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Engenharia

**Radial Basis Functions Based Identification of
Longitudinal Flight Dynamic Model
(Versão Corrigida Após Defesa)**

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Dedication

To my parents

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Resumo

No presente trabalho são apresentados dois métodos distintos de identificação de modelos de voo longitudinal, na representação no espaço de estados. A dissertação começa por introduzir as diferentes etapas que levam ao desenvolvimento e obtenção das diferentes equações do movimento que regem o voo sob uma forma genérica. Seguidamente apresenta-se a teoria na qual os métodos de identificação assentam. De forma a ilustrar os métodos propostos consideram-se o caso de uma aeronave comercial, suficientemente genérica, com o modelo conhecido de modo a poder validar os conceitos apresentados. Os resultados obtidos mostraram que os métodos propostos conseguem identificar com sucesso o modelo da aeronave alvo.

Palavras-chave

Identificação de Sistemas, Equações do Movimento, Estimação de Parâmetros, Filtro de Savitzky-Golay, Funções de Base Radial, Regressão Linear

Abstract

In the present work two distinct methods for the identification of the longitudinal flight model, in the state-space representation, are presented. Firstly, this dissertation introduces the different steps that lead up to the development and attainment of the flight equations of motion for a generic aircraft. Next, the theoretical background on which both identification methods are based upon is presented. In order to describe the proposed methods a sufficiently generic commercial aircraft is considered, which model is known, so the validation of the proposed concepts may be performed. The results obtained show that both proposed identification methods successfully achieve the model of the chosen aircraft.

Keywords

System Identification, Equations of Motion, Parameter Estimation, Savitzky-Golay Smoothing Filter, Radial Basis Functions, Linear Regression

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Nomenclature

List of Acronyms

ANOVA *Analysis of Variance*

DOF *Degrees of Freedom*

EOM *Equations of Motion*

i.i.d *independent and identically distributed*

LTI *Linear time-invariant*

LTV *Linear time-varying*

NED *North-East-Down*

ODE *Ordinary Differential Equation*

OLS *Ordinary Least Squares*

PDE *Partial Differential Equation*

Latin variables

\vec{V}_1	Velocity vector of the aircraft with respect the Earth axes	m/s
\vec{X}_1	A vector represented in Earth axes	—
\vec{X}_B	A vector represented in Body axes	—
a	Aerodynamic disturbances	—
AR	Aspect Ratio	—
c	Aerodynamic control disturbances	—
d	Atmospheric disturbances	—
g	Gravitational disturbances	—
L	Rolling moment	$N \cdot m$
M	Pitching moment	$N \cdot m$
Ma	Mach Number	—
$MAPE$	Mean Absolute Percent Error	—
MSE	Mean Square Error	—
MSE	Root Mean Square Error	—
N	Yawing	$N \cdot m$
p	Perturbed roll rate	rad/s
p	Power disturbances	—

Nomenclature

q	Perturbed pitch rate	rad/s
R	Transformation matrix equivalent to a rotation from Body axes to Earth Axes	—
r	Perturbed yaw rate	rad/s
S	Wing Area	m^2
SS_E	Error Sum of Squares	—
SS_R	Regression Sum of Squares	—
SS_T	Total Sum of Squares	—
T_ϕ	Transformation matrix equivalent to a rotation about the roll angle ϕ	—
T_ψ	Transformation matrix equivalent to a rotation about the yaw angle ψ	—
T_θ	Transformation matrix equivalent to a rotation about the pitch angle θ	—
U	Longitudinal speed	m/s
u	Perturbed longitudinal speed	m/s
U_e	Longitudinal speed at equilibrium conditions	m/s
V	Lateral speed	m/s
v	Perturbed lateral speed	m/s
V_e	Lateral speed at equilibrium conditions	m/s
W	Vertical speed	m/s
w	Perturbed vertical speed	m/s
W_E	Vertical speed at equilibrium conditions	m/s
X	Axial component of the total force acting on the aircraft	N
Y	Lateral component of the total force acting on the aircraft	N
Z	Normal component of the total force acting on the aircraft	N

Greek variables

α	Angle of attack	rad
α_e	Angle of attack at equilibrium conditions	rad
β	Sideslip angle	rad
β_e	Sideslip angle at equilibrium conditions	rad
η	Elevator deflection	rad
ϕ	Roll angle	rad
ψ	Yaw angle	rad
τ	Thrust	N

Nomenclature

θ	Pitch angle	<i>rad</i>
$\tilde{\sigma}_0(y, \hat{y})$	Relative error computed on the training set	—
$\tilde{\sigma}_{II}(y, \hat{y})$	Relative error computed on the validation set	—
$\tilde{\sigma}_I(y, \hat{y})$	Relative error computed on the test set	—
ξ	Aileron deflection	<i>rad</i>
ζ	Rudder deflection	<i>rad</i>

Chapter 1

Introduction

1.1 Motivation

In the design process of an aircraft, the longitudinal and lateral-directional models that describe the aircraft dynamics are determined. Alongside the stability and control derivatives which constitute a measure of the change in the aerodynamic forces and moments acting on aircraft when there is a small change in a flight condition parameter or when there is a deflection of control surface, respectively, the geometric properties of the aircraft (e.g.: the aspect ratio (AR), wing area (S), ...) and the flight conditions (e.g., flight Mach number (Ma)) are considered in order to obtain said models. In other words, in the design of an aircraft system identification, which is discussed in the next subsection, is performed.

Whether for prediction or inference purposes, system identification constitutes a powerful tool since it allows the determination of a model for describing the dynamics of an aircraft. However, in order to perform system identification flight data is required and it is often the case that it is recorded at equally spaced time intervals. Consequently, it is being performed discretization of a continuous physical process. Furthermore, as will be seen later, the determination of the previously discussed models requires the determination of time derivatives of the state variables involved. Due to the discretization mentioned above, there aren't analytical formulas representing the state variables against time which implies that time derivatives must be determined in a discrete fashion thus allowing this way errors to occur.

Furthermore, the flight data is quite often corrupted by disturbances, commonly known as noise, and if nothing is done to suppress it any prediction or inference done based upon this flight data set will be biased. Although system identification offers filtering and smoothing techniques in order to overcome the noise it is, however, restrained to a discrete setting.

Having all the above into consideration, it is desirable to have an approach to the problem of system identification of an aircraft in which the derivation of the state variables involved may be obtained in a continuous setting thus eliminating the errors incurred otherwise and also to have an approach which allows performing system identification in the continuous domain.

1.2 Identification

When presented before a physical system it is often necessary to identify it, that is, to construct a mathematical model, based upon (sometimes disrupted) observations capable of describing it, whether prediction or inference be the final goal. This activity is what is known as system identification [1, 2]. A definition of system identification might be the following:

"System identification is the determination, on the basis of observation of input and output, of a system within a specified class of systems to which the system under test is equivalent" [1].

1.2. Identification

Considering the above citation it follows that system identification concerns itself with the determination of a system (consequently, a mathematical model) equivalent to the system under observation which implies that broad possibilities of different mathematical models may describe the system. Consequently, for the same physical system, different models may be obtained and a question that naturally arises may be which model represents the system best. A widely used concept underlying theory building and model selection is that of the principle of parsimony [1]. The principle of parsimony aids to discard variables that may not be that useful to explain a given phenomenon (system), having this way less probability of the developed mathematical model having inconsistencies and ambiguities [3], or in other words, whenever between two competing equivalent models choose the one which is simpler [1]. A system is an object where different variables interplay with each other and as a result produce signals that may be observed [2, 4].

The variables commonly present in a system are defined as:

- u : represents the external and measurable input (signal) given to the system and can be manipulated by the observer;
- w : is a disturbance and represents an exogenous signal which may be directly measurable or its influence can be detected in the output of the system, but it cannot be manipulated;
- x : represents the state of the system and describes all the effects which the inputs u and the disturbances w had had on the system;
- v : represents the noise introduced by the sensor. It is an exogenous signal that cannot be manipulated by the user;
- y : represents the output of the sensors.

A diagram representing a generic system and the variables defined above is shown in Figure 1.1.

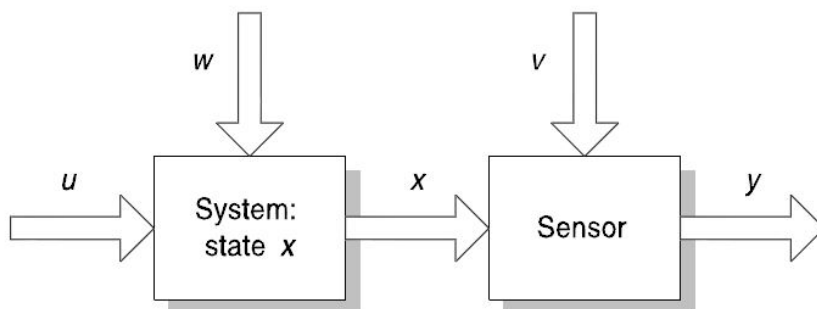


Figure 1.1: Representation of a generic system [4].

The particular case of an aircraft system identification constitutes one of three main problems with which the subject of aircraft dynamics and control is concerned with [1]. Observing Figure 1.2 one notices that at the left of the system an input u , in this case, the elevator is deflected (as depicted by the small graph right above the input arrow), the system S (the aircraft) is known but the output (change in the angle of attack as shown by the small graph right above the output arrow) is unknown (depicted by the question mark) and is required to be determined.

1.2. Identification

Another subject problem is that of the input determination, i.e., the system S is known as well as the output y but the input u is unknown, as depicted in Figure 1.3, and therefore this subject area is concerned with the determination of the latter.

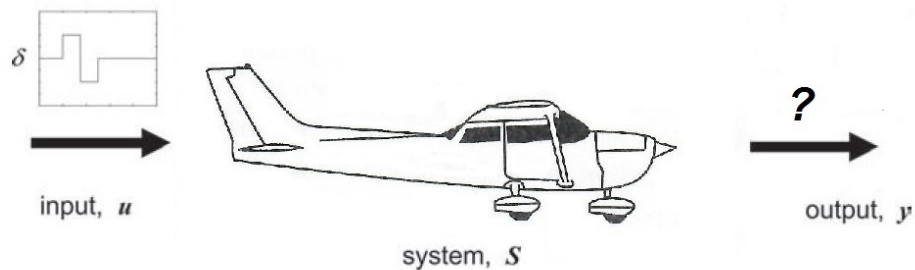


Figure 1.2: Representation of a simulation problem for the case of an airplane, adapted from [1, 5].

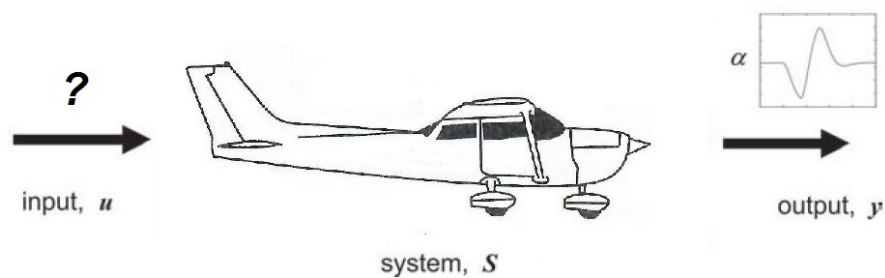


Figure 1.3: Representation of a control problem for the case of an airplane, adapted from [1, 5].

In the case of the problem of system identification (the subject matter of the present dissertation) both input, u , and output, y , are known but the system, S , is unknown as shown in Figure 1.4. It is then necessary to perform system identification on the aircraft in order to find the system S .

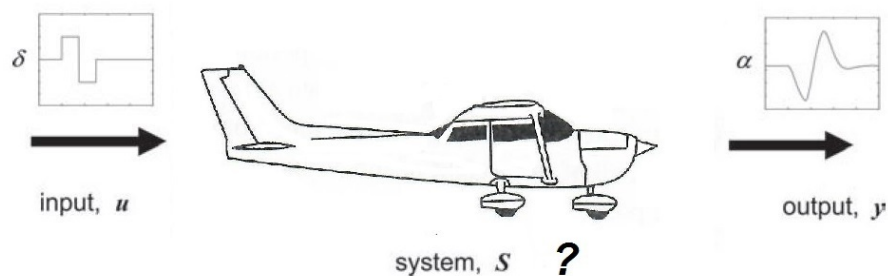


Figure 1.4: Representation of a system identification problem for the case of an airplane, adapted from [1, 5].

1.3 Objectives

This dissertation has henceforth as an objective the presentation of a method that allows the performance of system identification without incurring in errors due to the consideration of a continuous physical process as a discrete one.

1.4 Structures

The present dissertation is composed of five Chapters. The first one is the introduction where an explanation of the subject matter, motivation, objective and structure for the present work are presented.

In turn, the second chapter presents the derivation of the non-linear equations of motion (EOM) with six degrees of freedom (DOF) for a sufficiently generic aircraft, their linearisation, decoupling into longitudinal and lateral-directional motion and finally their representation in the state-space format in order to give the reader an overview of the role played by the several different variables and parameters that ultimately constitute de longitudinal and lateral-directional models that describe the dynamics of an aircraft.

Chapter 3 introduces the theoretical background needed to implement the two proposed different methods (e.g., numerical differentiation, regression analysis,...). Then, Chapter 4 presents a detailed explanation of both methods and the respective analysis.

Finally, in conclusion, an assessment of the overall works is performed and a discussion about possible future work is presented. Appendix A presents formulae for the stability and control derivatives, in a concise format, that build up the state and input matrices of the state-space representation of linearised 6 DOF EOM; additionally, these were retrieved and slightly modified from reference [6].

Chapter 2

Flight Dynamics Model

In the present Chapter, a derivation of the 6 DOF EOM for a general aircraft is presented. However, before proceeding to the development of these equations it is firstly required to define the frames of references, angles and sign conventions.

2.1 Reference Frames

Aerodynamic axes

The aerodynamic reference frame, $\mathfrak{R}_A = (O, \vec{x}_A, \vec{y}_A, \vec{z}_A)$, has its origin at the centre of gravity of the aircraft even though the reference point for the aerodynamic velocity of the aircraft being the aerodynamic centre. The Ox_A axis is parallel to the aircraft's aerodynamic velocity. If the atmosphere is assumed to be at rest relative to the earth (as is in the derivation of the 6 degrees of freedom equations of motion) the aerodynamic and body axes are considered to be the same [7, 8].

Body axes

The origin of the body axes is fixed at the centre of gravity of the aircraft; Ox points towards the front of the aircraft through the nose, Oz points downwards being orthogonal to the plane formed by Ox_By_B , that is the horizontal plane of the aircraft and Oy points towards the right through the right-wing. Furthermore, the plane Ox_Bz_B commonly constitutes the plane of symmetry of the aircraft. Then it can be written as $\mathfrak{R}_B = (O, \vec{x}_B, \vec{y}_B, \vec{z}_B)$ [1, 7].

Earth axes

Earth axes, $\mathfrak{R}_E = (O, \vec{x}_E, \vec{y}_E, \vec{z}_E)$, is a frame of reference, assumed to be inertial, whose origin can be set anywhere on earth's surface; axis Ox_E points toward the geographic north, Oy_E is pointing toward east and Oz_E points towards the centre of the earth. The plane Ox_Ey_E is designated as the local horizon plane. Due to its orientation this frame is also called NED (North-East-Down) [1, 7, 8].

Stability axes

The stability axes, $\mathfrak{R}_S = (O, \vec{x}_S, \vec{y}_S, \vec{z}_S)$ is a particular type of body axes and like the latter, it has its origin at the centre of gravity of the aircraft and its orientation is defined with respect to a reference flight condition. It can be obtained from the aircraft's body axes by rotating about the lateral axis by an angle of magnitude equal to the angle of attack α . Consequently, Ox_S has the same direction as the aerodynamic velocity vector, Oy_S is aligned with Oy_B , and the angle between Oz_S these and Oz_B equals the angle of attack α . If the aerodynamic velocity vector is in the plane of symmetry at the reference condition then the stability axes can be considered to be equal to the wind axes, however, if this condition is not verified and a disturbance occurs this equality won't persist [1, 7, 8].

Wind axes

Wind axes, $\mathfrak{R}_w = (O, x_{\vec{w}}, y_{\vec{w}}, z_{\vec{w}})$, is a frame of reference whose origin is fixed at the aircraft's centre of gravity. Axis Ox_w is directed along the aerodynamic velocity vector, Oy_w is directed along the right-wing of the aircraft and Oz_w points downwards. This frame of reference can be obtained from the stability axes by rotating about the vertical axis of the body axes by an angle equal to sideslip angle β . Assuming that the atmosphere is at rest, Ox_w depicts the aircraft's trajectory relative to the earth being always tangent to it [1, 7, 8].

For the rest of this dissertation and specifically for the development of the 6 DOF EOM the earth axes and body axes will be used as reference frames.

2.2 Aircraft Orientation

2.2.1 Sign Convention

The sign convention regarding the moments and angular velocities of the aircraft is that of the right-hand rule. Pointing the thumb in the positive direction of a given axis the fingers curling defines the positive rotation for moments and angular velocities (Figure 2.1). On the other hand, pointing in the thumb in the negative direction of an axis the fingers curling defines the negative rotation as depicted in Figure 2.2.

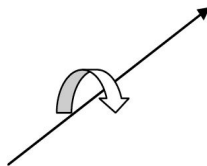


Figure 2.1: Positive rotation [7].

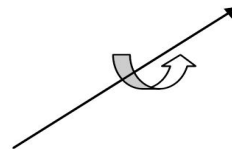


Figure 2.2: Negative rotation [7].

Figure 2.3 depicts the nomenclature used throughout the text to denote the several forces, moments, linear velocities as well as the angles described by the aircraft.

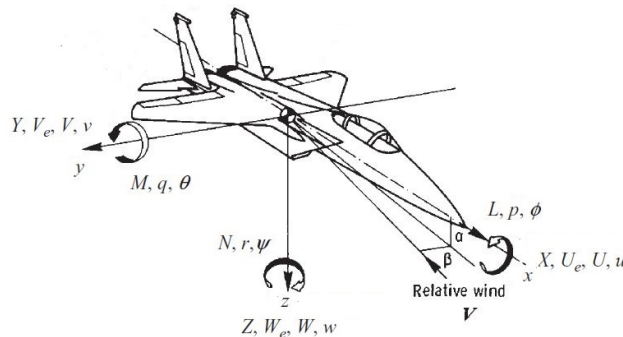


Figure 2.3: Aircraft nomenclature, adapted from [1, 6].

Additionally, Figure 2.4 depicts the different control surfaces. These encompass the rudder, elevators and the ailerons.

2.2. Aircraft Orientation

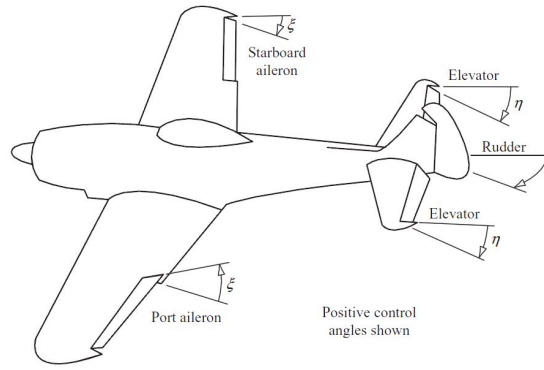


Figure 2.4: Notation of the aerodynamic controls [6].

2.2.2 Aerodynamic Angles

The aerodynamic forces and moments applied on the aircraft can be related to the aerodynamic velocity vector. To define this vector two angles are of fundamental importance: the sideslip angle β_e and the angle of attack α_e (where the subscript "e" denotes the equilibrium conditions) [9]. Observing Figures 2.3 and 2.5, one sees that the longitudinal (U_e), lateral (V_e) and vertical (W_e) speeds of the aircraft are functions of the components of the aerodynamic velocity vector and they can be written as:

$$U_e = V_0 \cos(\alpha_e) \cos(\beta_e) \quad (2.1)$$

$$V_e = V_0 \sin(\beta_e) \quad (2.2)$$

$$W_e = V_0 \sin(\alpha_e) \cos(\beta_e) \quad (2.3)$$

From the above equations, the sideslip angle (β_e) and the angle of attack (α_e) are defined by

$$\alpha_e = \arctan\left(\frac{W_e}{U_e}\right) \quad (2.4)$$

$$\beta_e = \arcsin\left(\frac{V_e}{V_0}\right) \quad (2.5)$$

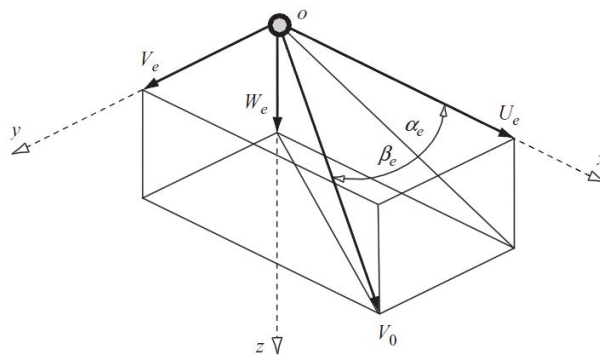


Figure 2.5: Aerodynamic velocity vector resolved into its components [6].

2.2. Aircraft Orientation

Since the aerodynamic velocity vector can be written as $V_0 = \sqrt{U_e^2 + V_e^2 + W_e^2}$, the sideslip angle β can be written as in equation 2.6.

$$\tan(\beta_e) = \frac{V_e}{\sqrt{U_e^2 + W_e^2}} \quad (2.6)$$

2.2.3 Euler Angles

The attitude of an aircraft is described by the angular orientation of the aircraft body axes with respect to earth axes, i.e., it is defined by the attitude angles, which are in turn a particular application of the Euler angles [6, 9]. This way the velocity components, measured relative to the body axes, can be determined with respect to the Earth axes [10]. The Euler angles are represented by the Greek letters ϕ , θ and ψ describing the roll angle, pitch angle and the yaw angle, respectively [7, 9]. They describe sequential rotations from the earth axis to the body axes and the other way around. The way the sequence is ordered is arbitrary but since these angles do not obey the commutative law once the sequence is defined it then must be retained [6, 9].

Consequently, the orientation of the aircraft can be determined by the following sequence (Figure 2.6):

- 1st Rotation: The aircraft's longitudinal and lateral axes, x_f and y_f , are rotated by a yaw angle ψ to x_1 and y_1 respectively;
- 2nd Rotation: The aircraft's longitudinal and vertical axes, x_1 and z_1 , are rotated by a pitch angle θ to x_2 and y_2 respectively;
- 3rd Rotation: The aircraft's lateral and vertical axes, y_2 and z_2 , are rotated by a roll angle ϕ to y_3 and z_3 respectively.

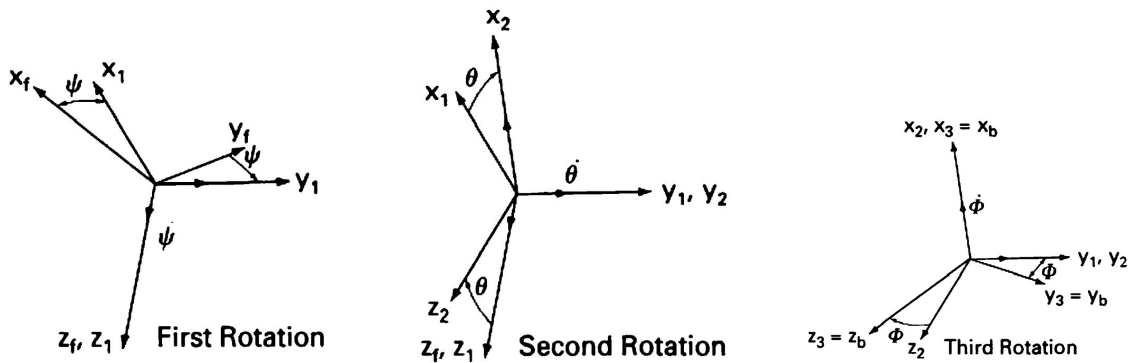


Figure 2.6: Sequential rotations from one frame of reference to the other [10].

The sequential rotations, i.e., Euler angles are defined, in radian units, on the intervals 2.7 through 2.9.

$$-\pi \leq \psi \leq \pi \quad (2.7)$$

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \quad (2.8)$$

$$-\phi_{min} \leq \phi \leq \phi_{max} \quad (2.9)$$

2.2. Aircraft Orientation

It is, however, important to notice that $\phi_{min} < 0$ and $\phi_{max} > 0$ depending on the individual aircraft. The sequential rotations (or transformations) depicted in Figure 2.6 can be described using the following matrices T_ϕ 2.10, T_θ 2.11 and T_ψ 2.12 representing the roll, pitch and yaw rotation matrices, respectively [7].

$$T_\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{bmatrix} \quad (2.10)$$

$$T_\theta = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \quad (2.11)$$

$$T_\psi = \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.12)$$

Subsequently, any given vector represented in Earth axes can be transformed into a vector in body axes as described by equation 2.13 [11],

$$\vec{X}_1 \Rightarrow T_\psi \Rightarrow \vec{X}_2 \Rightarrow T_\theta \Rightarrow \vec{X}_3 \Rightarrow T_\phi \Rightarrow \vec{X}_B \quad (2.13)$$

where \vec{X}_1 is a vector represented in earth axes, \vec{X}_B is the \vec{X}_1 vector transformed into body axes, \vec{X}_2 and \vec{X}_3 are vectors resulting from the transformations T_ψ and T_θ , respectively. The above transformations are computed in 3 steps. For instance, the transformation $\vec{X}_1 \Rightarrow T_\psi \Rightarrow \vec{X}_2$ is computed as $\vec{X}_2 = T_\psi \vec{X}_1$, described by matrix equation 2.14.

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \quad (2.14)$$

The transformation $\vec{X}_2 \Rightarrow T_\theta \Rightarrow \vec{X}_3$ is computed through matrix equation 2.15

$$\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \quad (2.15)$$

and the transformation $\vec{X}_3 \Rightarrow T_\phi \Rightarrow \vec{X}_B$ is computed as shown in matrix equation 2.16.

$$\begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} \quad (2.16)$$

2.2. Aircraft Orientation

In order to transform a vector represented in earth axes directly into body axes, it is required to compute $\vec{X}_B = T_\phi T_\theta T_\psi \vec{X}_1$. Defining a matrix R as begin the dot product between the rotation matrices then it follows that $\vec{X}_B = R\vec{X}_1$.

$$R = \begin{bmatrix} \cos(\psi) \cos(\theta) & \sin(\psi) \cos(\theta) & -\sin(\theta) \\ \cos(\psi) \sin(\theta) \sin(\phi) - \sin(\psi) \cos(\phi) & \sin(\psi) \sin(\theta) \sin(\phi) + \cos(\psi) \cos(\phi) & \cos(\theta) \sin(\phi) \\ \cos(\psi) \sin(\theta) \cos(\phi) + \sin(\psi) \sin(\phi) & \sin(\psi) \sin(\theta) \cos(\phi) - \cos(\psi) \sin(\phi) & \cos(\theta) \cos(\phi) \end{bmatrix} \quad (2.17)$$

Since the rotations matrices T_ϕ , T_θ and T_ψ are invertible [7] and orthogonal [11], then matrix R is also orthogonal and its inverse equals its transpose. Hence to transform a vector from body axes directly to earth axes it is only required to compute $\vec{X}_1 = R^T \vec{X}_B$. Consequently, if the aircraft's velocity vector is needed in Earth axes, it can be rotated from the Body axes as

$$\begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \end{bmatrix} = R^T \begin{bmatrix} U_e \\ V_e \\ W_e \end{bmatrix} \quad (2.18)$$

From the above matrix equation 2.18, the manoeuvrable rates p , q and r can determined as described by equation 2.19 through 2.21 [7].

$$p = \dot{\phi} - \dot{\psi} \sin(\theta) \quad (2.19)$$

$$q = \dot{\theta} \cos(\phi) + \dot{\psi} \cos(\theta) \sin(\phi) \quad (2.20)$$

$$r = \dot{\psi} \cos(\phi) \cos(\theta) - \dot{\theta} \sin(\phi) \quad (2.21)$$

Further manipulation of equations 2.19 to 2.21 yields $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$, equations 2.22, 2.23 and 2.24, respectively.

$$\dot{\phi} = p + (q \sin(\phi) + r \cos(\phi)) \cdot \tan(\theta) \quad (2.22)$$

$$\dot{\theta} = q \cos(\phi) - r \sin(\phi) \quad (2.23)$$

$$\dot{\psi} = \frac{q \sin(\phi) + r \cos(\phi)}{\cos(\theta)} \quad (2.24)$$

Perusing equations from 2.22 to equation 2.24, when θ assumes values near 90° the terms $\tan(\theta)$ and $1/\cos(\theta)$ become ∞ [7], i.e., there exists a singularity. Although the usage of Euler angles aids the visualisation of the physical state of the aircraft when $\theta \approx 90^\circ$ others methods must be employed such as quaternions [7] or direction cosines [12].

2.3. 6 Degree of Freedom Equations of Motion

2.3 6 Degree of Freedom Equations of Motion

The derivation of the 6 DOF EOM describing the dynamics of a general aircraft commonly considers flight in non-stationary air, varying mass, the elastic properties of the body, where the aerodynamic surfaces are subjected to bending and twisting motions, and the forces applied on the aircraft, which are in turn functions of its shape and motion. Such derivation would be very complex and outside the scope of this dissertation. Hence the derivation presented in this text is in accordance with the following simplifying assumptions [1, 10, 12]:

- the aircraft is assumed to be a rigid body where the mass remains constant throughout the flight;
- the air is at rest with respect to the earth and consequently wind shears and gusts are neglected;
- the flight's duration is small and close the earth's surface which allows the curvature of the earth to be dismissed;
- the gravity field is considered to be constant and non-varying with respect to the altitude thus the centre of gravity and centre of mass can be assumed to be coincident;
- the earth's rotation is neglected and it is considered to be an inertial space.

For the derivation of the equations of motion it is required the application of Newton's second law of motion which in turn can only be applied on an inertial frame of reference and consequently in the present case earth axes is chosen. However, the aerodynamic velocity and other parameters must be computed with respect to the body axes otherwise the complexity of the derivation would increase. Hence the following derivation of the aircraft's equations of motion requires two frames of reference namely the earth axes and the aircraft body axes (Figure 2.7)

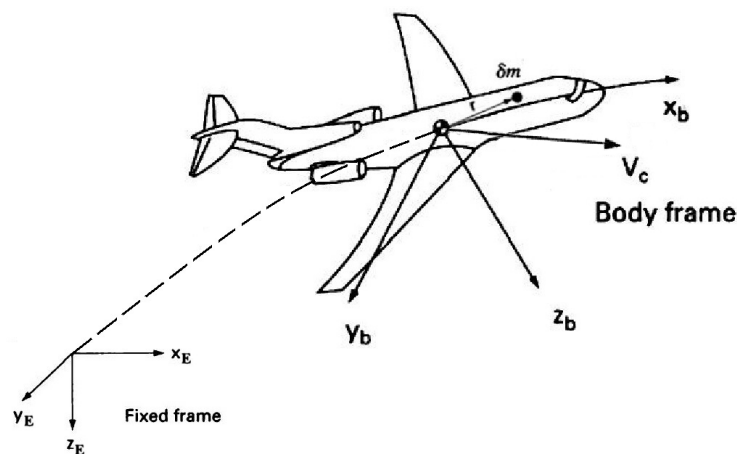


Figure 2.7: Definition of the aircraft body axes and the earth axes, adapted from [10].

2.3. 6 Degree of Freedom Equations of Motion

Newton's second law of motion states that 2.25

$$\sum \vec{F} = \frac{d}{dt}(m\vec{V}_1) \quad (2.25)$$

the summation of all external forces acting on a body is equal to the time derivative of its mass times its velocity (\vec{V}_1). As previously stated, it is assumed that the mass of the aircraft remains constant throughout the flight. Thus equation 2.25 reduces to equation 2.26.

$$\vec{F} = m \frac{d\vec{V}_1}{dt} \quad (2.26)$$

Decomposing equation 2.26 into the body axes yields equation 2.27 through 2.29.

$$X = \frac{d}{dt}(m \cdot U) \quad (2.27)$$

$$Y = \frac{d}{dt}(m \cdot V) \quad (2.28)$$

$$Z = \frac{d}{dt}(m \cdot W) \quad (2.29)$$

Considering the aircraft in Figure 2.7, δm denotes an infinitesimal element of mass of the aircraft. Then applying Newton's second law yields equation 2.30.

$$\delta \vec{F} = \delta m \frac{d\vec{V}_1}{dt} \quad (2.30)$$

The total external force applied to the aircraft equals the sum of the individual external forces applied on each element of mass

$$\vec{F} = \sum \delta \vec{F} \quad (2.31)$$

The velocity of δm with respect to earth axes, \vec{V}_1 , is described by equation 2.32 which states that the velocity of element of mass, δm , is the vectorial sum of the velocity of the aircraft's centre of mass and of the time derivative of the position vector \vec{r} .

$$\vec{V}_1 = \vec{V}_c + \frac{d\vec{r}}{dt} \quad (2.32)$$

Thus, the total external force applied to the aircraft can be written as described by equation 2.33.

$$\vec{F} = \frac{d}{dt} \sum \left(\vec{V}_c + \frac{d\vec{r}}{dt} \right) \delta m \quad (2.33)$$

2.3. 6 Degree of Freedom Equations of Motion

Expanding equation 2.33 yields equation 2.34

$$\vec{F} = m \frac{d\vec{V}_c}{dt} + \frac{d^2}{dt^2} \sum \vec{r} \delta m \quad (2.34)$$

and since $\sum \vec{r} \delta m = 0$ equation 2.34 can be rewritten as in 2.35 which gives a relationship between the total external force applied on the aircraft and the velocity of its centre of mass.

$$\vec{F} = m \frac{d\vec{V}_c}{dt} \quad (2.35)$$

Proceeding in a similar manner to that undertaken from equation 2.25 up until this point a relationship between the moments applied on the aircraft and the velocity of its centre of mass can be described using equation 2.36 where \vec{H} denotes the angular momentum.

$$\sum \vec{M} = \frac{d}{dt}(\vec{H}) \quad (2.36)$$

Since the angular moment can be written as $\vec{H} = \vec{r} \times \vec{V}_1$ and considering an infinitesimal element of mass equation 2.36 can be rewritten as in equation 2.37.

$$\delta \vec{M} = \frac{d}{dt}(\delta \vec{H}) = \frac{d}{dt}[(\vec{r} \times \vec{V}_1) \delta m] \quad (2.37)$$

Similarly to equation 2.32, the velocity of the element of mass can be expressed as a vectorial sum of the velocity of the aircraft's centre of mass but with the cross product between the angular velocity and position of the element of mass. Then, the velocity of the element of mass can be written as in equation 2.38.

$$\begin{aligned} \vec{V}_1 &= \vec{V}_c + \frac{d\vec{r}}{dt} \Leftrightarrow \\ \Leftrightarrow \vec{V}_1 &= \vec{V}_c + \vec{\omega} \times \vec{r} \end{aligned} \quad (2.38)$$

Substituting equation 2.38 into equation 2.37 yields equation 2.39.

$$\begin{aligned} \vec{H} &= \sum \delta \vec{H} \\ &= \sum (\vec{r} \times \vec{V}_c) \delta m + \sum [\vec{r} \times (\vec{\omega} \times \vec{r})] \delta m \quad (2.39) \\ &= \sum \vec{r} \delta m \times \vec{V}_c + \sum [\vec{r} \times (\vec{\omega} \times \vec{r})] \delta m \end{aligned}$$

Since, $\sum \vec{r} \delta m = 0$ equation 2.39 can be rewritten as in equation 2.40 which represents the angular momentum with respect to the earth (inertial) frame of reference.

$$\vec{H} = \sum \vec{r} \times (\vec{\omega} \times \vec{r}) \delta m \quad (2.40)$$

2.3. 6 Degree of Freedom Equations of Motion

Decomposing the position (\vec{r}) and the angular velocity ($\vec{\omega}$) vectors into the respective components yields equations 2.41 2.42, respectively.

$$\vec{\omega} = p \hat{i} + q \hat{j} + r \hat{k} \quad (2.41)$$

$$\vec{r} = x \hat{i} + y \hat{j} + z \hat{k} \quad (2.42)$$

Substituting equations 2.41 and 2.42 into equation 2.40 and computing the cross-products it can then be shown that the components of equation 2.40 can be written as in equation 2.43.

$$\begin{aligned} H_x &= p \sum (y^2 + z^2) \delta m - q \sum xy \delta m - r \sum xz \delta m \\ H_y &= -p \sum xy \delta m + q \sum (x^2 + z^2) \delta m - r \sum yz \delta m \\ H_z &= -p \sum xy \delta m - q \sum yz \delta m + r \sum (x^2 + y^2) \delta m \end{aligned} \quad (2.43)$$

As stated at the beginning of section 2.3 it is advantageous to work with the body axes because it is fixed, that is, it rotates as the aircraft rotates and consequently the moments of inertia won't vary. However, Newton's second law can only be applied to inertial frames of reference. To overcome this situation it is assumed that the Body axes are fixed to the aircraft, that is, as the aircraft moves so does this reference frame. However, due to such assumption, the time derivatives of the vector quantities corresponding to the velocity of the infinitesimal element of mass and the angular momentum must be taken with respect to Earth axes. Consequently, it is assumed that the derivative of an arbitrary vector quantity, \vec{A} , with respect to an inertial axes, in this case with respect to Earth axes, equals the derivative of that very same vector with respect to the Body axes plus the cross product of the angular velocity of the body times that very same vector which is clearly described by equation 2.44 (where the subscripts E and B refer to the Earth and Body fixed frames of reference).

$$\left. \frac{d\vec{A}}{dt} \right|_E = \left. \frac{d\vec{A}}{dt} \right|_B + \vec{\omega} \times \vec{A} \quad (2.44)$$

Thus applying identity 2.44 to the force and moment equations yields 2.45 and 2.46.

$$\vec{F} = m \left. \frac{d\vec{V}_c}{dt} \right|_B + m(\vec{\omega} \times \vec{V}_c) \quad (2.45)$$

$$\vec{M} = \left. \frac{d\vec{H}}{dt} \right|_B + \vec{\omega} \times \vec{H} \quad (2.46)$$

2.3. 6 Degree of Freedom Equations of Motion

From equation 2.45 it follows that the components of the total force applied to the aircraft are described by the force equations (equation 2.47 through 2.49).

$$X = m [\dot{U} - rV + qW] \quad (2.47)$$

$$Y = m [\dot{V} - pW + rU] \quad (2.48)$$

$$Z = m [\dot{W} - qU + pV] \quad (2.49)$$

And considering that $I_{xy} = I_{yz} = 0$, due the aircraft symmetry, from equation 2.46 the moment equations are constituted by equations 2.50 through 2.52.

$$L = I_{xx}\dot{p} - (I_{yy} - I_{zz})qr - I_{xz}(pq + \dot{r}) \quad (2.50)$$

$$M = I_{yy}\dot{q} + (I_{xx} - I_{zz})pr + I_{xz}(p^2 - r^2) \quad (2.51)$$

$$N = I_{zz}\dot{r} - (I_{xx} - I_{yy})pq + I_{xz}(qr + \dot{p}) \quad (2.52)$$

The above equations 2.47 and 2.52 constitute the six non-linear DOF EOM for a generic aircraft. These can be, however, further developed by the specification of the applied forces and moments on the aircraft. Considering the approach depicted in reference [6] it is assumed that the applied forces and moments are due to aerodynamic, gravitational, aerodynamic controls, power and atmospheric disturbances (henceforth the subscripts: a , g , c , p and d are used to denote each one of these effects, respectively). Hence the force equations (2.47 through 2.49) and the moment equations (2.50 through 2.52) can be rewritten as in equations 2.53 through 2.55 and as in 2.56 through 2.58, respectively.

$$m [\dot{U} + qW - rV] = X_a + X_g + X_c + X_p + X_d \quad (2.53)$$

$$m [\dot{V} + rU - pW] = Y_a + Y_g + Y_c + Y_p + Y_d \quad (2.54)$$

$$m [\dot{W} + pV - qU] = Z_a + Z_g + Z_c + Z_p + Z_d \quad (2.55)$$

$$I_{xx}\dot{p} - (I_{yy} - I_{zz})qr - I_{xz}(pq + \dot{r}) = L_a + L_g + L_c + L_p + L_d \quad (2.56)$$

$$I_{yy}\dot{q} + (I_{xx} - I_{zz})pr + I_{xz}(p^2 - r^2) = M_a + M_g + M_c + M_p + M_d \quad (2.57)$$

$$I_{zz}\dot{r} - (I_{xx} - I_{yy})pq + I_{xz}(qr + \dot{p}) = N_a + N_g + N_c + N_p + N_d \quad (2.58)$$

The above force and moment equations are non-linear and hence cannot be solved to obtain an analytical solution. One way to solve such type of equations is to linearise them. The linearisation of non-linear equations is restricted to an application where these represent motions of small displacement from the equilibrium point, which was assumed previously in the development of the non-linear 6 DOF EOM for a generic aircraft [6, 13].

2.4 Linearised Equations of Motion

In this section, the non-linear six DOF EOM of an aircraft, derived in the previous section are linearised using the small-disturbance theory. The application of this theory assumes that the aircraft is flying in a steady and level flight and that its motion consists of small deviations from the steady condition, as said previously, and consequently it cannot be applied to situations in which motions of large amplitude occur. Despite this constraint, it yields results with enough accuracy for engineering purposes [10, 12]. Furthermore, it is also considered rolling, yawing and sideslip motions do not occur and that the atmosphere is assumed to be stable and undisturbed and consequently equation 2.59 can be applied.

$$X_d = Y_d = Z_d = L_d = M_d = N_d = 0 \quad (2.59)$$

Considering that the aircraft deviates only a small amount from the steady flight condition due to a perturbation, the equations describing the linear velocity are written as in equations 2.60 through 2.62 where the subscript "e" denotes the equilibrium or steady condition and the lower-case letters represent the linear disturbance velocities.

$$U = U_e + u \quad (2.60)$$

$$V = V_e + v \quad (2.61)$$

$$W = W_e + w \quad (2.62)$$

In a similar manner, p , q and r in equations 2.63 through 2.65 represent the angular disturbance velocities and their time derivatives in equations 2.66 through 2.68. Since the perturbed quantities are small the product between these and its powers are assumed very small and are consequently discarded. Then substituting equations 2.59, 2.60, 2.61 and 2.62 into equations 2.53 through 2.58 yields the linearised equations of motion (equation 2.63 to equation 2.68).

$$m(\dot{u} + qW_e) = X_a + X_g + X_c + X_p \quad (2.63)$$

$$m(\dot{v} - pW_e + rU_e) = Y_a + Y_g + Y_c + Y_p \quad (2.64)$$

$$m(\dot{w} - qU_e) = Z_a + Z_g + Z_c + Z_p \quad (2.65)$$

$$I_{xx}\dot{p} - I_{xz}\dot{r} = L_a + L_g + L_c + L_p \quad (2.66)$$

$$I_{yy}\dot{q} = M_a + M_g + M_c + M_p \quad (2.67)$$

$$I_{zz}\dot{r} - I_{xz}\dot{p} = N_a + N_g + N_c + N_p \quad (2.68)$$

The further development of the equations 2.63 through 2.68 requires the determination of the several right-hand side terms which represent the applied forces and moments on the aircraft due to a variety of factors. Firstly, considering the effects due to the force of gravity, i.e., the weight force whose contribution can be added to equations 2.63 through 2.68 by decomposing it into the disturbed body axes. Perusing Figure 2.8 it can be seen that the origin of the aircraft body axes is coincident with its centre of gravity which implies that no moments are created

2.4. Linearised Equations of Motion

due to the aircraft's weight [6]. Hence it is assumed that

$$L_g = M_g = N_g = 0 \quad (2.69)$$

Considering that the aircraft is flying in steady and level flight the y -component of the weight force is consequently considered to be null.

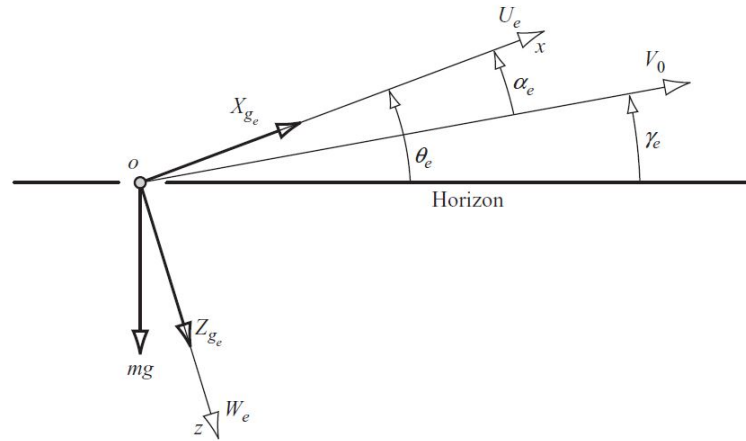


Figure 2.8: Weight force decomposition in the plane of symmetry [6].

Hence, the weight force is decomposed into body axes as described in equations 2.70 through 2.72.

$$X_{g_e} = -mg \sin(\theta_e) \quad (2.70)$$

$$Y_{g_e} = 0 \quad (2.71)$$

$$Z_{g_e} = mg \cos(\theta_e) \quad (2.72)$$

Considering the transformation matrix 2.17 the weight components described previously can be transformed into the disturbed body axes through matrix equation 2.73,

$$\begin{bmatrix} X_g \\ Y_g \\ Z_g \end{bmatrix} = \begin{bmatrix} 1 & \psi & -\theta \\ -\psi & 1 & \phi \\ \theta & -\phi & 1 \end{bmatrix} \begin{bmatrix} X_{g_e} \\ Y_{g_e} \\ Z_{g_e} \end{bmatrix} \quad (2.73)$$

thus yielding this way the equations 2.74 through 2.76.

$$X_g = -mg \sin(\theta_e) - mg\theta \cos(\theta_e) \quad (2.74)$$

$$Y_g = mg\psi \sin(\theta_e) + mg\phi \cos(\theta_e) \quad (2.75)$$

$$Z_g = mg \cos(\theta_e) - mg\theta \sin(\theta_e) \quad (2.76)$$

Secondly the terms corresponding to the aerodynamic effects also need to be considered. In order to specify the interactions present due to the aerodynamic effects, it is assumed that

2.4. Linearised Equations of Motion

in equations 2.53 through 2.58, the aerodynamic force and moment terms are only dependent on the disturbed motion variables (u, v, w, p, q, r) and their derivatives. For example, the axial force X_a (equation 2.77) is given by a constant X_{a_e} plus several Taylor series, each one involving a different variable [6].

$$\begin{aligned}
 X_a = X_{a_e} &+ \left(\frac{\partial X}{\partial u} u + \frac{\partial^2 X}{\partial u^2} \frac{u^2}{2!} + \frac{\partial^3 X}{\partial u^3} \frac{u^3}{3!} + \frac{\partial^4 X}{\partial u^4} \frac{u^4}{4!} + \dots \right) \\
 &+ \left(\frac{\partial X}{\partial v} v + \frac{\partial^2 X}{\partial v^2} \frac{v^2}{2!} + \frac{\partial^3 X}{\partial v^3} \frac{v^3}{3!} + \frac{\partial^4 X}{\partial v^4} \frac{v^4}{4!} + \dots \right) \\
 &+ \left(\frac{\partial X}{\partial w} w + \frac{\partial^2 X}{\partial w^2} \frac{w^2}{2!} + \frac{\partial^3 X}{\partial w^3} \frac{w^3}{3!} + \frac{\partial^4 X}{\partial w^4} \frac{w^4}{4!} + \dots \right) \\
 &+ \left(\frac{\partial X}{\partial p} p + \frac{\partial^2 X}{\partial p^2} \frac{p^2}{2!} + \frac{\partial^3 X}{\partial p^3} \frac{p^3}{3!} + \frac{\partial^4 X}{\partial p^4} \frac{p^4}{4!} + \dots \right) \\
 &+ \left(\frac{\partial X}{\partial q} q + \frac{\partial^2 X}{\partial q^2} \frac{q^2}{2!} + \frac{\partial^3 X}{\partial q^3} \frac{q^3}{3!} + \frac{\partial^4 X}{\partial q^4} \frac{q^4}{4!} + \dots \right) \quad (2.77) \\
 &+ \left(\frac{\partial X}{\partial r} r + \frac{\partial^2 X}{\partial r^2} \frac{r^2}{2!} + \frac{\partial^3 X}{\partial r^3} \frac{r^3}{3!} + \frac{\partial^4 X}{\partial r^4} \frac{r^4}{4!} + \dots \right) \\
 &+ \left(\frac{\partial X}{\partial \dot{u}} \dot{u} + \frac{\partial^2 X}{\partial \dot{u}^2} \frac{\dot{u}^2}{2!} + \frac{\partial^3 X}{\partial \dot{u}^3} \frac{\dot{u}^3}{3!} + \dots \right) \\
 &+ \left(\frac{\partial X}{\partial \dot{v}} \dot{v} + \frac{\partial^2 X}{\partial \dot{v}^2} \frac{\dot{v}^2}{2!} + \frac{\partial^3 X}{\partial \dot{v}^3} \frac{\dot{v}^3}{3!} + \dots \right) \\
 &+ \text{terms in } \dot{w}, \dot{p}, \dot{q} \text{ and } \dot{r} \\
 &+ \text{terms of higher order derivatives}
 \end{aligned}$$

Since the motion variables have very small magnitudes the higher terms can be ignored with the exception of the first term (on each Taylor series). Conversely, \dot{w} is considered since it is the only significantly higher-order derivative that usually comes across [6]. Hence equation 2.77 can be rewritten as in equation 2.78 where the several partial derivatives are known as the aerodynamic stability derivatives.

$$X_a = X_{a_e} + \frac{\partial X}{\partial u} u + \frac{\partial X}{\partial v} v + \frac{\partial X}{\partial w} w + \frac{\partial X}{\partial p} p + \frac{\partial X}{\partial q} q + \frac{\partial X}{\partial r} r + \frac{\partial X}{\partial \dot{w}} \dot{w} \quad (2.78)$$

In a similar way to that of the determination of the mathematical formula (above) for the axial force X due to aerodynamic effects, the remaining aerodynamic terms present in equations 2.53 through 2.58 are derived. Other terms to be specified are those relating to the deflections of aerodynamic surfaces, i.e., ailerons, elevator and rudder (Figure 2.4). Again, assuming that the force and moments are only dependent on the disturbed motion variables the case pitching moment can be described as in equation 2.79.

$$M_c = \frac{\partial M}{\partial \xi} \xi + \frac{\partial M}{\partial \eta} \eta + \frac{\partial M}{\partial \zeta} \zeta \quad (2.79)$$

2.4. Linearised Equations of Motion

Notice that the aileron, elevator and rudder angles (ξ, η, ζ) are measured with respect to the corresponding trim angles (ξ_e, η_e, ζ_e) . In a similar fashion the remaining control terms present in equations 2.53 through 2.58 are determined. Lastly, there only remains the power terms to be specified. The thrust of an aircraft is controlled by the throttle setting, that is, by setting the throttle lever angle (ϵ) .

The throttle contribution is commonly accounted for as in equation 2.80. As in the control terms, thrust is also measured with respect to the corresponding trim settings.

$$Z_p = \frac{\partial Z}{\partial \tau} \tau \quad (2.80)$$

The other power terms present in equations 2.53 through 2.58 are determined in the same way as in equation 2.80.

After the determination of all the remaining terms present in equations 2.53 through 2.58 it then remains to substitute the expressions obtained into these thus obtaining the linearised equations of motion accounting for the several effects discussed above thus yielding this way the EOM for small perturbations (equations 2.81 through 2.86).

$$\begin{aligned} m(\dot{u} + qW_e) = & X_{a_e} + \frac{\partial X}{\partial u}u + \frac{\partial X}{\partial v}v + \frac{\partial X}{\partial w}w + \frac{\partial X}{\partial p}p + \frac{\partial X}{\partial q}q + \frac{\partial X}{\partial r}r + \frac{\partial X}{\partial \dot{w}}\dot{w} \\ & - mg \sin(\theta_e) - mg\theta \cos(\theta_e) + \frac{\partial X}{\partial \xi}\xi + \frac{\partial X}{\partial \eta}\eta + \frac{\partial X}{\partial \zeta}\zeta + \frac{\partial X}{\partial \tau}\tau \end{aligned} \quad (2.81)$$

$$\begin{aligned} m(\dot{v} - pW_e + RU_e) = & Y_{a_e} + \frac{\partial Y}{\partial u}u + \frac{\partial Y}{\partial v}v + \frac{\partial Y}{\partial w}w + \frac{\partial Y}{\partial p}p + \frac{\partial Y}{\partial q}q + \frac{\partial Y}{\partial r}r + \frac{\partial Y}{\partial \dot{w}}\dot{w} \\ & + mg\psi \sin(\theta_e) + mg\phi \cos(\theta_e) + \frac{\partial Y}{\partial \xi}\xi + \frac{\partial Y}{\partial \eta}\eta + \frac{\partial Y}{\partial \zeta}\zeta + \frac{\partial Y}{\partial \tau}\tau \end{aligned} \quad (2.82)$$

$$\begin{aligned} m(\dot{w} - qU_e) = & Z_{a_e} + \frac{\partial Z}{\partial u}u + \frac{\partial Z}{\partial v}v + \frac{\partial Z}{\partial w}w + \frac{\partial Z}{\partial p}p + \frac{\partial Z}{\partial q}q + \frac{\partial Z}{\partial r}r + \frac{\partial Z}{\partial \dot{w}}\dot{w} \\ & + mg \cos(\theta_e) - mg\theta \sin(\theta_e) + \frac{\partial Z}{\partial \xi}\xi + \frac{\partial Z}{\partial \eta}\eta + \frac{\partial Z}{\partial \zeta}\zeta + \frac{\partial Z}{\partial \tau}\tau \end{aligned} \quad (2.83)$$

$$\begin{aligned} I_{xx}\dot{p} - I_{xz}\dot{r} = & L_{a_e} + \frac{\partial L}{\partial u}u + \frac{\partial L}{\partial v}v + \frac{\partial L}{\partial w}w + \frac{\partial L}{\partial p}p + \frac{\partial L}{\partial q}q \\ & + \frac{\partial L}{\partial r}r + \frac{\partial L}{\partial \dot{w}}\dot{w} + \frac{\partial L}{\partial \xi}\xi + \frac{\partial L}{\partial \eta}\eta + \frac{\partial L}{\partial \zeta}\zeta + \frac{\partial L}{\partial \tau}\tau \end{aligned} \quad (2.84)$$

$$\begin{aligned} I_{yy}\dot{q} = & M_{a_e} + \frac{\partial M}{\partial u}u + \frac{\partial M}{\partial v}v + \frac{\partial M}{\partial w}w + \frac{\partial M}{\partial p}p + \frac{\partial M}{\partial q}q \\ & + \frac{\partial M}{\partial r}r + \frac{\partial M}{\partial \dot{w}}\dot{w} + \frac{\partial M}{\partial \xi}\xi + \frac{\partial M}{\partial \eta}\eta + \frac{\partial M}{\partial \zeta}\zeta + \frac{\partial M}{\partial \tau}\tau \end{aligned} \quad (2.85)$$

2.4. Linearised Equations of Motion

$$\begin{aligned}
 I_{zz}\dot{r} - I_{xz}\dot{p} = & N_{a_e} + \frac{\partial N}{\partial u}u + \frac{\partial N}{\partial v}v + \frac{\partial N}{\partial w}w + \frac{\partial N}{\partial p}p + \frac{\partial N}{\partial q}q \\
 & + \frac{\partial N}{\partial r}r + \frac{\partial N}{\partial \dot{w}}\dot{w} + \frac{\partial N}{\partial \xi}\xi + \frac{\partial N}{\partial \eta}\eta + \frac{\partial N}{\partial \zeta}\zeta + \frac{\partial N}{\partial \tau}\tau
 \end{aligned} \quad (2.86)$$

However, in steady-state it is considered that the perturbation variables and their derivatives are zero (equation 2.87).

$$\begin{aligned}
 X_{a_e} = mg \sin(\theta_e) \quad Y_{a_e} = 0 \quad Z_{a_e} = -mg \cos(\theta_e) \\
 L_{a_e} = 0 \quad M_{a_e} = 0 \quad N_{a_e} = 0
 \end{aligned} \quad (2.87)$$

Considering the above assumptions equations 2.81 through 2.86 are rearranged into equations 2.88 through 2.93.

$$\begin{aligned}
 m\dot{u} - \frac{\partial X}{\partial u}u - \frac{\partial X}{\partial v}v - \frac{\partial X}{\partial \dot{w}}\dot{w} - \frac{\partial X}{\partial w}w - \frac{\partial X}{\partial p}p - \left(\frac{\partial X}{\partial q}q - mW_e \right) q \\
 - \frac{\partial X}{\partial r}r + mg\theta \cos(\theta_e) = \frac{\partial X}{\partial \xi}\xi + \frac{\partial X}{\partial \eta}\eta + \frac{\partial X}{\partial \zeta}\zeta + \frac{\partial X}{\partial \tau}\tau
 \end{aligned} \quad (2.88)$$

$$\begin{aligned}
 -\frac{\partial Y}{\partial u}u + m\dot{v} - \frac{\partial Y}{\partial v}v - \frac{\partial Y}{\partial \dot{w}}\dot{w} - \frac{\partial Y}{\partial w}w - \left(\frac{\partial Y}{\partial p}p + mW_e \right) p - \frac{\partial Y}{\partial q}q - \left(\frac{\partial Y}{\partial r}r + mU_e \right) r \\
 - mg\phi \cos(\theta_e) - mg\psi \sin(\theta_e) = \frac{\partial Y}{\partial \xi}\xi + \frac{\partial Y}{\partial \eta}\eta + \frac{\partial Y}{\partial \zeta}\zeta + \frac{\partial Y}{\partial \tau}\tau
 \end{aligned} \quad (2.89)$$

$$\begin{aligned}
 -\frac{\partial Z}{\partial u}u - \frac{\partial Z}{\partial v}v + \left(m - \frac{\partial Z}{\partial \dot{w}} \right) \dot{w} - \frac{\partial Z}{\partial w}w - \frac{\partial Z}{\partial p}p - \left(\frac{\partial Z}{\partial q} + mU_e \right) q \\
 - \frac{\partial Z}{\partial r}r + mg\theta \sin(\theta_e) = \frac{\partial Z}{\partial \xi}\xi + \frac{\partial Z}{\partial \eta}\eta + \frac{\partial Z}{\partial \zeta}\zeta + \frac{\partial Z}{\partial \tau}\tau
 \end{aligned} \quad (2.90)$$

$$\begin{aligned}
 -\frac{\partial L}{\partial u}u - \frac{\partial L}{\partial v}v - \frac{\partial L}{\partial \dot{w}}\dot{w} - \frac{\partial L}{\partial w}w + I_{xx}\dot{p} - \frac{\partial L}{\partial p}p - \frac{\partial L}{\partial q}q \\
 - I_{xz}\dot{r} - \frac{\partial L}{\partial r}r = -\frac{\partial L}{\partial \xi}\xi + \frac{\partial L}{\partial \eta}\eta + \frac{\partial L}{\partial \zeta}\zeta + \frac{\partial L}{\partial \tau}\tau
 \end{aligned} \quad (2.91)$$

$$\begin{aligned}
 -\frac{\partial M}{\partial u}u - \frac{\partial M}{\partial v}v - \frac{\partial M}{\partial \dot{w}}\dot{w} - \frac{\partial M}{\partial w}w - \frac{\partial M}{\partial p}p + I_{yy}\dot{q} - \frac{\partial M}{\partial q}q \\
 - \frac{\partial M}{\partial r}r = \frac{\partial M}{\partial \xi}\xi + \frac{\partial M}{\partial \eta}\eta + \frac{\partial M}{\partial \zeta}\zeta + \frac{\partial M}{\partial \tau}\tau
 \end{aligned} \quad (2.92)$$

$$\begin{aligned}
 -\frac{\partial N}{\partial u}u - \frac{\partial N}{\partial v}v - \frac{\partial N}{\partial \dot{w}}\dot{w} - \frac{\partial N}{\partial w}w - I_{xz}\dot{p} - \frac{\partial N}{\partial p}p - \frac{\partial N}{\partial q}q + I_{zz}\dot{r} \\
 - \frac{\partial N}{\partial r}r = \frac{\partial N}{\partial \xi}\xi + \frac{\partial N}{\partial \eta}\eta + \frac{\partial N}{\partial \zeta}\zeta + \frac{\partial N}{\partial \tau}\tau
 \end{aligned} \quad (2.93)$$

2.4. Linearised Equations of Motion

The above linear differential equations, from 2.88 through 2.93, constitute the small perturbations equations of motion, which describe the small amplitude response of the aircraft, at a trim condition, to a perturbation [6, 11]. Although these allow the description of longitudinal and lateral coupled motions of the aircraft since only small disturbances are considered a common simplification is to rearrange these into two different groups, one describing longitudinal motion and the other describing lateral-directional motion [6, 12].

2.4.1 Decoupled Equations of Motion

The decoupled longitudinal equations of motions describe the motion restricted the $Ox_B Oz_B$ plane of symmetry. Therefore only the axial force X , the normal force Z and the pitching moment M are considered, which in turn implies that lateral motion variables, aerodynamic coupling derivatives and aerodynamic control derivatives can be assumed to be zero [6] as described in equation 2.94.

$$\begin{aligned} \frac{\partial X}{\partial v} = \frac{\partial X}{\partial p} = \frac{\partial X}{\partial r} = \frac{\partial Z}{\partial v} = \frac{\partial Z}{\partial p} = \frac{\partial Z}{\partial r} = \frac{\partial M}{\partial v} = \frac{\partial M}{\partial p} = \frac{\partial M}{\partial r} = \frac{\partial X}{\partial \xi} = \frac{\partial X}{\partial \zeta} = \\ \frac{\partial Z}{\partial \xi} = \frac{\partial Z}{\partial \zeta} = \frac{\partial M}{\partial \xi} = \frac{\partial M}{\partial \zeta} = 0 \end{aligned} \quad (2.94)$$

Thus, the dimensional decoupled equations of longitudinal symmetric motion (equations 2.95 through 2.97) , with respect to the aircraft body axes, are obtained by considering the several equalities of equation 2.94 and by plucking out the equations corresponding the axial force X , the normal force Z and pitching moment M from equations 2.88 through 2.93.

$$m\dot{u} - \frac{\partial X}{\partial u}u - \frac{\partial X}{\partial \dot{w}}\dot{w} - \frac{\partial X}{\partial w}w - \left(\frac{\partial X}{\partial q}q - mW_e \right) q + mg\theta \cos(\theta_e) = \frac{\partial X}{\partial \eta}\eta + \frac{\partial X}{\partial \tau}\tau \quad (2.95)$$

$$-\frac{\partial Z}{\partial u}u + \left(m - \frac{\partial Z}{\partial \dot{w}} \right) \dot{w} - \frac{\partial Z}{\partial w}w - \left(\frac{\partial Z}{\partial q} + mU_e \right) q + mg\theta \sin(\theta_e) = \frac{\partial Z}{\partial \eta}\eta + \frac{\partial Z}{\partial \tau}\tau \quad (2.96)$$

$$-\frac{\partial M}{\partial u}u - \frac{\partial M}{\partial \dot{w}}\dot{w} - \frac{\partial M}{\partial w}w + I_{yy}\dot{q} - \frac{\partial M}{\partial q}q = \frac{\partial M}{\partial \eta}\eta + \frac{\partial M}{\partial \tau}\tau \quad (2.97)$$

Opposed to the decoupled longitudinal equations of motions the decoupled lateral-directional motion takes into account the rolling, yawing and sideslip motion alone. Therefore only the side force Y , the rolling moment L and the yawing moment N are considered whilst longitudinal motion variables and aerodynamic coupling derivatives and aerodynamic coupling control derivatives are set to be equal to zero [6].

$$\begin{aligned} \frac{\partial Y}{\partial u} = \frac{\partial Y}{\partial \dot{w}} = \frac{\partial Y}{\partial w} = \frac{\partial Y}{\partial q} = \frac{\partial L}{\partial u} = \frac{\partial L}{\partial \dot{w}} = \frac{\partial L}{\partial w} = \frac{\partial L}{\partial q} = \frac{\partial N}{\partial u} = \frac{\partial N}{\partial \dot{w}} = \\ = \frac{\partial N}{\partial w} = \frac{\partial N}{\partial q} = \frac{\partial Y}{\partial \eta} = \frac{\partial Y}{\partial \tau} = \frac{\partial L}{\partial \eta} = \frac{\partial L}{\partial \tau} = \frac{\partial N}{\partial \eta} = \frac{\partial N}{\partial \tau} = 0 \end{aligned} \quad (2.98)$$

2.4. Linearised Equations of Motion

Similarly to the decoupled longitudinal motion, the dimensional decoupled equations of lateral directional asymmetric motion (equations 2.99 through 2.101), with respect to body axes, are obtained by considering equation 2.98 and by plucking out the equations corresponding to the side force Y , the rolling moment L and yawing moment N from equations 2.88 through 2.93 [6].

$$\left(m\dot{v} - \frac{\partial Y}{\partial v}v - \left(\frac{\partial Y}{\partial p} + mW_e \right) p - \left(\frac{\partial Y}{\partial r} - mU_e \right) r - mg\phi \cos(\theta_e) - mg\psi \sin(\theta_e) \right) = \frac{\partial Y}{\partial \xi}\xi + \frac{\partial Y}{\partial \zeta}\zeta \quad (2.99)$$

$$-\frac{\partial L}{\partial v}v + I_{xx}\dot{p} - \frac{\partial L}{\partial p}p - I_{xz}\dot{r} - \frac{\partial L}{\partial r}r = \frac{\partial L}{\partial \xi}\xi + \frac{\partial L}{\partial \zeta}\zeta \quad (2.100)$$

$$-\frac{\partial N}{\partial v}v + I_{xz}\dot{p} - \frac{\partial N}{\partial p}p - I_{zz}\dot{r} - \frac{\partial N}{\partial r}r = \frac{\partial N}{\partial \xi}\xi + \frac{\partial N}{\partial \zeta}\zeta \quad (2.101)$$

2.4.2 State Space Representation of the Linearised Equations of Motion

In the previous sections, a mathematical model representing the dynamics of a generic aircraft with six DOF was developed, which culminated with the determination of the decoupled longitudinal and lateral-directional EOM. If only small disturbances are to be considered, as was in the development of the linearised EOM, the aircraft can be considered as a linear dynamic system. Hence it can then be classified as being a linear time-invariant system (LTI) or as a linear time-varying system (LTV) depending on whether the differential equations which describe the system have constant or time-dependent coefficients. Since the decoupled equations of motion (2.95, 2.96, 2.97, 2.99, 2.100 and 2.101) are linear and have constant coefficients the dynamic system (aircraft) is said to be LTI. The motion (state) of a linear dynamic system is described by state variables, that is, it is described by a set with the least amount of variables with which the state of a dynamic system can be described. The motion of a dynamic system is represented in a n -dimensional space called state space and each state of the system is represented by a point in this space. The state-space equations model the dynamics of the system and differ upon whether the system is considered to be LTI or LTV. If the system is LTV the state equations are written as in 2.102 and 2.103 [6, 14].

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad (2.102)$$

$$y(t) = C(t)x(t) + D(t)u(t) \quad (2.103)$$

Conversely, equations 2.104 and 2.105 represent a LTI system.

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (2.104)$$

$$y(t) = Cx(t) + Du(t) \quad (2.105)$$

The equations 2.102 and 2.104 are the state equations whereas equations 2.103 and 2.105 are the output equations of the respective dynamic systems where:

- $x(t)$ is the state vector constituted by n state variables;
- $u(t)$ is the input vector constituted by m input variables;
- $y(t)$ is the output vector constituted by r output variables;

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- A or $A(t)$ is the $(n \times n)$ state matrix;
- B or $B(t)$ is the $(n \times m)$ input matrix;
- C or $C(t)$ is the $(r \times n)$ output matrix;
- D or $D(t)$ is the $(r \times m)$ direct matrix.

Since the aircraft is a LTI dynamic system, equations 2.104 and 2.105 describe its dynamics. It now remains to rearrange the decoupled EOM to such format. For convenience purposes, it is common practice to choose for the output variables, for most aircraft, to be the state variables as described by equation 2.106

$$y(t) = x(t) \quad \text{and} \quad r = n \quad (2.106)$$

which implies that the output matrix C equals to the $(n \times n)$ identity matrix and the direct matrix D is equal to the $(n \times m)$ zero matrix. Consequently, the output equation can be written as in 2.107. Perusing the latter it can be concluded that it is only required to determine the state equation.

$$y(t) = Ix(t) = x(t) \quad (2.107)$$

In order to rewrite the decoupled longitudinal EOM (equations 2.95, 2.96 and 2.97) in matrix format an additional differential equation needs to be considered since the aircraft's longitudinal motion is described by four state variables. As such it considered that the time derivative of the pitch angle θ is equal to the pitch rate q (equation 2.108).

$$\dot{\theta} = q \quad (2.108)$$

Hence, considering equations 2.95, 2.96, 2.97 and 2.108 we have

$$M\dot{x}(t) = A'x(t) + B'u(t) \quad (2.109)$$

with

$$x(t) = \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} \quad (2.110)$$

$$u(t) = \begin{bmatrix} \eta \\ \tau \end{bmatrix} \quad (2.111)$$

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$$M = \begin{bmatrix} m & -\frac{\partial X}{\partial \dot{w}} & 0 & 0 \\ 0 & (m - \frac{\partial Z}{\partial \dot{w}}) & 0 & 0 \\ 0 & -\frac{\partial M}{\partial \dot{w}} & I_{yy} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.112)$$

$$A' = \begin{bmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial w} & (\frac{\partial X}{\partial q} - mW_e) & -mg \cos(\theta_e) \\ \frac{\partial Z}{\partial u} & \frac{\partial Z}{\partial w} & (\frac{\partial Z}{\partial q} + mU_e) & -mg \sin(\theta_e) \\ \frac{\partial M}{\partial u} & \frac{\partial M}{\partial w} & \frac{\partial M}{\partial q} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (2.113)$$

$$B' = \begin{bmatrix} \frac{\partial X}{\partial \eta} & \frac{\partial X}{\partial \tau} \\ \frac{\partial Z}{\partial \eta} & \frac{\partial Z}{\partial \tau} \\ \frac{\partial M}{\partial \eta} & \frac{\partial M}{\partial \tau} \\ 0 & 0 \end{bmatrix} \quad (2.114)$$

Multiplying equation 2.109 by the M^{-1} yields equation 2.115.

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (2.115)$$

Consequently the state matrix and input matrix are determined as in equations 2.116 and 2.117, respectively. The terms that form matrices A and B are the aerodynamic stability and control derivatives, respectively, with respect the body axes written in a concise form. The complete expressions of these derivatives, adapted from reference [6], are in appendix A.

$$A = M^{-1}A' = \begin{bmatrix} x_u & x_w & x_q & x_\theta \\ z_u & z_w & z_q & z_\theta \\ m_u & m_w & m_q & m_\theta \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (2.116)$$

$$B = M^{-1}B' = \begin{bmatrix} x_\eta & x_\tau \\ z_\eta & z_\tau \\ m_\eta & m_\tau \\ 0 & 0 \end{bmatrix} \quad (2.117)$$

Consequently, the longitudinal state equation is written as given by matrix equation 2.118.

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} x_u & x_w & x_q & x_\theta \\ z_u & z_w & z_q & z_\theta \\ m_u & m_w & m_q & m_\theta \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} x_\eta & x_\tau \\ z_\eta & z_\tau \\ m_\eta & m_\tau \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta \\ \tau \end{bmatrix} \quad (2.118)$$

2.4. Linearised Equations of Motion

In turn, the output equation is written as described in matrix equation 2.119.

$$y(t) = Ix(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} \quad (2.119)$$

In a similar manner to the longitudinal state equation, the lateral-direction state equation is obtained from the decoupled lateral-directional EOM (equation 2.99 through equation 2.101) with respect to body axes thus yielding matrix equation 2.120.

$$\begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} y_v & y_p & y_r & y_\phi & y_\psi \\ l_v & l_p & l_r & l_\phi & l_\psi \\ n_v & n_p & n_r & n_\phi & n_\psi \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \phi \\ \psi \end{bmatrix} + \begin{bmatrix} y_\xi & y_\zeta \\ l_\xi & l_\zeta \\ n_\xi & n_\zeta \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \zeta \end{bmatrix} \quad (2.120)$$

2.4. Linearised Equations of Motion

Chapter 3

Parameters Estimation Methods

3.1 Numerical Differentiation

As previously stated in sub-subsection 2.4.2 the aircraft is an LTI system whose state equation is described by equation 2.104 (here rewritten in equation 3.1 for convenience purposes).

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (3.1)$$

Later on Chapter 4 it will be shown the necessity of having the time derivative of the flight data. However, the flight data is recorded (in the present dissertation it will be generated) at discrete time intervals and consequently, an analytical derivative for each state variable that constitutes the term \dot{x} in equation 3.1 is not available. To overcome this predicament, in the present section, it will be derived three formulae for computing the first time derivative at a given for discrete flight.

It is often necessary to compute the value of a derivative at some point of a generic function whose expression is difficult to differentiate or is not available. As an alternative, an approach to obtain an approximation to the value of the derivative at some point of a generic function is through the usage of Taylor series expansions. Given the value of a function and the value of its derivatives at some point, a , the Taylor series allows to determine a function value at another point, x [13, 15].

Taylor Polynomial *If the function f and its $n+1$ first derivatives exist and are continuous within an interval that contains x and a then there exists an ξ between x and a such that the value of f at x is given by equation 3.2*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x) \quad (3.2)$$

where $R_n(x)$ is the rest of n -th order and it is obtained by computing equation 3.3

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \quad (3.3)$$

The Taylor series expansion or Taylor polynomial, equation 3.2, will only yield an exact estimate of the function value at some point a if an infinite number of terms is added. This is however not feasible and for practical purposes, the inclusion of only a few terms in the Taylor series expansion will return a good approximation of the true value at some point of the function. Substituting in equation 3.2 $a = x_i$ and $x = x_{i+1}$ yields equation 3.4.

3.1. Numerical Differentiation

$$\begin{aligned}
 f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \dots \\
 + \frac{f^{(n)}(x_i)}{n!}(x_{i+1} - x_i)^n \\
 + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x_{i+1} - x_i)^{n+1}
 \end{aligned} \quad (3.4)$$

Let $h = x_{i+1} - x_i$, equation 3.4 is then simplified to

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \dots + \frac{f^{(n)}(x_i)}{n!}h^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}h^{n+1} \quad (3.5)$$

If equation 3.5 is truncated at $n = 1$ it then assumes the form of equation 3.6 with $\xi \in]x_i, x_{i+1}[$.

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(\xi)}{2!}h^2 \quad (3.6)$$

Then, rearranging 3.6 yields the formula for the first derivative at x_i as is described by equation 3.7.

$$f'(x_i) = \frac{1}{h} (f(x_{i+1}) - f(x_i)) - \frac{f''(\xi)}{2!}h \quad (3.7)$$

The second term of the right-hand side of equation 3.7 is the truncation error of order h , $\mathcal{O}(h)$. Using equation 3.7, the value of the derivative at some point x_i can be determined by knowing the value of the function at a point x_i and at the following point x_{i+1} . This is shown by Figure 3.1(a). Similarly, a formula can be derived using a previous point to x_i . Substituting in equation 3.2 $a = x_i$ and $x = x_{i-1}$, letting $h = x_{i-1} - x_i$ and truncating the resulting equation at $n = 1$ yields equation 3.8 with $\xi \in]x_{i-1}, x_i[$.

$$f(x_{i-1}) = f(x_i) - f'(x_i)h + \frac{f''(\xi)}{2!}h^2 \quad (3.8)$$

Rearranging equation 3.8 yields equation 3.9.

$$f'(x_i) = \frac{1}{h} (f(x_i) - f(x_{i-1})) - \frac{f''(\xi)}{2!}h \quad (3.9)$$

Using equation 3.9 the value of the first derivative of the function is computed using the value of the function at the previous point the actual point as shown in 3.1(b). The error estimates associated with equations 3.7 and 3.9 are of order $\mathcal{O}(h)$, however, better estimates can be determined if when deriving the expression of the derivatives of the function terms of higher-order in the Taylor series expansion are considered. Proceeding in a similar manner, if this time equation 3.5 is truncated at $n = 2$ we have:

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f'''(\xi_1)}{3!}h^3 \quad (3.10)$$

3.1. Numerical Differentiation

with $\xi_1 \in]x_i, x_{i+1}[$, and

$$f(x_{i-1}) = f(x_i) - f'(x_i)h + \frac{f''(x_i)}{2!}h^2 - \frac{f'''(\xi_2)}{3!}h^3 \quad (3.11)$$

with $\xi_2 \in]x_{i-1}, x_i[$.

Rearranging equations 3.10 and 3.11 yields equations 3.12 and 3.13, respectively.

$$f'(x_i) = \frac{1}{2h} [-3f(x_i) + 4f(x_{i+1}) - f(x_{i+2})] + \frac{h^2}{3} f'''(\xi) \quad (3.12)$$

$$f'(x_i) = \frac{1}{2h} [f(x_{i-2}) - 4f(x_{i-1}) + 3f(x_i)] + \frac{h^2}{3} f'''(\xi) \quad (3.13)$$

Similarly to equations 3.7 and 3.9, equations 3.12 and 3.13 allow to compute the derivative at some point x_i of the function however equations 3.12 and 3.13 require the knowledge of one more point hence reducing therefore the error committed in the approximation from order $\mathcal{O}(h)$ to order $\mathcal{O}(h^2)$ thus improving the accuracy of the approximation. Additionally, another type of finite-divided-difference formula can be derived from subtracting equation 3.11 from equation 3.10 gives equation 3.14.

$$f(x_{i+1}) - f(x_{i-1}) = 2f'(x_i)h + \frac{h^2}{6} f'''(\xi) \quad (3.14)$$

Hence,

$$f'(x_i) = \frac{1}{2h} [-f(x_{i-1}) + f(x_{i+1})] - \frac{h^2}{6} f'''(\xi) \quad (3.15)$$

Equation 3.15 gives the centered finite-divided-difference formula of order $\mathcal{O}(h^2)$ depicted in Figure 3.1(c).

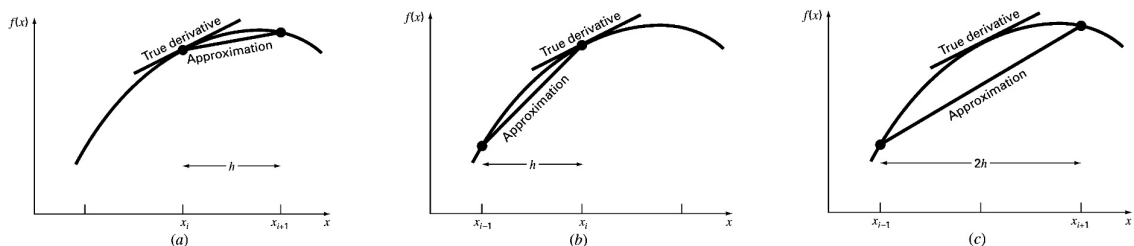


Figure 3.1: Visual representation of the forward (a), backward (b) and centered (c) finite-divided difference approximation of the first derivative of a generic function [13].

3.2 Numerical Methods for Solving Ordinary Differential Equations

A differential equation is defined as begin an equation that involves an unknown function and its derivatives. If only one independent variable is present, the differential equation would be classified as an ordinary differential equation (ODE) whereas if more than one independent variable was present the equation would be called a partial differential equation (PDE).

A differential equation can often be solved through integration techniques, in which case, an analytical exact solution (a function with the independent variable and parameters as arguments) that satisfies the equality of the differential equation is found. However, many problems involve differential equations that are non-linear and thus an exact analytical solution isn't available. As such a different approach may be to solve it through linearisation or through numerical methods. Although linearisation can be used to solve differential equations its applicability is constrained to problems where the dependent variable only assumes small values.

For example, consider the pendulum of Figure 3.2. Applying Newton's second law to the above system yields equation 3.16 which is a non-linear ODE.

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin(\theta) = 0 \quad (3.16)$$

However, 3.16 can be solved through linearisation if it is considered that the pendulum only moves a small amount from the equilibrium, in which case, it can be assumed that $\sin(\theta) \approx \theta$ and consequently equation 3.16 can be rewritten as in equation 3.17

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \quad (3.17)$$

thus obtaining, this way, a linear ODE which can now be solved through analytical methods. However, the application linearisation to non-linear differential equations is constrained when only small displacements (from the equilibrium position) are considered. When that is not the case numerical methods must be employed [13]. There is a wide range of numerical methods to solve a differential equation but, in the present work, the 6th order Butcher algorithm is used since it yields high accuracy.

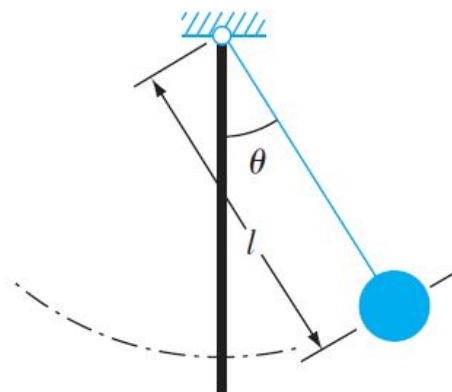


Figure 3.2: Pendulum and the geometrical descriptive parameters of its motion [13].

3.3. Savitzky-Golay Smoothing Filter

Butcher's Algorithm[16]

Given the initial value x_0 the states of the system in $t_1, t_2, \dots, t_k, \dots$ are determined by computing equation 3.18.

$$x_{k+1} = x_k + \frac{1}{90} (7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6) \quad (3.18)$$

where

$$k_1 = h \cdot f(x_k, u_k)$$

$$k_2 = h \cdot f(x_k + k_1/4, u_k)$$

$$k_3 = h \cdot f(x_k + k_1/8 + k_2/8, u_k)$$

$$k_4 = h \cdot f(x_k - k_2/2 + k_3, u_k)$$

$$k_5 = h \cdot f(x_k + 3k_1/16 + 9k_4/16, u_k)$$

$$k_6 = h \cdot f(x_k - 3k_1/7 + 2k_2/7 + 12k_3/7 - 12k_4/7 + 8k_5/7, u_k)$$

3.3 Savitzky-Golay Smoothing Filter

When in an experiment, data has to be collected, whether by direct measurements or through analogue or electronic measuring instruments, it is often the case that random errors, commonly known as noise, are superimposed on the collected data. Hence, to analyse the data and draw conclusions from it one wants to reduce as much noise as possible. One method that may be employed to reduce the noise is the moving average.

From the set of collected data one chooses an uneven numbered interval of data points ranging from $-n_L$ to n_R , where n_L and n_R represent the number of points to the left and to the right of the central data point, respectively. Then their values are added, $f_{i-n_L} + \dots + f_{i+n_R}$, and divided by the number of points, N . The value of middle point of the interval, f_i , is then replaced by the computed average, g_i , given by equation 3.19 [17, 18]

$$g_i = \sum_{n=-n_L}^{n_R} c_n f_{i+n} \quad (3.19)$$

where $c_n = 1/(n_L + n_R + 1)$. To obtain the next interval of data points, the first interval is slid in such a way that the first data point of the first interval is dropped and the data point adjacent to the last data point of the first interval is added thus creating a new interval and then the averaging process is repeated.

Consequently, the moving average procedure is based on the concept of convolution. Each data value that is added in the interval is multiplied by one and then divided by the number of points added. The set of ones multiplied is what is known as the convolution function whereas the number of points by which the sum of products is divided by is what is known as the normalizing factor. It is however important to notice that the experimental data, when plotted a continuous and slowly varying smooth function, must be depicted otherwise the results obtained would be

3.3. Savitzky-Golay Smoothing Filter

biased, that is, although the moving average will reduce the noise present in the data, if a peak (a maximum or minimum) is present its height will be reduced since the same weight is given to each data point. Furthermore, the data points must be equally-spaced contrarily some data points would be left unchanged [17, 18].

Alternatively, in order to avoid the introduction of bias into the data set the Savitzky-Golay Smoothing Filter may be considered. Similarly to the moving average, a convolution function, i.e., a set of coefficients c_n is determined but in such a way that the fitted function is considered to be the best fit to the data set in accordance to the least-squares criterion. For example, if a given cubic polynomial is to be fitted to a data set its coefficients are determined, through the Ordinary Least Squares (OLS), in such a way that the sum of the squared values between each observed value and the data point obtained with the estimated polynomial, at the same abscissa, is minimized. After the polynomial coefficients are determined, the value at the centre of the interval is computed and it then replaces the corresponding observed value. The window is then slid as in the moving average procedure. Hence, in this way for every interval, it is necessary to compute a new set of coefficients for a given polynomial which would be a burdensome and time-consuming process.

Since performing OLS is just a means of replacing the observed value f_i at the centre of the interval by the value given by the fitted polynomial g_i , a procedure using OLS can be derived in such way that a set of coefficients is determined and then convoluted with the data values as in the moving average procedure [17] [18]. In this way, each observed value is replaced by a data point, using the least-squares criteria, without determining a new polynomial for every interval thus making this procedure more efficient.

Consider a matrix $A_{(n_L+n_R+1) \times (M+1)}(\mathbb{R})$, the vector of unknowns $a_{(M+1) \times 1}(\mathbb{R})$ and the vector of independent terms $f_{(n_L+n_R+1) \times 1}(\mathbb{R})$

$$A = \begin{bmatrix} 1 & -n_L & \cdots & (-n_L)^M \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & n_R & \cdots & (n_R)^M \end{bmatrix} \quad a = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_M \end{bmatrix} \quad f = \begin{bmatrix} f_{i-n_L} \\ \vdots \\ f_i \\ \vdots \\ f_{i+n_R} \end{bmatrix}$$

the normal equations are given by 3.20

$$(A^T \cdot A) \cdot a = A^T \cdot f \quad (3.20)$$

or by equation 3.21

$$a = (A^T \cdot A)^{-1} \cdot (A^T \cdot f) \quad (3.21)$$

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The dot product between the transpose of matrix A and itself, depicted in equation 3.20 and 3.21 , is given by equation 3.22.

$$\{A^T \cdot A\}_{ij} = \sum_{k=-n_L}^{n_R} A_{ki}A_{kj} = \sum_{k=-n_L}^{n_R} k^{i+j} \quad (3.22)$$

In a similar manner, the dot product between the transpose of matrix A and the vector of observations f , depicted in equation 3.20 and 3.21 , is given by equation 3.23.

$$\{A^T \cdot f\}_j = \sum_{k=-n_L}^{n_R} A_{kj}f_k = \sum_{k=-n_L}^{n_R} k^j f_k \quad (3.23)$$

Since the coefficient c_n is the component a_0 when f is replaced by the unit vector e_n , $-n_L \leq n \leq n_R$, then it can be computed as in equation 3.24.

$$c_n = \left\{ (A^T \cdot A)^{-1} \cdot (A^T \cdot e_n) \right\}_0 = \sum_{m=0}^M \left\{ (A^T \cdot A)^{-1} \right\}_{0m} n^m \quad (3.24)$$

Equation 3.24 states that only one row from the inverse matrix is then necessary. A question that may arise when applying the Savitzky-Golay Smoothing Filter is that of which length to define the window with and that of which order to choose for the polynomial to be fitted to the data set. Accordingly to reference [19], the window length q is always uneven numbered, as previously stated since this algorithm always requires a data point to be at the centre of the window. Letting p denote the order of the polynomial the window length q can then be computed as in equation 3.25.

$$q = 2p + 1 \quad (3.25)$$

Furthermore, the window length q should always assume a value between seven and eleven since if it was to assume a value less than seven the noise would not be filtered by the Savitzky-Golay smoothing filter procedure. Conversely, if q was to assume a value greater than eleven this procedure would filter too much noise, that is, it would distort the "true signal".

3.4 Linear Regression Analysis

In science and engineering it is often needed to determine, through the analysis of data, the relationship between variables and moreover generate a model that describes and predicts the process's outcome where those variables interplay [20]. The analysis of data, and consequently the attainment of a model, can be performed aided by a subfield of statistics named statistical learning.

Statistical learning is a series of tools used for drawing conclusions from a set of data and is divided into two different groups: supervised and unsupervised statistical learning. The first involves the derivation of a statistical model for prediction or estimation of a quantitative or qualitative output given one or more inputs, a problem known as regression or classification,

3.4. Linear Regression Analysis

respectively; whereas the latter comprises the development of a statistical model where there are one or more inputs but there isn't any output and consequently no prediction or estimation is made, instead relationships are drawn from the data set by the organisation of data into clusters based upon a given criteria, or by the reduction of the data's dimensionality, a problem known as clustering [21].

As stated previously, supervised statistical learning provides the tools with which a relationship between an observed response Y and one or more predictors X_1, X_2, \dots, X_K can be estimated as in equation 3.26

$$Y = f(x_1, \dots, x_k, \beta_0, \dots, \beta_k) + \varepsilon \quad (3.26)$$

where f represents an unknown function and the error term is a stochastic term, independent of x and with zero mean, which implies the inexact relationship between the dependent and independent variables.

The two main reasons for trying to assess the functional form of f is to make predictions or inference. In the first case the main objective is to make accurate predictions of the values which the independent variable Y can assume in the future, for example, to predict the resale price of a vehicle given the number of kilometres travelled, the initial price and so on; in the latter case the objective is to understand what effect a given predictor has on the outcome variable, i.e., using the above example, what is the effect on resale price of a vehicle if the initial price of the same vehicle is increased by a thousand euros, other things being equal.

Furthermore, statistical learning methods encompass a broad scope of methods for estimating the functional form of f and this can be divided into two different categories: parametric and non-parametric. The main difference between these is that parametric methods make an assumption about the functional form of f , that is, it simplifies the problem of estimating the functional form by reducing it to the estimation of a set of parameters. A potential disadvantage of this approach is that the estimated functional form of f won't match perfectly its true form. On the other hand, if a non-parametric functional form were to be chosen the assumption of the functional form of f is discarded and a wider variety of functional forms are available and consequently, a better match to the data may be achieved. There are however major disadvantages regarding non-parametric models: a far larger number of observations is required, compared with the parametric model, in order to estimate an accurate form of f and since a more flexible form of f may be chosen a phenomenon named overfitting may occur.

Overfitting occurs when the estimated function f represents very accurately the training data and the noise that it contains and therefore it will fail to accurately represent new observations made on the system of interest. For these reasons, in the present work, a parametric form for f is chosen [21]. Additionally, since the present work only focuses on problems whose answers are quantitative values then only regression is discussed.

Regression analysis is a set of statistical techniques used to model relationships between a response variable (or dependent variable) and one or more explanatory variables (or independent variables) [1, 3, 20]. Suppose that data is collected on a system of interest (commonly called training data) where Y is the outcome random variable that depends on k independent variables $x = (x_1, x_2, \dots, x_k)$.

3.4. Linear Regression Analysis

Since the chosen statistical learning method was regression analysis the relationship between the dependent and independent variables can be written as described by 3.27.

$$Y = \beta_0 + \beta_1 x_1 + \dots + \beta_k x_k + \varepsilon \quad (3.27)$$

The relationship is assumed to be linear in the parameters, i.e., the partial derivatives of the function f with respect to the parameters, $\frac{\partial f}{\partial \beta_i}$ do not depend on the parameters, β_i for $i = 0, \dots, k$, themselves. Then the statistical model given by equation 3.27 is therefore known when the parameters β_i are determined. The estimation of the parameters can be performed through several different methods such as the principle of least squares, the maximum likelihood, method of moments and others [22]. For the present work, only the principle of least squares is employed in order to estimate the parameters mentioned above.

The principle of least squares, also known as the ordinary least squares (OLS), is a method for estimating the parameters β_0, \dots, β_k by minimizing the squared distance between the observations and corresponding values predicted by the fitted model (estimated regression line) as shown in Figure 3.3 [3, 22]. Henceforth, fitted model, regression line or estimated regression line denote the same concept and are used interchangeably.

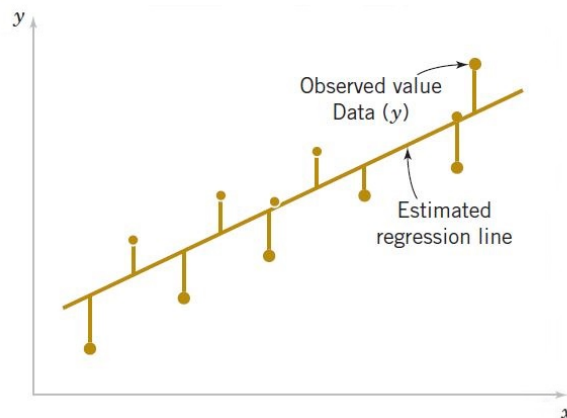


Figure 3.3: Deviation between the observations and the estimated regression model, adapted from [20].

3.4.1 Simple Linear Regression

In this subsection, it is discussed the simple linear regression model which considers a model with one dependent variable and only one independent variable [20, 23]. Hence it is assumed that the true relationship between the dependent variable Y and the independent variable X may be described by a straight line (equation 3.28).

$$E(Y|x) = \beta_0 + \beta_1 x \quad (3.28)$$

where $E(Y|x)$, depicted in Figure 3.4, represents the expected value of Y as a linear function of x and the parameters β_0 and β_1 are known as the regression coefficients, representing the interception of the straight line with the y -axis and the slope of line, respectively. However

3.4. Linear Regression Analysis

every observation of the training data do not fall precisely on fitted line (equation 3.28) and consequently, every data point of the training data can be obtained as in equation 3.29.

$$Y = \beta_0 + \beta_1 x + \varepsilon \quad (3.29)$$

The term ε in equation 3.29 is the a random error term assumed to be independent and identically distributed (i.i.d) from a normal distribution with zero mean and unknown variance σ^2 , i.e., $N(0, \sigma^2)$ [20, 22, 23].

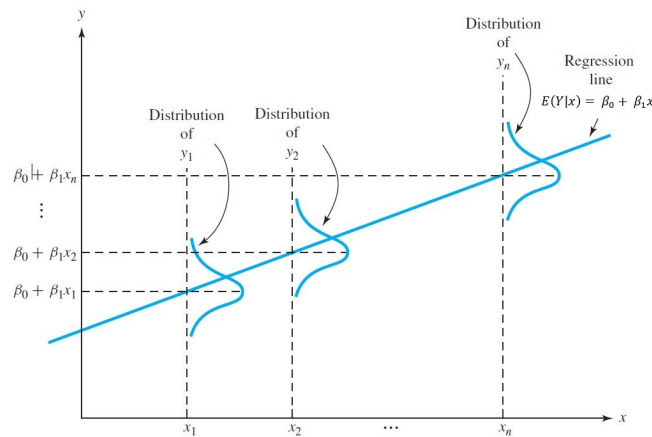


Figure 3.4: Distribution of Y as a function of x_i for the simple linear regression model, adapted from [23].

As previously stated at the beginning of section 3.4, in order to estimate the line (equation 3.28), that is, the line that represents the best fit to the training data OLS is used. Using the least-squares principle the sum of the squared vertical deviations between the observations and the line in Figure 3.5 is minimized. Hence to estimate the parameters the partial derivatives of the equation describing the sum of squares of the deviations of the observations with respect to β_0 and β_1 are taken and set equal to zero as described by equation 3.30 [20, 23].

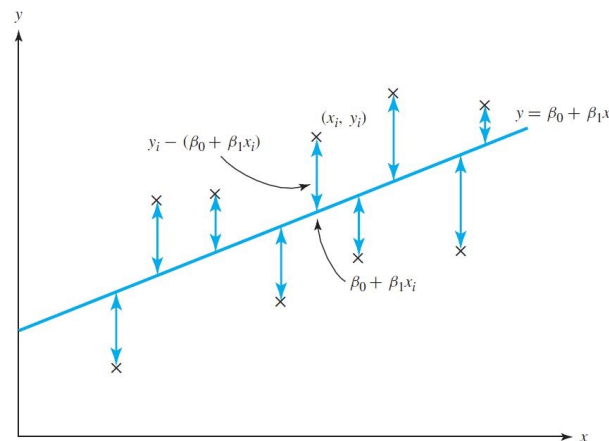


Figure 3.5: Distances between the data points of the training set and the corresponding estimated data point by the model, adapted from [23].

3.4. Linear Regression Analysis

$$L = \sum_{i=1}^n \varepsilon_i^2 = \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2 \quad (3.30)$$

Taking the partial derivatives of equation 3.30 with respect to the model's parameters yields equations 3.31 and 3.32.

$$\frac{\partial L}{\partial \beta_0} = -2 \sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0 \quad (3.31)$$

$$\frac{\partial L}{\partial \beta_1} = -2 \sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) x_i = 0 \quad (3.32)$$

Simplifying equations 3.31 and 3.32 results in what are known as the least squares normal equations (equations 3.33 and 3.34).

$$n\hat{\beta}_0 + \hat{\beta}_1 \sum_{i=1}^n x_i = \sum_{i=1}^n y_i \quad (3.33)$$

$$\hat{\beta}_0 \sum_{i=1}^n x_i + \hat{\beta}_1 \sum_{i=1}^n x_i^2 = \sum_{i=1}^n y_i x_i \quad (3.34)$$

The solution to the above equations are the least squares estimators, equations 3.35 and 3.36,

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \quad (3.35)$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n y_i x_i - \frac{(\sum_{i=1}^n y_i)(\sum_{i=1}^n x_i)}{n}}{\sum_{i=1}^n x_i^2 - \frac{(\sum_{i=1}^n x_i)^2}{n}} \quad (3.36)$$

where $\bar{y} = (1/n) \sum_{i=1}^n y_i$ and $\bar{x} = (1/n) \sum_{i=1}^n x_i$.

Assuming, for convenience purposes, the notation described in equations 3.37 and 3.38

$$S_{xx} = \sum_{i=1}^n (x_i - \bar{x})^2 = \sum_{i=1}^n x_i^2 - \frac{(\sum_{i=1}^n x_i)^2}{n} \quad (3.37)$$

$$S_{xy} = \sum_{i=1}^n (y_i - \bar{y})(x_i - \bar{x}) = \sum_{i=1}^n x_i y_i - \frac{(\sum_{i=1}^n x_i)(\sum_{i=1}^n y_i)}{n} \quad (3.38)$$

equation 3.36 can be rewritten as

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} \quad (3.39)$$

3.4. Linear Regression Analysis

Then, the estimated fitted line on the training data assumes the functional form of equation 3.40.

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x \quad (3.40)$$

For particular values of x_i and y_i equation 3.40 is written as in equation 3.41.

$$y_i = \hat{\beta}_0 + \hat{\beta}_1 x_i + e_i \quad (3.41)$$

3.4.2 Multiple Linear Regression

The multiple linear regression model is an extension of the simple linear regression model where two or more independent variables are considered. The general form of a model with k independent variables is given by

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_k x_k + \varepsilon \quad (3.42)$$

where β_0, \dots, β_k are the regression coefficients. The coefficient β_0 is the bias term of the model and the remaining regressions coefficients, β_1, \dots, β_k , are called the partial regression coefficients which measure the expected change in the dependent variable Y per unit change in a particular independent variable while all the other independent variables remain constant [20]. The equation 3.42 produces an hyperplane in the k -dimensional space of the independent variables. The shape of the fitted regression function may be non-linear but the multiple regression model is considered to be linear as long as the model is linear in the parameters as stated at the beginning of section 3.4.

In the previous subsection 3.4.1 the OLS method was derived from a calculus point of view. However, in the case of multiple regression, it will be derived from the perspective of linear algebra since computations are greatly eased using matrix notation [20].

For n observations the equation 3.42 can be written in matrix form as [20]

$$Y = X\beta + \varepsilon$$

with

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad X = \begin{bmatrix} 1 & x_{11} & x_{12} & \dots & x_{1k} \\ 1 & x_{21} & x_{22} & \dots & x_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & x_{n2} & \dots & x_{nk} \end{bmatrix} \quad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} \quad \varepsilon = \begin{bmatrix} \varepsilon_0 \\ \varepsilon_1 \\ \vdots \\ \varepsilon_k \end{bmatrix}$$

where X is the design matrix, β is the vector of parameters and Y is the vector of observations [24]. The system of equations $Y = X\beta$ is consistent if and only if a vector y is in the span spanned by the columns of X , i.e., if y is a linear combination of the columns X . However, when the matrix X has more rows than columns, that is, there are more equations than unknowns this matrix equation doesn't have a solution. Nevertheless there are still problems that require a

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solution even though the corresponding matrix equation $Y = X\beta$ is inconsistent so in order to solve such a system of equations it is needed to find a vector in the column space of the matrix X that is the best approximation of the vector y , i.e., it is necessary to find the least-squares solution of the system of equations. Consequently the least-squares problem consists in minimizing $\|Y - X\beta\|$ [24, 25]. However, before proceeding any further the concept of best approximation needs to be first defined.

The Best Approximation Theorem *Let W be a subspace of \mathbb{R}^n , let y be any vector in \mathbb{R}^n , and let \hat{y} be the orthogonal projection of y onto W . Then \hat{y} is the closest point in W to y , in the sense that*

$$\|y - \hat{y}\| < \|y - v\|$$

for all v in W distinct from \hat{y} [26].

Hence, the best approximation theorem states that the vector in W that is the closest, i.e., which is the best approximation to the vector y is the unique vector in W that is the orthogonal projection of the vector y onto W , denoted by \hat{y} in Figure 3.6.

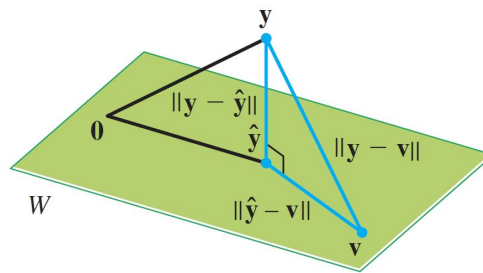


Figure 3.6: The orthogonal projection of y onto W , \hat{y} , is the closest point in W to y , adapted from [26].

Since \hat{y} is in the $Col X$ the matrix equation $X\beta = \hat{y}$ is consistent and the vector $\hat{\beta}$ is the least squares solution of matrix equation $Y = X\beta$ if and only if $X\hat{\beta} = \hat{y}$ as depicted in Figure 3.7 [26]. Also notice that $y - X\hat{\beta}$ is orthogonal to each column of matrix X and considering that x_k and x_k^T represent the k^{th} column and a row of matrix X , respectively, then $x_k \cdot (y - X\hat{\beta}) = 0$ and $x_k^T (y - X\hat{\beta}) = 0$. Since x_k^T is a row of the transpose matrix X then it can be written that

$$X^T (y - X\hat{\beta}) = 0 \quad (3.43)$$

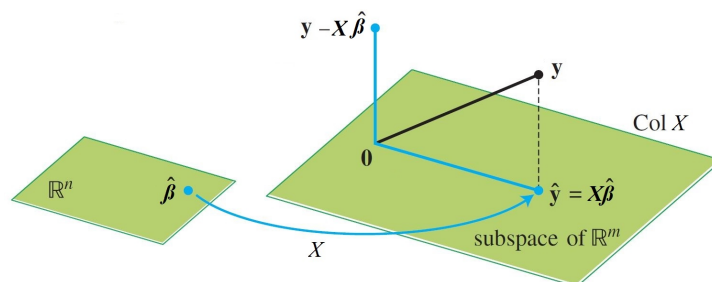


Figure 3.7: The least-squares solution $\hat{\beta}$ is in \mathbb{R}^n [26].

Multiplying the terms on the left-hand side of equation 3.43 yields ultimately equation 3.46.

$$X^T y - X^T X \hat{\beta} = 0 \quad (3.44)$$

$$X^T X \hat{\beta} = X^T y \quad (3.45)$$

$$X^T X \beta = X^T y \quad (3.46)$$

The matrix equation 3.46 is a system of equations known as the normal equations. Further developing equation 3.46 yields the least squares solution of $Y = X\beta$, denoted by $\hat{\beta}$, and given by equation 3.47.

$$\hat{\beta} = (X^T X)^{-1} X^T y \quad (3.47)$$

3.4.3 Radial Basis Functions

At the beginning of this chapter statistical learning methods and the choice of parametric and non-parametric forms were briefly discussed. The choice of a parametric form may compromise predictive power since it restricts the functional form of a chosen model to be linear. On the other hand, if a non-parametric functional form were to be chosen higher predictive power may be achieved, however, at the expense of loss of interpretation ability. Nonetheless, a parametric model can be extended so a higher predictive model may be attained while still retaining the most of the interpretation ability [21]. In order to implement this approach the independent variables in equation 3.42 can be replaced, that is, transformed by basis functions, $\varphi_1(\cdot), \varphi_2(\cdot), \dots, \varphi_k(\cdot)$, described by equation 3.48.

$$Y = \beta_0 + \beta_1 \varphi_1(x) + \beta_2 \varphi_2(x) + \dots + \beta_k \varphi_k(x) + \varepsilon \quad (3.48)$$

Once the basis functions are chosen the coefficients of the regression model can be computed using OLS just as discussed in the previous subsections. For the present work as a basis function it will be chosen a specific type of radial basis functions. Radial basis functions were first used in the nineteen-sixties to interpolate scattered data [27] and display a good performance when the interpolation function is itself a function of many parameters or variables [28]. These are defined by a large dataset giving the same values as the original data at the same location being its behaviour between the original scattered data dependent on the type of radial basis function chosen to perform the interpolation [27]. The interpolation function to be fitted into the scattered data, s , can be written as

$$s(x) = \sum_{i=1}^N \gamma_i \varphi_i(\|x - x_k\|) \quad (3.49)$$

where

- $s(\cdot)$ is a scalar function;
- γ_i is the relative weight of the source points;

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- $\varphi(\cdot)$ is a function depicting the radial interaction between the evaluation points and source points;
- x is an evaluation point;
- x_k is a source point.

The radial interaction function $\varphi(\cdot)$ evaluates therefore the euclidean distance between a point x with a data centre (source point), x_k . In a two-dimensional plane this interaction could be described as in Figure 3.8.

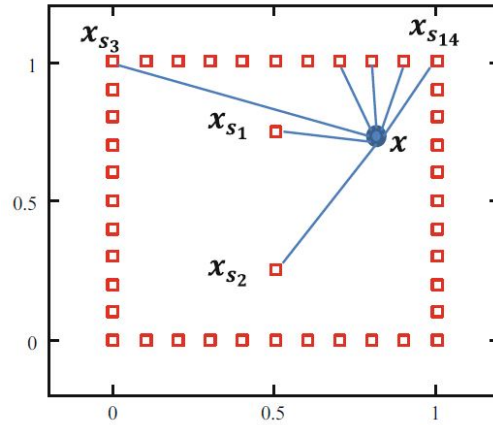


Figure 3.8: Interaction between a point x and several source points x_k [27].

For the scalar function $s(\cdot)$ it can be chosen one of several types of radial basis functions. For example, a Gaussian 3.50, an exponential 3.51, a multiquadric 3.52, an inverse multiquadric 3.53, among others radial basis functions.

Gaussian

$$\varphi(x) = \exp\left(-\frac{\|x - x_k\|^2}{2\sigma^2}\right) \quad (3.50)$$

Exponential

$$\varphi(x) = \exp\left(-\frac{\|x - x_k\|}{2\sigma^2}\right) \quad (3.51)$$

Multiquadric

$$\varphi(x) = \sqrt{\sigma^2 + \|x - x_k\|^2} \quad (3.52)$$

Inverse Multiquadric

$$\varphi(x) = \frac{1}{\sqrt{\sigma^2 + \|x - x_k\|^2}} \quad (3.53)$$

Although radial basis function started out as an interpolation method there are a few situations in which they can be employed in a regression set-up. In the case of interpolation the number of centres (source points), x_k , of radial basis function and the number of scattered points in the n -dimensional plane are the same whereas in the case of regression the number of centres is usually inferior to that of scattered points which consequently implies an error in the approximation. Despite this error, regression may be preferred over interpolation since it displays a better performance when compared to interpolation when the data set is corrupted by noise and, additionally, it also reduces computability effort since the size of the design matrix would be reduced compared with its interpolation counter-part [27]. For these reasons only regression is considered and the multiquadric radial basis functions are chosen for basis function. Consequently equation 3.48 can be rewritten as

$$Y = \beta_0 + \beta_1 \sqrt{\sigma^2 + \|x - x_1\|^2} + \beta_2 \sqrt{\sigma^2 + \|x - x_2\|^2} + \dots + \beta_k \sqrt{\sigma^2 + \|x - x_k\|^2} + \varepsilon \quad (3.54)$$

or in matrix equation form

$$\begin{bmatrix} 1 & \sqrt{\sigma^2 + (x_0 - x_1)^2} & \sqrt{\sigma^2 + (x_0 - x_2)^2} & \dots & \sqrt{\sigma^2 + (x_0 - x_N)^2} \\ 1 & \sqrt{\sigma^2 + (x_1 - x_1)^2} & \sqrt{\sigma^2 + (x_1 - x_2)^2} & \dots & \sqrt{\sigma^2 + (x_1 - x_N)^2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \sqrt{\sigma^2 + (x_N - x_1)^2} & \sqrt{\sigma^2 + (x_N - x_2)^2} & \dots & \sqrt{\sigma^2 + (x_N - x_N)^2} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{N+1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \quad (3.55)$$

3.4.3.1 Choice of the Shape Parameter

After the selection of the particular radial basis functions, in the present work the multiquadric is chosen, perusing equation 3.52 one realizes that the term σ is yet to be defined. The latter is known as the shape parameter of the multiquadric RBF. For example, Figure 3.9 depicts equation 3.52 for different values of σ . It can be seen that as the value of σ increases the multiquadric RBF gets narrower. In conclusion, different values of σ will ultimately yield different predicted values. Having this, one may choose to base the choice of the shape parameter upon given criteria. For the present work, σ will be chosen accordingly to the minimization of the RMSE, i.e., the σ chosen is the one which minimizes the distance between the data values and the data values predicted by 3.54.

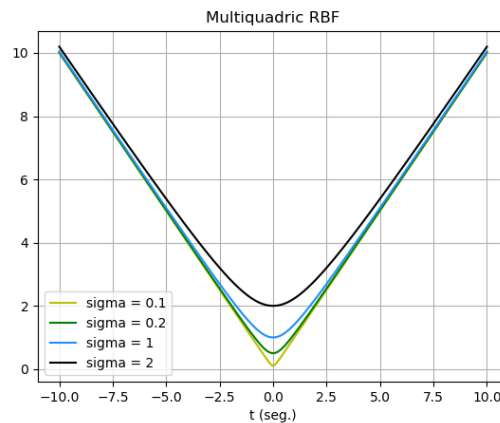


Figure 3.9: Effect of sigma in the multiquadric RBF's shape.

3.4. Linear Regression Analysis

3.4.4 Accessing the Model's Accuracy

In order to assess the accuracy of the estimated model, it is required that a formal and pragmatic analysis are performed.

3.4.4.1 Formal Analysis

The significance of the regression, i.e., the quality of adjustment of the fitted model can be determined through a method named analysis of variance (ANOVA) which is based upon on the partition of the total variation present in the dependent variable of the fitted model. Figure 3.10 depicts a data set whereupon the mean of the data is fitted as a model. The total sum of the squared distances, SS_T , between each observation (data point) and the corresponding data point on the model describes the total variation present in the dependent variable Y [20, 29].

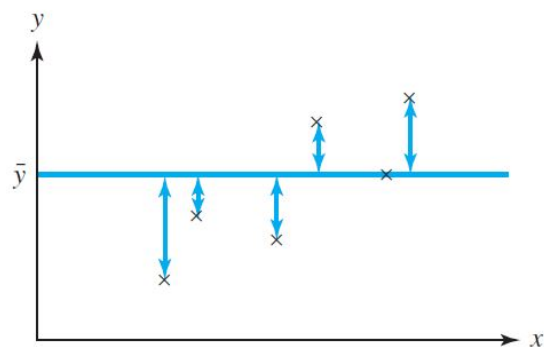


Figure 3.10: Total sum of squares [23].

Perusing Figure 3.11 it is seen that another model (in this case a first-order polynomial) is fitted to the data set. Similarly as before, the sum of squared distances between the data points and the corresponding data points of the first-order polynomial describes the amount of variability present in Y that is accounted by this model, that is, the amount of variability by which the variability in Y is eliminated when this model is used to predict Y [20, 29].

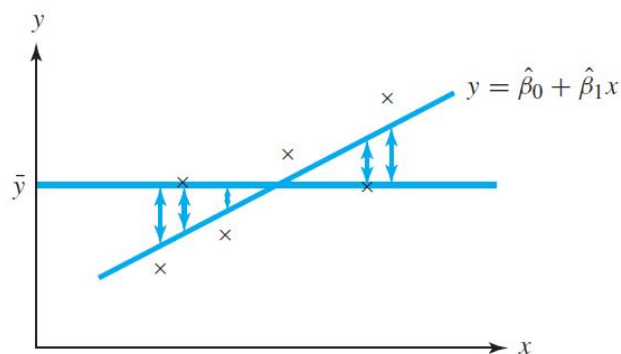


Figure 3.11: Regression sum of squares [23].

3.4. Linear Regression Analysis

On the other hand Figure 3.12 depicts the sum of squared distances, SS_E , between the first-order polynomial of the mean of the data set, or in other words, it accounts for the amount of variability in Y that is unknown, i.e., that cannot be predicted by the first-order polynomial [20, 29].

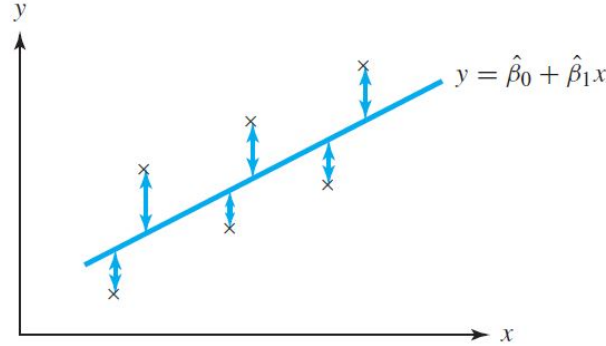


Figure 3.12: Error sum of squares [23].

Therefore, the total amount of variability in the independent variable Y equals the sum between the variability in Y explained by the regression model and the variability in Y that is not accounted by the regression model. Hence, the previously described relationship can be written mathematically as described by equation 3.56. This equation is known as the analysis of variance identity [20].

$$SS_T = SS_R + SS_E \quad (3.56)$$

The individual terms in equation 3.56 are computed as in equations 3.57 and 3.58.

$$SS_R = \sum_{i=0}^N (\hat{y}^{(i)} - \bar{y})^2 = (\hat{y}^{(0)} - \bar{y})^2 + (\hat{y}^{(1)} - \bar{y})^2 + (\hat{y}^{(2)} - \bar{y})^2 + \dots + (\hat{y}^{(N)} - \bar{y})^2 \quad (3.57)$$

$$SS_T = \sum_{i=0}^N (y^{(i)} - \bar{y})^2 = (y^{(0)} - \bar{y})^2 + (y^{(1)} - \bar{y})^2 + (y^{(2)} - \bar{y})^2 + \dots + (y^{(N)} - \bar{y})^2 \quad (3.58)$$

The mean of the observations, \bar{y} , in equations 3.57 and 3.58 are computed as described by equation 3.59.

$$\bar{y} = \frac{1}{N+1} \sum_{i=0}^N y^{(i)} = \frac{y^{(0)} + y^{(1)} + y^{(2)} + \dots + y^{(N)}}{N+1} \quad (3.59)$$

The assessment of the adequacy of the regression model is then measured, using the above quantities, by the coefficient of determination R^2 defined as in equation 3.60.

$$R^2 = \frac{SS_R}{SS_T} = 1 - \frac{SS_E}{SS_T} \quad (3.60)$$

3.4. Linear Regression Analysis

The coefficient of determination measures the amount of variation present in the data set that is explained by the regression line, plane or hyperplane with respect to the whole data set [20, 29]. Since R^2 denotes a proportion it assumes a value between zero and one; if it assumes a value close to one then the fitted model explains a large amount of variability present in the dependent variable whereas a value near zero suggests that the regression model only explains a very small amount of the variance present in Y [21]. Apart from this statistic another commonly used measure of the accuracy of prediction of the regression model is the mean square error (MSE) given by equation 3.61.

$$MSE = \frac{1}{N+1} \sum_{i=0}^N (y^{(i)} - \hat{y}^{(i)})^2 \quad (3.61)$$

The MSE represents the average value of the squared distances between the observed values and the correspondingly predicted ones. Hence the lower values of MSE correspond to more accurate predictions. Conversely, MSE will assume larger values as the predicted values and the values of the observations diverge [21]. However, if the individual errors were to double the MSE would quadruple due to the squaring operation in equation 3.61 thus making this measure more difficult to interpret. On the other hand, if MSE is squared rooted then a new measure named root mean square error (RMSE), equation 3.62, can be used instead of using the MSE [30].

$$RMSE = \sqrt{\frac{1}{N+1} \sum_{i=0}^N (y^{(i)} - \hat{y}^{(i)})^2} \quad (3.62)$$

The RMSE, as opposed to the MSE, exhibits a linear relationship between itself and the individual errors in the sense that doubling the individual errors doubles the RMSE thus making it a more interpretable measure relative to the MSE. However, the RMSE is constrained to the units of measure used in the problem where it is computed from, i.e., if two different values of RMSE are to be compared then the units must be the same [30]. In order to overcome this constrain the RMSE can be divided by the mean of the observations as in equation 3.63.

$$\tilde{\sigma}(y, \hat{y}) = \frac{RMSE}{\bar{y}} \quad (3.63)$$

Equation 3.63 represents the relative RMSE, denoted by $\tilde{\sigma}(y, \hat{y})$, but it can only be used if every single observation is greater than zero. If at least one observation equals or is less than zero equation 3.64 is used instead.

$$\tilde{\sigma}(y, \hat{y}) = \sqrt{\frac{1}{N+1} \frac{\sum_{i=0}^N (y^{(i)} - \hat{y}^{(i)})^2}{\sum_{i=0}^N (y^{(i)} - \bar{y})^2}} \quad (3.64)$$

3.4. Linear Regression Analysis

As an alternative to the relative RMSE the mean absolute percent error (MAPE), equation 3.65, can be used.

$$\tilde{\sigma}(y, \hat{y}) = \sqrt{\frac{1}{\tilde{N}} \sum_{i:y^{(i)} \neq 0} \left| \frac{y^{(i)} - \hat{y}^{(i)}}{y^{(i)}} \right|} \quad (3.65)$$

If these above defined measures verify the conditions described by equation 3.66 the formal analysis is considered to be "very good".

$$R^2 > 0.8 \quad \text{or} \quad \tilde{\sigma}(y, \hat{y}) \leq 0.1 \quad (3.66)$$

However, in order to fully conclude that the regression model is adequate, the formal analysis must be complemented by the pragmatic analysis and, additionally, it shouldn't be used as a standalone form of assessing the model's adequacy [19].

3.4.4.2 Pragmatic Analysis

The pragmatic analysis of the linear regression procedure consists of computing a previously chosen performance measure on different data sets. In the present work three data sets are used: a training set, a test set and a global set. The reason why this is so is as follows.

Let us consider that an experiment is conducted and data is recorded every five seconds and it is required to fit a model to the collected data. However, the fitted model must have a good prediction capability, i.e., not only it is required that the model performs well on the available data but that it also performs well on yet unknown data (future data). Consequently, the data set of collected data may be partitioned into a training set and a test set, where the model can be trained on the first set and its prediction ability can be tested on the latter. However, if two different models were to be fitted and compete among themselves a question that may arise is that of which of the previously mentioned data set (training and test sets) to compare the model's performance measure on. If the performance measures of the respective models were computed and compared based upon the training set then the performance measure computed on the test set would be considered as an unbiased estimate of the expected future performance, which constitutes a very good property. But, if this approach were to be followed and if the observed data were corrupted by noise then a problem might occur.

Consider that a model A will perform averagely and a model B will have an excellent performance. This naturally leads up to the conclusion that the second model is better with respect to the first model. However, by definition the noise patterns are not repeatable, that is, they aren't expected to occur again in the future. Consequently, since model B will learn the noise it will overfit the data. As a consequence, its performance, in the future, will probably be inferior to the performance of the model A. This fact introduces a training bias. On the other hand, if the comparison of the performances measures were based upon the test set a selection bias would be introduced instead.

Consequently, if the models are compared on the training set overfitting will probably occur and if they are compared based upon the test set then we no longer will have an unbiased estimate of the expected future performance. Then, in order to overcome both training and selection bias, the original set of observed data is partitioned not into two but three data set: a training

3.4. Linear Regression Analysis

set, a test set and a validation (global) set. This way the models are trained on the training set which will introduce training bias as previously discussed; then, unbiased estimates of the performance measure can be computed on the validation set and the model which presents the best figure is the chosen one. However, its measure will still be amplified by the selection bias. So to eliminate the selection bias the selected model is evaluated on the test set. The resulting figure of this analysis is an unbiased estimate of the future capabilities of the selected model [31].

Thus to conduct the pragmatic analysis the relative errors, previously discussed in this section, are computed on all three data sets as described in equation 3.67 through 3.69.

$$0 \leq \tilde{\sigma}_0 (y, \hat{y}) \leq 0.1 \quad (3.67)$$

$$0 \leq \tilde{\sigma}_I (y, \hat{y}) \leq 0.2 \quad (3.68)$$

$$0 \leq \tilde{\sigma}_{II} (y, \hat{y}) \leq 0.1 \quad (3.69)$$

In the above equations, $\tilde{\sigma}_0$ represents the relative error computed on the training set, $\tilde{\sigma}_I$ is the relative error computed on the test set and, finally, $\tilde{\sigma}_{II}$ is the one computed on the global (validation) set. It is, however, important to notice that the validation set corresponds to the original data set of observed data whereas the training set will represent from 70% up to 80% and the test set represents from 20% up to 30% [19]. For the present work, a proportion of 70% and 30% is chosen for performing the pragmatic analysis in the next chapter.

3.4. Linear Regression Analysis

Chapter 4

Validation

In the present chapter numerical simulations are performed using Python 3.6.7 and the results obtained are presented. Python is a general-purpose programming language, clear and object-oriented, which includes a standard library that supports common programming tasks, e.g., connecting to web servers, reading and modifying files and others. It can also be extended through the addition of modules in compiled language [32]. A few modules were used in the development of the numerical simulations presented in this work such as "NumPy", "SciPy library" and the "matplotlib". "NumPy" is a scientific package used for scientific computing where n -dimensional arrays objects, linear algebra, among other tools are available [33]. In turn, "SciPy library" is a collection of numerical algorithms with applications in fields like signal processing, statistics and others [34]. The module "matplotlib" is used in Python to produce publication-quality 2D Figures [35]. As previously mentioned in the introduction, the main objective of this dissertation is the development of two identification methods for determining the longitudinal equations of motion in state-space form for a sufficiently generic airplane given that flight data is available or, in other words, the objective is to find all elements of the state matrix A and of the input matrix B in matrix equation 4.1.

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 0 \end{bmatrix} \eta \quad (4.1)$$

In the next couple sections different methods will be presented where, from flight data for a given airplane, the above state matrix A and the input B are determined.

4.1 Method 1: Based on Savitzky-Golay Smoothing Filter

As previously stated, the state and control matrices for a given airplane are to be determined from existing flight data. Hence, from reference [6] the longitudinal equations of motion in state-space form for a particular airplane are selected and depicted in matrix equation 4.2.

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 7.181 \times 10^{-4} & 4.570 \times 10^{-3} & -29.072 & -9.678 \\ -0.0687 & -0.2953 & 174.868 & -1.601 \\ 1.73 \times 10^{-3} & -0.0105 & -0.4462 & 1.277 \times 10^{-3} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 1.041 \\ -6.294 \\ -4.888 \\ 0 \end{bmatrix} \eta \quad (4.2)$$

The above matrix equation 4.2 is composed of four ordinary differential equations whose solution can be found through the application of a numerical method for solving ODEs. However, to specify the solution four initial conditions are needed. For demonstration purposes, it is assumed the initial conditions in equation 4.3.

4.1. Method 1: Based on Savitzky-Golay Smoothing Filter

$$x(0) = \begin{bmatrix} u(0) \\ w(0) \\ q(0) \\ \theta(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0.8 \\ 0 \end{bmatrix} \quad (4.3)$$

Furthermore, different inputs on the control variable η are entered as times elapses, as described on the piecewise function 4.4 and as depicted on Figure 4.1.

$$\eta(t) = \begin{cases} 0^\circ & \text{if } 0 \leq t < 20 \\ 10^\circ & \text{if } 20 \leq t < 40 \\ -10^\circ & \text{if } 40 \leq t < 60 \\ 0^\circ & \text{if } 60 \leq t < 80 \\ 10^\circ & \text{if } 80 \leq t < 100 \\ -10^\circ & \text{if } 100 \leq t < 110 \\ 0^\circ & \text{if } 110 \leq t < 120 \end{cases} \quad (4.4)$$

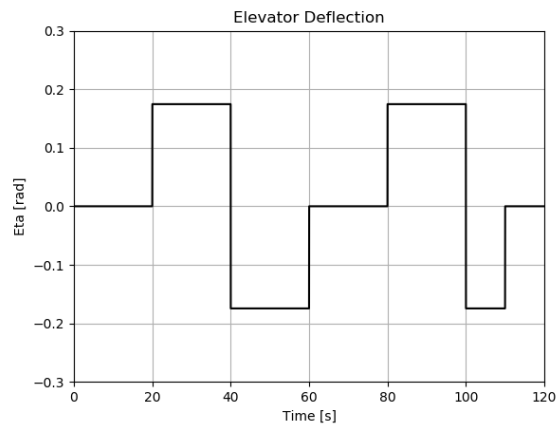


Figure 4.1: Elevator deflection as function of time.

Consequently, a numerical simulation is carried out, using Butcher's algorithm (equation 3.18), which yields the flight data in table 4.1. Note that the data depicted in table 4.1 is only partially shown due to space constraints and the same applies to subsequent tables throughout this work.

t	u	w	q	θ
0	5	0	0.8	0
0.01	4.76762	1.39038	0.79645	0.00798
0.02	4.53558	2.77039	0.79277	0.01593
\vdots	\vdots	\vdots	\vdots	\vdots
119.98	153.09193	24.00607	0.08779	2.08373
119.99	152.86693	23.95012	0.08756	2.08461

Table 4.1: Flight Data

4.1. Method 1: Based on Savitzky-Golay Smoothing Filter

Plotting the flight data in table 4.1 against time generates the four different graphs shown in Figures 4.2 through 4.5.

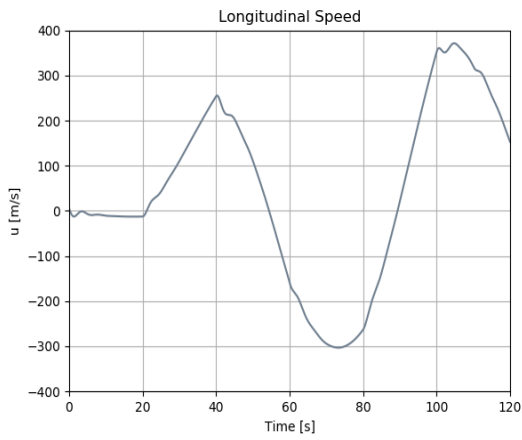


Figure 4.2: Longitudinal Speed, u .

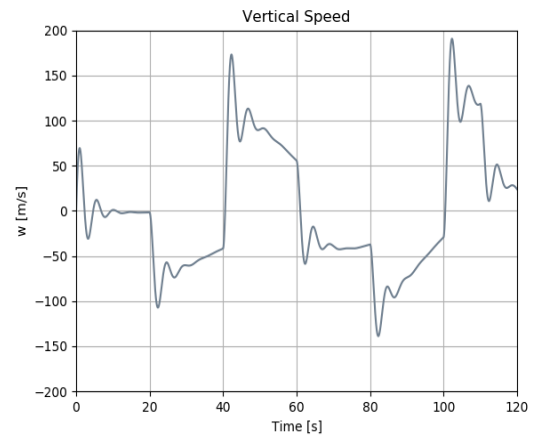


Figure 4.3: Vertical Speed, w .

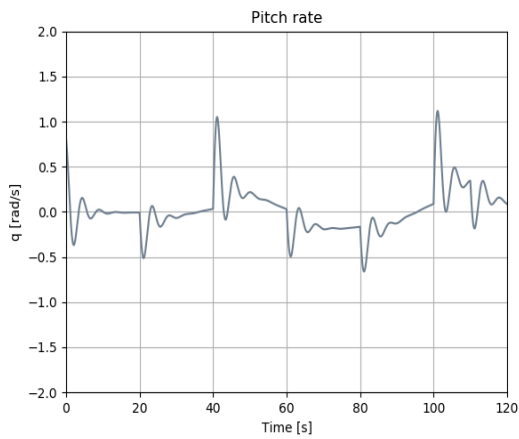


Figure 4.4: Pitch Rate, q .

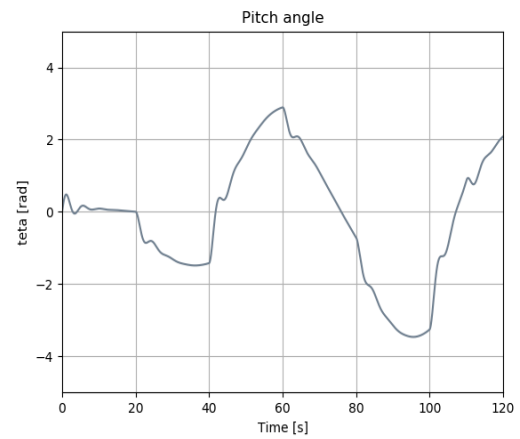


Figure 4.5: Pitch Angle, θ .

However, the acquisition of data in a real-world environment would yield data corrupted by noise. In order to introduce noise, it is imported from the Python's module "Numpy" the following function $random.normal(\mu, \sigma, size)$. For instance, let a vector v be defined as in equation 4.5 where μ represents the mean of a normal distribution, σ is the standard deviation and the $size$ denotes the length of the of vector v .

$$v = random.normal(\mu, \sigma, size) \quad (4.5)$$

For the present work, it is assumed that the noise comes from a normal distribution with zero mean and standard deviation of one. Consequently, it follows that equation 4.5 can be rewritten as equation 4.6.

$$v = random.normal(0, 1, 12000) \quad (4.6)$$

4.1. Method 1: Based on Savitzky-Golay Smoothing Filter

Then, four different vectors, appropriately scaled, are created as described by equations 4.7 through 4.10.

$$v_1 = 0.1 \times v \quad (4.7)$$

$$v_2 = 0.2 \times v \quad (4.8)$$

$$v_3 = \frac{2\pi}{180} \times v \quad (4.9)$$

$$v_4 = \frac{2\pi}{180} \times v \quad (4.10)$$

These are then added to their corresponding state variable's data on table 4.1, that is, $u + v_1$, $w + v_2$, $q + v_3$ and $\theta + v_4$ thus obtaining, in table 4.2, flight data corrupted by noise.

t	u_N	w_N	q_N	θ_N
0	4.92335	-0.15330	0.77324	-0.01338
0.01	4.86374	1.58262	0.83000	0.02476
0.02	4.68121	3.06166	0.84360	0.04135
\vdots	\vdots	\vdots	\vdots	\vdots
119.98	153.11636	24.05492	0.09632	2.08799
119.99	152.82513	23.86651	0.07297	2.07731

Table 4.2: Noisy Flight Data

In order to grasp 'how much' noise is added into the flight data four different plots, Figure 4.6 through 4.9, are given, each one representing a state variable against time and the corrupted state variable's data against time.

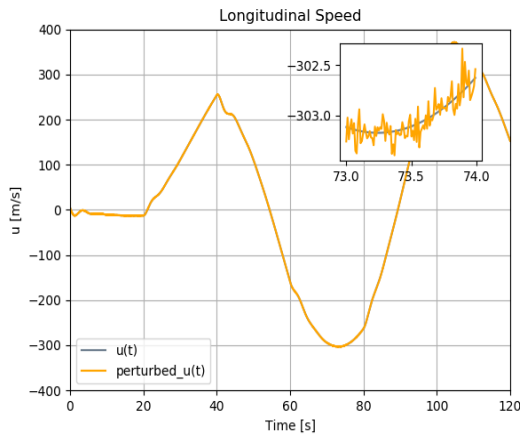


Figure 4.6: Longitudinal Speed, u , and Noisy Longitudinal Speed, u_N .

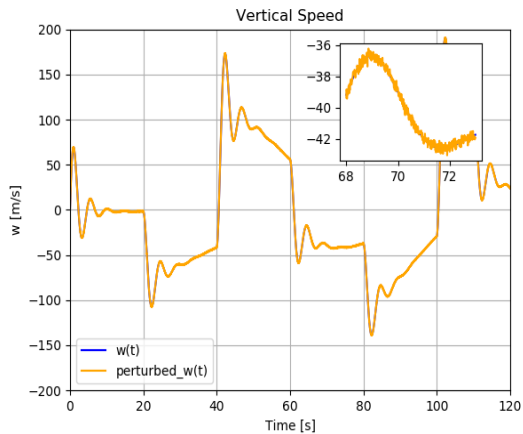


Figure 4.7: Vertical Speed, w , and Noisy Vertical Speed, w_N .

Having obtained the noisy flight data, it is now necessary to filter out the noise. To do so the "Savitzky-Golay Smoothing Filter" is applied. Consequently, from the module "SciPy library", the function `savgol_filter` is used to filter out the noise previously added to the flight data. In order to do so, this function is applied to the last four columns of table 4.2. Besides the selection of data to be filtered, the `savgol_filter` function requires the user to insert which window length to be used as well as the degree of the polynomial. In order to choose both parameters, the information presented in the final part of section 3.3 is recalled.

4.1. Method 1: Based on Savitzky-Golay Smoothing Filter

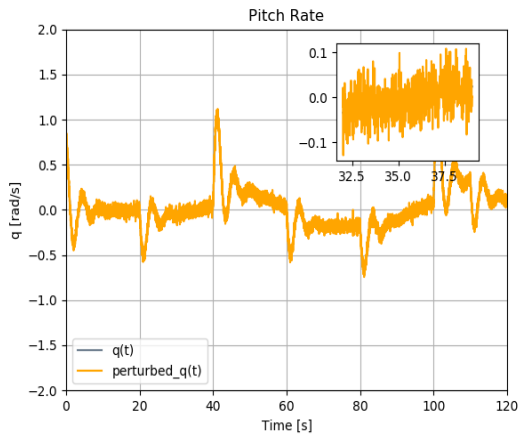


Figure 4.8: Pitch Rate, q , and Noisy Pitch Rate, q_N .

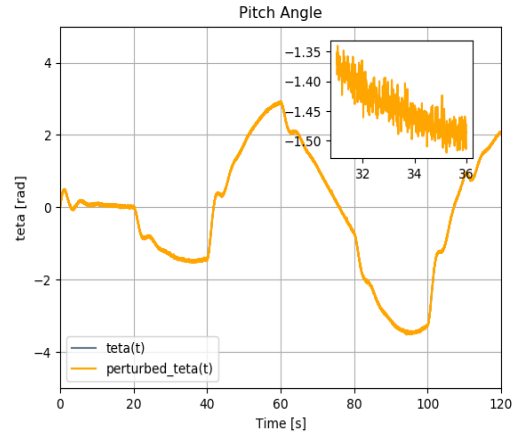


Figure 4.9: Pitch Angle, θ , and Noisy Pitch Angle, θ_N .

It is stated that the window length should contain between seven and eleven data points and that the window length value is twice the degree of the polynomial's value plus one. As such and considering the constraint imposed on the number of data points enclosed by the window the degree of the polynomial can be readily determined. Consequently, there are three different choices:

- 1st: Choose a window of length 7 and a polynomial of degree 3;
- 2nd: Choose a window of length 9 and a polynomial of degree 4;
- 3rd: Choose a window of length 11 and a polynomial of degree 5.

q	p	u_{RMSE}	w_{RMSE}	q_{RMSE}	θ_{RMSE}
7	3	0.05795	0.11590	0.02023	0.01011
9	4	0.06471	0.12941	0.02259	0.01129
11	9	0.05794	0.11587	0.02022	0.01011

Table 4.3: RMSE for different combinations of window length and polynomial degree (the minimum RMSE are written in red)

Hence a criterion must be established in order to choose between the three above possibilities. Since the noiseless flight data is available in table 4.1 one criteria could be to minimize the distance between the noiseless flight data and the filtered flight data. As such, a metric previously defined in sub-subsection 3.4.4.1 is chosen, namely the RMSE metric. Consequently, each state variable's data in table 4.2 is filtered three times, each time corresponding each possible choice of the values of the parameters as presented above resulting in twelve RMSE values which are then displayed in table 4.3 where the minimum values of RMSE are written in red. The final choice of parameters for each state variable's data is the one that minimizes the chosen criteria. For the present problem, the filtered flight data u_F , w_F , q_F and θ_F , where the "F" subscript denotes filtered data, is obtained by choosing a window length containing eleven data points and a polynomial of degree 9 for all the state variables u , w , q and θ . The filtered flight data is shown in table 4.4.

4.1. Method 1: Based on Savitzky-Golay Smoothing Filter

t	u_F	w_F	q_F	θ_F
0	4.90389	-0.19221	0.76645	-0.01677
0.01	4.92887	1.71288	0.85274	0.03613
0.02	4.62058	2.94040	0.82244	0.03076
\vdots	\vdots	\vdots	\vdots	\vdots
119.98	153.07937	23.98094	0.08341	2.08154
119.99	152.83247	23.88118	0.07553	2.07859

Table 4.4: Filtered Flight Data

Plotting the filtered flight data against time yields Figures 4.10 through 4.13. Afterwards, it is necessary to differentiate the filtered data as follows: for $t = 0$ s the equation 3.12 is applied, the derivative at $t = 119.99$ s is computed through equation 3.13 and for the rest of data between equation 3.15 is employed. This yields the time derivative flight data as shown in table 4.5.

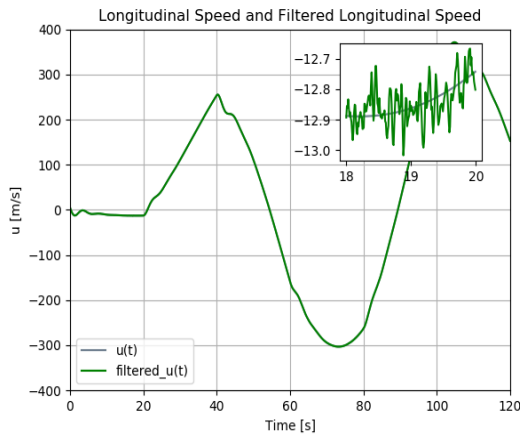


Figure 4.10: Longitudinal Speed, u , and Filtered Longitudinal Speed, u_F .

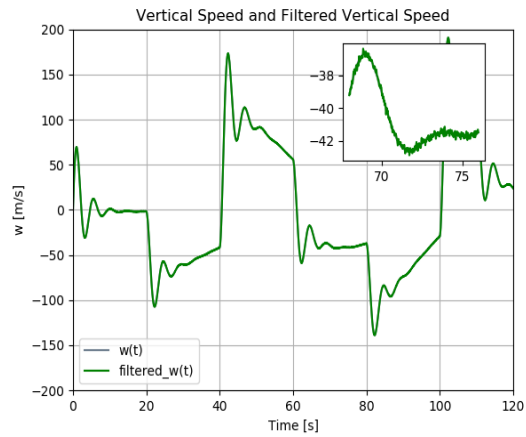


Figure 4.11: Vertical Speed, w , and Filtered Vertical Speed, w_F .

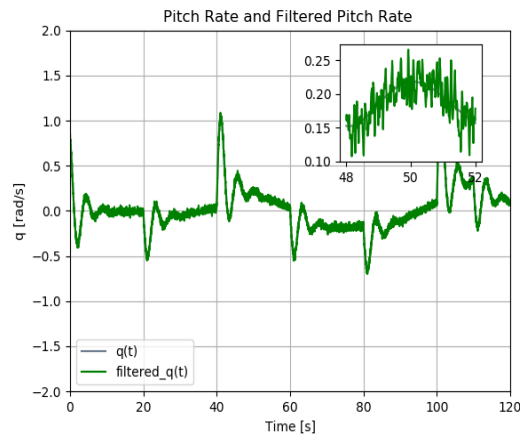


Figure 4.12: Pitch Rate, q , and Filtered Pitch Rate, q_F .

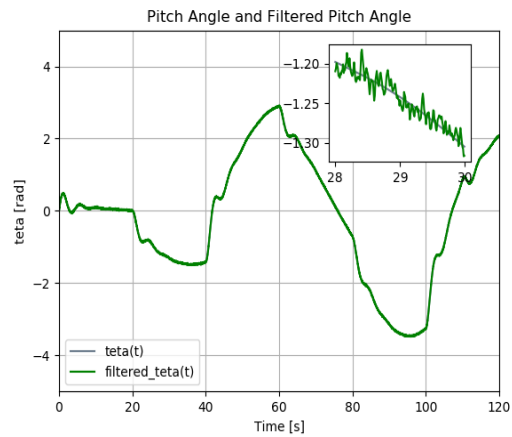


Figure 4.13: Pitch Angle, θ , and Filtered Pitch Angle, θ_F .

4.1. Method 1: Based on Savitzky-Golay Smoothing Filter

t	\dot{u}	\dot{w}	\dot{q}	$\dot{\theta}$
0	19.16037	181.78979	14.45753	8.20299
0.01	-14.16552	154.64518	2.79933	2.37694
0.02	-33.03124	132.94064	-3.81196	-0.92577
\vdots	\vdots	\vdots	\vdots	\vdots
119.98	-22.34215	-3.09595	0.03105	0.11518
119.99	-27.03853	0.67917	-1.60729	-0.70438

Table 4.5: Time Derivatives of the Filtered Flight Data

It is now necessary to do three different linear regressions using OLS. Each regression has a common design matrix $M_{12000 \times 5}$, matrix (equation 4.11), having the filtered flight data as the first four columns and as for the fifth column the input vector η , i.e., the elevator deflection's data.

$$M = \begin{bmatrix} 4.90389 & -0.12839 & 0.76645 & -0.01677 & 0 \\ 4.92887 & 1.55378 & 0.85274 & 0.03613 & 0 \\ 4.62058 & 2.96451 & 0.82244 & 0.03076 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 153.07937 & 23.92288 & 0.08341 & 2.08154 & 0 \\ 152.83247 & 23.91079 & 0.07553 & 2.07859 & 0 \end{bmatrix} \quad (4.11)$$

Having defined the matrix M , the three unknown vectors to be determined are composed by the elements of the state matrix A and of the input matrix B . Let x_i denote the unknown column vectors with $i = 1, \dots, 3$. Then the three coefficient column vectors are represented as

$$\begin{aligned} x_1 &= [a_{11} \ a_{12} \ a_{13} \ a_{14} \ b_1]^T \\ &\vdots \\ x_3 &= [a_{31} \ a_{32} \ a_{33} \ a_{34} \ b_3]^T \end{aligned}$$

Subsequently, three linear regressions are performed, as discussed in section 3.4.2, using as observation vectors the columns \dot{u} , \dot{w} , \dot{q} and $\dot{\theta}$ in the table 4.5. The unknown vectors are computed as follows:

$$\begin{aligned} x_1 &= (M^T M)^{-1} M^T [\dot{u}] \\ &\vdots \\ x_3 &= (M^T M)^{-1} M^T [\dot{q}] \end{aligned}$$

where $[\dot{u}]$ and $[\dot{q}]$ represent the vectors, whose values correspond to \dot{u} and \dot{q} for all samples, thus yielding all the elements of matrix equation 4.12

4.1. Method 1: Based on Savitzky-Golay Smoothing Filter

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 1.910 \times 10^{-4} & 5.379 \times 10^{-3} & -28.436 & -9.658 \\ -0.0667 & -0.2985 & 172,410 & -1.692 \\ 1.69 \times 10^{-3} & -0.0104 & -0.3989 & 2.216 \times 10^{-3} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 2.152 \\ -10.795 \\ -4.807 \\ 0 \end{bmatrix} \eta \quad (4.12)$$

Comparing matrices equations 4.2 and 4.12 it can be seen that some elements assume a numerical value closer to their corresponding values in matrix equation 4.2 than others. However, a proper analysis must be carried out and, consequently, the analysis procedure presented in subsection 3.4.4 is performed. In sum, there are three separate analysis for the three linear regressions just performed to be conducted, each one subdivided into a formal and a pragmatic analysis as described in subsection 3.4.4.

4.1.1 First Regression Analysis

Formal Analysis

The results obtained through performing the formal analysis are the following:

$$\begin{aligned} \bar{y} &= 1.233 & R^2 &= 0.968 \\ SS_R &= 3967447.141 & RMSE &= \text{Not Applicable} \\ SS_T &= 4096587.586 & \tilde{\sigma} &= 0.002 \end{aligned}$$

The measure RMSE assumes 'Not Applicable' due to the fact that at least one observation values equal or is less than zero which implies, as seen in subsection 3.4.4, 3.64 is computed rather than equation 3.63 thus defeating the purpose of computing RMSE. The same reasoning is applied in the forthcoming analysis whenever RMSE depicts "Not Applicable". Analysing the results obtained above it can be seen that both conditions are shown in 3.66 are verified and as such the first regression passes the formal analysis and it is considered to be "very good". Subsequently, pragmatic analysis is performed.

Pragmatic Analysis

Since that the first regression performed previously passed the formal analysis, that is, it was considered as "very good" linear regression then the pragmatic analysis is readily conducted and the following results were obtained:

$$\begin{aligned} \tilde{\sigma}_0(y, \hat{y}) &= 0.002 \\ \tilde{\sigma}_I(y, \hat{y}) &= 0.002 \\ \tilde{\sigma}_{II}(y, \hat{y}) &= 0.002 \end{aligned}$$

Comparing these results with the conditions given in sub-subsection 3.4.4.2 one sees that all the three relative errors above verify the established conditions of equations 3.67 through 3.69. Consequently, the first linear regression also passes the pragmatism analysis and is therefore considered to be "excellent".

4.1. Method 1: Based on Savitzky-Golay Smoothing Filter

4.1.2 Second Regression Analysis

Formal Analysis

The results obtained through performing the formal analysis are as follows.

$$\begin{aligned}\bar{y} &= 0.211 & R^2 &= 0.943 \\ SS_R &= 10671806.785 & RMSE &= \text{Not Applicable} \\ SS_T &= 11313725.592 & \tilde{\sigma} &= 0.002\end{aligned}$$

It can be seen that the results of second linear regression agree with the conditions given in 3.66. As such the formal analysis is considered to be "very good".

Pragmatic Analysis

The pragmatic analysis was conducted the following results were obtained:

$$\begin{aligned}\tilde{\sigma}_0(y, \hat{y}) &= 0.003 \\ \tilde{\sigma}_I(y, \hat{y}) &= 0.004 \\ \tilde{\sigma}_{II}(y, \hat{y}) &= 0.002\end{aligned}$$

These also agree with the conditions given sub-subsection 3.4.4.2 and consequently the second linear regression is considered to be "excellent".

4.1.3 Third Regression Analysis

Formal Analysis

The results obtained through performing the formal analysis are the following:

$$\begin{aligned}\bar{y} &= -0.005 & R^2 &= 0.045 \\ SS_R &= 709.386 & RMSE &= \text{Not Applicable} \\ SS_T &= 15845.536 & \tilde{\sigma} &= 0.009\end{aligned}$$

It can be seen that the results of third linear regression agree with those stated in 3.66. As such the formal analysis is considered to be very good.

Pragmatic Analysis

The pragmatic analysis was conducted the following results were obtained:

$$\begin{aligned}\tilde{\sigma}_0(y, \hat{y}) &= 0.011 \\ \tilde{\sigma}_I(y, \hat{y}) &= 0.006 \\ \tilde{\sigma}_{II}(y, \hat{y}) &= 0.009\end{aligned}$$

4.2. Method 2: Based on Multiquadric Radial Basis Functions

The above results are also in agreement with the conditions previously defined which consequently leads the conclusion that the third linear regression is excellent.

4.2 Method 2: Based on Multiquadric Radial Basis Functions

As for the first method, presented earlier, the objective of the second method is also the estimation of the elements of the state and input matrices of matrix equation 4.2 (here reproduced for convenience purposes).

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 7.181 \times 10^{-4} & 4.570 \times 10^{-3} & -29.072 & -9.678 \\ -0.0687 & -0.2953 & 174.868 & -1.601 \\ 1.73 \times 10^{-3} & -0.0105 & -0.4462 & 1.277 \times 10^{-3} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 1.041 \\ -6.294 \\ -4.888 \\ 0 \end{bmatrix} \eta \quad (4.13)$$

As stated previously the above matrix equation is composed by four ordinary differential equations whose solution is found by applying Butcher's algorithm (equation 3.18) with the initial conditions described in equation 4.14,

$$x(0) = \begin{bmatrix} u(0) \\ w(0) \\ q(0) \\ \theta(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0.8 \\ 0 \end{bmatrix} \quad (4.14)$$

and with inputs (elevator deflections) defined by by the piecewise function 4.15.

$$\eta(t) = \begin{cases} 0^\circ & \text{if } 0 \leq t < 20 \\ 10^\circ & \text{if } 20 \leq t < 40 \\ -10^\circ & \text{if } 40 \leq t < 60 \\ 0^\circ & \text{if } 60 \leq t < 80 \\ 10^\circ & \text{if } 80 \leq t < 100 \\ -10^\circ & \text{if } 100 \leq t < 110 \\ 0^\circ & \text{if } 110 \leq t < 120 \end{cases} \quad (4.15)$$

Additionally Figure 4.14 shows the control variable η against time.

Similarly to the first method, a numerical simulation Butcher's algorithm (equation 3.18) is promptly applied and the flight data is generated. Conversely, however, the numerical simulation here performed considers a time step $h = 0.5s$ instead of the time step $h = 0.01s$ used in the first method. The results of the numerical simulation, that is, the flight data is shown in table 4.6

4.2. Method 2: Based on Multiquadric Radial Basis Functions

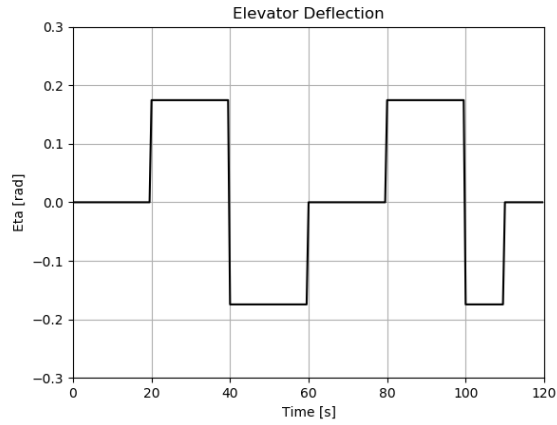


Figure 4.14: Elevator deflection as function of time.

t	u	w	q	θ
0	5	0	0.8	0
0.5	-5.48118	53.70860	0.49483	0.33300
1.0	-11.57355	69.34135	0.08312	0.47722
\vdots	\vdots	\vdots	\vdots	\vdots
119.0	163.86582	26.46482	0.10283	2.03759
119.5	152.62890	23.91640	0.08742	2.08485

Table 4.6: Flight Data

and the four state variables are graphed against time in Figures 4.15 through 4.18.

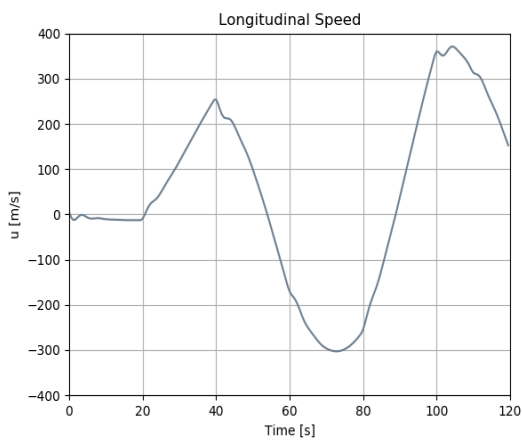


Figure 4.15: Longitudinal Speed, u .

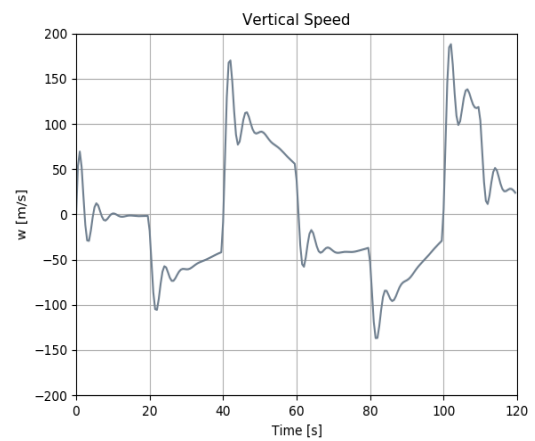


Figure 4.16: Vertical Speed, w .

4.2. Method 2: Based on Multiquadric Radial Basis Functions

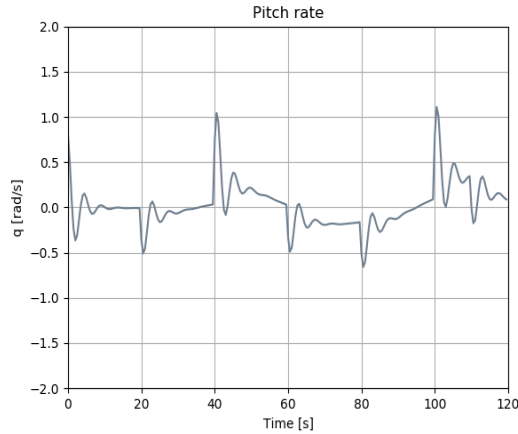


Figure 4.17: Pitch Rate, q .

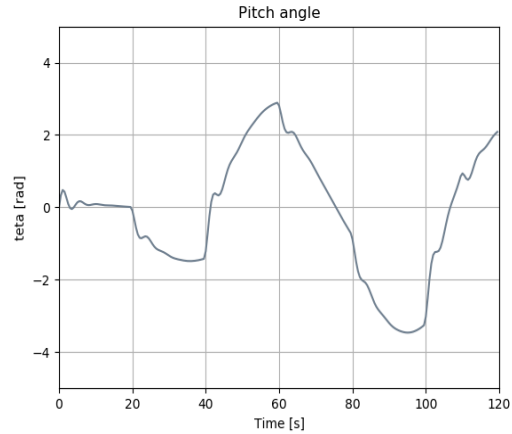


Figure 4.18: Pitch Angle, θ .

To simulate a real-world scenario white noise is added to the flight data. In order to do so from Numpy's module the function $random.normal(u, \sigma, size)$ is retrieved and used to define the vector v as

$$v = random.normal(0, 1, 240) \quad (4.16)$$

Similarly, four different vectors (equations 4.17), appropriately scaled, are created as:

$$\begin{aligned} v_1 &= 0.1 \times v \\ v_2 &= 0.2 \times v \\ v_3 &= \frac{2\pi}{180} \times v \\ v_4 &= \frac{2\pi}{180} \times v \end{aligned} \quad (4.17)$$

and are added to each state variable's corresponding vector thus yielding this way the flight data corrupted by noise in table 4.7.

t	u_N	w_N	q_N	θ_N
0	4.92335	-0.15330	0.77324	-0.02676
0.5	-5.38506	53.90084	0.52838	0.36655
1.0	-11.42792	69.63262	0.13396	0.52806
\vdots	\vdots	\vdots	\vdots	\vdots
119.0	163.78636	26.30589	0.07510	2.00985
119.5	152.65862	23.97584	0.09780	2.09523

Table 4.7: Noisy Flight Data

The flight data in the above table 4.7 is plotted against time, Figures 4.19 through 4.22, so a visual comparison between the noiseless and corrupted flight data can be performed.

4.2. Method 2: Based on Multiquadric Radial Basis Functions

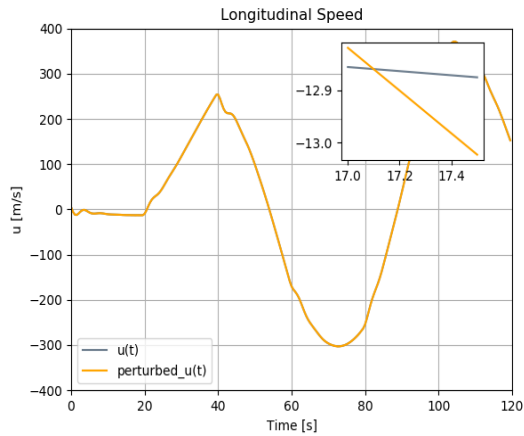


Figure 4.19: Longitudinal Speed, u , and Noisy Longitudinal Speed, u_N .

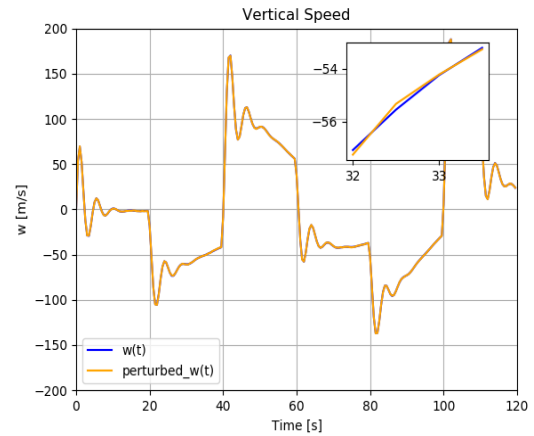


Figure 4.20: Vertical Speed, w , and Noisy Vertical Speed, w_N .

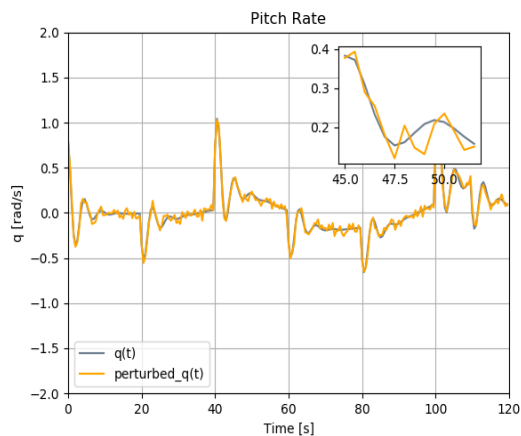


Figure 4.21: Pitch Rate, q , and Noisy Pitch Rate, q_N .

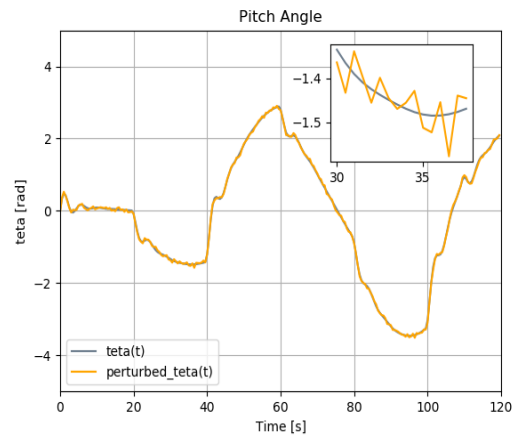


Figure 4.22: Pitch Angle, θ , and Noisy Pitch Angle, θ_N .

Up until this point, the sequence of steps performed for both the first and second methods are fairly similar. However, going further along the present method diverges from the first one. Being the flight data, at this point, corrupted by the noise it is crucial to eliminate as much noise as possible. On the previous method, this was achieved by using the "Savitzky-Golay Smoothing Filter". Here, however, in order to remove all the noise from the flight data RBF, namely, multiquadric RBF are used. As stated previously, in subsection 3.4.3, performing regression using radial basis functions in order to filter out the noise of data is preferred over interpolation using radial basis functions. However, instead of using all data points as data centres only a smaller part (for the present problem it will only be used half of the data points) of them will be used while the number of observations remains the same. That is, instead of using the matrix equation 4.18 corresponding to centring a multiquadric on each data centre

4.2. Method 2: Based on Multiquadric Radial Basis Functions

$$\begin{bmatrix} 1 & \sqrt{\sigma^2 + (0 - 0)^2} & \sqrt{\sigma^2 + (0 - 0.5)^2} & \dots & \sqrt{\sigma^2 + (0 - 119.5)^2} \\ 1 & \sqrt{\sigma^2 + (0.5 - 0)^2} & \sqrt{\sigma^2 + (0.5 - 0.5)^2} & \dots & \sqrt{\sigma^2 + (0.5 - 119.5)^2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \sqrt{\sigma^2 + (119.5 - 0)^2} & \sqrt{\sigma^2 + (119.5 - 0.5)^2} & \dots & \sqrt{\sigma^2 + (119.5 - 119.5)^2} \end{bmatrix} \quad (4.18)$$

as depicted in Figure 4.23

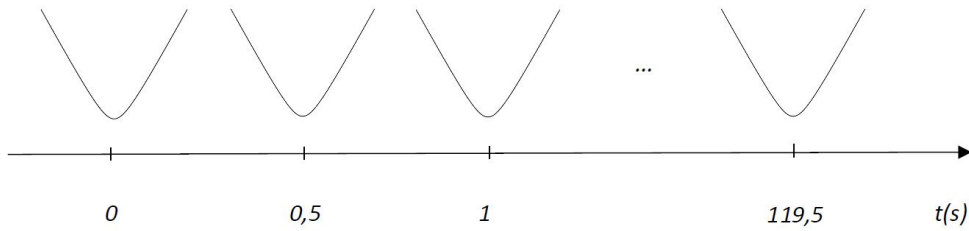


Figure 4.23: Multiquadric centred on every data point.

the matrix equation 4.19 is preferred corresponding to centring on only half of the data set

$$\begin{bmatrix} 1 & \sqrt{\sigma^2 + (0 - 0.5)^2} & \sqrt{\sigma^2 + (0 - 1.5)^2} & \dots & \sqrt{\sigma^2 + (0 - 119.5)^2} \\ 1 & \sqrt{\sigma^2 + (0.5 - 0.5)^2} & \sqrt{\sigma^2 + (0.5 - 1.5)^2} & \dots & \sqrt{\sigma^2 + (0.5 - 119.5)^2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \sqrt{\sigma^2 + (119.5 - 0.5)^2} & \sqrt{\sigma^2 + (119.5 - 1.5)^2} & \dots & \sqrt{\sigma^2 + (119.5 - 119.5)^2} \end{bmatrix} \quad (4.19)$$

as depicted in Figure 4.24.

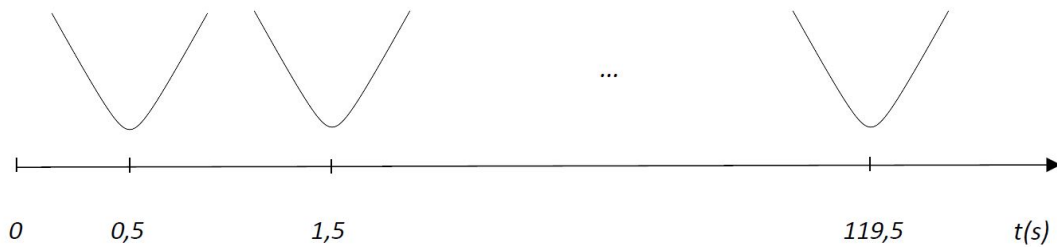


Figure 4.24: Multiquadric centred on half of data points.

After the choice of the design matrix, the shape parameter σ has still to be defined. Since changing the value of sigma changes the shape of the multiquadric function as shown in subsection 3.4.3.1 it is desirable to define a criterion so that the choice of sigma is based upon the minimization of the distance between each the noiseless flight data and the filtered flight data. Consequently, it is defined that the sigma to be chosen in the four different regressions are those which display the smallest RMSE and, consequently, the RMSE is computed for different values of σ and plotted in Figures 4.25 through 4.28. Perusing these it can be seen that the RMSE values do not change rapidly especially in the last two Figures but they decrease as σ increases except for the case of the third regression in which the RMSE is minimum when σ assumes a value of 0.4.

4.2. Method 2: Based on Multiquadric Radial Basis Functions

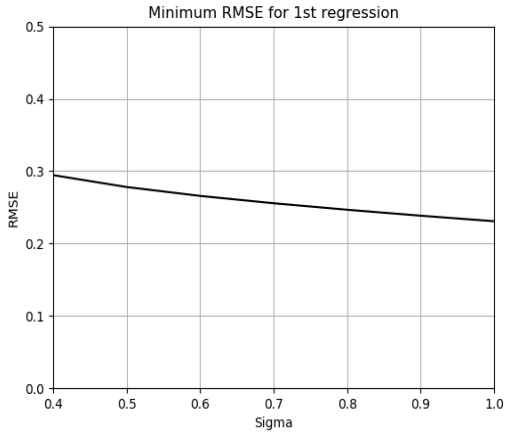


Figure 4.25: RMSE for different values of sigma for the case of the longitudinal speed u .

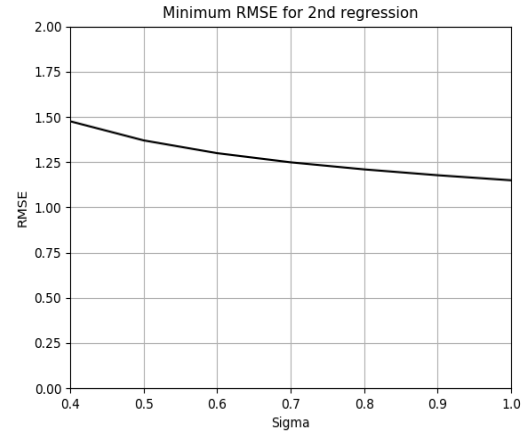


Figure 4.26: RMSE for different values of sigma for the case of the vertical speed w .

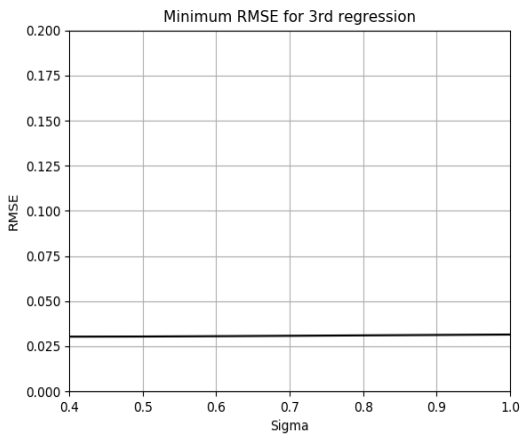


Figure 4.27: RMSE for different values of sigma for the case of the pitch rate q .

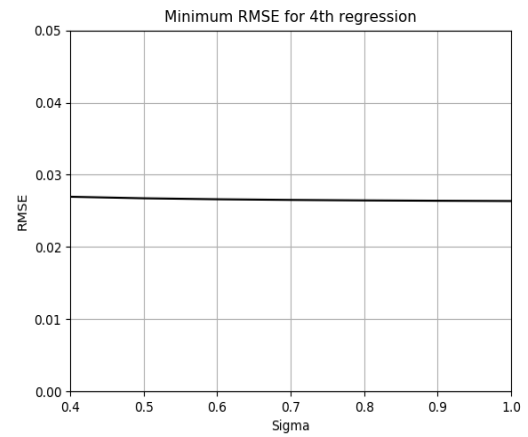


Figure 4.28: RMSE for different values of sigma for the case of the pitch angle θ .

In table 4.8 a correspondence is made for the RMSE values for different values of σ for all the state variables. The RMSE values in red represent the minimum value for each column. Hence, in order to have the least RMSE possible, for the first, second and fourth regression a $\sigma = 1.0$ is chosen whereas for the third regression $\sigma = 0.4$ is chosen instead.

σ	u_{RMSE}	w_{RMSE}	q_{RMSE}	θ_{RMSE}
0.4	0.29451	1.47579	0.03032	0.02695
0.5	0.27809	1.37006	0.03039	0.02674
0.6	0.26583	1.29948	0.03057	0.02660
0.7	0.25567	1.24893	0.03080	0.02651
0.8	0.24669	1.20986	0.03103	0.02645
0.9	0.23847	1.17759	0.03127	0.02640
1.0	0.23085	1.14955	0.03149	0.02636

Table 4.8: RMSE for different sigmas for each one of the state variables

4.2. Method 2: Based on Multiquadric Radial Basis Functions

Considering the definition of the design matrices and that of the shape parameter σ the matrix equations for the four different regressions to be assumed in order to remove as much noise as possible are as follows:

• First Regression

$$\begin{bmatrix} 1 & \sqrt{1.0^2 + (0 - 0.5)^2} & \sqrt{1.0^2 + (0 - 1.5)^2} & \dots & \sqrt{1.0^2 + (0 - 119.5)^2} \\ 1 & \sqrt{1.0^2 + (0.5 - 0.5)^2} & \sqrt{1.0^2 + (0.5 - 1.5)^2} & \dots & \sqrt{1.0^2 + (0.5 - 119.5)^2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \sqrt{1.0^2 + (119.5 - 0.5)^2} & \sqrt{1.0^2 + (119.5 - 1.5)^2} & \dots & \sqrt{1.0^2 + (119.5 - 119.5)^2} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{120} \end{bmatrix} = \begin{bmatrix} u_{N_0} \\ u_{N_1} \\ \vdots \\ u_{N_{239}} \end{bmatrix}$$

• Second Regression

$$\begin{bmatrix} 1 & \sqrt{1.0^2 + (0 - 0.5)^2} & \sqrt{1.0^2 + (0 - 1.5)^2} & \dots & \sqrt{1.0^2 + (0 - 119.5)^2} \\ 1 & \sqrt{1.0^2 + (0.5 - 0.5)^2} & \sqrt{1.0^2 + (0.5 - 1.5)^2} & \dots & \sqrt{1.0^2 + (0.5 - 119.5)^2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \sqrt{1.0^2 + (119.5 - 0.5)^2} & \sqrt{1.0^2 + (119.5 - 1.5)^2} & \dots & \sqrt{1.0^2 + (119.5 - 119.5)^2} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{120} \end{bmatrix} = \begin{bmatrix} w_{N_0} \\ w_{N_1} \\ \vdots \\ w_{N_{239}} \end{bmatrix}$$

• Third Regression

$$\begin{bmatrix} 1 & \sqrt{0.4^2 + (0 - 0.5)^2} & \sqrt{0.4^2 + (0 - 1.5)^2} & \dots & \sqrt{0.4^2 + (0 - 119.5)^2} \\ 1 & \sqrt{0.4^2 + (0.5 - 0.5)^2} & \sqrt{0.4^2 + (0.5 - 1.5)^2} & \dots & \sqrt{0.4^2 + (0.5 - 119.5)^2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \sqrt{0.4^2 + (119.5 - 0.5)^2} & \sqrt{0.4^2 + (119.5 - 1.5)^2} & \dots & \sqrt{0.4^2 + (119.5 - 119.5)^2} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{120} \end{bmatrix} = \begin{bmatrix} q_{N_0} \\ q_{N_1} \\ \vdots \\ q_{N_{239}} \end{bmatrix}$$

• Fourth Regression

$$\begin{bmatrix} 1 & \sqrt{1.0^2 + (0 - 0.5)^2} & \sqrt{1.0^2 + (0 - 1.5)^2} & \dots & \sqrt{1.0^2 + (0 - 119.5)^2} \\ 1 & \sqrt{1.0^2 + (0.5 - 0.5)^2} & \sqrt{1.0^2 + (0.5 - 1.5)^2} & \dots & \sqrt{1.0^2 + (0.5 - 119.5)^2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \sqrt{1.0^2 + (119.5 - 0.5)^2} & \sqrt{1.0^2 + (119.5 - 1.5)^2} & \dots & \sqrt{1.0^2 + (119.5 - 119.5)^2} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{120} \end{bmatrix} = \begin{bmatrix} \theta_{N_0} \\ \theta_{N_1} \\ \vdots \\ \theta_{N_{239}} \end{bmatrix}$$

It is now necessary to find the coefficients β for all four regressions above and as such OLS is applied. Afterwards four different functions, each corresponding to one state variable, are formed (equation 4.20).

$$\begin{cases} u_F(t) = \beta_0 + \beta_1 \sqrt{1.0^2 + (t - 0.5)^2} + \beta_2 \sqrt{1.0^2 + (t - 1.5)^2} + \dots + \beta_{120} \sqrt{1.0^2 + (t - 119.5)^2} \\ w_F(t) = \beta_0 + \beta_1 \sqrt{1.0^2 + (t - 0.5)^2} + \beta_2 \sqrt{1.0^2 + (t - 1.5)^2} + \dots + \beta_{120} \sqrt{1.0^2 + (t - 119.5)^2} \\ q_F(t) = \beta_0 + \beta_1 \sqrt{0.4^2 + (t - 0.5)^2} + \beta_2 \sqrt{0.4^2 + (t - 1.5)^2} + \dots + \beta_{120} \sqrt{0.4^2 + (t - 119.5)^2} \\ \theta_F(t) = \beta_0 + \beta_1 \sqrt{1.0^2 + (t - 0.5)^2} + \beta_2 \sqrt{1.0^2 + (t - 1.5)^2} + \dots + \beta_{120} \sqrt{1.0^2 + (t - 119.5)^2} \end{cases} \quad (4.20)$$

Consequently, through these equations, the filtered flight data is obtained and is shown in table 4.9.

4.2. Method 2: Based on Multiquadric Radial Basis Functions

t	u_F	w_F	q_F	θ_F
0	4.72098	0.80464	0.74750	-0.01244
0.5	-5.22441	53.06721	0.50442	0.35371
1.0	-11.21208	68.63189	0.22790	0.50669
\vdots	\vdots	\vdots	\vdots	\vdots
119.0	162.55428	31.65384	0.17641	2.07469
119.5	153.25591	21.31285	0.20743	2.05944

Table 4.9: Filtered Flight Data

In order to reach the final step of this method, it is necessary to determine the time derivative of all equations given by equation 4.20. Since the general form of the used basis functions is given by equation 4.21

$$\varphi_k(t) = \sqrt{\sigma^2 + (t - t_k)^2} \quad (4.21)$$

deriving the above equation yields equation 4.22.

$$\frac{d\varphi_k(t)}{dt} = \frac{t - t_k}{\sqrt{\sigma^2 + (t - t_k)^2}} \quad (4.22)$$

Hence the time derivatives of equations 4.20 are defined as:

$$\begin{cases} \dot{u}_F(t) = \beta_1 \frac{t-0.5}{\sqrt{1.0^2+(t-0.5)^2}} + \beta_2 \frac{t-1.5}{\sqrt{1.0^2+(t-1.5)^2}} + \dots + \beta_{120} \frac{t-119.5}{\sqrt{1.0^2+(t-119.5)^2}} \\ \dot{w}_F(t) = \beta_1 \frac{t-0.5}{\sqrt{1.0^2+(t-0.5)^2}} + \beta_2 \frac{t-1.5}{\sqrt{1.0^2+(t-1.5)^2}} + \dots + \beta_{120} \frac{t-119.5}{\sqrt{1.0^2+(t-119.5)^2}} \\ \dot{q}_F(t) = \beta_1 \frac{t-0.5}{\sqrt{0.4^2+(t-0.5)^2}} + \beta_2 \frac{t-1.5}{\sqrt{0.4^2+(t-1.5)^2}} + \dots + \beta_{120} \frac{t-119.5}{\sqrt{0.4^2+(t-119.5)^2}} \\ \dot{\theta}_F(t) = \beta_1 \frac{t-0.5}{\sqrt{1.0^2+(t-0.5)^2}} + \beta_2 \frac{t-1.5}{\sqrt{1.0^2+(t-1.5)^2}} + \dots + \beta_{120} \frac{t-119.5}{\sqrt{1.0^2+(t-119.5)^2}} \end{cases} \quad (4.23)$$

From equations 4.23 the time derivative of the filtered flight data are obtained and shown in table 4.10.

t	\dot{u}	\dot{w}	\dot{q}	$\dot{\theta}$
0	-22.98257	136.27300	-0.39782	0.90881
0.01	-16.29028	68.66303	-0.65864	0.53028
0.02	-7.44779	-3.94567	-0.86268	0.08494
\vdots	\vdots	\vdots	\vdots	\vdots
119.98	-24.90770	4.79971	-0.08969	0.14356
119.99	-10.19205	-54.19671	0.09415	-0.25786

Table 4.10: Time Derivatives of the Flight Data

The last step is to obtain all elements of the matrix equation 4.1 for this particular problem. Consequently, it is necessary to do three different linear regressions. Each regression has a common design matrix $M_{240 \times 5}$ (equation 4.24) having the filtered flight data as the first four columns and as for the fifth column the input vector η , i.e., the elevator deflection's data.

4.2. Method 2: Based on Multiquadric Radial Basis Functions

$$M = \begin{bmatrix} 4.72098 & 0.80464 & 0.74750 & -0.01244 & 0 \\ -5.22441 & 53.06721 & 0.50442 & 0.35371 & 0 \\ -11.21208 & 68.63189 & 0.22790 & 0.50669 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 162.55428 & 31.65384 & 0.17641 & 2.07469 & 0 \\ 153.25591 & 21.31285 & 0.20743 & 2.05944 & 0 \end{bmatrix} \quad (4.24)$$

Having now defined the matrix M , the three unknown vectors to be determined are composed by the elements of the state matrix A and of the input matrix B . Let x_i denote the unknown column vectors with $i = 1, \dots, 3$. Then the three coefficient column vectors are represented as:

$$\begin{aligned} x_1 &= [a_{11} \ a_{12} \ a_{13} \ a_{14} \ b_1]^T \\ &\vdots \\ x_3 &= [a_{31} \ a_{32} \ a_{33} \ a_{34} \ b_3]^T \end{aligned}$$

Using OLS in order to perform the linear regressions, as discussed in section 3.4.2, using as observation vectors the columns \dot{u} , \dot{w} , \dot{q} and $\dot{\theta}$ on the table 4.10 the unknown vectors are given by computing

$$\begin{aligned} x_1 &= (M^T M)^{-1} M^T [\dot{u}] \\ &\vdots \\ x_3 &= (M^T M)^{-1} M^T [\dot{q}] \end{aligned}$$

where $[\dot{u}]$ and $[\dot{q}]$ represent the vectors, whose values correspond to \dot{u} and \dot{q} for all samples. The coefficients obtained by performing OLS are the resulting elements are depicted in matrix equation 4.25.

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 3.590 \times 10^{-3} & 3.650 \times 10^{-2} & -37.638 & -9.353 \\ -0.0868 & -0.4927 & 229.080 & -3.377 \\ 1.79 \times 10^{-3} & -0.0107 & -0.3684 & 6.664 \times 10^{-3} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 14.536 \\ -86.699 \\ -4.784 \\ 0 \end{bmatrix} \eta \quad (4.25)$$

Looking at the above matrix equation it is seen that a few elements are more close to the corresponding elements in matrix equation 4.13 than others. However, to draw any conclusions as to the quality of any of the regressions a formal and a pragmatic analysis are carried out as in the first method.

4.2. Method 2: Based on Multiquadric Radial Basis Functions

4.2.1 First Regression Analysis

Formal Analysis

The results obtained through performing the formal analysis are the following:

$$\begin{aligned}\bar{y} &= 1.169 & R^2 &= 0.990 \\ SS_R &= 78752.209 & RMSE &= \text{Not Applicable} \\ SS_T &= 79548.656 & \tilde{\sigma} &= 0.007\end{aligned}$$

Perusing the results obtained above it can be seen that both condition shown in 3.66 are verified. Therefore, it is concluded that the first regression passes the formal analysis and it is considered to be "very good".

Pragmatic Analysis

The pragmatic analysis was conducted and the following results were obtained:

$$\begin{aligned}\tilde{\sigma}_0(y, \hat{y}) &= 0.008 \\ \tilde{\sigma}_I(y, \hat{y}) &= 0.013 \\ \tilde{\sigma}_{II}(y, \hat{y}) &= 0.007\end{aligned}$$

Comparing these results with the conditions given in sub-subsection 3.4.4.2 it is seen that all three relative errors above verify the conditions describe by the equation 3.67 through 3.69. Consequently, the first linear regression also passes the pragmatistical analysis and is therefore considered to be excellent.

4.2.2 Second Regression Analysis

Formal Analysis

The results obtained through performing the formal analysis are the following:

$$\begin{aligned}\bar{y} &= 0.331 & R^2 &= 0.889 \\ SS_R &= 200638.014 & RMSE &= \text{Not Applicable} \\ SS_T &= 225589.203 & \tilde{\sigma} &= 0.021\end{aligned}$$

It can be seen that the results of the second linear regression analysis agree with those stated in 3.66. As such the formal analysis is considered to be very good.

Pragmatic Analysis

The pragmatic analysis was conducted the following results were obtained:

$$\begin{aligned}\tilde{\sigma}_0(y, \hat{y}) &= 0.023 \\ \tilde{\sigma}_I(y, \hat{y}) &= 0.031 \\ \tilde{\sigma}_{II}(y, \hat{y}) &= 0.021\end{aligned}$$

4.2. Method 2: Based on Multiquadric Radial Basis Functions

These also agree with the conditions given sub-subsection 3.4.4.2 and consequently the second linear regression is considered to be excellent.

4.2.3 Third Regression Analysis

Formal Analysis

The results obtained through performing the formal analysis are as follows:

$$\begin{array}{ll} \bar{y} = -0.006 & R^2 = 0.823 \\ SS_R = 11.489 & RMSE = \text{Not Applicable} \\ SS_T = 13.958 & \tilde{\sigma} = 0.027 \end{array}$$

Similarly to the first and second linear regressions cases, the third formal analysis also verifies the conditions stated in 3.66. Hence, formal analysis is considered to be very good.

Pragmatic Analysis

The pragmatic analysis was conducted the following results were obtained:

$$\begin{array}{l} \tilde{\sigma}_0(y, \hat{y}) = 0.031 \\ \tilde{\sigma}_I(y, \hat{y}) = 0.049 \\ \tilde{\sigma}_{II}(y, \hat{y}) = 0.027 \end{array}$$

The above results are also in agreement with the conditions previously defined which consequently leads the conclusion that the third linear regression is excellent.

Chapter 5

Conclusion

The present work started out with the objective of building a method which allowed to perform system identification without incurring in errors due to the way in which flight data is obtained, that is, given the fact of considering a natural continuous system as a discrete one. To address this problem two distinct methods were proposed.

Firstly, it was necessary to have access to a model of an aircraft with which one could generate flight data. After the attainment of said model, flight data was promptly generated through an application of a method for solving ODE's, namely Butcher's algorithm. Subsequently, it was necessary to reproduce as much as possible a real-life situation, meaning that when recording flight data it is almost certain that it will be corrupted by disturbances (noise) however how little that it may be. Consequently, a disturbance is introduced into the generated flight data. Up until this point, both methods are similar, however, a different approach is undertaken from this point on.

As for the first method, any analysis of the flight data requires that the noise is eliminated as much as possible. To do so, the Savitzky-Golay Smoothing Filter was applied thus obtaining this way the filtered flight data. All that was left was to obtain the first time derivative of the latter. Thus, the derivation is conducted aided by numerical methods based upon the Taylor polynomial. Consequently, linear regression was conducted and the longitudinal dynamic model was obtained and its accuracy accessed. Afterwards, it is concluded, that even though the derivation and smoothing were conducted in a discrete-time setting the method proposed successfully achieved the objective.

The second method proposed diverges from the first one in a way that from the discrete filtered flight data continuous functions were obtained, through the usage of multiquadric radial basis functions, achieving this way the possibility of differentiating on a continuous setting thus eliminating the errors mentioned in the introduction. Next, the second model was accessed regarding its accuracy and it was concluded that it also successfully achieves the objective of the present work.

As for future works, it would be interesting to undertake a similar approach considering non-linear methods as opposed to the linear ones considered in the present dissertation. Furthermore, it would also be compelling to implement these methods in a real-time setting, that is, aboard a flying aircraft.

References

- [1] V. Klein e E. A. Morelli, *Aircraft System Identification*. American Institute of Aeronautics and Astronautics, 2006. 1, 2, 3, 5, 6, 11, 34
- [2] L. Ljung, *System Identification: Theory for the User*. Prentice Hall, Inc., 1987. 1, 2
- [3] B. S. Everitt e A. Skrondal, *The Cambridge Dictionary of Statistics*. Press, Cambridge University, 2010. 2, 34, 35
- [4] K. J. Keesman, *System Identification: An Introduction*. Springer-Verlag London Limited, 2011. 2
- [5] D. F. Anderson e S. Eberhardt, *Understanding Flight*. McGraw-Hill, 2000. 3
- [6] M. V. Cook, *Flight Dynamics Principles*, 2nd ed. Butterworth-Heinemann, 2007. 4, 6, 7, 8, 15, 17, 18, 21, 22, 24, 49
- [7] K. Bousson, “Dinâmica e Controlo de Voo (Lecture Notes),” Universidade da Beira Interior, “Departamento de Ciências Aeroespaciais”, Covilhã, 2018. 5, 6, 8, 9, 10
- [8] B. Etkin, *Dynamics of Atmospheric Flight*. Dover Publications, 2005. 5, 6
- [9] R. F. Stengel, *Flight Dynamics*. Press, Princeton University, 2004. 7, 8
- [10] R. C. Nelson, *Flight Stability and Automatic Control*, 2nd ed. McGraw-Hill, 1998. 8, 11, 16
- [11] L. V. Schmidt, *Introduction to Aircraft Flight Dynamics*. American Institute of Aeronautics and Astronautics, 1998. 9, 10, 21
- [12] B. Etkin e L. D. Reid, *Dynamics of Flight: Stability and Control*, 3rd ed. John Wiley & Sons Inc., 1996. 10, 11, 16, 21
- [13] S. C. Chapra e R. C. Canale, *Numerical Methods for Engineers*, 7th ed. Education, McGraw-Hill, 2010. 15, 27, 29, 30
- [14] K. Ogata, *Modern Control Engineering*, 5th ed. Hall, Prentice. 22
- [15] J. Duque, “Sebenta de Matemática Computacional,” Universidade da Beira Interior, “Departamento de Matemática”, Covilhã, 2013. 27
- [16] K. Bousson, “Resolução Numérica de Equações Diferenciais,” Universidade da Beira Interior, “Departamento de Ciências Aeroespaciais”, Covilhã, 2016. 31
- [17] W. H. Press, S. A. Teukolsky, W. T. Vetterling, e B. P. Flannery, *Numerical Recipes: The Art of Scientific Computing*, 3rd ed. Press, Cambridge University, 2007. 31, 32
- [18] A. Savitzky e M. J. E. Golay, “Smoothing and Differentiation of Data by Simplified Least Squares Procedures,” *Anal. Chem.*, vol. 36, no. 8, pp. 1627-1639, 1964. 31, 32
- [19] K. Bousson, “Curso de Fundamentos de Python & Ciência de Dados,” CFIUTE, Covilhã, 2018. 33, 46, 47

- [20] D. C. Montgomery e G. C. Runger, *Applied Statistics and Probability for Engineers*, 6th ed. Wiley, 2013. 33, 34, 35, 36, 38, 43, 44, 45
- [21] T. Hastie, G. James, e R. Tibshirani, *An Introduction to Statistical Learning: With Applications in R*. Inc., Springer-Verlag New York, 2013. 34, 40, 45
- [22] C. R. Rao, C. Heumann, C. Shalabh, e H. Toutenburg, *Linear Models and Generalizations: Least Squares and Alternatives*, 3rd ed. KG, Springer-Verlag Berlin and Heidelberg GMBH & CO., 2007. 35, 36
- [23] A. Hayter, *Probability and Statistics: For Engineers and Scientists*, 4th ed. Publishing, Cengage Learning Custom, 2012. 35, 36, 43, 44
- [24] I. Cabral, C. Perdigão, e C. Saiago, *Álgebra Linear: Teoria, Exercícios resolvidos e Exercícios propostos com soluções*, 2nd ed. Editora, Escolar, 2010. 38, 39
- [25] G. Strang, *Introduction to Linear Algebra*, 4th ed. Press, Wellesley-Cambridge, 2009. 39
- [26] D. C. Lay, S. R. Lay, e J. J. McDonald, *Linear Algebra and its Applications*, 5th ed. Limited, Pearson Education, 2015. 39
- [27] M. E. Biancolini, *Fast Radial Basis Functions for Engineering Applications*. Springer, 2017. 40, 41, 42
- [28] M. D. Buhmann, *Radial Basis Functions*. Cambridge University Press, 2003. 40
- [29] A. C. Pedrosa e S. M. A. Gama, *Introdução Computacional à Probabilidade e Estatística com Excel®*, 3rd ed. Porto Editora, 2018. 43, 44, 45
- [30] T. Masters, *Practical Neural Network Recipes in C++*. Academic Press, Inc., 1993. 45
- [31] T. Masters, *Assessing and Improving Prediction and Classification: Theory and Algorithms in C++*. Academic Press, Inc., 2018. 47
- [32] “BeginnersGuide/Overview - Python Wiki.” [Online]. Disponível: <https://wiki.python.org/moin/BeginnersGuide/Overview> (Acedido em: 2019-03-13). 49
- [33] “Numpy.” [Online]. Disponível: <http://www.numpy.org/> (Acedido em: 2019-03-13). 49
- [34] “Scientific Computing Tools for Python.” [Online]. Disponível: <https://www.scipy.org/about.html> (Acedido em: 2019-03-13). 49
- [35] “matplotlib.” [Online]. Disponível: <https://matplotlib.org/> (Acedido em: 2019-03-13). 49

Appendix A

Aerodynamic Stability and Control Derivatives In Concise Format

$$x_u = \frac{\frac{\partial X}{\partial u}}{m} + \frac{\frac{\partial X}{\partial \dot{w}} \frac{\partial Z}{\partial u}}{m \left(m - \frac{\partial Z}{\partial \dot{w}} \right)}$$

$$z_u = \frac{\frac{\partial Z}{\partial u}}{m - \frac{\partial Z}{\partial \dot{w}}}$$

$$m_u = \frac{\frac{\partial M}{\partial u}}{I_{yy}} + \frac{\frac{\partial Z}{\partial u} \frac{\partial M}{\partial \dot{w}}}{I_{yy} \left(m - \frac{\partial Z}{\partial \dot{w}} \right)}$$

$$x_w = \frac{\frac{\partial X}{\partial w}}{m} + \frac{\frac{\partial X}{\partial \dot{w}} \frac{\partial Z}{\partial w}}{m \left(m - \frac{\partial Z}{\partial \dot{w}} \right)}$$

$$z_w = \frac{\frac{\partial Z}{\partial w}}{m - \frac{\partial Z}{\partial \dot{w}}}$$

$$\frac{\frac{\partial M}{\partial w}}{I_{yy}} + \frac{\frac{\partial Z}{\partial w} \frac{\partial M}{\partial \dot{w}}}{I_{yy} \left(m - \frac{\partial Z}{\partial \dot{w}} \right)}$$

$$x_q = \frac{\left(\frac{\partial X}{\partial q} - mW_e \right)}{m} + \frac{\left(\frac{\partial Z}{\partial q} + mU_e \right) \frac{\partial X}{\partial \dot{w}}}{m \left(m - \frac{\partial Z}{\partial \dot{w}} \right)}$$

$$z_q = \frac{\left(\frac{\partial Z}{\partial q} + mU_e \right)}{m - \frac{\partial Z}{\partial \dot{w}}}$$

$$m_q = \frac{\frac{\partial M}{\partial q}}{I_{yy}} + \frac{\left(\frac{\partial Z}{\partial q} + mU_e\right) \frac{\partial M}{\partial \dot{w}}}{I_{yy} \left(m - \frac{\partial Z}{\partial \dot{w}}\right)}$$

$$x_\theta = -g \cos(\theta_e) - \frac{\frac{\partial X}{\partial \dot{w}} g \sin(\theta_e)}{m - \frac{\partial Z}{\partial \dot{w}}}$$

$$z_\theta = -\frac{mg \sin(\theta_e)}{m - \frac{\partial Z}{\partial \dot{w}}}$$

$$m_\theta = \frac{\frac{\partial M}{\partial \dot{w}} mg \sin(\theta_e)}{I_{yy} \left(m - \frac{\partial Z}{\partial \dot{w}}\right)}$$

$$x_\eta = \frac{\frac{\partial X}{\partial \eta}}{m} + \frac{\frac{\partial X}{\partial \dot{w}} \frac{\partial Z}{\partial \eta}}{m \left(m - \frac{\partial Z}{\partial \dot{w}}\right)}$$

$$z_\eta = \frac{\frac{\partial Z}{\partial \eta}}{m - \frac{\partial Z}{\partial \dot{w}}}$$

$$m_\eta = \frac{\frac{\partial M}{\partial \eta}}{I_{yy}} + \frac{\frac{\partial M}{\partial \dot{w}} \frac{\partial Z}{\partial \eta}}{I_{yy} \left(m - \frac{\partial Z}{\partial \dot{w}}\right)}$$

$$x_\tau = \frac{\frac{\partial X}{\partial \tau}}{m} + \frac{\frac{\partial X}{\partial \dot{w}} \frac{\partial Z}{\partial \tau}}{m \left(m - \frac{\partial Z}{\partial \dot{w}}\right)}$$

$$z_\tau = \frac{\frac{\partial Z}{\partial \tau}}{m - \frac{\partial Z}{\partial \dot{w}}}$$

$$m_\tau = \frac{\frac{\partial M}{\partial \tau}}{I_{yy}} + \frac{\frac{\partial M}{\partial \dot{w}} \frac{\partial Z}{\partial \tau}}{I_{yy} \left(m - \frac{\partial Z}{\partial \dot{w}} \right)}$$

