3(s)} - \beta(s)e^{u_2(s)} - \mu(s)\, ds \\[2ex] \int_0^\omega \beta(s)e^{u_1(s)} - \eta(s)e^{u_3(s)} - c(s)\, ds \\[2ex] \int_0^\omega \gamma(s)a(s)f(e^{u_1}, e^{u_2}, e^{u_3})e^{u_3(s)} + \theta(s)\eta(s)e^{u_2(s)} - b(s)e^{u_3(s)} + r(s)\, ds \end{bmatrix}.$$

The next lemma shows that $\mathcal{N}$ is $\mathcal{L}$-compact in the closure of any open bounded subset of its domain.

LEMMA 2.7. *The map $\mathcal{N}$ is $\mathcal{L}$-compact in the closure of any open bounded set $U \subseteq X$.*

PROOF. Let $U \subseteq X$ be an open bounded set and $\overline{U}$ its closure in X. Then, there is $M > 0$ such that, for any $u = (u_1, u_2, u_3) \in \overline{U}$, we have that $|u_i(t)| \leqslant M$, $i = 1, 2, 3$. Letting $Q\mathcal{N}u = ((Q\mathcal{N})_1 u, (Q\mathcal{N})_2 u, (Q\mathcal{N})_3 u)$, we have

$$|(Q\mathcal{N})_1 u(t)| \leqslant \mathrm{e}^M \left(\bar{\Lambda} + \bar{a} f(e^M, 0, 0) + \bar{\beta} \right) + \bar{\mu},$$

$$|(Q\mathcal{N})_2 u(t)| \leqslant e^M (\bar{\beta} + \bar{\eta}) + \bar{c}$$

$$|(Q\mathcal{N})_3 u(t)| \leqslant \mathrm{e}^M \left(\overline{\gamma a} f(e^M, 0, 0) + \overline{\theta \eta} + \bar{b} \right) + \bar{r}$$

and we conclude that $Q\mathcal{N}(\overline{U})$ is bounded.

Let now

$$\mathcal{K}(I - Q)\mathcal{N}u = ((\mathcal{K}(I - Q)\mathcal{N})_1 u, (\mathcal{K}(I - Q)\mathcal{N})_2 u, (\mathcal{K}(I - Q)\mathcal{N})_3 u).$$

Let $B \subset X$ be a bounded set. Note that the boundedness of B implies that there is M such that $|u_i| < M$, for all $i = 1, 2, 3$, and all $u = (u_1, u_2, u_3) \in B$. It is immediate that $\{\mathcal{K}(I - Q)\mathcal{N}u : u \in B\}$ is pointwise bounded. Given $u = (u_1, u_2, u_3)_{n \in \mathbb{N}} \in B$ we have

$$\mathcal{K}(I - Q)\mathcal{N})_1 u(t) - (\mathcal{K}(I - Q)\mathcal{N})_1 u(v)$$

$$= \int_v^t \Lambda(s) e^{-u_1(s)} - a(s) f(e^{u_1(s)}, e^{u_2(s)}, e^{u_3(s)}) e^{u_2(s)} - \beta(s) e^{u_2(s)} - \mu(s) \, ds$$

$$- \frac{t - v}{\omega} \int_0^\omega \Lambda(s) e^{-u_1(s)} - a(s) f(e^{u_1(s)}, e^{u_2(s)}, e^{u_3(s)}) e^{u_2(s)} - \beta(s) e^{u_2(s)} - \mu(s) \, ds$$

$$\leqslant 2(t - v) \left[e^M (\Lambda^u + a^u f(e^M, 0, 0) + \beta^u e^M) + \mu^M \right],$$

$$(2.1.15)$$

and similarly

$$(\mathcal{K}(I - Q)\mathcal{N})_2 u(t) - (\mathcal{K}(I - Q)\mathcal{N})_2 u(v) \leqslant 2(t - v) \left[e^M (\beta^u + \eta^u) + c^u \right] \qquad (2.1.16)$$

and

$$(\mathcal{K}(I - Q)\mathcal{N})_3 u(t) - (\mathcal{K}(I - Q)\mathcal{N})_3 u(v))$$

$$\leqslant 2(t - v) \left[(\gamma^u a^u f(e^M, 0, 0) + \theta^u \eta^u + b^u) e^M + r^u \right]. \qquad (2.1.17)$$

By (2.1.15), (2.1.16) and (2.1.17), we conclude that $\{\mathcal{K}(I-Q)\mathcal{N}u : u \in B\}$ is equicontinuous. Therefore, by Ascoli-Arzela's theorem, $\mathcal{K}(I-Q)\mathcal{N}(B)$ is relatively compact. Thus the operator $\mathcal{K}(I-Q)\mathcal{N}$ is compact.

We conclude that $\mathcal{N}$ is $\mathcal{L}$-compact in the closure of any bounded set contained in X. $\qquad\square$

2.1.4. Application of Mawhin's continuation theorem. In this section we will construct the set where, applying Mahwin's continuation theorem, we will find the periodic orbit in the statement of our result.

Consider the system of algebraic equations:

$$\begin{cases} \overline{\Lambda}e^{-u_1} - \overline{a}f(e^{u_1}, e^{u_2}, e^{u_3})e^{u_3-u_1} - \overline{\beta}e^{u_2} - \overline{\mu} = 0 \\ \overline{\beta}e^{u_1} - \overline{\eta}e^{u_3} - \overline{c} = 0 \\ \overline{\gamma a}f(e^{u_1}, e^{u_2}, e^{u_3}) + \overline{\theta\eta}e^{u_2} - \overline{b}e^{u_3} + \overline{r} = 0 \end{cases} . \qquad (2.1.18)$$

Note that, by hypothesis, the system above has a unique solution on the interior of the first octant. Denote this solution by $p^*(t) = (p_1^*, p_2^*, p_3^*)$. Note also that, by the second equation, we get

$$\overline{\eta}e^{u_3} = \overline{\beta}e^{u_1} - c. \qquad (2.1.19)$$

By Lemmas 2.3, 2.4 and 2.5, there is a constant $M_0 > 0$ such that $\|u_\lambda(t)\| < M_0$, for any $t \in [0, \omega]$ and any periodic solution $u_\lambda(t)$ of (2.1.8). Let

$$U = \{(u_1, u_2, u_3) \in X : \|(u_1, u_2, u_3)\| < M_0 + \|p^*\|\}. \qquad (2.1.20)$$

Conditions M1) and M2) in Mawhin's continuation theorem are fulfilled in the set U defined in (2.1.20).

Using the notation $v = (e^{p_1^*}, e^{p_2^*}, e^{p_3^*})$, the Jacobian matrix of the vector field corresponding to (2.1.18) computed in (p_1^*, p_2^*, p_3^*) is

$$J = \begin{bmatrix} -\overline{a}\frac{\partial f}{\partial S}(v)\,e^{p_3^*} - \overline{\beta}\,e^{p_2^*} - \overline{\mu} & -\overline{\beta}\,e^{p_2^*} - \overline{a}\frac{\partial f}{\partial I}(v)\,e^{p_3^*+p_2^*-p_1^*} & -\overline{a}\frac{\partial f}{\partial P}(v)\,e^{2p_3^*-p_1^*} - \overline{a}f(v)\,e^{p_3^*-p_1^*} \\ \overline{\beta}\,e^{p_1^*} & 0 & -\overline{\eta}\,e^{p_3^*} \\ \overline{\gamma a}\frac{\partial f}{\partial S}(v)\,e^{p_1^*} & \overline{\theta\eta}\,e^{p_2^*} + \overline{\gamma a}\frac{\partial f}{\partial I}(v)\,e^{p_2^*} & \overline{\gamma a}\frac{\partial f}{\partial P}(v)\,e^{p_3^*} - \overline{b}\,e^{p_3^*} \end{bmatrix}.$$

Thus

$$\det J(p_1^*, p_2^*, p_3^*)$$

$$= -\overline{\beta}\,e^{p_1^*}\left(-\overline{\beta}\,e^{p_2^*}\left(\overline{\gamma a}\frac{\partial f}{\partial P}(v)\,e^{p_3^*} -\overline{b}\,e^{p_3^*}\right) + \left(\overline{a}\frac{\partial f}{\partial P}(v)\,e^{2p_3^*-p_1^*} +\overline{a}f(v)\,e^{p_3^*-p_1^*}\right)\overline{\theta\eta}\,e^{p_2^*}\right)$$

$$-\overline{\beta}\,e^{p_1^*}\left(-\overline{a}\frac{\partial f}{\partial I}(v)\,e^{p_2^*+p_3^*-p_1^*}\left(\overline{\gamma a}\frac{\partial f}{\partial P}(v)\,e^{p_3^*} -\overline{b}\,e^{p_3^*}\right) + \left(\overline{a}\frac{\partial f}{\partial P}(v)\,e^{2p_3^*-p_1^*} +\overline{a}f(v)\,e^{p_3^*-p_1^*}\right)\overline{\gamma a}\frac{\partial f}{\partial I}(v)\,e^{p_2^*}\right)$$

$$+\overline{\eta}\,e^{p_3^*}\left(\left(-\overline{a}\frac{\partial f}{\partial S}(v)\,e^{p_3^*} -\overline{\beta}\,e^{p_2^*} -\overline{\mu}\right)\overline{\theta\eta}\,e^{p_2^*} +\overline{\beta}\,e^{p_2^*}\,\overline{\gamma a}\frac{\partial f}{\partial S}(v)\,e^{p_1^*}\right)$$

$$+\overline{\eta}\,e^{p_3^*}\left(\left(-\overline{a}\frac{\partial f}{\partial S}(v)\,e^{p_3^*} -\overline{\beta}\,e^{p_2^*} -\overline{\mu}\right)\overline{\gamma a}\frac{\partial f}{\partial I}(v)\,e^{p_2^*} +\overline{a}\frac{\partial f}{\partial I}(v)\,e^{p_2^*+p_3^*-p_1^*}\,\overline{\gamma a}\frac{\partial f}{\partial S}(v)\,e^{p_1^*}\right)$$

$$= -\overline{\beta}\,e^{p_1^*}\left(-\left(\overline{\beta} +\overline{a}\frac{\partial f}{\partial I}(v)\,e^{p_3^*-p_1^*}\right)e^{p_2^*}\left(\overline{\gamma a}\frac{\partial f}{\partial P}(v)\,e^{p_3^*} -\overline{b}\,e^{p_3^*}\right)\right.$$

$$+ \left(\overline{a}\frac{\partial f}{\partial P}(v)\,e^{2p_3^*-p_1^*} +\overline{a}f(v)\,e^{p_3^*-p_1^*}\right)\left(\overline{\theta\eta} +\overline{\gamma a}\frac{\partial f}{\partial I}(v)\right)e^{p_2^*}\right)$$

$$+\overline{\eta}\,e^{p_3^*}\left(\left(-\overline{a}\frac{\partial f}{\partial S}(v)\,e^{p_3^*} -\overline{\beta}\,e^{p_2^*} -\overline{\mu}\right)\left(\overline{\theta\eta} +\overline{\gamma a}\frac{\partial f}{\partial I}(v)\right)e^{p_2^*}\right.$$

$$\left.+ \left(\overline{\beta}\,e^{p_2^*} +\overline{a}\frac{\partial f}{\partial I}(v)\,e^{p_2^*+p_3^*-p_1^*}\right)\overline{\gamma a}\frac{\partial f}{\partial S}(v)\,e^{p_1^*}\right).$$

Taking into account P5) and (2.1.19), we have

$$\det J(p_1^*, p_2^*, p_3^*)$$

$$= -\overline{\beta}\,e^{p_1^*}\left(-\frac{\overline{\beta}}{\overline{\eta}}\left(\overline{\eta} +\overline{a}\frac{\partial f}{\partial I}(v) -\frac{\overline{ac}}{\overline{\beta}}\frac{\partial f}{\partial I}(v)\,e^{-p_1^*}\right)e^{p_2^*}\left(\overline{\gamma a}\frac{\partial f}{\partial P}(v)\,e^{p_3^*} -\overline{b}\,e^{p_3^*}\right)\right.$$

$$+\overline{a}\,e^{p_3^*-p_1^*}\left(\frac{\partial f}{\partial P}(v)\,e^{p_3^*} +f(v)\right)\left(\overline{\theta\eta} +\overline{\gamma a}\frac{\partial f}{\partial I}(v)\right)e^{p_2^*}\right)$$

$$+\overline{\eta}\,e^{p_3^*}\left(\left(-\overline{a}\frac{\partial f}{\partial S}(v)\,e^{p_3^*} -\overline{\beta}\,e^{p_2^*} -\overline{\mu}\right)\left(\overline{\theta\eta} +\overline{\gamma a}\frac{\partial f}{\partial I}(v)\right)e^{p_2^*}\right.$$

$$\left.+\frac{\overline{\beta}}{\overline{\eta}}\left(\overline{\eta} +\overline{a}\frac{\partial f}{\partial I}(v) -\frac{\overline{ac}}{\overline{\beta}}\frac{\partial f}{\partial I}(v)\,e^{-p_1^*}\right)e^{p_2^*}\,\overline{\gamma a}\frac{\partial f}{\partial S}(v)\,e^{p_1^*}\right) < 0.$$

Let $\mathcal{I} : \mathrm{Im}Q \to \ker\mathcal{L}$ be an isomorphism. Thus

$$\deg(\mathcal{IQN}, U \cap \ker\mathcal{L}, 0) = \det J(p_1^*, p_2^*, p_3^*) \neq 0 \qquad (2.1.21)$$

and condition M3) in Mawhin's continuation theorem holds. Taking into account Lemma 2.5, the proof of Theorem 2.1 is completed.

2.2. Examples

In this section we present some examples to illustrate the main result in the previous section.

2.2.1. Holling-type I functional response. Letting $f(S, I, P) = S$ (Holling-type I functional response) in system (2.1.1), we obtain the model:

$$\begin{cases} S' = \Lambda(t) - \mu(t)S - a(t)SP + \beta(t)SI \\ I' = \beta(t)SI - \eta(t)PI - c(t)I \\ P' = (r(t) - b(t)P)P + \gamma(t)a(t)SP + \theta\eta(t)PI \end{cases} \qquad . \qquad (2.2.1)$$

Since $f(S, I, P) = S$, conditions P2) to P5) are trivially satisfied and P7) is satisfied with $K = \alpha = 1$. We obtain the following corollary:

COROLLARY 8. *Assume that that conditions P1), P6) and P8) hold. If $\widetilde{R}_0 > 1$, $\overline{\beta}\,\overline{b} - \overline{\gamma}\,\overline{a\eta} > 0$ and*

$$\overline{\mathcal{R}}_0 > 1 + \overline{a\eta} \frac{\overline{\gamma}\,\overline{\Lambda}}{\overline{\mu}(\overline{r}\,\overline{\eta} + \overline{b}\,\overline{c})} + a\frac{\overline{\beta}\,\overline{r} + \overline{\gamma a}\,\overline{c}}{\overline{\mu}(\overline{b}\,\overline{\beta} - \overline{\gamma}\,\overline{a\eta})} \qquad (2.2.2)$$

then system (2.2.1) possesses an endemic periodic orbit of period ω.

PROOF. Consider the system of algebraic equations:

$$\begin{cases} \overline{\Lambda}e^{-u_1} - \overline{a}e^{u_3} - \overline{\beta}e^{u_2} - \overline{\mu} = 0 \\ \overline{\beta}e^{u_1} - \overline{\eta}e^{u_3} - \overline{c} = 0 \\ \overline{\gamma a}e^{u_1} + \overline{\theta\eta}e^{u_2} - \overline{b}e^{u_3} + \overline{r} = 0 \end{cases} \qquad . \qquad (2.2.3)$$

By the second and third equations we get

$$e^{u_1} = \frac{\overline{\eta}\,e^{u_3} + \overline{c}}{\overline{\beta}} \quad \text{and} \quad e^{u_2} = \frac{\overline{\beta}\,\overline{b} - \overline{\gamma}\,\overline{a\eta}}{\overline{\beta}\,\overline{\theta\eta}}e^{u_3} - \frac{\overline{\beta}\,\overline{r} + \overline{\gamma a}\,\overline{c}}{\overline{\beta}\,\overline{\theta\eta}}$$

Notice that by hypothesis $\overline{\beta}\,\overline{b} - \overline{\gamma}\,\overline{a\eta} > 0$ and the right hand side of the second equation is positive as long as $e^{u_3} > (\overline{\beta}\,\overline{r} + \overline{c}\,\overline{\gamma a})/(\overline{\beta}\,\overline{b} - \overline{\gamma}\,\overline{a\eta})$. Using the first equation we get

$$\frac{\overline{\beta}\,\overline{\Lambda}}{\overline{\eta}\,e^{u_3} + \overline{c}} - \left(\overline{a} + \frac{\overline{\beta}\,\overline{b} - \overline{\gamma}\,\overline{a\eta}}{\theta\,\eta}\right)e^{u_3} + \frac{\overline{\beta}\,\overline{r} + \overline{\gamma a}\,\overline{c}}{\theta\,\eta} - \overline{\mu} = 0.$$

Taking into account that we must have $e^{u_3} > (\overline{\beta}\,\overline{r} + \overline{c}\,\overline{\gamma a})/(\overline{\beta}\,\overline{b} - \overline{\gamma}\,\overline{a\eta})$, we consider the function $F : [(\overline{\beta}\,\overline{r} + \overline{c}\,\overline{\gamma a})/(\overline{\beta}\,\overline{b} - \overline{\gamma}\,\overline{a\eta}), +\infty[\to \mathbb{R}$ given by

$$F(x) = \frac{\overline{\beta}\,\overline{\Lambda}}{\overline{\eta}x + \overline{c}} - \left(\overline{a} + \frac{\overline{\beta}\,\overline{b} - \overline{\gamma}\,\overline{a\eta}}{\theta\,\eta}\right)x + \frac{\overline{\beta}\,\overline{r} + \overline{\gamma a}\,\overline{c}}{\theta\,\eta} - \overline{\mu}.$$

It is immediate that F is decreasing and that, by the hypothesis in our corollary, we have

$$F\left(\frac{\overline{\beta}\,\overline{r} + \overline{c}\,\overline{\gamma a}}{\overline{\beta}\,\overline{b} - \overline{\gamma}\,\overline{a\eta}}\right) = \overline{\mu}\left(\overline{\mathcal{R}}_0 - 1 - \frac{\overline{a\eta}\,\overline{\gamma}\,\overline{\Lambda}}{\overline{\mu}(\overline{r}\,\overline{\eta} + \overline{b}\,\overline{c})} - a\frac{\overline{\beta}\,\overline{r} + \overline{\gamma a}\,\overline{c}}{\overline{\mu}(\overline{b}\,\overline{\beta} - \overline{\gamma}\,\overline{a\eta})}\right) > 0$$

and $\lim\limits_{x \to +\infty} F(x) = -\infty$. We conclude that there is $x_0 \in [(\overline{\beta}\,\overline{r} + \overline{c}\,\overline{\gamma a})/(\overline{\beta}\,\overline{b} - \overline{\gamma}\,\overline{a\eta}), +\infty[$ such that $F(x_0) = 0$. This implies that there is a unique solution of (2.2.3). The result follows now from Theorem 2.1. $\hfill\square$

We now assume that the real valued functions Λ, μ, r, b, γ and a are constant and positive. Model (2.2.1) becomes

$$\begin{cases} S' = \Lambda - \mu S - aSP + \beta(t)SI \\ I' = \beta(t)SI - \eta(t)PI - c(t)I \\ P' = (r - bP)P + \gamma aSP + \theta\eta(t)PI \end{cases} \qquad . \qquad (2.2.4)$$

We have the following corollary:

COROLLARY 9. *Assume that that conditions P1) and P6) hold. If $\widetilde{R}_0 > 1$, $b\overline{\beta} - \gamma a\overline{\eta} > 0$, $\Lambda < \mu^2/a$ and*

$$\overline{\mathcal{R}}_0 > 1 + \frac{a}{\mu}\left(\frac{\overline{\eta}\gamma\,\Lambda}{r\,\overline{\eta} + b\,\overline{c}} + \frac{\overline{\beta}\,r + \gamma a\,\overline{c}}{b\,\overline{\beta} - \gamma\,a\overline{\eta}} \right)$$

then system (2.2.4) possesses an endemic periodic orbit of period ω.

PROOF. We begin by noticing that systems (2.1.4) and (2.1.5) become in our context

$$\begin{cases} x' = \lambda(\Lambda - \mu x - axz^*_{2,\varepsilon,\lambda}(t) - \varepsilon_1 x) \\ z' = \lambda(r - bz + \gamma ax)z \end{cases} \qquad (2.2.5)$$

where $(x^*_{2,\varepsilon,\lambda}(t), z^*_{2,\varepsilon,\lambda}(t))$ is a solution of

$$\begin{cases} x' = \lambda(\Lambda - \mu x) \\ z' = \lambda(r - bz + \gamma ax + \varepsilon)z \end{cases} \qquad (2.2.6)$$

System (2.2.6) has two equilibriums:

$$E_1 = \left(\frac{\Lambda}{\mu}, 0 \right) \quad \text{and} \quad E_2 = \left(\frac{\Lambda}{\mu}, \frac{r(\mu + \varepsilon) + \gamma a\Lambda}{b\mu} \right).$$

Letting $(x^*_{2,\varepsilon,\lambda}(t), z^*_{2,\varepsilon,\lambda}(t))$ be the solution corresponding to E_2, it is easy to check that system (2.2.5) also has two equilibriums:

$$\widetilde{E}_1 = (\Theta, 0) \quad \text{and} \quad \widetilde{E}_2 = \left(\Theta, \frac{r}{b} + \frac{\gamma a}{b}\Theta \right),$$

where

$$\Theta = \frac{b\mu\Lambda}{(b\mu + ar)(\mu + \varepsilon) + \gamma a^2\Lambda}.$$

One can check that E_2 is locally attractive when $\varepsilon > 0$ is sufficiently small and that E_1 is a saddle point whose stable manifold coincides with the x-axis. The line $x = \Lambda/\mu$ is invariant and by the first equations we can conclude that the ω-limit of any orbit must be on this line. Thus, according to the behavior of solutions on that line, the ω-limit of any orbit on that line must be the equilibrium E_2 and we conclude that we have global asymptotic stability of E_2 on the region $\mathbb{R}^+ \times \mathbb{R}_0^+$. We have a similar situation for system (2.2.5) when $(x_{2,\varepsilon,\lambda}^*(t), z_{2,\varepsilon,\lambda}^*(t))$ is the solution corresponding to E_2. In fact, we can easily see that $\widetilde{E}_2$ is locally attractive and that $\widetilde{E}_1$ is a saddle point whose stable manifold coincides with the x-axis. The line $x = \Theta$ is invariant and by the first equations we can conclude that the ω-limit of any orbit must be on this line. Again, the ω-limit of any orbit on that line must be the equilibrium $\widetilde{E}_2$ and we conclude that we have global asymptotic stability of $\widetilde{E}_2$ on the region $\mathbb{R}^+ \times \mathbb{R}_0^+$.

We conclude that condition P8) holds. $\square$

To do some simulation, we consider the following particular set of parameters: $\Lambda = 0.1$; $\mu = 0.6$; $\beta(t) = 20(1 + 0.9\cos(2\pi t))$; $\eta(t) = 0.7(1 + 0.7\cos(\pi + 2\pi t))$; $c(t) = 0.1$; $r = 0.2$; $b = 0.3$; $\theta = 10$, $\gamma(t) = 0.1$ and $a = 3$. We obtain the model:

$$\begin{cases} S' = 0.1 - 0.6S - 20(1 + 0.9\cos(2\pi t))SI - 3SP \\ I' = 20(1 + 0.9\cos(2\pi t))SI - 0.7(1 + 0.7\cos(\pi + 2\pi t))PI - 0.1I \\ P' = (0.2 - 0.3P)P + 7(1 + 0.7\cos(\pi + 2\pi t))PI + 0.3SP \end{cases} \quad (2.2.7)$$

Notice that, for our model, $\Lambda = 0.1 > 0.012 = \mu^2/a$, $b\overline{\beta} - \gamma a\overline{\eta} = 3.99 > 0$, $\overline{R}_0 \approx 5.88 > 1 + 1.86$ and $\widetilde{R}_0 \approx 24.8 > 1$, and thus the conditions in Corollary 8 are fulfilled. Considering the initial condition $(S_0, I_0, P_0) = (0.03567, 0.02047, 0.88021)$ we obtain the periodic orbit in figure 2.1. Although our theoretical result doesn't imply the attractivity
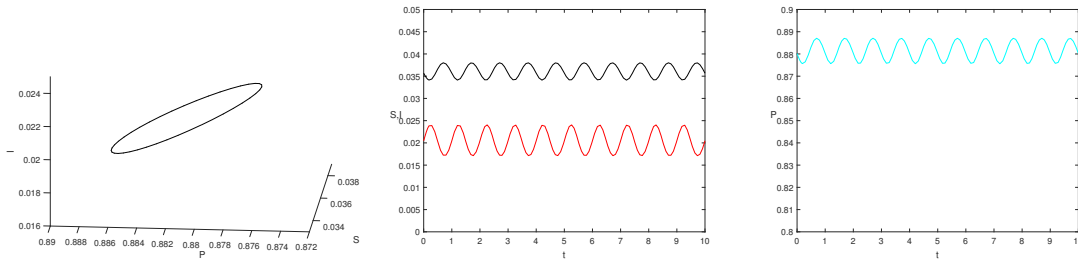

FIGURE 2.1.   Periodic orbit for model (2.2.7)

of the periodic solution, the simulations carried out suggest that this is the case.

2.2.2. No predation on susceptible preys. Letting $f \equiv 0$ in system (2.1.1), and still assuming that the real valued functions Λ, μ, β, η, c, γ, r, θ and b are periodic with period ω, nonnegative, continuous and also that $\bar{\Lambda} > 0$, $\bar{\mu} > 0$, $\bar{r} > 0$ and $\bar{b} > 0$, we obtain the periodic model considered in [**90**]:

$$\begin{cases} S' = \Lambda(t) - \mu(t)S - \beta(t)SI \\ I' = \beta(t)SI - \eta(t)PI - c(t)I \\ P' = (r(t) - b(t)P)P + \theta(t)\eta(t)PI \end{cases} \qquad . \qquad (2.2.8)$$

In [**75**], the authors refer that the assumption that predator mainly eats the infected prey (that is modelled by assuming that no predation on uninfected preys occur) is in accordance with the fact that the infected individuals are less active and can be caught more easily, or that infection modifies the behavior of the preys in such a way that they start living in parts of the habitat which are accessible to the predator. Some examples available in the literature are also provided in [**75**]: as an example of a situation where infected individuals can be caught more easily, the authors cite [**77**], where it is showed that wolf attacks on moose on Isle Royale in Lake Superior are more successful if the moose are heavily infected with a lungworm; as an example of a situation where the behavior of the prey individuals is modified, favoring predation, the authors cite [**49**].

Note that conditions P2) to P5) and P7) are trivially satisfied since $f \equiv 0$. Also note that systems (2.1.4) and (2.1.5) become in this context

$$\begin{cases} x' = \lambda(\Lambda(t) - \mu(t)x - \varepsilon_1 x) \\ z' = \lambda(r(t) - b(t)z)z \end{cases} \qquad (2.2.9)$$

and

$$\begin{cases} x' = \lambda(\Lambda(t) - \mu(t)x) \\ z' = \lambda(r(t) - b(t)z + \varepsilon)z \end{cases} \qquad (2.2.10)$$

and, by Lemmas 1 to 4 in [**75**] we conclude that condition P8) holds in this setting. Note also that condition (2.2.2) becomes $\overline{\mathcal{R}}_0 > 1$ and condition $\bar{b}\,\bar{\beta} - \overline{\gamma}\,\overline{a\eta} \leqslant 0$ is trivially satisfied since we can take $\gamma = 0$ or $a = 0$. We obtain the following corollary that recovers the result in [**90**]:

COROLLARY 10. *If $\widetilde{\mathcal{R}}_0 > 1$ and $\overline{R}_0 > 1$ hold, then system (2.2.8) possesses an endemic periodic orbit of period ω.*

2.3. Linear vital dynamics for predators

In this section we let $f \equiv 0$, $G(t,S) = \Lambda(t) - \mu(t)S$ and $H(t,P) = \Upsilon(t) - \zeta(t)P$ in (0.0.2), obtaining the following model:

$$\begin{cases} S' = \Lambda(t) - \mu(t)S - \beta(t)SI \\ I' = \beta(t)SI - \eta(t)g(S,I,P)I - c(t)I \\ P' = \Upsilon(t) - \zeta(t)P + \theta(t)\eta(t)g(S,I,P)I \end{cases} \quad . \quad (2.3.1)$$

To establish the existence of an endemic periodic orbit for system (2.3.1) we assume the following natural conditions:

P1*) The real valued functions Λ, μ, β, η, c, Υ, ζ and θ are periodic with period ω, nonnegative and continuous;

P2*) Functions $x \mapsto g(x,y,z)$, $y \mapsto g(x,y,z)$ are nonincreasing; function $z \mapsto g(x,y,z)$ is nondecreasing;

P3*) $\bar{\Lambda} > 0$, $\bar{\mu} > 0$, $\bar{\Upsilon} > 0$ and $\bar{\zeta} > 0$.

Like in the first scenario in section 1.1, the present setting includes several of the most common functional responses for function g, including Holling-type I, Holling-type II, Holling-type III, (Holling-type IV, Beddington-De Angelis and Crowley-Martin. Also note that conditions P1*) to P3*) are natural from a biological perspective and naturally are satisfied by the usual functional responses considered in the literature.

We also need to consider the following auxiliary system that corresponds to perturbations of the disease-free system for (2.3.1):

$$\begin{cases} x' = \Lambda(t) - \mu(t)x - \varepsilon_1 x \\ z' = \Upsilon(t) - \zeta(t)z + \theta(t)\eta(t)g(x,\varepsilon_2,z)\varepsilon_2 \end{cases} \quad . \quad (2.3.2)$$

satisfying

P4*) For each $\varepsilon_1, \varepsilon_2 \geqslant 0$ sufficiently small, system (2.3.2) has a unique ω-periodic solution, $(x^*_{\varepsilon_1,\varepsilon_2}(t), z^*_{\varepsilon_1,\varepsilon_2}(t))$, with

$$x^*_{\varepsilon_1,\varepsilon_2}(t) > 0 \quad \text{and} \quad z^*_{\varepsilon_1,\varepsilon_2}(t)) > 0,$$

that is globally asymptotically stable in the set

$$\{(x,z) \in (\mathbb{R}_0^+)^2 : x \geqslant 0 \ \wedge \ z > 0\}.$$

We assume that $(\varepsilon_1, \varepsilon_2) \mapsto (x^*_{\varepsilon_1,\varepsilon_2}(t), z^*_{\varepsilon_1,\varepsilon_2}(t))$ is continuous.

We write $x^* = x_{0,0}$ and $z^* = z_{0,0}^*$ for the components of the solutions in (2.3.2) corresponding to $\varepsilon = 0$.

To obtain the basic reproductive number for our model we consider the ordering (I, S, P) instead of (S, I, P), so that the infected compartment becomes the first one and the uninfected compartments became the last ones. Our new notation corresponds to the one in [79]. With this ordering, the functions $\mathcal{F}$, $\mathcal{V}^+$ and $\mathcal{V}^-$ in [79] become respectively

$$\mathcal{F}(t, (I, S, P)) = (\beta(t)SI, 0, 0),$$

$$\mathcal{V}^+(t, (I, S, P)) = (0, 0, \Upsilon(t) + \theta(t)\eta(t)g(S, I, P)I)$$

and

$$\mathcal{V}^-(t, (I, S, P)) = (\eta(t)g(S, I, P)I + c(t)I, \mu(t)S + \beta(t)SI, \zeta(t)P).$$

Having identified $\mathcal{F}$ and $\mathcal{V}$ we can compute the matrices $F(t)$ and $V(t)$ in [79] that in our context reduce to one dimensional matrices (that we identify with real numbers). In fact, we have

$$F(t) = \frac{\partial}{\partial I}(\beta(t)SI)|_{(x^*(t), 0, z^*(t))} = \beta(t)x^*(t)$$

and

$$V(t) = \frac{\partial}{\partial I}(\eta(t)g(S, P, I)I + c(t)I)|_{(x^*(t), 0, z^*(t))} = \eta(t)g(x^*(t), 0, z^*(t)) + c(t).$$

The evolution operator $W(s, t, \lambda)$ associated with the linear ω-periodic parametric system $w' = (-V(t) + F(t)/\lambda)w$ is easily seen to be given by

$$W(s, t, \lambda) = \mathrm{e}^{-\int_s^t \beta(r)x^*(r)/\lambda - c(r) - \eta(r)g(x^*(r), 0, z^*(r))\, dr}$$

and thus

$$W(\omega, 0, \lambda) = 1 \quad \Leftrightarrow \quad \overline{\beta x^*}/\lambda - \overline{c} - \overline{\eta g(x^*, 0, z^*)} = 0 \quad \Leftrightarrow \quad \lambda = \frac{\overline{\beta x^*}}{\overline{c} + \overline{\eta g(x^*, 0, z^*)}}.$$

Define

$$\mathcal{R}_0 = \frac{\overline{\beta x^*}}{\overline{c} + \overline{\eta g(x^*, 0, z^*)}}. \tag{2.3.3}$$

Note that our system satisfies conditions (A_1) to (A_7) in [38].

THEOREM 2.3. *Assume conditions P1*) to P3*). If $\mathcal{R}_0 > 1$, then model (2.3.1) has an endemic periodic orbit in $(\mathbb{R}_0^+)^3$.*

The proof of our theorem adapts to our situation the strategy in [79, 38]. It will be developed in two steps: using a result derived in [79], we obtain persistence of the infective

prey in subsection 2.3.1 and then, using a Poincaré map, we establish the existence of a periodic orbit in subsection 2.3.2.

2.3.1. Uniform strong persistence. The first step in the proof of Theorem 2.3 is to establish the persistence of all the compartments in our model. To do so we will use Theorem 2 in [**79**]. Note first that, as long as $\alpha_3\theta < \alpha_2 < \alpha_1$, we have

$$\langle (S', I', P'), (\alpha_1, \alpha_2, \alpha_3) \rangle = \alpha_1 \left(\Lambda(t) - \mu(t)S - \beta(t)SI \right) +$$

$$+ \alpha_2 \left(\beta(t)SI - \eta(t)g(S, I, P)I - c(t)I \right) +$$

$$+ \alpha_3 \left(\Upsilon(t) - \zeta(t)P + \theta(t)\eta(t)g(S, I, P)I \right)$$

$$< \alpha_1 \Lambda^u + \alpha_3 \Upsilon^u - \min\{\mu^\ell + c^\ell + \zeta^\ell\}(\alpha_1 S + \alpha_2 I + \alpha_3 P).$$

$$(2.3.4)$$

Thus, defining

$$K = \frac{\alpha_1 \Lambda^u + \alpha_3 \Upsilon^u}{\min\{\mu^\ell + c^\ell + \zeta^\ell\}},$$

we conclude $\langle (S', I', P'), (\alpha_1, \alpha_2, \alpha_3) \rangle < 0$ when $\alpha_1 S + \alpha_2 I + \alpha_3 P < K$ and that the set

$$\mathcal{K} = \{ (S, I, P) \in (\mathbb{R}_0^+)^3 : \alpha_1 S + \alpha_2 I + \alpha_3 P \leq K \} \qquad (2.3.5)$$

is forward invariant for the flow of system (2.3.1). Additionally, letting $W = \alpha_1 S + \alpha_2 I + \alpha_3 P$, $t_0 \geq 0$ and $W_0 = \alpha_1 S(t_0) + \alpha_2 I(t_0) + \alpha_3 P(t_0)$, by (2.3.4) we have for $t \geq t_0$

$$W(t) < K - (K - W_0) \, e^{-\min\{\mu^\ell + c^\ell + \zeta^\ell\}(t - t_0)}$$

and thus $\limsup\limits_{t \to +\infty} W(t) < K$. We conclude that $\mathcal{K}$ is an absorbing set for the flow. Thus the set $\mathcal{K}$ satisfies assumption (A_8) in [**38**].

Let now $(S(t), I(t), P(t))$ be a solution of (2.3.1) such that $I(t) \leq \varepsilon$, for $t \geq 0$. Since, by the first and third equations in (2.3.1), we have

$$\begin{cases} S' \geq \Lambda(t) - \mu(t)S - \beta^u S\varepsilon \\ P' \geq \Upsilon(t) - \zeta(t)P \end{cases}$$

and

$$\begin{cases} S' \leq \Lambda(t) - \mu(t)S \\ P' \leq \Upsilon(t) - \zeta(t)P + \theta^u \eta^u P\varepsilon \end{cases},$$

condition P4*), allows us to conclude that for sufficiently large $t > 0$ we have $S(t) \geq x^*_{\beta^u \varepsilon, 0}(t) \geq x^*(t) - \sigma_1(\varepsilon)$ and $P(t) \leq z^*_{0, \theta^u \eta^u \varepsilon}(t) \leq z^*(t) + \sigma_2(\varepsilon)$ with $\sigma_1(\varepsilon), \sigma_2(\varepsilon) \to 0$ as

$\varepsilon \to 0$. Thus, if $I(t) \leqslant \varepsilon$ we have

$$
\begin{aligned}
I' &= \beta(t)SI - \eta(t)g(S, I, P)I - c(t)I \\
&\geqslant \Big(\beta(t)x^*(t) - \beta^\ell \sigma_1(\varepsilon) - \eta(t)g(x^*(t) - \sigma_1(\varepsilon), 0, z^*(t) + \sigma_2(\varepsilon)) - c(t) \Big) I \\
&\geqslant (F(t)/\lambda(\varepsilon) - V(t)) I
\end{aligned}
$$

where $\lambda :]0, \varepsilon^*[\to \mathbb{R}$, well-defined when we take $\varepsilon^* > 0$ sufficiently small, is given by

$$
\lambda(\varepsilon) = \max_{t \in]0, \varepsilon[} \frac{\beta(t)x^*(t)}{\beta(t)x^*(t) - \beta^\ell \sigma_1(\varepsilon) + \eta(t)g(x^*(t), 0, z^*(t)) - \eta(t)g(x^*(t) - \sigma_1(\varepsilon), 0, z^*(t) + \sigma_2(\varepsilon))}
$$

and we can immediately see that $\lambda(\varepsilon) \to 1$ as $\varepsilon \to 0$.

By Theorem 2 in [**79**], we conclude that the infective prey is uniformly strong persistent in system (2.3.1). The uniform strong persistence of the susceptible prey and the predator, in our situation, is an immediate consequence of the uniform strong persistence of the infectives.

2.3.2. Existence of a periodic orbit. Next, to establish the existence of a positive periodic orbit for (2.3.1) we use the following result:

THEOREM 2.4. [**106**, Theorem 1.3.6] *Let $\tau : X \to X$ be a continuous map with $\tau(X_0) \subset X_0$ that is point dissipative, compact and uniform persistent with respect to $(X_0, \partial X_0)$. Then there exists a global attractor A_0 for S in X_0 that attracts strongly bounded sets in X_0 and S has a coexistence state $x_0 \in A_0$.*

To apply this result to our model we let $X = (\mathbb{R}_0^+)^3$, $X_0 = \mathcal{K}$ and $S = \tau$, where $\tau : (\mathbb{R}_0^+)^3 \to (\mathbb{R}_0^+)^3$ ia a time-ω map associated to our system and given by $\tau(S_0, I_0, P_0) = (S(\omega), I(\omega), P(\omega))$, where $(S(t), I(t), P(t))$ is the solution of (2.3.1) such that $(S(0), I(0), P(0)) = (S_0, I_0, P_0)$.

Since the bounded set $\mathcal{K}$ is an absorbing set for the flow of (2.3.1), we conclude that τ is point dissipative. It is immediate that τ is compact and, by the discussion in subsection 2.3.1, we conclude that τ is uniformly persistent with respect to $(\mathcal{K}, \partial \mathcal{K})$. Therefore, Theorem 2.4 allows us to conclude that τ has a coexistence state in $\mathcal{K}$. This coexistence state is a periodic orbit of our system contained in $\mathcal{K}$. This established our result.

2.4. Comments

In this chapter we discussed the existence of periodic orbits for periodic eco-epidemiological system with disease in the prey for two distinct families of models.

For the first family we used Mawhin's continuation theorem and for the second family we obtained a sharp result using a recent strategy that relies on the uniqueness of periodic orbits in the disease-free space. In the first case we proved that if $\widetilde{R}_0 > 1$, conditions P1) to P9) and there is a unique equilibrium of the averaged system (2.1.7) in $(\mathbb{R}^+)^3$ then system (2.1.1) possesses an endemic periodic orbit. In the second situation we proved that if $\mathcal{R}_0 > 1$ and conditions P1*) to P4*) hold then model (2.3.1) possesses an endemic periodic orbit. Somehow, the requirement that there is a unique equilibrium of the averaged system (2.1.7) in $(\mathbb{R}^+)^3$ is artificial, in the sense that it appears to be only there to allow the use of Mawhin's continuation theorem. Thus, a natural question that can be asked is if we can remove this condition and still be able to obtain the conclusion in Theorem 1.

Another interesting question is if we can still obtain the existence of the endemic periodic orbit for the first family of models when we assume other type of dynamical behavior for the uninfected subsystem. For instance, if we assume the uninfected subsystem to be a center like in the original Lotka-Volterra model.

Discrete Eco-Epidemiological Model

In the previous chapters the models involved are continuous. In contrast, in this chapter, using Mickens nonstandard method on model (0.0.2) with $G(t, S) = \Lambda(t) - \mu(t)S$ and $H(t, P) = (r(t) - b(t)P)P$, we obtain a discrete family of non-autonomous eco-epidemiological models . We obtain results on the persistence and extinction of the infected preys assuming that the bi-dimensional predator-prey subsystem that describes the dynamics in the absence of the infection satisfies some assumptions.

In [**52**] a discrete eco-epidemiological model was studied. In contrast with our non-autonomous model, in that paper the model considered is autonomous and assumes no predation on uninfected preys. Additionally, the discretization method is very different from ours, resulting in a very different form for the equations obtained. Some examples and simulations are undertaken to illustrate our results.

The structure of this chapter is the following: in section 3.1 we derive our model from the corresponding continuous model using Mickens nonstandard discretization scheme, establish our setting and some preliminary results; in section 3.2 we obtain our main result on extinction and persistence of the infective prey; finally, in section 3.3, we consider some particular models that illustrate our results.

3.1. A family of discrete models

We consider the following non-autonomous eco-epidemiological model that derives from model (0.0.2) by taking $G(t, S) = \Lambda(t) - \mu(t)S$ and $H(t, P) = (r(t) - b(t)P)P$:

$$\begin{cases} S' = \Lambda(t) - \mu(t)S - a(t)f(S, I, P)P - \beta(t)SI \\ I' = \beta(t)SI - \eta(t)g(S, I, P)I - c(t)I \\ P' = (r(t) - b(t)P)P + \gamma(t)a(t)f(S, I, P)P + \theta(t)\eta(t)g(S, I, P)I \end{cases} \quad . \quad (3.1.1)$$

The aim of this chapter is to discuss the uniform strong persistence and extinction of the infectives I of the discrete counterpart of the system (3.1.1). A possible discretization of the above model, with stepsize h, derived by applying Mickens' nonstandard finite

difference method [**72**], that leads to the following set of equations:

$$
\begin{cases}
S(nh+h) - S(nh) = h\Lambda(nh) - h\mu(nh)S(nh+h) \\
\qquad\qquad - ha(nh)f(S(nh+h), I(nh), P(nh))P(nh) \\
\qquad\qquad - h\beta(nh)S(nh+h)I(nh) \\
I(nh+h) - I(nh) = h\beta(nh)S(nh+h)I(nh) \\
\qquad\qquad - h\eta(nh)g(S(nh), I(nh), P(nh))I(nh+h) \\
\qquad\qquad - hc(nh)I(nh+h) \\
P(nh+h) - P(nh) = h(r(nh) - b(nh)P(nh+h))P(nh) + h\gamma(nh)a(nh)\times \\
\qquad\qquad \times f(S(nh+h), I(nh), P(nh))P(nh) \\
\qquad\qquad + h\theta(nh)\eta(nh)g(S(nh), I(nh), P(nh))I(nh+h)
\end{cases}
.
$$

Using the notation $\xi_n = h\xi(nh)$ for $\xi = \Lambda, \mu, a, \beta, \eta, c, r, b$ and also $\zeta_n = \zeta(nh)$ for $\zeta = \gamma, \theta$, we obtain the following system of difference equations:

$$
\begin{cases}
S_{n+1} - S_n = \Lambda_n - \mu_n S_{n+1} - a_n f(S_{n+1}, I_n, P_n)P_n - \beta_n S_{n+1} I_n \\
I_{n+1} - I_n = \beta_n S_{n+1} I_n - \eta_n g(S_n, I_n, P_n)I_{n+1} - c_n I_{n+1} \\
P_{n+1} - P_n = (r_n - b_n P_{n+1})P_n + \gamma_n a_n f(S_{n+1}, I_n, P_n)P_n + \theta_n \eta_n g(S_n, I_n, P_n)I_{n+1}
\end{cases}
.
$$

$$\tag{3.1.2}$$

We will assume that

D1) (Λ_n), (a_n), (β_n), (η_n), (c_n), (r_n), (b_n), (γ_n) and (θ_n) are bounded and nonnegative sequences and $0 < \mu_n \leqslant c_n$;

D2) (Λ_n), (r_n) and (b_n) are bounded away from zero;

D3) $f, g : (\mathbb{R}_0^+)^3 \to \mathbb{R}$ are C^1 nonnegative; for fixed $x, z \geqslant 0$, $y \mapsto f(x, y, z)$ and $y \mapsto g(x, y, z)$ are nonincreasing; for fixed $y, z \geqslant 0$, $x \mapsto f(x, y, z)$ is nondecreasing and $x \mapsto g(x, y, z)$ is nonincreasing; for fixed $x, y \geqslant 0$, $z \mapsto f(x, y, z)$ is nonincreasing and $z \mapsto g(x, y, z)$ is nondecreasing;

D4) there is $\omega \in \mathbb{N}$ such that

$$
\limsup_{n \to +\infty} \prod_{k=n}^{n+\omega} \frac{1}{1+\mu_k} < 1.
$$

It follows from D4) that there are constants $K > 0$ and $\theta \in]0, 1[$ such that

$$
\prod_{k=m}^{n-1} \frac{1}{1+\mu_k} < K\theta^{n-m}, \tag{3.1.3}
$$

for any $m, n \in \mathbb{N}_0$ with $n > m$.

D5) Given $p \in \mathbb{N}$ there is a unique solution $((S_n, I_n, P_n))_{n \geqslant p}$ of system (3.1.2) with initial condition $(S_p, I_p, P_p) \in (\mathbb{R}_0^+)^3$.

D6) Any solution of system (3.1.2) with nonnegative (resp. positive) initial condition, (S_q, I_q, P_q) is nonnegative (resp. positive) for all $n \geqslant q$.

Note that when $f(S_{n+1}, I_n, P_n) = S_{n+1}$ and $g(S_n, I_n, P_n) = P_n$ in (3.1.2), the equation can be rewritten in explicit form:

$$
\begin{cases}
S_{n+1} = \dfrac{\Theta_n}{\Psi_n} \\[2ex]
I_{n+1} = \dfrac{\beta_n \Theta_n + \Psi_n}{\Psi_n \Phi_n} I_n \\[2ex]
P_{n+1} = \dfrac{(1 + r_n)\Psi_n \Phi_n + \gamma_n a_n \Theta_n \Phi_n + \theta_n \eta_n (\Psi_n + \beta_n \Theta_n) I_n}{\Psi_n \Phi_n (1 + b_n P_n)} P_n
\end{cases}
, \qquad (3.1.4)
$$

where $\Psi_n = 1 + \mu_n + \beta_n I_n + a_n P_n$, $\Phi_n = 1 + \eta_n P_n + c_n$ and $\Theta_n = \Lambda_n + S_n$. From (3.1.4), we conclude that when $f(S_{n+1}, I_n, P_n) = S_{n+1}$ system (3.1.2) is well defined and D6) holds. Let us introduce the notation $f^\ell = \inf f_n$ and $f^u = \sup f_n$.

To proceed, we need to consider two auxiliary equations. The first one corresponds to the dynamics of preys in the absence of infected preys and predators:

$$
s_{n+1} - s_n = \Lambda_n - \mu_n s_{n+1}.
$$

Rearranging terms, the equation above becomes:

$$
s_{n+1} = \frac{\Lambda_n}{1 + \mu_n} + \frac{s_n}{1 + \mu_n}. \qquad (3.1.5)
$$

We have the following lemma that was essentially proved in [**70**]:

LEMMA 3.1. *We have the following:*

i) *The solution of equation (3.1.5) with $\Lambda_n = 0$, $n \geqslant p$, and initial condition $s_p = 0$ is the identically null sequence;*

ii) *All solutions (s_n) of equation (3.1.5) with initial condition $s_0 > 0$ are positive for all $n \in \mathbb{N}$;*

iii) *Given a solution (s_n) of equation (3.1.5) with initial condition $s_0 \in [\Lambda^\ell/\mu^u, \Lambda^u/\mu^\ell]$ we have*

$$
\frac{\Lambda^\ell}{\mu^u} \leqslant s_n \leqslant \frac{\Lambda^u}{\mu^\ell}
$$

for all $n \in \mathbb{N}$;

iv) Each fixed solution (s_n) of (3.1.5) with initial condition $s_0 \geqslant 0$ is bounded and globally uniformly attractive on $[0, +\infty)$;

v) There is a constant $D > 0$ such that if $\varphi_n \geqslant 0$, $n \in \mathbb{N}$, (s_n) is a solution of (3.1.5) and $(\tilde{s}_n)$ is a solution of the system

$$s_{n+1} = \frac{\Lambda_n + s_n + \varphi_n}{1 + \mu_n} \qquad (3.1.6)$$

with $\tilde{s}_0 = s_0$ then

$$\sup_{n \geqslant 0} |\tilde{s}_n - s_n| \leqslant D \sup_{n \geqslant 0} |\varphi_n|.$$

vi) There is a constant $E > 0$ such that if $\psi_n \geqslant 0$, $n \in \mathbb{N}$, (s_n) is a solution of (3.1.5) and $(\tilde{s}_n)$ is a solution of the system

$$s_{n+1} = \frac{\Lambda_n + s_n}{1 + \mu_n + \psi_n} \qquad (3.1.7)$$

with $\tilde{s}_0 = s_0$ then there is N_1 sufficiently large such that

$$\sup_{n \geqslant N_1} |\tilde{s}_n - s_n| \leqslant E \sup_{n \geqslant 0} |\psi_n|.$$

PROOF. Properties i) to v) follow from Lemma 1 in [**70**]. To prove vi), let (s_n) be a solution of (3.1.5) and $(\tilde{s}_n)$ be a solution of (3.1.7) with $\tilde{s}_0 = s_0$. By (3.1.5) and (3.1.7), we have

$$(\tilde{s}_{n+1} - s_{n+1})(1 + \mu_n) = \tilde{s}_n - s_n - \psi_n \tilde{s}_{n+1}$$

Therefore, letting $w_n = |\tilde{s}_n - s_n|$, we have

$$w_{n+1}(1 + \mu_n) \leqslant w_n + |\psi_n| \tilde{s}_{n+1}$$

and thus

$$w_{n+1} \leqslant \frac{w_n}{1 + \mu_n} + \frac{|\psi_n| \tilde{s}_{n+1}}{1 + \mu_n}$$

Fix $\varepsilon > 0$. By iii) and iv) we get, for n sufficiently large, say $n \geqslant N_1$,

$$w_{n+1} \leqslant \frac{w_n}{1 + \mu_n} + \frac{|\psi_n|}{1 + \mu_n} \left[\frac{\Lambda^u}{\mu^\ell + \psi^\ell} + \varepsilon \right]$$

and thus, for $n \geqslant N_1$,

$$
\begin{aligned}
w_n &\leqslant \left[\frac{\Lambda^u}{\mu^\ell + \psi^\ell} + \varepsilon \right] \sum_{m=0}^{n-1} |\psi_m| \left(\prod_{k=m}^{n-1} \frac{1}{1 + \mu_k} \right) \\
&\leqslant \left[\frac{\Lambda^u}{\mu^\ell + \psi^\ell} + \varepsilon \right] \sup_{n \geqslant 0} |\psi_n| \, K \sum_{m=0}^{n-1} \theta^{n-m} \\
&\leqslant \left[\frac{\Lambda^u}{\mu^\ell + \psi^\ell} + \varepsilon \right] \frac{K\theta}{1 - \theta} \sup_{n \geqslant 0} |\psi_n|.
\end{aligned}
$$

Defining $E = K\theta \left[\Lambda^u / (\mu^\ell + \psi^\ell) + \varepsilon \right] / (1 - \theta)$, we get

$$
\sup_{n \geqslant N_1} |\tilde{s}_n - s_n| = \sup_{n \geqslant N_1} w_n \leqslant E \sup_{n \geqslant 0} |\psi_n|,
$$

and the result follows. $\qquad \square$

We also need to consider the equation:

$$
y_{n+1} - y_n = (r_n - b_n y_{n+1}) y_n.
$$

Rearranging terms, we get:

$$
y_{n+1} = \frac{r_n y_n + y_n}{1 + b_n y_n} \tag{3.1.8}
$$

The following lemma holds.

LEMMA 3.2. *We have the following:*

i) *The solution of equation (3.1.8) with $y_p = 0$ is the identically null sequence;*

ii) *All solutions (y_n) of equation (3.1.8) with initial condition $y_0 > 0$ are positive for all $n \in \mathbb{N}$;*

iii) *Given a solution (y_n) of equation (3.1.8) with initial condition $y_0 \in [r^\ell/b^u, r^u/b^\ell]$ we have*

$$
\frac{r^\ell}{b^u} \leqslant y_n \leqslant \frac{r^u}{b^\ell}
$$

for all $n \in \mathbb{N}$;

iv) *Each fixed solution (y_n) of (3.1.8) with initial condition $y_0 > 0$ is bounded and globally uniformly attractive on $]0, +\infty)$;*

v) *There is a constant $E > 0$ such that, if $g_n \geqslant 0$, $n \in \mathbb{N}$, (y_n) is a solution of (3.1.8) and $(\tilde{y}_n)$ is a solution of the system*

$$
y_{n+1} = \frac{r_n y_n + y_n}{1 + (b_n + g_n) y_n}, \quad n = 0, 1, \ldots \tag{3.1.9}
$$

with $\tilde{y}_0 = y_0$ then there is N_1 sufficiently large such that

$$\sup_{n \geqslant N_1} |\tilde{y}_n - y_n| \leqslant E \sup_{n \geqslant N_1} |g_n|.$$

vi) There is a constant $G > 0$ such that, if $h_n \geqslant 0$, $n \in \mathbb{N}$, (y_n) is a solution of (3.1.8) and $(\tilde{y}_n)$ is a solution of the system

$$y_{n+1} = \frac{(r_n + h_n)y_n + y_n}{1 + b_n y_n}, \quad n = 0, 1, \dots \tag{3.1.10}$$

with $\tilde{y}_0 = y_0$ then there is N_2 sufficiently large such that

$$\sup_{n \geqslant N_2} |\tilde{y}_n - y_n| \leqslant G \sup_{n \geqslant N_2} |h_n|.$$

PROOF. With the change of variable $w_n = 1/y_n$, equation (3.1.8) becomes

$$w_{n+1} = \frac{b_n}{r_n + 1} + \frac{w_n}{r_n + 1},$$

equation (3.1.9) becomes

$$w_{n+1} = \frac{w_n + b_n + g_n}{r_n + 1}.$$

and equation (3.1.10) becomes

$$w_{n+1} = \frac{w_n + b_n}{1 + r_n + h_n}.$$

Using Lemma 3.1, we obtain ii) to vi). Property i) is immediate. □

We must assume the following:

D7) Each solution of (3.1.2) with positive initial condition is bounded and there is a bounded region $\mathcal{R}$ that contains the ω-limit of all solutions of (3.1.2) with positive initial conditions.

Notice in particular that condition D7) implies that there is $L > 0$ such that, for each solution (S_n, I_n, P_n) we have

$$\limsup_{t \to +\infty} (S_n + I_n + P_n) < L. \tag{3.1.11}$$

The next lemma shows that, when $g(S, I, P) = g_0(S, I)P$, there is an invariant region that attracts all orbits of system (3.1.2).

LEMMA 3.3. *Assume that conditions D1) to D6) hold and that $g(S, I, P) = g_0(S, I)P$. Then, there is $L > 0$ such that, for any solution (S_n, I_n, P_n) of (3.1.2), with nonnegative*

initial conditions, there is $T \in \mathbb{N}$ such that

$$S_n + I_n + P_n \leqslant L \quad for \quad n \geqslant T.$$

PROOF. Let (S_n, I_n, P_n) be a solution of (3.1.2) with nonnegative initial conditions $S_q = s_q$, $I_q = i_q$ and $P_q = p_q$. Adding the first two equations in (3.1.2) and writing $N_n = S_n + I_n$, we get

$$N_{n+1} - N_n = \Lambda_n - \mu_n S_{n+1} - c_n I_{n+1} - a_n f(S_{n+1}, I_n, P_n)P_n$$

$$- \eta_n g_0(S_{n+1}, I_n)P_n I_{n+1}$$

$$\leqslant \Lambda_n - \mu_n N_{n+1},$$

since $\mu_n = \min\{\mu_n, c_n\}$. Thus

$$N_{n+1} \leqslant \frac{\Lambda_n}{1 + \mu_n} + \frac{N_n}{1 + \mu_n}.$$

By iii) and iv) in Lemma 3.1, we conclude that, for any given $\varepsilon > 0$, we have $S_n + I_n = N_n \leqslant s_n \leqslant \Lambda^u/\mu^\ell + \varepsilon$, where s_n is a solution of (3.1.5) with initial condition $s_q = N_q$, for n sufficiently large, say $n \geqslant N_1$.

By the third equation in (3.1.2) we obtain

$$P_{n+1} = \frac{P_n + r_n P_n + \gamma_n a_n f(S_{n+1}, I_n, P_n)P_n + \theta_n \eta_n g_0(S_{n+1}, I_n)P_n I_{n+1}}{1 + b_n P_n}$$

$$\leqslant \frac{\left[r_n + \gamma_n a_n f(\Lambda^u/\mu^\ell + \varepsilon, 0, 0) + \theta_n \eta_n g_0(0, \Lambda^u/\mu^\ell + \varepsilon)(\Lambda^u/\mu^\ell + \varepsilon)\right] P_n + P_n}{1 + b_n P_n}$$

for $n \geqslant N_1$. By iii) and iv) in Lemma 3.2, we conclude that, for any given $\delta > 0$, there is $N_2 \geqslant N_1$ such that, for all $n \geqslant N_2$

$$P_n \leqslant \frac{r^u}{\beta^\ell} + \delta + G \sup_{n \geqslant q} \left(\gamma_n a_n f(\Lambda^u/\mu^\ell + \varepsilon, 0, 0) + \theta_n \eta_n g_0(0, \Lambda^u/\mu^\ell + \varepsilon)(\Lambda^u/\mu^\ell + \varepsilon) \right)$$

Thus

$$S_n + I_n + P_n \leqslant \frac{\Lambda^u}{\mu^\ell} + \varepsilon + \frac{r^u}{\beta^\ell} + \delta$$

$$+ G \sup_{n \geqslant q} \left(\gamma_n a_n f(\Lambda^u/\mu^\ell + \varepsilon, 0, 0) + \theta_n \eta_n g_0(0, \Lambda^u/\mu^\ell + \varepsilon)(\Lambda^u/\mu^\ell + \varepsilon) \right),$$

and the result follows. $\square$

To formulate our next assumption we need to consider the system

$$\begin{cases} x_{n+1} - x_n = \Lambda_n - \mu_n x_{n+1} - a_n f(x_{n+1}, 0, z_n)z_n \\ z_{n+1} - z_n = (r_n - b_n z_{n+1})z_n + \gamma_n a_n f(x_{n+1}, 0, z_n)z_n \end{cases} \quad . \quad (3.1.12)$$

which corresponds to the dynamics of the susceptible preys and the predators in the absence of infected preys. We also need to consider the two families of auxiliary systems:

$$
\begin{cases}
x_{n+1} - x_n = \Lambda_n - \mu_n x_{n+1} - a_n f(x_{n+1}, 0, 0) z^*_{2,\varepsilon,n} - \varepsilon x_n \\
z_{n+1} - z_n = (r_n - b_n z_{n+1}) z_n + \gamma_n a_n f(x_{n+1}, \varepsilon, z_n) z_n
\end{cases}
\tag{3.1.13}
$$

where $(x^*_{2,\varepsilon,n}, z^*_{2,\varepsilon,n})$ is a solution of

$$
\begin{cases}
x_{n+1} - x_n = \Lambda_n - \mu_n x_{n+1} \\
z_{n+1} - z_n = (r_n - b_n z_{n+1}) z_n + \gamma_n a_n f(x_{n+1}, 0, z_n) z_n + \theta_n \eta_n g(x_n, 0, z_n) z_n \varepsilon
\end{cases}
\tag{3.1.14}
$$

We make the following assumptions concerning systems (3.1.13) and (3.1.14):

D8) There is a family of nonnegative solutions $(x^*_{1,\varepsilon,n}, z^*_{1,\varepsilon,n})$ of system (3.1.13), one for each $\varepsilon > 0$ sufficiently small, depending on a solution $(x^*_{2,\varepsilon,n}, z^*_{2,\varepsilon,n})$ of system (3.1.14), such that each solution in the family is globally asymptotically stable in a set containing $\{(x, z) \in (R_0^+)^2 : x, z > 0\}$ and the function $\varepsilon \mapsto (x^*_{1,\varepsilon,n}, z^*_{1,\varepsilon,n})$ is continuous.

D9) The family of nonnegative solutions $(x^*_{2,\varepsilon,n}, z^*_{2,\varepsilon,n})$ of system (3.1.14), one for each $\varepsilon > 0$ sufficiently small, is such that each solution in the family is globally asymptotically stable in a set containing $\{(x, z) \in (R_0^+)^2 : x, z > 0\}$ and the function $\varepsilon \mapsto (x^*_{2,\varepsilon,n}, z^*_{2,\varepsilon,n})$ is continuous.

We denote the element of the family of solutions in D8) and D9) with $\varepsilon = 0$, by $(x^*_{1,n}, z^*_{1,n})$ and $(x^*_{2,n}, z^*_{2,n})$, respectively. For each solution $(x^*_{1,n}, z^*_{1,n})$ of (3.1.13) associated to a solution $(x^*_{2,n}, z^*_{2,n})$ of (3.1.14), with $\varepsilon = 0$, and initial conditions (x_0, z_0) with $x_0 > 0$ and $z_0 > 0$, and each $\lambda \in \mathbb{N}$, define the number

$$
\mathcal{R}^\ell(\lambda) = \liminf_{n \to +\infty} \prod_{k=n}^{n+\lambda} \frac{1 + \beta_k x^*_{1,k+1}}{1 + c_k + \eta_k g(x^*_{1,k}, 0, z^*_{2,k})}
\tag{3.1.15}
$$

and for each solution (s^*_n) of (3.1.5) with $s_0 > 0$, each solution (y^*_n) of (3.1.8) with $y_0 > 0$ and each $\lambda \in \mathbb{N}$, define the number

$$
\mathcal{R}^u(\lambda) = \limsup_{n \to +\infty} \prod_{k=n}^{n+\lambda} \frac{1 + \beta_k s^*_{k+1}}{1 + c_k + \eta_k g(s^*_k, 0, y^*_k)}
\tag{3.1.16}
$$

These numbers will be useful in obtaining conditions for permanence and extinction and, in some sense, play the role of upper and lower bounds for the basic reproductive number in this general context. In the following lemma we prove that the numbers above are independent of the particular positive solutions of (3.1.5), (3.1.8) and (3.1.13) considered.

LEMMA 3.4. *The numbers $\mathcal{R}^{\ell}(\lambda)$ and $\mathcal{R}^{u}(\lambda)$ are independent of the particular solutions (s_n^*) of* (3.1.5) *with $s_0 > 0$, (y_n^*) of* (3.1.8) *with $y_0 > 0$ and $(x_{1,n}^*, z_{1,n}^*)$ and $(x_{2,n}^*, z_{2,n}^*)$ of* (3.1.13) *and* (3.1.14), *with $x_{i,0} > 0$ and $z_{i,0} > 0$, for $i = 1, 2$.*

PROOF. Let $v = (x_{2,n}^*, z_{2,n}^*)_{n \in \mathbb{N}}$ and $\bar{v} = (\bar{x}_{2,n}^*, \bar{z}_{2,n}^*)_{n \in \mathbb{N}}$ be distinct solutions of (3.1.14) and $u = (x_{1,n}^*, z_{1,n}^*)_{n \in \mathbb{N}}$ and $\bar{u} = (\bar{x}_{1,n}^*, \bar{z}_{1,n}^*)_{n \in \mathbb{N}}$ be the corresponding solution of (3.1.13) with $w_{i,0} > 0, \bar{w}_{i,0} > 0$, for $w = x, z$ and $i = 1, 2$. Denote by $\mathcal{R}^{\ell}(\lambda, u, v)$ and $\mathcal{R}^{\ell}(\lambda, \bar{u}, \bar{v})$ the number in (3.1.15) corresponding to u, v and $\bar{u}, \bar{v}$.

Let $\delta > 0$ be sufficiently small. By assumptions D8) and D9), for $k \geqslant N$ (where $N \in \mathbb{N}$) sufficiently large, we have

$$x_{1,k}^* - \delta \leqslant \bar{x}_{1,k}^* \leqslant x_{1,k}^* + \delta \quad \text{and} \quad z_{2,k}^* - \delta \leqslant \bar{z}_{2,k}^* \leqslant z_{2,k}^* + \delta.$$

Additionally, since g is C^1 and therefore locally Lipschitz, by D3), there is $c > 0$ such that, for sufficiently large k,

$$|g(x_{1,k}^*, 0, z_{2,k}^*) - g(x_{1,k}^*, 0, \bar{z}_{2,k}^* - \delta)| \leqslant c|z_{2,k}^* - \bar{z}_{2,k}^* + \delta| \leqslant 2c\delta$$

and

$$|g(\bar{x}_{1,k}^*, 0, z_{2,k}^*) - g(x_{1,k}^*, 0, z_{2,k}^*)| \leqslant c|\bar{x}_{1,k}^* - x_{1,k}^*| \leqslant c\delta.$$

Thus, for $n \geqslant N$

$$
\begin{aligned}
&\prod_{k=n}^{n+\lambda} \frac{1 + \beta_k x_{1,k+1}^*}{1 + c_k + \eta_k g(x_{1,k}^*, 0, z_{2,k}^*)} \\
&\leqslant \prod_{k=n}^{n+\lambda} \frac{1 + \beta_k \bar{x}_{1,k+1}^* + \delta \beta_k}{1 + c_k + \eta_k g(x_{1,k}^*, 0, \bar{z}_{2,k}^* - \delta)} \\
&\leqslant \prod_{k=n}^{n+\lambda} \frac{1 + \beta_k \bar{x}_{1,k+1}^* + \delta \beta_k}{1 + c_k + \eta_k g(\bar{x}_{1,k}^*, 0, \bar{z}_{2,k}^*)} \frac{1 + c_k + \eta_k g(\bar{x}_{1,k}^*, 0, \bar{z}_{2,k}^*)}{1 + c_k + \eta_k g(x_{1,k}^*, 0, \bar{z}_{2,k}^* - \delta)} \\
&= \prod_{k=n}^{n+\lambda} \left(\frac{1 + \beta_k \bar{x}_{1,k+1}^*}{1 + c_k + \eta_k g(\bar{x}_{1,k}^*, 0, \bar{z}_{2,k}^*)} + \frac{\delta \beta_k}{1 + c_k + \eta_k g(\bar{x}_{1,k}^*, 0, \bar{z}_{2,k}^*)} \right) \times \\
&\quad \times \left(1 + \frac{3c\delta\eta_k}{1 + c_k + \eta_k g(x_{1,k}^*, 0, \bar{z}_{2,k}^* - \delta)} \right) \\
&\leqslant (1 + \delta B)^{\lambda+1} \prod_{k=n}^{n+\lambda} \left(\frac{1 + \beta_k \bar{x}_{1,k+1}^*}{1 + c_k + \eta_k g(\bar{x}_{1,k}^*, 0, \bar{z}_{2,k}^*)} + \delta A \right) \\
&\leqslant (1 + \delta B)^{\lambda+1} \left(\prod_{k=n}^{n+\lambda} \frac{1 + \beta_k \bar{x}_{1,k+1}^*}{1 + c_k + \eta_k g(\bar{x}_{1,k}^*, 0, \bar{z}_{2,k}^*)} + \sum_{j=1}^{\lambda+1} \binom{\lambda+1}{j} \delta^j C^{\lambda+1-j} A^j \right),
\end{aligned}
\tag{3.1.17}
$$

where

$$A = \frac{\beta^u}{1 + c^\ell + \eta^\ell(g(\bar{x}_{1,k}^*, 0, \bar{z}_{2,k}^*))^\ell}, \quad B = \frac{3c\eta^u}{1 + c^\ell + \eta^\ell(g(x_{1,k}^*, 0, \bar{z}_{2,k}^* - \delta))^\ell}$$

and

$$C = \frac{1 + \beta^u(\bar{x}_1^*)^u}{1 + c^\ell + \eta^\ell(g(\bar{x}_{1,k}^*, 0, \bar{z}_{2,k}^*))^\ell}.$$

By (3.1.15) and (3.1.17), taking limits we conclude that

$$\mathcal{R}^\ell(\lambda, u, v) \leqslant (1 + \delta B)^{\lambda+1} \left(\mathcal{R}^\ell(\lambda, \bar{u}, \bar{v}) + \sum_{j=1}^{\lambda+1} \binom{\lambda+1}{j} \delta^j C^{\lambda+1-j} A^j \right).$$

By the arbitrariness of $\delta > 0$, we conclude that $\mathcal{R}^\ell(\lambda, u, v) \leqslant \mathcal{R}^\ell(\lambda, \bar{u}, \bar{v})$ and, interchanging the roles of u, v and $\bar{u}, \bar{v}$ it is immediate that $\mathcal{R}^\ell(\lambda, \bar{u}, \bar{v}) \geqslant \mathcal{R}^\ell(\lambda, u, v)$. Thus $\mathcal{R}^\ell(\lambda, \bar{u}, \bar{v}) = \mathcal{R}^\ell(\lambda, u, v)$.

Now write $\mathcal{R}^u(\lambda, s, y)$ for the number in (3.1.16) corresponding to the solutions $s = (s_n^*)$ of (3.1.5) with $s_0 > 0$ and $y = (y_n^*)$ of (3.1.8) with $y_0 > 0$.

Let again $\delta > 0$ be sufficiently small. Additionally, let $s_1^* = (s_{1,n}^*)$ and $s_2^* = (s_{2,n}^*)$ be distinct solutions of (3.1.5) and $y_1^* = (y_{1,n}^*)$ and $y_2^* = (y_{2,n}^*)$ be distinct solutions of (3.1.8). By iv) in Lemma 3.1 and iv) in Lemma 3.2, we have

$$s_{1,k}^* - \delta \leqslant s_{2,k}^* \leqslant s_{2,k}^* + \delta \quad \text{and} \quad y_{1,k}^* - \delta \leqslant y_{2,k}^* \leqslant y_{1,k}^* + \delta$$

for $k \geqslant N$ sufficiently large. There is $c > 0$ such that

$$|g(s_{1,k}^*, 0, y_{1,k}^*) - g(s_{1,k}^*, 0, y_{2,k}^* - \delta)| \leqslant c|y_{1,k}^* - y_{2,k}^* + \delta| \leqslant 2c\delta$$

and

$$|g(s_{2,k}^*, 0, y_{1,k}^*) - g(s_{1,k}^*, 0, y_{1,k}^*)| \leqslant c|s_{2,k}^* - s_{1,k}^*| \leqslant c\delta.$$

Therefore

$$\prod_{k=n}^{n+\lambda} \frac{1 + \beta_k s_{2,k+1}^*}{1 + c_k + \eta_k g(s_{2,k}^*, 0, y_{2,k}^*)}$$

$$\leqslant \prod_{k=n}^{n+\lambda} \frac{1 + \beta_k s_{1,k+1}^* + \delta\beta_k}{1 + c_k + \eta_k g(s_{2,k}^*, 0, y_{1,k}^* - \delta)}$$

$$\leqslant \prod_{k=n}^{n+\lambda} \frac{1 + \beta_k s_{1,k+1}^* + \delta\beta_k}{1 + c_k + \eta_k g(s_{1,k}^*, 0, y_{1,k}^*)} \frac{1 + c_k + \eta_k g(s_{1,k}^*, 0, y_{1,k}^*)}{1 + c_k + \eta_k g(s_{2,k}^*, 0, y_{1,k}^* - \delta)}$$

$$\leqslant \prod_{k=n}^{n+\lambda} \left(\frac{1 + \beta_k s_{1,k+1}^*}{1 + c_k + \eta_k g(s_{1,k}^*, 0, y_{1,k}^*)} + \frac{\delta\beta_k}{1 + c_k + \eta_k g(s_{1,k}^*, 0, y_{1,k}^*)} \right) \times \qquad (3.1.18)$$

$$\times \left(1 + \frac{3c\delta\eta_k}{1 + c_k + \eta_k g(s_{2,k}^*, 0, y_{1,k}^* - \delta)} \right)$$

$$\leqslant (1 + \delta B)^{\lambda+1} \prod_{k=n}^{n+\lambda} \left(\frac{1 + \beta_k s_{1,k+1}^*}{1 + c_k + \eta_k g(s_{1,k}^*, 0, y_{1,k}^*)} + \delta A \right)$$

$$\leqslant (1 + \delta B)^{\lambda+1} \left(\prod_{k=n}^{n+\lambda} \frac{1 + \beta_k s_{1,k+1}^*}{1 + c_k + \eta_k g(s_{1,k}^*, 0, y_{1,k}^*)} + \sum_{j=1}^{\lambda+1} \binom{\lambda+1}{j} \delta^j C^{\lambda+1-j} A^j \right),$$

for $n \geqslant N$, where

$$A = \frac{\beta^u}{1 + c^\ell + \eta^\ell (g(s_{1,k}^*, 0, y_{1,k}^*))^\ell}, \quad B = \frac{2c\eta^u}{1 + c^\ell + \eta^\ell (g(s_{2,k}^*, 0, y_{1,k}^* - \delta))^\ell}$$

and

$$C = \frac{1 + \beta^u (s_1^*)^u}{1 + c^\ell + \eta^\ell (g(s_{1,k}^*, 0, y_{1,k}^*))^\ell}.$$

By (3.1.16) and (3.1.18), taking limits, we conclude that

$$\mathcal{R}^\ell(\lambda, s_2^*, y_2^*) \leqslant (1 + \delta B)^{\lambda+1} \left(\mathcal{R}^\ell(\lambda, s_1^*, y_1^*) + \sum_{j=1}^{\lambda+1} \binom{\lambda+1}{j} \delta^j C^{\lambda+1-j} A^j, \right)$$

By the arbitrariness of $\varepsilon > 0$, we conclude that $\mathcal{R}^\ell(\lambda, s_2^*, y_2^*) \leqslant \mathcal{R}^\ell(\lambda, s_1^*, y_1^*)$ and, interchanging the roles of (s_1^*, y_1^*) and (s_2^*, y_2^*) it is immediate that $\mathcal{R}^\ell(\lambda, s_2^*, y_2^*) \geqslant \mathcal{R}^\ell(\lambda, s_1^*, y_1^*)$. Thus $\mathcal{R}^\ell(\lambda, s_2^*, y_2^*) = \mathcal{R}^\ell(\lambda, s_1^*, y_1^*)$.

The result is proved. $\qquad\square$

3.2. Extinction and uniform strong persistence

In this section we establish our main results on extinction and persistence. To obtain our result on extinction we must make some additional assumptions on the function g. In spite of this, it is easy to see that the usual growth rates still fulfill these assumptions.

THEOREM 3.1. *Assume conditions D1) to D7). Assume further that $g(S+I,0,P) \leqslant g(S,I,P)$. If there is $\lambda \in \mathbb{N}$ such that $\mathcal{R}^u(\lambda) < 1$ then the infectives (I_n) go to extinction in system (3.1.2). Furthermore, if $a \equiv 0$ and $g(S,I,P) = g_0(S,I)P$, any disease-free solution $(s_n^*, 0, y_n^*)$ of (3.1.2), where (s_n^*) is a solution of (3.1.5) and (y_n^*) is a solution of (3.1.8), is globally asymptotically attractive.*

PROOF. Since $\mathcal{R}^u(\lambda) < 1$, given $\delta_1 > 0$ sufficiently small, there are $\delta_0 > 0$ and $N \in \mathbb{N}$ such that

$$\prod_{k=n}^{n+\lambda} \frac{1 + \beta_k(s_{k+1}^* + \delta)}{1 + c_k + \eta_k g(s_k^* + \delta, 0, y_k^* - \delta))} < 1 - \delta_1, \tag{3.2.1}$$

for $n \geqslant N$ and all positive $\delta \leqslant \delta_0$. Let $N_n = S_n + I_n$. Since $\mu_n \leqslant c_n$, by the first two equations in (3.1.2), we conclude that

$$N_{n+1} - N_n \leqslant \Lambda_n - \mu_n N_{n+1} \quad \Leftrightarrow \quad N_{n+1} \leqslant \frac{\Lambda_n}{1 + \mu_n} + \frac{N_n}{1 + \mu_n}$$

and thus $S_n, I_n \leqslant N_n \leqslant s_n$, where (s_n) is any solution of (3.1.5) with $s_0 = S_0$. By iv) in Lemma 3.1 we have $|s_n - s_n^*| \leqslant \delta_0$ for sufficiently large n, say $n \geqslant N_1 \geqslant N$. Thus

$$S_n, I_n \leqslant S_n + I_n = N_n \leqslant s_n \leqslant s_n^* + \delta_0,$$

for $n \geqslant N_1$.

By the third equation in (3.1.2), we conclude that

$$P_{n+1} - P_n \geqslant (r_n - b_n P_{n+1})P_n \quad \Leftrightarrow \quad P_{n+1} \geqslant \frac{r_n P_n + P_n}{1 + b_n P_n}$$

and thus $P_n \geqslant y_n$, where (y_n) is any solution of (3.1.8) with $y_0 = P_0$. By iv) in Lemma 3.2 we have $|y_n - y_n^*| \leqslant \delta_0$ for sufficiently large n, say $n \geqslant N_2 \geqslant N_1$. Thus $P_n \geqslant y_n \geqslant y_n^* - \delta_0$, for $n \geqslant N_2$. Using our hypothesis, by the second equation in (3.1.2) and (3.2.1),

$$\begin{aligned}
I_{n+1} &= \frac{\beta_n S_{n+1} I_n + I_n}{1 + \eta_n g(S_n, I_n, P_n) + c_n} \\
&\leqslant \frac{\beta_n S_{n+1} I_n + I_n}{1 + \eta_n g(S_n + I_n, 0, P_n) + c_n} \\
&\leqslant \frac{\beta_n(s_{n+1}^* + \delta_0) + 1}{1 + c_n + \eta_n g(s_n^* + \delta_0, 0, y_n^* - \delta_0)} I_n \\
&< (1 - \delta_1) I_{n-\lambda-1} \\
&< \cdots < (1 - \delta_1)^{\lfloor (n-N_2)/(\lambda+1) \rfloor} I_{n - \lfloor (n-N_2)/(\lambda+1) \rfloor (\lambda+1)} \\
&\leqslant d \left((1 - \delta_1)^{1/(\lambda+1)} \right)^{n - N_2},
\end{aligned}$$

for $n \geqslant N_2$, where $d = \max\limits_{j=0,\dots,N_2} I_j$. We conclude that $I_n \to 0$ as $n \to +\infty$ and we have extinction of the infectives.

Assume now that $a \equiv 0$ and $g(S, I, P) = g_0(S, I)P$, let $((S_n, I_n, P_n))$ be any solution of (3.1.2) and consider the sequence $((s_n^*, 0, y_n^*))$, where (s_n^*) is a solution of (3.1.5) and (y_n^*) is a solution of (3.1.8).

Since $I_n \to 0$ as $n \to +\infty$, given $\delta > 0$ there is $T \in \mathbb{N}$ such that $I_n < \delta$ for $n \geqslant T$. Letting $U_n = S_n - s_n^*$, we have, by the first equation in (3.1.2),

$$U_{n+1} - U_n = -\mu_n U_{n+1} - \beta_n S_{n+1} I_n,$$

for $n \geqslant T$. Thus, by iv) in Lemma 3.1 and by Lemma 3.3, we have

$$-\beta^u L \delta < (1 + \mu_n) U_{n+1} - U_n < 0$$

for n sufficiently large.

We get, for $\delta > 0$ sufficiently small

$$
\begin{aligned}
U_{n+1} &> -\frac{\beta^u L \delta}{1 + \mu_n} + \frac{1}{1 + \mu_n} U_n \\
&> -\frac{\beta^u L \delta}{1 + \mu_n} + \frac{1}{1 + \mu_n}\left(-\frac{\beta^u K \delta}{1 + \mu_{n-1}} + \frac{1}{1 + \mu_{n-1}} U_{n-1} \right) \\
&> \cdots \\
&> \left(\prod_{m=0}^{n} \frac{1}{1 + \mu_m} \right) U_0 - \sum_{m=0}^{n} (\beta^u L \delta)^{m+1} \left(\prod_{k=m}^{n} \frac{1}{1 + \mu_k} \right) \\
&> \left(\prod_{m=0}^{n} \frac{1}{1 + \mu_m} \right) U_0 - \delta \beta^u L \sum_{m=0}^{n} K \theta^{n-m} \\
&> \left(\prod_{m=0}^{n} \frac{1}{1 + \mu_m} \right) U_0 - \frac{\beta^u L K \theta}{1 - \theta} \delta \\
&> -\frac{\beta^u L K \theta}{1 - \theta} \delta.
\end{aligned}
$$

Similarly,

$$U_{n+1} < \frac{1}{1 + \mu_n} U_n < \left(\prod_{m=0}^{n} \frac{1}{1 + \mu_m} \right) U_0.$$

Since

$$\prod_{m=0}^{n} \frac{1}{1 + \mu_m} \to 0 \quad \text{as} \quad n \to +\infty,$$

given $\delta > 0$, we have $|U_{n+1}| < M\delta$, where $M = \beta^u L K \theta/(1-\theta)$, for sufficiently large n. We conclude that $|U_n| \to 0$ as $n \to +\infty$ and thus

$$S_n \to s_n^* \quad \text{as} \quad n \to +\infty. \tag{3.2.2}$$

By the third equation in (3.1.2), we have, for sufficiently large n,

$$P_{n+1} - P_n = (r_n - b_n P_{n+1})P_n + \theta_n \eta_n g_0(S_n, I_n)P_n I_{n+1}$$

$$\leqslant (r_n - b_n P_{n+1})P_n + \theta_n \eta_n g_0(0,0)P_n \delta$$

and thus

$$(r_n - b_n P_{n+1})P_n \leqslant P_{n+1} - P_n \leqslant (r_n + \theta^u \eta^u g_0(0,0)\delta - b_n P_{n+1})P_n.$$

We conclude that

$$\frac{r_n P_n + P_n}{1 + b_n P_n} \leqslant P_{n+1} \leqslant \frac{(r_n + \theta^u \eta^u g_0(0,0)\delta) P_n + P_n}{1 + b_n P_n}.$$

By iv) in Lemma 3.2, we have $|P_n - y_n^*| \to 0$ as $n \to +\infty$. The result follows since $(S_n, I_n, P_n) \to (s_n^*, 0, y_n^*)$ as $n \to +\infty$. $\qquad\square$

THEOREM 3.2. *Assume conditions D1) to D3) and D5) to D9). If there is a constant* $\lambda \in \mathbb{N}$ *such that* $\mathcal{R}^\ell(\lambda) > 1$ *then the infectives* (I_n) *are strong persistent in system (3.1.2).*

PROOF. Assume that there is a constant $\lambda > 0$ such that $\mathcal{R}^\ell(\lambda) > 1$. Then, there is a function ψ such that, for all $\delta > 0$ sufficiently small we have

$$\prod_{k=n}^{n+\lambda} \frac{1 + \beta_k(x_{1,k+1}^* - \delta_0)}{1 + c_k + \eta_k g(x_{1,k}^* - \delta_0, 0, z_{2,k}^* + \delta_0)} > 1 + \psi(\delta), \tag{3.2.3}$$

with $\psi(\delta) > 0$ for all $\delta > 0$ and $\psi(\delta) \to 0$ as $\delta \to 0$. Let $N_1 \in \mathbb{N}$ and (S_n, I_n, P_n) be a solution of (3.1.2) with $I_n > 0$ for all $n \geqslant N_1$. We will use a contradiction argument to prove that there is $\varepsilon_1 > 0$ such that

$$\limsup_{n \to +\infty} I_n > \varepsilon_1. \tag{3.2.4}$$

We may assume that $\varepsilon_1 > 0$ is sufficiently small so that D8) and D9) hold for ε_1. Assuming that (3.2.4) does not hold, there is $N_2 \geqslant N_1$ such that $I_n < \varepsilon_1$ for all $n \geqslant N_2$. By the first and third equation in (3.1.2), we conclude that

$$\begin{cases} S_{n+1} - S_n \leqslant \Lambda_n - \mu_n S_{n+1} \\ P_{n+1} - P_n \leqslant (r_n - b_n P_{n+1})P_n + \gamma_n a_n f(S_{n+1}, 0, P_n)P_n + \theta_n \eta_n g(S_n, 0, P_n)P_n \varepsilon_1 \end{cases},$$

for all $n \geqslant N_2$. Considering system (3.1.14) with $\varepsilon = \varepsilon_1$, we have $S_n \leqslant x_{2,\varepsilon_1,n}$ and $P_n \leqslant z_{2,\varepsilon_1,n}$ for sufficiently large n. By D9) we also have, for sufficiently large n,

$$x_{2,\varepsilon_1,n} \leqslant x^*_{2,\varepsilon_1,n} + \varepsilon_1 \quad \text{and} \quad z_{2,\varepsilon_1,n} \leqslant z^*_{2,\varepsilon_1,n} + \varepsilon_1$$

and by the continuity properties in D8) and D9), we have

$$x_{2,\varepsilon_1,n} \leqslant x^*_{2,\varepsilon_1,n} + \varepsilon_1 \leqslant x^*_{2,n} + \chi_1(\varepsilon_1) \quad \text{and} \quad z_{2,\varepsilon_1,n} \leqslant z^*_{2,\varepsilon_1,n} + \varepsilon_1 \leqslant z^*_{2,n} + \chi_2(\varepsilon_1),$$

with $\chi_1(\varepsilon_1), \chi_2(\varepsilon_1) \to 0$ as $\varepsilon_1 \to 0$. Thus, in particular, for sufficiently large n,

$$S_n \leqslant x_{2,\varepsilon_1,n} \leqslant x^*_{2,n} + \chi_1(\varepsilon_1) \quad \text{and} \quad P_n \leqslant z_{2,\varepsilon_1,n} \leqslant z^*_{2,n} + \chi_2(\varepsilon_1), \tag{3.2.5}$$

Again by the first and third equation in (3.1.2), we conclude that

$$\begin{cases} S_{n+1} - S_n \geqslant \Lambda_n - \mu_n S_{n+1} - a_n f(S_{n+1}, 0, 0) z_{2,\varepsilon_1,n} - \beta_n S_{n+1} \varepsilon_1 \\ P_{n+1} - P_n \geqslant (r_n - b_n P_{n+1}) P_n + \gamma_n a_n f(S_{n+1}, \varepsilon_1, P_n) P_n \end{cases},$$

for all $n \geqslant N_2$.

Consider system (3.1.13) with $\varepsilon = \varepsilon_1$. We have $S_n \geqslant x_{1,\varepsilon_1,n}$ and $P_n \geqslant z_{1,\varepsilon_1,n}$ for sufficiently large n. By D8) we also have, for sufficiently large n,

$$x_{1,\varepsilon_1,n} \geqslant x^*_{1,\varepsilon_1,n} - \varepsilon_1 \quad \text{and} \quad z_{1,\varepsilon_1,n} \geqslant z^*_{1,\varepsilon_1,n} - \varepsilon_1.$$

and by the continuity properties in D8) and D9), we have

$$x_{1,\varepsilon_1,n} \geqslant x^*_{1,\varepsilon_1,n} - \varepsilon_1 \geqslant x^*_{1,n} - \varphi_1(\varepsilon_1) \quad \text{and} \quad z_{1,\varepsilon_1,n} \geqslant z^*_{1,\varepsilon_1,n} - \varepsilon_1 \geqslant z^*_{1,n} - \varphi_2(\varepsilon_1),$$

with $\varphi_1(\varepsilon_1), \varphi_2(\varepsilon_1) \to 0$ as $\varepsilon_1 \to 0$. Thus, in particular, for sufficiently large n,

$$S_n \geqslant x_{1,\varepsilon,n} \geqslant x^*_{1,n} - \varphi_1(\varepsilon_1) \quad \text{and} \quad P_n \geqslant z_{1,\varepsilon,n} \geqslant z^*_{1,n} - \varphi_2(\varepsilon_1). \tag{3.2.6}$$

From the second equation in (3.1.2), (3.2.6), (3.2.5) and (3.2.3), we conclude that

$$\begin{aligned} I_{n+1} &= \frac{\beta_n S_{n+1} I_n + I_n}{1 + \eta_n g(S_n, I_n, P_n) + c_n} \\ &\geqslant \frac{\beta_n(x^*_{1,n+1} - \varphi_1(\varepsilon_1)) + 1}{1 + \eta_n g(x^*_{1,n+1} - \varphi_1(\varepsilon_1), 0, z^*_{2,n} + \chi_2(\varepsilon_1)) + c_n} I_n \\ &> (1 + \psi(\varepsilon_1)) I_{n-\lambda-1} \\ &> \cdots > (1 + \psi(\varepsilon_1))^{\lfloor n/(\lambda+1) \rfloor} I_{n-\lfloor n/(\lambda+1) \rfloor(\lambda+1)}, \end{aligned} \tag{3.2.7}$$

for all $n \geqslant N_3$ with $N_3 \geqslant N_2$. Therefore, by (3.2.3) and (3.2.7), we conclude that $I_n \rightarrow +\infty$. A contradiction to Lemma 3.3. We have (3.2.4) and the infectives in system (3.1.2) are weak persistent.

Using again a contradiction argument, we will prove that we have strong persistence of the infectives. We may assume, with no loss of generality, that there are $\delta, \delta_0 > 0$ such that

$$\prod_{k=n}^{n+\lambda} \frac{1 + \beta_k(x^*_{1,k+1} - \delta_0)}{1 + c_k + \eta_k g(x^*_{1,k} - \delta_0, 0, z^*_{2,k} + \delta_0)} > 1 + \delta, \tag{3.2.8}$$

for all sufficiently large $n \in \mathbb{N}$. For each $z_0 = (S_0, I_0, P_0)$, denote by $((S_{n,z_0}, I_{n,z_0}, P_{n,z_0}))$ the solution of (3.1.2) with $(S_{0,z_0}, I_{0,z_0}, P_{0,z_0}) = (S_0, I_0, P_0)$.

Proceeding by contradiction, if the system is not strong persistent, then there is a sequence of initial values $z_{0,k} = (S_{0,k}, I_{0,k}, P_{0,k})$, $k \in \mathbb{N}$, such that

$$\liminf_{n \to +\infty} I_{n,z_{0,k}} < \frac{\varepsilon_0}{k^2}. \tag{3.2.9}$$

From (3.2.4) and (3.2.8), for each $k \in \mathbb{N}$ there are sequences $(s_{m,k})$ and $(t_{m,k})$ such that

$$0 < s_{1,k} < t_{1,k} < s_{2,k} < t_{2,k} < \cdots < s_{m,k} < t_{m,k} < \cdots, \tag{3.2.10}$$

$$s_{m,k} \to +\infty \quad \text{as} \quad m \to +\infty, \tag{3.2.11}$$

$$I_{s_{m,k},z_{0,k}} > \frac{\varepsilon_0}{k}, \quad I_{t_{m,k},z_{0,k}} < \frac{\varepsilon_0}{k^2}, \tag{3.2.12}$$

and

$$\frac{\varepsilon_0}{k^2} \leqslant I_{n,z_{0,k}} \leqslant \frac{\varepsilon_0}{k}, \quad \text{for all} \quad n \in [s_{m,k}, t_{m,k} - 1] \cap \mathbb{N}. \tag{3.2.13}$$

For any $n \in [s_{m,k}, t_{m,k} - 1] \cap \mathbb{N}$ sufficiently large, we have, using (3.1.11),

$$\begin{aligned}
I_{n+1,z_{0,k}} &= \frac{1 + \beta_n S_{n+1,z_{0,k}}}{1 + c_n + \eta_n g(S_{n,z_{0,k}}, I_{n,z_{0,k}}, P_{n,z_{0,k}})} I_{n,z_{0,k}} \\
&> \frac{1}{1 + c_n + \eta_n g(S_{n,z_{0,k}}, 0, P_{n,z_{0,k}})} I_{n,z_{0,k}} \\
&\geqslant \frac{1}{1 + a} I_{n,z_{0,k}},
\end{aligned}$$

where $a = c^u + \eta^u g(0, 0, L + \delta) > 0$. Therefore, by (3.2.12), we obtain

$$\frac{\varepsilon_0}{k^2} > I_{t_{m,k},z_{0,k}} \geqslant \left(\frac{1}{1+a}\right)^{t_{m,k}-s_{m,k}} I_{s_{m,k},z_{0,k}} > \left(\frac{1}{1+a}\right)^{t_{m,k}-s_{m,k}} \frac{\varepsilon_0}{k},$$

and therefore we get

$$t_{m,k} - s_{m,k} > \frac{\ln k}{\ln(1 + a)} \to +\infty \quad \text{as} \quad k \to +\infty.$$

Given $n_1 \in \mathbb{N}$, we conclude that we can choose $k_1 \in \mathbb{N}$ such that

$$t_{m,k} - s_{m,k} > n_1 + \lambda + 1,$$

for all $k \geqslant k_1$.

Now, for all $k \geqslant k_1$ and $n \in [s_{m,k} + 1, t_{m,k}] \cap \mathbb{N}$, we have

$$\begin{cases} S_{n+1,z_{0,k}} - S_{n,z_{0,k}} & \leqslant \Lambda_n - \mu_n S_{n+1,z_{0,k}} \\ P_{n+1,z_{0,k}} - P_{n,z_{0,k}} & \leqslant (r_n - b_n P_{n+1,z_{0,k}}) P_{n,z_{0,k}} \\ & \quad + \gamma_n a_n f(S_{n+1,z_{0,k}}, I_{n,z_{0,k}}, P_{n,z_{0,k}}) P_{n,z_{0,k}} \\ & \quad + \theta_n \eta_n g(S_{n,z_{0,k}}, I_{n,z_{0,k}}, P_{n,z_{0,k}}) \varepsilon_1 \end{cases},$$

Let $(\bar{x}_{1,n}, \bar{z}_{1,n})$ be a solution of (3.1.13) with initial condition $\bar{x}_{1,s_{m,k}+1} = S_{s_{m,k}+1}$ and $\bar{z}_{1,s_{m,k}+1} = P_{s_{m,k}+1}$. By D8), for sufficiently large $k \in \mathbb{N}$ we have

$$|S_{n,z_{0,k}} - x^*_{1,n}| \leqslant |S_{n,z_{0,k}} - \bar{x}_{1,n}| + |\bar{x}_{1,n} - x^*_{1,n}| < \varepsilon_0/2 + \varepsilon_0/2 = \varepsilon_0$$

for all $n \in [s_{m,k} + 1, t_{m,k}] \cap \mathbb{N}$. In particular

$$S_{n,z_{0,k}} \geqslant x^*_{1,n} - \varepsilon_0, \tag{3.2.14}$$

for all $n \in [s_{m,k} + 1, t_{m,k}] \cap \mathbb{N}$. In a similar way, using D9), we conclude that, for sufficiently large $k \in \mathbb{N}$ we have

$$P_{n,z_{0,k}} \leqslant z^*_{2,n} + \varepsilon_0, \tag{3.2.15}$$

for all $n \in [s_{m,k} + 1, t_{m,k}] \cap \mathbb{N}$.

Finally, we have

$$\begin{aligned} I_{n+1,z_{0,k}} &= \frac{1 + \beta_n S_{n+1,z_{0,k}}}{1 + c_n + \eta_n g(S_{n,z_{0,k}}, I_{n,z_{0,k}}, P_{n,z_{0,k}})} I_{n,z_{0,k}} \\ &\geqslant \frac{1 + \beta_n (x^*_{1,n} - \varepsilon_0)}{1 + c_n + \eta_n g(x^*_{1,n} - \varepsilon_0, 0, z^*_{2,n} + \varepsilon_0)} I_{n,z_{0,k}} \end{aligned} \tag{3.2.16}$$

for all $n \in [s_{m,k} + n_1 + 1, t_{m,k}] \cap \mathbb{N}$ and $k \geqslant n_4$. By (3.2.9) and (3.2.16) we get

$$\frac{\varepsilon_0}{k^2} > I_{t_{m,k},z_{0,k}} \geqslant I_{t_{m,k}-\lambda,z_{0,k}} \prod_{n=t_{m,k}-\lambda}^{t_{m,k}} \frac{1 + \beta_n(x^*_{1,n} - \varepsilon_0)}{1 + c_n + \eta_n g(x^*_{1,n} - \varepsilon_0, 0, z^*_{2,n} + \varepsilon_0)} I_{n,z_{0,k}} > \frac{\varepsilon_0}{k^2},$$

a contradiction. Thus we conclude that the infectives are strong persistent and the result follow. $\qquad \square$

3.3. Examples

3.3.1. No predation of uninfected preys. Letting $a \equiv 0$ and $g(x, y, z) = z$ in (3.1.2), we obtain the model below that corresponds to the discrete counterpart of the model in [**75**].

$$
\begin{cases}
S_{n+1} - S_n = \Lambda_n - \mu_n S_{n+1} - \beta_n S_{n+1} I_n \\[2mm]
I_{n+1} - I_n = \beta_n S_{n+1} I_n - \eta_n P_n I_{n+1} - c_n I_{n+1} \\[2mm]
P_{n+1} - P_n = (r_n - b_n P_{n+1}) P_n + \theta_n \eta_n P_n I_{n+1}
\end{cases}
\qquad . \qquad (3.3.1)
$$

For model (3.3.1) we assume conditions D1), D2) and D4). Notice that D3) is trivial, D5) and D6) follow from the discussion on (3.1.4) with $a_n = 0$, D7) follows from Lemma 3.3 and D8) and D9) follow from Lemma 3.1 and Lemma 3.2, respectively.

For each solution (s_n^*) of (3.1.5) with $s_0 > 0$, each solution (y_n^*) of (3.1.8) with $y_0 > 0$ and each $\lambda \in \mathbb{N}$, in this context of no predation (of uninfected preys) we set

$$
\mathcal{R}_{NP}^{\ell}(\lambda) = \liminf_{n \to +\infty} \prod_{k=n}^{n+\lambda} \frac{1 + \beta_k s_{k+1}^*}{1 + c_k + \eta_k y_k^*}
$$

and

$$
\mathcal{R}_{NP}^{u}(\lambda) = \limsup_{n \to +\infty} \prod_{k=n}^{n+\lambda} \frac{1 + \beta_k s_{k+1}^*}{1 + c_k + \eta_k y_k^*}.
$$

The next theorems correspond to discrete counterparts of the results in [**75**].

THEOREM 3.3. *If there is $\lambda \in \mathbb{N}$ such that $\mathcal{R}_{NP}^{u}(\lambda) < 1$ then the infectives (I_n) go to extinction in system (3.3.1) and any disease-free solution $((s_n^*, 0, y_n^*))$ of (3.3.1), where (s_n^*) is a solution of (3.1.5) and (y_n^*) is a solution of (3.1.8), is globally asymptotically attractive.*

THEOREM 3.4. *If there is $\lambda \in \mathbb{N}$ such that $\mathcal{R}_{NP}^{\ell}(\lambda) > 1$ then the infectives (I_n) are strongly persistent in system (3.3.1).*

To do some simulation, we consider the particular solutions $s_n^* = \Lambda/\mu$, $y_n^* = r/b$ and the following particular set of parameters in system (3.3.1): $\Lambda_n = 0.3$, $\mu_n = 0.1$, $\beta_n = \beta_0(1 + 0.7\cos(\pi n/5))$, $\eta_n = 0.3(1 + 0.7\cos(\pi n/5))$, $c_n = 0.18$, $r_n = 0.3$, $b_n = 0.2$, $\theta_n = 0.9$. This example is based the continuous time example from Chapter 1.

When $\beta_0 = 0.17$ we obtain $\mathcal{R}_{NP}^{u}(\lambda) \approx 0.89 < 1$ and we conclude that we have the extinction (figure 3.1). When $\beta_0 = 0.29$ we obtain $\mathcal{R}_{NP}^{\ell}(\lambda) \approx 1.24 > 1$ and we conclude that the infectives are strongly persistent (figure 3.2).
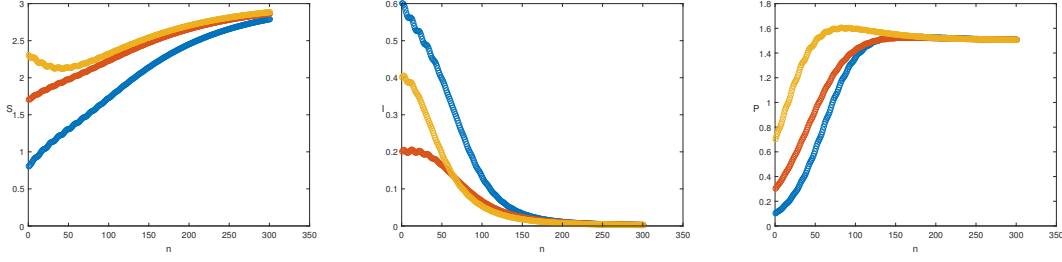
FIGURE 3.1.   Extinction; no predation uninfected preys; $\beta_0 = 0.17$.
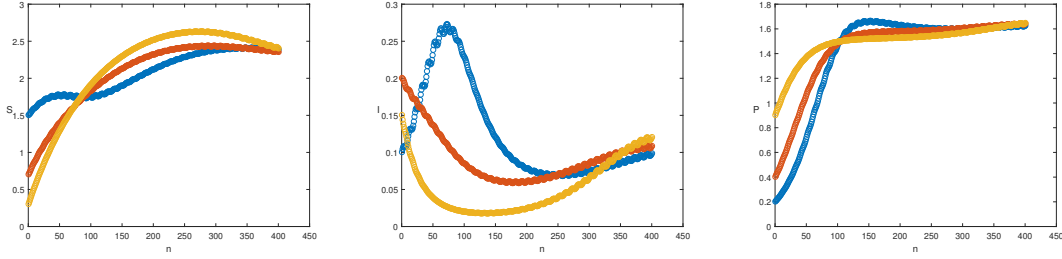
FIGURE 3.2.   Persistence; no predation on uninfected preys; $\beta_0 = 0.29$.

In extinction and uniform strong persistence scenario we considered, respectively, the following initial conditions: $(S_0, I_0, P_0) = (0.8, 0.6, 0.1)$, $(S_0, I_0, P_0) = (1.7, 0.2, 0.3)$ and $(S_0, I_0, P_0) = (2.3, 0.4, 0.7)$; $(S_0, I_0, P_0) = (1.5, 0.1, 0.2)$, $(S_0, I_0, P_0) = (0.7, 0.2, 0.4)$ and $(S_0, I_0, P_0) = (0.3, 0.15, 0.9)$.

3.3.2. Periodic coefficients. Consider the system (3.1.2) and assume that there is $\omega \in \mathbb{N}$ such that $\Lambda_{n+\omega} = \Lambda_n$, $\mu_{n+\omega} = \mu_n$, $a_{n+\omega} = a_n$, $\beta_{n+\omega} = \beta_n$, $\eta_{n+\omega} = \eta_n$, $c_{n+\omega} = c_n$, $r_{n+\omega} = r_n$, $b_{n+\omega} = b_n$, $\gamma_{n+\omega} = \gamma_n$ and $\theta_{n+\omega} = \theta_n$, for all $n \in \mathbb{N}$. Conditions D1) to D3) and D5) to D8) are assumed; condition D4) is trivial.

For each solution (s_n^*) of (3.1.5) with $s_0 > 0$, each solution (y_n^*) of (3.1.8) with $y_0 > 0$ and for each solution $((x_{1,n}^*, z_{1,n}^*))$ of (3.1.13) and $((x_{2,n}^*, z_{2,n}^*))$ of (3.1.14) with $\varepsilon = 0$ and initial conditions $x_0 > 0$ and $z_0 > 0$, and each $\lambda \in \mathbb{N}$, we set

$$\mathcal{R}_{PER}^{\ell} = \prod_{k=1}^{\omega} \frac{1 + \beta_k x_{1,k+1}^*}{1 + c_k + \eta_k g(x_{1,k}^*, 0, z_{2,k}^*)}$$

and

$$\mathcal{R}_{PER}^{u} = \prod_{k=1}^{\omega} \frac{1 + \beta_k s_{k+1}^*}{1 + c_k + \eta_k g(s_k^*, 0, y_k^*)},$$

COROLLARY 11. *If $\mathcal{R}_{PER}^{u} < 1$ then the infective (I_n) go to extinction in system (3.1.2) and any disease-free solution $((s_n^*, 0, y_n^*))$ of (3.1.2), where (s_n^*) is a solution of the periodic version of (3.1.5) and (y_n^*) is a solution of the periodic version of (3.1.8), is globally asymptotically attractive.*
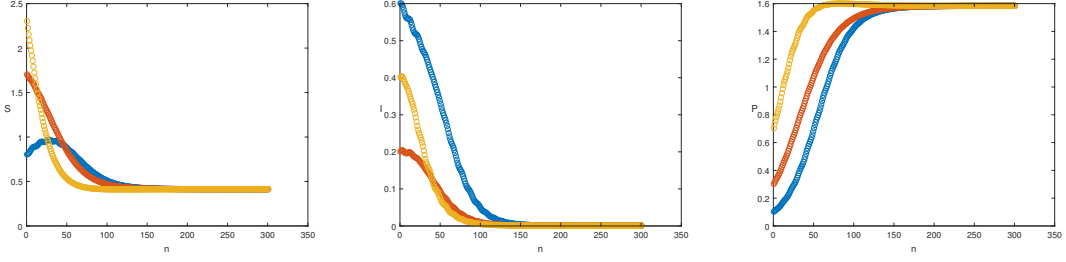
FIGURE 3.3.   Extinction; periodic coefficients; $\beta_0 = 0.17$.
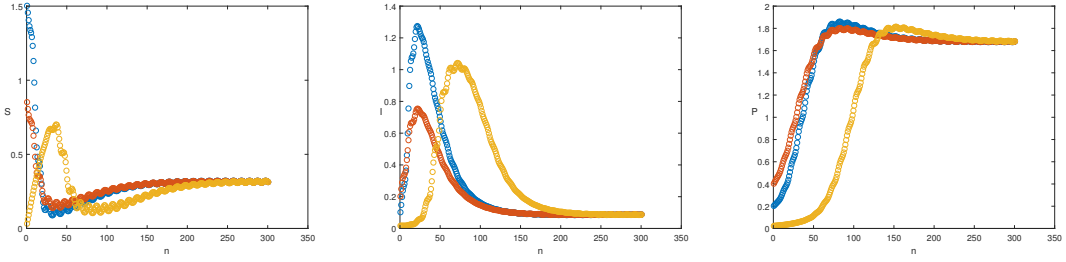
FIGURE 3.4.   Persistence; periodic coefficients; $\beta_0 = 2.2$.

COROLLARY 12. *If $\mathcal{R}^{\ell}_{PER} > 1$ then the infective (I_n) is strongly persistent in system* (3.1.2), *where $x^*_{1,n}, z^*_{1,n}$ and $x^*_{2,n}, z^*_{3,n}$ are the components of solutions of* (3.1.13) *and* (3.1.14), *respectively. Moreover, there exist a periodic orbit of period ω.*

To do some simulation, we consider $f(x, y, z) = x$, $g(x, y, z) = z$. We also considered the following particular set of parameters, and with the exception of β we assume that they are all constants: $\Lambda_n = 0.3$, $\mu_n = 0.1$, $a_n = 0.4$, $\beta_n = \beta_0(1 + 0.7\cos(\pi n/5))$, $\eta_n = 0.3$, $c_n = 0.18$, $r_n = 0.3$, $b_n = 0.2$, $\gamma_n = 0.1$ and $\theta = 0.9$. We have the particular solutions $s^*_k = \Lambda/\mu$, $y^*_k = r/b$,

$$(x^*_{2,\varepsilon,n}, z^*_{2,\varepsilon,n}) = \left(\frac{\Lambda}{\mu}, \Theta\right) \quad \text{and} \quad (x^*_{1,\varepsilon,n}, z^*_{1,\varepsilon,n}) = \left(\frac{\Lambda}{\mu + a\Theta + \varepsilon}, \frac{r}{b} + \frac{\gamma a \Lambda}{b(\mu + a\Theta + \varepsilon)}\right),$$

where

$$\Theta = \frac{r\mu + \gamma a \Lambda + \theta \eta \mu \varepsilon}{b\mu}.$$

When $\beta_0 = 0.17$, we obtain $\mathcal{R}^u_{PER} \approx 0.44 < 1$ and we conclude that we have the extinction (figure 3.3). When $\beta_0 = 2.2$, we obtain $\mathcal{R}^{\ell}_{PER}(\lambda) \approx 2.72 > 1$ and we conclude that the infectives are uniformly strong persistent (figure 3.4).   In extinction and uniform strong persistence scenario we considered, respectively, the following initial conditions: $(S_0, I_0, P_0) = (0.8, 0.6, 0.1)$, $(S_0, I_0, P_0) = (1.7, 0.2, 0.3)$ and $(S_0, I_0, P_0) = (2.3, 0.4, 0.7)$; $(S_0, I_0, P_0) = (1.5, 0.1, 0.2)$, $(S_0, I_0, P_0) = (0.7, 0.2, 0.4)$ and $(S_0, I_0, P_0) = (0.3, 0.15, 0.9)$.

3.3.3. Autonomous model. Consider the system (3.1.2), and assume now that $f(x, y, z) = x$, $g(x, y, z) = z$, $\Lambda_n = \Lambda$, $\mu_n = \mu$, $a_n = a$, $\beta_n = \beta$, $\eta_n = \eta$, $c_n = c$, $r_n = r$, $b_n = b$ and $\gamma_n = \gamma$, $\theta_n = \theta$. Then we obtain following the model:

$$\begin{cases} S_{n+1} - S_n = \Lambda - \mu S_{n+1} - a S_{n+1} P_n - \beta S_{n+1} I_n \\ I_{n+1} - I_n = \beta S_{n+1} I_n - \eta I_{n+1} P_n - c I_{n+1} \\ P_{n+1} - P_n = (r - b P_{n+1}) P_n + \gamma a S_{n+1} P_n + \theta \eta I_{n+1} P_n \end{cases} . \qquad (3.3.2)$$

Conditions D1) to D4) are immediate. Conditions D5) and D6) follow from the discussion on (3.1.4). Condition D7) follows from Lemma 3.3 and D8) and D9) follow from Lemma 3.1 and Lemma 3.2, respectively.   For each solution (s_n^*) of (3.1.5) with $s_0 > 0$, each solution (y_0^*) of (3.1.8) with $y_0 > 0$ and each solution $((x_n^*, z_n^*))$ of (3.1.13) with $\varepsilon = 0$ and initial conditions $x_0 > 0$ and $z_0 > 0$, and each $\lambda \in \mathbb{N}$, we set

$$\mathcal{R}_A^\ell = \frac{\mu + a\Theta + \beta}{(\mu + a\Theta)(1 + c + \eta\Theta)},$$

where $\Theta = (r\mu + \gamma a\Lambda)/(b\mu)$, and

$$\mathcal{R}_A^u = \frac{1 + \beta(\Lambda/\mu)}{1 + c + \eta(r/b)},$$

COROLLARY 13. *If $\mathcal{R}_A^u < 1$ then the infective (I_n) in system (3.3.2) go to extinction.*

COROLLARY 14. *If $\mathcal{R}_A^\ell > 1$ then the infective (I_n) in system (3.3.2) are strongly persistent.*

To do some simulation, we consider the following particular set of parameters: $\Lambda = 0.3$, $\mu = 0.1$, $a = 0.4$, $\eta = 0.3$, $c = 0.18$, $r = 0.3$, $b = 0.2$, $\gamma = 0.1$ and $\theta = 0.9$.

When $\beta = 0.17$ we obtain $\mathcal{R}_A^u \approx 0.93 < 1$ and we conclude that we have the extinction. When $\beta = 2.2$ we obtain $\mathcal{R}_A^\ell \approx 1.85 > 1$ and we conclude that the infectives are strongly persistent.

In uniform strong persistence and extinction scenario we considered, respectively, the following initial conditions: $(S_0, I_0, P_0) = (0.8, 0.6, 0.1)$, $(S_0, I_0, P_0) = (1.7, 0.2, 0.3)$ and $(S_0, I_0, P_0) = (2.3, 0.4, 0.7)$; $(S_0, I_0, P_0) = (1.5, 0.1, 0.2)$, $(S_0, I_0, P_0) = (0.7, 0.2, 0.4)$ and $(S_0, I_0, P_0) = (0.3, 0.15, 0.9)$.

3.4. Comments

In this chapter we used Mickens nonstandard method on a version of model (0.0.2) to obtain a discrete family of non-autonomous eco-epidemiological models and discuss the persistence and extinction of the infected preys.
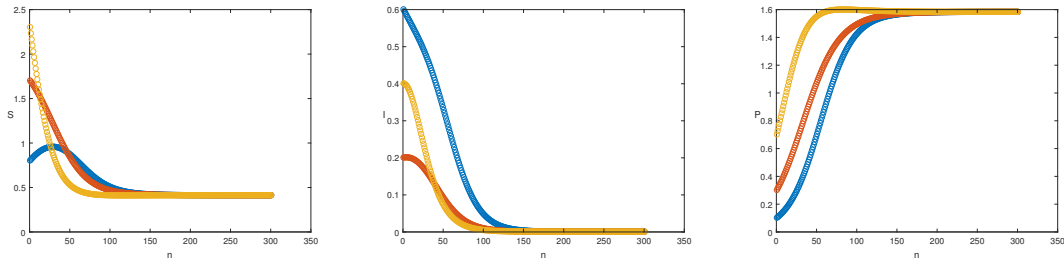
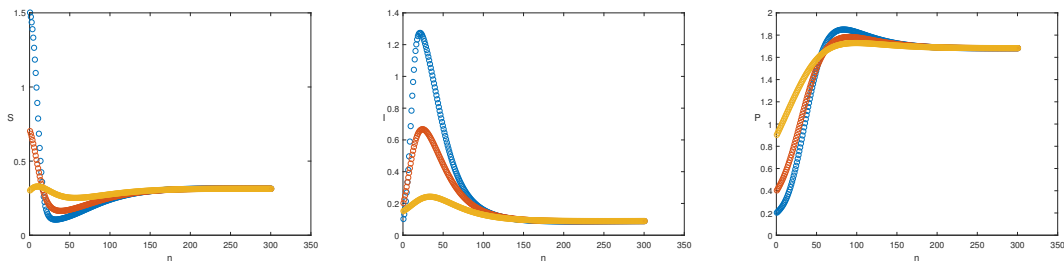FIGURE 3.5.   Extinction; autonomous model; $\beta_0 = 0.17$.

FIGURE 3.6.   Persistence; autonomous model; $\beta_0 = 2.2$.

A natural extension of our work would be to obtain persistence and extinction results when we have general functions for the vital dynamics of uninfected preys and predators, in the spirit of Chapter 1. It would also be interesting to obtain a discrete version of Theorem 3 in Chapter 1 for the general model.

CHAPTER 4

Random Eco-Epidemiological Model

In this chapter we consider random perturbations of system (0.0.2) by introducing a random noise in a parameter. We prove the existence of a global random attractor, the persistence of susceptibles preys and provide conditions for the simultaneous extinction of infectives and predators. We also discuss the dynamics of the corresponding random epidemiological SI and predator-prey models. We obtain for this cases a global random attractor, prove the prevalence of susceptibles preys and provide conditions for the extinctions of infectives/predators.

As already mentioned in the Introduction, the understanding of asymptotic behavior of eco-epidemiological models is an important problem in the mathematical biology. However, from the very beginning of the theory it became clear that even models for a ultra simplified version of real world phenomena exhibit intricate and complex behaviours, despite their simple formulation. In view of this, several mathematical tools were developed with great success in order to understand as much as possible the features and properties of these models.

Motivated, in one hand, by the attempts to approximate the mathematical models to real world phenomena as much as possible and, on the other hand, by the mathematical challenge to provide a deep and general knowledge on the theory constructed due to this formulations, the models and the related theory have been reconstructed and evolved in many directions. One of these directions aimed to consider nonautonomous elements in the mathematical models, such as the seasonal dynamics, and also random elements in order to deal with the presence of noise or complicated fluctuations, in contrast to a completely deterministic situation.

As mentioned in the Introduction, in the nondeterministic situation there are two main approaches to incorporate randomness by considering stochastic and random perturbations, which, roughly speaking, can be expressed throughout stochastic and random differential equations. There are techniques to transform a system with stochastic perturbation to a random dynamical system, being that this can lead to unbounded random

coefficients on the system and can substantially change the structure of the model; see for instance [14, 15].

Random attractors are a central concept in the analysis of random models. Since their introduction there are several improvements regarding the existence and properties of such attractors, but there are questions that are still open in this theory; see [22, 23, 24, 25, 26, 46]. The main strategy adopted to ensure the existence of a random attractor for a given family of random sets is to find a compact absorbing set. Moreover, since the family of random sets we are interested contains every compact deterministic set, the random attractor is actually unique (cf. Remark 4.6).

In this work we consider random perturbations of the general eco-epidemiological model introduced in Chapter 1, that generalizes the model in [75] by adding a general function corresponding to predation on uninfected and infected preys, that is, we aim to perturb the model (0.0.2) with $G(t, S) = \Lambda(t) - \mu S(t)$ and $H(t, P) = -\delta_1 P(t) - \delta_2 P(t)^2$.

We consider a random coefficient (real noise), establish a framework of Random Dynamical Systems (RDS) and discuss the asymptotic behaviour of the solutions of the model considered. Namely, we focus on the existence of a global random attractor which can be understood as a random counterpart of a deterministic global attractor. We moreover prove the prevalence of susceptibles preys and provide conditions for the simultaneous extinction of infectives and predators.

We also discuss the dynamics of the corresponding random epidemiological SI and predator-prey models, by considering the infectives (I) and predators (P) identically equal to zero in the main model (4.2.1). For both subsystems we obtain a global random attractor, prove the persistence of susceptibles/preys and provide conditions for the extinctions of infectives/predators. Random perturbations for a SI model, albeit slightly different, were discussed in [14, 15].

This chapter is organized as follows: in section 4.1 we recall basic facts from RDS and random attractors; in section 4.2 we introduce a random perturbation in an eco-epidemiological model and establish a RDS framework; in section 4.3 we prove existence, uniqueness and boundedness of solutions; in section 4.4 we prove the existence of a unique global random attractor and provide a threshold for the extinction of predators and infected preys, which is illustrated with a particular example; in section 4.5 we discuss the partial dynamics of the perturbed model in the absence of predators or infected preys.

4.1. Random attractors

We start by recalling some basic concepts about Random Dynamical Systems (RDS) and random attractors. For details on RDS we refer to the reference monograph by Arnold [1] and for random attractors see e.g. the survey [25] and reference therein.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where $\mathcal{F}$ is the σ-algebra of measurable subsets of Ω and $\mathbb{P}$ is a probability measure on $\mathcal{F}$. Given a topological space S we denote by $\mathcal{B}(S)$ the Borel σ-algebra of S. Consider a *metric dynamical system* $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ in the sense that

(i) $\theta \colon \mathbb{R} \times \Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$-measurable;

(ii) $\theta_t \colon \Omega \to \Omega$ given by $\theta_t \omega = \theta(t, \omega)$ satisfies:

    (a) $\theta_0 = \mathrm{Id}_\Omega$ and $\theta_{t+s} = \theta_t \circ \theta_s$, for all $t, s \in \mathbb{R}$;

    (b) $\mathbb{P}(\theta_t A) = \mathbb{P}(A)$, for all $A \in \mathcal{F}$, that is, θ_t preserves the probability measure $\mathbb{P}$ for all $t \in \mathbb{R}$.

The non-intuitive term *metric* is present in the literature for historical reasons.

A (*measurable*) *random dynamical system* (RDS) φ on $X = \mathbb{R}^d$ over θ (with time $\mathbb{R}_0^+$) is a map

$$\varphi : \mathbb{R}_0^+ \times \Omega \times X \to X$$

satisfying

(i) measurability: $(t, \omega, x) \mapsto \varphi(t, \omega, x)$ is $(\mathcal{B}(\mathbb{R}_0^+) \otimes \mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(X))$-measurable;

(ii) cocycle property: $\varphi(t, \omega, x)$ forms a cocycle over θ, i.e.,

    (a) $\varphi(0, \omega, x) = x$, for all $\omega \in \Omega, x \in X$;

    (b) $\varphi(t + s, \omega, x) = \varphi(t, \theta_s \omega, \varphi(s, \omega, x))$, for all $s, t \in \mathbb{R}_0^+$, $\omega \in \Omega$ and $x \in X$.

We, moreover, assume a continuity condition:

(iii) $x \to \varphi(t, \omega, x)$ is continuous for all $\omega \in \Omega$ and $t \in \mathbb{R}_0^+$.

To simplify we refer to such a RDS as the pair (θ, φ).

REMARK 4.1.

1. *The cocycle property is assumed to hold for all $\omega \in \Omega$, or at least in a $\mathbb{P}$-full measure subset. This can be a delicate issue to ensure in specific examples of RDS generated from a stochastic or a random differential equation.*

2. *Often joint continuity $(t, x) \to \varphi(t, \omega, x)$ is assumed, but the continuity in time t will not have a role in the following theory of random attractors. Nevertheless, the RDS to be considered here is induced by random differential equations, which provide joint continuity in time and space. In this case, the mapping $\varphi(t, \omega, x_0)$*

corresponds to the solution mapping with noise realization ω and initial condition $x_0 = \varphi(0, \omega, x_0)$; cf. Theorem 4.1.

3. *We consider the phase space $X = \mathbb{R}^d$ because we have in mind specific RDS related to random versions of eco-epidemiological models that evolve on $\mathbb{R}^3$. However, it is typical to assume X to be a Polish space, i.e., a separable topological space for which there is a complete metric which induces the topology. This, in particular, includes open non-empty subsets of Euclidean spaces as well as separable Hilbert and Banach spaces.*

DEFINITION 4.1. *A random set C is a measurable subset of $X \times \Omega$ with respect to the product σ-algebra $\mathcal{B}(X) \otimes \mathcal{F}$.*

Given $\omega \in \Omega$, the ω-section of a random set $C \subseteq X \times \Omega$ is defined by

$$C(\omega) = \{x : (x, \omega) \in C\}.$$

If a set $C \subseteq X \times \Omega$ has closed or compact ω-sections $C(\omega)$ it is a random set as soon as the mapping $\omega \mapsto d(x, C(\omega))$ is measurable (from Ω to $[0, \infty[$) for every $x \in X$ (see [**16**]). In this case C will be said to be a *closed* or a *compact* random set, respectively. We say that a random set C has deterministic components or, for short, that is a deterministic set (as subset of $X \times \Omega$) if its ω-sections are constant: there is $\hat{C} \subseteq X$ such that $C(\omega) = \hat{C}$ for all (or, at least, almost all) $\omega \in \Omega$. We define $\mathcal{D}(X)$ as the set of all deterministic compact random sets. For any set $C \subseteq X \times \Omega$, we define $\overline{C} := \{(x, \omega) : x \in \overline{C(\omega)}\}$. We say that a random set $C(\omega)$ is *bounded* if $C(\omega) \subseteq X$ is bounded for all (or, at least, almost all) $\omega \in \Omega$.

REMARK 4.2. *In general, having $\omega \mapsto d(x, C(\omega))$ measurable for every $x \in X$, does not guarantee that $C \subseteq X \times \Omega$ is a $(\mathcal{B}(X) \otimes \mathcal{F})$- measurable set; see [**25**, Remark 4].*

DEFINITION 4.2. *A bounded random set K is said to be* tempered *with respect to θ if for $\mathbb{P}$-a.e. $\omega \in \Omega$,*

$$\lim_{t \to \infty} e^{-\beta t} \sup_{x \in K(\theta_{-t}\omega)} \|x\| = 0, \ \text{ for all } \beta > 0.$$

A random variable $r \colon \Omega \to \mathbb{R}$ is said to be tempered *with respect to θ if for $\mathbb{P}$-a.e. $\omega \in \Omega$,*

$$\lim_{t \to \infty} e^{-\beta t} \sup_{t \in \mathbb{R}} |r(\theta_t \omega)| = 0, \ \text{ for all } \beta > 0.$$

We denote by $\mathcal{T}(X)$ the set of all tempered sets of X (i.e, tempered bounded random sets with fibers on X) with respect to θ. Notice that $\mathcal{D}(X) \subseteq \mathcal{T}(X)$. In our perspective, the underlying dynamics θ is given, so that we will often omit the reference to θ.

DEFINITION 4.3. *Consider a RDS (θ, φ) on X and an arbitrary family $\mathcal{R}$ of random sets. A random set Γ is called a* random absorbing set *in $\mathcal{R}$ if for any $K \in \mathcal{R}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, there exists $T_K(\omega) > 0$ such that*

$$\varphi(t, \theta_{-t}\omega, K(\theta_{-t}\omega)) \subseteq \Gamma(\omega) \ \text{ for all } t \geqslant T_K(\omega).$$

If, in addition, Γ is a closed random set, then we say that Γ is a closed absorbing set.

DEFINITION 4.4. *Consider a RDS (θ, φ) on X and an arbitrary family $\mathcal{R}$ of random sets. A compact random set $\mathcal{A}$ is called a* pullback $\mathcal{R}$ attractor *if:*

(i) *invariance: for $\mathbb{P}-$a.e. $\omega \in \Omega$ and all $t \geq 0$ it holds*

$$\varphi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega);$$

(ii) *attracting property: for any $K \in \mathcal{R}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,*

$$\lim_{t \to \infty} \text{dist}(\varphi(t, \theta_{-t}\omega, K(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0, \tag{4.1.1}$$

where

$$dist(G, H) = \sup_{g \in G} \inf_{h \in H} \|g - h\| \tag{4.1.2}$$

is the Hausdorff semi-metric for $G, H \subseteq X$.

If $\mathcal{R} = \mathcal{T}(X)$ we say in this conditions that $\mathcal{A}$ is a global random attractor.

REMARK 4.3. *Notice that a global random attractor is also a pullback $\mathcal{D}(X)$ attractor.*

REMARK 4.4. *Unless stated otherwise, a random attractor will be understood always as a pullback attractor. There are other notions of attraction in this context, such as forward attraction, weak attraction and attraction in probability but it is not our purpose to investigate those behaviours. Notice also that the notion of pullback attractor does not depend on the choice of the metric dist, which is not the case when we consider a forward attractor $\mathcal{A}$ which instead condition (4.1.1) satisfies*

$$\lim_{t \to \infty} \text{dist}(\varphi(t, \omega, K(\omega)), \mathcal{A}(\theta_t\omega)) = 0;$$

*see [**26**, Section 5].*

PROPOSITION 4.1. [**15**, Proposition 1] *Consider a RDS (θ, φ) on X and an arbitrary family $\mathcal{R}$ of random sets containing $\mathcal{D}(X)$. If there is a compact absorbing set $\Gamma \in \mathcal{R}$ then*

there is a unique global random attractor $\mathcal{A}$ with component subsets

$$\mathcal{A}(\omega) = \bigcap_{\tau \geq T_\Gamma(\omega)} \overline{\bigcup_{t \geq \tau} \varphi(t, \theta_{-t}\omega, \Gamma(\theta_{-t}\omega))}. \tag{4.1.3}$$

If the pullback absorbing set is positively invariant, i.e., $\varphi(t, \omega, \Gamma(\omega)) \subset \Gamma(\theta_t\omega)$ for all $t \geq 0$, then

$$\mathcal{A}(\omega) = \bigcup_{t > T_\Gamma(\omega)} \overline{\varphi(t, \theta_{-t}\omega, \Gamma(\theta_{-t}\omega))}. \tag{4.1.4}$$

REMARK 4.5. *The original statement requires an asymptotic compactness property which is trivially satisfied in our context since $X = R^d$. Note also that this path-wise attracting in the pullback sense does not need to be path-wise attracting in the forward sense, although it is forward attracting in probability: for any $\varepsilon > 0$,*

$$\mathbb{P}(\{\omega \in \Omega \colon \mathrm{dist}(\varphi(t, \theta_{-t}\omega, K(\theta_{-t}\omega)), \mathcal{A}(\omega)) \geq \varepsilon\})$$

$$= \mathbb{P}(\{\omega \in \Omega \colon \mathrm{dist}(\varphi(t, \omega, K(\omega)), \mathcal{A}(\theta_t\omega)) \geq \varepsilon\})$$

which goes to 0 as $t \to \infty$. That is,

$$\lim_{t \to \infty} d\big(\varphi(t, \omega)K(\omega), A(\theta_t\omega)\big) = 0 \quad \text{in probability}$$

for every $K \in \mathcal{R}$. In particular, this allows individual realizations along sample paths to have large deviations from the attractor, but still to converge in this probabilistic sense.

REMARK 4.6. *The attractor need not be unique for a general family $\mathcal{R}$. However, as soon as $\mathcal{R}$ contains every compact deterministic set, if a random attractor for $\mathcal{R}$ exists then it is unique (cf. [26]). Notice this is the case if $\mathcal{R} \in \{\mathcal{D}(X), \mathcal{T}(X)\}$.*

4.2. Random eco-epidemiological model with real noise

In this section we consider random perturbations of a particular case of model (0.0.2), by introducing a random noise in a parameter, meanwhile the remaining parameters are assumed to be all positive constants.

We consider the eco-epidemiological model (0.0.2), with $G(t, S) = \Lambda(t) - \mu S(t)$, $H(t, P) = -\delta_1 P(t) - \delta_2 P(t)^2$, $a(t) = 1$, with constant parameter functions $\mu, \beta, \eta, c, \gamma$ and θ (that we rewrite in this chapter as r, once θ is used for the auxiliary metric dynamical system), and with a random birth rate Λ of the prey population modelled by a

random variable, as follows:

$$\begin{cases} S'(t,\omega) = \Lambda(\theta_t\omega) - \mu S(t) - f(S(t), I(t), P(t))P(t) - \beta S(t)I(t) \\ I'(t,\omega) = \beta S(t)I(t) - \eta g(S(t), I(t), P(t))I(t) - cI(t) \\ P'(t,\omega) = \gamma f(S(t), I(t), P(t))P(t) + r\eta g(S(t), I(t), P(t))I(t) - \delta_1 P(t) - \delta_2 P(t)^2 \end{cases}$$
(4.2.1)

where $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is a metric dynamical system that drives the noise and:

H1) $\mu, \beta, \eta, c, \gamma, r, \delta_1$ and δ_2 are all positive constants, and we assume that $\mu < c$;

H2) functions $f, g : (\mathbb{R}_0^+)^3 \to \mathbb{R}_0^+$ are locally Lipschitz and satisfy

   a) $S \to f(S, I, P)$ and $P \mapsto g(S, I, P)$ are nondecreasing,

   b) $I \to f(S, I; P)$, $P \to f(S, I, P)$, $S \mapsto g(S, I, P)$ and $I \mapsto g(S, I; P)$ are nonincreasing,

   c) $f(0, I, P) = 0$ and $g(S, I, 0) = 0$;

   d) $f(S, 0, 0) > 0$ whenever $S > 0$;

H3) $\Lambda : \Omega \to \mathbb{R}^+$ is a measurable function such that

$$\Lambda(\omega) \in [\Lambda^\ell, \Lambda^u] := q_0[1 - \varepsilon; 1 + \varepsilon],$$
(4.2.2)

with $q_0 > 0, \varepsilon \in (0, 1)$, for all $\omega \in \Omega$, and such that the function $t \mapsto \Lambda(\theta_t\omega)$ is continuous.

Typically, the functional response of the predator to prey is given by some particular function. Besides the population compartments, given by S, I and P that correspond, respectively, to the susceptible prey, infected prey and predator, we may understood Λ and μ as the (random) recruitment rate and the natural death rate of prey population, respectively, β as the incidence rate of the disease, η as the predation rate of infected prey, c as the death rate in the infective class, γ as the rate converting susceptible prey into predator (biomass transfer), r as the rate of converting infected prey into predator, $f(S, I, P)$ is the predation of susceptible prey and $g(S, I, P)$ is the predation of infected prey. It is assumed that only susceptible preys S are capable of reproducing, i.e, the infected prey is removed by death (including natural and disease-related death) or by predation before having the possibility of reproducing.

Like in Chapter 1, the present setting includes several of the most common functional responses for both functions f and g, including Holling-type I, Holling-type II, Holling-type III, Beddington-De Angelis and Crowley-Martin. Also note that conditions in H2)

are natural from a biological perspective and they are satisfied by the usual functional responses considered in the literature.

REMARK 4.7. *Although it can be possible to consider a more generalised model in which some others coefficients can also be random, in this particular case we consider just one to highlight the technique and specificities of this model. For examples of bounded* real noise *as considered in this model (in particular, as in 3) see for instance* [**4, 13**].

4.3. Existence and properties of solutions

In this section we prove the existence, uniqueness and boundedness of solutions to (4.2.1) with nonnegative initial conditions on the populations. Moreover, we prove that the solution mapping gives rise to a RDS. We start by showing that nonnegative initial conditions for the populations remains nonnegative, avoiding meaningless solutions in biological contexts.

LEMMA 4.1. *The set*

$$\mathbb{R}_+^3 = \{(S, I, P) \in \mathbb{R}^3 : S \geq 0, I \geq 0, P \geq 0\} \tag{4.3.1}$$

is positively invariant for the system (4.2.1) *for each fixed* $\omega \in \Omega$.

PROOF. The planes $I = 0$ and $P = 0$ are invariant since on it we have $I'(t, \omega) = 0$ and $P'(t, \omega) = 0$, respectively, and $S'(t, \omega) > 0$ on the plane $S = 0$.

If we start on the positive P-semi axes we have $S'(t, \omega) > 0$ and $I'(t, \omega) = 0$, so that the solution remains on $\mathbb{R}_+^3 \cap \{I = 0\}$, while if we start on the positive I-semi axes we have $S'(t, \omega) > 0$ and $P'(t, \omega) = 0$, so that solution does not leave $\mathbb{R}_+^3$.

Finally, we claim that the positive S-semi axes is invariant. Indeed on this semi axe we have $I'(t, \omega) = P'(t, \omega) = 0$ and

$$S'(t, \omega) = \Lambda(\theta^t \omega) - \mu(t)S(t). \tag{4.3.2}$$

That is, writing $S_0 = S(t_0, \omega)$ for the initial condition of population S on time t_0, we have for the corresponding solution

$$S(t; t_0, \omega, S_0) = S_0 e^{-\mu(t-t_0)} + e^{-\mu t} \int_{t_0}^t \Lambda(\theta_s \omega) e^{\mu s} ds.$$

Since $\Lambda^\ell \leq \Lambda(\theta_t \omega) \leq \Lambda^u$, the last term is bounded:

$$\Lambda^\ell e^{-\mu t} \int_{t_0}^t e^{\mu s} ds \leq e^{-\mu t} \int_{t_0}^t \Lambda(\theta_s \omega) e^{\mu s} ds \leq \Lambda^u e^{-\mu t} \int_{t_0}^t e^{\mu s} ds,$$

hence

$$\Lambda^\ell(1 - e^{-\mu(t-t_0)}) \le \mu e^{-\mu t} \int_{t_0}^t \Lambda(\theta_s\omega)e^{\mu s}ds \le \Lambda^u(1 - e^{-\mu(t-t_0)}).$$

In particular, for nonnegative initial condition S_0 the population $S(t; t_0, \omega, S_0)$ remains nonnegative. We conclude that the vector field at the boundary of $\mathbb{R}_+^3$ never points outwards.

$\square$

Set $a^u = \max\{\gamma, r\}$, $a^\ell = \min\{\gamma, r\}$. To simplify the notation, unless stated otherwise, given $(S_0, I_0, P_0) \in \mathbb{R}_+^3$ we write $S = S(t; t_0, \omega, S_0)$, $I = I(t; t_0, \omega, I_0)$ and $P = P(t; t_0, \omega, P_0)$ to be the components of the solution $u(t; t_0, \omega, u_0)$ of system (4.2.1) with initial state $u_0 = (S_0, I_0, P_0) \in \mathbb{R}_+^3$ at time $t = t_0$ and fixed $\omega \in \Omega$. Moreover, define $M_0 = a^\ell S_0 + r I_0 + P_0$, $N_0 = a^u S_0 + r I_0 + P_0$, $M = M(t; t_0, \omega, M_0) = a^\ell S + rI + P$ and $N = N(t; t_0, \omega, N_0) = a^u S + rI + P$. This notation will also be used for the particular situation $t_0 = 0$, which should become clear from the context. In this case we should write $h = h(t; 0, \omega, h_0) = h(t; \omega, h_0)$, for $h = S, I, P, M$ and N (and also for $h = V$ and $h = W$ to be defined later).

In the following we provide thresholds for forward invariant subsets of $\mathbb{R}_+^3$. Set

$$\Theta^u = \frac{a^u \Lambda^u}{\min\{\mu, \delta_1\}},$$

and, for $\delta \ge 0$,

$$\Theta_\delta^\ell = \max\left\{0, \frac{a^\ell \Lambda^\ell - \delta_2(\Theta^u + \delta)^2}{\max\{c, \delta_1\}}\right\},$$

and set $\Theta^\ell = \Theta_0^\ell$.

PROPOSITION 4.2. *For each $\delta \ge 0$ the region*

$$\mathcal{K}_\delta = \left\{(S_0, I_0, P_0) \in \mathbb{R}_+^3 : \Theta_\delta^\ell \le M_0 \le N_0 \le \Theta^u + \delta\right\} \tag{4.3.3}$$

is positively invariant for the system (4.2.1) for each $\omega \in \Omega$.

PROOF. Let $\delta > 0$ and $\omega \in \Omega$ be fixed. For $t_0 \ge 0$, we have

$$
\begin{aligned}
N' &= a^u \Lambda(\theta_t\omega) - a^u \mu S - a^u f(S, I, P)P - a^u \beta SI \\
&\quad + r\beta SI - r\eta g(S, I, P)I - rcI \\
&\quad + \gamma f(S, I, P)P + r\eta g(S, I, P)I - \delta_1 P - \delta_2 P^2 \\
&\le a^u \Lambda^u - a^u \mu S - rcI - \delta_1 P + (\gamma - a^u)f(S, I, P)P \\
&\quad + \beta(r - a^u)SI.
\end{aligned}
\tag{4.3.4}
$$

Since $\gamma - a^u \leq 0$ and $r - a^u \leq 0$, we have

$$N' \leq a^u \Lambda^u - \min\{\mu, \delta_1\} N. \tag{4.3.5}$$

This implies

$$N \leq \Theta^u + (N_0 - \Theta^u) e^{-\min\{\mu, \delta_1\}(t - t_0)}. \tag{4.3.6}$$

Similarly,

$$\begin{aligned} M' &\geq a^\ell \Lambda^\ell - \max\{c, \delta_1\} M \\ &+ (\gamma - a^\ell) f(S, I, P) P + \beta(r - a^\ell) S I \\ &- \delta_2 P^2. \end{aligned} \tag{4.3.7}$$

Recall that $\gamma - a^\ell \geq 0$ and $r - a^\ell \geq 0$, and if $N_0 \leq \Theta^u + \delta$, from (4.3.6) we have $N \leq \Theta^u + \delta$ for all $t \geq t_0$ and

$$M' \geq a^\ell \Lambda^\ell - \delta_2(\Theta^u + \delta)^2 - \max\{c, \delta_1\} M, \tag{4.3.8}$$

which implies, in this situation,

$$M \geq \Theta_\delta^\ell + \left(M_0 - \Theta_\delta^\ell \right) e^{-\max\{c, \delta_1\}(t - t_0)}. \tag{4.3.9}$$

Thus if $u_0 = (S_0, I_0, P_0) \in \mathcal{K}_\delta$ then $u(t; t_0, \omega, u_0) \in \mathcal{K}_\delta$ for all $t \geqslant t_0$. $\qquad \square$

COROLLARY 15. *For all $\omega \in \Omega$, $t_0 \in \mathbb{R}_0^+$ and $(S_0, I_0, P_0) \in \mathbb{R}_+^3$ we have*

$$\lim_{t \to \infty} N(t; t_0, \omega, N_0) \in [\Theta^\ell, \Theta^u] \quad and \quad \lim_{t \to \infty} M(t; t_0, \omega, M_0) \in [\Theta^\ell, \Theta^u].$$

PROOF. From (4.3.6) we have

$$\begin{aligned} \lim_{t \to \infty} M(t; t_0, \omega, M_0) &\leq \lim_{t \to \infty} N(t; t_0, \omega, N_0) \\ &\leq \lim_{t \to \infty} \Theta^u + (N_0 - \Theta^u) e^{-\min\{\mu, \delta_1\}(t - t_0)} \\ &= \Theta^u. \end{aligned}$$

In particular, for any $\delta > 0$, if t is sufficiently large we have $N < \Theta^u + \delta$, and from (4.3.9) follows that

$$
\begin{aligned}
\lim_{t \to \infty} N(t; t_0, \omega, N_0) &\geq \lim_{t \to \infty} M(t; t_0, \omega, M_0) \\
&\geq \lim_{t \to \infty} \Theta_\delta^\ell + (M_0 - \Theta_\delta^\ell) e^{-\max\{c, \delta_1\}(t - t_0)} \\
&= \Theta_\delta^\ell.
\end{aligned}
$$

$\square$

THEOREM 4.1. *For any $\omega \in \Omega$, $t_0 \in \mathbb{R}_0^+$ and any initial condition $u_0 = (S_0, I_0, P_0) \in \mathbb{R}_+^3$ the system (4.2.1) admits a unique bounded solution $u(\cdot) = u(\cdot; t_0, \omega, u_0) \in \mathcal{C}([t_0, +\infty), \mathbb{R}_+^3)$, with $u(t_0; t_0, \omega, u_0) = u_0$. Moreover, the solution generates a RDS (θ, φ) defined as*

$$
\varphi(t, \omega, u_0) = u(t; 0, \omega, u_0), \quad \text{for all } t \geq 0, u_0 \in \mathbb{R}_+^3 \text{ and } \omega \in \Omega. \tag{4.3.10}
$$

PROOF. The system (4.2.1) can be rewritten in the following form

$$
u'(t) = F(\theta_t \omega, u). \tag{4.3.11}
$$

Since $t \mapsto \Lambda(\theta_t \omega)$ is continuous, the map $F_\omega(t, u) = F(\theta_t \omega, u) \in \mathcal{C}([t_0, +\infty) \times \mathbb{R}_+^3, \mathbb{R}_+^3)$ is locally Lipschitz respect to u. From Corollary 15, all the solutions $u(t)$ are bounded. Thus for each $\omega \in \Omega$ the system (4.2.1) possesses a unique global solution $u(t; t_0, \omega, u_0)$ with initial condition $u(t_0) = u_0$.

Since $F(\theta_t \omega, u) = F(t, \omega, u)$ is also measurable in ω, the map

$$
u(\cdot; t_0, \cdot, \cdot) : [t_0, \infty) \times \Omega \times \mathbb{R}_+^3 \to \mathbb{R}_+^3
$$

is $(\mathcal{B}([t_0, \infty)) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+^3), \mathcal{B}(\mathbb{R}_+^3))$-measurable. From [**1**, Theorem 2.2.2], the solutions of system (4.2.1) generate a RDS via (4.3.10). We remark in particular the cocycle property of (θ, φ) that can be obtained through

$$
u(t + t_0; t_0, \omega, u_0) = u(t; 0, \theta_t \omega, u_0),
$$

for all $t \geq t_0 \geq 0, \omega \in \Omega$ and $u_0 \in \mathbb{R}_+^3$. $\square$

4.4. Global random attractor

In this section we prove the existence of a global random attractor, analyze the vital dynamics of susceptible preys and discuss thresholds for extinction of infectives of (4.2.1).

4.4.1. Global random attractor. We establish now the existence of a pullback $\mathcal{T}(\mathbb{R}_+^3)$ attractor for system (4.2.1).

THEOREM 4.2. *The RDS (θ, φ) generated by (4.2.1) possesses a unique global random attractor.*

The proof follows straightforward from Proposition 4.1 and from the existence of a compact random absorbing set $\Gamma \in \mathcal{T}(\mathbb{R}^3_+)$ given by Proposition 4.3 below.

PROPOSITION 4.3. *There exists a compact random absorbing set $\Gamma \in \mathcal{T}(\mathbb{R}^3_+)$ of the RDS (θ, φ) generated by (4.2.1). Moreover, for any $\delta > 0$ the sets $\Gamma(\omega)$ can be chosen as the deterministic $\mathcal{K}_\delta$ for any $\omega \in \Omega$.*

PROOF. Consider $A \in \mathcal{T}(\mathbb{R}^3_+)$ and $\delta > 0$. We want to prove that for each $\omega \in \Omega$ there exists $T_A(\omega) > 0$ such that for all $t \geq T_A(\omega)$

$$\varphi(t, \theta_{-t}\omega, A(\theta_{-t}\omega)) \subseteq \mathcal{K}_\delta.$$

From Proposition 4.2 the set $\mathcal{K}_\delta$ is positively invariant, which means that for all $t \geq 0$

$$\varphi(t, \omega, K_\delta) \subseteq K_\delta.$$

To simplify, we write $N(t; 0, \omega, N_0) = N(t; \omega, N_0)$ and $M(t; 0, \omega, N_0) = M(t; \omega, N_0)$. Recall that given $u_0 = (S_0, I_0, P_0)$ we write $M_0 = a^\ell S_0 + r I_0 + P_0$ and $N_0 = a^u S_0 + r I_0 + P_0$. From (4.3.6) we have

$$N(t; \theta_{-t}\omega, N_0) \leq \Theta^u + \sup_{u_0 \in A(\theta_{-t}\omega)} (N_0 - \Theta^u) e^{-\min\{\mu, \delta_1\}t}.$$

Since A is tempered,

$$\lim_{t \to \infty} \sup_{u_0 \in A(\theta_{-t}\omega)} (N_0 - \Theta^u) e^{-\min\{\mu, \delta_1\}t} = 0 \tag{4.4.1}$$

and thus

$$\lim_{t \to \infty} N(t; \theta_{-t}\omega, N_0) \leq \Theta^u. \tag{4.4.2}$$

Assume that $\Theta^\ell_\delta > 0$, otherwise the result follows from Lemma 4.1. From (4.4.2), for any $0 < \delta' < \delta$ and t sufficiently large we have $N(t; \theta_{-t}\omega, N_0) < \Theta^u + \delta'$ and from (4.3.9), in this situation we get

$$M(t; \theta_{-t}\omega, M_0) \geq \Theta^\ell_{\delta'} + \inf_{u_0 \in A(\theta_{-t}\omega)} \left(M_0 - \Theta^\ell_{\delta'} \right) e^{-\max\{c, \delta_1\}t}.$$

Since

$$\lim_{t \to \infty} \inf_{u_0 \in A(\theta_{-t}\omega)} \left(M_0 - \Theta^\ell_{\delta'} \right) e^{-\max\{c, \delta_1\}t} = 0 \tag{4.4.3}$$

we have

$$\lim_{t\to\infty} M(t;\theta_{-t}\omega, M_0) \geq \Theta_{\delta'}^{\ell} > \Theta_{\delta}^{\ell}. \tag{4.4.4}$$

Henceforth there is $T_A(\omega)$ such that for all $t \geq T_A(\omega)$ we have for all $u_0 \in A(\theta_{-t}\omega)$ that

$$\Theta_{\delta}^{\ell} \leq M(t;\theta_{-t}\omega, M_0) \leq N(t;\theta_{-t}\omega, N_0) \leq \Theta^u + \delta$$

and the conclusion holds. $\qquad\square$

From Remark 4.6, the global random attractor is unique. From Remark 4.5, (θ, φ) possesses a forward attractor in probability and from Remark 4.3, it also possesses global random $\mathcal{D}(\mathbb{R}_+^3)$ attractor.

4.4.2. Susceptible dynamics.

4.4.2.1. *Random attractor for susceptible vital dynamics.* If we have no predators neither infected preys, from (4.2.1) the dynamics of susceptible preys is given by

$$S'(t,\omega) = \Lambda(\theta_t\omega) - \mu S(t). \tag{4.4.5}$$

For each $\omega \in \Omega$, the solution of (4.4.5) with initial condition $S_0 \geq 0$ at $t = t_0$ is

$$S(t;t_0,\omega,S_0) = S_0 e^{-\mu t} + \int_{t_0}^{t} \Lambda(\theta_s\omega)e^{-\mu(t-s)}ds.$$

Replacing ω by $\theta_{-t}\omega$, and taking $t_0 = 0$ we have, denoting $S(t;0,\omega,S_0)$ by $S(t;\omega,S_0)$,

$$S(t;\theta_{-t}\omega,S_0) = S_0 e^{-\mu t} + \int_{-t}^{0} \Lambda(\theta_s\omega)e^{-\mu s}ds.$$

For any $K \in \mathcal{T}([0,+\infty[)$ we have

$$\lim_{t\to\infty} \sup_{S_0 \in K(\theta_{-t}\omega)} S_0 e^{-\mu t} = 0$$

so that we may define

$$\lim_{t\to\infty} S(t;\theta_{-t}\omega,S_0) = \int_{-\infty}^{0} \Lambda(\theta_s\omega)e^{-\mu s}ds := S^*(\omega) \tag{4.4.6}$$

The equation (4.4.5) generates a RDS (θ, φ_S), with $\varphi_S(t,\omega,S_0) = S(t;\omega,S_0)$, which possesses a singleton global random attractor $\mathcal{A}(\omega) = S^*(\omega)$. Moreover, it follows from (4.2.2) and (4.4.6) that $S^*(\omega) \in \left[\frac{\Lambda^{\ell}}{\mu}, \frac{\Lambda^u}{\mu}\right]$.

4.4.2.2. *Persistence of susceptible preys.* We give conditions to ensure the prevalence of susceptible preys. We do not discuss conditions for prevalence of infected preys neither

predators. To simplify the following computations, for a given $\delta > 0$ we set

$$\xi_\delta = \Lambda^\ell - f\left(\frac{\Theta^u + \delta}{a^u}, 0, 0\right)(\Theta^u + \delta) \quad \text{and} \quad \zeta_\delta = \mu + \beta\left(\frac{\Theta^u + \delta}{r}\right).$$

PROPOSITION 4.4. *The global random attractor $\mathcal{A}$ for the RDS generated by (4.2.1) possesses nontrivial components on the ω-sections: $\mathcal{A}(\omega) = (A_S(\omega), A_I(\omega), A_P(\omega))$ with $A_S(\omega) \geq \xi_\delta/\zeta_\delta$, for all $\omega \in \Omega$. In particular, susceptible preys are prevalent if $\xi_\delta > 0$ for some $\delta > 0$.*

PROOF. To simplify, in the following we write

$$h = h(t; 0, \theta_{-t}\omega, h_0) = h(t; \theta_{-t}\omega, h_0),$$

for $h = S, I$ and P. From Proposition 4.3, for any $\delta > 0$ and any $K \in \mathcal{T}(\mathbb{R}_+^3)$ there exists $T'_K(\omega)$ such that, for $t \geq T'_K(\omega)$ and $(S_0, I_0, P_0) \in K(\theta_{-t}\omega)$ we have

$$a^u S + rI + P \leq \Theta^u + \delta.$$

From (4.2.1) we therefore have

$$S' \geq \Lambda^\ell - \mu S - f\left(\frac{\Theta^u + \delta}{a^u}, 0, 0\right)(\Theta^u + \delta) - \beta\left(\frac{\Theta^u + \delta}{r}\right)S$$

$$= \xi_\delta - \zeta_\delta S.$$

Thus, for all $t \geq T'_K(\omega)$

$$S \geq \frac{\xi_\delta}{\zeta_\delta} + \left(S(T'_K(\omega); \theta_{-t}\omega, S_0) - \frac{\xi_\delta}{\zeta_\delta}\right)e^{-\zeta_\delta(t - T'_K(\omega))}.$$

Hence for any $\delta > 0$ and $K \in \mathcal{T}(\mathbb{R}_+^3)$ and large t, we have for all $(S_0, I_0, P_0) \in K(\theta_{-t}\omega)$

$$S = S(t; \theta_{-t}\omega, S_0) \geq \frac{\xi_\delta}{\zeta_\delta}. \tag{4.4.7}$$

$\square$

4.4.3. Extinction of predators and infected preys. We discuss now conditions that lead to the vanish of infectious and predators.

PROPOSITION 4.5. *The global random attractor $\mathcal{A}$ for the RDS generated by (4.2.1) has singleton components $\mathcal{A}(\omega) = (S^*(\omega), 0, 0)$ for every $\omega \in \Omega$, provided that*

$$\frac{\beta\Theta^u}{a^u} < c \quad \text{and} \quad \gamma f\left(\frac{\Theta^u}{a^u}, 0, 0\right) < \delta_1.$$

PROOF. The last two equations in system (4.2.1) yields to

$$(rI + P)' = r\beta SI - rcI + \gamma f(S, I, P)P - \delta_1 P - \delta_2 P^2$$
$$= (\beta S - c)rI + (\gamma f(S, I, P) - \delta_1 - \delta_2 P)P. \tag{4.4.8}$$

We will see that our hypothesis imply that both factors

$$\beta S - c \quad \text{and} \quad \gamma f(S, I, P) - \delta_1 - \delta_2 P$$

are negative for large t. If $\frac{\beta \Theta^u}{a^u} < c$ we can choose a $\delta' > 0$ small enough such that taking $\delta = \beta \delta'/a^u$ we have

$$\frac{\beta \Theta^u}{a^u} + \delta < c.$$

From Proposition 4.3 we have that $\mathcal{K}_{\delta'} \times \Omega$ is an absorbing set in $\mathcal{T}(\mathbb{R}^3_+)$, so that for any $K \in \mathcal{T}(\mathbb{R}^3_+)$ and $\omega \in \Omega$ there exists $T'_K(\omega)$ such that for $t \geq T'_K(\omega)$ and $(S_0, I_0, P_0) \in K(\theta_{-t}\omega)$ we have

$$\beta S = \beta S(t; \theta_{-t}\omega, S_0) \leq \frac{\beta \Theta^u}{a^u} + \delta < c,$$

which implies that

$$\beta S - c < 0, \quad \text{for all } t \geqslant T'_K(\omega). \tag{4.4.9}$$

Now, if $\gamma f\left(\frac{\Theta^u}{a^u}, 0, 0\right) < \delta_1$, since f is continuous, by taking $\delta' > 0$ even smaller, if necessary, we also have that

$$\gamma f\left(\frac{\Theta^u}{a^u} + \delta, 0, 0\right) < \delta_1.$$

Again, since $\mathcal{K}_\delta \times \Omega$ is also an absorbing set in $\mathcal{T}(\mathbb{R}^3_+)$, for any $K \in \mathcal{T}(\mathbb{R}^3_+)$ and $\omega \in \Omega$ there exists $T_K(\omega) \geq T'_K(\omega)$ such that, for $t \geq T_K(\omega)$ and $(S_0, I_0, P_0) \in K(\theta_{-t}\omega)$ we have $S \leq \frac{\Theta^u}{a^u} + \delta$ and, setting $P = P(t; \theta_{-t}\omega, P_0)$, by Lemma 4.1 we have $I, P \geq 0$. By the monotonicity of f,

$$\gamma f(S, I, P) \leq \gamma f\left(\frac{\Theta^u}{a^u} + \delta, 0, 0\right) \leq \delta_1,$$

which implies

$$\gamma f(S, I, P) - \delta_1 - \delta_2 P < 0 \tag{4.4.10}$$

for all $t \geqslant T_K(\omega)$. Setting $I = I(t; \theta_{-t}\omega, I_0)$, from (4.4.8), (4.4.9) and (4.4.10) we have for $t \geq T_K(\omega)$

$$(rI + P)' \leq \max\left\{\frac{\beta \Theta^u}{a^u} - c + \delta, \gamma f\left(\frac{\Theta^u}{a^u} + \delta, 0, 0\right) - \delta_1\right\}(rI + P).$$

This implies that for all $K \in \mathcal{T}(\mathbb{R}^3_+)$,

$$\lim_{t \to +\infty} (rI + P) = \lim_{t \to +\infty} (rI(t; \theta_{-t}\omega, I_0) + P(t; \theta_{-t}\omega, P_0)) = 0,$$

for all $(S_0, I_0, P_0) \in K(\theta_{-t}\omega)$. Moreover, from (4.4.6) if $I, P = 0$ we have

$$\lim_{t \to +\infty} S(t; \theta_{-t}\omega, S_0) = S^*(\omega).$$

Thus the global random attractor $\mathcal{A}$ for the RDS generated by (4.2.1) has singleton components sets $\mathcal{A}(\omega) = \{(S^*(\omega), 0, 0)\}$ for every $\omega \in \Omega$. $\square$

EXAMPLE 4.1. *To illustrate this result in a model we consider Holling-type I functional responses $f(S, I, P) = S$ and $g(S, I, P) = P$. Our model is in this specific case is*

$$\begin{cases} S'(t, \omega) = \Lambda(\theta_t \omega) - \mu S(t) - S(t)P(t) - \beta S(t)I(t) \\ I'(t, \omega) = \beta S(t)I(t) - \eta I(t)P(t) - cI(t) \\ P'(t, \omega) = \gamma S(t)P(t) + r\eta I(t)P(t) - \delta_1 P(t) - \delta_2 P(t)^2 \end{cases}.$$

Thus we have simultaneous extinction of infected preys and predators, in the sense that the global random attractor has ω-sections of type $(S^*(\omega), 0, 0)$, if

$$\frac{\beta \Lambda^u}{c \min\{\mu, \delta_1\}} < 1 \quad and \quad \frac{\gamma \Lambda^u}{\delta_1 \min\{\mu, \delta_1\}} < 1.$$

This can also be interpreted in the deterministic setting by considering $\Lambda(\omega) = \Lambda_0$ for all ω and some $\Lambda_0 > 0$.

4.5. Random attractors for partial dynamics

We discuss now the dynamics of the corresponding random epidemiological SI and predator-prey subsystems of model (4.2.1).

In section 4.4.2.1 we analysed the vital dynamics of susceptible population, in the simultaneous absence of disease and predators, for which we concluded the existence of a singleton random global attractor with sections $(S^*(\omega), 0, 0)$. We discuss now the existence of random global attractors in other subsystems, namely either in the absence of predators or infectious, respectively. Set $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$.

4.5.0.1. *The case without predator.* Let us now consider the system (4.2.1) when we do not have predators, by making $P = 0$. This case reduces to

$$\begin{cases} S'(t, \omega) = \Lambda(\theta_t \omega) - \mu S(t) - \beta S(t)I(t) \\ I'(t, \omega) = \beta S(t)I(t) - cI(t) \end{cases}. \tag{4.5.1}$$

In this situation we have a (random) epidemiological SI model. This SI model is a slightly different model from de SI model corresponding to the first two equations (SI) of the SIR

models considered in [**14, 15**]. In our work, we obtain global random attractor, prove the persistence of susceptibles and provide conditions for the extinctions of infectives.

Let us consider now $v(t; t_0, \omega, v_0) = (S(t; t_0, \omega, S_0), I(t; t_0, \omega, I_0))$ as a solution of the system (4.5.1) with initial conditions $S(t_0, \omega) = S_0$ and $I(t_0, \omega) = I_0$, and $v_0 = (S_0, I_0)$. Let us define $V = S + I$ and $V_0 = S_0 + I_0$. Similarly to Lemma 4.1, we easily conclude that the region $\mathbb{R}^2_+$ is positively invariant for system (4.5.1). In the following we provide thresholds for forward invariant subsets of $\mathbb{R}^2_+$.

PROPOSITION 4.6. *For each $0 < \delta \leq \Lambda^\ell / c$ the region*

$$\mathcal{V}_\delta = \left\{ (S_0, I_0) \in \mathbb{R}^2_+ : \frac{\Lambda^\ell}{c} - \delta \leq V_0 \leq \frac{\Lambda^u}{\mu} + \delta \right\}$$

is positively invariant for the system (4.5.1).

PROOF. Recall that we assume $\mu < c$. Adding the two equations in (4.5.1) we have

$$\begin{aligned} V'(t, \omega) &= \Lambda(\theta_t \omega) - \mu S - cI \\ &\leq \Lambda^u - \mu V \end{aligned} \tag{4.5.2}$$

and

$$\begin{aligned} V'(t, \omega) &= \Lambda(\theta_t \omega) - \mu S - cI \\ &\geq \Lambda^\ell - cV. \end{aligned} \tag{4.5.3}$$

Writing $V = V(t; 0, \theta_{-t}\omega, V_0) = V(t; \theta_{-t}\omega, V_0)$, this implies

$$\frac{\Lambda^\ell}{c} + \left(V_0 - \frac{\Lambda^\ell}{c} \right) e^{-ct} \leq V \leq \frac{\Lambda^u}{\mu} + \left(V_0 - \frac{\Lambda^u}{\mu} \right) e^{-\mu t}. \tag{4.5.4}$$

If $(S_0, I_0) \in \mathcal{V}_\delta$, the solution remains in this region.

$\square$

We notice that from (4.5.2) and (4.5.3) we have

$$\lim_{t \to \infty} V \leq \lim_{t \to \infty} \frac{\Lambda^u}{\mu} + (V_0 - \frac{\Lambda^u}{\mu}) e^{-\mu t} = \frac{\Lambda^u}{\mu}$$

and

$$\lim_{t \to \infty} V \geq \lim_{t \to \infty} \frac{\Lambda^\ell}{c} + (U_0 - \frac{\Lambda^\ell}{c}) e^{-ct} = \frac{\Lambda^\ell}{c}$$

From the previous estimates we get easily that the solutions v are bounded. The following result follows straightforward as in the proof of Theorem (4.1).

THEOREM 4.3. *For any $\omega \in \Omega$, $t_0 \in \mathbb{R}_0^+$ and any initial condition $v_0 = (S_0, I_0) \in \mathbb{R}_+^2$ the system (4.5.1) admits a unique bounded solution $v(\cdot) = v(\cdot; t_0, \omega, v_0) \in \mathcal{C}([t_0, +\infty), \mathbb{R}_+^2)$, with $v(t_0; t_0, \omega, v_0) = v_0$. Moreover, the solution generates a RDS (θ, φ_{SI}) defined as*

$$\varphi_{SI}(t, \omega, u_0) = v(t; 0, \omega, u_0), \quad \text{for all } t \geq 0, v_0 \in \mathbb{R}_+^2 \text{ and } \omega \in \Omega. \tag{4.5.5}$$

In the following we establish the existence of a random global attractor for the partial dynamics with no predators.

THEOREM 4.4. *The RDS (θ, φ_{SI}) generated by (4.5.1) possesses a global random attractor $\mathcal{A}_{SI}$.*

We will prove that exists a closed random absorbing set $\Gamma \in \mathcal{T}(\mathbb{R}_+^2)$. The result follows then from Proposition 4.1.

PROPOSITION 4.7. *There exists a closed random absorbing set $\Gamma \in \mathcal{T}(\mathbb{R}_+^2)$ of the RDS (θ, φ_{SI}) generated by (4.5.1). Moreover, for $0 < \delta \leq \frac{\Lambda^\ell}{c}$, the sets $\Gamma(\omega)$ can be chosen as the deterministic $\mathcal{V}_\delta$ for any $\omega \in \Omega$.*

PROOF. Consider $A \in \mathcal{T}(\mathbb{R}_+^2)$ and $\delta > 0$. We want to prove that for each $\omega \in \Omega$ there exists $T_A(\omega) > 0$ such that

$$\varphi(t, \theta_{-t}\omega, A(\theta_{-t}\omega)) \subseteq \mathcal{V}_\delta \text{ for all } t \geq T_A(\omega).$$

From (4.5.4), we have

$$V(t; \theta_{-t}\omega, V_0) \leq \sup_{v_0 \in A(\theta_{-t}\omega)} \left(V_0 - \frac{\Lambda^u}{\mu} \right) e^{-\mu t} + \frac{\Lambda^u}{\mu},$$

and since A is tempered, we have

$$\lim_{t \to \infty} \sup_{v_0 \in A(\theta_{-t}\omega)} \left(V_0 - \frac{\Lambda^u}{\mu} \right) e^{-\mu t} = 0$$

and thus

$$\lim_{t \to \infty} V(t; \theta_{-t}\omega, V_0) \leq \frac{\Lambda^u}{\mu}. \tag{4.5.6}$$

Similarly,

$$\lim_{t \to \infty} V(t; \theta_{-t}\omega, V_0) \geq \frac{\Lambda^\ell}{c}. \tag{4.5.7}$$

Considering the inequalities (4.5.6) and (4.5.7), there exists a $T_A(\omega)$ such that for $t \geq T_A(\omega)$, we have $\varphi_{SI}(t, \theta_{-t}\omega, V_0) \in \mathcal{V}_\delta$ for all $V_0 \in A(\theta_{-t}\omega)$. If $\delta \leq \frac{\Lambda^\ell}{c}$ then $\mathcal{V}_\delta \subseteq \mathbb{R}_+^2$. □

PROPOSITION 4.8. *The global random attractor $\mathcal{A}_{SI}$ for the RDS (θ, φ_{SI}) generated by (4.5.1) possesses nontrivial component sets on the ω-sections: $\mathcal{A}_{SI}(\omega) = (A_S(\omega), A_I(\omega))$ with*

$$A_S(\omega) \geq \frac{\Lambda^\ell}{\mu + \frac{\beta \Lambda^u}{\mu}}.$$

PROOF. Since for any $\delta > 0$ the random set $\mathcal{V}_\delta \times \Omega$ is an absorbing set in $\mathcal{T}(\mathbb{R}_+^2)$, for any $K \in \mathcal{T}(\mathbb{R}_+^2)$ there exists $T_K(\omega)$ such that for $t \geq T_K(\omega)$ and $(S_0, I_0) \in K(\theta_{-t}\omega)$ we have

$$S + I = S(t; \theta_{-t}\omega, S_0) + I(t; \theta_{-t}\omega, I_0) \leq \frac{\Lambda^u}{\mu} + \delta.$$

From (4.5.1) we have

$$S' \geq \Lambda^\ell - \mu S - \beta \left(\frac{\Lambda^u}{\mu} + \delta \right) S$$

$$= \Lambda^\ell - \left(\mu + \beta \left(\frac{\Lambda^u}{\mu} + \delta \right) \right) S.$$

Hence for any $\delta > 0$ and $K \in \mathcal{T}(\mathbb{R}_+^2)$, $(S_0, I_0) \in K(\theta_{-t}\omega)$ and large t we have

$$S(t; \theta_{-t}\omega, S_0) \geq \frac{\Lambda^\ell}{\mu + \beta \left(\frac{\Lambda^u}{\mu} + \delta \right)}.$$

$\square$

In the following we give condition for an attractor without infectious component.

PROPOSITION 4.9. *The global random attractor $\mathcal{A}_{SI}$ for the RDS (θ, φ_{SI}) generated by (4.5.1) has singleton components $\mathcal{A}_{SI}(\omega) = (S^*(\omega), 0)$ for every $\omega \in \Omega$, provided that $\frac{\beta \Lambda^u}{\mu} < c$.*

PROOF. From the second equation in (4.5.1) we have that

$$I'(t, \omega) = (\beta S - c)I. \tag{4.5.8}$$

Consider $\delta > 0$ small enough such that

$$\frac{\beta \Lambda^u}{\mu} + \beta \delta < c.$$

From Proposition 4.7, for any $A \in \mathcal{T}(\mathbb{R}_+^2)$ there exists $T_A(\omega)$ such that for all $t \geq T_A(\omega)$, and $(S_0, I_0) \in A(\theta_{-t}\omega)$ we have

$$\beta S = \beta S(t; \theta_{-t}\omega, S_0) \leq \frac{\beta \Lambda^u}{\mu} + \beta \delta < c$$

which implies

$$\beta S - c \leq \frac{\beta \Lambda^u}{\mu} + \beta \delta - c < 0, \quad \text{for all } t \geq T_A(\omega).$$

From (4.5.8) we have

$$\lim_{t \to +\infty} I(t; \theta_{-t}\omega, I_0) \leq \lim_{t \to +\infty} I(T_A(\omega); \theta_{-t}\omega, I_0) e^{\left(\frac{\beta \Lambda^u}{\mu} + \beta \delta - c\right)(t - T_A(\omega))} = 0.$$

Moreover, if $I = 0$ we have from (4.4.6) that $S(t; \theta_{-t}\omega, S_0)$ converges to $S^*(\omega)$, as $t \to +\infty$, for each $\omega \in \Omega$. $\qquad \square$

The case without infectious. We consider now the case that we have no infected preys in system (4.2.1), by making $I = 0$, which becomes

$$\begin{cases} S'(t, \omega) = \Lambda(\theta_t \omega) - \mu S - \bar{f}(S, P)P \\ P'(t, \omega) = \gamma \bar{f}(S, P)P - \delta_1 P - \delta_2 P^2. \end{cases} \tag{4.5.9}$$

where $\bar{f}(S, P) = f(S, 0, P)$. This models corresponds to a random perturbation of a predator-prey model. We obtain global random attractor, prove the persistence of preys and provide conditions for the extinctions of predators.

Let $w(t; t_0, \omega, w_0)$ be the solution of system (4.5.9) with initial condition $w_0 = (S_0, P_0)$ and let $W = \gamma S + P$ and $W_0 = \gamma S_0 + P_0$. Define

$$\hat{\Theta}^u = \frac{\gamma \Lambda^u}{\min\{\mu, \delta_1\}}$$

and, for $\delta \geq 0$

$$\hat{\Theta}^\ell_\delta = \max \left\{ 0, \frac{\gamma \Lambda^\ell - \delta_2(\hat{\Theta}^u + \delta)^2}{\max\{\mu, \delta_1\}} \right\}.$$

PROPOSITION 4.10. *For each $\delta > 0$ the region*

$$\mathcal{W}_\delta = \left\{ (S_0, P_0) \in \mathbb{R}^2_+ : \hat{\Theta}_\delta \leq W_0 \leq \hat{\Theta}^u + \delta \right\}$$

is positively invariant for the system (4.5.9).

PROOF. From (4.5.9) we have

$$W'(t, \omega) = \gamma \Lambda(\theta_t \omega) - \gamma \mu S - \delta_1 P - \delta_2 P^2$$

$$\leq \gamma \Lambda^u - \min\{\mu, \delta_1\}W, \tag{4.5.10}$$

which, writing $W = W(t; 0, \theta_{-t}\omega, W_0) = W(t; \theta_{-t}\omega, W_0)$, implies

$$W \leq \hat{\Theta}^u + \left(W_0 - \hat{\Theta}^u \right) e^{-\min\{\mu, \delta_1\}t}. \tag{4.5.11}$$

We also have

$$W'(t, \omega) \geq \gamma \Lambda^\ell - \delta_2 P^2 - \max\{\mu, \delta_1\} W. \tag{4.5.12}$$

Notice that if the initial condition (S_0, P_0) belongs to $\mathcal{W}_\delta$ then $P \leq \hat{\Theta}^u + \delta$, and in this situation we have

$$W'(t, \omega) \geq \gamma \Lambda^\ell - \delta_2 (\hat{\Theta}^u + \delta)^2 - \max\{\mu, \delta_1\} W. \tag{4.5.13}$$

Hence, in this case we have

$$W \geq \hat{\Theta}_\delta^\ell + \left(W_0 - \hat{\Theta}_\delta^\ell\right) e^{-\max\{\mu, \delta_1\} t}. \tag{4.5.14}$$

If $(S_0, P_0) \in \mathcal{W}_\delta$, the solution remains in that region. $\qquad \square$

From (4.5.11) we have

$$\lim_{t \to \infty} W \leq \hat{\Theta}^u.$$

Notice that for any $\delta > 0$, for large t we have $P \leq W < \hat{\Theta}^u + \delta$, and then (4.5.14) implies

$$\lim_{t \to \infty} W \geq \hat{\Theta}_\delta^\ell.$$

This implies that the solutions w are bounded.

PROPOSITION 4.11. *For any $\omega \in \Omega$, $t_0 \in \mathbb{R}_0^+$ and any initial condition $w_0 = (S_0, P_0) \in (\mathbb{R}_0^+)^2$ the system (4.5.9) admits a unique bounded solution $w(\cdot) = w(\cdot; t_0, \omega, w_0) \in \mathcal{C}([t_0, +\infty), (\mathbb{R}_0^+)^2)$ with $w(t_0; t_0, \omega, w_0) = w_0$. Moreover, the solution generates a random dynamical system (θ, φ_{SP}) defined as*

$$\varphi_{SP}(t, \omega, v_0) = w(t; 0, \omega, w_0), \quad \textit{for all } t \geq 0, w_0 \in (\mathbb{R}_0^+)^2 \textit{ and } \omega \in \Omega.$$

THEOREM 4.5. *The RDS (θ, φ_{SP}) generated by (4.5.9) possesses a global random attractor $\mathcal{A}_{SP}$.*

As before, the proof follows from Proposition 4.1 and from the fact that there exists a closed random absorbing set $\Gamma \in \mathcal{T}(\mathbb{R}_+^2)$ given by Proposition 4.12 below.

PROPOSITION 4.12. *There exists a closed random absorbing set $\Gamma \in \mathcal{T}(\mathbb{R}_+^2)$ of the RDS (θ, φ_{SP}) generated by (4.5.9). Moreover, for any $\delta > 0$ the sets $\Gamma(\omega)$ can be chosen as the deterministic $\mathcal{W}_\delta$ for any $\omega \in \Omega$.*

PROOF. Consider $A \in \mathcal{T}(\mathbb{R}_+^2)$. We will prove that for any $\omega \in \Omega$ there exists $T_A(\omega) > 0$ such that

$$\varphi_{SP}(t, \theta_{-t}\omega, A(\theta_{-t}\omega)) \subseteq \mathcal{W}_\delta \text{ for all } t \geq T_A(\omega).$$

From Proposition 4.10 the region $\mathcal{W}_\delta$ is positively invariant for $\delta > 0$. From (4.5.11) we have for $w_0 = (S_0, P_0) \in A(\theta_{-t}\omega)$ that

$$W(t; \theta_{-t}\omega, W_0) \leq \hat{\Theta}^u + \sup_{w_0 \in A(\theta_{-t}\omega)} \left(W_0 - \hat{\Theta}^u\right) e^{-\min\{\mu, \delta_1\}t}.$$

Since $\mathcal{A}$ is tempered, we have

$$\lim_{t \to \infty} \sup_{w_0 \in A(\theta_{-t}\omega)} \left(W_0 - \hat{\Theta}^u\right) e^{-\min\{\mu, \delta_1\}t} = 0$$

and thus

$$\lim_{t \to \infty} W(t; \theta_{-t}\omega, W_0) \leq \Theta^u. \tag{4.5.15}$$

Similarly, for any $0 < \delta' < \delta$ and large t we have

$$W(t; \theta_{-t}\omega, W_0) \geq \hat{\Theta}_{\delta'}^\ell + \inf_{w_0 \in A(\theta_{-t}\omega)} \left(W_0 - \hat{\Theta}_{\delta'}^\ell\right) e^{-\max\{\mu, \delta_1\}t}.$$

Since

$$\lim_{t \to \infty} \inf_{w_0 \in A(\theta_{-t}\omega)} \left(W_0 - \hat{\Theta}_{\delta'}^\ell\right) e^{-\max\{\mu, \delta_1\}t} = 0$$

we have

$$\lim_{t \to \infty} W(t; \theta_{-t}\omega, W_0) \geq \Theta_{\delta'}^\ell, \tag{4.5.16}$$

and the result follows.   □

PROPOSITION 4.13. *The global random attractor $\mathcal{A}_{SP}$ for the RDS generated by (4.5.9) possesses component sets on the ω-sections $\mathcal{A}_{SP}(\omega) = (A_S(\omega), A_P(\omega))$ with*

$$\mu A_S(\omega) \geq \Lambda^\ell - \hat{\Theta}^u \bar{f}\left(\hat{\Theta}^u / \gamma, 0\right).$$

In particular, susceptible preys are prevalent provided the right side of inequality is positive.

PROOF. Since for any $\delta > 0$ the random set $\mathcal{W}_\delta \times \Omega$ is an absorbing set in $\mathcal{T}(\mathbb{R}_+^2)$, for any $K \in \mathcal{T}(\mathbb{R}_+^2)$ there exists $T_K(\omega)$ such that for $t \geq T_K(\omega)$ and $(S_0, P_0) \in K(\theta_{-t}\omega)$ we have

$$\gamma S + P = \gamma S(t; \theta_{-t}\omega, S_0) + P(t; \theta_{-t}\omega, P_0) \leq \hat{\Theta}^u + \delta.$$

From (4.5.9) we then have for any $\delta > 0$

$$S' > \Lambda^\ell - \mu S - \bar{f}\left(\hat{\Theta}^u/\gamma + \delta, 0\right)\left(\hat{\Theta}^u + \delta\right).$$

Hence for any $\delta > 0$ and $K \in \mathcal{T}(\mathbb{R}_+^2)$, $(S_0, I_0) \in K(\theta_{-t}\omega)$ and large t we have

$$S(t; \theta_{-t}\omega, S_0) > \frac{\Lambda^\ell - \bar{f}\left(\hat{\Theta}^u/\gamma + \delta, 0\right)\left(\hat{\Theta}^u + \delta\right)}{\mu}$$

and the result follows. $\qquad\square$

We give now a condition leading to the extinction of predators.

PROPOSITION 4.14. *The global random attractor $\mathcal{A}_{SP}$ for the RDS (θ, φ_{SP}) generated by (4.5.9) has a singleton components $\mathcal{A}_{SP} = (S^*(\omega), 0)$ for every $\omega \in \Omega$, provided that*

$$\gamma \bar{f}\left(\hat{\Theta}^u/\gamma, 0\right) < \delta_1.$$

PROOF. We can choose $\delta > 0$ small enough, such that

$$\gamma \bar{f}\left(\frac{\hat{\Theta}^u}{\gamma} + \delta, 0\right) < \delta_1. \tag{4.5.17}$$

Since $\mathcal{W}_\delta \times \Omega$ is an absorbing set in $\mathcal{T}(\mathbb{R}_+^2)$, for any $K \in \mathcal{T}(\mathbb{R}_+^2)$, there is $T_K(\omega)$ such that, for $t > 0$ sufficiently large we have

$$S = S(t; \theta_{-t}\omega, S_0) \leq \frac{\hat{\Theta}^u}{\gamma} + \delta.$$

From the monotonicity of f (and thus of $\bar{f}$) we have for all $t \geq T_K(\omega)$

$$\begin{aligned}
P' = P'(t, \theta_{-t}\omega, P_0) &= (\gamma \bar{f}(S, P) - \delta_1 - \delta_2 P)P \\
&\leq \left(\gamma \bar{f}\left(\hat{\Theta}^u/\gamma + \delta, 0\right) - \delta_1\right) P.
\end{aligned} \tag{4.5.18}$$

From (4.5.17) we conclude that $P(t; \theta_{-t}\omega, P_0)$ decreases to zero as t goes to infinity. Moreover, if $P = 0$, from the vital dynamics of susceptible preys (4.4.6) we have for all $\omega \in \Omega$

$$\lim_{t \to +\infty} S(t; \theta_{-t}\omega, S_0) = S^*(\omega).$$

$\qquad\square$

4.6. Comments

In Chapter 4 it was discussed the existence and properties of solutions of the eco-epidemiological model with random perturbations (4.2.1) (Section 4.3), the existence of

a global random attractor (Section 4.4), the vital dynamics of susceptible preys (Section 4.4.2.1) and thresholds for extinction of infectives (Secion 4.4.3). It would be interesting to understand under which conditions the uniform strong persistence occur. The asymmetry in the equations of system (4.2.1) with respect to biomass transfer gives rise to significant obstacles and some other ideas should be incorporated to accomplish this objective. On the other hand, a natural extension of this chapter is to consider perturbations in more parameters than just the Λ parameter. One should not expect significant changes in the main strategy in this situation.

Another question that is pertinent from both mathematical and biological viewpoints is to consider perturbations with white noise, i.e., to consider systems derived from stochastic differential equations. However, as mentioned before, these perturbations may change the nature of the original model and, in some circumstances, can even lead to situations where one looses the biological meaning of the system. Still, it would be interesting to study this type of perturbations considering the general model (0.0.2). Moreover, numerical simulations for this random dynamical systems should be considered taken into account the theory of numerical analysis for stochastic dynamical systems (see [**46**]).

Motivated by the seasonal behaviour of dynamics that are considered in Biology, one could aim to consider periodic random perturbations and look for the existence of periodic random attractors, in the spirit of Chapter 2, but here for systems with random perturbations. However, there is a substantial lack in the theory of periodic random dynamical systems which can imply a great effort to move on in this direction.

Finally, there are few works on random perturbations for discrete eco-epidemiological models. It would be very interesting to do research along this line. Namely, to consider random perturbations of the discrete model described in Chapter 3 and provide information about random attractors in this context.

Bibliography

[1] L. Arnold, *Random Dynamical systems*, Springer Monographs in Mathematics Springer-Verlag, Berlin (1998), xvi+586.
https://doi.org/10.1007/978-3-662-12878-7

[2] H. Alfred; S. Emanuel *Analysis of an eco-epidemiological model under optimal control measures for infected prey*, Appl. Appl. Math. 14 (2019), no. 1, 117–138.
https://doi.org/10.2140/jomms.2019.14.97

[3] H. Alfred, M. O. Daniel, K. Santosh, F. C. Fred, *Optimal control and cost effectiveness analysis for Newcastle disease eco-epidemiological model in Tanzania.* J. Biol. Dyn. 11 (2017), no. 1, 190–209.
https://doi.org/10.1080/17513758.2016.1258093

[4] Y. Asai, P. E. Kloeden, *Numerical schemes for random ODEs via stochastic differential equations*, Commun. Appl. Anal. 17 (2013), no. 3-4 511-528.

[5] N. Bacaër, S. Guernaoui, *The epidemic Threshold of vector-borne diseases with seasonality*, J. Math. Biol. 53 (2006), 421-436.
https://doi.org/10.1007/s00285-006-0015-0

[6] H. Bai, R. Xo, *Global stability of a delayed eco-epidemiological model with holling type III functional response*, Springer, Singapore, Proc. math. Stat. 225 (2018), 119-130.
https://doi.org/10.1007/978-981-10-7814-9

[7] I. Bashkirtseva, L. Ryashko, T. Ryazanova, *Analysis of regular and chaotic dynamics in a stochastic eco-epidemiological model*, Chaos, Solitons & Fractals 131 (2020), 109549.
https://doi.org/10.1016/j.chaos.2019.109549

[8] A. M. Bate, F. M. Hilker, *Predator-prey oscillations can shift when diseases become endemic*, J. Theoret. Biol. 316 (2013), 1-8.
https://doi.org/10.1016/j.jtbi.2012.09.013

[9] A. M. Bate, F. M. Hilker, *Complex Dynamics in an Eco-epidemiological Model.* Bull. Math. Biol. 75 (2013), 2059–2078.
https://doi.org/10.1007/s11538-013-9880-z

[10] M. Biswas, N. Bairagi, *Discretization of an eco-epidemiological model and its dynamic consistency*, J. Difference Equ. Appl., 23 (2017), 860-877.
https://doi.org/10.1080/10236198.2017.1304544

[11] S. S. Biswas, S. Samanta, S. Sarkar, J. Chattopadhyay, *Complex dynamics of an eco-epidemiological model with different competition coefficients and weak Allee in the predator*, Chaos, Solitons & Fractals 91 (2016), 270-285.
https://doi.org/10.1016/j.chaos.2016.06.009

[12] T. Caraballo, C. Colucci, X. Han, *Predation with indirect effects in fluctuating environments*, Nonlinear Dynam. 84 (2016), no. 1, 115-126.
https://doi.org/10.1007/s11071-015-2238-3

[13] T. Caraballo, R. Colucci, J. López-de-la-Cruz, A. Rapaport, *A way to model stochastic perturbations in population dynamics models with bounded realizations*, Commun. Nonlinear Sci. Numer. Simul. 77 (2019), 239-257.
https://doi.org/10.1016/j.cnsns.2019.04.019

[14] T. Caraballo, X. Han, *Applied Nonautonomous and Random Dynamical systems*, Applied dynamical systems. SpringerBriefs in Mathematics. Springer, Cham, (2016).
https://doi.org/10.1007/978-3-319-49247-6

[15] T. Caraballo, C. Colucci, *A comparison between random and stochastic modeling for a SIR model*, Commun. Pure Appl. Anal. 16 (2017), 151-162.
https://doi.org/10.3934/cpaa.2017007

[16] C. Castaing, M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Math. Springer-Verlag, Berlin-New York, (1977), Vol. 580.
https://doi:10.1007/bfb0087685

[17] K. Chakraborty, K. Das, S. Haldar, T. K. Kar, *A mathematical study of an eco-epidemiological system on disease persistence and extinction perspective*, Appl. Math. and Comput. 254 (2015), 99-112.
https://doi.org/10.1016/j.amc.2014.12.109

[18] J. Chattopadhyay, O. Arino, *A predator-prey model with disease in the prey*. Nonlinear Anal. 36 (1999), no. 6, Ser. B: Real World Appl., 747–766.
https://doi.org/10.1016/S0362-546X(98)00126-6

[19] S. Chatterjee, J. Chattopadhyay, *Role of migratory bird population in a simple eco-epidemiological model*, Mathematical and Computer Modelling of Dynamical Systems 13 (2007), 99-114.
https://doi.org/10.1080/13873950500303352

[20] J. W. Cholewa, T. Dloko, *Global attractors in the Abstract Parabolic Problems*, London Mathematical Society Lecture Note Series, (2000), vol. 278.
https://doi.org/10.1017/CBO9780511526404

[21] I. Chueshov, *Monotone random systems theory and applications*, Lecture Notes in Math, Springer-Verlag, Berlin, (2002), Vol. 580.
https://doi.org/10.1007/b83277

[22] H. Crauel, *Global random attractors are uniquely determined by attracting deterministic compact sets*, Ann. Mat. Pura Appl. 176(4) (1999), 57-72.
https://doi.org/10.1007/BF02505989

[23] H. Crauel, *Random point attractors versus random set attractors*. J. London Math. Soc.(2) 63 (2001), no. 2, 413-427.
https://doi.org/10.1017/S0024610700001915

[24] H. Crauel, F. Flandoli, *Attractors for random dynamical systems*, Probab. Theory Related Fields 100 (1994), no. 3, 365-393.
https://doi.org/10.1007/BF01193705

[25] H. Crauel, P. E. Kloeden, *Nonautonomous and random attractors*, jahresber. Dtsch. Math.-Ver. 117 (2015), no. 3, 173-206.
https://doi.org/10.1365/s13291-015-0115-0

[26] H. Crauel, M. Scheutzow, *Minimal random attractors*, J. Differential Equations 265 (2018),no. 2 702–718.
https://doi.org/10.1016/j.jde.2018.03.011

[27] H. Crauel, *Random probability measures on polish spaces Stochastics Monographs*, London, (2002),V.11.

[28] D. Greenhalgh, Q. J. A. Khan, F. A. Al-Kharousi, *Eco-epidemiological model with fatal disease in the prey*. Nonlinear Anal. Real World Appl. 53 (2020), 19.
https://doi.org/10.1016/j.nonrwa.2019.103072

[29] D. Greenhalgh, Qamar J. A. Khan Joseph S. Pettigrew, *An eco-epidemiological predator–prey model where predators distinguish between susceptible and infected prey*, Math. Meth. Appl. Sci., 40 (2017), 146–166.
https://doi.org/10.1002/mma.3974

[30] O. Diekmann, J. A. P. Heesterbeek, J. A. J. Metz , *On the definition and the computation of the basic reproduction ratio $\mathcal{R}_0$ in models for infectious diseases in heterogeneous population*, J. Math. Biol 28 (1990), 365.
https://doi.org/10.1007/BF00178324

[31] Q. Din, W. Ishaque, *Bifurcation analysis and chaos control in discrete-time eco–epidemiological models of pelicans at risk in the Salton Sea*, Int. J. Dyn. Control 8 (2020), 132-148.
https://doi:10.1007/s40435-019-00508-x

[32] T. Dondé, *Uniform persistence in a prey-predator model with a diseased predator*, J. of Math. Biol. 80 (2020), 1077-1093.
https://doi.org/10.1007/s00285-019-01451-3

[33] V. den Driessche, W. James, *Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission*, Math. Biosci. 180 (2002), 29-48.
https://doi.org/10.1016/S0025-5564(02)00108-6

[34] Y. Enatsu, Y. Nakata, Y. Muroya, G. Izzo, A. Vecchio, *Global dynamics of difference equations for SIR epidemic models with a class of nonlinear incidence rates*, J. Difference Equ. Appl. 18 (2012), 1163–1181.
https://doi.org/10.1080/10236198.2011.555405

[35] K. Fan, Y. Zhang, S. Gao, *On a new eco-epidemiological model for migratory birds with modified Leslie-Gower functional schemes*, Adv. Differ. Equ. 2016, 97 (2016).
https://doi.org/10.1186/s13662-016-0825-3

[36] D. Feng; L. Bin, *Optimal control problem for a general reaction-diffusion eco-epidemiological model with disease in prey*, Appl. Math. Model., 88 (2020), 1–20.
https://doi.org/10.1016/j.apm.2020.06.040

[37] S. Gao, F. Zhang, Y. He, *The effects of migratory bird population in a nonautonomous eco-epidemiological model*, Appl. Math. Model. 37 (2013), 3903-3916.
https://doi.org/10.1016/j.apm.2012.07.051.

[38] M. Garrione, C. Rebelo, *Persistence in seasonally varying predator-prey systems via the basic reproduction*, Nonlinear Anal. Real World Appl. 30 (2016), 73-98.
https://doi.org/10.1016/j.nonrwa.2015.11.007

[39] R. E. Gaines, J. L. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Lecture Notes in Mathematics 568 (1977), Springer-Verlag, Berlin Heidelberg.
https://doi:10.1007/bfb0089537

[40] B. S. Goh, *Global stability in two species interactions*, J. Math. Biol. 3 (1976), 313-318.
https://doi.org/10.1007/BF00275063

[41] K. Ghosh, T. Sardar, S. Biswas, S. Samanta, J. Chattopadhyay, *An eco-epidemiological model with periodic transmission rate*, Nonlinear Stud. 23 (2016), 345-363.

[42] D. Greenhalgh, Q. J. A. Khan, F. A. Al-Kharousi, *Eco-epidemiological model with fatal disease in the prey*, Nonlinear Anal. Real World Appl. 53 (2020), 103072.
https://doi.org/10.1016/j.nonrwa.2019.103072

[43] K. P. Hadeler, H. I. Freedman, *Predator-prey populations with parasitic infection*, J. Math. Biol. 27 (1989), 609-631.
https://doi.org/10.1007/BF00276947

[44] S. Haldar, K. Chakraborty, K. Das, et al. *Bifurcation and control of an eco-epidemiological system with environmental fluctuations: a stochastic approach*, Nonlinear Dyn 80 (2015), 1187–1207.
https://doi.org/10.1007/s11071-015-1935-2

[45] M. Haque, J. Chattopadhyay, *Influences of non-linear incidence in an eco-epidemiological model of the Salton Sea*, Nonlinear Stud. 10, (2003), 373-388.

[46] X. Han, P. Kloeden, *Random Ordinary Differential Equations and Their Numerical Solution*, Probability Theory and Stochastic Modelling, 85 Springer, Singapore, (2017). https://doi.org/10.1007/978-981-10-6265-0

[47] H. W. Hethcote, W. Wang, L. Han, Z. Ma, *A predator-prey model with infected prey*, Theor. Popul. Biol. 66 (2004), 259-268. https://doi:10.1016/j.tpb.2004.06.010

[48] F. M. Hilker, K. Schmitz, *Disease-induced stabilization of predator–prey oscillations*, J. Theoret. Biol. 255 (2008), no. 3, 299–306. https://doi.org/10.1016/j.jtbi.2008.08.018

[49] J. C. Holmes, W. M. Bethel, *Modification of intermediate host behavior by parasites, in: E.U. Canning, C.A. Wright, (Eds.), Behavioural Aspects of Parasite Transmission*, Zool. F. Linnean Soc., vol. 51 (1972), 123–149.

[50] S. B. Hsu, T. W. Hwang, Y. Kuang, *Global analysis of the Michaelis-Menten-type ratio-dependent predator-prey system*, J. Math. Biol. 42 (2001), 489-506. https://doi.org/10.1007/s002850100079

[51] Z. Hu, Z. Teng, T. Zhang, Q. Zhou, X. Chen, *Globally asymptotically stable analysis in a discrete time eco-epidemiological system*, Chaos, Solitons and Fractals 99 (2017), 20-31. https://doi.org/10.1016/j.chaos.2017.03.042.

[52] Z. Hu, Z. Teng, C. Jia, L. Zhang, X. Chen *Complex dynamical behaviors in a discrete eco-epidemiological model with disease in prey*, Adv. Difference Equ. (2014), 2014:265. https://doi.org/10.1186/1687-1847-2014-265

[53] A. O. Ignat'ev, *On the Global Asymptotics Stability of the Equilibrium of the Lotka-Volterra Equations in a Varying Environment*, Differ. Equ. 3 (2014), 286-291. https://doi.org/10.1134/S0012266114030021

[54] L. F. de Jesus, C. M. Silva and Helder Vilarinho, *An Eco-epidemiological model with general functional response of predator to prey*, preprint.

[55] L. F. de Jesus, C. M. Silva and Helder Vilarinho, *Periodic orbits for periodic eco-epidemiological systems with infected prey*, Electron. J. Qual. Theory Differ. Equ. (2020), No. 54, 1-20. https://doi.org/10.14232/ejqtde.2020.1.54

[56] L. F. de Jesus, C. M. Silva and Helder Vilarinho, *Dynamics of a discrete eco-epidemiological model with disease in the prey*, preprint.

[57] L. F. de Jesus, C. M. Silva and Helder Vilarinho, *Random perturbations of an eco-epidemiological model*, preprint.

[58] W. O. Kermack, A. G. McKendrick, *A Contribution to the Mathematical Theory of Epidemics*, Proceedings of the Royal Society of London, series A (1927), vol. 115, no. 772.

[59] W. O. Kermack, A. G. McKendrick, *A Contribution to the Mathematical Theory of Epidemics II-The Problem of endemicity*, From the Laboratory of the Royal College of Physicians, Edinburgh (1932).

[60] W. O. Kermack, A. G. McKendrick, *Contributions to the Mathematical Theory of Epidemics. III-Further Studies of the Problem of Endemicity*, From the Laboratory of the Royal College of Physicians, Edinburgh (1933).

[61] A. P. Krishchenko, K. E. Starkov, *Convergence dynamics in one eco-epidemiological model: Self-healing and some related results*, Communications in Nonlinear Science and Numerical Simulation 85 (2020), 105223.
https://doi.org/10.1016/j.cnsns.2020.105223

[62] Y. Kuang, H. I. Freedman, *Uniqueness of limit cycles in Gause-type models of predator-prey systems*, Math. Biosc. 88 (1988), 67-84.
https://doi.org/10.1016/0025-5564(88)90049-1

[63] K. Kundu, J. Chattopadhyay, *A ratio-dependent eco-epidemiological model of the Salton Sea*, Math. Meth. Appl. Sci., 29 (2006), 191-207.
https://doi.org/10.1002/mma.671

[64] J. López-Gómez, R. Ortega, A. Tineo, *The periodic predator-prey Lotka-Volterra model*, Adv. Differential Equations 1 (1996), 403-423.

[65] W. Lingshu, F. Guanghui, *Global stability of an eco-epidemiological predator-prey model with saturation incidence*, J. Appl. Math. Comput. 53 (2017), no. 1-2, 303–319.
https://doi.org/10.1007/s12190-015-0969-4

[66] W. Lingshu, X. Rui, *Modelling and analysis of an eco-epidemiological model with time delay and stagestructure*, J. Appl. Math. Comput. 50 (2016), no. 1-2, 175–197.
https://doi.org/10.1007/s12190-014-0865-3

[67] X. Liu, *Bifurcation of an eco-epidemiological model with a nonlinear incidence rate*, Applied Mathematics and Computation 218 (2011), 2300-2309.
https://doi.org/10.1016/j.amc.2011.07.050

[68] Y. Lu, X. Wang, S. Liu, *A non-autonomous predator-prey model with infected prey*, Discrete Contin. Dyn. Syst. B 23 (2018), 3817-3836.
https://doi.org/10.3934/dcdsb.2018082

[69] J. Lu, X. Zhang, R. Xu, *Global stability and Hopf bifurcation of an eco-epidemiological model with time delay*, Int. J. Biomath. 12 (2019), no. 6, 1950062, 21 pp. 92D30.
https://doi.org/10.1142/S1793524519500621

[70] J. Mateus, *A Nonautonomous Discrete Epidemic Model with Isolation*, Int. J. of Difference Equ., 11 (2016), 105-121.

[71] H. Mainul, P. Nikhil, S. Sudip, *Impact of fear on an eco-epidemiologicalmodel*, Chaos Solitons Fractals 134 (2020), 17.
https://doi.org/10.1016/j.chaos.2020.109718

[72] R. E. Mickens, *Discretizations of nonlinear differential equations using explicit nonstandard methods*, J. Comput. Appl. Math. 110 (1999), 181-185.
https://doi.org/10.1016/S0377-0427(99)00233-2

[73] B. B. Mukhopadhyay, *On an epidemiological model with nonlinear infection incidence. Local and global perspective*, Appl. Math. Model. 35 (2011) 3166–3174.
https://doi.org/10.1016/j.apm.2010.12.014

[74] B. Nayyereh, R. Z. Z. Hamid, M. G. Mojtaba, *Multiple bifurcation analysis in a diffusive eco-epidemiological model with time delay*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 29 (2019), no. 3, 1950033, 23 pp. 92D30.
https://doi.org/10.1142/S0218127419500330

[75] X. Niu, T. Zhang, Z. Teng, *The asymptotic behavior of a nonautonomous eco-epidemic model with disease in the prey*, Appl. Math. Model. 35 (2011), 457-470.
https://doi.org/10.1016/j.apm.2010.07.010

[76] N. M. Oliveira, F. M. Hilker, *Modelling Disease Introduction as Biological Control of Invasive Predators to Preserve Endangered Prey*, Bull. Math. Biol. 72 (2010), no. 2, 444–468.
https://doi.org/10.1007/s11538-009-9454-2.

[77] R. O. Peterson, R. E. Page, *The rise and fall of Isle Royale wolves*, J. Mammal. 69 (1988), 89–99.

[78] A. S. Purnomo, I. Darti, A. Suryanto, *Dynamics of eco-epidemiological model with harvesting*, AIP Conference Proceeding 1913 (2017), 020018.
https://doi.org/10.1063/1.5016652

[79] C. Rebelo, A. Margheri, N. Bacaër, *Persistence in seasonally forced epidemiological models*, J. Math. Biol. 64 (6) (2012), 933-949.
https://doi.org/10.1007/s00285-011-0440-6

[80] Y. Saito, J. Sugie , Y. H. Lee , *Global asymptotic stability for predator-prey models with environmental time-variations*, Appl. Math. Lett. 24 (2011), 1973-1980.
https://doi.org/10.1016/j.aml.2011.05.015

[81] Md. Saifuddin, S. Biswas, S. Samanta, S. Sarkar, J. Chattopadhyay, *Complex dynamics of an eco-epidemiological model with different competition coefficients and weak Allee in the predator*, Chaos, Solitons & Fractals 91, 2016, 270-285.
https://doi.org/10.1016/j.chaos.2016.06.009

[82] B. Santosh, S. Sudip, C. Joydev, *A cannibalistic eco-epidemiological model with disease in predator population*, J. Appl. Math. Comput. 57 (2018), no. 1-2, 161–197.
https://doi.org/10.1007/s12190-017-1100-9

[83] ,S. Samantaa, P. K. Tiwarib, A. K. Alzahranic, A. S. Alshomranid, *Chaos in a nonautonomous eco-epidemiological model with delay*, Appl. Math. Model. 79 (2020), 865–880.
https://doi.org/10.1016/j.apm.2019.11.006

[84] B. Sahoo, *Role of additional food in eco-epidemiological system with disease in the prey*, Applied Mathematics and Computation 259 (2015), 61-79.
https://doi.org/10.1016/j.amc.2015.02.038

[85] Md. Saifuddin, S. Sourav, S. Biswas, S. Sarkar, M. Alquran, J. Chattopadhyay, *Effect of emergent carrying capacity in an eco-epidemiological system*, Math. Meth. Appl. Sci., 39 (2016), 806-823.
https://doi.org/10.1002/mma.3523

[86] S. K. Sasmal and J. Chattopadhyay, *An eco-epidemiological system with infected prey and predator subject to the weak Allee effect*, Math. Biosci. 246 (2013), 260-271.
https://doi.org/10.1016/j.mbs.2013.10.005

[87] A. Sha, S. Samanta, M. Martcheva, J. Chattopadhyay, *Backward bifurcation, oscillations and chaos in an eco-epidemiological model with fear effect*, J. Biol. Dyn. 13 (2019), 301-327.
https://doi.org/10.1080/17513758.2019.1593525

[88] A. Saikh, N. H. Gazi, *Mathematical analysis of a predator–prey eco-epidemiological system under the reproduction of infected prey*, J. Appl. Math. Comput. 58 (2018), 621–646.
https://doi.org/10.1007/s12190-017-1160-x

[89] A. A. Shaikh, H. Das, S. Sarwardi, *Dynamics of an eco-epidemiological system with disease in competitive prey species*, J. Appl. Math. Comput. 62 (2020), 525–545.
https://doi.org/10.1007/s12190-019-01295-6

[90] C. M. Silva, *Existence of Periodic Solutions for Eco-Epidemic Model with Disease in the Prey*, J. Math. Anal. Appl. 53 (2017), 383-397.
https://doi.org/10.1016/j.jmaa.2017.03.074

[91] J. Sugie, Y. Saito, M. Fan, *Global asymptotic stability for predator-prey systems whose prey receives time-variation of the environment*, Proc. Amer. Math. Soc. 139 (2011), 3475–3483.
https://doi.org/10.1090/S0002-9939-2011-11124-9

[92] S. L. Sun, C. D. Yuan, *Analysis of an eco-epidemiological SIS model with epidemic in the predator*, Gongcheng Shuxue Xuebao 22 (2005), no. 1, 30–34.

[93] R. K. Upadhyay, N. Bairagi, K. Kundu, J. Chattopadhyay, *Chaos in eco-epidemiological problem of the Salton Sea and its possible control*, Appl. Math. Comput. 196 (2008), 392-401.
https://doi.org/10.1016/j.amc.2007.06.007

[94] R. K. Upadhyay, P. Roy, *Spread of a disease and its effect on population dynamics in an eco-epidemiological system*, Commun. Nonlinear Sci. Numer. Simul. , 19 (2014), 4170-4184.
https://doi.org/10.1016/j.cnsns.2014.04.016.

[95] R. K. Upadhyay, S. N. Raw, P. Roy and V. Rai, *Restoration and recovery of damaged eco-epidemiological systems: Application to the Salton Sea, California, USA*, Math. Biosci. 242 (2013), 172 - 187.
https://doi.org/10.1016/j.mbs.2013.01.002

[96] P. Van den Driessche, J. Watmough, *Reproduction numbers and sub-threshould endemic equilibia for compartmental models of disease transmission*, Math. Biosci. 180 (2002), 29-48.
https://doi.org/10.1016/S0025-5564(02)00108-6

[97] E. Venturino, *The influence of diseases on Lotka-Volterra systems*, Rocky Mountain J. Math. 24 (1994), 381-402.
https://doi.org/10.1216/rmjm/1181072471

[98] W. Wang, X.-Q. Zhao, *Threshold dynamics for compartmental epidemic models in periodic environments*, J. Dynam. Differential Equations 20 (3) (2008), 699-717.
https://doi.org/10.1007/s10884-008-9111-8

[99] L. Wang, P. Yao, G. Feng, *Mathematical analysis of an eco-epidemiological predator–prey model with stage-structure and latency*, J. Appl. Math. Comput. 57 (2018), 211–228.
https://doi.org/10.1007/s12190-017-1102-7

[100] S. Wang, Z. Ma, W. Wang, *Dynamical behavior of a generalized eco-epidemiological system with prey refuge*, Adv. Difference Equ. (2018), 2018:244.
https://doi.org/10.1186/s13662-018-1704-x

[101] J. Waldvogel, *The Period in the Lotka-Volterra System is Monotonic*, J. Math. Anal. Appl. 114 (1986), 78-184.
https://doi.org/10.1016/0022-247X(86)90076-4

[102] W. Yaping, C. Wenyan, *Existence for positive steady states of an eco-epidemiological model*, J. Southeast Univ. (English Ed.) 27 (2011), 119-122.

[103] X. Yanni, C. Lansun, *Analysis of a three species eco-epidemiological model*, J. Math. Anal. Appl. 258 (2001), 733-754.
https://doi.org/10.1006/jmaa.2001.7514

[104] Q. Zhang, D. Jiang, L. Zu, *The stability of a perturbed eco-epidemiological model with Holling type II functional response by white noise*, Discrete Contin. Dyn. Syst. Ser. B 20 (2015), 295-321.
https://doi.org/10.3934/dcdsb.2015.20.295

[105] Q. Zhang, D. Jiang, Z. Liu, D. O'Regan, *Asymptotic behavior of a three species eco-epidemiological model perturbed by white noise*, J. Math. Anal. Appl. 433 (2016), 121-148.
https://doi.org/10.1016/j.jmaa.2015.07.025

[106] X.-Q. Zhao, *Dynamical Systems in Population Biology*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 16 (2003), Springer-Verlag, New York.
https://doi.org/10.1007/978-0-387-21761-1

[107] W. Zheng, J. Sugie, *A necessary and sufficient condition for global asymptotic stability of time-varying Lotka-Volterra predator-prey systems*, Nonlinear Anal. 127 (2015), 128-142. https://doi.org/10.1016/j.na.2015.06.031

[108] W. Zheng, J. Sugie, *Uniform global asymptotic stability of time-varying Lotka-Volterra predator-prey systems*, Appl. Math. Lett. 87 (2019), 125-133. https://doi.org/10.1016/j.aml.2018.07.030

[109] Y. Zhou, Z. Ma, F. Brauer, *A discrete epidemic model for SARS transmission and control in China*, Math. Comput. Model. 40 (2004), 1491–1506. https://doi.org/10.1016/j.mcm.2005.01.007

APPENDIX A

Mathlab code for figures

Figures in chapter 1

Figure 1. clear all

global Lamb mu beth c eta r b Thet delt1 delt2

format short

t=0;

Lamb=0.7;
mu=0.6;
beth=0.075;
eta=0.7;
c=0.1;
delt1=0.2;
Thet=0.9;
delt2=0.3;
S0=2.66
I0=0.51
P0=0.09

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('sistem51',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S1=S; I1=I; P1=P;
t1=t;

S0=1.6
I0=0.2
P0=0.3

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('sistem51',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S2=S; I2=I; P2=P;
t2=t;

S0=0.15
I0=0.7
P0=0.6

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('sistem51',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);

```
S3=S; I3=I; P3=P;
t3=t;

figure
plot(t1,S1,'k')
xlabel ('t');
ylabel ('S','Rotation',0.0)
axis ([0,35,0,1.3])
hold on
plot(t2,S2,'r')
hold on
plot(t3,S3,'c')

figure
plot(t1,I1,'k')
xlabel ('t');
ylabel ('I','Rotation',0.0)
axis ([0,40,0,.2])
hold on
plot(t2,I2,'r')
hold on
plot(t3,I3,'c')

figure
plot(t1,P1,'k')
xlabel ('t');
ylabel ('P','Rotation',0.0)
axis ([0,50,0,1])
hold on
plot(t2,P2,'r')
hold on
plot(t3,P3,'c')
```

Figure 2. clear all

global Lamb mu beth c eta r b Thet delta1 delta2

format short

t=0;

```
Lamb=0.7;
mu=0.6;
beth=0.09;
eta=0.7;
c=0.1;
delta1=0.2;
Thet=0.9;
delta2=0.3;

S0=0.5
I0=0.1
P0=0.4

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('sistem51',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S1=S;
I1=I;
```

```
P1=P;
t1=t;

S0=0.4
I0=0.8
P0=0.7

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('sistem51',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S2=S;
I2=I;
P2=P;
t2=t;

S0=1.036
I0=0.387
P0=0.153

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('sistem51',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S3=S;
I3=I;
P3=P;
t3=t;

figure
plot(t1,S1,'k')
xlabel ('t');
ylabel ('S','Rotation',0.0)
axis ([0,35,0,1.3])
hold on
plot(t2,S2,'r')
hold on
plot(t3,S3,'c')

figure
plot(t1,I1,'k')
xlabel ('t');
ylabel ('I','Rotation',0.0)
axis ([0,40,0,.2])
hold on
plot(t2,I2,'r')
hold on
plot(t3,I3,'c')

figure
plot(t1,P1,'k')
xlabel ('t');
ylabel ('P','Rotation',0.0)
axis ([0,50,0,1])
hold on
plot(t2,P2,'r')
hold on
plot(t3,P3,'c')
```

Figure 3. clear all

global Lamb mu beth c eta r b Thet gamm a

format short

t=0;

Lamb=0.7;
mu=0.18;
beth=.2;
eta=0.7;
c=0.1;
r=0.6;
Thet=0.9;
b=0.8;
gamm=0.1;
a=0.4;

S0=.811
I0=0.0624
P0=1.388

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('sistem52',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S1=S;
I1=I;
P1=P;
t1=t;

S0=0.6
I0=0.16
P0=0.46

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('sistem52',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S2=S;
I2=I;
P2=P;
t2=t;

S0=1.0975
I0=0.044
P0=0.76

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('sistem52',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S3=S;
I3=I;
P3=P;
t3=t;
S3=S3

```
figure plot(t1,S1,'k')
xlabel ('t');
ylabel ('S','Rotation',0.0)
axis ([0,150,0,1.4])
hold on
plot(t2,S2,'r')
hold on
plot(t3,S3,'c')

figure plot(t1,I1,'k')
xlabel ('t');
ylabel ('I','Rotation',0.0)
axis ([0,150,0,.3])
hold on
plot(t2,I2,'r')
hold on
plot(t3,I3,'c')

figure plot(t1,P1,'k')
xlabel ('t');
ylabel ('P','Rotation',0.0)
axis ([0,50,0,1.6])
hold on
plot(t2,P2,'r')
hold on
plot(t3,P3,'c')
```

Figure 4. clear all

global Lamb mu beth c eta r b Thet gamm a

format short

t=0;

Lamb=0.7;
mu=0.18;
beth=1.4;
eta=0.7;
c=0.1;
r=0.6;
Thet=0.9;
b=0.8;
gamm=0.1;
a=0.4;

S0=1.388
I0=0.426
P0=1.388

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('sistem52',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S1=S;
I1=I;
P1=P;
t1=t;

S0=.5
I0=.1

```
P0=.4

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('sistem52',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S2=S;
I2=I;
P2=P;
t2=t;

S0=.4
I0=.04
P0=.7

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('sistem52',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S3=S;
I3=I;
P3=P;
t3=t;
S3=S3
figure plot(t1,S1,'k')
xlabel ('t');
ylabel ('S','Rotation',0.0)
axis ([0,150,0,1.4])
hold on
plot(t2,S2,'r')
hold on
plot(t3,S3,'c')

figure plot(t1,I1,'k')
xlabel ('t');
ylabel ('I','Rotation',0.0)
axis ([0,150,0,.3])
hold on
plot(t2,I2,'r')
hold on
plot(t3,I3,'c')

figure
plot(t1,P1,'k')
xlabel ('t');
ylabel ('P','Rotation',0.0)
axis ([0,50,0,1.6])
hold on
plot(t2,P2,'r')
hold on
plot(t3,P3,'c')
```

   Figure 5. clear all

global Lamb mu beth c eta m Thet d a gamm

format short

```
t=0;

Lamb=0.7;
mu=0.6;
beth=0.07;
eta=0.7;
c=0.1;
m=2;
Thet=0.9;
d=0.3;
a=0.978;
gamm=0.9;

S0=1.66
I0=0.514
P0=0.9

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system3',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S1=S;
I1=I;
P1=P;
t1=t;

S0=.6
I0=.2
P0=.3

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system3',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S2=S;
I2=I;
P2=P;
t2=t;

S0=2.45
I0=.7
P0=.6

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system3',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S3=S;
I3=I;
P3=P;
t3=t;

figure
plot(t1,S1,'k')
xlabel ('t');
ylabel ('S','Rotation',0.0)
axis ([0,100,0,3])
hold on
```

```
plot(t2,S2,'r')
hold on
plot(t3,S3,'c')

figure
plot(t1,I1,'k')
xlabel ('t');
ylabel ('I','Rotation',0.0)
axis ([0,25,0,.8])
hold on
plot(t2,I2,'r')
hold on
plot(t3,I3,'c')

figure
plot(t1,P1,'k')
xlabel ('t');
ylabel ('P','Rotation',0.0)
axis ([0,150,0,1.4])
hold on
plot(t2,P2,'r')
hold on
plot(t3,P3,'c')
```

    Figure 6. clear all

```
global Lamb mu beth c eta m Thet d a gamm

format short

t=0;

Lamb=0.7;
mu=0.6;
beth=0.6;
eta=0.7;
c=0.1;
m=2;
Thet=0.9;
d=0.3;
a=0.978;
gamm=0.9;

S0=1.0357
I0=0.387
P0=.153

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system3',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S1=S;
I1=I;
P1=P;
t1=t;

S0=.5
I0=.1
P0=.4
```

```
OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system3',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S2=S;
I2=I;
P2=P;
t2=t;

S0=.4
I0=.04
P0=.7

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system3',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S3=S;
I3=I;
P3=P;
t3=t;

figure
plot(t1,S1,'k')
xlabel ('t');
ylabel ('S','Rotation',0.0)
axis ([0,75,0,1])
hold on
plot(t2,S2,'r')
hold on
plot(t3,S3,'c')

figure
plot(t1,I1,'k')
xlabel ('t');
ylabel ('I','Rotation',0.0)
axis ([0,70,0,.8])
hold on
plot(t2,I2,'r')
hold on
plot(t3,I3,'c')

figure
plot(t1,P1,'k')
xlabel ('t');
ylabel ('P','Rotation',0.0)
axis ([0,75,0,.8])
hold on
plot(t2,P2,'r')
hold on
plot(t3,P3,'c')
```

Figure 7. clear all

global Lamb mu beth c eta m Thet d a gamm

format short

```
t=0;

Lamb=0.7;
mu=0.6;
beth=0.25;
eta=0.7;
c=0.1;
m=2;
Thet=0.9;
d=0.4;
a=0.4;
gamm=0.8;

S0=1.0357
I0=0.387
P0=0.1525

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system4',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S1=S;
I1=I;
P1=P;
t1=t;

S0=.5
I0=.1
P0=.4

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system4',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S2=S;
I2=I;
P2=P;
t2=t;

S0=.4
I0=.04
P0=.7

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system4',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S3=S;
I3=I;
P3=P;
t3=t;

figure
plot(t1,S1,'k')
xlabel ('t');
ylabel ('S','Rotation',0.0)
axis ([0,150,0,1.2])
hold on
```

```
plot(t2,S2,'r')
hold on
plot(t3,S3,'c')

figure
plot(t1,I1,'k')
xlabel ('t');
ylabel ('I','Rotation',0.0)
axis ([0,150,0,.6])
hold on
plot(t2,I2,'r')
hold on
plot(t3,I3,'c')

figure
plot(t1,P1,'k')
xlabel ('t');
ylabel ('P','Rotation',0.0)
axis ([0,100,0,.8])
hold on
plot(t2,P2,'r')
hold on
plot(t3,P3,'c')
```

Figure 8. clear all

global Lamb mu beth c eta m Thet d a gamm

format short

t=0;

Lamb=0.7;
mu=0.6;
beth=0.08;
eta=0.7;
c=0.1;
m=2;
Thet=0.9;
d=0.4;
a=0.4;
gamm=0.8;

S0=2.66
I0=0.514
P0=0.9

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system4',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S1=S;
I1=I;
P1=P;
t1=t;

S0=1.2
I0=.2
P0=.3
```

```
OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system4',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S2=S;
I2=I;
P2=P;
t2=t;

S0=.45
I0=.7
P0=.6

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system4',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S3=S;
I3=I;
P3=P;
t3=t;

figure
plot(t1,S1,'k')
xlabel ('t');
ylabel ('S','Rotation',0.0)
axis ([0,50,0,2.5])
hold on
plot(t2,S2,'r')
hold on
plot(t3,S3,'c')

figure
plot(t1,I1,'k')
xlabel ('t');
ylabel ('I','Rotation',0.0)
axis ([0,50,0,.8])
hold on
plot(t2,I2,'r')
hold on
plot(t3,I3,'c')

figure
plot(t1,P1,'k')
xlabel ('t');
ylabel ('P','Rotation',0.0)
axis ([0,75,0,1.2])
hold on
plot(t2,P2,'r')
hold on
plot(t3,P3,'c')
```

Figure 9. clear all

global p q d h beth c eta b Thet

format short

```
t=0;

p=0.7;
q=0.7;
b=0.3;
d=b*0.6;
h=0.5;
beth=0.01;
eta=0.7;
c=0.1;
Thet=0.9;

S0=7.16
I0=.15
P0=4.5

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system5',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S1=S;
I1=I;
P1=P;
t1=t;

S0=.5
I0=.1
P0=.4

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system5',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S2=S;
I2=I;
P2=P;
t2=t;

S0=.4
I0=.04
P0=.7

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system5',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S3=S;
I3=I;
P3=P;
t3=t;

figure
plot(t1,S1,'k')
xlabel ('t');
ylabel ('S','Rotation',0.0)
axis ([0,50,0,11])
hold on
plot(t2,S2,'r')
```

```
hold on
plot(t3,S3,'c')

figure
plot(t1,I1,'k')
xlabel ('t');
ylabel ('I','Rotation',0.0)
axis ([0,15,0,.16])
hold on
plot(t2,I2,'r')
hold on
plot(t3,I3,'c')

figure
plot(t1,P1,'k')
xlabel ('t');
ylabel ('P','Rotation',0.0)
axis ([0,50,0,5])
hold on
plot(t2,P2,'r')
hold on
plot(t3,P3,'c')
```

Figure 10. clear all

```
global p q d h beth c eta b Thet

format short

t=0;

p=0.7;
q=0.7;
b=0.3;
d=b*0.6;
h=0.5;
beth=0.5;
eta=0.7;
c=0.1;
Thet=0.9;

S0=2.48
I0=0.38
P0=1.95

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system5',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S1=S;
I1=I;
P1=P;
t1=t;

S0=.5
I0=.1
P0=.4

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system5',[0,300],w0,OPTIONS);
S=w(:,1);
```

```
I=w(:,2);
P=w(:,3);
S2=S;
I2=I;
P2=P;
t2=t;

S0=.4
I0=.04
P0=.7

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system5',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S3=S;
I3=I;
P3=P;
t3=t;

figure
plot(t1,S1,'k')
xlabel ('t');
ylabel ('S','Rotation',0.0)
axis ([0,80,0,5.5])
hold on
plot(t2,S2,'r')
hold on
plot(t3,S3,'c')

figure
plot(t1,I1,'k')
xlabel ('t');
ylabel ('I','Rotation',0.0)
axis ([0,75,0,2.5])
hold on
plot(t2,I2,'r')
hold on
plot(t3,I3,'c')

figure
plot(t1,P1,'k')
xlabel ('t');
ylabel ('P','Rotation',0.0)
axis ([0,75,0,4])
hold on
plot(t2,P2,'r')
hold on
plot(t3,P3,'c')
```

Figure 11. clear all

global p q d h beth c eta b Thet

format short

```
t=0;
p=0.7;
q=0.9;
b=0.3;
```

```
d=b*0.6;
h=0.5;
beth=0.01;
eta=0.7;
c=0.1;
Thet=0.9;

S0=3.342
I0=0.15
P0=2.23

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system6',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S1=S;
I1=I;
P1=P;
t1=t;

S0=.5
I0=.1
P0=.4

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system6',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S2=S;
I2=I;
P2=P;
t2=t;

S0=.4
I0=.04
P0=.7

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system6',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S3=S;
I3=I;
P3=P;
t3=t;

figure
plot(t1,S1,'k')
xlabel ('t');
ylabel ('S','Rotation',0.0)
axis ([0,50,0,11])
hold on
plot(t2,S2,'r')
hold on
plot(t3,S3,'c')

figure
plot(t1,I1,'k')
```

```
xlabel ('t');
ylabel ('I','Rotation',0.0)
axis ([0,15,0,.16])
hold on
plot(t2,I2,'r')
hold on
plot(t3,I3,'c')

figure
plot(t1,P1,'k')
xlabel ('t');
ylabel ('P','Rotation',0.0)
axis ([0,50,0,5])
hold on
plot(t2,P2,'r')
hold on
plot(t3,P3,'c')
```

Figure 12. clear all

global p q d h beth c eta b Thet

format short

t=0;

```
p=0.7;
q=0.9;
b=0.3;
d=b*0.6;
h=0.5;
beth=0.5;
eta=0.7;
c=0.1;
Thet=0.9;
```

```
S0=3.889
I0=.15
P0=2.334
```

```
OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system6',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S1=S;
I1=I;
P1=P;
t1=t;
```

```
S0=.5
I0=.1
P0=.4
```

```
OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system6',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S2=S;
I2=I;
```

```
P2=P;
t2=t;

S0=.4
I0=.04
P0=.7

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode113('system6',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S3=S;
I3=I;
P3=P;
t3=t;

figure
plot(t1,S1,'k')
xlabel ('t');
ylabel ('S','Rotation',0.0)
axis ([0,80,0,5.5])
hold on
plot(t2,S2,'r')
hold on
plot(t3,S3,'c')

figure
plot(t1,I1,'k')
xlabel ('t');
ylabel ('I','Rotation',0.0)
axis ([0,75,0,2.5])
hold on
plot(t2,I2,'r')
hold on
plot(t3,I3,'c')

figure
plot(t1,P1,'k')
xlabel ('t');
ylabel ('P','Rotation',0.0)
axis ([0,75,0,4])
hold on
plot(t2,P2,'r')
hold on
plot(t3,P3,'c')
```

Figures 13 and 14. clear all

global Alp b gamm bt a et c Tt

format short

t=0;

```
Alp=0.7;
b=0.7;
a=1.2;
gamm=.5;
bt=.9;
et=0.5;
```

```
c=.1;
Tt=0.9;

S0=.8
I0=1.7
P0=.7

OPTIONS=odeset('Reltol',1e-13,'AbsTol',1e-30);
w0=[S0,I0,P0]; [t,w]=ode113('sistemLVC1',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S1=S;
I1=I;
P1=P;
t1=t;

S0=.6
I0=1.7
P0=.5

OPTIONS=odeset('Reltol',1e-13,'AbsTol',1e-30);
w0=[S0,I0,P0]; [t,w]=ode113('sistemLVC1',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S2=S;
I2=I;
P2=P;
t2=t;

S0=.4
I0=1.3
P0=.3

OPTIONS=odeset('Reltol',1e-13,'AbsTol',1e-18);
w0=[S0,I0,P0]; [t,w]=ode113('sistemLVC1',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S3=S;
I3=I;
P3=P;
t3=t;

S0=0.0041
I0=0.3531
P0=0

OPTIONS=odeset('Reltol',1e-13,'AbsTol',1e-18);
w0=[S0,I0,P0]; [t,w]=ode113('sistemLVC1',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S4=S;
I4=I;
P4=P;
t4=t;

S0=0.0065
I0=1.2949
```

```
P0=0

OPTIONS=odeset('Reltol',1e-13,'AbsTol',1e-18);
w0=[S0,I0,P0]; [t,w]=ode113('sistemLVC1',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S5=S;
I5=I;
P5=P;
t5=t;

S0=0.0845
I0=0.4234
P0=0

OPTIONS=odeset('Reltol',1e-13,'AbsTol',1e-18);
w0=[S0,I0,P0]; [t,w]=ode113('sistemLVC1',[0,300],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S6=S;
I6=I;
P6=P;
t6=t;

figure
plot(t1,S1,'k')
xlabel ('t');
ylabel ('S','Rotation',0.0)
axis ([0,150,0,1.2])
hold on
plot(t2,S2,'r')
hold on
plot(t3,S3,'c')
hold on
plot(t4,S4,'c')

figure
plot(t1,I1,'k')
xlabel ('t');
ylabel ('I','Rotation',0.0)
axis ([0,100,0,2.5])
hold on
plot(t2,I2,'r')
hold on
plot(t3,I3,'c')
hold on
plot(t4,I4,'c')
hold on
plot(t5,I5,'c')
hold on
plot(t6,I6,'c')

figure
plot(t1,P1,'k')
xlabel ('t');
ylabel ('P','Rotation',0.0)
axis ([0,180,0,.9])
hold on
```

```
plot(t2,P2,'r')
hold on
plot(t3,P3,'c')
hold on
plot(t4,P4,'c')

figure
plot3(S1,P1,I1,'k')
axis ([0,1,0,2,0,2])
xlabel ('S');
ylabel ('P');
zlabel ('I')
box on
hold on
plot3(S2,P2,I2,'k')
hold on
plot3(S3,P3,I3,'k')
hold on
plot3(S4,P4,I4,'b')
hold on
plot3(S5,P5,I5,'r')
hold on
plot3(S6,P6,I6,'c')
```

Figure in chapter 2

Figure 1. clear all

global Lamb mu beth c eta r b Thet a gamm

format short

t=0;

Lamb=0.1;
mu=0.6;
beth=20;
eta=0.7;
c=0.1;
r=0.2;
Thet=4;
b=0.3;
a=2;
gamm=.1;

S0=0.03567
I0=0.02047
P0=0.88021

OPTIONS=odeset('AbsTol',1e-10);
w0=[S0,I0,P0]; [t,w]=ode45('system7',[0,10],w0,OPTIONS);
S=w(:,1);
I=w(:,2);
P=w(:,3);
S1=S;
I1=I;
P1=P;
t1=t;

figure
plot(t1,S1,'k')
```

```
xlabel ('t');
ylabel ('S,I','Rotation',0.0)
axis ([0,10,0,.05])
hold on
plot(t1,I1,'r')

figure
plot(t1,P1,'c')
xlabel ('t');
ylabel ('P','Rotation',0.0)
axis ([0,10,0.8,.9])

figure
plot3(S1,P1,I1,'k')
axis ([.033,.039,.872,.89,0.016,.025])
xlabel ('S');
ylabel ('P');
zlabel ('I')
```

Figures in chapter 3

Figure 1. clear all

```
function sistemD_{NZT}
mu=0.1;
Lambda=0.3;
betaPAR=0.17;
c=0.18;
theta=0.9;
eta=0.3;
b=0.2;
r=0.3;
gammaPar=0.1;
a=0.4;
lambda=0.4;

function betaPAR1=betaPAR1(tp) betaPAR1=betaPAR*(1+0.7*cos(2*pi*tp));
end

function etaPAR1=etaPAR1(tp) etaPAR1=eta*(1+0.7*cos(2*pi*tp));
end

S10=1.5;
I10=0.1;
P10=0.2;

S20=0.7;
I20=0.2;
P20=0.4;

S30=0.3;
I30=0.15;
P30=0.9;

m=400;
h=0.1;

S1(1)=S10;
I1(1)=I10;
P1(1)=P10;
```

```
for n=1:m
PSIN=1+h*mu+h*betaPAR1(n*h)*I1(n)
PHIN=1+h*etaPAR1(n*h)*P1(n)+h*c
THETAN=h*Lambda+S1(n)
S1(n+1)=(THETAN)/(PSIN);
I1(n+1)=I1(n)*(h*betaPAR1(n*h)*THETAN+PSIN)/(PSIN*PHIN)
P1(n+1)=P1(n)*((1+h*r)*PHIN*PSIN+h*theta*etaPAR1(n*h)*(...)
(PSIN+h*betaPAR1(n*h)*THETAN)*I1(n))/(PSIN*PHIN*(1+h*b*P1(n)));

end

S2(1)=S20;
I2(1)=I20;
P2(1)=P20;

for n=1:m
PSIN=1+h*mu+h*betaPAR1(n*h)*I2(n)
PHIN=1+h*etaPAR1(n*h)*P2(n)+h*c
THETAN=h*Lambda+S2(n)
S2(n+1)=(THETAN)/(PSIN);
I2(n+1)=I2(n)*(h*betaPAR1(n*h)*THETAN+PSIN)/(PSIN*PHIN)
P2(n+1)=P2(n)*((1+h*r)*PHIN*PSIN+h*theta*etaPAR1(n*h)*(...)
(PSIN+h*betaPAR1(n*h)*THETAN)*I2(n))/(PSIN*PHIN*(1+h*b*P2(n)));

end

S3(1)=S30;
I3(1)=I30;
P3(1)=P30;

for n=1:m
PSIN=1+h*mu+h*betaPAR1(n*h)*I3(n)
PHIN=1+h*etaPAR1(n*h)*P3(n)+h*c
THETAN=h*Lambda+S3(n)
S3(n+1)=(THETAN)/(PSIN);
I3(n+1)=I3(n)*(h*betaPAR1(n*h)*THETAN+PSIN)/(PSIN*PHIN)
P3(n+1)=P3(n)*((1+h*r)*PHIN*PSIN+h*theta*etaPAR1(n*h)*(...)
(PSIN+h*betaPAR1(n*h)*THETAN)*I3(n))/(PSIN*PHIN*(1+h*b*P3(n)));

end

n=1:m+1;

S=[S1(n).',S2(n).',S3(n).'];
figure(1)
stem (S,'LineStyle','none');
xlabel ('n');
ylabel ('S','Rotation',0.0);
figure(2)
I=[I1(n).',I2(n).',I3(n).'];
stem (I,'LineStyle','none');
xlabel ('n');
ylabel ('I','Rotation',0.0)
figure(3)
P=[P1(n).',P2(n).',P3(n).'];
stem (P,'LineStyle','none');
xlabel ('n');
ylabel ('P','Rotation',0.0)
```

end

Figure 2. clear all

```
function sistemD_NZT
mu=0.1;
Lambda=0.3;
betaPAR=0.29;
c=0.18;
theta=0.9;
eta=0.3;
b=0.2;
r=0.3;
gammaPar=0.1;
a=0.4;
lambda=0.4;

function betaPAR1=betaPAR1(tp)
betaPAR1=betaPAR*(1+0.7*cos(2*pi*tp));

end

function etaPAR1=etaPAR1(tp)
etaPAR1=eta*(1+0.7*cos(2*pi*tp));

end

S10=1.5;
I10=0.1;
P10=0.2;

S20=0.7;
I20=0.2;
P20=0.4;

S30=0.3;
I30=0.15;
P30=0.9;

m=400;
h=0.1;

S1(1)=S10;
I1(1)=I10;
P1(1)=P10;

for n=1:m
PSIN=1+h*mu+h*betaPAR1(n*h)*I1(n)
PHIN=1+h*etaPAR1(n*h)*P1(n)+h*c
THETAN=h*Lambda+S1(n)
S1(n+1)=(THETAN)/(PSIN);
I1(n+1)=I1(n)*(h*betaPAR1(n*h)*THETAN+PSIN)/(PSIN*PHIN)
P1(n+1)=P1(n)*((1+h*r)*PHIN*PSIN+h*theta*etaPAR1(n*h)*(...)
(PSIN+h*betaPAR1(n*h)*THETAN)*I1(n))/(PSIN*PHIN*(1+h*b*P1(n)));

end

S2(1)=S20;
I2(1)=I20;
P2(1)=P20;
```

```
for n=1:m
PSIN=1+h*mu+h*betaPAR1(n*h)*I2(n)
PHIN=1+h*etaPAR1(n*h)*P2(n)+h*c
THETAN=h*Lambda+S2(n)
S2(n+1)=(THETAN)/(PSIN);
I2(n+1)=I2(n)*(h*betaPAR1(n*h)*THETAN+PSIN)/(PSIN*PHIN)
P2(n+1)=P2(n)*((1+h*r)*PHIN*PSIN+h*theta*etaPAR1(n*h)*(...)
(PSIN+h*betaPAR1(n*h)*THETAN)*I2(n))/(PSIN*PHIN*(1+h*b*P2(n)));

end

S3(1)=S30;
I3(1)=I30;
P3(1)=P30;

for n=1:m
PSIN=1+h*mu+h*betaPAR1(n*h)*I3(n)
PHIN=1+h*etaPAR1(n*h)*P3(n)+h*c
THETAN=h*Lambda+S3(n)
S3(n+1)=(THETAN)/(PSIN);
I3(n+1)=I3(n)*(h*betaPAR1(n*h)*THETAN+PSIN)/(PSIN*PHIN)
P3(n+1)=P3(n)*((1+h*r)*PHIN*PSIN+h*theta*etaPAR1(n*h)*(...)
(PSIN+h*betaPAR1(n*h)*THETAN)*I3(n))/(PSIN*PHIN*(1+h*b*P3(n)));

end

n=1:m+1;

S=[S1(n).',S2(n).',S3(n).'];
figure(1)
stem (S,'LineStyle','none');
xlabel ('n');
ylabel ('S','Rotation',0.0);
figure(2)
I=[I1(n).',I2(n).',I3(n).'];
stem (I,'LineStyle','none');
xlabel ('n');
ylabel ('I','Rotation',0.0)
figure(3)
P=[P1(n).',P2(n).',P3(n).'];
stem (P,'LineStyle','none');
xlabel ('n');
ylabel ('P','Rotation',0.0)

end
```

Figure 3. clear all

```
function sistemD_{PER}
mu=0.1;
Lambda=0.3;
betaPAR=0.17;
c=0.18;
theta=0.9;
eta=0.3;
b=0.2;
r=0.3;
gammaPar=0.1;
a=0.4;
```

```
function betaPAR1=betaPAR1(tp)
betaPAR1=betaPAR*(1+0.7*cos(2*pi*tp));

end

function etaPAR1=etaPAR1(tp)
etaPAR1=eta*(1+0.7*cos(2*pi*tp));

end

S10=1.5;
I10=0.1;
P10=0.2;

S20=0.85;
I20=0.2;
P20=0.4;

S30=0.03;
I30=0.015;
P30=0.02;

m=300;
h=0.1;

S1(1)=S10;
I1(1)=I10;
P1(1)=P10;

for n=1:m PSIN=1+h*mu+h*betaPAR1(n*h)*I1(n)+h*a*P1(n)
PHIN=1+h*etaPAR1(n*h)*P1(n)+h*c
THETAN=h*Lambda+S1(n)
S1(n+1)=(THETAN)/(PSIN);
I1(n+1)=I1(n)*(h*betaPAR1(n*h)*THETAN+PSIN)/(PSIN*PHIN)
P1(n+1)=P1(n)*((1+h*r)*PHIN*PSIN+h*gammaPar*a*THETAN*PHIN
+h*theta*etaPAR1(n*h)*(PSIN+h*betaPAR1(n*h)*THETAN)*I1(n))/(PSIN*PHIN*(1+h*b*P1(n)));

end

S2(1)=S20;
I2(1)=I20;
P2(1)=P20;

for n=1:m
PSIN=1+h*mu+h*betaPAR1(n*h)*I2(n)+h*a*P2(n)
PHIN=1+h*etaPAR1(n*h)*P2(n)+h*c
THETAN=h*Lambda+S2(n)
S2(n+1)=(THETAN)/(PSIN);
I2(n+1)=I2(n)*(h*betaPAR1(n*h)*THETAN+PSIN)/(PSIN*PHIN)
P2(n+1)=P2(n)*((1+h*r)*PHIN*PSIN+h*gammaPar*a*THETAN*PHIN
+h*theta*etaPAR1(n*h)*(PSIN+h*betaPAR1(n*h)*THETAN)*I2(n))/(PSIN*PHIN*(1+h*b*P2(n)));

end

S3(1)=S30;
I3(1)=I30;
P3(1)=P30;

for n=1:m
PSIN=1+h*mu+h*betaPAR1(n*h)*I3(n)+h*a*P3(n)
PHIN=1+h*etaPAR1(n*h)*P3(n)+h*c
```

```
THETAN=h*Lambda+S3(n)
S3(n+1)=(THETAN)/(PSIN);
I3(n+1)=I3(n)*(h*betaPAR1(n*h)*THETAN+PSIN)/(PSIN*PHIN)
P3(n+1)=P3(n)*((1+h*r)*PHIN*PSIN+h*gammaPar*a*THETAN*PHIN
+h*theta*etaPAR1(n*h)*(PSIN+h*betaPAR1(n*h)*THETAN)*I3(n))/(PSIN*PHIN*(1+h*b*P3(n))));

end

n=1:m+1;

S=[S1(n).',S2(n).',S3(n).'];
figure(1)
stem (S,'LineStyle','none');
xlabel ('n');
ylabel ('S','Rotation',0.0);
figure(2)
I=[I1(n).',I2(n).',I3(n).'];
stem (I,'LineStyle','none');
xlabel ('n');
ylabel ('I','Rotation',0.0)
figure(3)
P=[P1(n).',P2(n).',P3(n).'];
stem (P,'LineStyle','none');
xlabel ('n');
ylabel ('P','Rotation',0.0)

end
```

Figure 4. clear all

```
function sistemD_PER
mu=0.1;
Lambda=0.3;
betaPAR=2.2;
c=0.18;
theta=0.9;
eta=0.3;
b=0.2;
r=0.3;
gammaPar=0.1;
a=0.4;

function betaPAR1=betaPAR1(tp)
betaPAR1=betaPAR*(1+0.7*cos(2*pi*tp));

end

function etaPAR1=etaPAR1(tp)
etaPAR1=eta*(1+0.7*cos(2*pi*tp));
end

S10=1.5;
I10=0.1;
P10=0.2;

S20=0.85;
I20=0.2;
P20=0.4;

S30=0.03;
I30=0.015;
```

```
P30=0.02;

m=300;
h=0.1;

S1(1)=S10;
I1(1)=I10;
P1(1)=P10;

for n=1:m PSIN=1+h*mu+h*betaPAR1(n*h)*I1(n)+h*a*P1(n)
PHIN=1+h*etaPAR1(n*h)*P1(n)+h*c
THETAN=h*Lambda+S1(n)
S1(n+1)=(THETAN)/(PSIN);
I1(n+1)=I1(n)*(h*betaPAR1(n*h)*THETAN+PSIN)/(PSIN*PHIN)
P1(n+1)=P1(n)*((1+h*r)*PHIN*PSIN+h*gammaPar*a*THETAN*PHIN
+h*theta*etaPAR1(n*h)*(PSIN+h*betaPAR1(n*h)*THETAN)*I1(n))/(PSIN*PHIN*(1+h*b*P1(n)));

end

S2(1)=S20;
I2(1)=I20;
P2(1)=P20;

for n=1:m
PSIN=1+h*mu+h*betaPAR1(n*h)*I2(n)+h*a*P2(n)
PHIN=1+h*etaPAR1(n*h)*P2(n)+h*c
THETAN=h*Lambda+S2(n)
S2(n+1)=(THETAN)/(PSIN);
I2(n+1)=I2(n)*(h*betaPAR1(n*h)*THETAN+PSIN)/(PSIN*PHIN)
P2(n+1)=P2(n)*((1+h*r)*PHIN*PSIN+h*gammaPar*a*THETAN*PHIN
+h*theta*etaPAR1(n*h)*(PSIN+h*betaPAR1(n*h)*THETAN)*I2(n))/(PSIN*PHIN*(1+h*b*P2(n)));

end

S3(1)=S30;
I3(1)=I30;
P3(1)=P30;

for n=1:m
PSIN=1+h*mu+h*betaPAR1(n*h)*I3(n)+h*a*P3(n)
PHIN=1+h*etaPAR1(n*h)*P3(n)+h*c
THETAN=h*Lambda+S3(n)
S3(n+1)=(THETAN)/(PSIN);
I3(n+1)=I3(n)*(h*betaPAR1(n*h)*THETAN+PSIN)/(PSIN*PHIN)
P3(n+1)=P3(n)*((1+h*r)*PHIN*PSIN+h*gammaPar*a*THETAN*PHIN
+h*theta*etaPAR1(n*h)*(PSIN+h*betaPAR1(n*h)*THETAN)*I3(n))/(PSIN*PHIN*(1+h*b*P3(n)));

end

n=1:m+1;

S=[S1(n).',S2(n).',S3(n).'];
figure(1)
stem (S,'LineStyle','none');
xlabel ('n');
ylabel ('S','Rotation',0.0);
figure(2)
I=[I1(n).',I2(n).',I3(n).'];
stem (I,'LineStyle','none');
```

```
xlabel ('n');
ylabel ('I','Rotation',0.0)
figure(3)
P=[P1(n).',P2(n).',P3(n).'];
stem (P,'LineStyle','none');
xlabel ('n');
ylabel ('P','Rotation',0.0)

end
```

Figure 5. clear all

```
function sistemD_AUT
mu=0.1;
Lambda=0.3;
betaPAR=0.17;
c=0.18;
theta=0.9;
eta=0.3;
b=0.2;
r=0.3;
gammaPar=0.1;
a=0.4;

S10=1.5;
I10=0.1;
P10=0.2;

S20=0.7;
I20=0.2;
P20=0.4;

S30=0.3;
I30=0.15;
P30=0.9;

m=300;
h=0.1;

S1(1)=S10;
I1(1)=I10;
P1(1)=P10;

for n=1:m
PSIN=1+h*mu+h*betaPAR*I1(n)+h*a*P1(n)
PHIN=1+h*eta*P1(n)+h*c
THETAN=h*Lambda+S1(n)
S1(n+1)=(THETAN)/(PSIN);
I1(n+1)=I1(n)*(h*betaPAR*THETAN+PSIN)/(PSIN*PHIN)
P1(n+1)=P1(n)*((1+h*r)*PHIN*PSIN+h*gammaPar*a*THETAN*PHIN
+h*theta*eta*(PSIN+h*betaPAR*THETAN)*I1(n))/(PSIN*PHIN*(1+h*b*P1(n)));

end

S2(1)=S20;
I2(1)=I20;
P2(1)=P20;

for n=1:m
PSIN=1+h*mu+h*betaPAR*I2(n)+h*a*P2(n)
PHIN=1+h*eta*P2(n)+h*c
```

```
THETAN=h*Lambda+S2(n)
S2(n+1)=(THETAN)/(PSIN);
I2(n+1)=I2(n)*(h*betaPAR*THETAN+PSIN)/(PSIN*PHIN)
P2(n+1)=P2(n)*((1+h*r)*PHIN*PSIN+h*gammaPar*a*THETAN*PHIN
+h*theta*eta*(PSIN+h*betaPAR*THETAN)*I2(n))/(PSIN*PHIN*(1+h*b*P2(n)));

end

S3(1)=S30;
I3(1)=I30;
P3(1)=P30;

for n=1:m
PSIN=1+h*mu+h*betaPAR*I3(n)+h*a*P3(n)
PHIN=1+h*eta*P3(n)+h*c
THETAN=h*Lambda+S3(n)
S3(n+1)=(THETAN)/(PSIN);
I3(n+1)=I3(n)*(h*betaPAR*THETAN+PSIN)/(PSIN*PHIN)
P3(n+1)=P3(n)*((1+h*r)*PHIN*PSIN+h*gammaPar*a*THETAN*PHIN
+h*theta*eta*(PSIN+h*betaPAR*THETAN)*I3(n))/(PSIN*PHIN*(1+h*b*P3(n)));

end

n=1:m+1;

S=[S1(n).',S2(n).',S3(n).'];
figure(1)
stem (S,'LineStyle','none');
xlabel ('n');
ylabel ('S','Rotation',0.0);
figure(2)
I=[I1(n).',I2(n).',I3(n).'];
stem (I,'LineStyle','none');
xlabel ('n');
ylabel ('I','Rotation',0.0)
figure(3)
P=[P1(n).',P2(n).',P3(n).'];
stem (P,'LineStyle','none');
xlabel ('n');
ylabel ('P','Rotation',0.0)

end
```

Figure 6. clear all

```
function sistemD_AUT
mu=0.1;
Lambda=0.3;
betaPAR=2.2;
c=0.18;
theta=0.9;
eta=0.3;
b=0.2;
r=0.3;
gammaPar=0.1;
a=0.4;

S10=1.5;
I10=0.1;
```

```
P10=0.2;

S20=0.7;
I20=0.2;
P20=0.4;

S30=0.3;
I30=0.15;
P30=0.9;

m=300;
h=0.1;

S1(1)=S10;
I1(1)=I10;
P1(1)=P10;

for n=1:m
PSIN=1+h*mu+h*betaPAR*I1(n)+h*a*P1(n)
PHIN=1+h*eta*P1(n)+h*c
THETAN=h*Lambda+S1(n)
S1(n+1)=(THETAN)/(PSIN);
I1(n+1)=I1(n)*(h*betaPAR*THETAN+PSIN)/(PSIN*PHIN)
P1(n+1)=P1(n)*((1+h*r)*PHIN*PSIN+h*gammaPar*a*THETAN*PHIN
+h*theta*eta*(PSIN+h*betaPAR*THETAN)*I1(n))/(PSIN*PHIN*(1+h*b*P1(n)));

end

S2(1)=S20;
I2(1)=I20;
P2(1)=P20;

for n=1:m
PSIN=1+h*mu+h*betaPAR*I2(n)+h*a*P2(n)
PHIN=1+h*eta*P2(n)+h*c
THETAN=h*Lambda+S2(n)
S2(n+1)=(THETAN)/(PSIN);
I2(n+1)=I2(n)*(h*betaPAR*THETAN+PSIN)/(PSIN*PHIN)
P2(n+1)=P2(n)*((1+h*r)*PHIN*PSIN+h*gammaPar*a*THETAN*PHIN
+h*theta*eta*(PSIN+h*betaPAR*THETAN)*I2(n))/(PSIN*PHIN*(1+h*b*P2(n)));

end

S3(1)=S30;
I3(1)=I30;
P3(1)=P30;

for n=1:m
PSIN=1+h*mu+h*betaPAR*I3(n)+h*a*P3(n)
PHIN=1+h*eta*P3(n)+h*c
THETAN=h*Lambda+S3(n)
S3(n+1)=(THETAN)/(PSIN);
I3(n+1)=I3(n)*(h*betaPAR*THETAN+PSIN)/(PSIN*PHIN)
P3(n+1)=P3(n)*((1+h*r)*PHIN*PSIN+h*gammaPar*a*THETAN*PHIN
+h*theta*eta*(PSIN+h*betaPAR*THETAN)*I3(n))/(PSIN*PHIN*(1+h*b*P3(n)));

end

n=1:m+1;
```

```
S=[S1(n).',S2(n).',S3(n).'];
figure(1)
stem (S,'LineStyle','none');
xlabel ('n');
ylabel ('S','Rotation',0.0);
figure(2)
I=[I1(n).',I2(n).',I3(n).'];
stem (I,'LineStyle','none');
xlabel ('n');
ylabel ('I','Rotation',0.0)
figure(3)
P=[P1(n).',P2(n).',P3(n).'];
stem (P,'LineStyle','none');
xlabel ('n');
ylabel ('P','Rotation',0.0)

end
```