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ИНСТИТУТ ПРИКЛАДНОЙ МАТЕМАТИКИ
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Dynamics of gyrostat satellite subject to gravitational torque. Investigation of equilibria

Москва — 2012

Сарычев В.А., Гутник С.А., Силва А., Сантуш Л.

Динамика спутника-гиростата под действием гравитационного момента. Исследование положений равновесия

Исследована динамика спутника-гиростата, движущегося в центральном ньютоновом силовом поле по круговой орбите. Предложен метод определения всех положений равновесия спутника-гиростата в орбитальной системе координат при заданных значениях вектора гиростатического момента и главных центральных моментов инерции и получены условия их существования в зависимости от четырех безразмерных параметров системы. Найдены бифуркационные значения параметров, при которых изменяется число положений равновесия. Проведен детальный численный анализ эволюции областей существования различного числа равновесий в пространстве безразмерных параметров. Рассмотрена взаимосвязь полученных областей существования с областями существования равновесий осесимметричного спутника. Показано, что число положений равновесия спутника-гиростата в общем случае не превышает 24 и не может быть меньше 8.

Ключевые слова: спутник-гиростат, гравитационный момент, положения равновесия, устойчивость, точки бифуркации.

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Sarychev V.A., Gutnik S.A., Silva A., Santos L.

Dynamics of gyrostat satellite subject to gravitational torque. Investigation of equilibria

Dynamics of gyrostat satellite moving along a circular orbit in the central Newtonian gravitational field is investigated. A symbolic-numerical method for determining all equilibrium orientations of gyrostat satellite in the orbital coordinate system with given gyrostatic torque and given principal central moments of inertia is proposed. Conditions of equilibriums existence are obtained depending on four dimensionless parameters of the system. All bifurcational values of parameters at which there is a change of numbers of equilibrium orientations are determined. Evolution of domains in the space of parameters which correspond to various numbers of equilibria are carried out in detail. Relationship with axisymmetrical cases of satellite gyrostat is considered. It is shown that the number of equilibria of the gyrostat satellite in general case not be less than 8 and not more than 24.

Key words: gyrostat satellite, gravitational torque, equilibria, stability, bifurcation points.

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1. Equations of motion

Let us consider the attitude motion of a gyrostat satellite which is a rigid body with statically and dynamically balanced rotors inside the satellite body. The angular velocities of rotors relative to the satellite body are constant. The center of mass O of gyrostat satellite is situated in a circular orbit.

Let us introduce two right-hand Cartesian coordinate systems with origin in the center of mass O of the gyrostat satellite.

$OXYZ$ is the orbital coordinate system whose OZ axis is directed along the radius vector connecting the centers of mass of the Earth and of the gyrostat satellite; the OX axis is directed along the vector of linear velocity of the center of mass O.

$Oxyz$ is the gyrostat-fixed coordinate system; Ox, Oy, Oz are the principal central axes of inertia of the gyrostat satellite.

Let us define the orientation of the $Oxyz$ coordinate system with respect to the orbital coordinate system by Euler angles ψ, ϑ and φ. Now the direction cosines of Ox, Oy, Oz axes in the orbital coordinate system are represented by the following expressions [1]:

$$\begin{aligned}
a_{11} &= \cos(x,X) = \cos\psi\cos\varphi - \sin\psi\cos\vartheta\sin\varphi,\\
a_{12} &= \cos(y,X) = -\cos\psi\sin\varphi - \sin\psi\cos\vartheta\cos\varphi,\\
a_{13} &= \cos(z,X) = \sin\psi\sin\vartheta,\\
a_{21} &= \cos(x,Y) = \sin\psi\cos\varphi + \cos\psi\cos\vartheta\sin\varphi,\\
a_{22} &= \cos(y,Y) = -\sin\psi\sin\varphi + \cos\psi\cos\vartheta\cos\varphi,\\
a_{23} &= \cos(z,Y) = -\cos\psi\sin\vartheta,\\
a_{31} &= \cos(x,Z) = \sin\vartheta\sin\varphi,\\
a_{32} &= \cos(y,Z) = \sin\vartheta\cos\varphi,\\
a_{33} &= \cos(z,Z) = \cos\vartheta.
\end{aligned} \tag{1}$$

Then, the equations of motion of the gyrostat satellite relative to its center of mass are written in the following form [1, 2]:

$$\begin{aligned}
&A\dot{p} + (C-B)qr - 3\omega_0^2(C-B)a_{32}a_{33} - \bar{H}_2 r + \bar{H}_3 q = 0,\\
&B\dot{q} + (A-C)rp - 3\omega_0^2(A-C)a_{33}a_{31} - \bar{H}_3 p + \bar{H}_1 r = 0,\\
&C\dot{r} + (B-A)pq - 3\omega_0^2(B-A)a_{31}a_{32} - \bar{H}_1 q + \bar{H}_2 p = 0;
\end{aligned} \tag{2}$$

$$p = \dot{\psi} a_{31} + \dot{\vartheta}\cos\varphi + \omega_0 a_{21} = \bar{p} + \omega_0 a_{21},$$
$$q = \dot{\psi} a_{32} - \dot{\vartheta}\sin\varphi + \omega_0 a_{22} = \bar{q} + \omega_0 a_{22}, \qquad (3)$$
$$r = \dot{\psi} a_{33} + \dot{\varphi} + \omega_0 a_{23} = \bar{r} + \omega_0 a_{23}.$$

In equations (2), (3) $\bar{H}_1 = \sum_{k=1}^{n} J_k \alpha_k \dot{\varphi}_k$, $\bar{H}_2 = \sum_{k=1}^{n} J_k \beta_k \dot{\varphi}_k$, $\bar{H}_3 = \sum_{k=1}^{n} J_k \gamma_k \dot{\varphi}_k$; J_k is the axial moment of inertia of k-th rotor; $\alpha_k, \beta_k, \gamma_k$ are the constant direction cosines of the symmetry axis of the k-th rotor in the coordinate system $Oxyz$; $\dot{\varphi}_k$ is the constant angular velocity of the k-th rotor relative to the gyrostat; A, B, C are the principal central moments of inertia of the gyrostat; p, q, r are the projections of the absolute angular velocity of the gyrostat satellite onto the Ox, Oy, Oz axes; ω_0 is the angular velocity of motion of the center of mass of the gyrostat satellite along a circular orbit. Dots designate differentiation with respect to time t.

Further it will be more convenient to use parameters

$$H_1 = \bar{H}_1 / \omega_0,\ H_2 = \bar{H}_2 / \omega_0,\ H_3 = \bar{H}_3 / \omega_0. \qquad (4)$$

For the systems of equations (2) and (3) the generalized energy integral exists in the form

$$\frac{1}{2}(A\bar{p}^2 + B\bar{q}^2 + C\bar{r}^2) + \frac{3}{2}\omega_0^2[(A-C)a_{31}^2 + (B-C)a_{32}^2] +$$
$$+\frac{1}{2}\omega_0^2[(B-A)a_{21}^2 + (B-C)a_{23}^2] - \omega_0^2(H_1 a_{21} + H_2 a_{22} + H_3 a_{23}) = const. \qquad (5)$$

2. Equilibrium orientations

Setting in (2) and (3) $\psi = \psi_0 = const$, $\vartheta = \vartheta_0 = const$, $\varphi = \varphi_0 = const$, we obtain at $A \neq B \neq C$ the equations

$$(C-B)(a_{22}a_{23} - 3a_{32}a_{33}) - H_2 a_{23} + H_3 a_{22} = P = 0,$$
$$(A-C)(a_{23}a_{21} - 3a_{33}a_{31}) - H_3 a_{21} + H_1 a_{23} = Q = 0, \qquad (6)$$
$$(B-A)(a_{21}a_{22} - 3a_{31}a_{32}) - H_1 a_{22} + H_2 a_{21} = R = 0,$$

allowing us to determine the gyrostat satellite equilibria in the orbital coordinate system. Actually, it is more convenient to use in subsequent investigation the equivalent system

$$
\begin{aligned}
&Pa_{11} + Qa_{12} + Ra_{13} = 0,\\
&Pa_{21} + Qa_{22} + Ra_{23} = 0,\\
&Pa_{31} + Qa_{32} + Ra_{33} = 0.
\end{aligned} \tag{7}
$$

System (6) depends on four dimensionless parameters

$$
h_1 = \frac{H_1}{B-C},\ h_2 = \frac{H_2}{B-C},\ h_3 = \frac{H_3}{B-C},\ \nu = \frac{B-A}{B-C}. \tag{8}
$$

Equations (7) are equivalent to equations (6) and can be rewritten in the form

$$
\begin{aligned}
&4(Aa_{21}a_{31} + Ba_{22}a_{32} + Ca_{23}a_{33}) + (H_1a_{31} + H_2a_{32} + H_3a_{33}) = 0,\\
&Aa_{11}a_{31} + Ba_{12}a_{32} + Ca_{13}a_{33} = 0,\\
&(Aa_{11}a_{21} + Ba_{12}a_{22} + Ca_{13}a_{23}) + (H_1a_{11} + H_2a_{12} + H_3a_{13}) = 0
\end{aligned} \tag{9}
$$

or using dimensionless parameters (8) in the form

$$
\begin{aligned}
&-4(\nu a_{21}a_{31} + a_{23}a_{33}) + (h_1a_{31} + h_2a_{32} + h_3a_{33}) = 0,\\
&\nu a_{11}a_{31} + a_{13}a_{33} = 0,\\
&\nu a_{11}a_{21} + a_{13}a_{23} - (h_1a_{11} + h_2a_{12} + h_3a_{13}) = 0.
\end{aligned} \tag{10}
$$

Taking into account expressions (1), system (6) or system (9) can be considered as a system of three equations with unknowns $\psi_0, \vartheta_0, \varphi_0$. The second more convenient method to close equations (9) consists in adding six conditions of orthogonality for the direction cosines (1)

$$
\begin{aligned}
&a_{11}^2 + a_{12}^2 + a_{13}^2 = 1,\\
&a_{21}^2 + a_{22}^2 + a_{23}^2 = 1,\\
&a_{31}^2 + a_{32}^2 + a_{33}^2 = 1,\\
&a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = 0,\\
&a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} = 0,\\
&a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0.
\end{aligned} \tag{11}
$$

Further, we will study the equilibrium orientations of the gyrostat satellite using systems (9) and (11).

As it was shown in [1, 2], the system of second equation in (9) and first, second, fourth, fifth and sixth equations in (11) can be solved for $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$ if $A \neq B \neq C$:

$$a_{11}=\frac{4(C-B)a_{32}a_{33}}{F},\quad a_{21}=\frac{4(I_3-A)a_{31}}{F},$$
$$a_{12}=\frac{4(A-C)a_{33}a_{31}}{F},\quad a_{22}=\frac{4(I_3-B)a_{32}}{F},\tag{12}$$
$$a_{13}=\frac{4(B-A)a_{31}a_{32}}{F},\quad a_{23}=\frac{4(I_3-C)a_{33}}{F}.$$

Here $F=H_1a_{31}+H_2a_{32}+H_3a_{33},\quad I_3=Aa_{31}^2+Ba_{32}^2+Ca_{33}^2.$

Substituting equations (12) in to the first and third equations (9) and adding the third equating (11) we get three equations

$$\begin{aligned}&16[(B-C)^2a_{32}^2a_{33}^2+(C-A)^2a_{33}^2a_{31}^2+(A-B)^2a_{31}^2a_{32}^2]=(H_1a_{31}+H_2a_{32}+H_3a_{33})^2,\\&4(B-C)(C-A)(A-B)a_{31}a_{32}a_{33}+\\&+[H_1(B-C)a_{32}a_{33}+H_2(C-A)a_{33}a_{31}+H_3(A-B)a_{31}a_{32}](H_1a_{31}+H_2a_{32}+H_3a_{33})=0,\\&a_{31}^2+a_{32}^2+a_{33}^2=1\end{aligned}\tag{13}$$

for determining direction cosines a_{31},a_{32},a_{33}. If system (13) will be solved then relations (12) allow us to find the other six direction cosines.

Solutions (12) and equations (13) in dimensionless parameters will have the form

$$a_{11}=\frac{-4a_{32}a_{33}}{h_1a_{31}+h_2a_{32}+h_3a_{33}},\qquad a_{21}=\frac{4[\nu a_{32}^2-(1-\nu)a_{33}^2]a_{31}}{h_1a_{31}+h_2a_{32}+h_3a_{33}},$$
$$a_{12}=\frac{4(1-\nu)a_{33}a_{31}}{h_1a_{31}+h_2a_{32}+h_3a_{33}},\qquad a_{22}=\frac{-4(\nu a_{31}^2+a_{33}^2)a_{32}}{h_1a_{31}+h_2a_{32}+h_3a_{33}},\tag{14}$$
$$a_{13}=\frac{4\nu a_{31}a_{32}}{h_1a_{31}+h_2a_{32}+h_3a_{33}},\qquad a_{23}=\frac{4[(1-\nu)a_{31}^2+a_{32}^2]a_{33}}{h_1a_{31}+h_2a_{32}+h_3a_{33}};$$

$$\begin{aligned}&16[a_{32}^2a_{33}^2+(1-\nu)^2a_{33}^2a_{31}^2+\nu^2a_{31}^2a_{32}^2]=(h_1a_{31}+h_2a_{32}+h_3a_{33})^2(a_{31}^2+a_{32}^2+a_{33}^2),\\&4\nu(1-\nu)a_{31}a_{32}a_{33}+[h_1a_{32}a_{33}-h_2(1-\nu)a_{33}a_{31}-h_3\nu a_{31}a_{32}]\times\\&\times(h_1a_{31}+h_2a_{32}+h_3a_{33})=0,\\&a_{31}^2+a_{32}^2+a_{33}^2=1.\end{aligned}\tag{15}$$

The right part of first equation in (15) was multiplied by $a_{31}^2+a_{32}^2+a_{33}^2=1$.

Let us introduce the values $x = \frac{a_{31}}{a_{33}}$, $y = \frac{a_{32}}{a_{33}}$ and divide all terms of first equation in (15) by a_{33}^4 and second equation by a_{33}^3. Then we will have the system of two equations with unknown values x, y:

$$\begin{aligned} &16[y^2 + (1-\nu)^2 x^2 + \nu^2 x^2 y^2] = (h_1 x + h_2 y + h_3)^2 (1 + x^2 + y^2), \\ &4\nu(1-\nu)xy + [h_1 y - h_2(1-\nu)x - h_3 \nu xy](h_1 x + h_2 y + h_3) = 0. \end{aligned} \tag{16}$$

Now substituting expressions $a_{31} = xa_{33}$, $a_{32} = ya_{33}$ in the last equation of the system (15) we receive

$$a_{33}^2 = \frac{1}{1 + x^2 + y^2}. \tag{17}$$

The system of equations (16) can be presented in such form:

$$\begin{aligned} &a_0 y^2 + a_1 y + a_2 = 0, \\ &b_0 y^4 + b_1 y^3 + b_2 y^2 + b_3 y + b_4 = 0. \end{aligned} \tag{18}$$

Here

$$\begin{aligned} &a_0 = h_2(h_1 - \nu h_3 x), \\ &a_1 = h_1 h_3 + [4\nu(1-\nu) + h_1^2 - (1-\nu)h_2^2 - \nu h_3^2]x - \nu h_1 h_3 x^2, \\ &a_2 = -(1-\nu)h_2(h_1 x + h_3)x, \\ &b_0 = h_2^2, \\ &b_1 = 2h_2(h_1 x + h_3), \\ &b_2 = (h_2^2 + h_3^2 - 16) + 2h_1 h_3 x + (h_1^2 + h_2^2 - 16\nu^2)x^2, \\ &b_3 = 2h_2(h_1 x + h_3)(1 + x^2), \\ &b_4 = (h_1 x + h_3)^2 (1 + x^2) - 16(1-\nu)^2 x^2. \end{aligned} \tag{19}$$

Resultant $R(x)$ of equations (18) has the form

$$R(x) = \begin{bmatrix} a_0 & a_1 & a_2 & 0 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & 0 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & 0 \\ 0 & 0 & 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & b_3 & b_4 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & b_4 \end{bmatrix}$$

Let us consider equation $R(x)=0,$ which can be presented with the help of Mathematica symbolic matrix function as the twelfth order algebraic equation

$$\begin{aligned} &p_0x^{12}+p_1x^{11}+p_2x^{10}+p_3x^9+p_4x^8+p_5x^7+p_6x^6+\\ &+p_7x^5+p_8x^4+p_9x^3+p_{10}x^2+p_{11}x+p_{12}=0, \end{aligned} \tag{20}$$

where

$$p_0=-h_1^4h_3^4\nu^6,$$

$$p_1=2h_1^3h_3^3\nu^5\left[2h_1^2-h_2^2(\nu-1)-2\nu\left(h_3^2+2\nu-2\right)\right],$$

$$\begin{aligned} p_2=&-h_1^2h_3^2\nu^4\Big\{6h_1^4+h_2^4(\nu-1)^2-h_2^2(\nu-1)\left[16\left(\nu^3-\nu^2\right)+(\nu-1)+h_3^2(1-7\nu)\right]+\\ &+h_1^2\left[\left(-25\nu^2+26\nu-1\right)+h_3^2\left(\nu^2-16\nu+1\right)+h_2^2\left(\nu^2-8\nu+7\right)\right]+\\ &+2\nu^2\left[3h_3^4+8(\nu-1)^2-4h_3^2\left(2\nu^2-7\nu+5\right)\right]\Big\}, \end{aligned}$$

$$\begin{aligned} p_3=&\ 2h_1h_3\nu^3\Big\{2h_1^6+h_1^4\left[\left(-13\nu^2+14\nu-1\right)+2h_3^2\left(\nu^2-6\nu+1\right)+h_2^2\left(\nu^2-5\nu+4\right)\right]+\\ &+h_3^2\Big[-h_2^4(\nu-1)^2(2\nu-1)+h_2^2(\nu-1)\nu\left[h_3^2(1-4\nu)+\left(16\nu^3-16\nu^2+\nu-1\right)\right]+\\ &+2\nu^3\left[-h_3^4+8(\nu-1)^2(4\nu-5)+2h_3^2\left(7-11\nu+4\nu^2\right)\right]\Big]-\\ &-h_1^2\Big[h_2^4(\nu-2)(\nu-1)^2+h_2^2(\nu-1)\left[\left(16\nu^3-16\nu^2+\nu-1\right)+h_3^2\left(3\nu^2-13\nu+3\right)\right]+\\ &+2\nu\left[-2(\nu-1)^2(5\nu-1)+h_3^4\left(\nu^2-6\nu+1\right)+h_3^2\left(18\nu^3-53\nu^2+38\nu-3\right)\right]\Big]\Big\}, \end{aligned}$$

$$\begin{aligned} p_4=&-\nu^2\Big\{h_1^8+h_1^6\left[-1+10\nu-9\nu^2+h_2^2\left(3-4\nu+\nu^2\right)+2h_3^2\left(3-8\nu+3\nu^2\right)\right]+\\ &+h_3^2\Big[h_2^6(\nu-1)^4+\left\{h_3^2-16(\nu-1)^2\right\}\nu^4\left(-4+h_3^2+4\nu\right)^2+\\ &+h_2^4(\nu-1)^2\nu\left\{-8(\nu-1)^2(2\nu+1)+h_3^2(3\nu-2)\right\}+\\ &+h_2^2(\nu-1)\nu^2\left\{h_3^4(3\nu-1)+16(\nu-1)^3(1+8\nu)+h_3^2\left(17-49\nu+64\nu^2-32\nu^3\right)\right\}\Big]+\\ &+h_1^2\Big[h_2^6(\nu-1)^4+h_2^4(\nu-1)^2\left\{-(\nu-1)^2(1+8\nu)+2h_3^2\left(4-9\nu+4\nu^2\right)\right\}+\\ &+h_2^2(\nu-1)\nu\Big\{8(\nu-1)^3(1+2\nu)+h_3^4\left(14-33\nu+13\nu^2\right)+\\ &+h_3^2\left(-4+38\nu-98\nu^2+64\nu^3\right)\Big\}+2\nu^2\Big\{-8(\nu-1)^4+h_3^6\left(3-8\nu+3\nu^2\right)+\\ &+4h_3^2(\nu-1)^2\left(7-63\nu+52\nu^2\right)+h_3^4\left(-23+134\nu-187\nu^2+76\nu^3\right)\Big\}\Big]-\\ &-h_1^4\Big[h_2^4(\nu-1)^2(2\nu-3)+h_2^2(\nu-1)\left\{4\nu-19\nu^2-2+17\nu^3+h_3^2\left(13-33\nu+14\nu^2\right)\right\}+\\ &+\nu\left\{h_3^4\left(16-37\nu+16\nu^2\right)-8(\nu-1)^2(3\nu-1)+h_3^2\left(209\nu-298\nu^2+121\nu^3-32\right)\right\}\Big]\Big\}, \end{aligned}$$

$$
\begin{aligned}
p_5 = {} & 2h_1h_3\nu\{-2h_2^6(\nu-1)^5+2h_1^6(\nu^2-\nu+1)-h_2^4(\nu-1)^2\nu[(\nu-1)^2(11+4\nu)+ \\
& +h_3^2(5-10\nu+6\nu^2)]- \\
& -2\nu^3[40(\nu-1)^4(4\nu-1)+h_3^6(\nu^2-\nu+1)+2h_3^2(\nu-1)^2(27-79\nu+56\nu^2)+ \\
& +h_3^4(-15+46\nu-53\nu^2+22\nu^3)]- \\
& -h_2^2(\nu-1)\nu^2[-8(\nu-1)^3(16\nu-1)+h_3^2(-33+131\nu-146\nu^2+48\nu^3)]- \\
& -h_2^2(\nu-1)\nu^2h_3^4(6\nu^2-8\nu+5)+ \\
& +h_1^4[h_2^2(6-14\nu+13\nu^2-5\nu^3)+\nu\{14-61\nu+92\nu^2-45\nu^3-2h_3^2(6-7\nu+6\nu^2)\}]+ \\
& +h_1^2[h_2^4(\nu-1)^2(6-10\nu+5\nu^2)+ \\
& +h_2^2(\nu-1)\nu\{-3+41\nu-56\nu^2+18\nu^3+h_3^2(17-26\nu+17\nu^2)\}+ \\
& +2\nu^2h_3^4(6\nu^2-7\nu+6)+ \\
& +2\nu^2\{2(\nu-1)^2(57\nu^2-64\nu+11)+h_3^2(67\nu^3-155\nu^2+125\nu-37)\}]\},
\end{aligned}
$$

$$
\begin{aligned}
p_6 = {} & -\{h_2^8(\nu-1)^6-2h_2^6[h_3^2(1-2\nu)+8(\nu-1)^2]\nu(\nu-1)^4+h_1^8(\nu^2+1)- \\
& -2h_2^2(\nu-1)\nu^2[-84h_3^2(\nu-1)^2+128(\nu-1)^5+17h_3^4(\nu-1)\nu+h_3^6(-1+\nu-2\nu^2)]+ \\
& +\nu^4(-4+h_3^2+4\nu)^2[-17h_3^2(\nu-1)^2+16(\nu-1)^4+h_3^4(1+\nu^2)]+ \\
& +h_2^4(\nu-1)^2\nu^2[96(\nu-1)^4-3h_3^2(\nu-1)^2(3+8\nu)+h_3^4(2-6\nu+6\nu^2)]+ \\
& +h_1^6[-2h_2^2(-2+3\nu-2\nu^2+\nu^3)+\nu\{8-25\nu+42\nu^2-25\nu^3-4h_3^2(4-3\nu+4\nu^2)\}]+ \\
& -h_1^2[2h_2^6(\nu-2)(\nu-1)^4+h_2^4(\nu-1)^2\nu\{3(\nu-1)^2(8+3\nu)+h_3^2(20-34\nu+20\nu^2)\}- \\
& +h_2^2(\nu-1)\nu^2\{h_3^4(30-40\nu+34\nu^2)-168(\nu-1)^3\nu+17h_3^2(-7+27\nu-27\nu^2+7\nu^3)\} \\
& +2\nu^3\{8(\nu-1)^4(25\nu-8)+h_3^6(8-6\nu+8\nu^2)+4h_3^2(\nu-1)^2(67-122\nu+67\nu^2)+ \\
& +h_3^4(-104+247\nu-230\nu^2+87\nu^3)\}]+ \\
& +2h_1^4[h_2^4(\nu-1)^2(3-3\nu+\nu^2)+h_2^2(\nu-1)\nu\{-17\nu(\nu-1)+h_3^2(17-20\nu+15\nu^2)\}+ \\
& +\nu^2 2h_3^4(9-8\nu+9\nu^2)+ \\
& +\nu^2\{4(\nu-1)^2(4-21\nu+21\nu^2)+h_3^2(-87+230\nu-247\nu^2+104\nu^3)\}]\},
\end{aligned}
$$

$$p_7=-2h_1h_3\Big\{-2h_2^6(\nu-1)^5+2h_1^6(\nu^2-\nu+1)-$$
$$-h_2^4(\nu-1)^2\nu\Big[-(\nu-1)^2(4+11\nu)+h_3^2(5-10\nu+6\nu^2)\Big]-$$
$$-h_2^2(\nu-1)\nu^2h_3^4(5-8\nu+6\nu^2)-$$
$$-h_2^2(\nu-1)\nu^2\Big[-8(\nu-16)(\nu-1)^3+h_3^2(-18+56\nu-41\nu^2+3\nu^3)\Big]-$$
$$-\nu^3\Big[80(\nu-4)(\nu-1)^4+2h_3^6(\nu^2-\nu+1)+4h_3^2(\nu-1)^2(57-64\nu+11\nu^2)+$$
$$+h_3^4(-45+92\nu-61\nu^2+14\nu^3)\Big]+$$
$$+h_1^4\Big[h_2^2(6-14\nu+13\nu^2-5\nu^3)-2\nu\{53\nu-46\nu^2+15\nu^3-22+h_3^2(6-7\nu+6\nu^2)\}\Big]+$$
$$+h_1^2h_2^4(\nu-1)^2(6-10\nu+5\nu^2)+$$
$$+h_1^2\Big[h_2^2(\nu-1)\nu\{-48+146\nu-131\nu^2+33\nu^3+h_3^2(17-26\nu+17\nu^2)\}+$$
$$+2\nu^2h_3^4(6-7\nu+6\nu^2)+$$
$$+2\nu^2\Big\{2(\nu-1)^2(56-79\nu+27\nu^2)+h_3^2(-67+155\nu-125\nu^2+37\nu^3)\Big\}\Big]\Big\},$$

$$p_8=-\Big\{h_1^8+h_1^6\Big[h_2^2(3-4\nu+\nu^2)+2\{h_3^2(3-8\nu+3\nu^2)-4(2-5\nu+3\nu^2)\}\Big]+$$
$$+h_3^2\Big[h_2^6(\nu-1)^4+\nu^4\{h_3^2-(\nu-1)^2\}(-4+h_3^2+4\nu)^2-$$
$$-h_2^4(\nu-1)^2\nu\{h_3^2(2-3\nu)+(\nu-1)^2(8+\nu)\}+$$
$$+h_2^2(\nu-1)\nu^2\{8(\nu-1)^3(2+\nu)+h_3^4(3\nu-1)+h_3^2(17-19\nu+4\nu^2-2\nu^3)\}\Big]-$$
$$-h_1^4\Big[h_2^2(\nu-1)\{-32+64\nu-49\nu^2+17\nu^3+h_3^2(13-33\nu+14\nu^2)\}+$$
$$+h_2^4(\nu-1)^2(2\nu-3)+$$
$$+\nu\{h_3^4(16-37\nu+16\nu^2)-16(\nu-1)^2(9\nu-8)+2h_3^2(187\nu-134\nu^2+23\nu^3-76)\}\Big]+$$
$$+h_1^2\Big[h_2^6(\nu-1)^4+2h_2^4(\nu-1)^2\{h_3^2(4-9\nu+4\nu^2)-4(2-3\nu+\nu^3)\}+$$
$$+h_2^2(\nu-1)\nu h_3^4(14-33\nu+13\nu^2)+$$
$$+h_2^2(\nu-1)\nu\{16(\nu-1)^3(8+\nu)+h_3^2(-64+98\nu-38\nu^2+4\nu^3)\}+$$
$$+\nu^2\{-256(\nu-1)^4+2h_3^6(3-8\nu+3\nu^2)+8h_3^2(\nu-1)^2(52-63\nu+7\nu^2)+$$
$$+h_3^4(-121+298\nu-209\nu^2+32\nu^3)\}\Big]\Big\},$$

$$p_9 = -2h_1h_3\Big\{2h_1^6 + h_3^2\Big[-h_2^4(\nu-1)^2(2\nu-1)+$$
$$+h_2^2(\nu-1)\nu\left\{-16+h_3^2(1-4\nu)+16\nu-\nu^2+\nu^3\right\}+$$
$$+\nu^3\left\{-2h_3^4+4(\nu-5)(\nu-1)^2+h_3^2\left(13-14\nu+\nu^2\right)\right\}\Big]+$$
$$+h_1^4\Big[h_2^2\left(4-5\nu+\nu^2\right)+2\left\{-8+22\nu-14\nu^2+h_3^2\left(1-6\nu+\nu^2\right)\right\}\Big]-$$
$$-h_1^2\Big[h_2^2(\nu-1)\left\{-16+16\nu-\nu^2+\nu^3+h_3^2\left(3-13\nu+3\nu^2\right)\right\}-$$
$$-h_1^2h_2^4(\nu-2)(\nu-1)^2+2\nu h_3^4\left(1-6\nu+\nu^2\right)+$$
$$+2\nu\left\{-8(\nu-1)^2(5\nu-4)+h_3^2\left(-18+53\nu-38\nu^2+3\nu^3\right)\right\}\Big]\Big\},$$

$$p_{10} = -h_1^2h_3^2\Big\{6h_1^4+h_2^4(\nu-1)^2-h_2^2(\nu-1)\Big[\left(\nu^3-\nu^2+16\nu-16\right)+h_3^2(1-7\nu)\Big]+$$
$$+\nu^2\Big[6h_3^4+16(\nu-1)^2-h_3^2\left(\nu^2-26\nu+25\right)\Big]+$$
$$+h_1^2\Big[h_3^2\left(\nu^2-16\nu+1\right)+h_2^2\left(\nu^2-8\nu+7\right)-8\left(5\nu^2-7\nu+2\right)\Big]\Big\},$$

$$p_{11} = -2h_1^3h_3^3\left\{2h_1^2-h_2^2(\nu-1)-2\nu\left(h_3^2+2\nu-2\right)\right\},$$

$$p_{12} = -h_1^4h_3^4.$$

Substituting the value of a real root x_1 of equation (20) into the equations (18) we can find coinciding root y_1 of these equations. For each solution x_1, y_1 one can determine two values of a_{33} from equation (17), and then the values $a_{31} = x_1a_{33}$ and $a_{32} = y_1a_{33}$. Thus, each real root of the algebraic equation (20) corresponds to two sets of values a_{31}, a_{32}, a_{33}, which, by virtue of (14), uniquely determine the remaining direction cosines $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$. It follows from these considerations that the satellite gyrostat in general case $(h_1 \neq 0,\ h_2 \neq 0,\ h_3 \neq 0)$ may have no more than 24 equilibrium orientations in orbital coordinate frame.

3. Analysis of equilibrium orientations

Using equation (20), it is possible to determine numerically all equilibrium orientations of the gyrostat satellite in orbital coordinate system and analyze their stability We will analyze numerically dependence of the number of real solutions of equation (20) on the parameters, using Mathematica factorization method. It is possible to provide the numerical calculations, without breaking a generality for the case when $B > A > C$. From these inequalities it follows that $0 < \nu < 1$. The parameters $h_1,\ h_2,\ h_3$ can take on any nonzero values.

For the limiting values $\nu = 0$ and $\nu = 1$ (cases of an axisymmetric satellite) it is possible to define analytically a boundary of regions with the fixed number of equilibrium orientations.

In the case $\nu = 0$ $(A = B)$ we have an axisymmetric satellite. For this case the system of stationary equations (9) becomes simpler. And it is possible to derive from these equations the equations of circles in the plane (h_1, h_2), which define the borders between the regions with the fixed number of equilibria:

$$\begin{aligned} h_1^2 + h_2^2 &= (4^{2/3} - h_3^{2/3})^3, \\ h_1^2 + h_2^2 &= (1 - h_3^{2/3})^3. \end{aligned} \tag{21}$$

In the case $\nu = 1$ $(A = C)$ we have also an axisymmetric satellite. For this case it is possible to derive from the equations (9) the equations of two astroids in the plane (h_1, h_2), which define the borders between the regions with the fixed number equilibria:

$$\begin{aligned} h_2^{2/3} + (h_1^2 + h_3^2)^{1/3} &= 4^{2/3}, \\ h_2^{2/3} + (h_1^2 + h_3^2)^{1/3} &= 1. \end{aligned} \tag{22}$$

The coefficients of equation (20) depend on 4 dimensionless parameters ν, h_1, h_2, h_3. The system of stationary equations (9) depends on 6 dimensional parameters H_1, H_2, H_3, A, B, C. For the numerical calculations decrease of the number of system parameters is very essential.

Let us consider the properties of the algebraic equation (20) in detail. It is possible to show that the number of real roots of equation (20) does not depend on the sign of the parameters h_1, h_2, h_3.

It is evident that coefficients of equation (20) with odd x degree p_{2k} $(k = 1, 2, 3, 4, 5, 6)$ depend only on odd degree of the parameters h_1, h_2, h_3. For the coefficients with even x degree p_{2k+1} $(k = 1, 2, 3, 4, 5)$ we can represent them, using factorization to the form $p_{2k+1} = h_1 h_3 P_{2k+1}$, where factor P_{2k+1} depends only on odd degree of the parameters h_1, h_2, h_3. Thus, changing sign of h_1, h_2, h_3, we will change only sign of the factor $h_1 h_3$ and therefore sign of real root of polynomial (20). Therefore the number of real roots does not change.

Hence, the numerical analysis of the number of real roots of the equation (20) is possible to do with positive values of h_1, h_2, h_3 and $0 < \nu < 1$ condition. Thus, the numerical investigation of real roots of equation (20) will be simplified.

The numerical calculations were made for fixed values of ν and h_3, the number of real roots was determined at the nodes of a uniform grid in the plane (h_1, h_2). The direct calculations for h_2 with step equals to 0.0001 are very complicated. In this case we have for the size $4x4(h_1, h_2)$ region about 10^9 nodes. The calculation task was divided in two parts. In the first place, the number of real roots in the 10^7 nodes (0.001 step value for h_2) was calculated. Secondly, the number of real roots was calculated in the vicinity of the border between two regions with the fixed number of real roots (0.0001 step value for h_2).

Then for the fixed value of h_2 it was defined more precise border value of h_1 between two regions with the fixed number of real roots with the determined accuracy, using half division method, realized in Mathematica language as a package.

The equation (20) was derived under the conditions $h_1 \neq 0,\ h_2 \neq 0,\ h_3 \neq 0$, so we use $h_1,\ h_2,\ h_3$ in the vicinity of zero with the higher accuracy equal to 0.00001.

Calculations were made for the inertia parameters $\nu = 0.01$ (near limit value $\nu = 0$), $\nu = 0.1$, $\nu = 0.2$, $\nu = 0.3$, $\nu = 0.4$, $\nu = 0.5$, $\nu = 0.6$, $\nu = 0.7$, $\nu = 0.8$, $\nu = 0.9$ and $\nu = 0.99$ (near limit value $\nu = 1$). The results of calculations are presented in the Figures 1-55.

From the analysis of all calculations for the mentioned above inertia parameters ν it follows that with increase of the h_3 values of several regions with the fixed number of equilibria become narrowed until they completely disappear. The point in the space of parameters where region with the fixed number of equilibria vanishes, was defined as bifurcation point. Calculated bifurcation values of the parameters are presented in the Table 1.

The figures 1-55 were calculated for the bifurcation values h_3, indicated in the Table 1, and for the values h_3, corresponding to middle points between two bifurcation values.

All bifurcation points from Table 1 were calculated numerically, and it is possible to see that the h_3 bifurcation values for regions with 24 equilibria (12 real roots) vanish in accordance with the equation $h_3 = 1 - \nu$.

For the regions with 20 equilibria (10 real roots) the h_3 bifurcation values increase with the increase of ν up to $\nu = 0.6$, and decrease after that with the further increase of ν.

For the regions with 16 equilibria (8 real roots) the bifurcation values always decrease with the decreasing of ν.

The regions with 12 equilibria become smaller with the increase of h_3 values. These regions are vanishing in the center of system of coordinates for $h_3 = 4$. For $h_3 \geq 4$ there are small regions of 12 equilibria near h_2 axis with the size along h_1 and h_2 axes less than 10^{-1} (Figs.13, 19). And the bigger the h_3 value, the further from the center of coordinate system these small regions take position along the h_2 axis.

Let us consider the example, when first inertia parameter $\nu = 0.01$ (near axisymmetric gyrostat, $\nu = 0$). In this case were calculated figures for the next h_3

values: $h_3 = 0.01$ (zero vicinity point, Fig.1), $h_3 = 0.495$ (middle point, Fig.3) and $h_3 = 0.99$ (bifurcation point, where region with 24 equilibria disappear, Fig.5). These figures are very similar to the corresponding figures for $\nu = 0$ (see figures 2, 4, 6), which are defined by the equations (21). As shown in [3], in axisymmetric case the number of equilibria of the gyrostat satellite is no less than 8 and no more than 16. There are only 3 regions in the space of parameters with fixed number of equilibria – 16, 12 and 8. There are two bifurcation values of h_3 in this case: $h_3 = 1$ and $h_3 = 4$.

For the inertia parameter $\nu = 0.99$, near axisymmetric limit value $\nu = 1$, were calculated figures for the next h_3 values: $h_3 = 0.005$ (zero vicinity point, Fig.48), $h_3 = 0.01$ (bifurcation point, where region with 24 equilibria disappear, Fig.49), $h_3 = 0.5$ (middle point, Fig.50) and $h_3 = 1.0$ (bifurcation point, where region with 16 equilibria disappear, Fig.51).

These figures are similar to the corresponding figures for $\nu = 1$ case (Figures 52-55), which are defined by the equations (22). There are two bifurcation points of h_3 in this case, $h_3 = 1$ and $h_3 = 4$.

For the interval $0.1 \le \nu \le 0.9$ we investigate numerically the evolution of regions with 24, 20, 16, 12 and 8 equilibria. It was used a small increment of the ν parameter equal to 0.1 (see Figures 7-47).

For example, if $\nu = 0.2$ then analysis of the numerical results shows that five regions with 24, 20, 16, 12 and 8 equilibria exist in the plane $(h_1,\, h_2)$ for the interval $h_3 < 0.8$ (Figures 14, 15). When we pass through the bifurcation value $h_3 = 0.8$ region with 24 equilibria vanishes (Fig.16) and in the interval $0.8 < h_3 < 1.048$ only four regions with 20, 16, 12 and 8 equilibria exist. The value $h_3 = 1.048$ is bifurcational (Fig.17). When we pass through the bifurcation value $h_3 = 1.048$ region with 20 equilibria vanishes.

In the interval $1.048 < h_3 < 3.264$ only three regions with 16, 12 and 8 equilibria exist.

The value $h_3 = 3.264$ is bifurcational (Fig.18). When we pass through the bifurcation value $h_3 = 3.264$ region with 16 equilibria vanishes. In the interval $3.264 < h_3 < 4$ only two regions with 12 and 8 equilibria exist near the center of coordinate system.

When the values of parameter h_3 of the gyrostatic torque are more then 4, the satellite has 8 equilibrium orientations, which correspond to four real roots of the equation (20). There are only small regions of 12 equilibria outside the center of the plane $(h_1,\, h_2)$ near h_2 axis.

Conclusions

In this work the attitude motion of a gyrostat satellite under the action of gravitational torque in a circular orbit has been investigated. The main attention was given to determination of a satellite equilibrium orientation in the orbital coordinate system. The symbolic - numerical method of determination of all satellite equilibria is suggested in general case when $A \neq B \neq C$ and $h_1 \neq 0,\ h_2 \neq 0,\ h_3 \neq 0$.

The evolution of regions with a fixed number of equilibrium orientations was investigated numerically in the plane of two parameters $(h_1,\ h_2)$ for different values of parameters ν and h_3.

It was shown that in general case the gyrostat satellite subjected to gravitational torques can have no more than 24 and no less than 8 equilibrium orientations in a circular orbit.

The obtained results may be used in the stage of preliminary projecting of the satellite with gravitational stabilization system.

References

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2. Sarychev V.A., Gutnik S.A. Investigations of the relative equilibria of a gyrostat satellite // *Keldysh Institute of Applied Mathematics Preprint.* 1990. №84. 31p.
3. Sarychev V.A., Dynamics of an axisymmetric gyrostat satellite under the action of gravitational moment // *Cosmic Research*. 2010. V.48. №2. P.188-193.

Table 1.

Bifurcation points

Regions of equilibria / v	24/20	20/16	16/12	12/8
0.01	h_1=0.00001 h_2=0.00001 h_3=0.990	h_1=0.0065 h_2=0.0001 h_3=0.999	h_1=0.0001 h_2=0.0006 h_3=3.959	h_1=0.0001 h_2=0.0001 h_3=4.0
0.1	h_1=0.00001 h_2=0.00001 h_3=0.900	h_1=0.0740 h_2=0.0001 h_3=1.021	h_1=0.0001 h_2=0.0682 h_3=3.610	h_1=0.0001 h_2=0.0001 h_3=4.0
0.2	h_1=0.00001 h_2=0.00001 h_3=0.800	h_1=0.1400 h_2=0.0001 h_3=1.048	h_1=0.0001 h_2=0.1809 h_3=3.264	h_1=0.0001 h_2=0.0001 h_3=4.0
0.3	h_1=0.00001 h_2=0.0001 h_3=0.700	h_1=0.197 h_2=0.0001 h_3=1.082	h_1=0.0001 h_2=0.3154 h_3=2.950	h_1=0.0001 h_2=0.0001 h_3=4.0
0.4	h_1=0.00001 h_2=0.00001 h_3=0.600	h_1=0.231 h_2=0.0001 h_3=1.124	h_1=0.0001 h_2=0.4603 h_3=2.669	h_1=0.0001 h_2=0.0001 h_3=4.0
0.5	h_1=0.00001 h_2=0.0001 h_3=0.500	h_1=0.224 h_2=0.0001 h_3=1.182	h_1=0.0001 h_2=0.6132 h_3=2.412	h_1=0.0001 h_2=0.0001 h_3=4.0
0.6	h_1=0.00001 h_2=0.00001 h_3=0.400	h_1=0.1236 h_2=0.0001 h_3=1.186	h_1=0.0001 h_2=0.7778 h_3=2.167	h_1=0.0001 h_2=0.0001 h_3=4.0
0.7	h_1=0.00001 h_2=0.00001 h_3=0.300	h_1=0.0144 h_2=0.0001 h_3=1.105	h_1=0.0001 h_2=0.9675 h_3=1.915	h_1=0.0001 h_2=0.0001 h_3=4.0
0.8	h_1=0.00001 h_2=0.0001 h_3=0.200	h_1=0.0001 h_2=0.0155 h_3=0.909	h_1=0.0001 h_2=1.2107 h_3=1.629	h_1=0.0001 h_2=0.0001 h_3=4.0
0.9	h_1=0.00001 h_2=0.00001 h_3=0.100	h_1=0.0004 h_2=0.0989 h_3=0.676	h_1=0.0001 h_2=1.5915 h_3=1.245	h_1=0.0001 h_2=0.0001 h_3=4.0
0.99	h_1=0.00001 h_2=0.00001 h_3=0.010	h_1=0.0001 h_2=0.2521 h_3=0.168	h_1=0.0030 h_2=0.0001 h_3=0.997	h_1=0.0001 h_2=0.0001 h_3=4.0

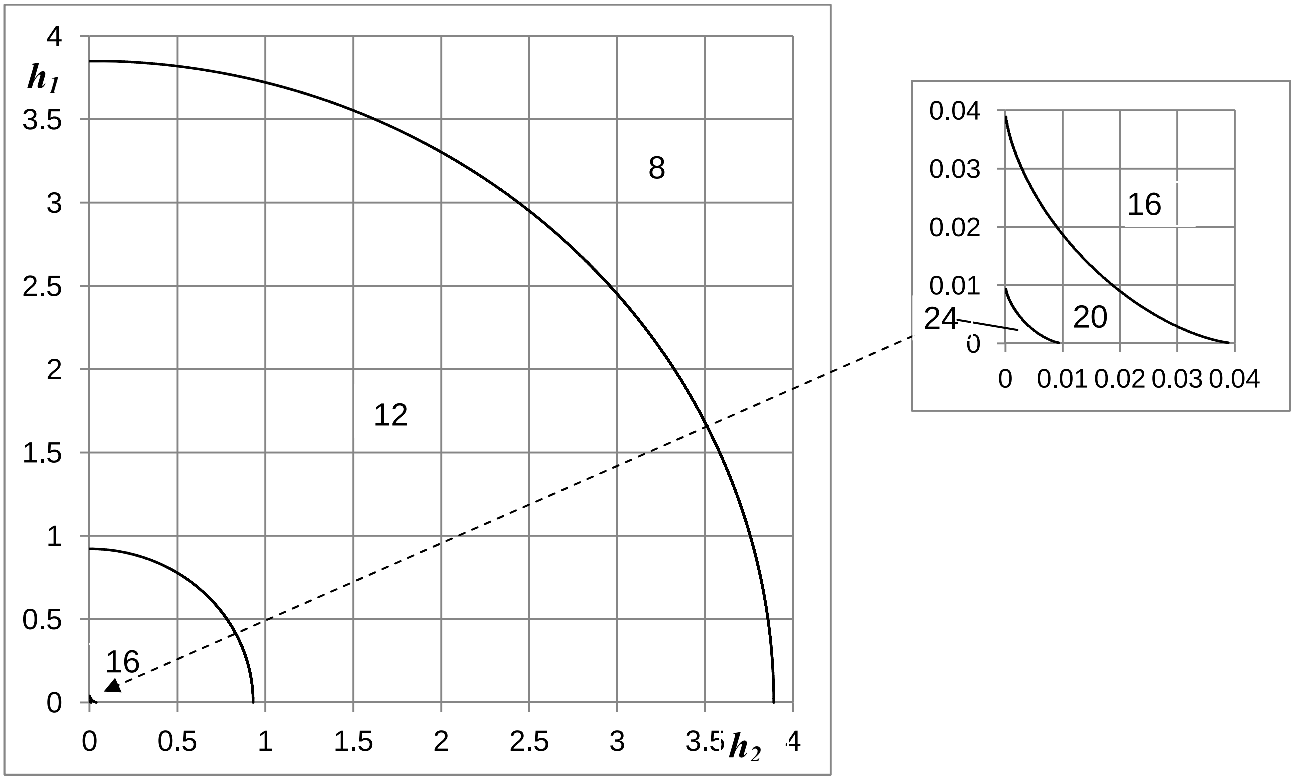


Fig. 1. v=0.01, h_3 = 0.01

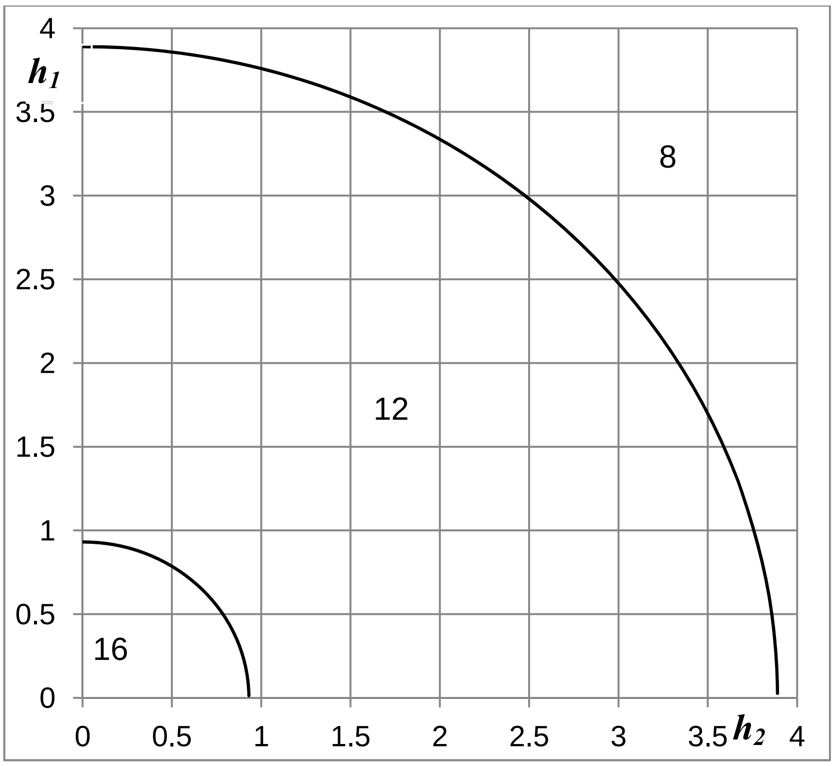


Fig. 2. v=0, h_3 = 0.01

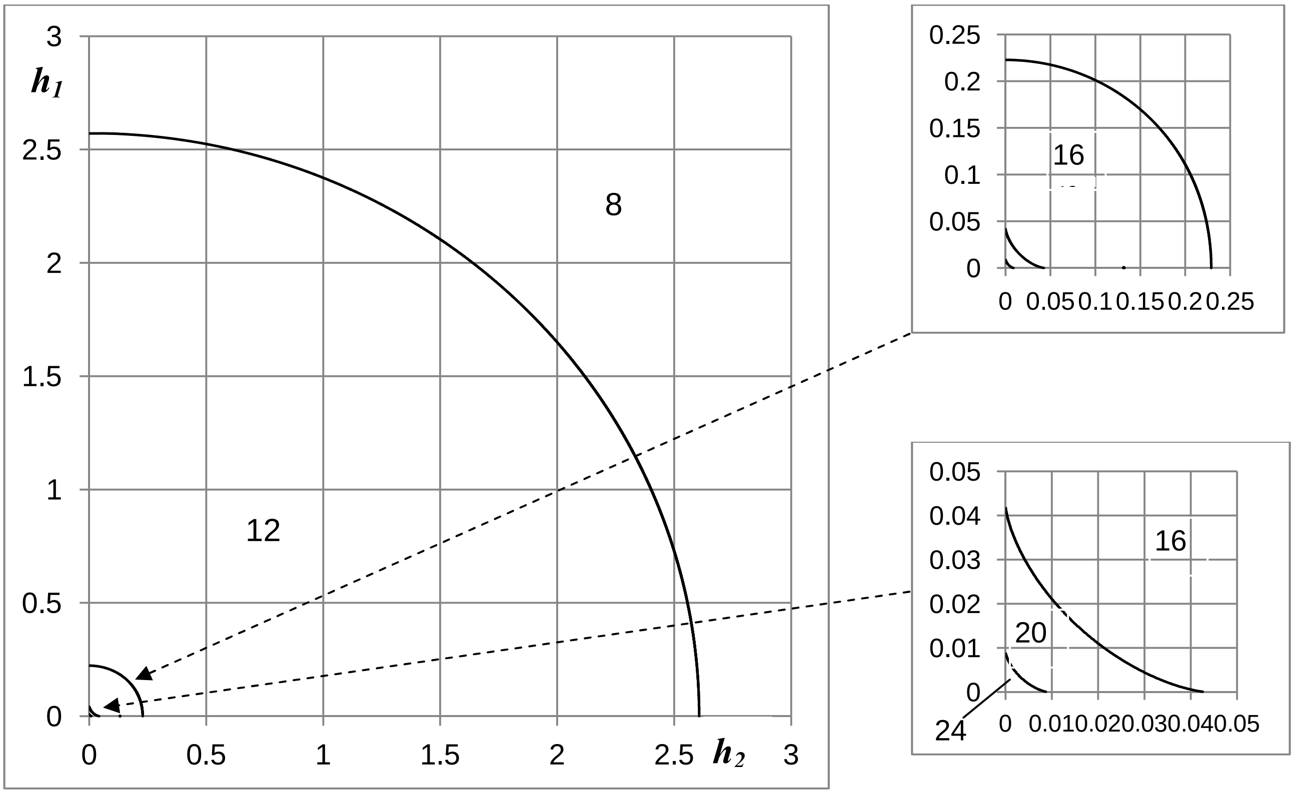


Fig. 3. v=0.01, h_3 = 0.495

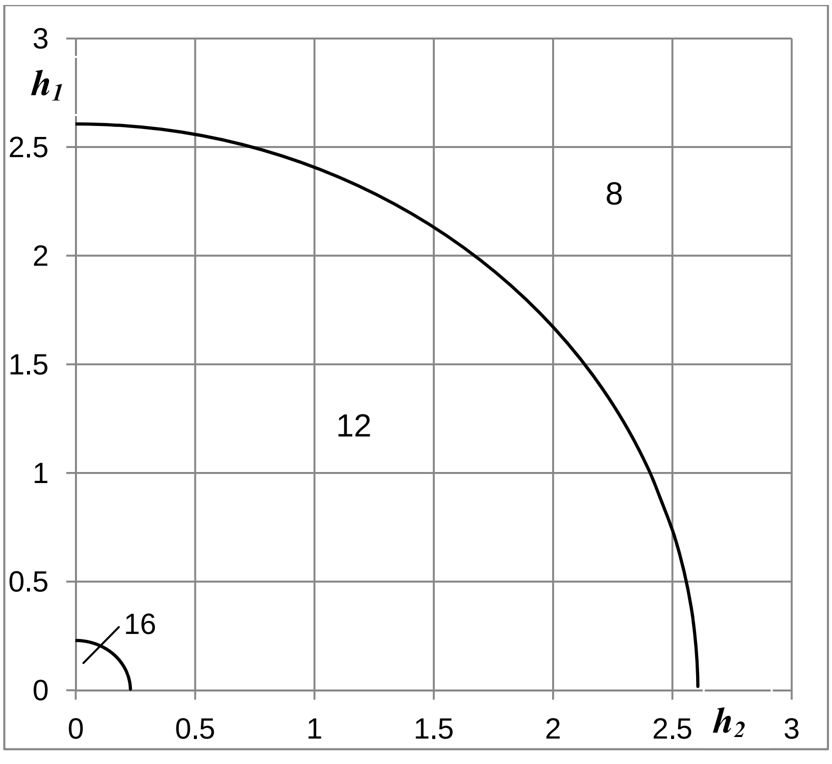


Fig. 4. v=0, h_3 = 0.495

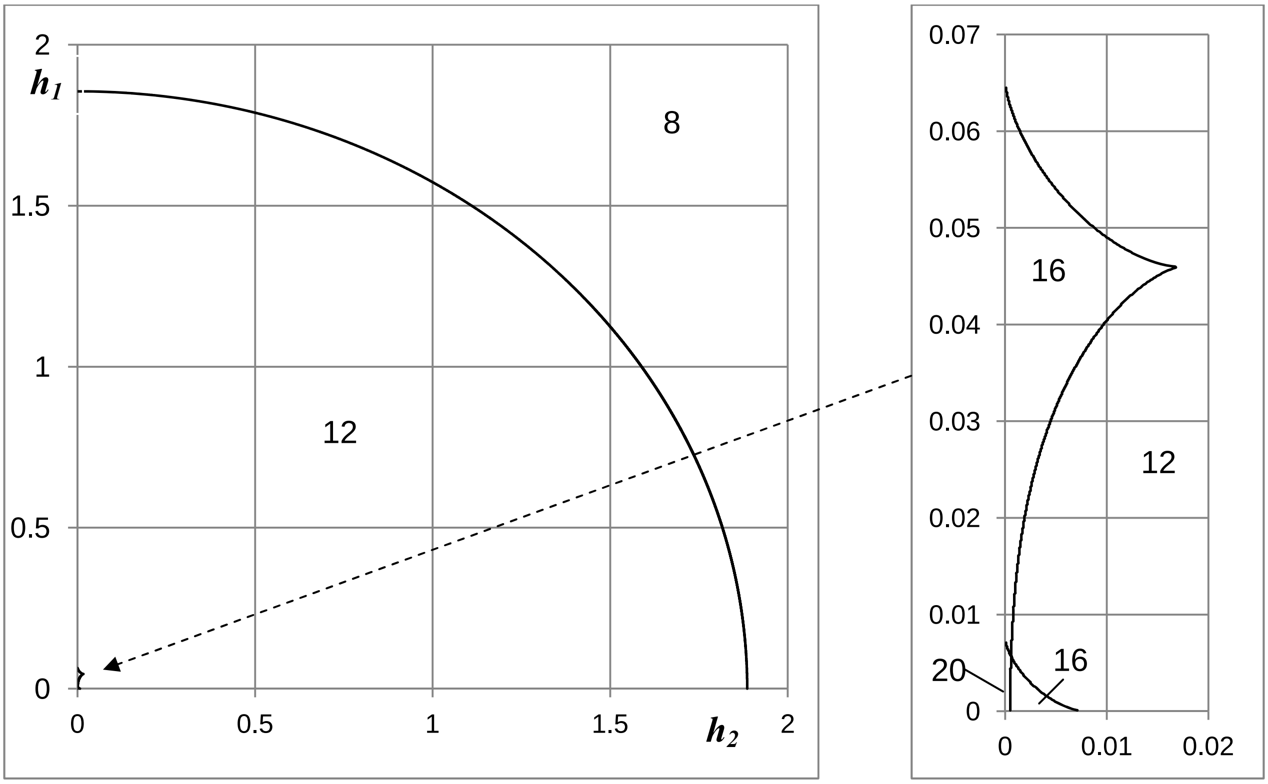


Fig. 5. v=0.01, h_3 = 0.99

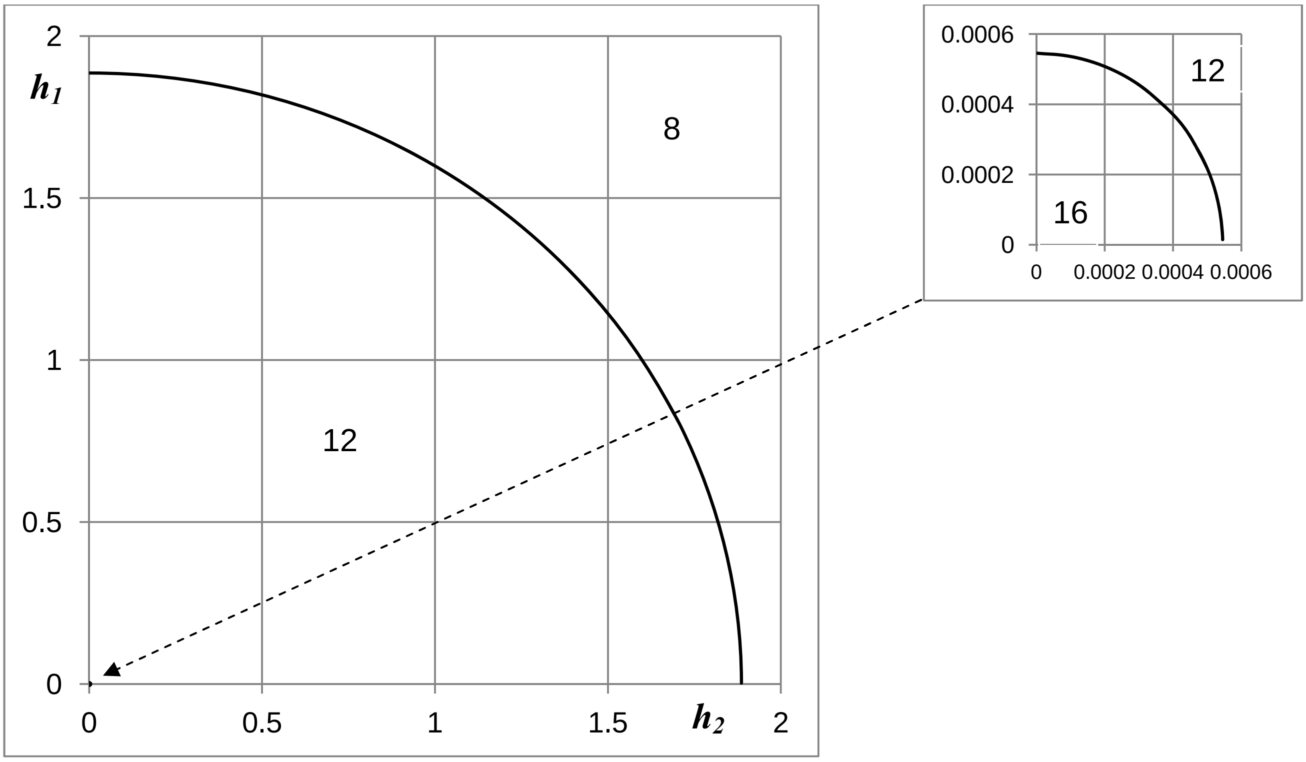


Fig. 6. v=0, h_3 = 0.99

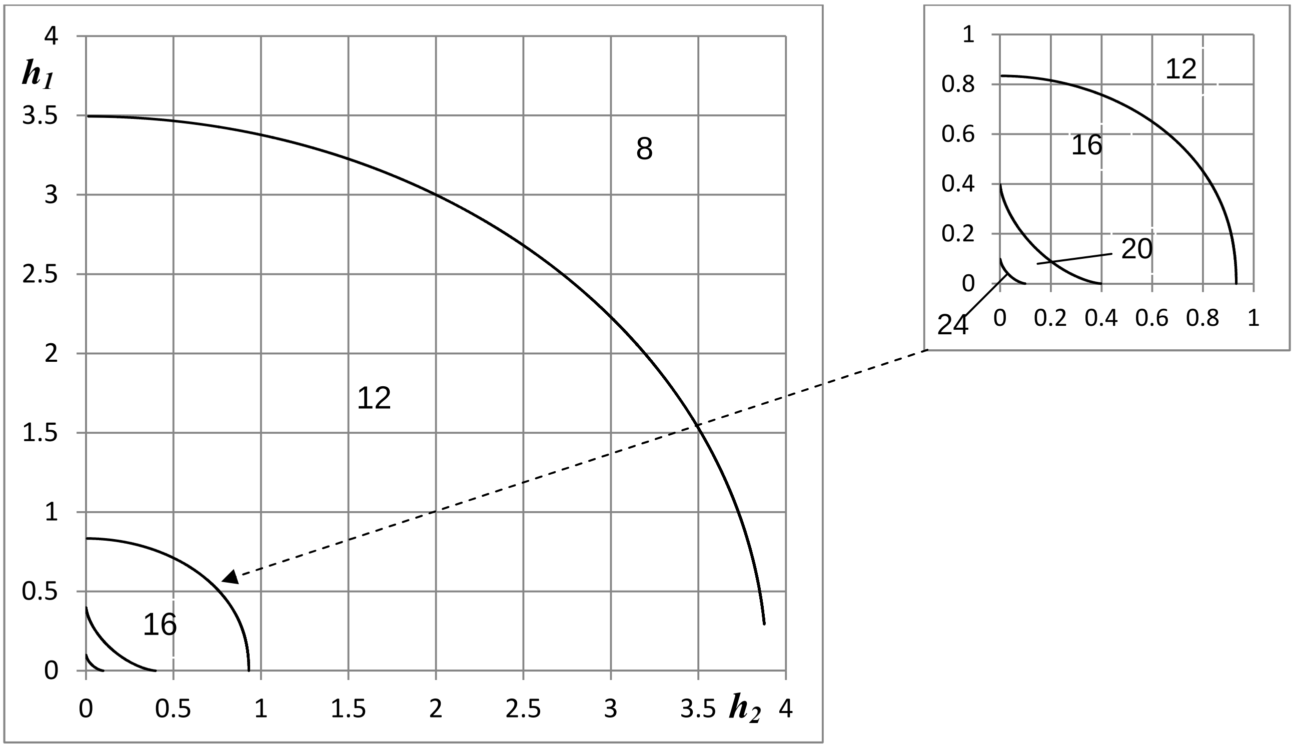


Fig. 7. v=0.1, h_3 = 0.01

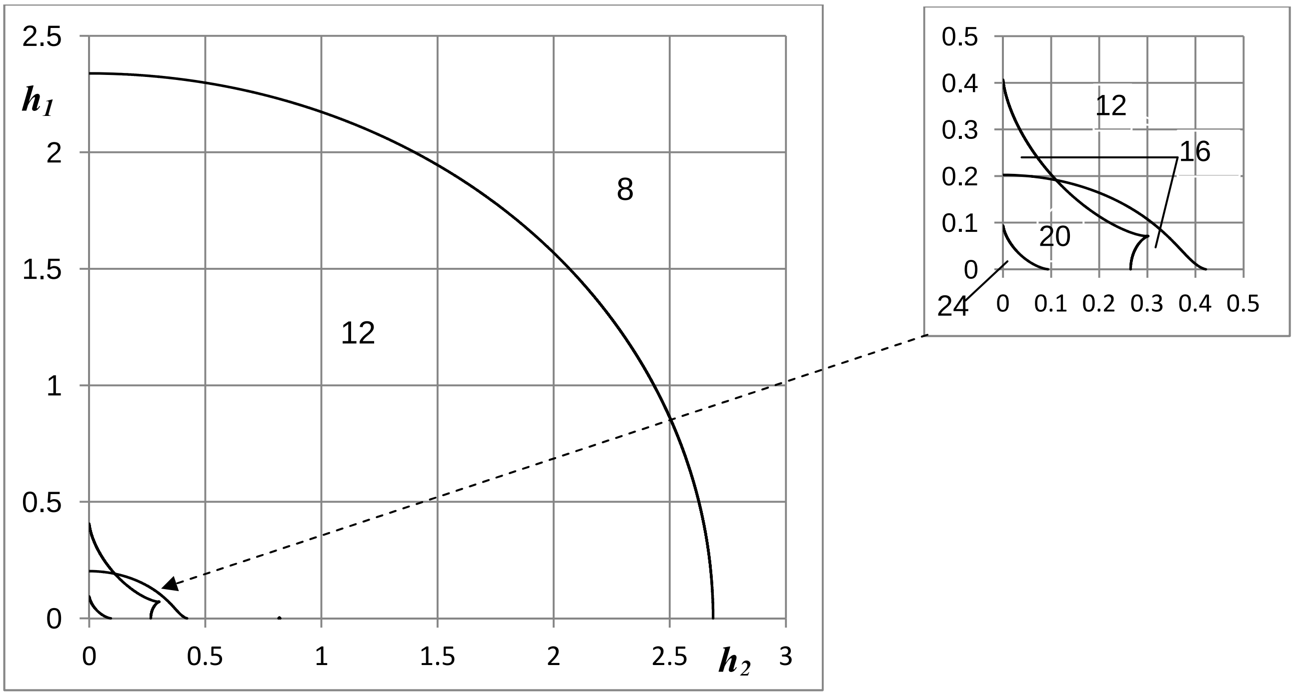


Fig. 8. v=0.1, h_3 = 0.495

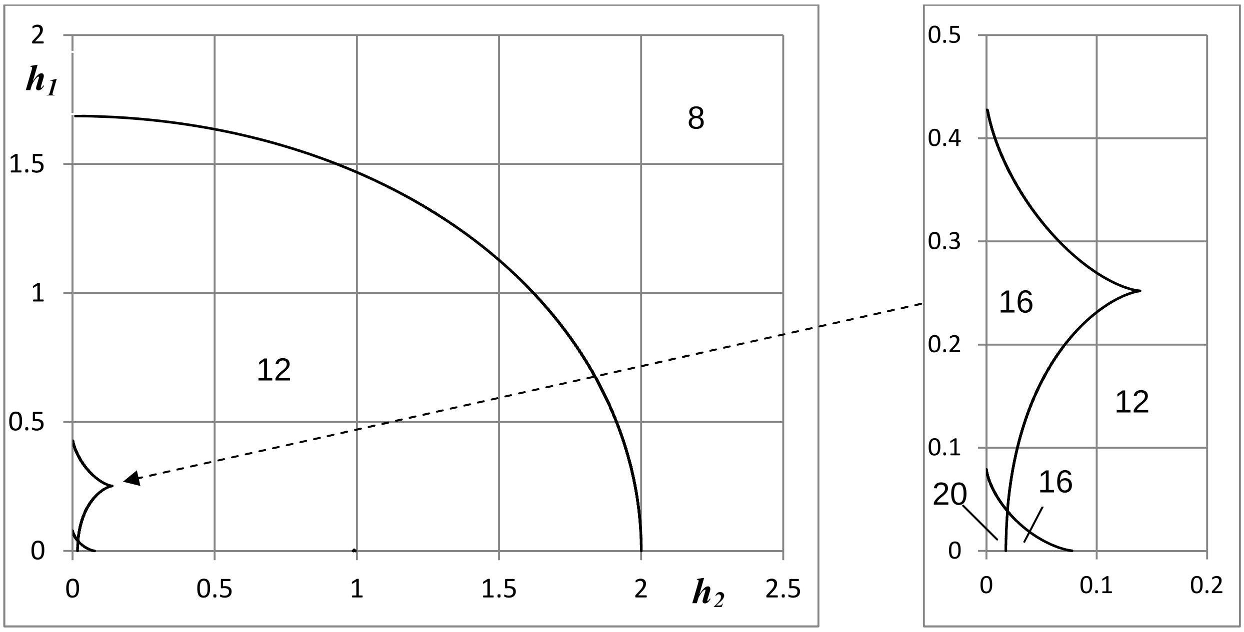


Fig. 9. v=0.1, h_3 = 0.9

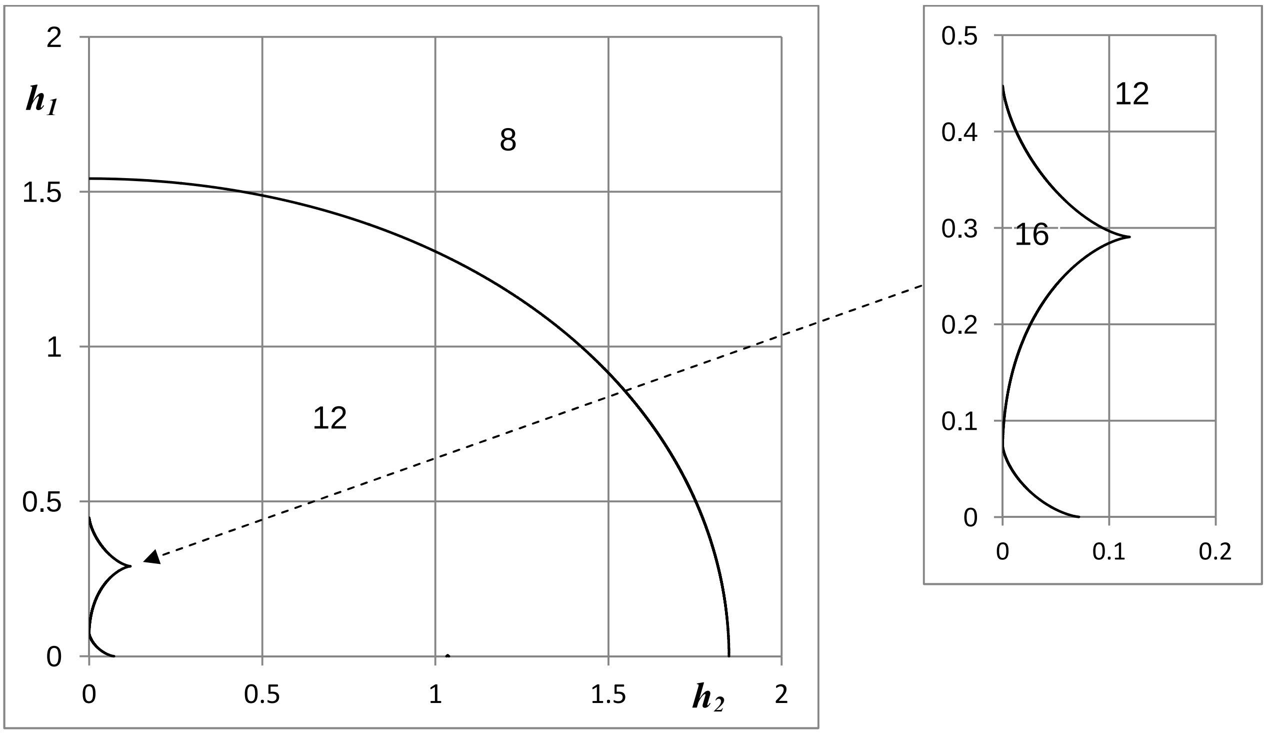


Fig. 10. v=0.1, h_3 = 1.021

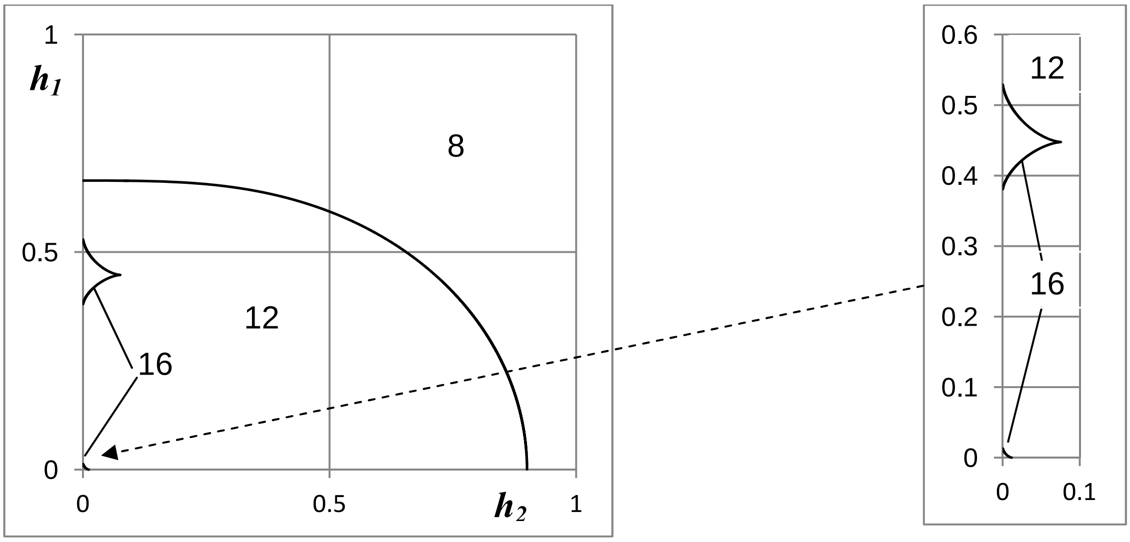


Fig. 11. v=0.1, h_3 = 2.0

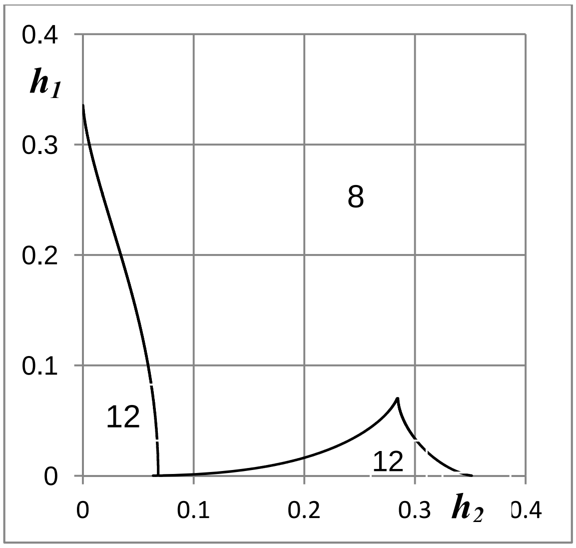


Fig. 12. v=0.1, h_3 = 3.61

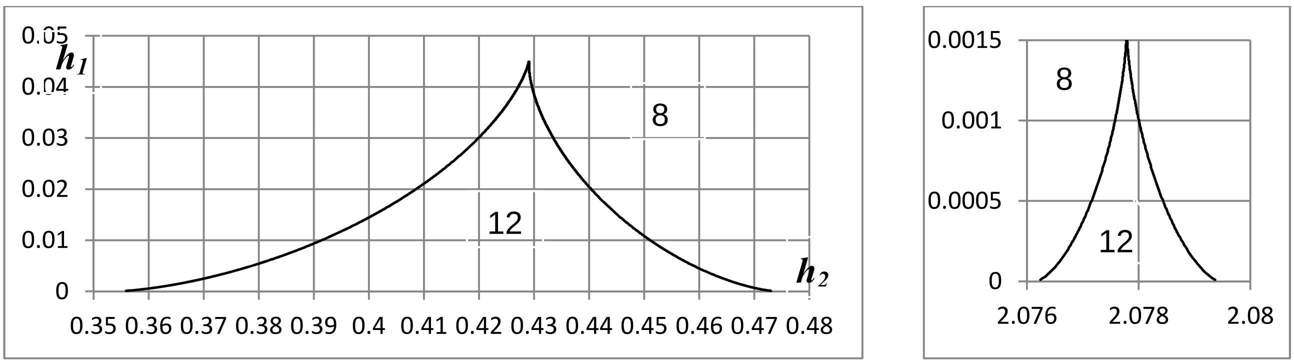


Fig. 13. v=0.1, h_3 = 4.0

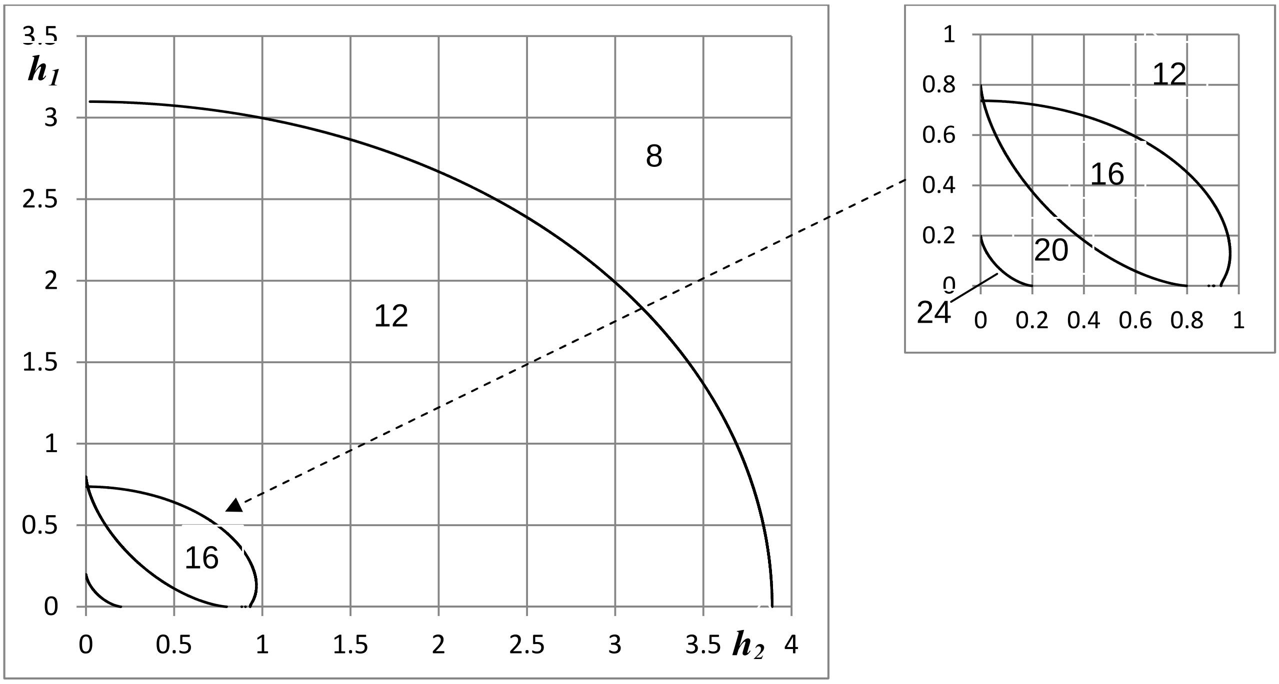


Fig. 14. v=0.2, h_3 = 0.01

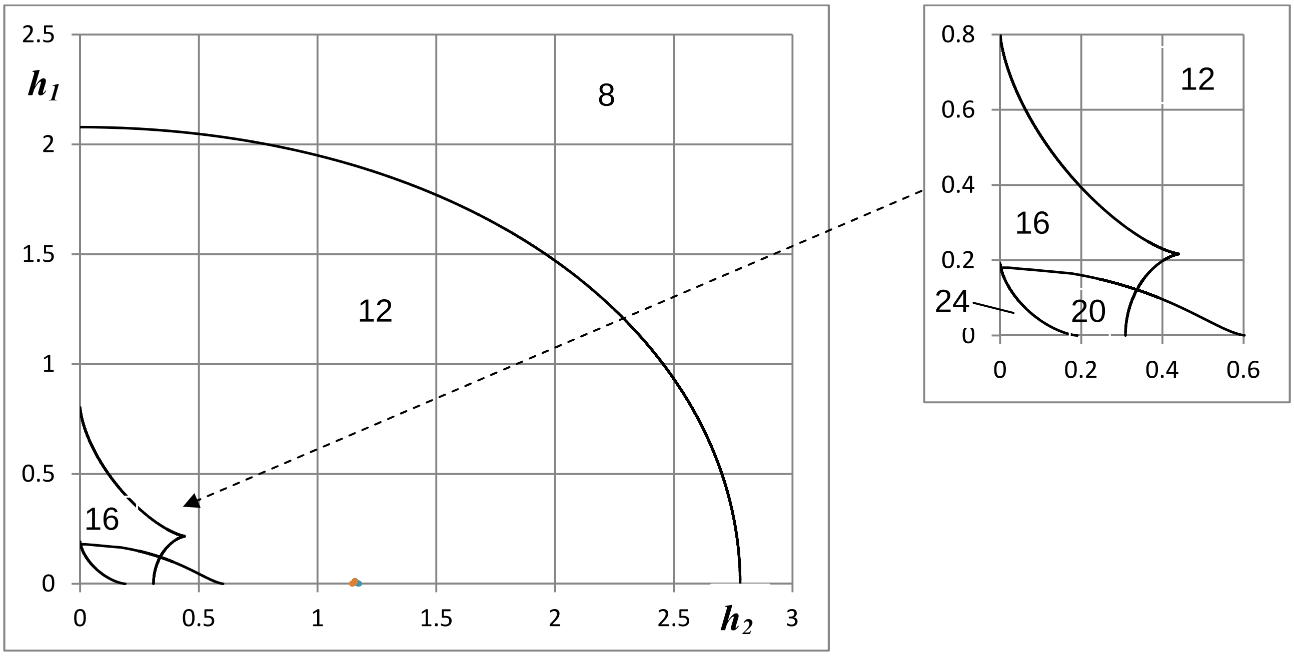


Fig. 15. v=0.2, h_3 = 0.4

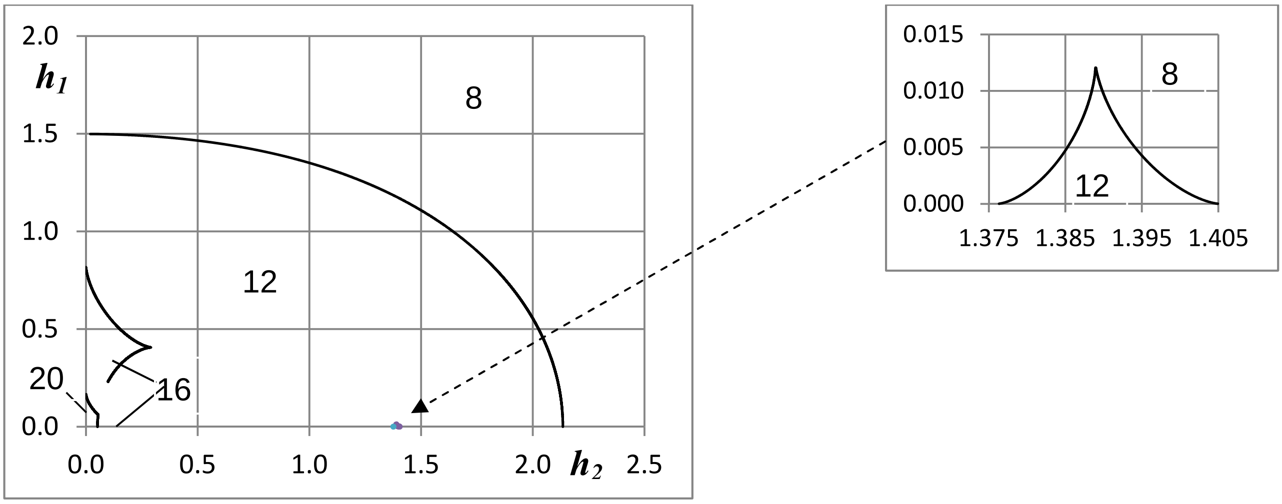


Fig. 16. v=0.2, h_3 = 0.8

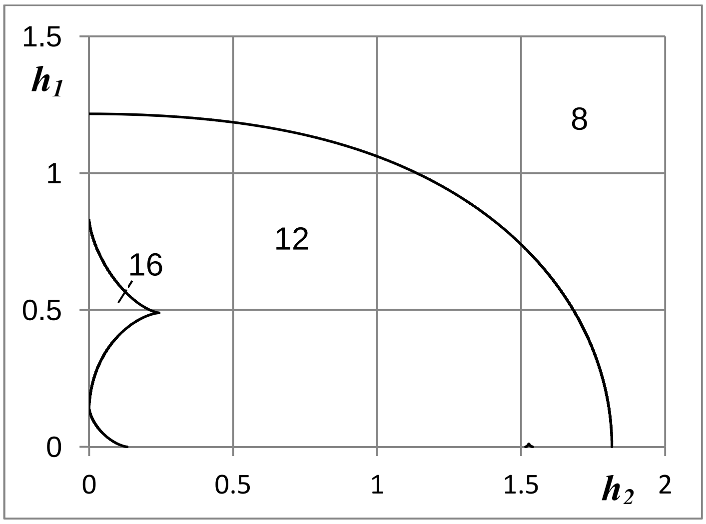


Fig. 17. v=0.2, h_3 = 1.048

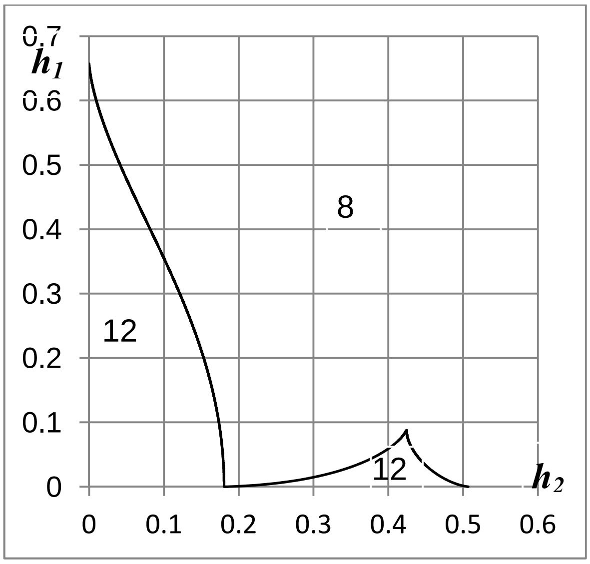


Fig. 18. v=0.2, h_3 = 3.264

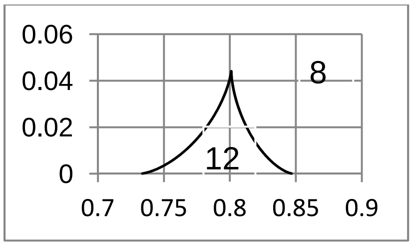


Fig. 19. v=0.2, h_3 = 4

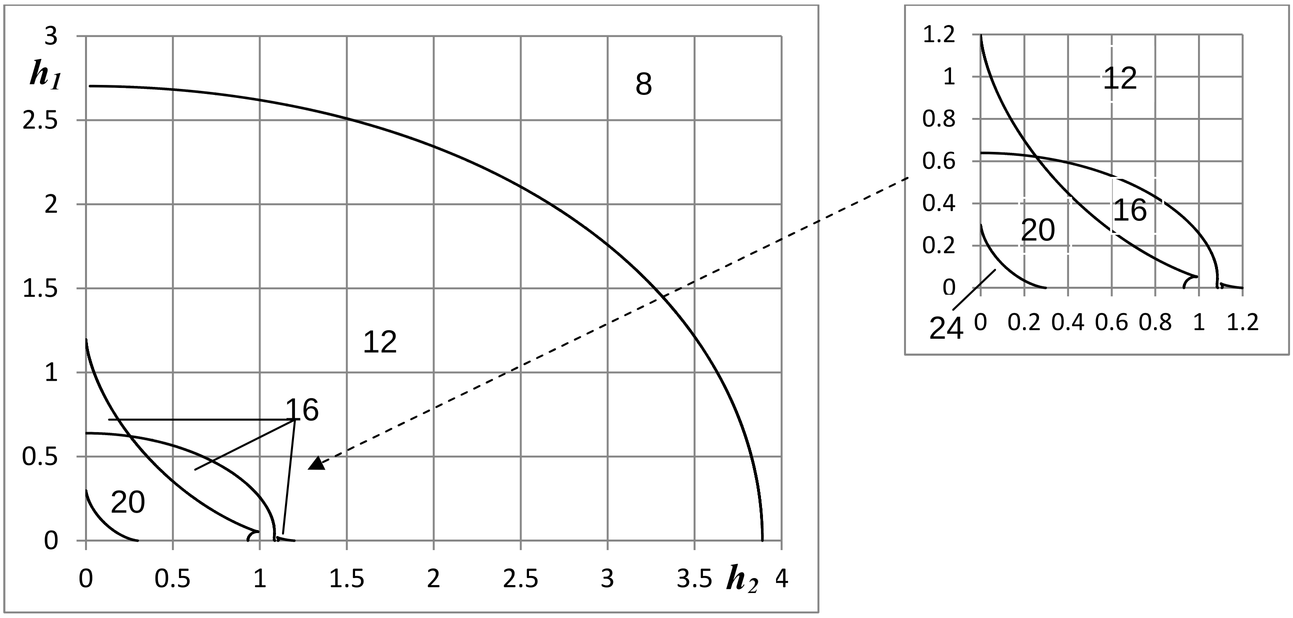


Fig. 20. ν=0.3, h_3 = 0.01

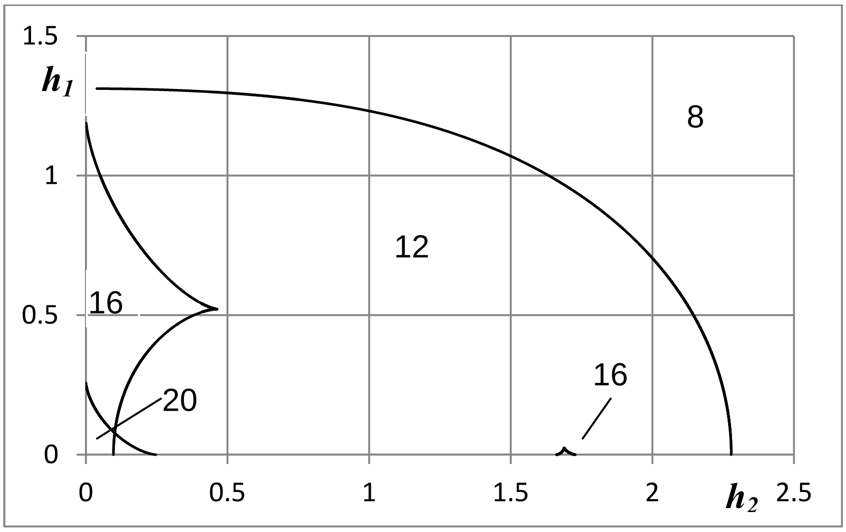


Fig. 21. ν=0.3, h_3 = 0.7

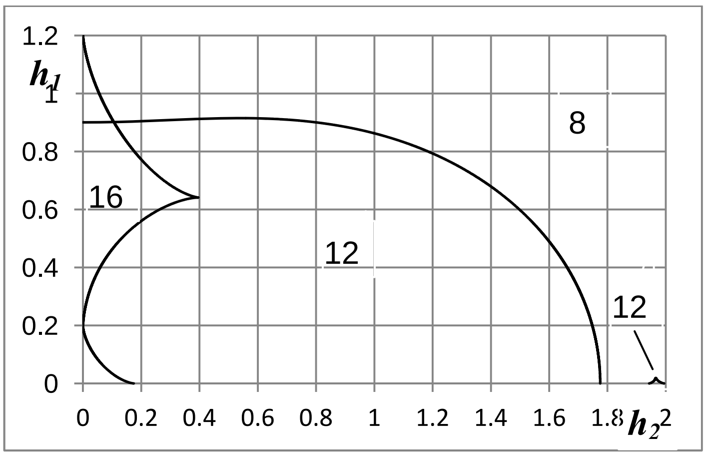


Fig. 22. ν=0.3, h_3 = 1.082

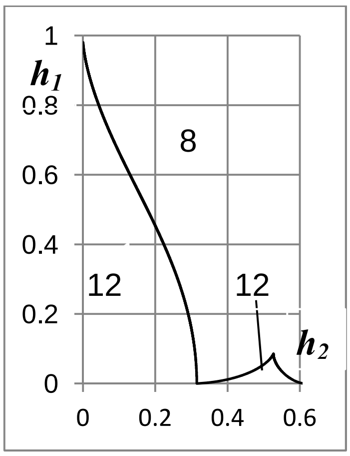


Fig. 23. ν=0.3, h_3 = 2.95

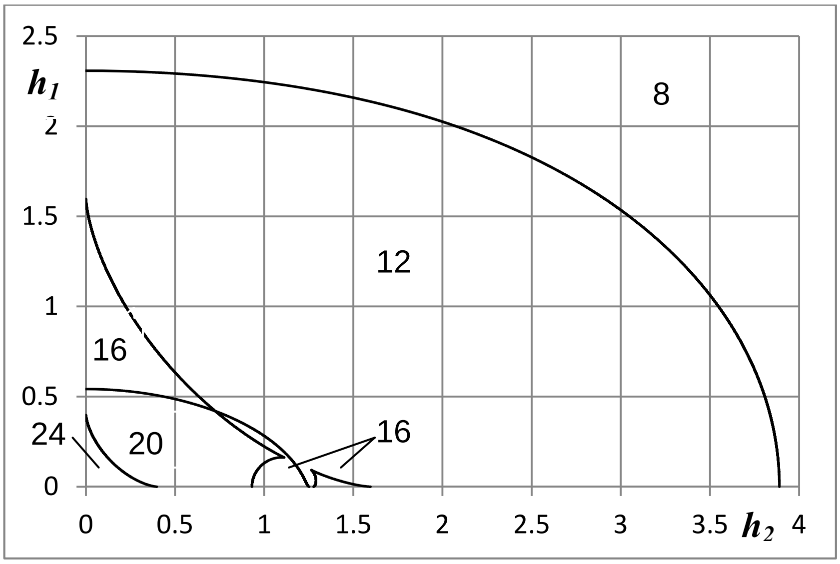


Fig. 24. v=0.4, h_3 = 0.01

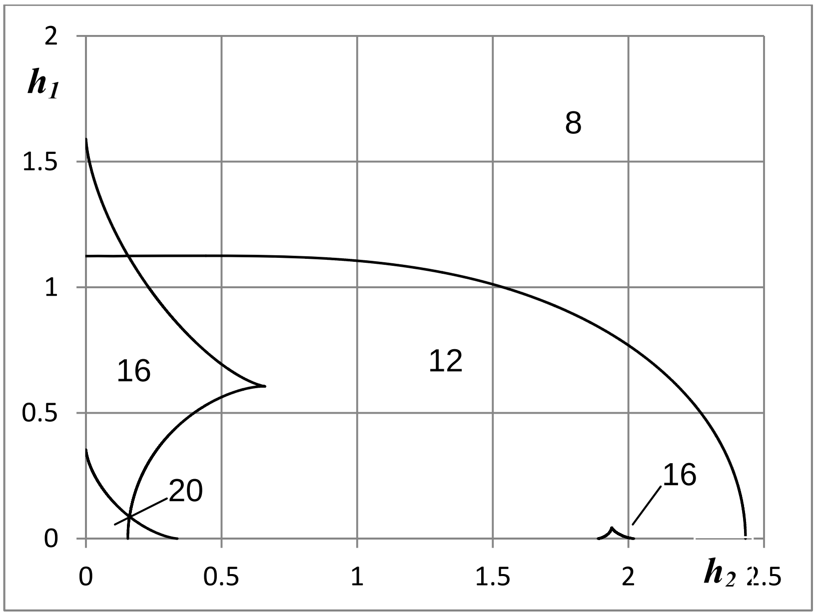


Fig. 25. v=0.4, h_3 = 0.6

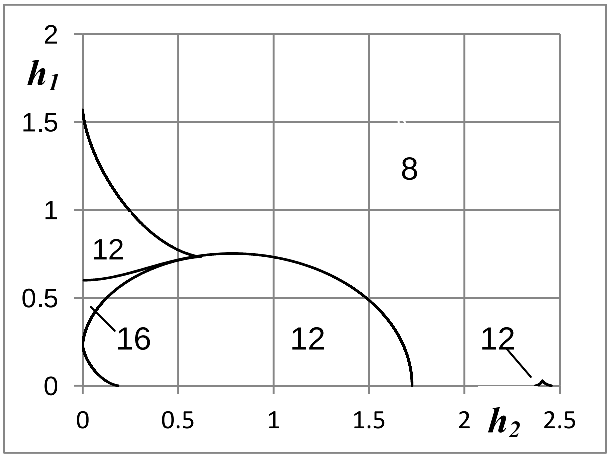


Fig. 26. v=0.4, h_3 = 1.124

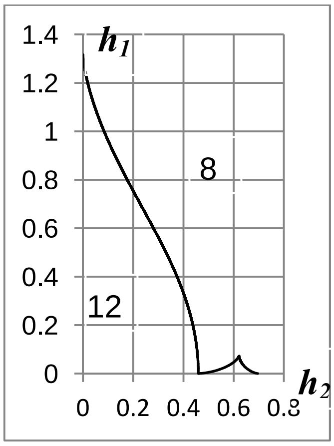


Fig. 27. v=0.4, h_3 =2.7

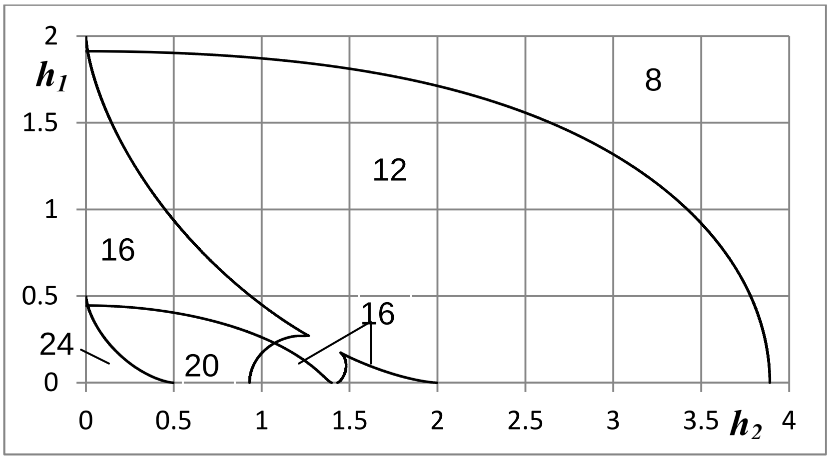


Fig. 28. v=0.5, h_3 = 0.01

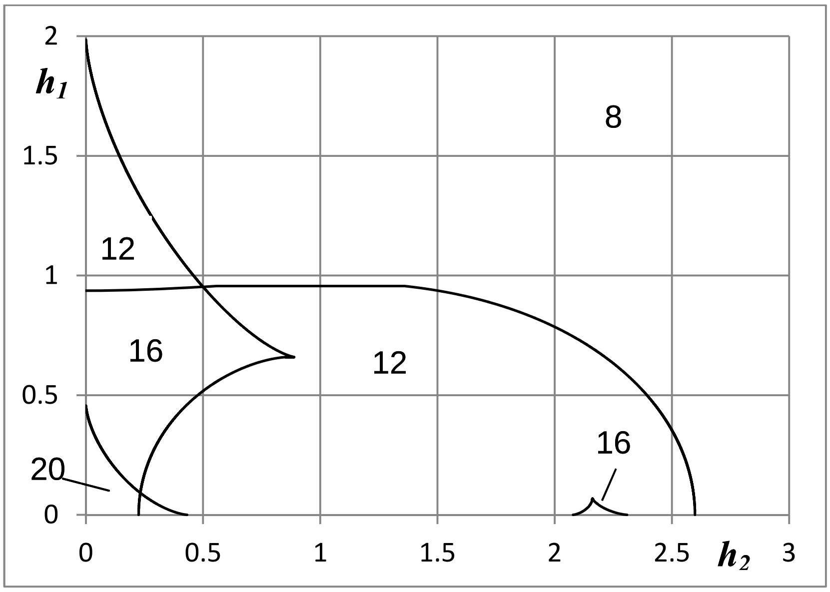


Fig. 29. v=0.5, h_3 = 0.5

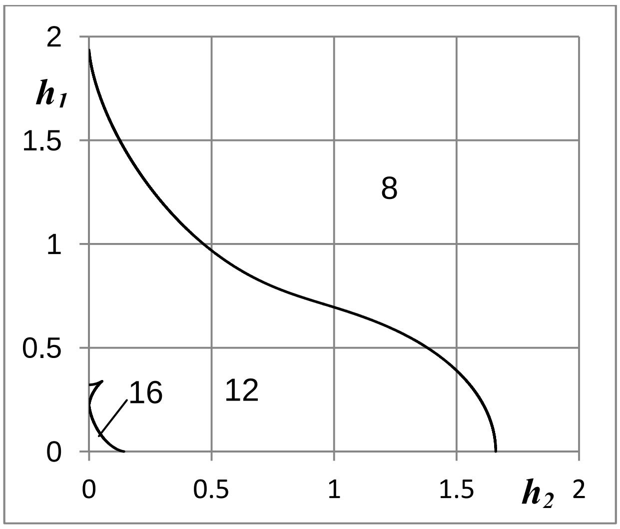


Fig. 30. v=0.5, h_3 =1.182

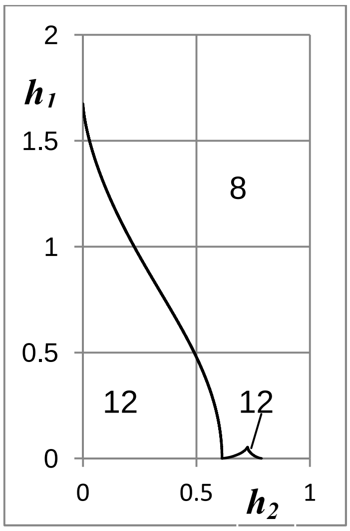


Fig.31. v=0.5, h_3=2.412

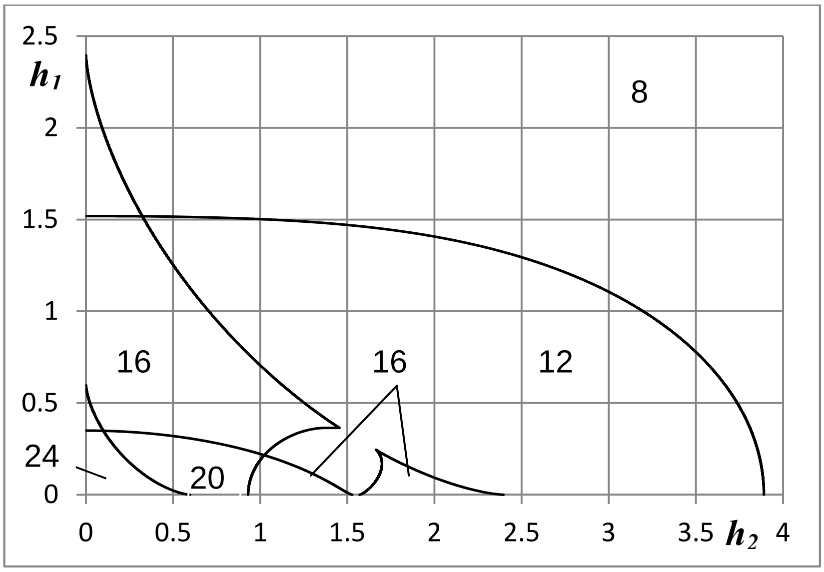


Fig. 32. v=0.6, h_3 = 0.01

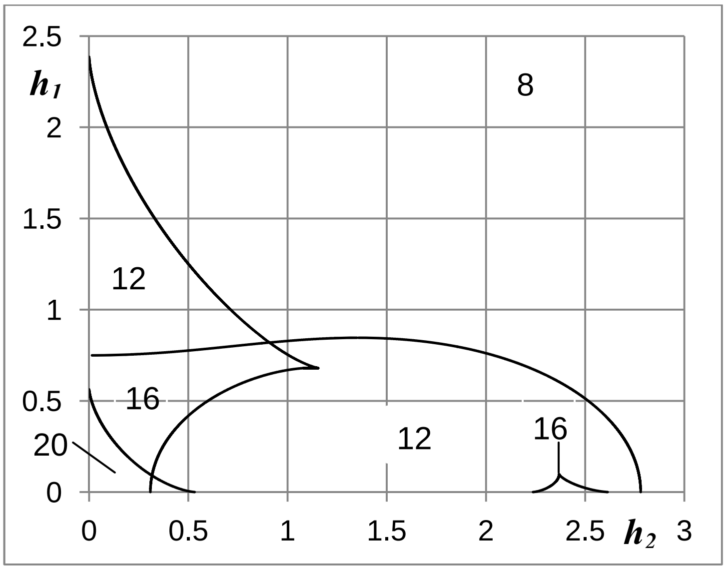


Fig. 33. v=0.6, h_3 = 0.4

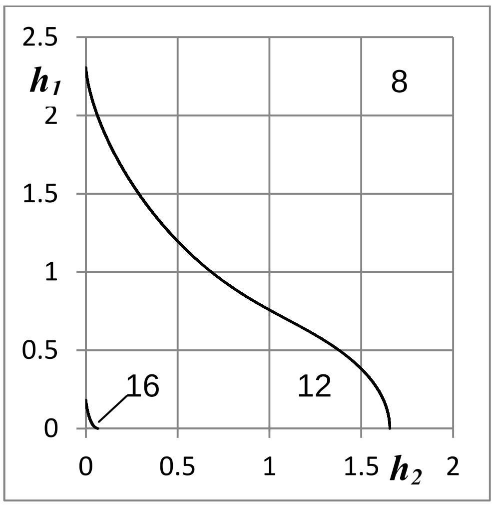


Fig. 34. v=0.6, h_3 = 1.186

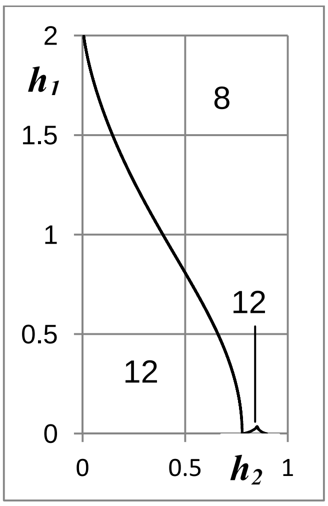


Fig. 35. v=0.6, h_3=2.167

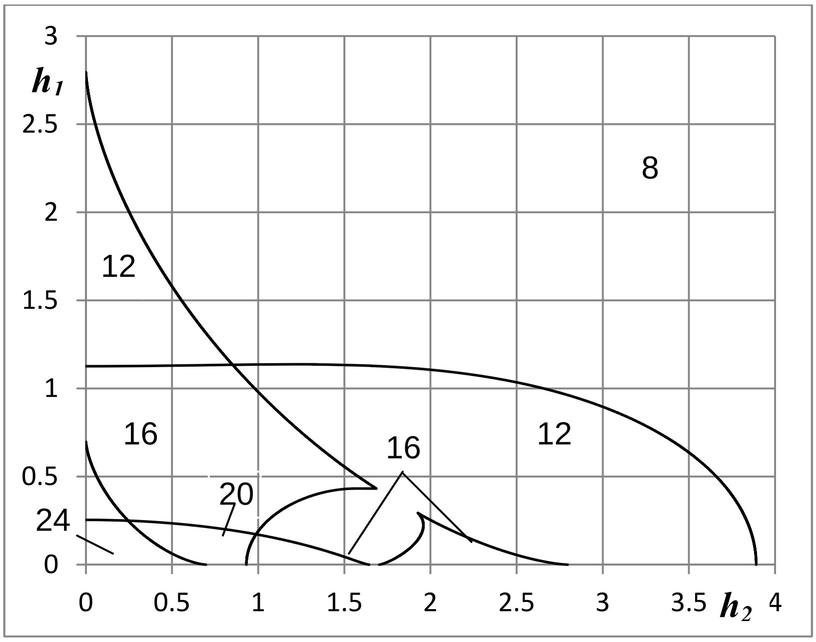


Fig. 36. v=0.7, h_3 = 0.01

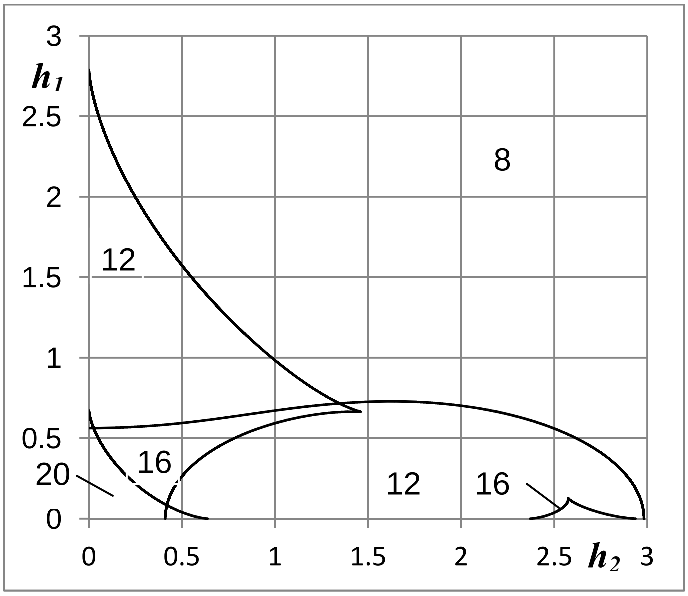
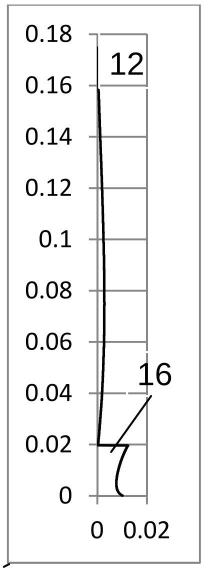


Fig. 37. v=0.7, h_3 = 0.3



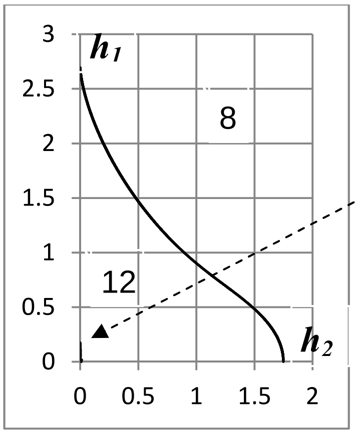


Fig. 38. v=0.7, h_3 = 1.105

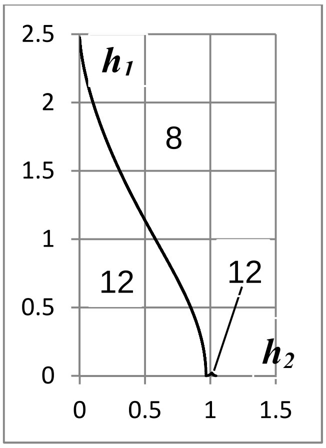


Fig. 39. v=0.7, h_3=1.915

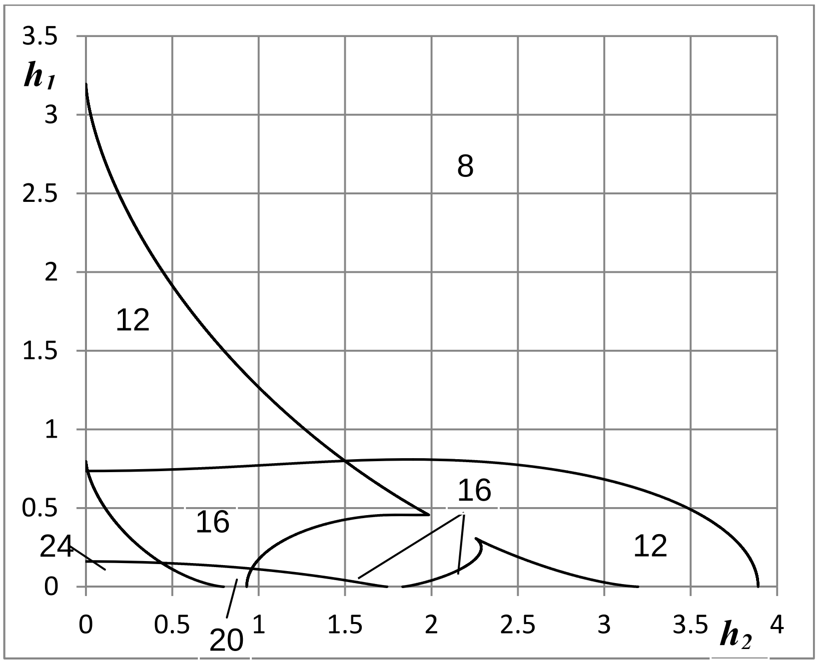


Fig. 40. v=0.8, h_3 = 0.01

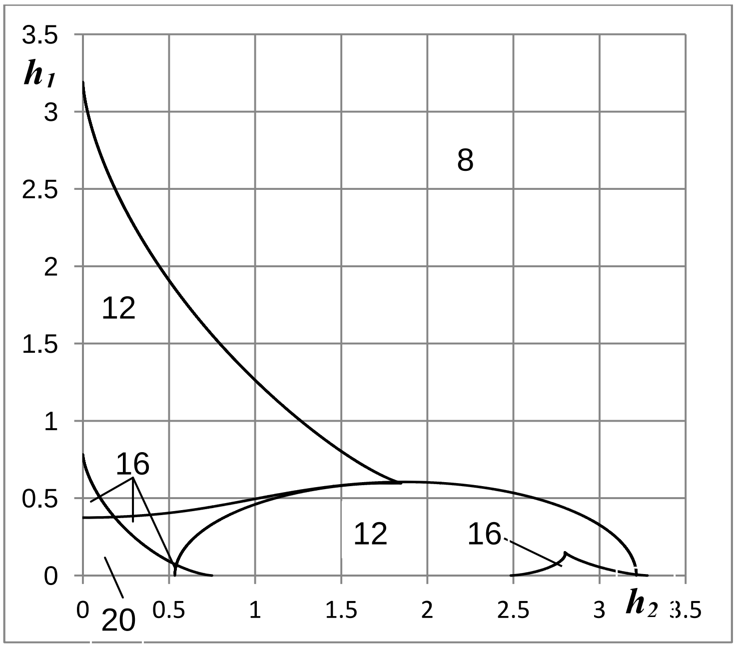


Fig. 41. v=0.8, h_3 = 0.2

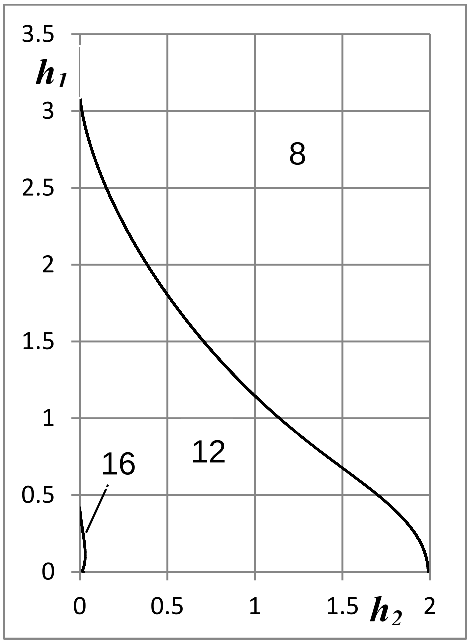


Fig. 42. v=0.8, h_3= 0.909

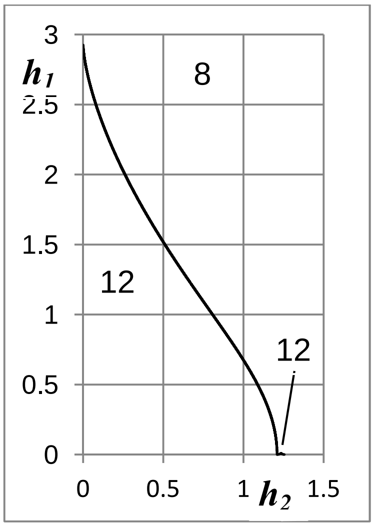


Fig. 43. v=0.8, h_3=1.629

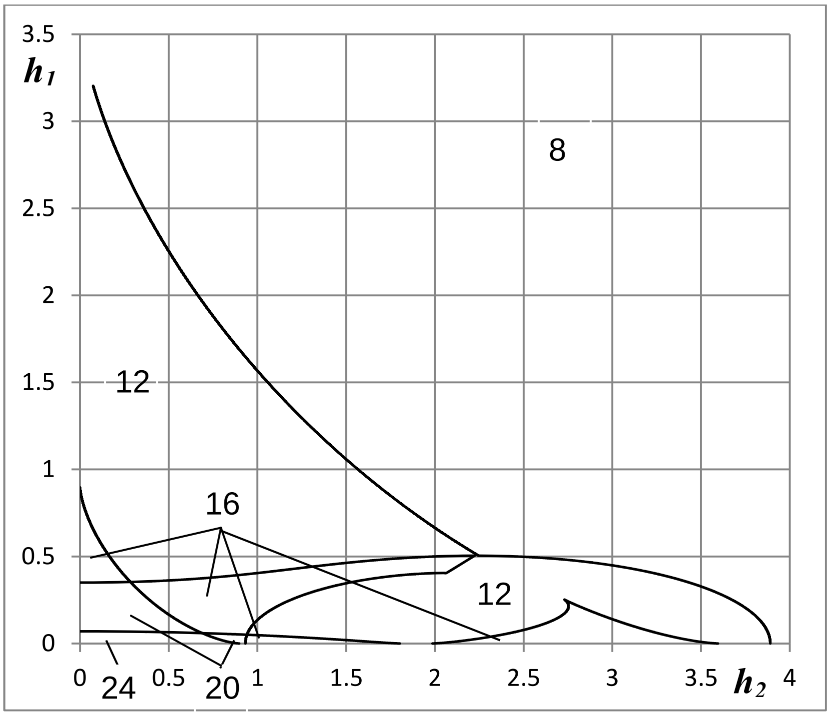


Fig. 44. v=0.9, h_3= 0.01

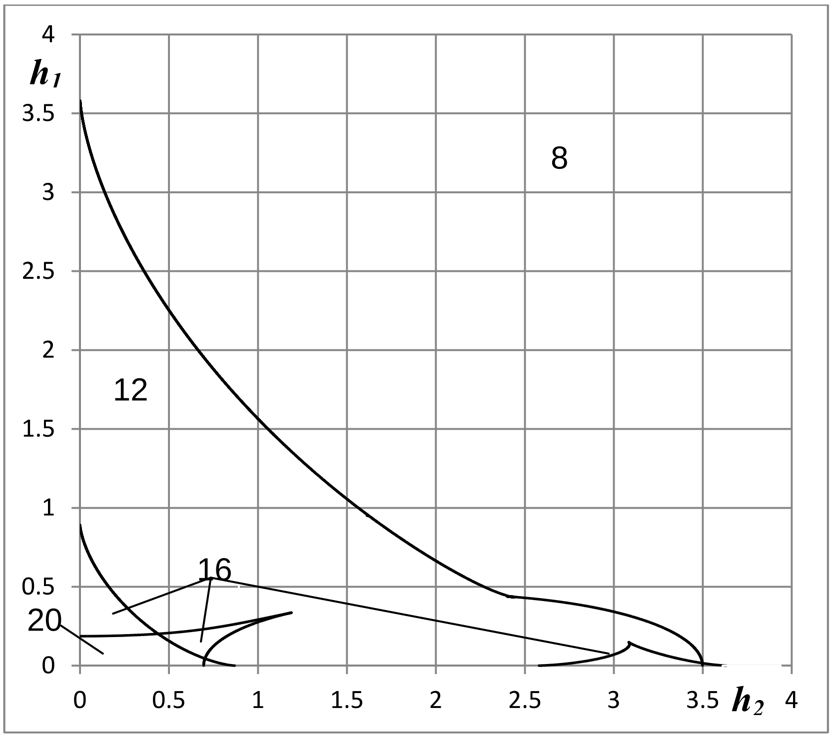


Fig. 45. v=0.9, h_3 = 0.1

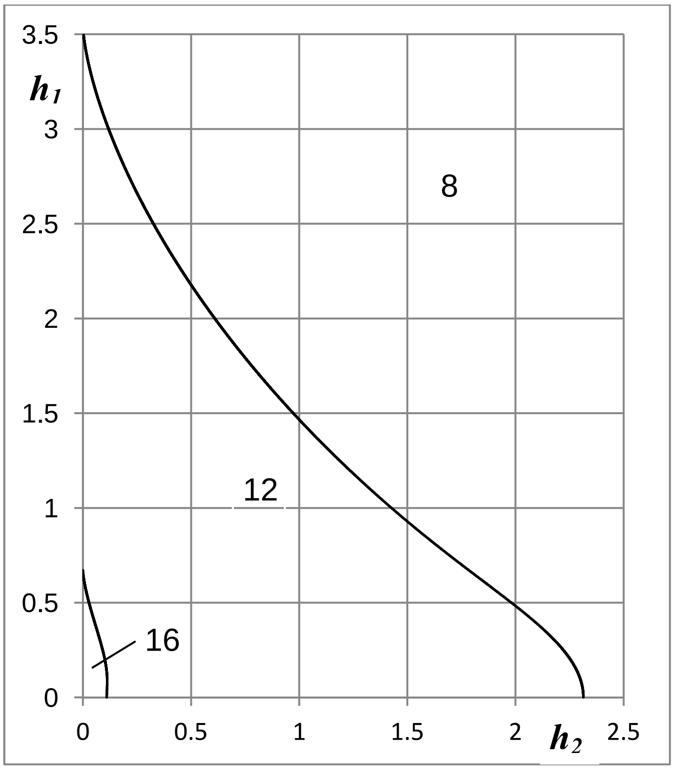


Fig. 46. v=0.9, h_3 = 0.676

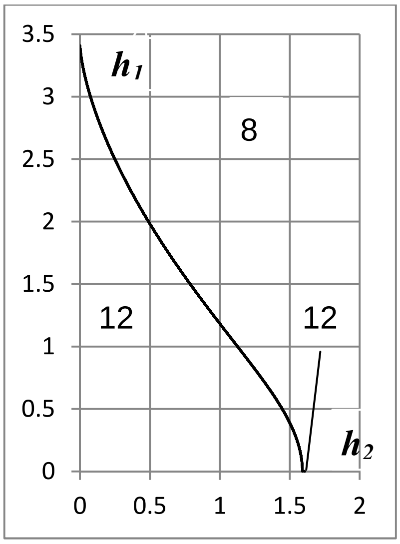


Fig. 47. v=0.9, h_3=1.245

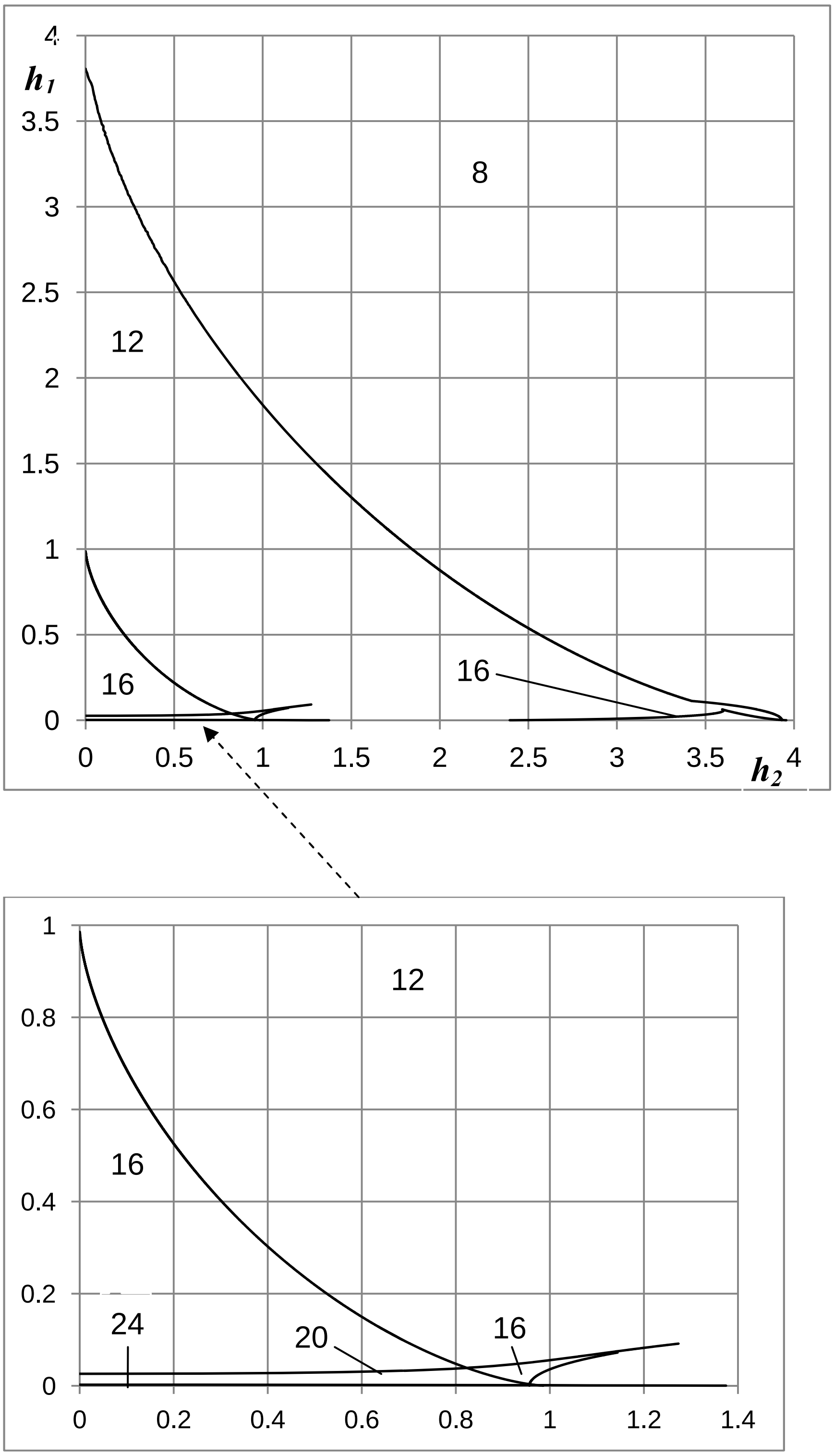


Fig.48. v=0.99, h_3 = 0.005

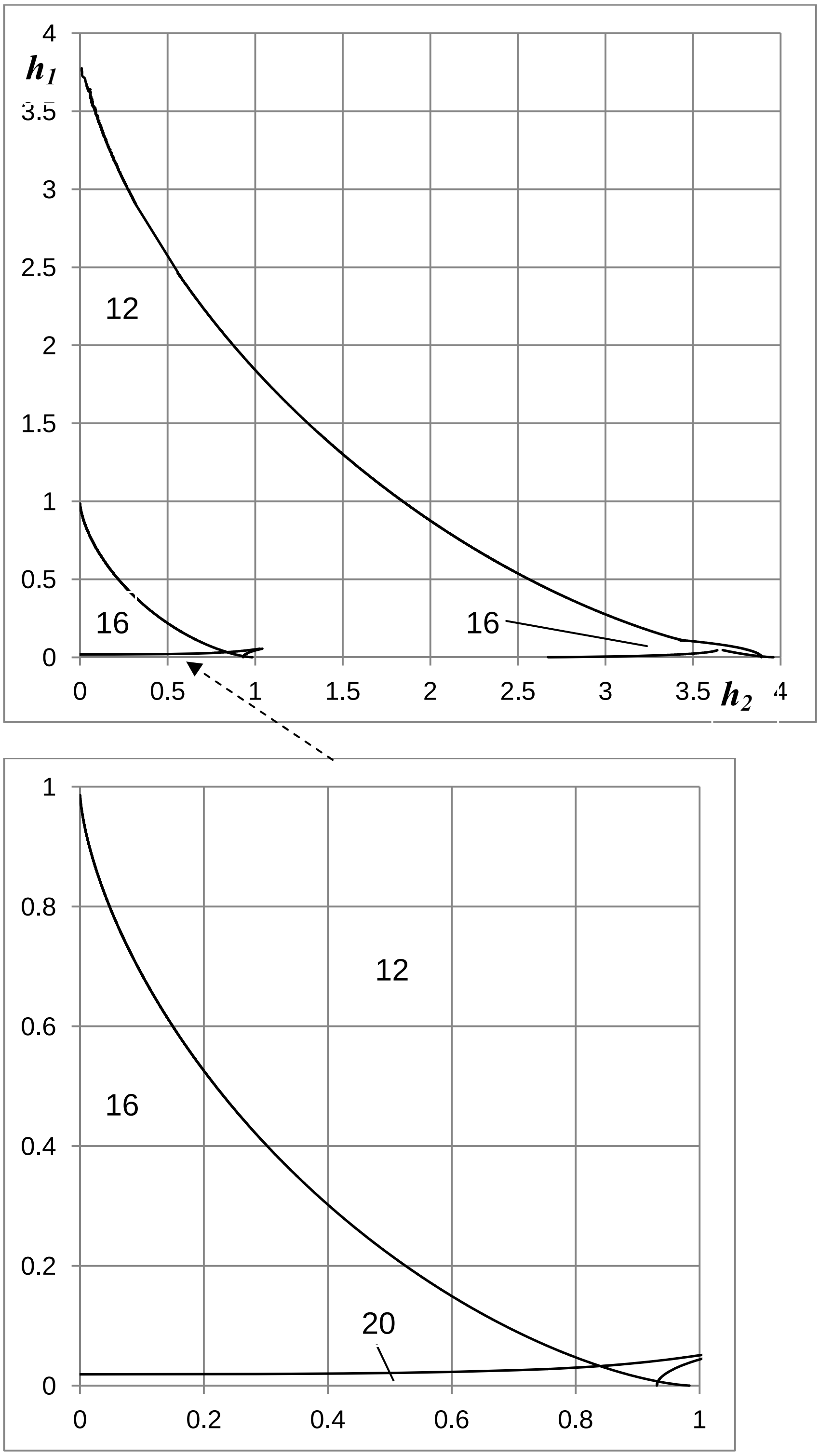


Fig. 49. v=0.99, h_3 = 0.01

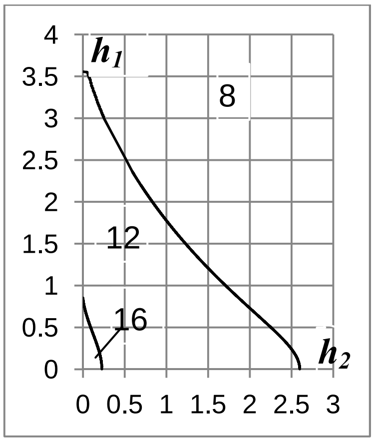


Fig. 50. v=0.99, h_3 = 0.5

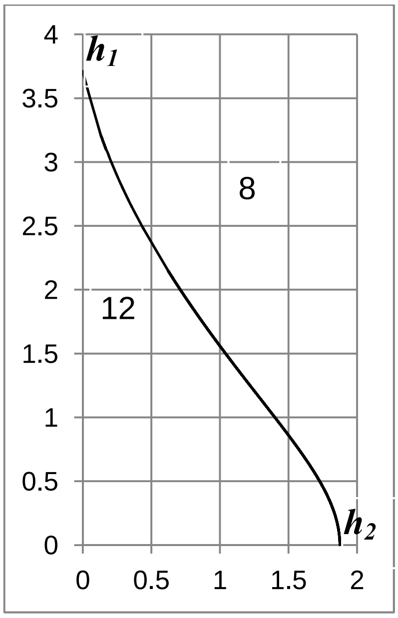


Fig. 51. v=0.99, h_3 =1.0

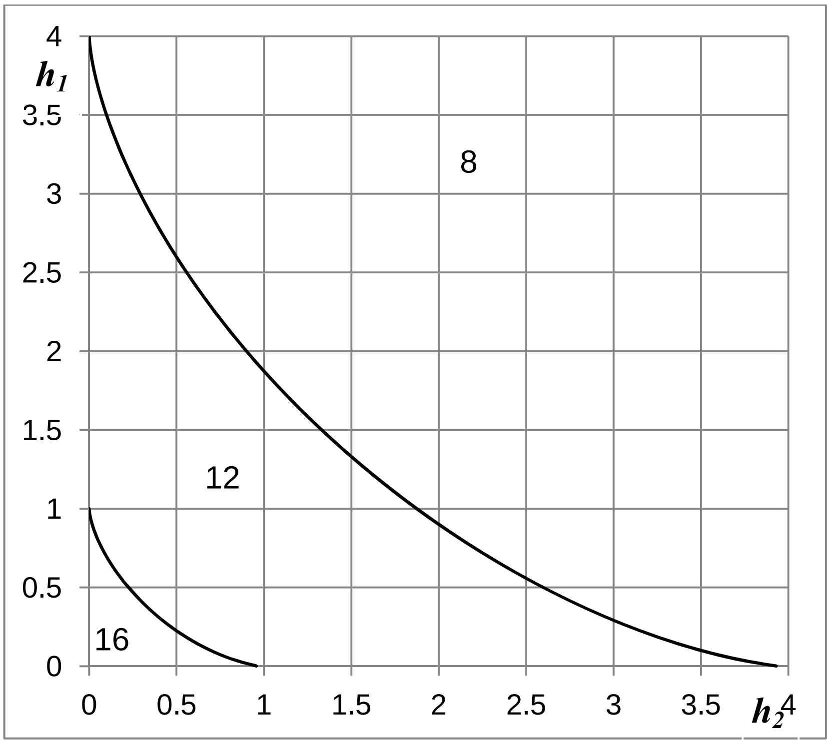


Fig. 52. v=1.0, h_3 = 0.005

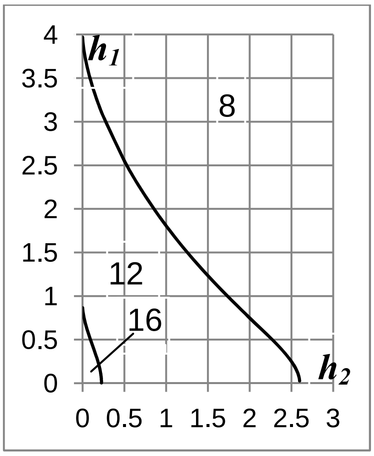


Fig. 53. v=1.0, h_3 =0.5

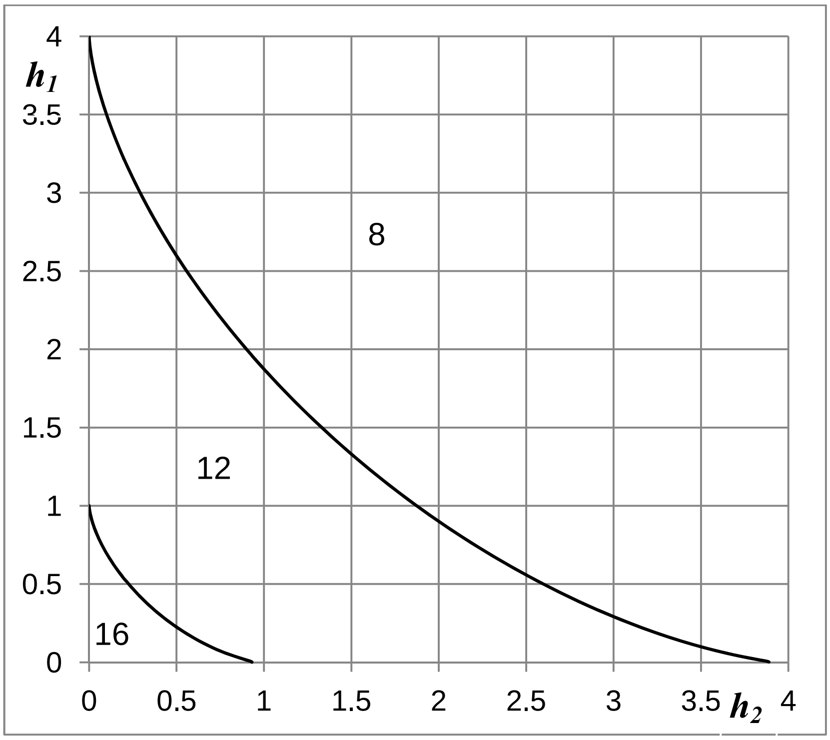


Fig. 54. v=1.0, h_3 = 0.01

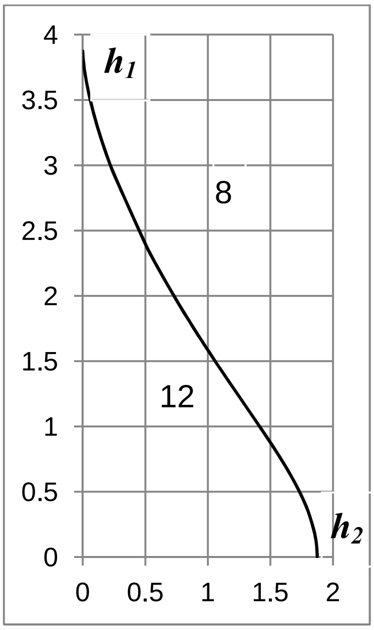


Fig. 55. v=1.0, h_3 = 1.0