Tensors with eigenvectors in a given subspace

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Abstract

The first author with B. Sturmfels studied in [16] the variety of matrices with eigenvectors in a given linear subspace, called the Kalman variety. We extend that study from matrices to symmetric tensors, proving in the tensor setting the irreducibility of the Kalman variety and computing its codimension and degree. Furthermore, we consider the Kalman variety of tensors having singular t-tuples with the first component in a given linear subspace and we prove analogous results, which are new even in the case of matrices. Main techniques come from Algebraic Geometry, using Chern classes for enumerative computations.

Keywords Eigenvectors · Tensors · Singular tuples · Vector bundles · chern classes

Mathematics subject classification 14N07 · 14N05 · 14N10 · 15A69 · 15A18

1 Introduction

We introduce the subject of this paper with a basic example. Let $L \subset \mathbb{C}^n$ be a linear space of dimension d. We are interested in the variety $\mathcal{K}_{d,n,m}(L)$ of matrices $A \in \mathbb{C}^n \otimes \mathbb{C}^m$ having a singular pair $(v, w) \in \mathbb{C}^n \times \mathbb{C}^m$ with $v \in L$. Namely $A \in \mathcal{K}_{d,n,m}(L)$ if and only if there exists $v \in \mathbb{C}^n \setminus \{0\}, w \in \mathbb{C}^m \setminus \{0\}, \lambda_1, \lambda_2 \in \mathbb{C}$ such that $Aw = \lambda_1 v$, $A^t v = \lambda_2 w$ and $v \in L$ (over $\mathbb{R}$ it is well known that the equality $\lambda_1 = \lambda_2$ can be assumed but already on $\mathbb{C}$ the situation is more subtle, see [2] and Example 3.10, note that we have not conjugated the transpose matrix A^t). The main Theorem of this paper applies in this case (see Theorem 3.4 and Remark 3.7) and it shows that the variety $\mathcal{K}_{d,n,m}(L)$ is algebraic, irreducible, has codimension $n - d$ and it has degree (see Remark 3.7)

$$\sum_{j=0}^{d-1} \sum_{k=d-j-1}^{\min(n-j-1,m-1)} \binom{n-j-1}{k}\binom{k}{d-1-j}. \tag{1.1}$$

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For $n \leq m$ the expression (1.1) simplifies to $2^{n-d} \binom{n}{d-1}$ (see Proposition 3.6), which does not depend on m, in other words it stabilizes for $m \gg 0$. Moreover, for $n \leq m$, we have that $A \in \mathcal{K}_{d,n,m}(L)$ if and only if the symmetric matrix AA^t has an eigenvector in L (see Lemma 3.8), so that the result is a consequence of [16, Prop. 1.2]. The paper [16], by the first author and B. Sturmfels, was indeed the main source for this paper. When $n > m$ the condition that AA^t has an eigenvector on L is necessary for $A \in \mathcal{K}_{d,n,m}(L)$ but is no longer sufficient (see Example 3.9) and [16, Prop. 1.2] cannot be invoked anymore.

The formula (1.1) can be expressed as the coefficient of the monomial

$$h^{n-d} v^{d-1} w^{m-1}$$

in the polynomial

$$\frac{(w+h)^n - v^n}{w+h-v} \cdot \frac{(v+h)^m - w^m}{v+h-w}.$$

Note that both fractions are indeed homogeneous polynomials, respectively of degree $n-1$ and $m-1$. We list the degree values for small d, n, m in Table 1 in §3. The above formulation generalizes to tensors and it suggests the way we have computed it as well as the formula to compute the number of singular t-tuples of a tensors obtained by S. Friedland and the first author in [5]. This was interpreted as the EDdegree of the Segre variety in [3]. The technique we use comes from Algebraic Geometry and corresponds to a Chern class computation of a certain vector bundle. For a short introduction to sections of vector bundles and Chern classes we refer the reader to [5, §2.1, 2.2, 2.3].

Singular pairs replace eigenvectors for non-symmetric matrices in optimization questions. Generalizing from matrices to tensors, we have two analogous concepts. For symmetric tensors in $Sym^k\mathbb{C}^n$, eigenvectors [12, 18] are used in best rank one approximation problems, while for general tensors in $\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_k}$ k-tuples of singular vectors [12] play the same role. Here are our main results.

Theorem 1.1 *Let $L \subset \mathbb{C}^n$, $\dim L = d$. The variety*

$$\kappa^s_{d,n,k}(L) = \{f \in \mathbb{P}(Sym^k(\mathbb{C}^n)) | f \text{ has an eigenvector in } L\}$$

is irreducible, it has codimension $n-d$ and degree $\Sigma_{i=0}^{d-1} \binom{n-d+i}{i} (k-1)^i$.

Theorem 1.2 *Let $L \subset \mathbb{C}^{n_1}$, $\dim L = d$. The variety*

$$\kappa_{d,n_1,\ldots,n_k}(L) = \{T \in \mathbb{P}(\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_k}) | T \text{ has a singular } k\text{-tuple } (v_1, \ldots, v_k) \text{ with } v_1 \in L\}$$

is irreducible, it has codimension $n_1 - d$ and degree given by the coefficient of the monomial $h^{n_1-d} v_1^{d-1} \prod_{i \geq 2} v_i^{(n_i-1)}$ in the polynomial

$$\prod_{i=1}^{k} \frac{[(\widetilde{v}_i + h)^{n_i} - v_i^{n_i}]}{(\widetilde{v}_i + h) - v_i},$$

where $\widetilde{v}_i = \Sigma_j v_j - v_i$.

In [16] it was proposed the name Kalman variety inspired from the Kalman controllability condition in PDE, after [11, Corollary 5.5] and [22, Remark 2.2, Theorems 2.2, 2.3].

The structure of this paper is as follows. In §2 we study Kalman varieties for symmetric tensors. In Proposition 2.1 we prove irreducibility and compute the dimension, in Theorem 2.3 we compute the degree. In §3 we study Kalman varieties for general tensors, in Proposition 3.2 we prove irreducibility and compute the dimension, we compute the degree in Theorem 3.4 and we prove a stabilization result when $n_k \gg 0$, off the boundary format, see Corollary 3.5.

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2 Symmetric tensors with eigenvectors in a given subspace

Let $L \subset \mathbb{C}^n$ be a linear subspace of dimension d. A symmetric tensor $T \in Sym^k(\mathbb{C}^n)$ can be represented by a homogeneous polynomial $f_T \in \mathbb{C}[x_1, \ldots, x_n]$ of degree k given by

$$f_T(x_1, \ldots, x_n) = T \cdot x^k := \Sigma_{i_1,\ldots,i_k=1}^n T_{i_1,\ldots,i_k} x_{i_1} x_{i_2} \ldots x_{i_k}.$$

The concept of eigenvectors for matrices was extended to symmetric tensors by Lim [12] and Qi [18]. A vector $v \in V$ is an eigenvector of T if there exist $\lambda \in \mathbb{C}$ such that

$$Tv^{k-1} := [\Sigma_{i_2,\ldots,i_k=1}^n T_{i,i_2,\ldots,i_k} v_{i_2} \ldots v_{i_k}]_i = \lambda v,$$

this is equivalent to $\bigtriangledown f_T(v) = k \cdot \lambda v$, i.e. $\bigtriangledown f_T(v)$ and v are dependent. This is the definition used in [1] and [13], while in [19] E-eigenvectors are defined with the additional requirement that they are not isotropic. The general tensor has no isotropic eigentensors [4, Lemma 4.2], so that for general tensors the two definitions coincide. We are interested in the scheme $\kappa_{d,n,k}^s(L)$ of all symmetric tensors $f \in Sym^k\mathbb{C}^n$ that have an eigenvector in L.

Proposition 2.1 *The variety $\kappa_{d,n,k}^s(L)$ is irreducible, it has codimension $n - d$.*

Proof We regard vectors $v \in \mathbb{C}^n \setminus \{0\}$ as points in the projective space $\mathbb{P}(\mathbb{C}^n)$, and we regard a polynomial of degree k as a point in $\mathbb{P}(Sym^k(\mathbb{C}^n))$. The product of these two projective spaces, $X = \mathbb{P}(Sym^k(\mathbb{C}^n)) \times \mathbb{P}(\mathbb{C}^n)$, has the two projections

$$\begin{array}{ccc} & X & \\ p \swarrow & & \searrow q \\ \mathbb{P}(Sym^k(\mathbb{C}^n)) & & \mathbb{P}(\mathbb{C}^n) \end{array} \tag{2.1}$$

Fix the incidence variety $W = \{(f, z) \in X | z \text{ is an eigenvector of } f\}$

The projection of W to the second factor

$$q : W \longrightarrow \mathbb{P}(\mathbb{C}^n)$$
$$(f, z) \longmapsto z$$

is surjective and every fiber is $q^{-1}(z) = \{f \in \mathbb{P}(Sym^k(\mathbb{C}^n)) | \nabla f \cdot z = \lambda \cdot z\}$ and it is the zero scheme of the 2×2 minors of the matrix $[\nabla f(z) | z]$. Hence it is a linear subspace of codimension $n - 1$ in $\mathbb{P}(Sym^k(\mathbb{C}^n))$.

These properties imply that W is irreducible and has codimension $n - 1$ in $\mathbb{P}(Sym^k(\mathbb{C}^n)) \times \mathbb{P}(\mathbb{C}^n)$.

The projection of the incidence variety W to the first factor

$$p : W \longrightarrow \mathbb{P}(Sym^k(\mathbb{C}^n))$$
$$(f, z) \longmapsto f$$

is surjective and $p^{-1}(f) = \{z \in \mathbb{P}(\mathbb{C}^n) | z \text{ is an eigenvector of f}\}$. This set is finite for generic f and its number is equal to $\dfrac{(k-1)^n - 1}{k-2}$ by [6] or [1] (this expression simplifies to n for $k = 2$).

We note that $\kappa^s_{d,n,k}(L) = p(W \cap q^{-1}(\mathbb{P}(L))) \subseteq \mathbb{P}(Sym^k(\mathbb{C}^n))$.

The $(d-1)$- dimensional subspace $\mathbb{P}(L)$ of $\mathbb{P}(\mathbb{C}^n) = \mathbb{P}^{n-1}$ specifies the following diagram:

$$
\begin{array}{ccc}
 & W \cap q^{-1}(\mathbb{P}(L)) & \\
\swarrow {\scriptstyle p} & & \searrow {\scriptstyle q} \\
\kappa^s_{d,n,k}(L) & & \mathbb{P}(L)
\end{array}
\qquad (2.2)
$$

Each fiber of the map q in above diagram is a linear space of codimension $n - 1$ in $\mathbb{P}(Sym^k(\mathbb{C}^n))$.

This implies that $W \cap q^{-1}(\mathbb{P}(L))$ is irreducible and its dimension equals

$$\dim(\mathbb{P}(L)) + \dim(\mathbb{P}(Sym^k(\mathbb{C}^n))) - (n-1) = (d-1) + \binom{n+k-1}{k} - (n-1) = \binom{n+k-1}{k} - (n-d).$$

Acting with $SO(n)$ we can find a tensor in $\kappa^s_{d,n,k}(L)$ having finitely many eigenvectors, it follows that the general fiber of the surjection p is finite, the variety $\kappa_{d,n,k}(L)$ is irreducible of the same dimension than $W \cap q^{-1}(\mathbb{P}(L))$. Hence $\kappa^s_{d,n,k}(L)$ has codimension $n - d$ in $\mathbb{P}^{\binom{k+n-1}{k}-1}$. $\qquad \square$

In order to compute the degree of $\kappa^s_{d,n,k}(L)$ we construct a vector bundle E on X with a section vanishing on W. We briefly recall the construction in [13, §3.1].

Let $\mathcal{O}(-1)$ be the universal bundle of rank 1 and Q the quotient bundle of rank $n - 1$. They appear in the following exact sequence, with $\mathcal{O}$ the structure sheaf of $\mathbb{P}^{n-1}$:

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \otimes \mathbb{C}^n \longrightarrow Q \longrightarrow 0. \qquad (2.3)$$

Tensoring by $\mathcal{O}(k-1)$ we get

$$0 \longrightarrow \mathcal{O}(k-2) \longrightarrow \mathcal{O}(k-1) \otimes \mathbb{C}^n \longrightarrow Q(k-1) \longrightarrow 0.$$

The fiber of $Q(k-1)$ at $v \in \mathbb{C}^n$, is isomorphic to $Hom(\langle v^{k-1} \rangle, \dfrac{\mathbb{C}^n}{\langle v \rangle})$.

Every $f \in Hom(Sym^{k-1}\mathbb{C}^n, \mathbb{C}^n)$ induces a section $s_f \in H^0(Q(k-1))$ which corresponds to the composition $\langle v^{k-1} \rangle \overset{i}{\longrightarrow} Sym^{k-1}\mathbb{C}^n \overset{f}{\longrightarrow} \mathbb{C}^n \overset{\pi}{\longrightarrow} \dfrac{\mathbb{C}^n}{\langle v \rangle}$ on the fiber of $\langle v \rangle$. The sec-

tion s_f vanishes on $\langle v \rangle$ if and only if $\pi(f(v^{k-1})) = 0$. In particular, every symmetric tensor $f \in Sym^k\mathbb{C}^n$ defines (by contraction) an element in $Hom(Sym^{k-1}\mathbb{C}^n, \mathbb{C}^n)$ and hence a section of $Q(k-1)$, which by abuse of notation we denote again with s_f. This implies

Lemma 2.2 [13, *Lemma 3.7 (2)*] *For $f \in Sym^k\mathbb{C}^n$, the section s_f vanishes in v iff v is an eigenvector of f.*

We recall that if a vector bundle E of rank r on X has a section vanishing on Z, and the codimension of Z is equal to r, then the class of $[Z]$ in the degree r component of the Chow ring $A^*(X) = \mathbb{C}[h, v]/(h^{\binom{n+k-1}{k}}, v^n)$ is computed by $[Z] = c_r(E)$. We shall apply this to the following vector bundle on the product variety X

$$E := p^*\mathcal{O}_{\mathbb{P}(Sym^k(\mathbb{C}^n))}(1) \otimes q^*Q(k-1).$$

Since $H^0(\mathcal{O}(1)) = Sym^k(\mathbb{C}^n)$ and $H^0(Q(k-1)) = \Gamma^{k,1^{n-2}}\mathbb{C}^n$, by the Künneth formula we get

$$H^0(E) = Hom(Sym^k(\mathbb{C}^n), \Gamma^{k,1^{n-2}}\mathbb{C}^n).$$

We have a section $I \in H^0(E)$ given by the map $f \mapsto s_f$. The section I vanishes exactly at the pairs (f, z) such that z is eigenvector of f, so we get that the zero locus $Z(I)$ of $I \in H^0(E)$ equals the incidence variety W.

Theorem 2.3 *The degree of $\kappa^s_{d,n,k}(L)$ equals $\Sigma_{i=0}^{d-1}\binom{n-d+i}{i}(k-1)^i$ in $\binom{n+k-1}{k}_{\mathbb{P}}^{-1}$.*

Proof Since $Z(I)$ has codimension $rkE = n - 1$ in X, the class of $Z(I)$ equals the top Chern class of E(By theorem 4.4). In symbols, $[Z(I)] = c_{n-1}(E)$.

The desired degree equals

$$\deg \kappa^s_{d,n,k}(L) = p^*c_1(\mathcal{O}(1))^{\binom{n-k-1}{k}-(n-d)-1} \cdot c_{(n-1)}(E) \cdot q^*(c_1(\mathcal{O}(1)))^{n-d}. \qquad (2.4)$$

The Chern class of $E = p^*\mathcal{O}_{\mathbb{P}(Sym^k(\mathbb{C}^n))}(1) \otimes q^*Q(k-1)$ decomposes as

$$c_{(n-1)}(E) = \Sigma_{i=0}^{n-1}p^*c_1(\mathcal{O}(1))^{(n-1)-i}.q^*c_i(Q(k-1)).$$

Hence the equality on the right of (2.4) can be written as

$$\deg \kappa^s_{d,n,k}(L) = p^*c_1(\mathcal{O}(1))^{\binom{n-k-1}{k}-(n-d)-1} \cdot (\Sigma_{i=0}^{n-1}p^*c_1(\mathcal{O}(1))^{(n-1)-i} \cdot q^*c_i(Q(k-1)) \cdot q^*c_1(\mathcal{O}(1))^{n-d})$$

$$= \Sigma_{i=0}^{n-1}p^*c_1(\mathcal{O}(1))^{\binom{n-k-1}{k}-1+(d-1)-i} \cdot q^*[c_i(Q(k-1)) \cdot c_1(\mathcal{O}(1))^{n-d}].$$

All summands are zero except for $i = d - 1$, and we remain with $\deg c_{d-1}(Q(k-1))$. Since $\deg c_i(Q) = 1$, $rk(Q) = n - 1$, the result follows from the formula

$$c_j(E \otimes L) = \sum_{i=0}^{j}\binom{rkE - j + i}{i}c_i(E)c_1(L)^i,$$

see [14, §1.2]. $\qquad\qquad\square$

Remark 2.4 The result generalizes to any complex vector space V equipped with a symmetric nondegenerate bilinear form. The construction works in the setting of $SO(V)$-actions, in particular the space of sections we have considered, like $H^0(Q(k-1))$, are $SO(V)$-modules. Note that in this setting V is isomorphic to its dual $V^\vee$.

3 Singular vector k-ples of tensors

Let $L \subset \mathbb{C}^{n_1}$ be a fixed d- dimensional linear subspace .

Definition 3.1 The k−ple $(v_1, \ldots, v_k)$ with $v_i \in \mathbb{C}^{n_i} \setminus \{0\}$ is called a singular k-tuple for a tensor $T \in \mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_k}$ if

$$T(v_1, \ldots, \widehat{v_i}, \ldots, v_k) = \lambda_i v_i, \text{ for some } \lambda_i \in \mathbb{C}, \quad i = 1, \ldots, k.$$

We are interested in the Kalman variety $\kappa_{d,n_1,\ldots,n_k}(L)$ of all tensors
$T \in \mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_k}$ that have a singular k−tuple $(v_1, \ldots, v_k)$ with $v_1 \in L$.

Proposition 3.2 *The variety $\kappa_{d,n_1,\ldots,n_k}(L)$ is irreducible and it has codimension $n_1 - d$.*

Proof The product $X = \mathbb{P}(\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_k}) \times \mathbb{P}(\mathbb{C}^{n_1}) \times \ldots \times \mathbb{P}(\mathbb{C}^{n_k})$, has the two projections

$$
\begin{array}{ccc}
 & X & \\
\swarrow^{p} & & \searrow^{q} \\
\mathbb{P}(\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_k}) & & \mathbb{P}(\mathbb{C}^{n_1}) \times \ldots \times \mathbb{P}(\mathbb{C}^{n_k})
\end{array}
\tag{3.1}
$$

Fix the variety (we use the same notation of previous section to ease the analogy)

$$W = \{(T, v_1, \ldots, v_k) \in X | v_1, \ldots, v_k) \text{ is singular } k - \text{tuple for } T\} \subseteq X.$$

The projection of W to the second factor

$$q : W \longrightarrow \mathbb{P}(\mathbb{C}^{n_1}) \times \ldots \times \mathbb{P}(\mathbb{C}^{n_k})$$
$$(T, v_1, \ldots, v_k) \longmapsto (v_1, \ldots, v_k)$$

is surjective, every fiber is

$$q^{-1}(v_1, \ldots, v_k) = \{T \in \mathbb{P}(\mathbb{C}^{n_1})) | T(v_1, \ldots, \widehat{v_i}, \ldots, v_k) = \lambda_i v_i\}$$

and this is the zero scheme of the ideal of 2×2 minors of the $n_i \times 2$ matrix $(T(v_1, \ldots, \widehat{v_i}, \ldots, v_k) | v_i)$. Hence it is a linear subspace of codimension $\Sigma_{i=1}^{k}(n_i - 1) = \Sigma_{i=1}^{k} n_i - k$ in $\mathbb{P}(\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_k})$.
The projection of the variety W to the first factor

$$p : W \longrightarrow \mathbb{P}(\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_k})$$
$$(T, v_1, \ldots, v_k) \longmapsto T$$

is surjective and $p^{-1}(T) = \{(v_1, \ldots, v_k) \in \mathbb{P}(\mathbb{C}^{n_1}) \times \ldots \times \mathbb{P}(\mathbb{C}^{n_k}) | (v_1, \ldots, v_k)$ is singular k-tuple for T}. This set is finite for generic T.

Consider the map

$$q_1 : \mathbb{P}(\mathbb{C}^{n_1}) \times \ldots \times \mathbb{P}(\mathbb{C}^{n_k}) \longrightarrow \mathbb{P}(\mathbb{C}^{n_1}).$$

The Kalman variety has the following description in terms of the above diagram:

$$\kappa_{d,n_1,\ldots,n_k}(L) = p(W \cap q^{-1}(q_1^{-1}(\mathbb{P}(L)))),$$

where $q_1^{-1}(\mathbb{P}(L)) = \mathbb{P}(L) \times \mathbb{P}(\mathbb{C}^{n_2}) \times \ldots \times \mathbb{P}(\mathbb{C}^{n_k})$. The $(d-1)$-dimensional subspace $\mathbb{P}(L)$ of $\mathbb{P}(\mathbb{C}^{n_1}) = \mathbb{P}^{n_1-1}$ specifies the following diagram:

$$
W \cap q^{-1}(q_1^{-1}(\mathbb{P}(L)))
$$

$$
\kappa_{d,n_1,\ldots,n_k}(L) \quad \swarrow^{p} \qquad\qquad\qquad \searrow^{q} \quad \mathbb{P}(L) \times \mathbb{P}(\mathbb{C}^{n_2}) \times \ldots \mathbb{P}(\mathbb{C}^{n_k})
$$

$$(3.2)$$

Each fiber of the map q in above diagram is a linear space of codimension $\Sigma_{i=1}^{k}(n_i - 1)$ in $\mathbb{P}((\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_k}))$, and we conclude exactly as in the proof of Proposition 2.1. $\qquad\square$

In order to compute the degree of $\kappa_{d,n_1,\ldots,n_k}(L)$ we construct a vector bundle E on X with a section vanishing exactly on W. We briefly recall the construction in [5, §3].

The fiber of $q_i^* Q_{\mathbb{P}(\mathbb{C}^{n_i})}(1,\ldots 1, \underbrace{0}_{i}, 1\ldots, 1)$ at $x = [v_1 \otimes \ldots \otimes v_k]$, is isomorphic to

$Hom(\langle v_1 \otimes \ldots \otimes \widehat{v}_i \otimes \ldots \otimes v_k \rangle, \dfrac{\mathbb{C}^{n_i}}{[v_i]})$. Every tensor

$T \in \mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_k} \simeq \mathrm{H}om(\mathbb{C}^{n_1} \otimes \ldots \otimes \widehat{\mathbb{C}^{n_i}} \otimes \ldots \mathbb{C}^{n_k}, \mathbb{C}^{n_i})$ induces a section s_T of $q_i^* Q_{\mathbb{P}(\mathbb{C}^{n_i})}(1,\ldots 1, \underbrace{0}_{i}, 1\ldots, 1)$ which corresponds to the composition

$$\langle v_1 \otimes \ldots \otimes \widehat{v}_i \otimes \ldots \otimes v_k \rangle \xrightarrow{i} \mathbb{C}^{n_1} \otimes \ldots \otimes \widehat{\mathbb{C}^{n_i}} \otimes \ldots \otimes \mathbb{C}^{n_k} \xrightarrow{T} \mathbb{C}^{n_i} \xrightarrow{\pi} \dfrac{\mathbb{C}^{n_i}}{\langle v_i \rangle}$$

on the fiber of $\langle v_1 \otimes \ldots \otimes v_k \rangle$. The section s_T vanishes in $\langle v_1 \otimes \ldots \otimes v_k \rangle$ if and only if $T(v_1, \ldots, \widehat{v}_i, \ldots, v_k) = \lambda_i v_i$, for some $\lambda_i \in \mathbb{C}$.

Define the vector bundle $\varepsilon = \sum_{i=1}^{k} q_i^* Q_{\mathbb{P}(\mathbb{C}^{n_i})}(1,\ldots 1, \underbrace{0}_{i}, 1\ldots, 1)$.

This implies

Lemma 3.3 ([5, Lemma 11]) *For $T \in \mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_k}$, the diagonal section $(s_T, \ldots, s_T) \in H^0(\varepsilon)$ vanishes in $v_1 \otimes \ldots \otimes v_k$ iff $(v_1, \ldots, v_k)$ is a singular k- tuple of T.*

Consider the following vector bundle on the product variety $X = \mathbb{P}(\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_k}) \times \mathbb{P}(\mathbb{C}^{n_1}) \times \ldots \times \mathbb{P}(\mathbb{C}^{n_k})$

$$E := p^* \mathcal{O}_{\mathbb{P}(\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_k})}(1) \otimes q^*(\varepsilon).$$

Hence

$$H^0(X, E) = H^0(p^* \mathcal{O}_{\mathbb{P}(\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_k})}(1) \otimes H^0(q^* \varepsilon),$$

we get

$$H^0(E) = (\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_k}) \otimes [(\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_k}) \oplus \ldots \oplus (\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_k})].$$

We have a section $I \in H^0(E)$ given by the diagonal map $T \mapsto (s_T, \ldots, s_T)$. The section I vanishes exactly at $(T, v_1, \ldots, v_k)$ such that $(v_1, \ldots, v_k)$ is a singular k-tuple of T, so we get that the zero locus $Z(I)$ of $I \in H^0(E)$ equals the incidence variety W.

Theorem 3.4 *The degree of Kalman variety $\kappa_{d,n_1,\ldots,n_k}(L)$ is the coefficient of $h^{n_1-d} v_1^{d-1} \prod_{i \geq 2} v_i^{(n_i-1)}$ in the polynomial*

$$\prod_{i=1}^{k} \frac{[(\widetilde{v}_i + h)^{n_i} - v_i^{n_i}]}{(\widetilde{v}_i + h) - v_i}$$

where $\widetilde{v}_i = \Sigma_j v_j - v_i$.

Proof As in the proof of Theorem 2.3, the degree is equal to

$$p^* c_1(\mathcal{O}(1))^{[\Sigma_{i=1}^k (n_i-1)-(n_1-d)]} . c_{(\Sigma_{i=1}^k (n_i-1))}(E) . q_{V_1}^*(\mathcal{O}(1))^{n_1-d}.$$

Denote $h = c_1(\mathcal{O}_{\mathbb{P}(\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_k})}(1))$, $v_i = c_1(\mathcal{O}_{\mathbb{P}(\mathbb{C}^{n_i})}(1))$, they are the generators of the Chow ring $A^*(X) = \mathbb{C}[h, v_1, \ldots, v_k]/(h^{1+\Sigma_i (n_i-1)}, v_1^{n_1}, \ldots, v_k^{n_k})$.

E is the direct sum of k summands, by the Euler sequence each of them has Chern polynomial $\frac{(1+v_1+\ldots+\widetilde{v}_i+\ldots v_k+h)^{n_i}}{(1+v_1+\ldots-v_i+\ldots v_k+h)}$ for $i = 1, \ldots, k$. So the degree is the coefficient of $h^{(\prod n_i-1)} v_1^{n_1-d} \prod_i v_i^{n_i-1}$ in the expansion of

$$h^{(\prod n_i-1)-(n_1-d)} \frac{(1 + v_2 + \ldots + v_k + h)^{n_1}}{(1 - v_1 + v_2 + \ldots + v_k + h)} \cdots \frac{(1 + v_1 + \ldots + v_{k-1} + h)^{n_k}}{(1 + v_1 + \ldots + v_{k-1} - v_k + h)} v_1^{n_1-d}.$$

Hence it is the coefficient of $h^{n_1-d} v_1^{d-1} \prod_{i \geq 2} v_i^{(n_i-1)}$ in the expansion of $\prod_{i=1}^{k} \frac{(1+\widetilde{v}_i+h)^{n_i}}{1+\widetilde{v}_i+h-v_i}$.

Since

$$\frac{(1 + \widetilde{v}_i + h)^{n_i}}{(1 + \widetilde{v}_i + h - v_i)} = \frac{(1 + \widetilde{v}_i + h)^{n_i-1}}{1 - x}; \quad x = \frac{v_i}{1 + \widetilde{v}_i + h}$$

$$= (1 + \widetilde{v}_i + h)^{n_i-1} \Sigma_{p=0}^{\infty} x^p = (1 + \widetilde{v}_i + h)^{n_i-1} \Sigma_{p=0}^{n_i-1} x^p$$

$$= \Sigma_{j=0}^{n_i-1} (1 + \widetilde{v}_i + h)^{n-1-j} v^j$$

We get that the degree sought is the coefficient of $h^{n_1-d} v_1^{d-1} \prod_{i \geq 2} v_i^{(n_i-1)}$ in the expansion of the polynomial

$$\prod_{i=1}^{k} \frac{[(1 + \widetilde{v}_i + h)^{n_i} - v_i^{n_i}]}{(1 + \widetilde{v}_i + h) - v_i}.$$

All terms of the last polynomial have degree $\leq \Sigma_i (n_i - 1)$ which equals the degree of the monomial $h^{n_1-d} v_1^{d-1} \prod_{i \geq 2} v_i^{(n_i-1)}$, hence it is enough to consider the homogeneous part of top degree. The thesis follows. $\square$

The following stabilization phenomenon is similar to the one observed in [17].

Corollary 3.5 (Stabilization) *Let* $(n_k - 1) = \sum_{i=1}^{k-1} (n_i - 1)$ (*boundary format, see* [7]). *For any* $m \geq n_k$ *we have* $\kappa_{d,n_1,\ldots,n_k}(L) = \kappa_{d,n_1,\ldots,n_{k-1},m}(L)$.

Denote $V^N = \prod_{i=2}^{k-1} v_i^{n_i-1}$. We need to compute the coefficient of $h^{n_1-d} v_1^{d-1} V^N v_k^{m-1}$ in the polynomial

$$\left(\prod_{i=1}^{k-1} \frac{[(\widetilde{v}_i + h)^{n_i} - v_i^{n_i}]}{(\widetilde{v}_i + h) - v_i} \right) \frac{[(\widetilde{v}_k + h)^m - v_k^m]}{(\widetilde{v}_k + h) - v_k},$$

when $m \geq n_k$. Compare indeed the coefficient of $h^{n_1-d} v_1^{d-1} V^N v_k^{n_k-1}$ in

$$\left(\prod_{i=1}^{k-1} \frac{[(\widetilde{v}_i + h)^{n_i} - v_i^{n_i}]}{(\widetilde{v}_i + h) - v_i} \right) (\Sigma_{j=0}^{n_k-1} (\widetilde{v}_k + h)^{n_k-1-j} v_k^j)$$

with the coefficient of $h^{n_1-d} v_1^{d-1} V^N v_k^{m-1}$ in the polynomial

$$\left(\prod_{i=1}^{k-1} \frac{[(\widetilde{v}_i + h)^{n_i} - v_i^{n_i}]}{(\widetilde{v}_i + h) - v_i} \right) (\Sigma_{j=0}^{m-1} (\widetilde{v}_k + h)^{m-1-j} v_k^j).$$

The first one is the coefficient of $h^{n_1-d} v_1^{d-1} V^N v_k^{m-1}$ in

$$\left(\prod_{i=1}^{k-1} \frac{[(\widetilde{v}_i + h)^{n_i} - v_i^{n_i}]}{(\widetilde{v}_i + h) - v_i} \right) (\Sigma_{j=0}^{n_k-1} (\widetilde{v}_k + h)^{n_k-1-j} v_k^{j+m-n_k}),$$

namely in

$$\left(\prod_{i=1}^{k-1} \frac{[(\widetilde{v}_i + h)^{n_i} - v_i^{n_i}]}{(\widetilde{v}_i + h) - v_i} \right) (\Sigma_{j=m-n_k}^{m-1} (\widetilde{v}_k + h)^{m-1-j} v_k^j).$$

With $j < m - n_k$ in the last sum we have

$$\left(\prod_{i=1}^{k-1} \frac{[(\widetilde{v}_i + h)^{n_i} - v_i^{n_i}]}{(\widetilde{v}_i + h) - v_i} \right) (\Sigma_{j=0}^{m-n_k-1} (\widetilde{v}_k + h)^{m-1-j} v_k^j),$$

which has zero coefficient of $h^{n_1-d} v_1^{d-1} V^N v_k^{m-1}$ since the maximum exponent of v_k is $\sum_{i=1}^{k-1} (n_i - 1) + (m - n_k - 1) = (n_k - 1) + (m - n_k - 1) = m - 2$. Hence the two coefficients are equal.

This concludes the proof of the Corollary.

The following Proposition covers the case of matrices, $k = 2$.

Proposition 3.6 *The degree of Kalman variety of $n \times m$ matrices for $n \leq m$ is*

$$\deg \kappa_{d,n,m}(L) = 2^{(n-d)} \binom{n}{d-1}$$

Proof By Corollary 3.5 we may assume $n = m$ (square case). By Theorem 3.4 we need to compute the coefficient of $h^{n-d} v^{d-1} w^{n-1}$ in the expansion of the polynomial

$$(\Sigma_{j=0}^{n-1}(w+h)^{n-1-j}v^j)(\Sigma_{j=0}^{n-1}(v+h)^{n-1-j}w^j).$$

By Newton expansion, this is equal to

$$\sum_{j=0}^{d-1}\sum_{h=d-j-1}^{n-j-1}\binom{n-j-1}{h}\binom{h}{d-1-j}=$$

$$\sum_{j=0}^{d-1}\sum_{h=d-j-1}^{n-j-1}\binom{n-j-1}{n-d}\binom{n-d}{n-1-j-h}=$$

$$\sum_{j=0}^{d-1}\binom{n-j-1}{n-d}2^{n-d}=\binom{n}{d-1}2^{n-d}.$$

$\square$

Remark 3.7 Following Theorem 3.4, the coefficient of $h^{n-d}v^{d-1}w^{m-1}$ in the expansion of the polynomial

$$(\Sigma_{j=0}^{n-1}(w+h)^{n-1-j}v^j)(\Sigma_{j=0}^{m-1}(v+h)^{m-1-j}w^j)$$

is computed as (1.1) by Newton expansion.

Lemma 3.8 *Let A be a $n \times m$ matrix with $n \leq m$. A has a singular pair (v, w) with $v \in L$ if and only if AA^t has an eigenvector in L.*

Proof If $Aw = \lambda_1 v$ and $A^t v = \lambda_2 w$ it follows $AA^t v = \lambda_1\lambda_2 v$. Conversely, let $AA^t v = \lambda^2 v$. If $\lambda \neq 0$ pose $w = \lambda^{-1}A^t v$. If $\lambda = 0$, in case $A^t v \neq 0$ we may still pose $w = A^t v$. In case $A^t v = 0$, since $n \leq m$ we have $\ker A \neq 0$ then with both $\lambda_i = 0$ any $w \in \ker A$ works. $\square$

Example 3.9 The assumption $n \leq m$ is necessary in Lemma 3.8. Let $A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, w \right)$ is a singular pair. We have $A^t v = v_1$, $Aw = \begin{pmatrix} w \\ 0 \end{pmatrix}$. Set $L = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$, so AA^t has an eigenvector in L but $Aw \in L$ only if $w = 0$.

Example 3.10 If A has a singular pair (v, w) with $v \in L$ then v is an eigenvector of AA^t, so AA^t has an eigenvector in L. The converse is true on real numbers but it is false on complex numbers if we wish the additional requirement $\lambda_1 = \lambda_2$. A simple counterexample is as follows. Let $A = \begin{pmatrix} 1 & \sqrt{-1} \\ 1 & \sqrt{-1} \end{pmatrix}$ and let $L = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$. The matrix A has no singular pair (v, w) with $v \in L$, but $AA^t = 0$ so AA^t has an eigenvector in L.

Proposition 3.11 *Let $d = 1$, so that $L = \langle v_0 \rangle$ is a line in V. Let C be a $(n-1) \times n$ matrix such that $L = \{v|Cv = 0\}$. Then*

$$\kappa_{1,n,m} = \{A|C(AA^t)v_0 = 0, \mathrm{rk}(CA) \leq m-1\}.$$

Note the condition on $\mathrm{rk}(CA)$ is empty when $n \leq m$.

Table 1 Description of Kalman variety for small values of (d, n, m)

d, n, m	deg $\kappa_{d,n,m}(L)$	Generators of ideal	Singular locus
(1,2,2)	2	One quadric	$\varnothing$
(1,3,2)	3	Three quadrics	$\varnothing$
(2,3,2)	4	One quartic	$\{A | ker(A) \supset L\} \cup \{A | ker(A) \supset L^\perp\} \cup (Q_0 \times \mathbb{P}^2)$
(1,3,3)	4	Two quadrics	$\{A | ker(A) \supset L, rk(A) \leq 1\}$
(2,3,3)	6	One sextic	$Codim = 2, deg = 4$
(1,4,2)	4	Six quadrics	$\varnothing$
(2,4,2)	6	One quadric and three quartics	$\{A | ker(A) \supset L, rk(A) \leq 1\} \cup \{A | ker(A) \supset L^\perp\} \cup (Q_0 \times \mathbb{P}^3)$
(3,4,2)	4	One quartic	Analogous to (3, 2, 2)
(1,4,3)	7	Three quadrics and one cubic	$Codim = 7, deg = 42$
(2,4,3)	13	One quartic, two quintics, three sextics	$Codim = 4, deg = 40$
(3,4,3)	9	One generator of degree nine	$Codim = 2, deg = 13$

Proof The pair $v_0 \otimes (A^t v_0)$ is a singular pair if $(CA)(A^t v_0) = 0$. In order to be a non-zero pair we have to exclude the case when $n > m$ and CA has maximal rank m. □

We list in Table 1 the first cases of degree of $\kappa_{d,n,m}(L)$ for $n \geq m$ and its singularities. The singular locus of $\kappa_{d,n,m}(L)$ contains the closure of the set of matrices having at least two distinct singular pairs with first component on L. The table shows that this containment is strict in many cases, contrary to the eigenvector case in [16, Theorem 4.4], where equality holds.

Remark 3.12 In the matrix case $d = 2$, equations for $\kappa_{d,n,m}^s(L)$ were found in [10, 21], even in the more general case of matrix eigenvectors. It should be interesting to extend that study to the tensor case.

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