Characterizations of Flip-Accessibility for Domino Tilings of the Whole Plane

Olivier Bodini, Thomas Fernique, Éric Rémila

Formal Power Series and Algebraic Combinatorics
Nankai University, Tianjin, China, 2007

Characterizations of Flip-Accessibility for Domino Tilings of the Whole Plane

Olivier Bodini, Thomas Fernique, and Éric Rémila

Abstract. It is known that any two domino tilings of a polygon are flip-accessible, *i.e.*, linked by a finite sequence of local transformations, called flips. This paper considers flip-accessibility for domino tilings of the *whole plane*, asking whether two of them are linked by a *possibly infinite* sequence of flips. The answer turning out to depend on tilings, we provide three equivalent *characterizations* of flip-accessibility.

Résumé. Étant donnés deux pavages par dominos d'un mme polygone, on sait qu'on peut toujours passer de l'un l'autre en effectuant un nombre fini de transformations locales, appelées *flips* ; ces pavages sont dits *flip-accessibles.* Dans ce papier, nous étendons cette notion de flip-accessibilité aux pavages par dominos du *plan entier*, en s'autorisant cette fois effectuer un nombre *infini* de flips. Dans ce cas, la flip-accessibilité dépend des pavages considérés et nous en donnons trois *caractérisations* équivalentes.

Introduction

In this paper, we study domino tilings. These tilings are of particular importance in theoretical physics, where a domino is seen as a dimer, that is a diatomic molecule (as the molecule of dihydrogen) and each tiling is a possible state of a solid, or a fluid. *Flips* are local transformations involving two tiles covering a 2×2 square (see Fig. 1). They induce a dynamic on tilings that plays a central role in domino tilings theory.

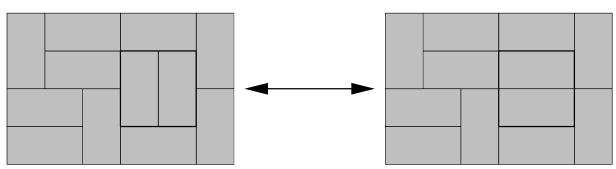

Figure 1. Two domino tilings differing from a flip.

Flips can be directed using the notion of *height function*, introduced by Thurston [**13**]. Then, given a hole-free bounded domain D, consider the directed graph whose vertices corresponds to tilings of D, with an edge from v to v' if the tiling v' can be obtained from v by performing an upwards flips: a central result is that this graph is the covering relation of a distributive lattice (see, for example, [**4, 12**], and also [**6**] for general lattice theory). Many applications rely on this result: tiling algorithm, computation of the distance (in number of flips) between tilings, random sampling [**9**] or listing [**8, 7**]. Some extensions for domains with holes can be found in [**2, 10**].

In particular, the previous result yields that, given any two domino tilings of a same given polygon, one can be transformed into the other one by performing flips: they are said to be flip-accessible. However, this

2000 *Mathematics Subject Classification.* Primary 52C20; Secondary 37A60.
Key words and phrases. domino tilings, flips, height functions, shadows.

no more holds, in general, for domino tilings of the *whole plane*, as considered here. Thus, we would like to *characterize* pairs of domino tilings of the plane which can be linked by flips. Moreover, we would like to allow *infinitely* many flips to be performed, for linking two such tilings, since it appears more natural in this new context. It is worth stressing that this relaxation changes the nature of flip-accessibility since, for example, we will see that it is no more a symmetric relation.

The main results of this paper consist in three equivalent characterizations of flip-accessibility for these domino tilings of the whole plane. Since we strongly rely on the background used for finite domino tilings, we first briefly recall it in Section 1, where it is also extended to the infinite case, in particular concerning the notion of flip-accessibility. This allows to state our characterizations in the three following sections: the first one (Section 2) uses special domino tilings, called *pyramids*; the second one (Section 3) reduces the problem to a set of one-dimensional problems of flip-accessibility; the third one (Section 4) relies on the new notion of *shadows* of domino tilings. Let us note that a similar study can be done for lozenge tilings (see [**1, 3**]) and for eulerian orientations of the grid.

1. General settings

1.1. Domino tilings and height functions. Here, we recall some basic settings which hold for both finite or infinite domino tilings. A *cell* is a unit square whose vertices belong to this grid $\mathbb{Z}^2$. A *Domino tiling* $\mathcal{T}$ can then be defined as an involutive function over a set of cells, such that a cell and its image always share a unit segment. In other words, each domino covers exactly two cells, and each cell is covered by exactly one domino. Without loss of generality, we consider only *connected* set of cells.

Given a set of cells, we assume that cells are colored in black and white as a chessboard, and we associate with this set a directed graph G as follows (see Fig. 2, left):

- vertices correspond to vertices of cells;
- edges correspond to boundary segments of cells;
- an edge (v, v') is directed from v to v' so that an ant moving from v to v' has a white cell on its left and a black cell on its right.

This allows to associate, with a domino tiling $\mathcal{T}$ of this set of cells, a function $h_{\mathcal{T}}$ defined over vertices of G as follows (see Fig. 2, right):

- we choose a vertex v_0 and we arbitrarily set $h_{\mathcal{T}}(v_0) \in \mathbb{R}$;
- for each edge (v, v'), one has $h_{\mathcal{T}}(v') = h_{\mathcal{T}}(v) - 3$ if (v, v') belongs to two cells covered by a domino (in other words, this edge *cuts* a domino), $h_{\mathcal{T}}(v') = h_{\mathcal{T}}(v) + 1$ otherwise.

The consistency of such a definition is easily checked.

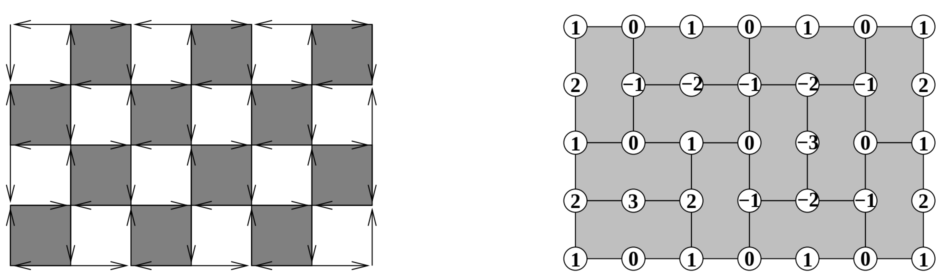

FIGURE 2. A set of chessboard-like colored cells and the directed edges of the associated graph (left). A domino tiling $\mathcal{T}$ of this set of cells and a corresponding function $h_{\mathcal{T}}$ (right).

Conversely, to any function h over the vertices of G such that $h(v') - h(v) \in \{1, -3\}$ for any edge (v, v'), with moreover $h(v') - h(v) = 1$ if (v, v') belongs to only one cell (that is, it is on the boundary of the set of cell), corresponds a domino tiling. Such function are called *height functions*: to any height function corresponds a unique domino tiling, and to any domino tiling correspond a family of height functions, identical up to a translation (that is, of the form $\{h_c : v \to h_0(v) + c \mid c \in \mathbb{R}\}$, where h_0 is a fixed height function). Note that domino tilings of the whole plane correspond to height functions defined over the whole $\mathbb{Z}^2$. Geometrically, one can represent height functions using a three-dimensional viewpoint, as depicted on Fig. 3.

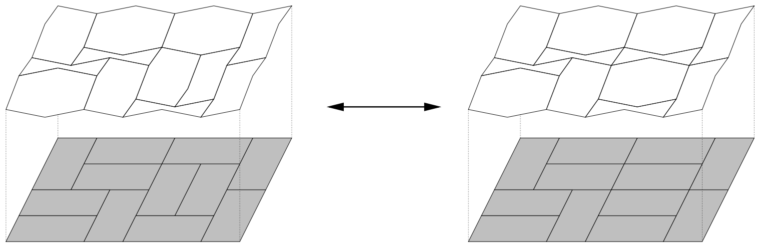

FIGURE 3. The domino tilings of Fig. 1 (bottom) and a lifted viewpoint, according to height functions (top). Note that performing a flip changes the height of only one vertex (by ± 4).

In order to perform operation over height functions, it is convenient to consider, for each domino tiling, only a restricted subset of the height functions which correspond to it. More precisely, let us define the function mod_4 from $\mathbb{Z}^2$ to $\mathbb{Z}/4\mathbb{Z}$ by:

$$mod_4(0,1) = 3 \quad mod_4(1,1) = 2$$
$$mod_4(0,0) = 0 \quad mod_4(1,0) = 1$$

and, for any $(x, y) \in \mathbb{Z}^2$:

$$mod_4(x+2, y) = mod_4(x, y+2) = mod_4(x, y).$$

We say that a point $(v, z) \in \mathbb{Z}^2 \times \mathbb{Z}$ is *admissible* if $z = mod_4(v) \mod 4$. Without loss of generality, we will consider in all what follows only height functions whose vertices are admissible (or, equivalently, which contains at least one admissible vertex). This allows to define the following useful operations (see [**12**]):

DEFINITION 1.1. Let h and h' be two height functions defined over the same set of cells. Then, $\max(h, h')$ and $\min(h, h')$ are height functions over this set of cells. They are respectively called the *supremum* and the *infimum* of h and h', and denoted by $h \vee h'$ and $h \wedge h'$.

These two operations naturally endow height functions over a given set of cells with a structure of *distributive lattice*, and will be useful in the following sections. Note that intuitively, following the previous three-dimensional viewpoint (recall Fig. 3), the height function $h \vee h'$ (resp. $h \wedge h'$) corresponds to the domino tiling that we would see by looking top-down (resp. bottom-up) height functions h and h' (both represented on the same picture); the admissibility restriction then ensures that domino tiles of h and h' cross only along edges.

1.2. Flip-accessibility. Now, we recall the classic notion of flip-accessibility, and we extend it to the case of domino tilings of the whole plane. Note that there is only two way to tile a 2×2 square by dominoes: two vertical dominoes, or two horizontal ones. If v denotes the central vertex of such a square in a domino tiling, then a *flip around* v is the operation which exchanges these two local configurations (recall Fig. 1).

Two *finite* domino tilings are said to be *flip-accessible* if one can transform the first one into the second one by performing a finite number of flips. It is known that any two finite domino tilings are flip-accessible (see *e.g.* []). Here, we are interested in the case of domino tilings of the whole plane. One cannot expect that a finite number of flips always suffices to link such tilings, since they can be different arbitrarily far from the origin. Thus, it is natural to consider possibly infinite sequence of flips. For this, we first need to define a notion of *convergence* over domino tilings of the whole plane. Let us define the distance $d(\mathcal{T}, \mathcal{T}')$ between two tilings $\mathcal{T}$ and $\mathcal{T}'$ by:

$$d(\mathcal{T}, \mathcal{T}') = \inf\{2^{-r} \mid \mathcal{T}_{|B(0,r)} = \mathcal{T}'_{|B(0,r)}\},$$

where $\mathcal{T}_{|B(0,r)}$ denotes the set of dominoes which belong to the ball of center 0 and radius r. This allows us to extend the previous definition of flip-accessibility:

DEFINITION 1.2. Let $\mathcal{T}$ and $\mathcal{T}'$ be two domino tilings of the whole plane. If there is a sequence $(\mathcal{T}_n)_{n\geq 0}$ of tilings such that $\mathcal{T}_0 = \mathcal{T}$, $d(\mathcal{T}_n, \mathcal{T}')$ tends toward 0, and either $\mathcal{T}_{n+1}$ is obtained from $\mathcal{T}_n$ by performing a flip, or $\mathcal{T}_{n+1} = \mathcal{T}_n$, then one says that $\mathcal{T}'$ is *flip-accessible* from $\mathcal{T}$, and one writes:

$$\mathcal{T} \xrightarrow{flips} \mathcal{T}'.$$

It is worth noting that this notion of flip-accessibility changes the situation. Indeed, domino tilings of the whole plane turn out to be not always flip-accessible. Moreover, flip-accessibility is no more a symmetric relation, *i.e.*, there is pairs of tilings such that the second one is flip-accessible from the first one, but not conversely. Fig. 4 illustrates this. Thus, we are interested in *characterizations* of pairs of domino tilings of the whole plane which are flip-accessible.

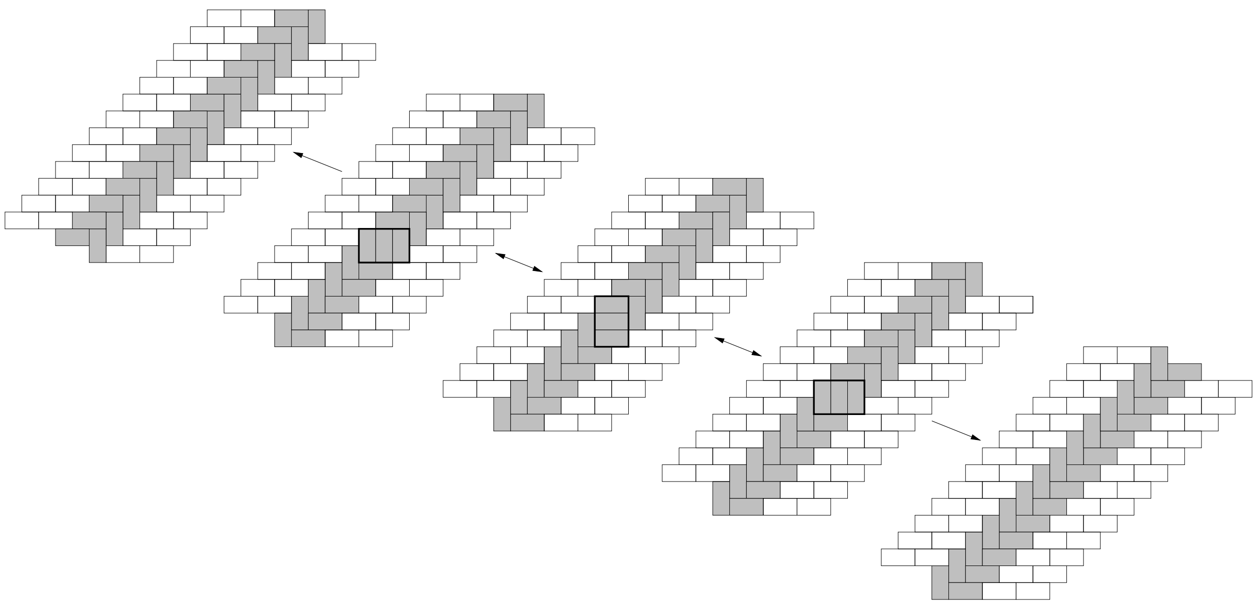

FIGURE 4. Some domino tilings of the whole plane which differ on a thin infinite diagonal (grey dominoes) and agree everywhere else (white dominoes, arranged as brickwalls up to infinity). Consider the central tiling: there is a local configuration of 3 similar dominoes which tiles a 2×3 rectangle (a "bubble"). This allows to perform two flips: the lower one leads to the tiling at its left (the bubble moved downward) and the upper one leads to the tiling at its right (the bubble moved upward). This can be infinitely repeated, up to obtain one of the extremal tilings (the bubble is rejected to infinity, either upward or downward), on which no more flip can be performed. Thus, the three central tilings are mutually flip-accessible, while the leftmost and rightmost tilings are sort of dead ends.

Last, note that it is straightforward to restate flip-accessibility for height functions corresponding to domino tilings and, clearly, flip-accessibility of height functions implies flip-accessibility of the corresponding domino tilings (with the converse being true only up to a translation). This is the viewpoint we use in the three following sections for characterizing flip-accessibility.

2. Characterization by pyramids

In this section, we give a first characterization of flip-accessibility for domino tilings of the whole plane. Let us first introduce, using Def. 1.1, specific domino tilings of the whole plane (see Fig. 5):

DEFINITION 2.1. Let (v, z) be an admissible point of $\mathbb{Z}^2 \times \mathbb{Z}$. The *pyramid* $\hat{h}_{v,z}$ (resp. $\check{h}_{v,z}$) is the infimum (resp. supremum) of all the height functions over $\mathbb{Z}^2$ that are equal to z in v:

$$\hat{h}_{v,z} = \bigwedge\{h \in \mathcal{H} \mid h(v) = z\} \qquad \text{and} \qquad \check{h}_{v,z} = \bigvee\{h \in \mathcal{H} \mid h(v) = z\}.$$

We now give an explicit formula for the value of $\hat{h}_{v,z}$ (resp. $\check{h}_{v,z}$) and, in the same time prove its existence. For each pair (v, v') of vertices, there exists a directed path of G from v to v'. The length of a shortest path from v to v' is denoted by $l_+(v, v')$.

Proposition 2.1. *Let (v_0, z) be a an admissible point of $\mathbb{Z}^2 \times \mathbb{Z}$. For each v of $\mathbb{Z}^2$, we have,*

$$\check{h}_{v_0,z}(v) = z + l_+(v_0, v)$$

and

$$\hat{h}_{v_0,z}(v) = z - l_+(v, v_0)$$

Proof. For each stepped surface h such that $h(v_0) = z$, for each vertex v, for each directed path $(v_0, v_1, ..., v_p)$ from v_0 to v, we have

$$h(v) - h(v_0) = \sum_{0 \le i < p} (h(v_{i+1}) - h(v_i)) \le \sum_{0 \le i < p} 1 = p$$

Thus, taking a shortest path, we have $h(v) - z \le l_+(v_0, v)$, which gives: $h(v) \le z + l_+(v_0, v)$. Especially, $\check{h}_{v_0,z}(v) \le z + l_+(v_0, v)$.

On the other hand, for each pair (v, v') of neighbor vertices in the directed graph G, we have: $l_+(v_0, v') \le l_+(v_0, v) + 1$, since a directed path from v_0 to v' can be obtained adding one edge to a directed path from v_0 to v. One the same way, we have: $l_+(v_0, v) \le l_+(v_0, v') + 3$ since a directed path from v_0 to v can be obtained adding three steps to a directed path from v_0 to v'. Thus we obtain :

$$l_+(v_0, v) - 3 \le l_+(v_0, v') \le l_+(v_0, v) + 1$$

Morover, remark that the length of any directed cycle of G is 0 modulo 4. Thus the lengths of two given paths from v_0 to v are equal modulo 4. This gives us that $l_+(v_0, v') = l_+(v_0, v) + 1$ modulo 4. Thus either $l_+(v_0, v') = l_+(v_0, v) + 1$ or $l_+(v_0, v') = l_+(v_0, v) - 3$, which gives the fact that the given expressions really are stepped surfaces, according to the characterization given in 1.1. Thus, by definition, we have $\check{h}_{v_0,z}(v) \ge z + l_+(v_0, v)$, which gives the result for $\check{h}_{v_0,z}$.

The stepped surface $\hat{h}_{v_0,z}$ is treated on a similar way. □

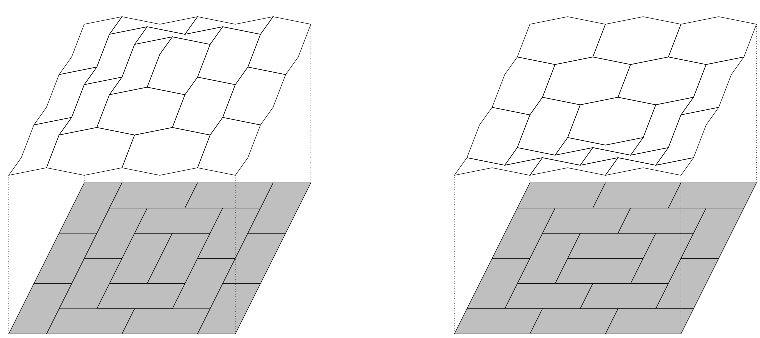

Figure 5. Patches of the pyramids $\hat{h}_{v,z}$ (left) and $\check{h}_{v,z}$ (right): domino tiling viewpoint (bottom) and lifted viewpoint (top). Note that the height is extremal in v in both cases.

Note that, for any height function h such that $h(v) = z$, this definition yields $\hat{h}_{v,z} \le h \le \check{h}_{v,z}$. Let us now consider a height function h and an admissible point (v, z). If $h(v) \le z$, we define the following subset of $\mathbb{Z}^2$:

$$S_{v,z,h} = \{u \in \mathbb{Z}^2 \mid \hat{h}_{v,z}(u) \ge h(u)\}.$$

Similarly, if $h(v) > z$, we define:

$$S_{v,z,h} = \{u \in \mathbb{Z}^2 \mid \check{h}_{v,z}(u) \le h(u)\}.$$

Following the three-dimensional viewpoint described in the previous section, one can see the set $S_{v,z,h}$ (or more precisely the set $\{(u, h(u)) \mid u \in S_{v,z,h}\}$) as the vertices of the dominoes of h that are below the pyramid $\hat{h}_{v,z}$ (or above the pyramid $\check{h}_{v,z}$, according to $h(v) \le z$ or $h(v) > z$). Intuitively, this is the minimal set of vertices whose height has to be changed for transforming h into a height function which is equal to z in v.

Fig. 6 (left) illustrates this.

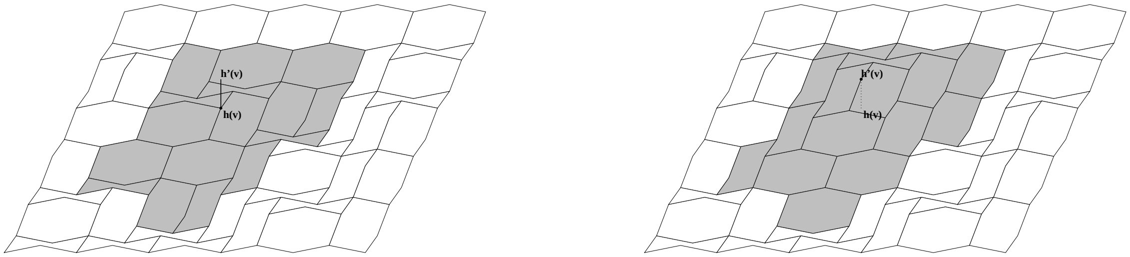

FIGURE 6. Three-dimensional viewpoint of a height function h, with the shaded dominoes being those which have vertices in the set $S_{v,h'(v),h(v)}$ (left). Here, there is a finite number of shaded dominoes: any other tiling over the same set of cells can thus be obtained by performing a finite number of flips. In particular, this holds for the restriction of the pyramid $\hat{h}_{v,h'(v)}$ to $S_{v,h'(v),h(v)}$ (the shaded dominoes, right). This transforms h into a height function that is equal to h' in v.

These sets are the main ingredients of the characterization of flip-accessibility provided in this section. Before we formally state a theorem, let us sketch leading ideas. Let h and h' be two height functions over $\mathbb{Z}^2$, and let $v \in \mathbb{Z}^2$. Without loss of generality, suppose $h'(v) \geq h(v)$. Suppose now that the set $S_{v,h'(v),h}$ is *finite*. This yields two finite domino tilings over the same set of cells, namely the tilings corresponding to the restrictions of h and $\hat{h}_{v,h'(v)}$ to $S_{v,h'(v),h}$. In particular, the first one can be transformed into the second one by performing a finite sequence of flips (according to a classic result recalled in Sec. 1), and this turns out to transform h into $h \vee \hat{h}_{v,h'(v)}$ (see Fig. 6, right). Thus, by performing a finite number of flips, we can transform h into a height function equals to $h'(v)$ in v (namely $h \vee \hat{h}_{v,h'(v)}$). If we can do this for any $v \in \mathbb{Z}^2$, it is not hard to lay end to end the obtained finite sequences of flips, thus defining a (possibly infinite) sequence of flips that links h to h'. This is the keypoint of the following theorem:

THEOREM 2.2. *Let h and h' be two height functions over $\mathbb{Z}^2$. Then, h' is flip-accessible from h if and only if, for any $v \in \mathbb{Z}^2$, the set $S_{v,h'(v),h}$ is finite:*

$$h \xrightarrow{flips} h' \;\Leftrightarrow\; \forall v \in \mathbb{Z}^2,\; Card(S_{v,h'(v),h}) < \infty.$$

PROOF. First assume that h' is accessible from h by a finite or infinite sequence $(h_0, h_1, ...,)$. Let v be a vertex such that $h'(v) < h(v)$. There exists an integer p such that $h_p(v) = h'(v)$, which yields that $h_p \leq \check{h}_{v,h'(v)}$. Thus, for each v' of $S_{v,h'(v),h}$, we have $h_p(v') \leq \check{h}_{v,h'(v)}$, which gives $h_p(v') \leq h'(v) < h(v')$. In other words, for each v' of $S_{v,h'(v),h}$, the finite sequence $(h_0(v'), h_1(v'), ..., h_p(v'))$ is not constant. Thus, there exists a stepped surface h_i, with $i < p$ such that one passes from h_i to h_{i+1} by a flip in v'. Thus, the number of elements of $S_{v,h'(v),h}$ is at most p, which gives the condition.

The $h'(v) > h(v)$ case is similar, and $S_{v,h'(v),h}$ is empty when $h'(v) = h(v)$.

Conversely, assume that, for any $v \in \mathbb{Z}^2$, $S_{v,h'(v),h}$ is finite. We use a fixed numbering $(v_j)_{j \in \mathbb{N}^*}$ of vertices of $\mathbb{Z}^2$ (*i.e.*, for each vertex v of $\mathbb{Z}^2$, there is a unique positive integer j such that $v_j = v'$). We recursively define a sequence of stepped surfaces by:

- $h = h_0$,
- $h_{i+1} = h_i \wedge \check{h}_{v_{i+1},h'(v_{i+1})}$ if $h'(v_{i+1}) \leq h(v_{i+1})$, and $h_{i+1} = h_i \vee \hat{h}_{v_{i+1},h'(v_{i+1})}$ if $h'(v_{i+1}) \geq h(v_{i+1})$.

For each non negative integer i, we claim that the set $S_{v_{i+1},h'(v_{i+1}),h_i}$ is finite: for initialization, this is true for $i = 0$, by hypothesis. Now, assume that this fact is true for any k such that $0 \leq k < i$, for a fixed $i > 0$. let $v \in S_{v_{i+1},h'(v_{i+1}),h_i}$.

- If $h_i(v) = h(v)$, then $v \in S_{v_{i+1},h'(v_{i+1}),h}$.
- If $h_i(v) \neq h(v)$, then there exists an integer k, with $0 \leq k < i$, such that $h_k(v) = h(v)$ and $h_{k+1}(v) \neq h(v)$. Thus v is element of $S_{v_{k+1},h'(v_{k+1}),h_k}$.

Therefore we have:

$$S_{v_{i+1},h'(v_{i+1}),h_i} \subseteq \cup_{k=0}^{i-1} S_{v_{k+1},h'(v_{k+1}),h_k} \cup S_{v_{i+1},h'(v_{i+1}),h}$$

which is a finite union of finite sets, by induction hypothesis.

Since $S_{v_{i+1},h'(v_{i+1}),h_i}$ is finite, h_i and h_{i+1} only differ on this finite set, and there is a finite domain D_i such that the restrictions $h_i|_{D_i}$ and $h_{i+1}|_{D_i}$ both induce domino tilings of D_i and $h_i(v') = h_{i+1}(v')$ for each $v' \in \mathbb{Z}^2 \setminus V_{D_i}$. One then can pass from h_i to h_{i+1} by a finite sequence of monotonic flips (by flip-accessibility of finite domino tilings). These finite sequences can be concatenated: precisely, there exists an infinite sequence $(h'_k)_{k\in\mathbb{N}}$ such that $h_0 = h'_0$ the sequence $(h_i)_{i\in\mathbb{N}}$ is a subsequence of $(h'_k)_{k\in\mathbb{N}}$, and if $h_i = h'_k$ and $h_{i+1} = h'_{k'}$, then the finite sequence $(h'_k, h'_{k+1}, ..., h'_{k'})$ is monotonic.

For each vertex v of $\mathbb{Z}^2$, we study the sequence of values: $|h'_j(v) - h'(v)|_{j\in\mathbb{N}}$. We claim that this sequence is non-increasing. It suffices to prove it for each finite subsequence $|h'_j(v) - h'(v)|_{k\leq j\leq k'}$, with $h_i = h'_k$ and $h_{i+1} = h'_{k'}$. Assume (without loss of generality) that $h'(v_{i+1}) \leq h_i(v_{i+1})$. The sequence of flips to pass from h_i to h_{i+1} is formed by downward flips (*i.e.*, which decrease heights). If v is not element of $S_{(v_{i+1},h'(v_{i+1}),h_i)}$, then $h_i(v) = h_{i+1}(v)$. Thus the sequence $|h'_j(v) - h'(v)|_{k\leq j\leq k'}$ is necessarily constant. For each vertex v of $S_{(v_{i+1},h'(v_{i+1}),h_i)}$, we have $h'(v) \leq h_{max(v_{i+1},h'(v_{i+1}))}(v) < h_i(v)$. Thus each downwards flip done around v used to pass from $h_i(v)$ to $h_{i+1}(v)$ reduces the value $|h'_j(v) - h'(v)|$ (flips done around another vertex do not change this quantity). Thus the sequence $|h'_j(v) - h'(v)|_{k\leq j\leq k'}$ is non-increasing.

On the other hand, for each integer i of $\mathbb{N}$, we have $h_i(v_i) = h'(v_i)$. Thus, since the sequence $|h'_j(v_i) - h'(v_i)|_{j\in\mathbb{N}}$ is non increasing, we have $h'_j(v_i) = h'(v_i)$, for j sufficiently large. To conclude the proof, take a finite domain D, let i_D be the largest integer such that v_{i_D} is in V_D, and k_D be the integer such that $h'_{k_D} = h_{i_D}$. Clearly, for $k \geq k_D$, we have $h'_k|_D = h'|_D$, which gives the result. □

3. Characterization by reduction of dimension

3.1. Bicolor unidimensional tilings. Our next characterization uses unidimensional tilings. We first need to define them and give main properties now. Informally, a bicolor tiling is a partition of $\mathbb{R}$ into unitary colored (red or green) segments (the colored tiles), with the endpoints of each segment being integers. This can be formalized as follows: a bicolor tiling is a mapping f from $\mathbb{Z}$ to the set $\{G, R\}$. Hence, $f(i)$ is the color of the segment $[i; i+1]$.

A flip is naturally defined as the inversion of colors of two consecutive segments: if f and f' are bicolored tilings such that $f = f'$ except for two consecutive values, i_0 and $i_0 + 1$ such that $f(i_0) = G$ and $f(i_0 + 1) = R$, then we say that one passes from f to f' by a downward flip around $i_0 + 1$. This yields a notion of *flip-accessibility* for stepped lines, similarly to Def. 1.2.

Given a bicolor tiling f, *stepped line* associated with f is a function h from $\mathbb{Z}$ to $\mathbb{Z}$ such that:

- if $f(i) = G$, then $h(i+1) - h(i) = 2$;
- if $f(i) = R$, then $h(i+1) - h(i) = -2$.

As we will see later, the value 2 (instead of 1, which seems more natural) is chosen for consistence with domino tilings. Conversely , a function h from $\mathbb{Z}$ to $\mathbb{Z}$ such that $h(i+1) - h(i) \in \{2, -2\}$ induces a unique bicolored tiling. Bicolored tilings and stepped lines can be seen as unidimensional analogue of, respectively, domino tilings of the whole plane and height functions over $\mathbb{Z}^2$. Note that a flip around $j \in \mathbb{Z}$ only changes the value of the stepped line in j.

Definition 3.1. Let (i_0, z) be a pair of $\mathbb{Z} \times \mathbb{Z}$. We define the stepped lines $\check{h}_{i_0,z}$ and $\hat{h}_{i_0,z}$ by:

$$\check{h}_{i_0,z}(i) = z + 2|i - i_0| \qquad \text{and} \qquad \hat{h}_{i_0,z}(i) = z - 2|i - i_0|.$$

It is not hard to see that, if h is a stepped line such that $h(i_0) = z$, then one has $\hat{h}_{i_0,z} \leq h \leq \check{h}_{i_0,z}$. Let us now consider a stepped line h and two integers $i, z \in \mathbb{Z}$ such that $h(i) = z \mod 4$. We define the set $S_{(i,z,h)}$ and the stepped line $h_{i,z,h}$ as follows:

- if $z \leq h(i)$, then $S_{i,z,h} = \{i' | i' \in \mathbb{Z}, \check{h}_{i,z}(i') < h(i')\}$ and $h_{i,z,h} = \check{h}_{i,z} \wedge h$;
- if $z > h(i)$, then $S_{i,z,h} = \{i' | i' \in \mathbb{Z}, \hat{h}_{i,z}(i') > h(i')\}$ and $h_{i,z,h} = \hat{h}_{i,z} \vee h$.

Let us now define *shadows* of a stepped line. Given a vertex $v = (x, y)$, we call *first shadow* of v (resp. *second shadow* of v) the $s_1(v) = \{v' \in \mathbb{Z}^2 \mid v' - v \in \mathbb{R}(1,2)\}$ (resp. $s_2(v) = \{v' \in \mathbb{Z}^2 \mid v' - v \in \mathbb{R}(1,-2)\}$). Then, the first shadow (resp. second shadow) of a stepped line h is the set $s_1(h) = \{s_1(i, h(i)), i \in \mathbb{Z}\}$ (resp. $s_2(h) = \{s_2(i, h(i)), i \in \mathbb{Z}\}$).

This allows to characterize flip-accessibility for stepped lines as follows:

PROPOSITION 3.1. *Let h and h' be stepped lines such that $h = h'$ modulo 4. The following conditions are equivalent:*

- *the line h' is accessible from h;*
- *for each integer i, the set $S_{i,h'(i),h}$ is finite;*
- *(shadow inclusion) $s_1(h') \subset s_1(h)$ and $s_2(h') \subset s_2(h)$.*

PROOF. (sketch) The proof that the two first items are equivalent is very similar (and easier, because of the dimension 1) of the proof of Theorem 2.2. The equivalence of the two last items follows by noting that the set $S_{i,z,h}$ is finite if and only if there exists a pair (j, j') of $\mathbb{Z}^2$ such that $s_1(i, z) = s_1(j, h(j))$ and $s_2(i, z) = s_2(j', h(j'))$. □

3.2. Interlaced stepped lines. We now show that a domino tiling can be seen as two interlaced sets of stepped lines, and we obtain a characterization of flip-accessibility for domino tilings which relies on flip-accessibility for stepped lines.

Informally, stepped lines are obtained from a height function by following diagonal directions along the corresponding domino tiling (*i.e.*, vectors $(1, 1)$ and $(1, -1)$). There are some technical difficulties since the origin has to be (more or less) arbitrarily chosen.

For each pair (k, i) of $\mathbb{Z}^2$, there exists a unique pair (a, b) of $\mathbb{Z}^2 \cup (\mathbb{Z}+1/2)^2$ such that $(k, i) = (a+b, -a+b)$ (precisely, we have $(a, b) = ((k-i)/2, (k+i)/2))$. Using the notations of the previous subsection, we state $f(k, i) = (a, b)$. Note that f is bijective from $\mathbb{Z}^2$ to $\mathbb{Z}^2 \cup (\mathbb{Z} + 1/2)^2$. Let h be a height function, and (a, b) be an element of $\mathbb{Z}^2 \cup (\mathbb{Z} + 1/2)^2$. We will especially use the fact that $f(k + j, i + j) = (a, b + j)$ and $f(k + j, i - j) = (a + j, b)$. We define $g(a, b) = h(k, i)$, with $f(k, i) = (a, b)$.

Let us now introduce the stepped lines we are more particularly interested in:

- for $a \in \mathbb{Z}$, let $h_{1,a}(b) = g(a, b)$;
- for $a \in \mathbb{Z} + 1/2$, let $h_{1,a}(b - 1/2) = g(a, b)$;
- for $b \in \mathbb{Z}$, let $h_{2,b}(a) = g(a, b)$;
- for $b \in \mathbb{Z} + 1/2$, let $h_{2,b}(a - 1/2) = g(a, b)$.

It is easily checked that these functions are stepped lines: they are called stepped lines *induced* by h. Conversely, the height function h is characterized by such stepped lines. Fig. 7 illustrates this.

Note that to a downward flip on h around a vertex $v = (i, k)$ such that $f(i, k) = (a, b)$ correspond two downward flips on the induced stepped lines $h_{1,a}$ and $h_{2,b}$, respectively around b or $b - 1/2$ (according to the parity of $(k - i)$) and a or $a - 1/2$.

We are now in a position to state our second characterization:

THEOREM 3.2. *Let h and h' be two height functions over $\mathbb{Z}^2$. Then, h' is flip-accessible from h if and only if, for each pair (x, a) of $\{1, 2\} \times \frac{1}{2}\mathbb{Z}$, the stepped line $h'_{x,a}$ is flip-accessible from $h_{x,a}$.*

PROOF. The direct part of the theorem is obvious.

For the converse part, we use the characterization given by Theorem 2.2. Let $v = (k, i)$ be a vertex such that $h'(v) \leq h(v)$, and (a, b) such that $f(k, i) = (a, b)$. By hypothesis, $h'_{1,a}$ is flip-accessible from $h_{1,a}$. Thus,

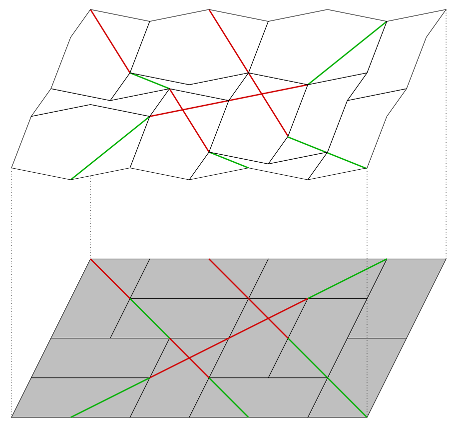

FIGURE 7. Three-dimensional viewpoint of a height function (top) and associated domino tiling (bottom). Three stepped lines are represented (top), with the corresponding bicolored unidimensional tilings (bottom). The other stepped lines are “parallel” to one of those (according to their form $h_{1,a}$ or $h_{2,b}$), and there are two stepped lines crossing in the middle of each cell (recall that two cells form a domino).

there is a positive integer j_1 such that:

$$\begin{array}{ll} h'_{1,a}(b+j_1) \geq h_{1,a}(b+j_1) & \text{for } (a,b)\in\mathbb{Z}^2, \\ h'_{1,a}(b-1/2+j_1) \geq h_{1,a}(b-1/2+j_1) & \text{for } (a,b)\in(\mathbb{Z}+1/2)^2. \end{array}$$

In both case, this yields $g'(a,b)+2j_1 \geq g(a,b+j_1)$. Then, according to Def. 1.1, we have $g(a,b+j_1) = h(k+j_1,i+j_1)$ and $g'(a,b)+2j_1 = h'(k,i)+2j_1 = \check{h}_{v,h'(v)}(k+j_1,i+j_1)$. This leads to $\check{h}_{v,h'(v)}(k+j_1,i+j_1) \geq h(k+j_1,i+j_1)$, that is:

$$\check{h}_{v,h'(v)}(v+(j_1,j_1)) \geq h(v+(j_1,j_1)).$$

Simarly, one shows that there is positive integers j_2, j_3 and j_4 such that:

$$\check{h}_{v,h'(v)}(v+(-j_2,j_2)) \geq h(v+(-j_2,j_2));$$

$$\check{h}_{v,h'(v)}(v+(-j_3,-j_3)) \geq h(v+(-j_3,-j_3));$$

$$\check{h}_{v,h'(v)}(v+(j_4,-j_4)) \geq h(v+(j_4,-j_4)).$$

We then set $v_1 = v+(j_1,j_1)$, $v_2 = v+(-j_2,j_2)$, $v_3 = v+(-j_3,-j_3)$, $v_4 = v+(j_4,-j_4)$, and we define the following height function:

$$h'' = \bigwedge_{k=1\to 4} \check{h}_{v_k,\check{h}_{v,h'(v)}(v_k)}.$$

By Lemma 3.3, stated and proved below, we can deduce from this that $h'' = \check{h}_{v,h'(v)}$ except on a finite domain, thus that the set $S_{v,h'(v),h''}$ is finite. Since $h \leq h''$ (this follows from Def. 1.1), we deduce that $S_{v,h'(v),h}$ is finite, and applying Theorem 2.2 then ends the proof. □

LEMMA 3.3. *Let $(v_0,z_0) = ((x_0,y_0),z_0)$ be an admissible point and j be a positive integer. For each vertex $v=(x,y)$ such that $x+y \geq x_0+y_0+2j$, we have: $\check{h}_{v_0,z}(v) = \check{h}_{v_0+(j,j),z+2j}(v)$.*

PROOF. First remark that the equality of this lemma is equivalent to $z+l_+(v_0,v) = z+2j+l_+(v_0+(j,j),v)$. We have $l_+(v_0,v_0+(j,j)) = 2j$: there exists a directed path of length $2j$ and this is a lower bound since the path necessarily contains at least j upwards steps and j rightwards steps. so we have to prove that $l_+(v_0,v) = l_+(v_0,v+(j,j))+l_+(v_0+(j,j),v)$, i. e. there exists a shortest directed path from v_0 to v passing through $(v_0+(j,j))$.

If $v=(x,y)$ with $x+y > x_0+y_0+2j$, each directed path (and, in particular, a shortest path) from v_0 to v passes through a vertex $v_1=(x_1,y_1)$ such that $x_1+y_1 = x_0+y_0+2j$.

We claim that

$$l_+(v_0,v_1) = l_+(v_0,v_0+(j,j)) + l_+(v_0+(j,j),v_1)$$

This equality proves the result. We have seen that $l_+(v_0, v_0 + (j,j)) = 2j$. Moreover, we can state $v_1 = v_0 + (j,j) + (j', -j')$ with $j' \in \mathbb{Z}$. There is no loss of generality to assume j' positive, up to symmetry. With the same argument, we have $l_+(v_0 + (j,j), v_1) = 2j'$.

On the other hand we have $l_+(v_0, v_1) = 2(j+j')$ since there exists a path from v_0 to v_1 of length $2(j+j')$ (by concatenation of the previous ones), and each path from v_0 to v_1 contains at least $(j+j')$ rightwards steps, not pairwise consecutive (and the path cannot simultaneously start and finish by a horizontal step, because $mod_4(v) - mod_4(v_0)$ is even). This proves the equality, and therefore the result. □

4. Characterization by inclusions of shadows

In the previous section, we defined shadows for stepped lines and use it for characterizing flip-accessibility of stepped lines. Here, we obtaine a similar characterization by extending the notion of shadows to domino tilings.

Let (v,z) be an element of $\mathbb{Z}^2 \times \mathbb{Z}$. For $u \in \mathbb{R}^3$, the *shadow* of $v(v,z)$ along u is the set defined by:

$$p_u(v,z) = \{(v',z') \in \mathbb{Z}^2 \times \mathbb{Z} \mid (v',z') - (v,z) \in \mathbb{R}u\}.$$

Then, the shadow along u of a height function h is the set $p_u(h)$. Here, we are especially interested in the shadows along the four following vectors:

$$u_1 = (1,1,2) \qquad u_2 = (1,1,-2) \qquad u_3 = (1,-1,2) \qquad u_4 = (1,-1,-2).$$

In what follows, we respectively denote by $p_1(h)$, $p_2(h)$, $p_3(h)$ and $p_4(h)$ these shadows. They allow to state our third characterization:

THEOREM 4.1. *Let h and h' be two height functions over $\mathbb{Z}^2$. Then, h' is flip-accessible from h if and only if the shadows of h' are included in the shadows of h:*

$$h \xrightarrow{flips} h' \;\Leftrightarrow\; \forall i \in \{1,2,3,4\},\; p_i(h') \subset p_i(h).$$

PROOF. We rely on the previous characterization (Th. 3.2) and the corresponding notations. First, note that, for any 6-uple (k,k',i,j,z,z') of $\mathbb{Z}^6$, we have $p_1(k,i,z') = p_1(k+j, i+j, z)$ if and only if $s_1(k',z') = s_1(k'+j, z)$.

Suppose now that h' is flip-accessible from h. Let (k,i) in $\mathbb{Z}^2$ and $(a,b) = f(k,i)$. By Theorem 3.2, $h'_{1,a}$ is flip-accessible from $h_{1,a}$. In particular, there exists and integer j such that $s_1(b, h'_{1,a}(b)) = s_1(b+j, h_{1,a}(b+j))$. According to the above preliminary remark, this yields $p_1(k,i,h'(k,i)) = p_1(k+j, i+j, h(k+j, i+j))$. Thus, $p_1(h') \subset p_1(h)$. Inclusions for the three other shadows are proved similarly.

Conversely, suppose that $p_i(h') \subset p_i(h)$ for $i = 1,\ldots,4$. Let $(a,b) \in \mathbb{Z}^2$ and (k,i) such that $(a,b) = f(k,i)$. There exists an integer j of $\mathbb{Z}$ such that $p_1(k,i,h'(k,i)) = p_1(k+j, i+j, h(k+j, i+j))$. Thus, $s_1(b, h'_{1,a}(b)) = s_1(b+j, h_{1,a}(b+j))$ According to the above preliminary remark, this yields $s_1(h'_{1,a}) \subset s_1(h_{1,a})$. Similarly, $s_i(h'_{1,a}) \subset s_i(h_{1,a})$ for $i = 1,\ldots,4$. The result then follows from Theorem 3.2. □

5. From stepped surfaces to tilings

We know have conditions for accessibility for stepped surfaces. How can they be used for the accessibility for tilings? A problem arises from the fact that a stepped surface induced by a tiling is defined up to a constant integer.

5.1. One dimensional tilings. We say that a one dimensional tiling is *ultimately monotonic* in $+\infty$ (respectively $-\infty$) if only one color appears in a semi-infinite interval of the form $[a, +\infty)$ (respectively $(-\infty, a]$). These definitions are naturally extended to stepped lines. Clearly, from our conditions, each stepped line is accessible from a fixed non ultimately monotonic stepped line. Thus, each tiling is accessible from a non ultimately fixed monotonic tiling.

Moreover, if a stepped line h' is accessible from a fixed ultimately monotonic in $+\infty$ stepped line h, then h' is also ultimately monotonic in $+\infty$, and moreover h and h' are *ultimately equal*, i. e there exists a semi-infinite interval $[a, +\infty)$ on which $h = h'$. This fact allows us to fix a convenient constant to study the accessibility for tilings: let (T', T) be a pair of tilings, with T ultimately monotonic in $+\infty$. If there exists no semi-infinite interval $[a, +\infty)$ on which $T = T'$, then T' is not accessible from T. Otherwise, consider two stepped lines $h_{T'}$ and h_T, respectively induced by T' and T, such that $h_{T'}(a) = h_T(a)$. The tiling T' is accessible from T if and only if $h_{T'}$ is accessible from h_T.

5.2. Two dimensional tilings. It is easy to extend the previous study to dimension 2. We say that a stepped surface is *ultimately broken* if at least one of the induced stepped lines is ultimately monotonic, and we say that a 2-dimensional tiling is ultimately broken if one (and therefore all) of the stepped surfaces induced is monotonic. Each stepped surface is accessible from a fixed non ultimately broken stepped surface, thus, each tiling is accessible from a fixed non ultimately broken tiling.

Moreover, if a stepped surface h' is accessible from a fixed ultimately broken stepped surface h then h' is also ultimately broken. Precisely, there necessarily exists two corresponding induced stepped lines $h'_{i,x}$ and $h_{i,x}$ which are ultimately monotonic, and, moreover, ultimately equal.

Let T be a (2 dimensional) tiling, with T ultimately broken. Let h be a stepped surface induced by T, and $h_{i,x}$ be an ultimately monotonic stepped line induced by h, in $+\infty$ (which can be assumed without loss of generality, up to symmetry). Let T' be another tiling and h' be one of its induced stepped surfaces.

If the difference $h_{i,x} - h'_{i,x}$ is not ultimately constant then T' is s not accessible from T (since, for each integer c, $h'_{i,x} + c$ is not accessible from $h_{i,x}$, and for each stepped surface h'' induced by T', there exists an integer c such that $h'' = h' + c$).

If $h_{i,x} - h'_{i,x}$ is is ultimately equal to a constant c, then $h'' = h' + c$ is a stepped surface induced by T'. The stepped surface h'' is accessible from h if and only if T' is accessible from T.

References

[1] P. Arnoux, V. Berthé, T. Fernique, D. Jamet, *Functional stepped surfaces, flips and generalized substitutions*, Theorical Computer Science (to appear).

[2] O. Bodini, T. Fernique, *Planar dimer tilings*, proc. of the 1st international Computer Science symposium in Russia (CSR'06), LNCS **3967** Springer (2006), pp. 104–113.

[3] O. Bodini, T. Fernique, E. Rémila, *A characterization of flip-accessibility for rhombus tilings of the whole plane*, proc. of the 1st international Conference on Language and Automata Theory and Applications (LATA'07), to appear in LNCS

[4] H. Cohn, N. Elkies, J. G. Propp, *Local Statistics for Random Domino Tilings of the Aztec Diamond*, Duke Mathematical Journal **85** (1996), pp. 117–166.

[5] J. H. Conway, J. C. Lagarias, *Tiling with polyominoes and combinatorial group theory*, Journal of Combinatorial Theory **53** (1990), pp. 183–208.

[6] P. A. Davey, H. A. Priestley, *An introduction to lattices and orders*, Cambridge University Press (1990).

[7] S. Desreux, E. Rémila, *An optimal algorithm to generate tilings*, Journal of Discrete Algorithms **4** (2006) pp. 168–180.

[8] S. Desreux, *An algorithm to generate exactly once every tiling with lozenges of a domain*, Theoretical Computer Science **303** (2003) pp. 375–408.

[9] M. Luby, D. Randall, and A. Sinclair, *Markov chain algorithms for planar lattice structures*, SIAM Journal of Computing **31** (2001), pp. 167–192.

[10] S. Desreux, M. Matamala, I. Rapaport, E. Rémila, *Domino tilings and related models: space of configurations of domains with holes*, Theorical Computer Science **319** (2004), pp. 83–101.

[11] J. G. Propp, *Lattice structure for orientations of graphs*, draft (available on James Propp's home page).

[12] E. Rémila, *On the lattice structure of the set of tilings of a simply connected figure with dominoes*, Theoretical Computer Science **322**,(2004), pp. 409–422 .

[13] W. P. Thurston, *Conway's tiling group*, American Mathematical Monthly **97** (1990), pp. 757–773.

LIP6 UMR 7606 (CNRS - Univ. Paris 6) 4 place Jussieu 75005 Paris - France
E-mail address: `Olivier.Bodini@ufr-info-p6.jussieu.fr`

LIRMM UMR 5506 (CNRS - Univ. Montpellier 2) 161 rue Ada 34392 Montpellier Cedex 5 - France.
E-mail address: `Thomas.Fernique@lirmm.fr`

LIP UMR 5668 (CNRS - INRIA - ENS Lyon - Univ. Lyon 1) 46 allée d'Italie 69364 Lyon Cedex 7 - France.
E-mail address: `Eric.Remila@ens-lyon.fr`