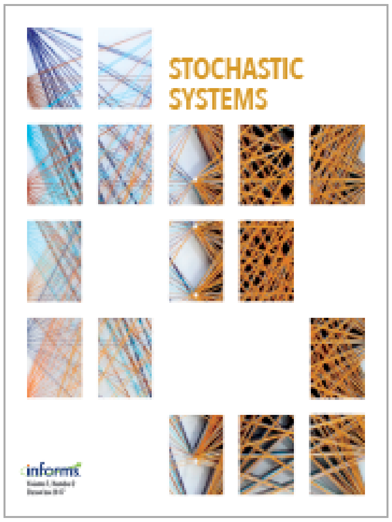

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A Fluid Model of an Electric Vehicle Charging Network

Angelos Aveklouris,[a,*] Maria Vlasiou,[b,c] Bert Zwart[b,d]

[a] The University of Chicago Booth School of Business, Chicago, Illinois 60637; [b] Mathematics and Computer Science, Eindhoven University of Technology, 5600 MB Eindhoven, Netherlands; [c] Faculty of Electrical Engineering, University of Twente, 7522 NB Twente, Netherlands; [d] Stochastics, Centrum Wiskunde en Informatica, 1098 XG Amsterdam, Netherlands

*Corresponding author

Contact: angelos.aveklouris@chicagobooth.edu, https://orcid.org/0000-0003-3358-5474 (AA); m.vlasiou@tue.nl (MV); bert.zwart@cwi.nl, https://orcid.org/0000-0001-9336-0096 (BZ)

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Abstract. We develop and analyze a measure-valued fluid model keeping track of parking and charging requirements of electric vehicles in a local distribution grid. We show how this model arises as an accumulation point of an appropriately scaled sequence of stochastic network models. Our analysis incorporates load-flow models that describe the laws of electricity. Specifically, we consider the alternating current (AC) and the linearized Distflow power flow models and show a continuity property of the associated power allocation functions.

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Keywords: electric vehicle charging • fluid approximation • measure-valued processes • AC power flow model • linearized Distflow

1. Introduction

To deal with the effects of climate change, many countries are in the process of implementing new policy measures to stimulate the use of electricity generated by renewable sources such as solar and wind. This comes with many societal challenges and opportunities for research. The supply of energy is less predictable, which makes the task of keeping high-voltage transmission networks reliable more challenging. In the local distribution grids, new products and services can be used to balance the grid emerge (such as smart devices), but also create more intermittency. In particular, electric vehicles (EVs) can cause a substantial additional load on local distribution grids (Hoogsteen et al. 2017).

The focus of the present paper is on analyzing congestion associated to *slow charging*, which happens when a car is parked while its owner is at home, at work, or shopping. In Carvalho et al. (2015), it was suggested to model the evolution of slowly charging EVs in a local grid by bandwidth sharing networks, approximating the instantaneous allocation of electricity to vehicles by proportional fairness. The main constraint that needs to be satisfied is that the voltage drop in the network needs to remain bounded. The focus in Carvalho et al. (2015) was solely on simulation, assuming a Markovian model and infinitely many parking spaces for EVs. Using simulations, the stability of proportional fairness and max-min fairness was examined.

In a recent paper (Aveklouris et al. 2019), we proposed an extension of Carvalho et al. (2015) by allowing for load limits, finitely many parking spaces, and deadlines (associated with parking times). The joint distribution of charging requirements and parking times was not restricted to Markovian or independence assumptions. Using heuristic arguments, Aveklouris et al. (2019) proposed a fluid model keeping track of the number of charged and uncharged cars in the system and an associated invariant point. This invariant point is shown to be computationally tractable in Aveklouris et al. (2019), as it is formulated in terms of an AC optimal power flow problem with an exact convex relaxation.

The goal of the present paper is to put the analysis of Aveklouris et al. (2019) on rigorous footing using measure-valued fluid limits. As in Aveklouris et al. (2019), we allow the parking times and the charging requirements of EVs to be dependent and generally distributed random variables. In addition, we consider general arrival processes with time-varying arrival rates and multiple EV types. The distribution grid is explicitly modeled, and we allow for multiple parking lots, each with finitely many parking spaces; the fluid approximation of Aveklouris et al. (2019) did not take the subtleties of dynamically rejecting vehicles at parking lots into consideration as we do here.

A measure-valued process that keeps track of the age in service or the residual service times of customers has proven to be a useful tool in proving asymptotic limits for systems with many-server queues in which the service discipline is not first-come-first-served. In Kang and Ramanan (2010) and Zhang (2013), the authors develop fluid limit approximations for a many-server queueing system with impatient customers, where the arrival, service, and abandonment times follow general distributions, and the assumption of the absolute continuity of the service and patience time distributions is removed in Zuñiga (2014). A fluid model for many-server queues with time-varying arrival rates are studied in Liu and Whitt (2011) and Mandelbaum and Momčilović (2017). In Puha and Ward (2019), the authors develop fluid limits for a multiclass many-server queue with impatient customers for a wide class of scheduling policies. Fluid limits for many-server retrial queues with nonpersistent customers are studied in Kang (2015), for processor-sharing queues in Zhang et al. (2009), and for processor-sharing queues with impatient customers in Gromoll et al. (2008). Fluid approximations for bandwidth-sharing networks with generally distributed service and patience times are developed in Gromoll and Williams (2009) and in Remerova et al. (2014).

Our work is also connected to the literature on bandwidth-sharing networks. Such networks have been successfully used to model communication networks where the set of feasible schedules is determined by the maximum amount of data a communication channel can carry per time unit Massoulié and Roberts (1999). The stochastic analysis of bandwidth-sharing networks was initially restricted to specific networks (Bonald and Proutiere 2003, Bonald et al. 2006). The application of fluid and diffusion approximations led to computationally tractable approximations of a large class of networks (Kang et al. 2009, Ye and Yao 2012, Borst et al. 2014, Reed and Zwart 2014, Remerova et al. 2014, Vlasiou et al. 2014).

In the context of communication networks, proportional fairness is a nontrivial but justified approximation of the transmission control protocol (TCP). A similar justification in the context of EV charging is performed in Ardakanian et al. (2013) and Fan (2012). In these papers, by using arguments similar to the seminal work (Kelly 1997), it is shown how algorithms like proportional fairness emerge in decentralized EV charging. Our class of controls contain proportional fairness as a special case.

Our analysis is mostly related to Remerova et al. (2014). The main difference is that, in the setting of EVs, an important constraint that needs to be satisfied is to keep the voltage drops bounded, making the bandwidth-sharing network proposed here different. This also causes new technical issues, as the capacity set can be nonpolyhedral or even nonconvex. In addition, arriving vehicles finding a full parking lot are discarded; we assume such cars park on a regular parking spot. This leads to the additional technical complication of a loss process in a measure-valued context. An extension of the Erlang loss system has been analyzed in a measure-valued framework in Kang (2015). However, in our setting, we use a different approach. We keep track of the residual service times of EVs instead of their age in service. Moreover, we consider a multidimensional (dependent) vector of processes that describes the number of total EVs and the number of uncharged EVs in each parking lot and uses different test functions to define the fluid model.

We now describe our contributions in more detail. We develop a measure-valued fluid model for the vector process which describes the number of total and uncharged EVs in each parking lot, allowing the dynamics of the stochastic model to be approximated with a deterministic model. This model depends on the joint distribution of the charging requirements and the parking times. We show that our measure-valued fluid model arises as a weak limit of a vector of measure-valued processes under an appropriate scaling. To prove properties for the solutions of the fluid model, we investigate the properties of the bandwidth allocation function in our setting where the capacity set is convex and establish similar continuity properties of the allocation function as in Reed and Zwart (2014). Although the structural properties of a linearized voltage model can be developed in full, for the AC power flow equations we were only able to show continuity of the allocation function. We conjecture that Lipschitz continuity holds as well but leave this problem open; we refer to Section 3 for more specific comments.

For our proofs we mostly use techniques from Kang (2015) and Remerova et al. (2014). We first prove that a fluid model solution is bounded away from zero. Then, we establish tightness of the processes by showing a compact containment property and an oscillation control; the Prokhorov metric of the corresponding measures is small enough. Last, we prove that any subsequential limit satisfies the fluid model equations.

The rest of this paper is organized as follows. In Section 2, we provide a detailed model description. In particular, we introduce our stochastic model, the power flow models that we use, and the definition of system dynamics. Next, in Section 3, we present a continuity property of the optimal power allocation. Then, we move to the stochastic model. A fluid model is presented in Section 4, where we also study its properties. Section 5 shows that the fluid model can arise as a weak limit of the fluid-scaled processes. All proofs are gathered in Sections 6–8.

2. Model Description

In this section, we provide a detailed formulation of our model and explain various notational conventions that are used in the remainder of this work. The model description in this section is nearly identical to that in Aveklouris et al. (2019). We include all details on the network structure and physical characteristic for completeness; the main difference is that the measure-valued state descriptor is fully developed and analyzed in the present paper. To this end, we also require more sophisticated notation, which we introduce first.

2.1. Preliminaries

We introduce the notational conventions that are used throughout the paper. All vectors and matrices are denoted by bold letters. Furthermore, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+$ is the set of nonnegative real numbers, and $\mathbb{N}$ is the set of strictly positive integers. For two real numbers x and y, define $x \vee y := \max\{x, y\}$, $x \wedge y := \min\{x, y\}$, and $x^+ := x \vee 0$. For two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^I$, define the coordinate-wise product $\boldsymbol{x} \circ \boldsymbol{y} := (x_1 y_1, \dots, x_I y_I)$ (i.e., the Hadamard product) and the maximum norm $\|\boldsymbol{x}\| := \max_{1 \le i \le I} |x_i|$. Vector inequalities hold coordinate-wise, namely $\boldsymbol{x} > \boldsymbol{y}$ implies that $x_i > y_i$ for all i. Furthermore, $\boldsymbol{I}$ represents the identity matrix and $\boldsymbol{e}$ and $\boldsymbol{e}_0$ are the vectors consisting of ones and zeros, respectively, the dimensions of which are clear from the context. Also, $\boldsymbol{e}_i$ is the vector whose i^{th} element is one and the rest are all zero.

Let Y be a metric space. We denote by $\mathcal{C}(Y, Y)$ the space of continuous functions $f : Y \to Y$ and by $\mathcal{C}_b(Y, Y)$ the space of continuous and bounded functions $f : Y \to Y$. By $\mathcal{D}(Y, Y)$ denote the space of functions $f : Y \to Y$ that are right continuous with left limits endowed with the J_1 topology; that is, the Skorokhod space. Furthermore, we write $X(\cdot) := \{X(t), t \ge 0\}$ to represent a stochastic process. Moreover, $\stackrel{d}{=}$ and $\stackrel{d}{\to}$ denote equality and convergence in distribution (weak convergence).

Let $\mathcal{M}(Y)$ be the space of Randon measures (i.e., locally finite and inner regular measures) on Y, endowed with the Borel σ-algebra denoted by $\mathcal{B}(Y)$. Furthermore, $\mathcal{M}_F(Y)$ is the space of the finite nonnegative measures in $\mathcal{M}(Y)$ equipped with the weak topology. We say that a sequence of measures μ^n in $\mathcal{M}_F(Y)$ converges to μ in the weak topology and we write $\mu^n \stackrel{\mathcal{W}}{\to} \mu$ if and only if for each $f \in \mathcal{C}_b(Y, Y)$,

$$\langle f, \mu_n \rangle \to \langle f, \mu \rangle, \text{ as } n \to \infty,$$

where $\langle f, \mu \rangle := \int_Y f(y) \mu(dy)$. Weak convergence in $\mathcal{M}_F(Y)$ is equivalent to convergence in the Prokhorov metric: for $\mu, \nu \in \mathcal{M}_F(Y)$ and $\epsilon > 0$,

$$d(\mu, \nu) := \inf\{\epsilon : \mu(B) \le \nu(B^\epsilon) + \epsilon \text{ and } \nu(B) \le \mu(B^\epsilon) + \epsilon \\ \text{for any nonempty closed } B \subseteq Y\},$$

where B^ϵ is the ϵ-neighborhood of B, that is, $B^\epsilon := \{y \in Y : dist(y, B) \le \epsilon\}$. When $Y = \mathbb{R}^k$, then $dist(y, B) := \inf_{\boldsymbol{x} \in B} \|\boldsymbol{y} - \boldsymbol{x}\|$. For $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{M}_F(Y)^k$, define

$$d_k(\boldsymbol{\mu}, \boldsymbol{\nu}) := \max_{1 \le i \le k} d(\mu_i, \nu_i).$$

It is known that $(\mathcal{M}_F(Y)^k, d_k)$ is a separate and complete space (Billingsley 1995); that is, a Polish space. When $Y = \mathbb{R}_+^n$, we simplify the notation to $\mathcal{M}_F$.

2.2. Network and Infrastructure

We consider the typical situation where a low-voltage distribution network has a tree structure. Thus, take a rooted tree $\mathcal{G} = (\mathcal{I}, \mathcal{E})$, where $\mathcal{I} = \{0, 1, \dots, I\}$, denotes its set of nodes (buses) and $\mathcal{E}$ is its set of directed edges, assuming that node 0 is the root node (known as *feeder*). Denote by $\epsilon_{ik} \in \mathcal{E}$ the edge that connects node i to node k, assuming that i is closer to the root node than k. Let $\mathcal{I}(k)$ and $\mathcal{E}(k)$ be the node and edge set of the subtree rooted in node $k \in \mathcal{I}$. The active and reactive power consumed by the subtree $(\mathcal{I}(k), \mathcal{E}(k))$ are $P_{\mathcal{I}(k)}$ and $Q_{\mathcal{I}(k)}$. The resistance, the reactance, and the active and reactive power losses along edge ϵ_{ik} are denoted by r_{ik}, x_{ik}, L_{ik}^P, and L_{ik}^Q, respectively. Moreover, V_i is the voltage at node i and V_0 is known. At any node, except for the root node, there is a charging station with $K_i > 0$, $i \in \mathcal{I} \setminus \{0\}$, parking spaces (each having an EV charger). Furthermore, we assume that there are $\mathcal{J} = \{1, \ldots, J\}$ different types of EVs indexed by j.

2.3. Stochastic Model for EVs

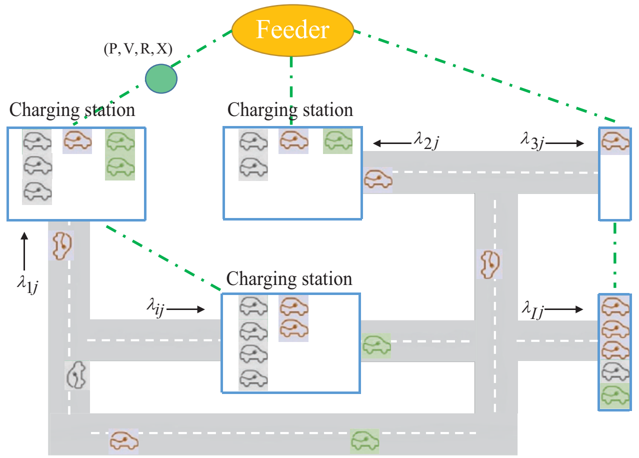
We start by giving a high-level description of the stochastic model and later we give a more detailed model description. We consider multiple types of EVs that arrive randomly at charging stations that have a finite number of parking spaces. If all spaces are occupied, a newly arriving EV does not enter the system but is assumed to leave immediately. Furthermore, it is assumed that each space has an EV charger that is connected to the power grid. Moreover, each EV has a random charging requirement and a random parking time (counted both in time). These depend on the type of the EV and the location that it is parked (i.e., the node), but are independent between EVs. Our framework is general enough to distinguish between types. For example, we can classify types according to intervals of the ratio of the charging requirement and parking time and/or according to the contract they have with the network provider. An EV leaves the system after its parking time expires and it may not be fully charged. We refer to these EVs as *uncharged EVs*. In contrast, if an EV finishes its charge, it remains at its parking space without consuming power until its parking time expires. EVs that have finished their charge are called *fully charged*. The model is illustrated in Figure 1.

2.4. Charging Control Rule

An important part of our framework is the way we specify how the charging of EVs takes place. Let the number of uncharged vehicles (of all types and in all nodes) be given by the vector $\boldsymbol{z} \in [0, \infty)^{I \times J}$; that is, z_{ij} is the number of uncharged vehicles of type j in node i. We assume the existence of a vector function $\boldsymbol{p}(\boldsymbol{z}) = (p_{ij}(\boldsymbol{z}) : i \in \mathcal{I} \setminus \{0\}, j \in \mathcal{J})$ that specifies the instantaneous rate of power each uncharged vehicle receives. Moreover, we assume that this function is obtained by optimizing a global function. Specifically, for a type j EV at node i, we associate a function $u_{ij}(\cdot)$, which is strictly increasing and concave in $\mathbb{R}_+$, twice differentiable in $(0, \infty)$ with $\lim_{x \to 0} u'_{ij}(x) = \infty$. The charging rate $\boldsymbol{p}(\boldsymbol{z})$ is then determined by $\max_{\boldsymbol{p}} \sum_{i=1}^{I} \sum_{j=1}^{J} z_{ij} u_{ij}(p_{ij})$ subject to a number of constraints that take into account physical limits on the charging of the batteries, load limits, and most importantly voltage drop constraints. An important example is the choice $u_{ij}(p_{ij}) = w_{ij} \log p_{ij}$, which is known as *weighted proportional fairness*. The greedy optimization problem we consider is an abstraction of what would happen in practice: optimizing the social welfare function acts as a proxy for a market mechanism. The field of network utility maximization provides ways on how to relate our abstract formulation to what happens in practice (e.g., by using the alternating direction method of multipliers; Ye and Yao 2012).

A limitation of our formulation is that it does not take into account the remaining time until the deadline expires and the remaining charging requirement. Our multiclass framework allows to us at least partially overcome this, for example, by letting the functions $u_{ij}(\cdot)$ depend on the joint distribution of parking and charging times. For instance, we can classify types j by the ratio of parking and charging times, and, in the context of proportional fairness, modify weights w_{ij} accordingly. Last, note that it is feasible to communicate an indication of parking and charging times by the owner of an EV at the parking lot Arif et al. (2016).

Figure 1. Network with $\mathcal{J}$ Types of EVs and Constant Arrival Rates



We next introduce the physical constraints of the network. The maximum electric power that can be consumed in total by all cars is $M_i > 0$ at node i. Each type j EV can be charged at a rate that is at most equal to $c_j^{\max}$. That is,

$$\sum_{j=1}^{J} z_{ij} p_{ij} \le M_i \quad \text{and} \quad 0 \le p_{ij} \le c_j^{max}. \tag{2.1}$$

We refer to (2.1) as *load constraints*. In addition, we impose *voltage drop constraints* that are important constraints in power systems and essentially say that the voltage magnitudes should not drop a lot. This helps the power system to avoid voltage collapse, which can lead to blackouts (Simpson-Porco et al. 2016). These constraints rely on the load-flow model used. Two of these models that we consider are described next where we give a brief overview. For more details, we refer to Bienstock (2015), which provides a connection between electrical transmission systems and operations research. Moreover, for a textbook on power systems with an emphasis on mathematical aspects, see Machowski et al. (2008).

2.4.1. AC Voltage Model. An electric grid is a connected network for transferring electricity from producers to consumers. It consists of *generating stations* that produce electric power, *high voltage transmission lines* that carry power from distant sources to demand centers, and *distribution lines* that connect individual customers, for example, houses and electric chargers. The nodes that produce electricity are called *generators* and the ones that consume power are called *load nodes*. In the sequel, we focus on a network with only distribution lines, which is called a *distribution network* or a *low voltage network*. A distribution network usually consists of a neighborhood or a small town.

The AC power flow equations are one of the most widely used modeling tools in power systems. They characterize the steady-state relationship between loads at each node, the voltage magnitudes, and the phase angles that are necessary to transmit power from generators to load nodes. As distribution networks are typically radial, we focus only on tree networks with load nodes and a feeder (or root node) that generates power.

The power flow equations essentially say that the power flow at any edge should be balanced, and there are several equivalent ways to describe them (Low 2014a, Gan et al. 2015). In the sequel, we focus on the so-called *bus injection model* which is a variation of Kirchhoff's law. We consider a simplification of the full AC power flow equations, based on the typical situation that voltage angle differences in distribution networks are negligible (Kersting 2012, chapter 3). Under this assumption, Kirchhoff's law for a tree network (Low 2014a, equation (1)) takes the form, for $\epsilon_{pk} \in \mathcal{E}$,

$$V_p V_k - V_k V_k - P_{\mathcal{I}(k)} r_{pk} - Q_{\mathcal{I}(k)} x_{pk} = 0, \tag{2.2}$$

where $p \in \mathcal{I}$ denotes the unique parent of node k. The previous equations are nonlinear. Applying the transformation,

$$\boldsymbol{W}(\epsilon_{pk}) = \begin{pmatrix} V_p^2 & V_p V_k \\ V_k V_p & V_k^2 \end{pmatrix} =: \begin{pmatrix} W_{pp} & W_{pk} \\ W_{kp} & W_{kk} \end{pmatrix}$$

leads to linear equations (in terms of $\boldsymbol{W}(\epsilon_{pk})$),

$$W_{pk} - W_{kk} - P_{\mathcal{I}(k)} r_{pk} - Q_{\mathcal{I}(k)} x_{pk} = 0,\ \epsilon_{pk} \in \mathcal{E}. \tag{2.3}$$

Note that $\boldsymbol{W}(\epsilon_{pk})$ are positive semidefinite matrices (denoted by $\boldsymbol{W}(\epsilon_{pk}) \succeq 0$) of rank one. The active and reactive power consumed by the subtree $(\mathcal{I}(k), \mathcal{E}(k))$ are given by

$$\begin{aligned} P_{\mathcal{I}(k)} &= \sum_{l \in \mathcal{I}(k)} \sum_{j=1}^{J} z_{lj} p_{lj} + \sum_{l \in \mathcal{I}(k)} \sum_{\epsilon_{ls} \in \mathcal{E}(k)} L_{ls}^P, \\ Q_{\mathcal{I}(k)} &= \sum_{l \in \mathcal{I}(k)} \sum_{\epsilon_{ls} \in \mathcal{E}(k)} L_{ls}^Q, \end{aligned} \tag{2.4}$$

whereby (Carvalho et al. 2015, appendix B),

$$L_{ls}^P = (W_{ll} - 2W_{ls} + W_{ss}) r_{ls} / (r_{ls}^2 + x_{ls}^2),$$
$$L_{ls}^Q = (W_{ll} - 2W_{ls} + W_{ss}) x_{ls} / (r_{ls}^2 + x_{ls}^2).$$

Note that W_{kk} are dependent on the vectors $\boldsymbol{p}$ and $\boldsymbol{z}$. We sometimes write $W_{kk}(\boldsymbol{p}, \boldsymbol{z})$ when we wish to emphasize the dependence. The AC power flow equations cannot be solved explicitly because the set of possible solutions (i.e., the feasible set) of the AC power flow equations is usually nonconvex and can be extremely complicated (Hiskens and Davy 2001, Lavaei et al. 2014). Because the power flow equations are nonlinear, solutions may not exist, and even when a solution exists, there may be multiple solutions. In more general and realistic distribution networks, there is typically a unique "high-voltage" solution (Molzahn et al. 2016), which is assumed to be a desired operating point for the network (Dörfler et al. 2013).

Now, the power allocation function $\boldsymbol{p}(\boldsymbol{z})$ is given by the following optimization problem:

$$\begin{aligned} \max_{\boldsymbol{p},\boldsymbol{W}} \quad & \sum_{i=1}^{I}\sum_{j=1}^{J} z_{ij}u_{ij}(p_{ij}) \\ \text{subject to} \quad & (2.1),(2.3),\ \underline{v}_i \le W_{ii} \le \overline{v}_i, \\ & \boldsymbol{W}(\epsilon_{ik}) \ \ge\ 0,\ \text{rank}(\boldsymbol{W}(\epsilon_{ik})) = 1,\ \epsilon_{ik} \in \mathcal{E}, i,k \in \mathcal{I}, \end{aligned} \tag{2.5}$$

for $z_{ij} > 0$. If $z_{ij} = 0$, then $p_{ij} = 0$. Because of our assumptions of the utility functions, we have that if $z_{ij} > 0$, then the optimal solution of (2.5) satisfies $p_{ij}(\boldsymbol{z}) > 0$. In addition, $0 < \underline{v}_k \le W_{00} \le \overline{v}_k$ are the voltage limits and the inequalities hold for all $k \in \mathcal{I}$. Observe that the optimization problem (2.5) is nonconvex and in general NP-hard because of the rank-one constraints. Removing the nonconvex constraints yields a convex relaxation, which is a second-order cone program, namely

$$\begin{aligned} \max_{\boldsymbol{p},\boldsymbol{W}} \quad & \sum_{i=1}^{I}\sum_{j=1}^{J} z_{ij}u_{ij}(p_{ij}) \\ \text{subject to} \quad & (2.1),(2.3),\ \underline{v}_i \le W_{ii} \le \overline{v}_i, \\ & \boldsymbol{W}(\epsilon_{ik}) \ge 0,,\ \epsilon_{ik} \in \mathcal{E}, i,k \in \mathcal{I}. \end{aligned} \tag{2.6}$$

We denote the feasible set of (2.6) by $\mathfrak{F}(\boldsymbol{z})$. Furthermore, by Remark 2.1 and Low (2014b, theorem 5), we obtain that the convex relaxation problem is exact; that is, both problems have the same set of solutions. In other words, a solution of the convex relaxation problem satisfies the rank-one constraints and so we can work with the relaxed problem. Defining the bandwidth allocation function $\boldsymbol{\Lambda}(\boldsymbol{z}) := \boldsymbol{p}(\boldsymbol{z}) \circ \boldsymbol{z}$, that is, $\Lambda_{ij}(\boldsymbol{z}) = p_{ij}(\boldsymbol{z})z_{ij}$ for $i, j \ge 1$, the optimization problem (OP) (2.6) takes the following equivalent form:

$$\begin{aligned} \max_{\boldsymbol{\Lambda},\boldsymbol{W}} \quad & \sum_{i=1}^{I}\sum_{j=1}^{J} z_{ij}u_{ij}(\Lambda_{ij}/z_{ij}) \\ \text{subject to} \quad & \sum_{j=1}^{J} \Lambda_{ij} \le M_i, \quad 0 \le \Lambda_{ij} \le z_{ij}c_j^{max}, \\ & (2.3),\ \underline{v}_i \le W_{ii}(\boldsymbol{\Lambda}) \le \overline{v}_i, \\ & \boldsymbol{W}(\epsilon_{ik}) \ge 0,\ \epsilon_{ik} \in \mathcal{E}, i,k \in \mathcal{I}. \end{aligned} \tag{2.7}$$

The constraints $\boldsymbol{W}(\epsilon_{ik}) \ge 0$ are equivalent to $W_{ii}W_{kk} - W_{ik}^2 \ge 0$, because we consider $W_{ii} > 0$ for any node $i \ge 1$. In the sequel, we freely use both formulations.

2.4.2. Linearized Distflow Model. As mentioned previously, the AC power flow equations are not linear, which is especially because of power losses. Although AC power flow model is tractable enough for a convex relaxation to be exact, it is rather complicated. In Baran and Wu (1989), a simplification of these equations is suggested, assuming that the active and reactive power losses on edges are small relative to the power flows. This is a linear approximation of the AC power flow model, called the *linearized (or simplified) Distflow model*. Moreover, this is a widely used approximation in distribution networks and leads to a good approximation of AC power flow model because the power losses are typically much smaller than the power flows along the edges (Low 2014a, section VI). In this case, the voltage magnitudes $W_{kk}^{lin} := |V_k^{lin}|^2$ have an analytic expression (Low 2014a, lemma 12):

$$W_{kk}^{lin}(\boldsymbol{p},\boldsymbol{z}) = W_{00} - 2\sum_{\epsilon_{ls}\in\mathcal{P}(k)} r_{ls} \sum_{m\in\mathcal{I}(s)}\sum_{j=1}^{J} z_{mj}p_{mj}, \tag{2.8}$$

where the $\mathcal{P}(k)$ is the unique path from the feeder to node k.

Remark 2.1. Note that $W_{kk}^{lin} \le W_{00}$ for all nodes k, as we assume that the nodes only consume power, and by Low (2014a, lemma 12) we obtain $W_{kk}(\boldsymbol{p},\boldsymbol{z}) \le W_{kk}^{lin}(\boldsymbol{p},\boldsymbol{z})$. That is, we can remove the constraints $W_{kk}(\boldsymbol{\Lambda}) \le \overline{v}_k$ from (2.5).

To derive the representation of the power allocation mechanism $\boldsymbol{p}(\boldsymbol{z})$ in this setting, one replaces the constraints in (2.5) by (2.1) and $\underline{v}_k \le W_{kk}^{lin}(\boldsymbol{p},\boldsymbol{z})$.

2.5. State Descriptor

In this section, we first give a detailed model description introducing the essential notation, and then we move to the definition of the state descriptor.

Type j EVs arrive at node i according to a counting process $E_{ij}(\cdot) := \{E_{ij}(t), t \geq 0\}$; that is, $E_{ij}(t)$ is the number of EVs that arrive to the parking lot in the time interval $(0, t]$. We assume that all $E_{ij}(\cdot)$ are finite, nondecreasing processes with $E_{ij}(0) = 0$, $E_{ij}(t) - E_{ij}(t^-) \in \{0, 1\}$, and $\mathbb{E}[E_{ij}(t)] = \int_0^t \lambda_{ij}(s)ds$, where $\lambda_{ij}(s) > 0$ are integrable functions. Moreover, let ζ_{ijl} denote the arrival time of the l^{th} type j EV at node i. If all spaces are occupied, a newly arriving EV does not enter the system, but is assumed to leave immediately.

Let B_{ijl} and D_{ijl} denote the charging requirement and the parking time of the l^{th} EV of type j at node i. In queueing terminology, these quantities are respectively called *service requirements* and *deadlines*. Moreover, we assume that the sequence $\{B_{ijl}, D_{ijl}, l \in \mathbb{N}\}$ is a sequence of independent and identically distributed (i.i.d.) copies of a random vector (B_{ij}, D_{ij}) with distribution law $F_{ij}(A) = \mathbb{P}((B_{ij}, D_{ij}) \in A)$ for any Borel set $A \in \mathcal{B}(\mathbb{R}_+^2)$. Furthermore, for $l = 1, \ldots, Q_{ij}(0)$ we denote by (B^0_{ijl}, D^0_{ijl}) the residual charging requirement and the residual parking time of the initial population of type j at node i, that is, $Q_{ij}(0)$. Moreover, we assume the probability density function (pdf) of the parking times $f_{D_{ij}}(\cdot)$ exists with $f_{D_{ij}}(0) > 0$ for any $i, j \geq 1$. We shall see later that the deadlines are associated with the stochastic process that describes the total EVs at each node and it represents the population of a loss system. The latter is studied in Kang (2015), where a similar assumption of existence of the pdf is made.

In the sequel, we introduce the dynamics that describe the evolution of the system. Specifically, we incorporate in the system dynamics all residual processes needed to obtain a Markovian system. Let $\mathcal{Q}_{ij}(\cdot)$ and $\mathcal{Z}_{ij}(\cdot)$ be nonnegative discrete measures for $i, j \geq 1$. The total number of type j EVs at node i at time $t > 0$ and the number of uncharged EVs are given by $Q_{ij}(t) = \langle 1, \mathcal{Q}_{ij}(t)\rangle$ and $Z_{ij}(t) = \langle 1, \mathcal{Z}_{ij}(t)\rangle$, respectively. Moreover, $Q_i(t) := \sum_{j=1}^J Q_{ij}(t)$ gives the total number of EVs at node $i \geq 1$.

Recall that ζ_{ijl} is the arrival time of the l^{th} EV of type j at node i. The residual parking time of the l^{th} newly arriving EV of type j can be written as $D_{ijl}(t) := (D_{ijl} - (t - \zeta_{ijl}))^+$, $l = 1, \ldots, E_{ij}(t)$ and for the initial population $D^0_{ijl}(t) := (D^0_{ijl} - t)^+$, $l = 1, \ldots, Q_{ij}(0)$. To define the residual charging requirements, we first introduce the following operators:

$$S_{ij}(z, s, t) = \int_s^t p_{ij}(z(u))du, \tag{2.9}$$

where $p_{ij}(z(u))$ is the optimal solution of (2.6) if there are $z(u)$ uncharged EVs at time $u \geq 0$. For $s \leq t$, $S_{ij}(\mathbf{Z}, s, t)$ is the *cumulative bandwidth* allocated per type j EV at node i during time interval $[s, t]$. The residual charging requirement of the l^{th} type j EV at node i at time $t \geq 0$ is given by

$$B_{ijl}(t) = \left(B_{ijl} - S_{ij}(\mathbf{Z}, \zeta_{ijl}, t)\right)^+,$$

for the newly arriving EVs and $B^0_{ijl}(t) = (B^0_{ijl} - S_{ij}(\mathbf{Z}, 0, t))^+$, $l = 1, \ldots, Z_{ij}(0)$, for the initially uncharged EVs. Now, we define the measure-valued state descriptor for any $t \geq 0$ and for any Borel set $B \subseteq \mathbb{R}$,

$$\mathcal{Q}_{ij}(t)(B) := \sum_{l=1}^{Q_{ij}(0)} \delta^+_{D^0_{ijl}(t)})B) + \sum_{l=1}^{E_{ij}(t)} \delta^+_{D_{ijl}(t)}(B)\, \mathbb{1}_{\{Q_i(\zeta^-_{ijl}) < K_i\}}. \tag{2.10}$$

The measure $\delta^+_{\cdot}(B)$ is the Dirac measure restricted on $(0, \infty)$; THAT IS, $\delta^+_x(B) := \delta_x(B \cap (0, \infty))$ and $\delta_x(B) = 1$ if $x \in B$. The measure $\mathcal{Q}_{ij}(t)(B)$ counts the total number of type j EVs in node i whose residual parking time belongs to the Borel set B.

The number of uncharged EVs for which the minimum between their residual charging requirement and their residual parking time belongs to any Borel set $B' \subseteq \mathbb{R}^2$ is given by

$$\mathcal{Z}_{ij}(t)(B') := \sum_{l=1}^{Z_{ij}(0)} \delta^+_{\left(B^0_{ijl}(t), D^0_{ijl}(t)\right)}(B') + \sum_{l=1}^{E_{ij}(t)} \delta^+_{\left(B_{ijl}(t), D_{ijl}(t)\right)}(B')\, \mathbb{1}_{\{Q_i(\zeta^-_{ijl}) < K_i\}}. \tag{2.11}$$

The measure $\delta^+_{(\cdot,\cdot)}(B')$ is the Dirac measure restricted on $(0, \infty)^2$; that is, $\delta^+_{(x_1,x_2)}(B') := \delta_{(x_1,x_2)}(B' \cap (0, \infty)^2)$ and $\delta_{(x_1,x_2)}(B') = 1$ if $x_1 \wedge x_2 \in B'$. Last, $\{Q_i(\zeta^-_{ijl}) < K_i\}$ represents the event that there is an idle EV charger right before the arrival of the l^{th} type j EV. As not all EVs enter the system, we naturally define the following stochastic processes. First, the number of accepted type j EVs at node i until time $t > 0$ is given by

$$A_{ij}(t) = \sum_{l=1}^{E_{ij}(t)} \mathbb{1}_{\{Q_i(\zeta_{ijl}^-)<K\}}. \tag{2.12}$$

Next, the number of rejected EVs until time $t > 0$ is given by

$$R_{ij}(t) = \sum_{l=1}^{E_{ij}(t)} \mathbb{1}_{\{Q_i(\zeta_{ijl}^-)=K\}}. \tag{2.13}$$

Observe that the following relation holds: $A_{ij}(t) + R_{ij}(t) = E_{ij}(t)$.

Having introduced the stochastic model, which is defined through Equations (2.10)–(2.13), we move to the main results of this paper. We first study some properties of the bandwidth allocation function in Section 3. We then define an appropriate fluid model in Section 4 and derive some of its properties.

3. Continuity of the Optimal Allocation Function

In this short section, we state some structural properties of the optimal allocation function, which may be of independent interest. In particular, we show that the optimal solution of (2.7) is continuous under the AC power flow model (2.3). This result is needed in Section 5 to show convergence of the fluid-scaled processes. Last, in power system analysis, rigorous proofs are typically difficult and require additional assumptions on the distribution system (Dvijotham et al. 2017), even if one ignores the stochastic dynamics. In the rest of this section, we make an additional assumption for the ratio of resistance and reactance. That is, $\frac{r_{pk}}{x_{pk}} \geq \frac{r_{kl}}{x_{kl}}$ for all $\epsilon_{pk}, \epsilon_{kl} \in \mathcal{E}$. This assumption is reasonable in distribution networks because in most networks the resistances are decreasing as we move away from the feeder since the power losses should be kept small. Similar assumptions are made in Dvijotham et al. (2017) and Low (2014b).

We show that the optimal aggregated power allocation $\boldsymbol{\Lambda}(z)$, $z \in (0, \infty)^{I\times J}$, is a continuous function in z. To establish this property, we first present a preliminary result.

Proposition 3.1. *Let $z \in [0, \infty)^{I\times J}$ and $\boldsymbol{\Lambda}(z)$ be a feasible point of* (2.7). *Given a point $\mathbf{0} \leq \boldsymbol{\Lambda}' \leq \boldsymbol{\Lambda}(z)$, we have that $\boldsymbol{\Lambda}'$ is also a feasible point of* (2.7).

Observe that in case the feasible set of (2.7) is polyhedral, the conclusion is immediate. The proof of the previous proposition is given in Section 6. The main idea of the proof is to construct a new solution $(\boldsymbol{W}', \boldsymbol{\Lambda}')$. Then, using the feasibility of the point $\boldsymbol{\Lambda}(z)$ and induction starting from the leaf nodes, we show that the point $(\boldsymbol{W}', \boldsymbol{\Lambda}')$ lies in the feasible set of (2.7). In the sequel, we present the main result of this section, which says that $\boldsymbol{\Lambda}(\cdot)$ is a continuous function.

Theorem 3.1 (Continuity). *Let $\boldsymbol{\Lambda}(z)$ for $z \in (0, \infty)^{I\times J}$ be the unique optimal solution of* (2.7). *We have that $\boldsymbol{\Lambda}(z)$ is a continuous function in $(0, \infty)^{I\times J}$.*

The proof of Theorem 3.1 is given in Section 6, and it combines Proposition 3.1, the continuity property of the voltages as functions of loads, and arguments from Remerova et al. (2014, lemma 1).

If we assume the linearized Distflow power model (see Section 2.4), then the feasible set of (2.7) is polyhedral, and we can show the stronger result that $\boldsymbol{\Lambda}(\cdot)$ is Lipschitz continuous by applying directly (Reed and Zwart 2014, theorems 3.1 and 3.2). In this case, we do not need any additional assumption of the ratio of resistance and reactance. The same Lipschitz continuity property holds in the case of the AC power flow model under an additional assumption that the strict complementary condition holds for some constraints as it is shown in Aveklouris (2020, proposition 5.11.2). We summarize these in the following lemma.

Lemma 3.1. *Suppose one of the following holds:*

(i) *Consider the linearized Distflow model.*

(ii) *Consider the AC power flow model and assume that the strict complementary condition holds for the constraints $W_{pk}^2(z) - W_{pp}(z)W_{kk}(z) \leq 0$ for any $\epsilon_{pk} \in \mathcal{E}$ and $z \in (0, \infty)^{I\times J}$, that is, the Lagrange multipliers that correspond to these constraints are strictly positive.*

Then, the function $\boldsymbol{\Lambda}(\cdot)$ is (locally) Lipschitz on $(0, \infty)^{I\times J}$.

Although in the case of AC flow model, we have not been able to establish this property without the aforementioned assumption, we conjecture that $\boldsymbol{\Lambda}(\cdot)$ is Lipschitz continuous and leave this question open. The Lipschitz continuity property of the power allocation function will be needed in Theorem 4.1, where we prove that the

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fluid model has a unique solution. However, we point out that the continuity property is enough to show that an accumulation point of the fluid-scaled state descriptor satisfies the fluid model equations in Section 5.

We now move to the original stochastic network and its fluid model.

4. Fluid Model Definition

In this section, we define and study the properties of a deterministic fluid model, associated with the stochastic model introduced in Section 2. All proofs of this section are gathered in Section 7.

Define the following classes

$$\mathcal{C} := \{[x,\infty),\ x \in \mathbb{R}_+\}$$

and

$$\mathcal{C}' := \{[x,\infty) \times [y,\infty),\ x, y \in \mathbb{R}_+\}.$$

Further, for any $A \in \mathcal{C}$ and $s \in \mathbb{R}$, define $A + s := \{y + s, [y,\infty) \in A\}$ and for any $A' \in \mathcal{C}'$ and $(s,t) \in \mathbb{R}^2$, define $A' + (s,t) := \{s + [x,\infty) \times t + [y,\infty), [x,\infty) \times [y,\infty) \in A'\}$.

Definition 4.1 (Fluid Model). Let the initial data for the fluid model be given by

$$\left(\overline{E}(\cdot), \overline{\mathcal{Q}}(0), \overline{\mathcal{Z}}(0)\right) \in C(\mathbb{R}_+,\mathbb{R}_+) \times \mathcal{M}_F^{I\times J} \times \mathcal{M}_F^{I\times J},$$

where $\overline{E}_{ij}(t) = \int_0^t \lambda_{ij}(s)ds$. We say that the vector

$$\left(\overline{\mathcal{Q}}(\cdot), \overline{\mathcal{Z}}(\cdot), \overline{Q}(\cdot), \overline{Z}(\cdot)\right) \in C(\mathbb{R}_+, \mathcal{M}_F^{I\times J})^2 \times C(\mathbb{R}_+, R_+^{I\times J})^2$$

is a fluid model solution if $\overline{Q}_{ij}(t) = \langle 1, \overline{\mathcal{Q}}_{ij}(t)\rangle$, $\overline{Z}_{ij}(t) = \langle 1, \overline{\mathcal{Z}}_{ij}(t)\rangle$, and if there exist nondecreasing nonnegative continuous functions $\overline{R}_i(\cdot)$, $\overline{R}_{ij}(\cdot)$ such that

$$\overline{R}_i(t) = \int_0^t 1_{\{\overline{Q}_i(s)=K_i\}} d\overline{R}_i(s) \text{ and } \overline{R}_{ij}(t) = \int_0^t \frac{\lambda_{ij}(s)}{\sum_{h=1}^J \lambda_{ih}(s)} d\overline{R}_i(s).$$

Furthermore, for any $t \geq 0$, $A \in \mathcal{C}$, and $A' \in \mathcal{C}'$ the following relations hold:

$$\begin{aligned}\overline{\mathcal{Q}}_{ij}(t)(A) = \overline{Q}_{ij}(0)\mathbb{P}(D_{ij}^0 \in A + t) + \int_0^t \mathbb{P}(D_{ij} \in A + (t-s))d\overline{E}_{ij}(s) \\ - \int_0^t \mathbb{P}(D_{ij} \in A + (t-s))d\overline{R}_{ij}(s),\end{aligned} \tag{4.1}$$

$$\begin{aligned}\overline{\mathcal{Z}}_{ij}(t)(A') = \overline{Z}_{ij}(0)\mathbb{P}((B_{ij}^0, D_{ij}^0) \in A' + (S_{ij}(z,0,t),t)) \\ + \int_0^t \mathbb{P}((B_{ij}, D_{ij}) \in A' + (S_{ij}(\overline{\mathbf{Z}},s,t), t-s))d\overline{E}_{ij}(s) \\ - \int_0^t \mathbb{P}((B_{ij}, D_{ij}) \in A' + (S_{ij}(\overline{\mathbf{Z}},s,t), t-s))d\overline{R}_{ij}(s).\end{aligned}$$

Moreover, the functions $\overline{Q}_{ij}(\cdot) = \langle 1, \overline{\mathcal{Q}}_{ij}(\cdot)\rangle = \overline{\mathcal{Q}}_{ij}(\cdot)(\mathbb{R}_+)$ *and* $\overline{Z}_{ij}(\cdot) = \langle 1, \overline{\mathcal{Z}}_{ij}(\cdot)\rangle = \overline{\mathcal{Z}}_{ij}(\cdot)(\mathbb{R}_+^2)$ are given by

$$\overline{Q}_{ij}(t) = \overline{Q}_{ij}(0)\mathbb{P}(D_{ij}^0 \geq t) + \int_0^t \mathbb{P}(D_{ij} \geq t-s)d\overline{E}_{ij}(s) - \int_0^t \mathbb{P}(D_{ij} \geq t-s)d\overline{R}_{ij}(s) \tag{4.2}$$

and

$$\begin{aligned}\overline{Z}_{ij}(t) = \overline{Z}_{ij}(0)\mathbb{P}\left(B_{ij}^0 \geq S_{ij}(\overline{\mathbf{Z}},0,t), D_{ij}^0 \geq t\right) + \int_0^t \mathbb{P}(B_{ij} \geq S_{ij}(\overline{\mathbf{Z}},s,t), D_{ij} \geq t-s)d\overline{E}_{ij}(s) \\ - \int_0^t \mathbb{P}(B_{ij} \geq S_{ij}(\overline{\mathbf{Z}},s,t), D \geq t-s)d\overline{R}_{ij}(s).\end{aligned}$$

We call the vectors $(\overline{\mathcal{Q}}(\cdot), \overline{\mathcal{Z}}(\cdot))$ and $(\overline{\mathbf{Q}}(\cdot), \overline{\mathbf{Z}}(\cdot))$ the *measure-valued fluid model solution* and the *numeric fluid model solution*, respectively.

The fluid model equations, although still rather complicated, have an intuitive meaning. For instance, the term $\mathbb{P}(B_{ij} \geq S_{ij}(\overline{\mathbf{Z}},s,t), D_{ij} \geq t-s)$ represents the fraction of EVs of type j admitted to the system at time s at node i that are still in the system at time t. For this to happen, their deadline needs to exceed $t - s$ and their service

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requirement needs to be bigger than the service allocated, which is $S_{ij}(\overline{\mathbf{Z}}, s, t)$. In addition, $\overline{R}_{ij}(t)$ represents the lost fluid of type j EVs at node i due to a full system until time $t \geq 0$.

Remark 4.1. The sets $\mathcal{C}$ *and* $\mathcal{C}'$ generate the Borel $\sigma-$algebra of $\mathbb{R}$ *and* $\mathbb{R}^2$, respectively. Then, by Dynkin's π-λ theorem, the fluid model solutions hold for any Borel set. See section 2.3 in Gromoll et al. (2008) for more details. Moreover, by Remerova et al. (2014, remark 3.2), fluid model solutions are invariant with respect to time shifts.

We next show that the total number of EVs in the fluid model can be rewritten in a familiar form for queueing systems and the departure process in the fluid model can be written as a function of the total number of EVs.

Proposition 4.1. *We have that for any* $i \geq 1$ *and* $j \geq 1$,

$$\overline{Q}_{ij}(t) = \overline{Q}_{ij}(0) + \overline{E}_{ij}(t) - \overline{R}_{ij}(t) - \overline{D}_{ij}(t), \tag{4.3}$$

where $\overline{D}_{ij}(t)$ *represents the amount of fluid that departs from the system in time interval* $[0, t)$, *and*

$$\overline{D}_{ij}(t) = \int_0^t \lim_{\epsilon \to 0} \frac{\overline{Q}_{ij}(s) - \overline{\mathcal{Q}}_{ij}(s)([\epsilon, \infty))}{\epsilon} ds < \infty. \tag{4.4}$$

The last proposition uses the assumption of existence of the density of the parking times to ensure that the limit in (4.4) exists, and this is the only point where we need this assumption. It follows from Proposition 4.1 that the total number of EVs can be written with the help of a one-dimensional reflection mapping. This result will be helpful when we show uniqueness of the fluid model solution in Theorem 4.1. The novelty in our setting is (4.4), where an intuitive explanation is as follows. The difference $\overline{Q}_{ij}(s) - \overline{\mathcal{Q}}_{ij}(s)([\epsilon, \infty))$ represents the amount of fluid of type j EVs at node i for which its residual parking time lies in the interval $(0, \epsilon)$. It is natural now to expect that by dividing the last difference by $\epsilon > 0$ and by allowing ϵ to be arbitrary small, the quantity $\lim_{\epsilon \to 0} \frac{\overline{Q}_{ij}(s) - \overline{\mathcal{Q}}_{ij}(s)([\epsilon, \infty))}{\epsilon}$ represents the departure rate of an EV from the parking lot at time $s > 0$. Observe that (4.4) corresponds to Kang (2015, equation (3.2)). However, in the latter, the authors use different test functions to define the fluid model and they write the departure rate in terms of the hazard rate function.

Before we continue our analysis, we present an example in case of a Markovian model.

Example 4.1 (Markovian Model). Consider a Markovian model (i.e., Poisson arrival process with constant arrival rate and exponential parking times), and take $J = 1$ and $\overline{Q}_i(0) = 0$ for convenience. We shall show that the departure process given in (4.4) can be written in the well-known form for a Markovian model (Pang et al. 2007), namely

$$\overline{D}_i(t) = \int_0^t \lim_{\epsilon \to 0} \frac{\overline{Q}_i(s) - \overline{\mathcal{Q}}_i(s)([\epsilon, \infty))}{\epsilon} ds = \frac{1}{\mathbb{E}[D_i]} \int_0^t \overline{Q}_i(s) ds. \tag{4.5}$$

To show (4.5), use (4.2) and $A = [\epsilon, \infty)$ in (4.1) to get

$$\overline{\mathcal{Q}}_i(t)([\epsilon, \infty)) = \overline{Q}_i(t) e^{-\epsilon/\mathbb{E}[D_i]}.$$

Observing that $\lim_{\epsilon \to 0} \frac{1 - e^{-\epsilon/\mathbb{E}[D_i]}}{\epsilon} = \frac{1}{\mathbb{E}[D_i]}$, we derive

$$\begin{aligned}\int_0^t \lim_{\epsilon \to 0} \frac{\overline{Q}_i(s) - \overline{\mathcal{Q}}_i(s)([\epsilon, \infty))}{\epsilon} ds &= \lim_{\epsilon \to 0} \frac{1 - e^{-\epsilon/\mathbb{E}[D_i]}}{\epsilon} \int_0^t \overline{Q}_i(s) ds \\ &= \frac{1}{\mathbb{E}[D_i]} \int_0^t \overline{Q}_i(s) ds.\end{aligned}$$

Two important questions are when a solution of the fluid model equations exists and if it exists when it is unique. The next theorem answers these questions.

Theorem 4.1. *Assume that* $\overline{Q}_{ij}(0) > 0$ *for all* $j \in \mathcal{J}$ *if* $\overline{Q}_i(0) = K_i$ *and that* (i) *or* (ii) *in Lemma* 3.1 *holds. Suppose that* $\overline{\mathbf{Z}}(0) = \mathbf{0}$ *or that* $\overline{\mathbf{Z}}(0) \in (0, \infty)^{I \times J}$ *and the first projection of* $\overline{\mathcal{Z}}(0)$ *is Lipschitz continuous; that is, there exists* $L > 0$ *such that for any* $i, j \geq 1$, $x < x'$, *and* $y > 0$,

$$\overline{\mathcal{Z}}_{ij}(0)([x, x'] \times [y, \infty)) \leq L(x' - x).$$

Then there exists a unique solution of the fluid model equations.

The proof of Theorem 4.1 is given in Section 7 and the main steps of the proof are as follows.

1. The first step is to show that each pair $(\overline{Q}_i(\cdot), \overline{R}_i(\cdot))$ satisfies a one-dimensional reflection mapping and (each pair) is unique.
2. Second, we show that $\overline{Z}_{ij}(t) > 0$ for any $i, j \geq 1$ and $t > 0$.
3. Last, we prove that $(\overline{\mathcal{Z}}(\cdot), \overline{\boldsymbol{Z}}(\cdot))$ is also unique using arguments from Remerova et al. (2014).

We conclude this section by presenting numerical results based on the fluid model. Next, in Section 5, we show that a fluid model solution arises as a weak limit of the original stochastic model under an appropriate scaling.

4.1. Numerical Analysis

Consider a line network with three nodes (one generator and two consumers) under the linearized Distflow model and Markovian assumptions. We assume a single type of EVs and a weighted proportional fairness power allocation, that is, $u_i(p) = w_i log(p)$ with $w_1 = 0.01$ and $w_2 = 0.02$. We allow the voltage drop to diverge at most 10% from its nominal value and denote by $1/\nu_i$ and $1/\mu_i$ the mean parking and charging times, respectively. Furthermore, we fix the resistance and reactance to 0.01 for both lines, $\boldsymbol{\nu} = (1,1)$, $\overline{\boldsymbol{Q}}(0) = (2,2)$, $\overline{\boldsymbol{Z}}(0) = (1,1)$, and for simplicity we remove the load constraints. In what follows, we focus on the performance metric that gives the fraction of fully charging EVs at any time (success probability).

We first study the system when both charging stations have the same number of parking spaces and the arrival rate is periodic, for example, $\lambda_1(t) = \lambda_2(t) = \sin(t) + 10$. Figure 2 shows the success probability as a function of time for different values of mean parking times. We note that the success probability is higher for the charging station with a smaller mean charging time. Furthermore, the success probability is symmetric to both stations the fact that can intuitively be explained as follows. The power grid can allocate the same amount of power to both nodes because of the weighted proportional fairness allocation and the higher weight for the second node.

The behavior of the system is similar if we increase the arrival rate (Figure 3). In this case, we observe that the success probability can vary, and it can be higher than the one in Figure 2. It can also be zero for the charging station with the higher mean charging time. That is, both the total number of EVs and the uncharged EVs reach the parking capacity K_i. In other words, the charging station cannot charge the EVs because of the high demand and the high charging times.

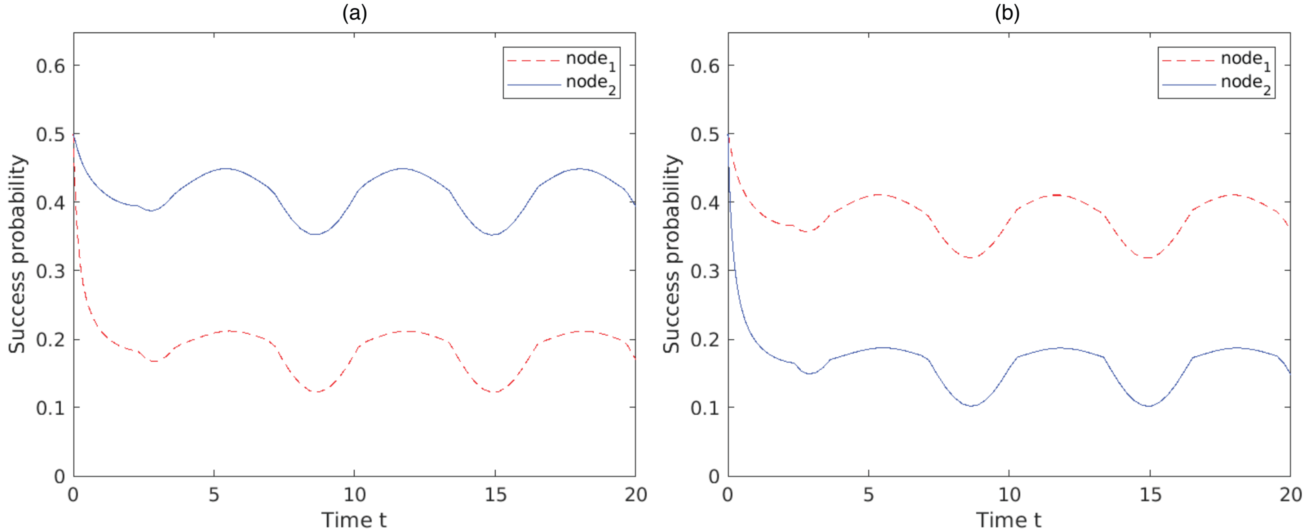
In Figure 4, we decrease the number of available parking spaces of the first charging station. This has an effect on its success probability which becomes smaller than the one in Figure 3 for both mean charging times.

Last, we note that the success probability (and hence the state descriptor) seems to converge to an invariant point when the arrival rate is constant as it is proven in Aveklouris (2020, section 5.6). We further observe that the success probability of the first node is smaller because its mean charging time is higher (Figure 5(a)) and its number of parking spaces is smaller (Figure 5(b)) than the second node.

5. Fluid Limit Theorem

In this section, we study the asymptotic behavior of the stochastic network described in Section 2. Consider a family of systems indexed by $n \in \mathbb{N}$, where n tends to infinity, with the same basic structure as that of the system

Figure 2. Success Probability for $\lambda_1(t) = \lambda_2(t) = \sin(t) + 10$ and $\boldsymbol{K} = (10, 10)$



Notes. (a) $\boldsymbol{\mu} = (1/2, 3/2)$. (b) $\boldsymbol{\mu} = (3/2, 1/2)$.

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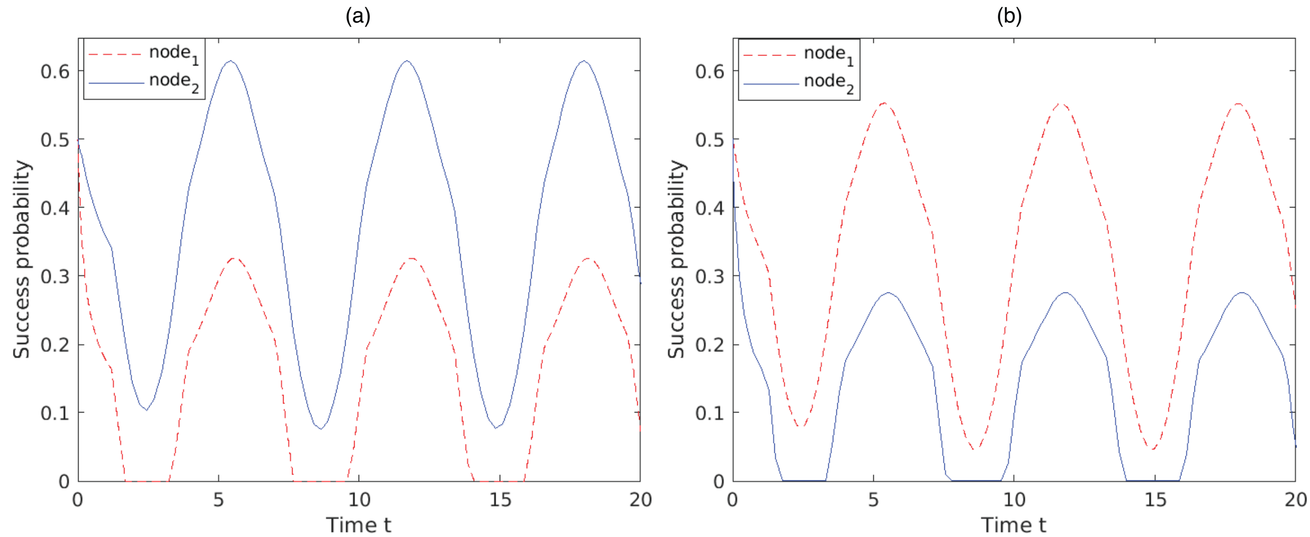
Figure 3. Success Probability for $\lambda_1(t) = \lambda_2(t) = 5\sin(t) + 10$ and $\boldsymbol{K} = (10, 10)$



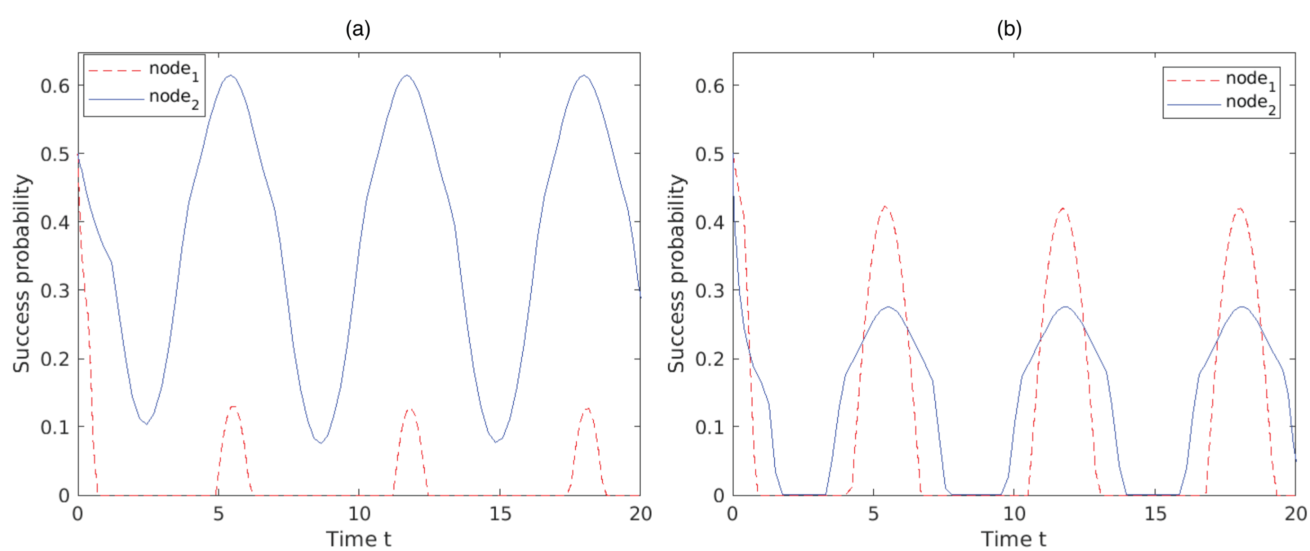
Notes. (a) $\boldsymbol{\mu} = (1/2, 3/2)$. (b) $\boldsymbol{\mu} = (3/2, 1/2)$.

described in Section 2. To indicate the position of the system in the sequence of systems, a superscript n will be appended to the system parameters and processes.

First, we introduce our asymptotic regime. We assume that the scaled capacity at node i is given by $M_i^n = nM$, the scaled number of EV chargers at node i is $K_i^n = nK$, and the scaled resistance and reactance online ϵ_{pk} are given by $r_{pk}^n = r_{pk}/n$ and $x_{pk}^n = x_{pk}/n$. In our setting, we need to scale the physical parameters of the system in contrast to the typical scalings in stochastic networks that arise in communication networks. The fluid-scaled measure-valued processes are given by $(\overline{\mathcal{Q}}^n(\cdot), \overline{\mathcal{Z}}^n(\cdot)) := \left(\frac{\mathcal{Q}^n(\cdot)}{n}, \frac{\mathcal{Z}^n(\cdot)}{n}\right)$ and the fluid-scaled counting processes are given by $(\overline{Q}^n(\cdot), \overline{Z}^n(\cdot)) := \left(\frac{Q^n(\cdot)}{n}, \frac{Z^n(\cdot)}{n}\right)$. We summarize the assumptions we make in this section.

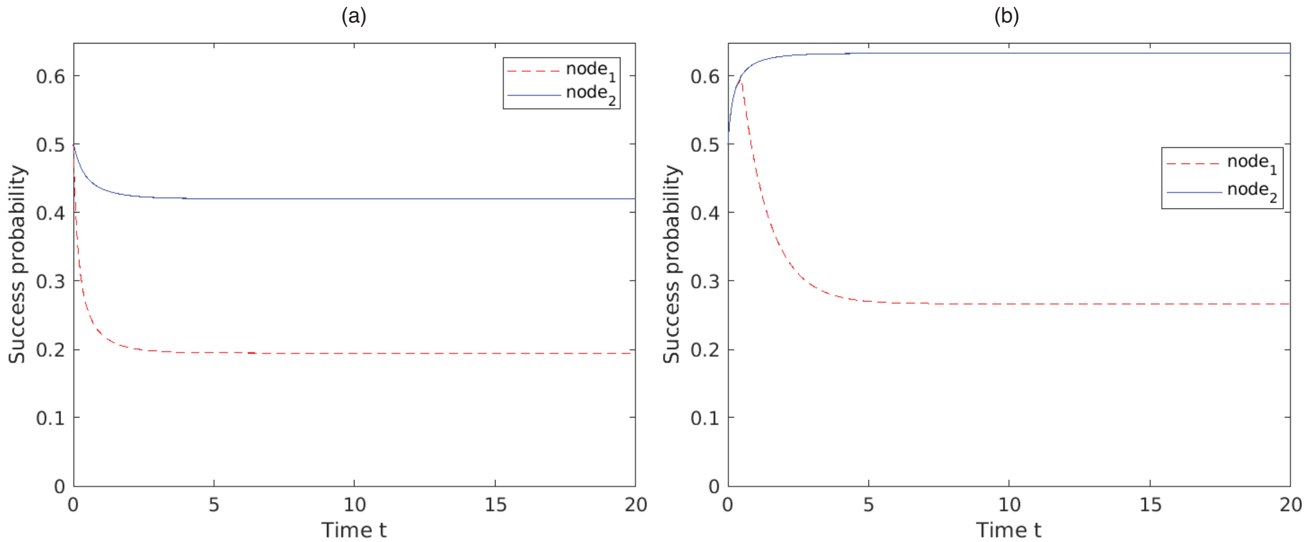
1. The scaling parameters are given by $K_i^n = nK$, $M_i^n = nM$, $r_{pk}^n = r_{pk}/n$, and $x_{pk}^n = x_{pk}/n$.
2. The external arrival process satisfies $\frac{E_{ij}^n(\cdot)}{n} \xrightarrow{d} \overline{E}_{ij}(\cdot)$, with $\overline{E}_{ij}(t) = \int_0^t \lambda_{ij}(s)ds$.

Figure 4. Success Probability for $\lambda_1(t) = \lambda_2(t) = 5\sin(t) + 10$ and $\boldsymbol{K} = (5, 10)$



Notes. (a) $\boldsymbol{\mu} = (1/2, 3/2)$. (b) $\boldsymbol{\mu} = (3/2, 1/2)$.

Figure 5. Success Probability for $\lambda_1(t) = \lambda_2(t) = 10$



Notes. (a) $\boldsymbol{\mu} = (1/2, 3/2)$ and $\boldsymbol{K} = (10, 10)$. (b) $\boldsymbol{\mu} = (2, 2)$ and $\boldsymbol{K} = (5, 10)$.

3. The limit of the external arrival process is Lipschitz continuous; that is, there exists $\eta_{ij} > 0$ such that $|\overline{E}_{ij}(t) - \overline{E}_{ij}(s)| \leq \eta_{ij}\,|t - s|$, for $t, s \geq 0$.

4. The scaled initial configurations converge to random vectors of finite measures, $\overline{\mathcal{Q}}^n_{ij}(0) \xrightarrow{d} \overline{\mathcal{Q}}_{ij}(0)$ and $\overline{\mathcal{Z}}^n_{ij}(0) \xrightarrow{d} \overline{\mathcal{Z}}_{ij}(0)$ as $n \to \infty$.

5. For any $i, j \geq 1$, $\overline{\mathcal{Q}}_{ij}(0)(\mathbb{R}_+)$ and the projections $\overline{\mathcal{Z}}_{ij}(0)(\cdot \times \mathbb{R}_+)$ and $\overline{\mathcal{Z}}_{ij}(0)(\mathbb{R}_+ \times \cdot)$ are almost surely free of atoms.

Moreover, our fluid scaling leads to the following relation $\boldsymbol{p}^n(\boldsymbol{z}) = \boldsymbol{p}(\frac{\boldsymbol{z}}{n})$. To see the latter, observe that under our scaling the feasible set of (2.6) can be written as follows:

$$\mathfrak{F}^n(\boldsymbol{z}) = \left\{ \begin{array}{l} \displaystyle\sum_{j=1}^{J} \frac{z_{ij}}{n} p_{ij} \leq M_i,\ 0 \leq p_{ij} \leq c_j^{max},\ W_{ii} \geq \underline{v}_i,\ W_{pp}W_{kk} - W_{pk}^2 \geq 0, \\ \displaystyle W_{pk} - W_{kk} - r_{pk} \sum_{l \in \mathcal{I}(k)} \sum_{j=1}^{J} \frac{z_{lj}}{n} p_{lj} \\ \displaystyle\quad + \sum_{\substack{l \in \mathcal{I}(k) \\ \epsilon_{ls} \in \mathcal{E}(k)}} \left((W_{ll} - 2W_{ls} + W_{ss}) \frac{r_{pk} r_{ls} + x_{pk} x_{ls}}{r_{ls}^2 + x_{ls}^2} \right) = 0. \end{array} \right\}.$$

It is clear now that $\mathfrak{F}^n(\boldsymbol{z}) = \mathfrak{F}(\frac{\boldsymbol{z}}{n})$, which leads to

$$\begin{aligned} \boldsymbol{p}^n(\boldsymbol{z}) &= \underset{(\boldsymbol{p},\boldsymbol{W}) \in \mathfrak{F}^n(\boldsymbol{z})}{\arg\max} \sum_{i=1}^{I} \sum_{j=1}^{J} z_{ij} u_{ij}(p_{ij}) \\ &= \underset{(\boldsymbol{p},\boldsymbol{W}) \in \mathfrak{F}\left(\frac{\boldsymbol{z}}{n}\right)}{\arg\max} \sum_{i=1}^{I} \sum_{j=1}^{J} \frac{z_{ij}}{n} u_{ij}(p_{ij}) = \boldsymbol{p}\left(\frac{\boldsymbol{z}}{n}\right). \end{aligned}$$

Furthermore, by (2.9), we have that

$$S^n_{ij}(\boldsymbol{Z}^n, s, t) = S_{ij}(\overline{\boldsymbol{Z}}^n, s, t).$$

The next theorem states that the fluid model arises as a limit of the fluid-scaled state descriptor under our assumptions.

Theorem 5.1 **(Fluid Limit).** *The sequence of the fluid-scaled measure-valued vector process $(\overline{\boldsymbol{\mathcal{Q}}}^n(\cdot), \overline{\boldsymbol{\mathcal{Z}}}^n(\cdot))$ is tight and every accumulation point $(\overline{\boldsymbol{\mathcal{Q}}}(\cdot), \overline{\boldsymbol{\mathcal{Z}}}(\cdot))$ is a fluid model solution.*

The proof of Theorem 5.1 is given in Section 8, which is organized as follows.

1. We establish tightness of the associated fluid-scaled measure-valued vector process $\left(\overline{\boldsymbol{\mathcal{Q}}}^n(\cdot), \overline{\boldsymbol{\mathcal{Z}}}^n(\cdot)\right)$.

2. We then show tightness for the fluid-scaled stochastic process describing the number of rejected customers, that is, $\overline{\boldsymbol{R}}^n(\cdot)$.

3. The last step is to show that the limit of any convergent subsequence of $\left(\overline{\mathcal{Q}}^n(\cdot), \overline{\mathcal{Z}}^n(\cdot)\right)$ satisfies the fluid model equations.

Remark 5.1. The fluid limit theorem holds even if the external arrival process is a process with a general mean $\overline{E}_{ij}(\cdot)$. In this case, we need to modify the definition of a fluid model solution such that $\overline{R}_{ij}(t) = \int_0^t \mathbb{1}_{\{\overline{Q}_i(s)=K_i\}} d\overline{R}_{ij}(s)$. However, it seems that the uniqueness of the fluid model solutions does not hold.

6. Proofs for Section 3

Proof of Proposition 3.1. First, note that point $\mathbf{0}$ lies in the feasible set by choosing $W_{pk} = W_{00}$. We now define a partition of the set $\mathcal{I}$. Recall that $\mathcal{I}(k)$ denotes the subtree rooted in node $k \in \mathcal{I}$ (including node k). Let us define the following sets $\mathcal{L}_0 := \{k \in \mathcal{I} : \mathcal{I}(k) = \{k\}\}$ and for any $m \geq 1$,

$$\mathcal{L}_m := \left\{k \in \mathcal{I} \setminus \bigcup_{n=0}^{m-1} \mathcal{L}_n : \mathcal{I}(k) \subseteq \bigcup_{n=0}^{m-1} \mathcal{L}_n \bigcup \{k\}\right\}.$$

As the number of nodes $I + 1$ is finite, there exists $I' \leq I + 1$ such that $\mathcal{L}_{I'} = \{0\}$, that is, $\mathcal{L}_{I'}$ contains only the feeder node. Note that $\mathcal{L}_0$ is the set of leaf nodes and the family $\mathcal{L} := \{\mathcal{L}_m, 0 \leq m \leq I'\}$ is a partition of the set $\mathcal{I}$. Indeed, we have that $\emptyset \notin \mathcal{L}$, $\cup_{m=0}^{I'} \mathcal{L}_m = \mathcal{I}$, and $\mathcal{L}_i \cap \mathcal{L}_k = \emptyset$ for $i \neq k$. In Figure 6, we depict an example of a partition with five sets.

Without loss of generality, we consider a single type of EVs; otherwise, set $\Lambda_k := \sum_{j=1}^J \Lambda_{kj}$. To simplify the notation, in the rest of the proof, we write $\boldsymbol{\Lambda}$ instead of $\boldsymbol{\Lambda}(\boldsymbol{z})$ and W_{kk} instead of $W_{kk}(\boldsymbol{\Lambda})$. Recalling that $\boldsymbol{\Lambda}$ is a feasible point of (2.7), we have that $\Lambda_k \leq M_k$, $\Lambda_k \leq z_k c^{\max}$ and for $k \geq 1$, $\epsilon_{pk} \in \mathcal{E}$,

$$\begin{aligned} W_{pk} - W_{kk} - P_{\mathcal{I}(k)} r_{pk} - Q_{\mathcal{I}(k)} x_{pk} &= 0, \\ \underline{v}_k \leq W_{kk} &\leq \overline{v}_k, \\ W_{pp} W_{kk} - W_{pk}^2 &\geq 0. \end{aligned} \tag{6.1}$$

Clearly, $\boldsymbol{\Lambda}'$ satisfies the linear constraints of (2.7), that is, $\Lambda'_k \leq \Lambda_k \leq M_k$ and $\Lambda'_k \leq \Lambda_k \leq z_k c^{\max}$. To show that $\boldsymbol{\Lambda}$ is a feasible point of (2.7), we need to construct W'_{il}, $i, l \geq 0$ such that the additional constraints of (2.7) are satisfied if we replace $\boldsymbol{\Lambda}$ by $\boldsymbol{\Lambda}'$. To this end, set $W'_{00} = W_{00}$, $W'_{pk} = W_{pk}$, for $\epsilon_{pk} \in \mathcal{E}$. Furthermore, W'_{kk} for $k \geq 1$, are given by the solution of

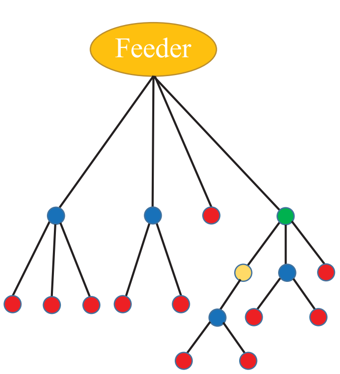
$$W'_{pk} - W'_{kk} - P'_{\mathcal{I}(k)} r_{pk} - Q'_{\mathcal{I}(k)} x_{pk} = 0, \quad \epsilon_{pk} \in \mathcal{E}. \tag{6.2}$$

We shall show that $W_{kk} \leq W'_{kk}$ for $k \in \mathcal{I}$. The proof is then concluded by observing that by the inequality $W_{kk} \leq W'_{kk}$, we have that $\underline{v}_k \leq W_{kk}(\boldsymbol{\Lambda}')$ for $k \geq 1$. Furthermore, by the third inequality of (6.1), we get for $\epsilon_{pk} \in \mathcal{E}$,

$$W'_{pp} W'_{kk} - W'^2_{pk} = W'_{pp} W'_{kk} - W^2_{pk} \geq W'_{pp} W'_{kk} - W_{pp} W_{kk} \geq W_{pp}(W'_{kk} - W_{kk}) \geq 0.$$

Thus, $\boldsymbol{\Lambda}'$ satisfies all the constraints of (2.7), and hence it is a feasible point.

Figure 6. Sets $\mathcal{L}_i$ in a Tree Network



Notes. In this case, $I' = 4$. The red nodes are in $\mathcal{L}_0$, the blue nodes are in $\mathcal{L}_1$, the yellow node is in $\mathcal{L}_2$, the green node is in $\mathcal{L}_3$, and $\mathcal{L}_4$ includes only the feeder.

We now proceed to the proof of the claim that $W_{kk} \le W'_{kk}$ for $k \in \mathcal{I}$. Define $\mathcal{I}(k)^- := \mathcal{I}(k) \setminus \{k\}$ and for $\epsilon_{pk}, \epsilon_{ls} \in \mathcal{E}$,

$$a_{pkls} := \frac{r_{pk} r_{ls} + x_{pk} x_{ls}}{r_{ls}^2 + x_{ls}^2}.$$

For some $0 \le m < I'$ and for $k \in \mathcal{L}_m$, we have that

$$\begin{aligned} W_{kk} - W'_{kk} &= r_{pk}(P'_{\mathcal{I}(k)} - P_{\mathcal{I}(k)}) + x_{pk}(Q'_{\mathcal{I}(k)} - Q_{\mathcal{I}(k)}) \\ &= r_{pk} \sum_{l \in \mathcal{I}(k)} (\Lambda'_l - \Lambda_l) + \sum_{l \in \mathcal{I}(k)} \sum_{\epsilon_{ls} \in \mathcal{E}(k)} a_{pkls}(W'_{ll} - W_{ll} + W'_{ss} - W_{ss}). \end{aligned}$$

The last equation can be rewritten as follows:

$$\begin{aligned} \Big(1 + \sum_{\epsilon_{ks} \in \mathcal{E}(k)} a_{pkks})(W_{kk} - W'_{kk}\Big) &= r_{pk} \sum_{l \in \mathcal{I}(k)^-} (\Lambda'_l - \Lambda_l) + r_{pk}(\Lambda'_k - \Lambda_k) + \sum_{l \in \mathcal{I}(k)^-} \sum_{\epsilon_{ls} \in \mathcal{E}(k)} a_{pkls}(W'_{ll} - W_{ll} + W'_{ss} - W_{ss}) \\ &\quad + \sum_{\epsilon_{ks} \in \mathcal{E}(k)} a_{pkks}(W'_{ss} - W_{ss}). \end{aligned} \tag{6.3}$$

We now show the inequality $W_{kk} \le W'_{kk}$ for each k by induction. Let $k \in \mathcal{L}_0$. By (6.3), we have that

$$W_{kk} - W'_{kk} = r_{pk}(\Lambda'_k - \Lambda_k) \le 0, \tag{6.4}$$

where p is the unique parent of node k. If $m = 1$ (i.e., $k \in \mathcal{L}_1$), then we have that $\mathcal{I}(k)^- = \mathcal{I}(k) \setminus \{k\} = \mathcal{L}_0 \cap \mathcal{I}(k) \setminus \{k\}$ and $\{\epsilon_{ls} \in \mathcal{E}(k) : l \in \mathcal{I}(k) \setminus \{k\}\} = \emptyset$. Furthermore, $\{s : \epsilon_{ks} \in \mathcal{E}(k)\} = \mathcal{L}_0 \cap \mathcal{I}(k) \setminus \{k\}$. By (6.3) and (6.4), we obtain

$$\Big(1 + \sum_{s \in \mathcal{L}_0 \cap \mathcal{I}(k)^-} a_{pkks}\Big)(W_{kk} - W'_{kk}) = r_{pk}(\Lambda'_k - \Lambda_k) + \sum_{l \in \mathcal{L}_0 \cap \mathcal{I}(k)^-} (r_{pk} - a_{pkkl} r_{kl})(\Lambda'_l - \Lambda_l). \tag{6.5}$$

Now, observe that

$$\begin{aligned} r_{pk} - a_{pkkl} r_{kl} &= r_{pk} - r_{kl} \frac{r_{pk} r_{kl} + x_{pk} x_{kl}}{r_{kl}^2 + x_{kl}^2} \\ &= (r_{kl}^2 + x_{kl}^2)^{-1}(r_{pk} r_{kl}^2 + r_{pk} x_{kl}^2 - r_{pk} r_{kl}^2 - r_{kl} x_{pk} x_{kl}) \\ &= x_{kl}(r_{kl}^2 + x_{kl}^2)^{-1}(r_{pk} x_{kl} - r_{kl} x_{pk}) \ge 0, \end{aligned}$$

where the last equation holds by the assumption $r_{pk}/x_{pk} \ge r_{kl}/x_{kl}$. That is, $W_{kk} \le W'_{kk}$, for $k \in \mathcal{L}_1$. Suppose now that $k \in \mathcal{L}_2$. By (6.3), we have that

$$\begin{aligned} \Big(1 + \sum_{\epsilon_{ks} \in \mathcal{E}(k)} a_{pkks})(W_{kk} - W'_{kk}\Big) &= r_{pk} \sum_{m=0}^{1} \sum_{l \in \mathcal{L}_m \cap \mathcal{I}(k)^-} (\Lambda'_l - \Lambda_l) + r_{pk}(\Lambda'_k - \Lambda_k) \\ &\quad + \sum_{l \in \mathcal{L}_1 \cap \mathcal{I}(k)^-} \sum_{\substack{\epsilon_{ls} \in \mathcal{E}(k) \\ s \in \mathcal{L}_0 \cap \mathcal{I}(l)}} a_{pkls}(W'_{ll} - W_{ll} + W'_{ss} - W_{ss}) + \sum_{m=0}^{1} \sum_{\substack{\epsilon_{ks} \in \mathcal{E}(k) \\ s \in \mathcal{L}_m}} a_{pkks}(W'_{ss} - W_{ss}). \end{aligned}$$

The last equation can be equivalently rewritten as follows:

$$\begin{aligned} \Big(1 + \sum_{\epsilon_{ks} \in \mathcal{E}(k)} a_{pkks}\Big)(W_{kk} - W'_{kk}) &= r_{pk} \sum_{m=0}^{1} \sum_{l \in \mathcal{L}_m \cap \mathcal{I}(k)^-} (\Lambda'_l - \Lambda_l) + r_{pk}(\Lambda'_k - \Lambda_k) \\ &\quad + \sum_{l \in \mathcal{L}_1 \cap \mathcal{I}(k)^-} \Bigg(\sum_{\substack{\epsilon_{ls} \in \mathcal{E}(k) \\ s \in \mathcal{L}_0 \cap \mathcal{I}(l)}} a_{pkls} + a_{pkkl} \Bigg)(W'_{ll} - W_{ll}) \\ &\quad + \sum_{l \in \mathcal{L}_1 \cap \mathcal{I}(k)^-} \sum_{\substack{\epsilon_{ls} \in \mathcal{E}(k) \\ s \in \mathcal{L}_0 \cap \mathcal{I}(l)}} a_{pkls}(W'_{ss} - W_{ss}) \\ &\quad + \sum_{\substack{\epsilon_{ks} \in \mathcal{E}(k) \\ s \in \mathcal{L}_0}} a_{pkks}(W'_{ss} - W_{ss}). \end{aligned}$$

Applying (6.5) in the last equation, we obtain the following relation:

$$\left(1+\sum_{\epsilon_{ks}\in\mathcal{E}(k)} a_{pkks}\right)(W_{kk}-W'_{kk}) = r_{pk}(\Lambda'_k-\Lambda_k) + \sum_{\substack{\epsilon_{ks}\in\mathcal{E}(k)\\ s\in\mathcal{L}_0}} (r_{pk}-r_{ks}a_{pkks})({\Lambda_s}'-\Lambda_s)$$
$$+\sum_{l\in\mathcal{L}_1\cap\mathcal{I}(k)^-}\left(r_{pk}-r_{kl}\left(1+\sum_{s\in\mathcal{L}_0\cap\mathcal{I}(l)^-} a_{klls}\right)^{-1}\left(\sum_{\substack{\epsilon_{ls}\in\mathcal{E}(k)\\ s\in\mathcal{L}_0\,\cap\,\mathcal{I}(l)}} a_{pkls}+a_{pkkl}\right)\right)(\Lambda'_l-\Lambda_l)$$
$$+\sum_{\substack{l\in\mathcal{L}_1\,\cap\,\mathcal{I}(k)^-\\ \epsilon_{ls}\in\mathcal{E}(l)\\ s\in\mathcal{L}_0}} (r_{pk}-r_{ls}a_{pkls})({\Lambda_s}'-\Lambda_s).$$

Now, observe that using the assumption $r_{pk}/x_{pk}\ge r_{kl}/x_{kl}$, we have that $r_{pk}-r_{ks}a_{pkks}\ge 0$. Furthermore, we have that

$$r_{pk}-r_{kl}\left(1+\sum_{s\in\mathcal{L}_0\cap\mathcal{I}(l)^-} a_{klls}\right)^{-1}\left(\sum_{\substack{\epsilon_{ls}\in\mathcal{E}(k)\\ s\in\mathcal{L}_0\,\cap\,\mathcal{I}(l)}} a_{pkls}+a_{pkkl}\right)$$
$$=\left(1+\sum_{s\in\mathcal{L}_0\cap\mathcal{I}(l)^-} a_{klls}\right)^{-1}\left(r_{pk}(1+\sum_{s\in\mathcal{L}_0\cap\mathcal{I}(l)^-} a_{klls})-r_{kl}\left(\sum_{\substack{\epsilon_{ls}\in\mathcal{E}(k)\\ s\in\mathcal{L}_0\,\cap\,\mathcal{I}(l)}} a_{pkls}+a_{pkkl}\right)\right)$$
$$=\left(1+\sum_{s\in\mathcal{L}_0\cap\mathcal{I}(l)^-} a_{klls}\right)^{-1}\left(r_{pk}-r_{kl}a_{pkkl}+\sum_{s\in\mathcal{L}_0\cap\mathcal{I}(l)^-}(r_{pk}a_{klls}-r_{kl}a_{pkls})\right)\ge 0.$$

Suppose now that for all $k\in\mathcal{L}_j$, $j=0,\dots,m$,

$$\left(1+\sum_{\epsilon_{ks}\in\mathcal{E}(k)} a_{pkks}\right)(W_{kk}-W'_{kk})\le r_{pk}(\Lambda'_k-\Lambda_k). \tag{6.6}$$

We shall show that the same holds for $k\in\mathcal{L}_{m+1}$. To this end, by (6.3) and (6.4), we have that

$$\left(1+\sum_{\epsilon_{ks}\in\mathcal{E}(k)} a_{pkks}\right)(W_{kk}-W'_{kk}) = r_{pk}(\Lambda'_k-\Lambda_k)$$
$$+\sum_{j=0}^{m}\sum_{\substack{s\in\mathcal{L}_j\,\cap\,\mathcal{I}(k)^-\\ l\,\in\,\cup_{b=j+1}^{m+1}\mathcal{L}_b}}\left(r_{pk}-r_{ls}\left(1+\sum_{\epsilon_{sf}\in\mathcal{E}(s)} a_{lssf}\right)^{-1}\left(\sum_{\substack{\epsilon_{sf}\in\mathcal{E}(s)\\ f\in\bigcup_{b=0}^{j-1}\mathcal{L}_b}} a_{pksf}+a_{pkls}\right)\right)({\Lambda_s}'-\Lambda_s).$$

Using again the assumption $r_{pk}/x_{pk}\ge r_{kl}/x_{kl}$ and adapting the previous steps, we obtain

$$\left(1+\sum_{\epsilon_{ks}\in\mathcal{E}(k)} a_{pkks}\right)(W_{kk}-W'_{kk})\le r_{pk}(\Lambda'_k-\Lambda_k),$$

for $k\in\mathcal{L}_{m+1}$. Thus, $W_{kk}\le W'_{kk}$ for any $k\in\mathcal{L}_m$, $0\le m\le I'$ or $k\in\bigcup_{m=0}^{I'}\mathcal{L}_m=\mathcal{I}$. This concludes the proof. □

Proof of Theorem 3.1. We follow the argument in Reed and Zwart (2014, lemma 7.1). Take a sequence $z^k\in(0,\infty)^{I\times J}$ such that $z^k\to z$ as $k\to\infty$. We proceed by contradiction. Let us assume that $\mathbf{\Lambda}(\cdot)$ is not continuous at point z. That is $\mathbf{\Lambda}(z^k)\to\mathbf{\Lambda}'$ and $\mathbf{\Lambda}'\ne\mathbf{\Lambda}(z)$. The limit $\mathbf{\Lambda}'$ exists as the sequence $\mathbf{\Lambda}(z^k)$ lives in a subset of the compact set $\{\mathbf{\Lambda}\in[0,\infty)^{I\times J}:\mathbf{\Lambda}\le\mathbf{M}\}$. First, we show that $\mathbf{\Lambda}'$ is a feasible point of (2.7). As $\mathbf{\Lambda}(z^k)$ is the optimal solution of (2.7), replacing z by z^k we have that $\sum_{j=1}^J\Lambda_{ij}(z^k)\le M_i$ and $0\le\Lambda_{ij}(z^k)\le c_j^{max}z_{ij}^k$. Taking the limit as $k\to\infty$, we derive $\sum_{j=1}^J{\Lambda_{ij}}'\le M_i$ and $0\le{\Lambda_{ij}}'\le c_j^{max}z_{ij}$. Furthermore, we have that $W_{ii}(\mathbf{\Lambda}(z^k))\ge\underline{v}_i$ and $\mathbf{W}(\epsilon_{il},\mathbf{\Lambda}(z^k))$

≥ 0, $\epsilon_{il} \in \mathcal{E}$. The latter is equivalent to $W_{ii}(\mathbf{\Lambda}(z^k))W_{ll}(\mathbf{\Lambda}(z^k)) - W_{il}(\mathbf{\Lambda}(z^k)) \geq 0$ (as we assume $\underline{v}_i > 0$). Now, by continuity of the voltage magnitudes (Dvijotham et al. 2017, theorem 3), we obtain $W_{ii}(\mathbf{\Lambda}') \geq \underline{v}_i$ and $W_{ii}(\mathbf{\Lambda}')W_{ll}(\mathbf{\Lambda}') - W_{il}(\mathbf{\Lambda}')^2 \geq 0$. That is, $\mathbf{\Lambda}'$ is a feasible point of (2.7). Recalling that $\mathbf{\Lambda}(z)$ is the optimal solution of (2.7), we have that

$$\sum_{i=1}^{I}\sum_{j=1}^{J} z_{ij}u_{ij}(\Lambda_{ij}(z)/z_{ij}) > \sum_{i=1}^{I}\sum_{j=1}^{J} z_{ij}u_{ij}(\Lambda'_{ij}/z_{ij}). \tag{6.7}$$

To derive the contradiction, we construct a point $\mathbf{\Lambda}^k$ that is feasible for (2.7) if we replace z by z^k. To this end, define for any $k \geq 1$,

$$\Lambda^k_{ij} := \Lambda_{ij}(z) \wedge c_j^{max} z^k_{ij}.$$

We have that $\mathbf{\Lambda}^k \to \mathbf{\Lambda}(z)$ and $\mathbf{\Lambda}^k \leq \mathbf{\Lambda}(z)$ for $k \geq k_0$. Observing that $\Lambda^k_{ij} \leq c_j^{max} z^k_{ij}$, by Proposition 3.1, we have that Λ^k is a feasible point of (2.7) by replacing z by z^k for $k \geq 1$. It follows that as $k \to \infty$,

$$\sum_{i=1}^{I}\sum_{j=1}^{J} z^k_{ij}u_{ij}(\Lambda_{ij}/z^k_{ij}) \to \sum_{i=1}^{I}\sum_{j=1}^{J} z_{ij}u_{ij}(\Lambda(z_{ij})/z_{ij})$$

and

$$\sum_{i=1}^{I}\sum_{j=1}^{J} z_{ij}u_{ij}(\Lambda(z_{ij})/z^k_{ij}) \to \sum_{i=1}^{I}\sum_{j=1}^{J} z_{ij}u_{ij}(\Lambda'_{ij}/z_{ij}).$$

That is, by (6.7), there exists a sufficiently large k such that

$$\sum_{i=1}^{I}\sum_{j=1}^{J} z^k_{ij}u_{ij}(\Lambda^k_{ij}/z^k_{ij}) > \sum_{i=1}^{I}\sum_{j=1}^{J} z_{ij}u_{ij}(\Lambda(z^k_{ij})/z^k_{ij}).$$

The last inequality yields a contradiction as $\mathbf{\Lambda}(z^k)$ is the optimal solution of (2.7) by replacing z by z^k. □

7. Proofs for Section 4

Proof of Proposition 4.1. Using the identity $\mathbb{P}(D_{ij} < t) + \mathbb{P}(D_{ij} \geq t) = 1$, (4.2) can be written as

$$\overline{Q}_{ij}(t) = \overline{Q}_{ij}(0) + \overline{E}_{ij}(t) - \overline{R}_{ij}(t) - \overline{D}_{ij}(t),$$

where

$$\overline{D}_{ij}(t) := \overline{Q}_{ij}(0)\mathbb{P}(D^0_{ij} < t) + \int_0^t \mathbb{P}(D_{ij} < t-s)d\overline{E}_{ij}(s) - \int_0^t \mathbb{P}(D_{ij} < t-s)d\overline{R}_{ij}(s). \tag{7.1}$$

In the sequel, we show that $\overline{D}_{ij}(t)$ can be written as in (4.4). By the definition of the fluid model, we have that

$$\begin{aligned}\overline{Q}_{ij}(t) - \overline{Q}_{ij}(t)([\epsilon,\infty)) &= \overline{Q}_{ij}(0)(\mathbb{P}(D^0_{ij} \geq t) - \mathbb{P}(D_{ij} \in t + [\epsilon,\infty])) \\ &\quad + \int_0^t (\mathbb{P}(D_{ij} \geq t-s) - \mathbb{P}(D_{ij} \in t-s+[\epsilon,\infty]))d\overline{E}_{ij}(s) \\ &\quad - \int_0^t (\mathbb{P}(D_{ij} \geq t-s) - \mathbb{P}(D_{ij} \in t-s+[\epsilon,\infty]))d\overline{R}_{ij}(s).\end{aligned}$$

Observing that $\mathbb{P}(D_{ij} \in t + [\epsilon,\infty]) = \mathbb{P}(D_{ij} \geq t+\epsilon)$ and

$$\mathbb{P}(D_{ij} \geq t) - \mathbb{P}(D_{ij} \geq t+\epsilon) = \mathbb{P}(t < D_{ij} < t+\epsilon),$$

we have that

$$\overline{Q}_{ij}(t)-\overline{\mathcal{Q}}_{ij}(t)([\epsilon,\infty))=\overline{Q}_{ij}(0)\quad \mathbb{P}(t<D^0_{ij}<t+\epsilon) + \int_0^t \mathbb{P}(t-s<D_{ij}<t-s+\epsilon)d\overline{E}_{ij}(s) - \int_0^t \mathbb{P}(t-s<D_{ij}<t-s+\epsilon)d\overline{R}_{ij}(s).$$

By the assumption of existence of the pdf $f_{D_{ij}}(\cdot)$, we have that

$$\overline{Q}_{ij}(t)-\overline{\mathcal{Q}}_{ij}(t)([\epsilon,\infty))=\overline{Q}_{ij}(0)\epsilon f_{D^0_{ij}}(t)+\int_0^t \epsilon f_{D_{ij}}(t-s)d\overline{E}_{ij}(s) - \int_0^t \epsilon f_{D_{ij}}(t-s)d\overline{R}_{ij}(s)+o(\epsilon).$$

Dividing the last equation by ϵ and letting ϵ go to zero, we have that

$$\lim_{\epsilon\to 0}\frac{\overline{Q}_{ij}(t)-\overline{\mathcal{Q}}_{ij}(t)([\epsilon,\infty))}{\epsilon}=\overline{Q}_{ij}(0)f_{D^0_{ij}}(t)+\int_0^t f_{D_{ij}}(t-s)d\overline{E}_{ij}(s) - \int_0^t f_{D_{ij}}(t-s)d\overline{R}_{ij}(s). \tag{7.2}$$

In other words, the limit of the left-hand side of (7.2) exists. Integrating (7.2) from zero to t and interchanging the integrals using Tonelli's theorem (Rudin 1987), we derive

$$\int_0^t \lim_{\epsilon\to 0}\frac{\overline{Q}_{ij}(s)-\overline{\mathcal{Q}}_{ij}(s)([\epsilon,\infty))}{\epsilon}ds=\overline{Q}_{ij}(0)\mathbb{P}(D^0_{ij}<t)+\int_0^t \mathbb{P}(D_{ij}<t-s)d\overline{E}_{ij}(s) - \int_0^t \mathbb{P}(D_{ij}<t-s)d\overline{R}_{ij}(s)=\overline{D}_{ij}(t).$$

Furthermore, the following inequality holds for any $t\geq 0$,

$$\overline{D}_{ij}(t)\leq \overline{Q}_{ij}(0)+\int_0^t \mathbb{P}(D_{ij}<t-s)d\overline{E}_{ij}(s)\leq \overline{Q}_{ij}(0)+\overline{E}_{ij}(t)<\infty.$$

That is, $\overline{D}_{ij}(t)$ represents the departure process that proves (4.3) and (4.4). □

The first step to prove Theorem 4.1 is to show that the fluid model solutions are bounded away from zero. This is stated in the following proposition.

Proposition 7.1. *Under the assumptions of Theorem 4.1, we have that for any* $\epsilon>0$,

$$\inf_{t\geq\epsilon}\min_{i,j}\overline{Z}_{ij}(t)>0.$$

Proof. Recall that an assumption of Theorem 4.1 is that $Q_{ij}(0)>0$ if $Q_i(0)=K_i$. Furthermore, by our assumptions, there exists the probability density function of parking times and $f_{D_{ij}}(0)>0$ for any $i,j\geq 1$. It is enough to show that $\overline{Z}_{ij}(\cdot)$ remains positive when the system is not full. Assume that $\overline{\boldsymbol{Z}}(0)=0$ and define $\tau:=\inf\{s\geq 0:\overline{Q}_i(s)=K_i\}$, where $\tau\in[0,\infty]$. Note that $\mathbb{P}\Big(\frac{B_{ij}}{c_j^{max}}\wedge D_{ij}\geq s\Big)\to\mathbb{P}\Big(\frac{B_{ij}}{c_j^{max}}\wedge D_{ij}\geq 0\Big)=1$ as $s\to 0$ and choose ϵ_1 such that $\mathbb{P}\Big(\frac{B_{ij}}{c_j^{max}}\wedge D_{ij}\geq s\Big)\geq\frac{1}{2}$ for $s\in[0,\epsilon_1]$. For $t\leq\tau$, we have that

$$\begin{aligned}\overline{Z}_{ij}(t)&\geq\int_0^t\lambda_{ij}(s)\mathbb{P}\left(\frac{B_{ij}}{c_j^{max}}\wedge D_{ij}\geq t-s\right)ds\\ &=\int_0^t\lambda_{ij}(t-s)\mathbb{P}\left(\frac{B_{ij}}{c_j^{max}}\wedge D_{ij}\geq s\right)ds\\ &=\inf_{0<s\leq\epsilon}\lambda_{ij}(s)\int_0^\epsilon\mathbb{P}\left(\frac{B_{ij}}{c_j^{max}}\wedge D_{ij}\geq s\right)ds\\ &\geq\inf_{0<s\leq\epsilon}\lambda_{ij}(s)\frac{\epsilon\wedge\epsilon_1}{2}>0,\end{aligned}$$

because $\lambda_{ij}(\cdot) > 0$. This also covers the case that $\tau = \infty$. If the arrival rate is constant, then the last bound coincides with the one in Remerova et al. (2014, lemma 3). Now, for $t > \tau$, we have that $\overline{Q}_i(t) = K_i$ and by the continuity of the fluid model solutions, we have that $\overline{Q}_{ij}(t) = \overline{Q}_{ij}(\tau)$. Furthermore, by (4.3), we have that

$$\overline{E}_{ij}(t) - \overline{R}_{ij}(t) = \overline{D}_{ij}(t) - \overline{D}_{ij}(\tau) + \overline{E}_{ij}(\tau),$$

and using (4.4), we obtain

$$\overline{Z}_{ij}(t) \geq \int_\tau^t \delta_{ij}(s)\mathbb{P}\left(\frac{B_{ij}}{c_j^{max}} \wedge D_{ij} \geq t-s\right)ds,$$

where we define $\delta_{ij}(s) := \lim_{\epsilon\to 0}\frac{\overline{Q}_{ij}(s)-\overline{Q}_{ij}(s)([\epsilon,\infty))}{\epsilon}$. By the fact that $\overline{Q}_{ij}(t) = \overline{Q}_{ij}(\tau) > 0$ for $t > \tau$ (this also covers the case $\tau = 0$), we have that $\delta_{ij}(t) = \delta_{ij}(\tau) = \delta_{ij}$. Furthermore, by the assumption $f_{D_{ij}}(0) > 0$, (7.2), and the fact that $\overline{R}_{ij}(s) = 0$ for $s \leq \tau$, we have that $\delta_{ij}(\tau) > 0$. Hence,

$$\begin{aligned}\overline{Z}_{ij}(t) &\geq \delta_{ij}\int_\tau^t \mathbb{P}\left(\frac{B_{ij}}{c_j^{max}} \wedge D_{ij} \geq t-s\right)ds\\ &= \delta_{ij}\int_0^{t-\tau} \mathbb{P}\left(\frac{B_{ij}}{c_j^{max}} \wedge D_{ij} \geq s\right)ds\\ &\geq \delta_{ij}\frac{(t-\tau)\wedge\epsilon_1}{2} > 0.\end{aligned}$$

□

Proof of Theorem 4.1. We first show that each pair $(K_i - \overline{Q}_i(\cdot), \overline{R}_i(\cdot))$ is unique for any $i \geq 1$. By Remark 4.1, fluid model solutions are invariant with respect to time shifts, and hence it suffices to show that $(K_i - \overline{Q}_i(\cdot), \overline{R}_i(\cdot))$ is unique on the time interval $[0, T]$ for $T > 0$.

By Proposition 4.1, we have that

$$K_i - \overline{Q}_i(t) = K_i - \overline{Q}_i(0) - \sum_{j=1}^J \overline{E}_{ij}(t) + \sum_{j=1}^J \overline{D}_{ij}(t) + \overline{R}_i(t), \tag{7.3}$$

where $\overline{R}_i(t) = \int_0^t \mathbb{1}_{\{\overline{Q}_i(s)=K_i\}} d\overline{R}_i(s) = \int_0^t \mathbb{1}_{\{K_i-\overline{Q}_i(s)=0\}} d\overline{R}_i(s)$. Now, by the one-dimensional reflection mapping (Chen and Yao 2001, chapter 6), we have that

$$K_i - \overline{Q}_i(t) = \Psi(\Phi_i)(t) := \Phi_i(t) + \sup_{0\leq s\leq t}(-\Phi_i(s) \vee 0), \tag{7.4}$$

where

$$\Phi_i(t) := K_i - \overline{Q}_i(0) - \sum_{j=1}^J \overline{E}_{ij}(t) + \sum_{j=1}^J \overline{D}_{ij}(t).$$

It is known that the reflection mapping $\Psi(\cdot)$ is Lipschitz continuous (Chen and Yao 2001). Now, for each $i \geq 1$, define the mapping B_i for each function $a(\cdot)$ on $[0, \infty)$,

$$\begin{aligned}B_i(a)(t) = \zeta_i(t) &- \sum_{j=1}^J \int_0^t \frac{\lambda_{ij}(s)}{\sum_{j=1}^J \lambda_{ij}(s)} a(s) f_{D_{ij}}(t-s)ds\\ &+ \sum_{j=1}^J \int_0^t \int_0^s a(u)d\frac{\lambda_{ij}(u)}{\sum_{j=1}^J \lambda_{ij}(u)} f_{D_{ij}}(t-s)ds,\end{aligned}$$

where

$$\zeta_i(t) = K_i - \overline{Q}_i(0) + \sum_{j=1}^{J} \overline{Q}_{ij}(0)\mathbb{P}(D_{ij}^0 < t) - \sum_{j=1}^{J} \overline{E}_{ij}(t) + \sum_{j=1}^{J} \int_0^t \overline{E}_{ij}(u) f_{D_{ij}}(t-u)du.$$

Observing that $\frac{\lambda_{ij}(\cdot)}{\sum_{h=1}^J \lambda_{ih}(\cdot)} \le 1$, we have that the mapping $B_i(\cdot)$ is locally Lipschitz continuous for any $i \ge 1$, namely

$$\sup_{0\le t\le T} |B_i(a_1)(t) - B_i(a_2)(t)| \le 2\sum_{j=1}^{J} \mathbb{P}(D_{ij} \le T) \sup_{0\le t\le T} |a_1(t) - a_2(t)|.$$

By Kang (2015, lemma 3), the following functional equation for any $i \ge 1$ has a unique solution on $[0, T]$:

$$a(t) = \Psi(B_i(a))(t) - B_i(a)(t). \tag{7.5}$$

The main idea now is to show that each function $R_i(\cdot)$ satisfies (7.5), and hence it is unique. To this end, by the proof of Proposition 4.1, the relation $\overline{R}_{ij}(t) = \int_0^t \frac{\lambda_{ij}(s)}{\sum_{h=1}^J \lambda_{ih}(s)} d\overline{R}_i(s)$, and the properties of the Riemann–Stieltjes integral, we obtain

$$\begin{aligned}\overline{D}_{ij}(t) &= \overline{Q}_{ij}(0)\mathbb{P}(D_{ij}^0 < t) + \int_0^t \mathbb{P}(D_{ij} < t-s)d\overline{E}_{ij}(s) - \int_0^t \mathbb{P}(D_{ij} < t-s)d\overline{R}_{ij}(s)\\ &= \overline{Q}_{ij}(0)\mathbb{P}(D_{ij}^0 < t) + \int_0^t \overline{E}_{ij}(s) f_{D_{ij}}(t-s)ds - \int_0^t \overline{R}_{ij}(s) f_{D_{ij}}(t-s)ds\end{aligned}$$

and

$$\begin{aligned}\int_0^t \overline{R}_{ij}(s) f_{D_{ij}}(t-s)ds &= \int_0^t \int_0^s \frac{\lambda_{ij}(u)}{\sum_{h=1}^J \lambda_{ih}(u)} d\overline{R}_i(u) f_{D_{ij}}(t-s)ds\\ &= \int_0^t \frac{\lambda_{ij}(s)}{\sum_{h=1}^J \lambda_{ih}(s)} \overline{R}_i(s) f_{D_{ij}}(t-s)ds\\ &\quad - \int_0^t \int_0^s \overline{R}_i(u) d\frac{\lambda_{ij}(u)}{\sum_{h=1}^J \lambda_{ih}(u)} f_{D_{ij}}(t-s)ds.\end{aligned}$$

Using the last equation and replacing $\overline{D}_{ij}(t)$ in (7.3), we have that

$$\begin{aligned}K_i - \overline{Q}_i(t) &= \zeta_i(t) - \sum_{j=1}^{J} \int_0^t \frac{\lambda_{ij}(s)}{\sum_{h=1}^J \lambda_{ih}(s)} \overline{R}_i(s) f_{D_{ij}}(t-s)ds\\ &\quad + \sum_{j=1}^{J} \int_0^t \int_0^s \overline{R}_i(u) d\frac{\lambda_{ij}(u)}{\sum_{h=1}^J \lambda_{ih}(u)} f_{D_{ij}}(t-s)ds + \overline{R}_i(t)\\ &= B_i(\overline{R}_i)(t) + \overline{R}_i(t).\end{aligned}$$

Using again the reflection mapping, we obtain

$$K_i - \overline{Q}_i(t) = \Psi(B_i(\overline{R}_i))(t).$$

The last equation and (7.4) yield

$$\Phi_i(t) = B_i(\overline{R}_i)(t). \tag{7.6}$$

Combining (7.3) and (7.4), we derive

$$\overline{R}_i(t) = \Psi(\Phi_i)(t) - \Phi_i(t).$$

Now, replacing $\Phi_i(\cdot)$ in the last equation by the right-hand side of (7.6) leads to

$$\overline{R}_i(t) = \Psi(B_i(\overline{R}_i))(t) - B_i(\overline{R}_i)(t).$$

Thus, $\overline{R}_i(\cdot)$ is a solution of (7.5) and hence unique. This implies that $\overline{R}_{ij}(\cdot)$ is unique for any $i, j \ge 1$, and hence $(\overline{\mathcal{Q}}_{ij}(\cdot), \overline{Q}_{ij}(\cdot))$ is unique for $i, j \ge 1$.

We now proceed to show the uniqueness of the $\overline{Z}_{ij}(\cdot)$. First, we show that $\overline{\mathcal{Z}}_{ij}(\cdot)$ has a Lipschitz continuous first projection. Indeed, let $x < x'$ and $y \geq 0$. For any $i, j \geq 0$, we have that

$$\overline{\mathcal{Z}}_{ij}(t)([x,x'] \times [y,\infty)\Big) \leq \overline{\mathcal{Z}}_{ij}(0)([x + S_{ij}(\mathbf{Z},0,t), x' + S_{ij}(\mathbf{Z},0,t)] \times [y,\infty)) + \int_0^t \mathbb{P}(x + S_{ij}(\mathbf{Z},s,t) \leq B_{ij} \leq x' + S_{ij}(\mathbf{Z},s,t)) d\overline{E}_{ij}(s).$$

By the Lipschitz continuity of $\overline{E}_{ij}(\cdot)$, the previous bound becomes

$$\overline{\mathcal{Z}}_{ij}(t)([x,x'] \times [y,\infty)) \leq \overline{\mathcal{Z}}_{ij}(0)([x + S_{ij}(\mathbf{Z},0,t), x' + S_{ij}(\mathbf{Z},0,t)] \times [y,\infty)) + \eta_{ij} \int_0^t \mathbb{P}(x + S_{ij}(\mathbf{Z},s,t) \leq B_{ij} \leq x' + S_{ij}(\mathbf{Z},s,t)) ds.$$

By the assumption of the Lipschitz continuity of the initial condition, the change of variable $v = \Theta(s) = S_{ij}(\mathbf{Z},s,t)$, and Remerova et al. (2014, lemma 5), we have that

$$\overline{\mathcal{Z}}_{ij}(t)([x,x'] \times [y,\infty)) \leq L(x' - x) + \eta_{ij} \int_0^{S_{ij}(\mathbf{Z},0,t)} \frac{\mathbb{P}(x + v \leq B_{ij} \leq x' + v)}{p_{ij}(\mathbf{Z}(\Theta^{-1}(v)))} dv \leq (L + \|\boldsymbol{\eta}\| \sup_{0 \leq s \leq t} \frac{1}{\mathbf{Z}(s)})(x' - x).$$

By Proposition 7.1, $\mathbf{Z}(\cdot)$ is bounded away from zero. Now, the property of the utility functions $\lim_{x \to 0} u'_{ij}(x) = \infty$ for $i \in \mathcal{I}, j \in \mathcal{J}$ guarantees that $\min_{ij} \inf_s p_{ij}(\mathbf{Z}(s)) > 0$. That is, the first projection of $\overline{\mathcal{Z}}_{ij}(\cdot)$ is Lipschitz continuous with constant

$$L + \|\boldsymbol{\eta}\| \sup_{0 \leq s \leq t} \frac{1}{\mathbf{Z}(s)} < \infty,$$

where the last inequality follows by Theorem 3.1. Now, point $\mathbf{0}$ is a feasible point of (2.5). Furthermore, for a vector z such that z_{ij} is small enough the power flow constraints are satisfied and hence $p_{ij}(z) = c_j^{max}$. Moreover, the power allocation function is Lipschitz continuous because we consider the linearized Distflow power flow model as we discussed in Section 3. Now, by the Lipschitz continuity of $\overline{E}_{ij}(\cdot)$ and by applying Remerova et al. (2014, theorem 1), we obtain that the fluid model solution $(\mathcal{Z}(\cdot), \mathbf{Z}(\cdot))$ is unique. □

8. Proof of Theorem 5.1

8.1. Establishing Tightness

The first step of the proof of Theorem 5.1 is to show that $(\overline{\mathcal{Q}}^n(\cdot), \overline{\mathcal{Z}}^n(\cdot))$ is C-tight, that is, tight with continuous weak limits. To do so, we follow the idea of proof of Remerova et al. (2014, theorem 5). First, we show that both processes satisfy the compact containment property. To this end, note that the following bounds hold almost surely

$$\mathcal{Q}_{ij}(t) \leq \sum_{l=1}^{Q_{ij}(0)} \delta^+_{D^0_{ijl}(t)} + \sum_{l=1}^{E_{ij}(t)} \delta^+_{D_{ijl}(t)} \tag{8.1}$$

and

$$\mathcal{Z}_{ij}(t) \leq \sum_{l=1}^{Z_{ij}(0)} \delta^+_{(B^0_{ijl}(t), D^0_{ijl}(t))} + \sum_{l=1}^{E_{ij}(t)} \delta^+_{(B_{ijl}(t), D_{ijl}(t))}. \tag{8.2}$$

Moreover, by our assumptions, $\frac{E^n_{ij}(\cdot)}{n} \xrightarrow{d} \overline{E}_{ij}(\cdot)$. Hence, all the bounds in Remerova et al. (2014, lemma 9) hold true for the measure-valued processes $\mathcal{Q}_{ij}(\cdot)$ and $\mathcal{Z}_{ij}(\cdot)$. That is, for any $T > 0$ and $\epsilon > 0$, there exist compact sets $C \in \mathcal{M}(\mathbb{R}_+)^{I \times J}$ and $C' \in \mathcal{M}(\mathbb{R}^2_+)^{I \times J}$ such that

$$\liminf_{n \to \infty} \mathbb{P}^n(\overline{\mathcal{Q}}^n(t) \in C \ \ \forall t \in [0,T]) \geq 1 - \epsilon, \tag{8.3}$$

and

$$\liminf_{n \to \infty} \mathbb{P}^n(\overline{\mathcal{Z}}^n(t) \in C' \ \ \forall t \in [0,T]) \geq 1 - \epsilon. \tag{8.4}$$

Next, we shall show the oscillation control. To do so, we first show a preliminary result. Define $H_a^b := \mathbb{R}_+ \times [a,b]$ and $V_a^b := [a,b] \times \mathbb{R}_+$. If $b = \infty$, then $H_a^\infty := \mathbb{R}_+ \times [a,\infty)$ and $V_a^b := [a,\infty) \times \mathbb{R}_+$.

Proposition 8.1. *For any $T > 0$, $\delta > 0$, and $\epsilon > 0$, there exist $\alpha > 0$ and $b > 0$ such that*

$$\liminf_{n\to\infty} \mathbb{P}^n(\sup_{0\le t\le T}\sup_{x\in\mathbb{R}_+}(\|\overline{\mathcal{Q}}^n(t)([x,x+\alpha])\|)\le\delta)\ge 1-\epsilon$$

and

$$\liminf_{n\to\infty} \mathbb{P}^n(\sup_{0\le t\le T}\sup_{x\in\mathbb{R}_+}(\|\overline{\mathcal{Z}}^n(t)(H_x^{x+b})\|\vee\|\overline{\mathcal{Z}}^n(t)(V_x^{x+b})\|)\le\delta)\ge 1-\epsilon.$$

Proof. By Remerova et al. (2014, lemma 10), we have that there exist $\alpha > 0$ and $b > 0$ such that

$$\liminf_{n\to\infty} \mathbb{P}^n(\sup_{x\in\mathbb{R}_+}(\|\overline{\mathcal{Q}}^n(0)([x,x+\alpha])\|)\le\delta)\ge 1-\epsilon \tag{8.5}$$

and

$$\liminf_{n\to\infty} \mathbb{P}^n(\sup_{x\in\mathbb{R}_+}(\|\overline{\mathcal{Z}}^n(0)(H_x^{x+b})\|\vee\|\overline{\mathcal{Z}}^n(0)(V_x^{x+b})\|)\le\delta)\ge 1-\epsilon. \tag{8.6}$$

Next, define

$$\mathcal{Q}_{ij}^\infty(t) := \sum_{l=1}^{E_{ij}(t)} \delta^+_{D_{ijl}(t)},\ \mathcal{Z}_{ij}^\infty(t) := \sum_{l=1}^{E_{ij}(t)} \delta^+_{(B_{ijl}(t),D_{ijl}(t))}.$$

We shall show that

$$\liminf_{n\to\infty} \mathbb{P}^n(\sup_{0\le t\le T}\sup_{x\in\mathbb{R}_+}(\|\overline{\mathcal{Q}}^{n,\infty}(t)([x,x+\alpha])\|)\le\delta)\ge 1-\epsilon \tag{8.7}$$

and

$$\liminf_{n\to\infty} \mathbb{P}^n(\sup_{0\le t\le T}\sup_{x\in\mathbb{R}_+}(\|\overline{\mathcal{Z}}^{n,\infty}(t)(H_x^{x+b})\|\vee\|\overline{\mathcal{Z}}^{n,\infty}(t)(V_x^{x+b})\|)\le\delta)\ge 1-\epsilon. \tag{8.8}$$

Then, the result follows. Indeed, by (8.1) and (8.2), we have that

$$\begin{aligned}\liminf_{n\to\infty} \quad & \mathbb{P}^n(\sup_{0\le t\le T}\sup_{x\in\mathbb{R}_+}(\|\overline{\mathcal{Q}}^n(t)([x,x+\alpha])\|)\le\delta)\\ & \ge \liminf_{n\to\infty}\mathbb{P}^n(\sup_{0\le t\le T}\sup_{x\in\mathbb{R}_+}(\|\overline{\mathcal{Q}}^{n,\infty}(t)([x,x+\alpha])\|)\le\delta)\end{aligned}$$

and

$$\begin{aligned}\liminf_{n\to\infty} \quad & \mathbb{P}^n(\sup_{0\le t\le T}\sup_{x\in\mathbb{R}_+}(\|\overline{\mathcal{Z}}^n(t)(H_x^{x+b})\|\vee\|\overline{\mathcal{Z}}^n(t)(V_x^{x+b})\|)\le\delta)\\ \ge \quad & \liminf_{n\to\infty}\mathbb{P}^n(\sup_{0\le t\le T}\sup_{x\in\mathbb{R}_+}(\|\overline{\mathcal{Z}}^{n,\infty}(t)(H_x^{x+b})\|\vee\|\overline{\mathcal{Z}}^{n,\infty}(t)(V_x^{x+b})\|)\le\delta).\end{aligned}$$

Now, Proposition 8.1 follows by using the last inequalities, (8.5)–(8.8), and Remerova et al. (2014, lemma 12).

We move now to the proof of (8.7) and (8.8). Denote by $\Omega_{0,q}^n$ and $\Omega_{0,z}^n$ the events for which (8.7) and (8.8) hold, respectively. Let $\Omega_{1,q}^n$ and $\Omega_{1,z}^n$ be the events for which (8.3) and (8.4) hold, respectively. By Remerova et al. (2014, proposition 1), C and C' are relatively compact. Hence, $\Xi := \sup_{\boldsymbol{m}\in C}\|\boldsymbol{m}(\mathbb{R}_+)\| < \infty$, $\Xi' := \sup_{\boldsymbol{m}\in C'}\|\boldsymbol{m}(\mathbb{R}_+^2)\| < \infty$, $\sup_{\boldsymbol{m}\in C}\|\boldsymbol{m}(\mathbb{R}_+ \setminus [0,L])\| \le \delta/4$, and $\sup_{\boldsymbol{m}\in C'}\|\boldsymbol{m}(\mathbb{R}_+^2 \setminus [0,L']^2)\| \le \delta/4$ for large L and L'. In addition, put $p_* := \min_{i,j}\{p_{ij} : z_{ij} > \delta/4, \|\boldsymbol{z}\| \le \Xi'\}$, $\beta := \frac{\delta}{8\|\boldsymbol{\eta}\|} \wedge T$, $\alpha = \frac{\beta}{3}$, and $b = \frac{\beta(p_*\wedge 1)}{3}$. Furthermore, take N and N' such that

$$N\alpha > L + T \text{ and } N'b > L' + (\|\boldsymbol{c}^{max}\| \vee 1)T,$$

and define the following sets

$$
\begin{aligned}
I_k &:= [(k-1)\alpha, k\alpha],\\
I^k &:= [(k-2)^+\alpha, (k+1)\alpha],\\
I_{k,k'} &:= [(k-1)b, kb] \times [(k'-1)b, k'b],\\
I^{k,k'} &:= [(k-2)^+ b, (k+1)b] \times [(k'-2)^+ b, (k'+1)b].
\end{aligned}
$$

Furthermore, pick functions $g_k \in \boldsymbol{C}(\mathbb{R}_+, [0,1])$ and $g_{k,k'} \in \boldsymbol{C}(\mathbb{R}_+^2, [0,1])$ such that

$$
\begin{aligned}
\mathbb{1}_{\{I_k\}}(\cdot) \le g_k(\cdot) \le 1_{\{I^k\}}(\cdot),\\
\mathbb{1}_{\{I_{k,k'}\}}(\cdot) \le g_{k,k'}(\cdot) \le 1_{\{I^{k,k'}\}}(\cdot),
\end{aligned}
$$

and note that

$$
\begin{aligned}
\sum_{k\in\mathbb{N}} \| < g_k, \boldsymbol{F_D} > \| \le \| \sum_{k\in\mathbb{N}} < g_k, \boldsymbol{F_D} > \| \le 3,\\
\sum_{k,k'\in\mathbb{N}} \| < g_{k,k'}, \boldsymbol{F} > \| \le \| \sum_{k,k'\in\mathbb{N}} < g_{k,k'}, \boldsymbol{F} > \| \le 9.
\end{aligned}
$$

Define the load processes for the nth system, and $t \ge 0$,

$$
\mathcal{L}_{ij}^{n,Q}(t) := \sum_{l=1}^{E_{ij}^n(t)} \delta_{D_{ijl}}, \mathcal{L}_{ij}^{n,Z}(t) := \sum_{l=1}^{E_{ij}^n(t)} \delta_{(B_{ijl}, D_{ijl})},
$$

and the corresponding scaled load processes

$$
\overline{\mathcal{L}}_{ij}^{n,Q}(t) := \frac{\mathcal{L}_{ij}^{n,Q}(nt)}{n}, \overline{\mathcal{L}}_{ij}^{n,Z}(t) := \frac{\mathcal{L}_{ij}^{n,Z}(nt)}{n}.
$$

By Gromoll and Williams (2009, theorem 5.1), we have that

$$
\lim_{n\to\infty} \mathbb{P}^n(\max_{1\le k\le N} \sup_{0\le t\le T} \| < g_k, \overline{\boldsymbol{\mathcal{L}}}^{n,Q} > -\overline{\boldsymbol{E}}(t) < g_k, \boldsymbol{F_D} > \| \le \frac{\delta}{16N^2}) = 1
$$

and

$$
\lim_{n\to\infty} \mathbb{P}^n(\max_{1\le k,k'\le N'} \sup_{0\le t\le T} \| < g_{k,k'}, \overline{\boldsymbol{\mathcal{L}}}^{n,Z} > -\overline{\boldsymbol{E}}(t) < g_{k,k'}, \boldsymbol{F} > \| \le \frac{\delta}{16N'2}) = 1,
$$

where we denote by $\Omega_{2,q}^n$ and $\Omega_{2,z}^n$ the corresponding events. Furthermore, by our assumptions

$$
\lim_{n\to\infty} \mathbb{P}^n(\sup_{0\le t\le T} \|\overline{\boldsymbol{E}}^n(t) - \overline{\boldsymbol{E}}(t)\| \le \delta/16) = 1,
$$

and we denote these events by Ω_3^n. Adapting the proof of Remerova et al. (2014, lemma 11), it follows that $\Omega_{1,q}^n \cap \Omega_{2,q}^n \cap \Omega_3^n \subseteq \Omega_{0,q}^n$ and $\Omega_{1,z}^n \cap \Omega_{2,z}^n \cap \Omega_3^n \subseteq \Omega_{0,z}^n$. This concludes the proof of Proposition 8.1. □

Proposition 8.2 (Oscillation Control). *For any $T > 0$, $\delta > 0$, and $\epsilon > 0$ there exist $h > 0$ and $h' > 0$* such that

$$
\liminf_{n\to\infty} \mathbb{P}^n(\omega(\overline{\boldsymbol{\mathcal{Q}}}^n(\cdot), h, T) \le \delta) \ge 1 - \epsilon \tag{8.9}
$$

and

$$
\liminf_{n\to\infty} \mathbb{P}^n(\omega(\overline{\mathcal{Z}}^n(\cdot), h', T) \le \delta) \ge 1 - \epsilon, \tag{8.10}
$$

where for a measure-valued process $\mathcal{X}(\cdot)$, we define

$$
\omega(\boldsymbol{\mathcal{X}}(\cdot), h, T) := \sup_{0\le s,t\le T} \{\boldsymbol{d}(\boldsymbol{\mathcal{X}}(t), \boldsymbol{\mathcal{X}}(s)) : |t-s| < h\}.
$$

Proof. We shall use the idea of proof of Remerova et al. (2014, lemma 13). Let Ω_q^n and Ω_z^n be the events such that (8.9) and (8.10) hold, respectively. Denote by Ω_1^n the following events:

$$\lim_{n\to\infty} \mathbb{P}^n\big(\sup_{0\le t\le T} \|\overline{\boldsymbol{E}}^n(t) - \overline{\boldsymbol{E}}(t)\| \le \delta/4\big) = 1.$$

Furthermore, by Proposition 8.1, there exist $a > 0$ and $b > 0$ such that

$$\liminf_{n\to\infty} \mathbb{P}^n\big(\sup_{0\le t\le T} \|\overline{\boldsymbol{\mathcal{Q}}}^n(t)([0,\alpha])\| \le \delta\big) \ge 1 - \epsilon$$

and

$$\liminf_{n\to\infty} \mathbb{P}^n\big(\sup_{0\le t\le T} \|\overline{\boldsymbol{\mathcal{Z}}}^n(t)(H_0^b \cup V_0^b)\| \le \delta\big) \ge 1 - \epsilon.$$

Denote the corresponding events by $\Omega_{2,q}^n$ and $\Omega_{2,z}^n$, respectively. Now, choose h and h' such that $h\|\eta\| \le \delta/2$, $h \le \delta \vee \alpha$ and $h'(\|c^{max}\| \vee 1) \le \delta \vee b$, $h'\|\eta\| \le \delta/2$.

We shall show that $\Omega_1^n \cap \Omega_{2,q}^n \subseteq \Omega_q^n$ and $\Omega_1^n \cap \Omega_{2,z}^n \subseteq \Omega_z^n$. Take $0 \le s < t \le T$ with $t - s < h$. Let $\omega \in \Omega_1^n \cap \Omega_{2,q}^n$, we shall show that for any nonempty closed Borel set $B \subseteq \mathbb{R}_+$,

$$\overline{\mathcal{Q}}_{ij}^n(s)(B) \le \overline{\mathcal{Q}}_{ij}^n(t)(B^\delta) + \delta, \tag{8.11}$$

$$\overline{\mathcal{Q}}_{ij}^n(t)(B) \le \overline{\mathcal{Q}}_{ij}^n(s)(B^\delta) + \delta, \tag{8.12}$$

where $B^\delta := \{x \in \mathbb{R}_+ : \inf_{y\in B}\|x - y\| \le \delta\}$. Then (8.9) follows. First, we prove (8.11). Define $\tau := \inf\{s \le u \le t : \overline{Q}_{ij}^n(u) = 0\} \wedge t$. Then, we have that

$$\overline{\mathcal{Q}}_{ij}^n(s)(B) \le \overline{\mathcal{Q}}_{ij}^n(s)(B \cap [\alpha,\infty)) + \overline{\mathcal{Q}}_{ij}^n(s)([0,\alpha)) \le \overline{\mathcal{Q}}_{ij}^n(s)(B \cap [\alpha,\infty)) + \delta,$$

where the last inequality holds because $\omega \in \Omega_{2,q}^n$. Now, observe that

$$\overline{\mathcal{Q}}_{ij}^n(s)(B \cap [\alpha,\infty)) \le \overline{\mathcal{Q}}_{ij}^n(\tau)(B^\delta),$$

because $\tau - s < h < \delta \wedge \alpha$. To see the last statement, observe that if for some EV in the system at time s, $D_{ijl} - (s - \zeta_{ijl}) \in B$, then $D_{ijl} - (s - \zeta_{ijl}) - D_{ijl} + (\tau - \zeta_{ijl}) \le \delta$, which yields $D_{ijl} + (\tau - \zeta_{ijl}) \in B^\delta$. Finally, we have that

$$\overline{\mathcal{Q}}_{ij}^n(s)(B) \le \overline{\mathcal{Q}}_{ij}^n(\tau)(B^\delta) + \delta.$$

Now, if $\tau = t$, then (8.11) follows. If $\tau < t$, then $0 \le \overline{\mathcal{Q}}_{ij}^n(\tau)(B^\delta) \le \overline{Q}_{ij}^n(\tau) = 0$, and (8.11) follows. To show (8.12), we write

$$\begin{aligned}\overline{\mathcal{Q}}_{ij}^n(t)(B) &\le \overline{\mathcal{Q}}_{ij}^n(s)(B^\delta) + \overline{E}_{ij}^n(t) - \overline{E}_{ij}^n(s) + \overline{R}_{ij}^n(s) - \overline{R}_{ij}^n(t)\\ &\le \overline{\mathcal{Q}}_{ij}^n(s)(B^\delta) + \overline{E}_{ij}^n(t) - \overline{E}_{ij}^n(s),\end{aligned}$$

where the second inequality follows because $\overline{R}_{ij}^n(s) - \overline{R}_{ij}^n(t) \le 0$. Now, (8.12) follows because $\omega \in \Omega_1^n$. We conclude that $\omega \in \Omega_q^n$. The proof of $\Omega_1^n \cap \Omega_{2,z}^n \subseteq \Omega_z^n$ follows by similar arguments. □

8.2. Fluid Limits Satisfy the Fluid Model Solutions

The total number of EVs can be written as follows:

$$\overline{Q}_{ij}^n(t) = \overline{Q}_{ij}^n(0) + \overline{E}_{ij}^n(t) - \overline{R}_{ij}^n(t) - \overline{D}_{ij}^n(t), \tag{8.13}$$

where the number of rejected EVs $\overline{R}_{ij}^n(\cdot)$ is given by (2.13) and

$$\overline{D}_{ij}^n(t) := \frac{1}{n}\sum_{l=1}^{n\overline{Q}_{ij}^n(0)} \mathbb{1}_{\{D_{ijl}^0 \le t\}} + \frac{1}{n}\sum_{l=1}^{n\overline{E}_{ij}^n(t)} \mathbb{1}_{\{\zeta_{ijl}+D_{ijl}\le t\}}\, \mathbb{1}_{\{\overline{Q}_i^n(\zeta_{ijl}^-) < K_i\}}.$$

Proposition 8.3. *The fluid-scaled stochastic processes* $\overline{\boldsymbol{D}}^n(\cdot)$ *and* $\overline{\boldsymbol{R}}^n(\cdot)$ *are tight.*

Proof. First, we shall show that $\overline{D}_{ij}^n(\cdot)$ is a relatively compact sequence using Kurtz's criteria (Kang and Ramanan 2010, proposition 6.2), and then by Prokhorov's theorem, it is tight. Observe that almost surely

$$\overline{D}_{ij}^n(t) \le \frac{1}{n}\sum_{l=1}^{n\overline{Q}_{ij}^n(0)} \mathbb{1}_{\{D_{ijl}^0 \le t\}} + \frac{1}{n}\sum_{l=1}^{n\overline{E}_{ij}^n(t)} \mathbb{1}_{\{\zeta_{ijl}+D_{ijl}\le t\}} =: \overline{D}_{ij}^{n,\infty}(t),$$

and by Reed (2009), the latter is a weakly convergent sequence in $(\mathcal{D}[0,\infty), J_1)$ and hence it is tight. By Prokhorov's theorem it is also relatively compact. That is,

$$\lim_{c\to\infty}\mathbb{P}(\overline{D}^n_{ij}(t)>c)\le\lim_{c\to\infty}\mathbb{P}(\overline{D}^{n,\infty}_{ij}(t)>c)=0.$$

In other words, $\overline{D}^n_{ij}(\cdot)$ is stochastically bounded and hence satisfies the first property of Kurtz's criteria. To show that it also satisfies the second property, we write

$$\begin{aligned}\overline{D}^n_{ij}(t+\delta)-\overline{D}^n_{ij}(t)&=\overline{D}^{n,\infty}_{ij}(t+\delta)-\overline{D}^{n,\infty}_{ij}(t)+\frac{1}{n}\sum_{l=1}^{n\overline{E}^n_{ij}(t+\delta)}\mathbb{1}_{\{\zeta_{ijl}+D_{ijl}\le t+\delta\}}\mathbb{1}_{\{\overline{Q}^n_i(\zeta^-_{ijl})=K_i\}}\\&\quad-\frac{1}{n}\sum_{l=1}^{n\overline{E}^n_{ij}(t)}\mathbb{1}_{\{\zeta_{ijl}+D_{ijl}\le t\}}\mathbb{1}_{\{\overline{Q}^n_i(\zeta^-_{ijl})=K_i\}}\\&=\overline{D}^{n,\infty}_{ij}(t+\delta)-\overline{D}^{n,\infty}_{ij}(t)+\frac{1}{n}\sum_{l=1}^{n\overline{E}^n_{ij}(t+\delta)}\mathbb{1}_{\{t<\zeta_{ijl}+D_{ijl}\le t+\delta\}}\mathbb{1}_{\{\overline{Q}^n_i(\zeta^-_{ijl})=K_i\}}.\end{aligned}$$

For any $t\ge 0$ and $n\ge 1$,

$$\begin{aligned}\frac{1}{n}\sum_{l=1}^{n\overline{E}^n_{ij}(t+\delta)}\mathbb{1}_{\{t<\zeta_{ijl}+D_{ijl}\le t+\delta\}}\mathbb{1}_{\{\overline{Q}^n_i(\zeta^-_{ijl})=K_i\}}&\le\frac{1}{n}\sum_{l=1}^{n\overline{E}^n_{ij}(t+\delta)}\mathbb{1}_{\{t<\zeta_{ijl}+D_{ijl}\le t+\delta\}}\\&\le\sup_n\overline{E}^n_{ij}(t+\delta)<\infty.\end{aligned}$$

Furthermore, by continuity of the random variables ζ_{ijl} and D_{ijl}, we have that as $\delta\to 0$,

$$\frac{1}{n}\sum_{l=1}^{n\overline{E}^n_{ij}(t+\delta)}\mathbb{1}_{\{t<\zeta_{ijl}+D_{ijl}\le t+\delta\}}\to 0. \tag{8.14}$$

Putting all the pieces together,

$$|\overline{D}^n_{ij}(t+\delta)-\overline{D}^n_{ij}(t)|\le|\overline{D}^{n,\infty}_{ij}(t+\delta)-\overline{D}^{n,\infty}_{ij}(t)|+\frac{1}{n}\sum_{l=1}^{n\overline{E}^n_{ij}(t+\delta)}\mathbb{1}_{\{t<\zeta_{ijl}+D_{ijl}\le t+\delta\}}.$$

By (8.14), the fact that $\overline{D}^{n,\infty}_{ij}(\cdot)$ is relatively compact and using the same arguments as in Kaspi and Ramanan (2011, lemma 5.10), we conclude that $\overline{D}^n_{ij}(\cdot)$ satisfies the second property of Kurtz's criteria. That is, $\overline{D}^n_{ij}(\cdot)$ is relatively compact and hence tight. The tightness of $\overline{\boldsymbol{R}}^n(\cdot)$ follows by (8.13) and by the tightness of $\overline{\boldsymbol{D}}^n(\cdot)$ and $\overline{\boldsymbol{Q}}^n(\cdot)$. □

Next, we show that the fluid limits are bounded away from zero.

Proposition 8.4. *Let $(\overline{\mathcal{Q}}(\cdot),\overline{\boldsymbol{Q}}(\cdot),\overline{\mathcal{Z}}(\cdot),\overline{\boldsymbol{Z}}(\cdot))$ be a fluid limit. Assume that if $\overline{Q}_i(0)=K_i$, then $0<\overline{Q}_{ij}(0)<K_i$ for any $i,j\ge 1$. For any $\delta>0$, there exist $C_\delta>0$ and $C_\delta'>0$ such that almost surely*

$$\inf_{t\ge\delta}\min_{i,j}\overline{Q}_{ij}(t)\ge C_\delta\ \text{ and }\ \inf_{t\ge\delta}\min_{i,j}\overline{Z}_{ij}(t)\ge C'_\delta.$$

Proof. First, we shall show that $\overline{Q}_{ij}(\cdot)$ is strictly positive. It is enough to show this inequality when the system is not full. Fix $\Delta>\delta$. It is enough to show the result for $t\in[\delta,\Delta]$. Define

$$\tau^0_i:=\inf\{\delta\le s\le\Delta:\overline{Q}_i(s)=K_i\},\ \tilde{\tau}^0_i:=\inf\{\tau^0_i\le s\le\Delta:\overline{Q}_i(s)<K_i\},$$
$$\tau^r_i:=\inf\{\tilde{\tau}^{r-1}\le s\le\Delta:\overline{Q}_i(s)=K_i\},\ \tilde{\tau}^r_i:=\inf\{\tau^r_i\le s\le\Delta:\overline{Q}_i(s)<K_i\}.$$

Take a partition

$$(0,\Delta]\setminus\bigcup_r[\tau^r_i,\tilde{\tau}^r_i)\subseteq\bigcup_{1\le m\le N(\Delta)}((m-1)b/2,mb/2].$$

By our assumptions for the external arrival process, we have that for any m,

$$\frac{1}{n}\sum_{l=E^n_{ij}((m-1)b/2)+1}^{E^n_{ij}(mb/2)}\mathbb{1}_{\{D_{ijl}\ge b\}}\xrightarrow{d}(\overline{E}_{ij}(mb/2)-\overline{E}_{ij}((m-1)b/2))\mathbb{P}(D_{ij}>b)>0,$$

where b is a continuity point for the distribution $F_{D_{ij}}(\cdot)$ with $\mathbb{P}(D_{ij} > b) > 0$, and the last inequality follows because $\overline{E}_{ij}(\cdot)$ is strictly increasing. Choose b such that $\max_{ij}(\overline{E}_{ij}(mb/2) - \overline{E}_{ij}((m-1)b/2))\mathbb{P}(D_{ij} > b) < K_i$, and pick C_δ such that $\max_{ij}(\overline{E}_{ij}(mb/2) - \overline{E}_{ij}((m-1)b/2))\mathbb{P}(D_{ij} > b) > C_\delta$. Then, for large enough n, we have that for any $i, j \geq 1$,

$$\mathbb{P}^n\left(\inf_{\delta\le t\le\Delta}\overline{Q}^n_{ij}(t) \geq C_\delta\right) \geq \mathbb{P}^n\left(\inf_{(m-1)b/2\le t\le mb/2}\overline{Q}^n_{ij}(t) \geq C_\delta \text{ for any } m\right)$$
$$\geq \mathbb{P}^n\left(\sum_{l=E^n_{ij}((m-1)b/2)+1}^{E^n_{ij}(mb/2)} 1_{\{D_{ijl}\geq b\}} \geq C_\delta \text{ for any } m\right) \to 1.$$

Furthermore, note that by continuity of the limit, we have $\overline{Q}^n_{ij}(t) = \overline{Q}^n_{ij}(\tau^r_i) \geq C_\delta$ for $t \in [\tau^r_i, \tilde{\tau}^r_i)$. Finally, we have that there exists $C_\delta > 0$, such that, for any $\Delta > \delta$,

$$\mathbb{P}^n(\inf_{\delta\le t\le\Delta}\min_{i,j}\overline{Q}^n_{ij}(t) \geq C_\delta) \to 1,$$

as $n \to \infty$. For any compact set $C \subseteq R_+$, define the mapping $\phi_C : D(\mathbb{R}_+, \mathbb{R}^{I\times J}) \to \mathbb{R}$, given by $\phi_C(\boldsymbol{y}) := \inf_{t\in C}\min_{i,j} y_{ij}(t)$. Note that $\phi_C(\boldsymbol{y})$ is continuous at a continuous $\boldsymbol{y}(\cdot)$, which implies that

$$\phi_{[\delta,\Delta]}(\overline{\boldsymbol{Q}}^n) \xrightarrow{d} \phi_{[\delta,\Delta]}(\overline{\boldsymbol{Q}}).$$

By the Portmanteau theorem (Billingsley 1999, theorem 2.1), we have that

$$\mathbb{P}^n(\phi_{[\delta,\Delta]}(\overline{\boldsymbol{Q}}) \geq C_\delta) \geq \limsup_{n\to\infty}\mathbb{P}^n(\phi_{[\delta,\Delta]}(\overline{\boldsymbol{Q}}^n) \geq C_\delta) = 1.$$

We now move to the proof of $\overline{Z}_{ij}(t) > 0$ for $t > 0$. We first note that $\boldsymbol{Q}(\cdot)$ is independent of $\boldsymbol{Z}(\cdot)$, and hence we can assume that the fluid limit $(\overline{\boldsymbol{\mathcal{Q}}}(\cdot), \overline{\boldsymbol{Q}}(\cdot))$ satisfies the fluid model equations as we shall show later. That is, $(\overline{\boldsymbol{\mathcal{Q}}}(\cdot), \overline{\boldsymbol{Q}}(\cdot))$ satisfies the equations in Proposition 4.1. By Proposition 8.3, we have that the fluid-scaled process that describes the number of accepted EVs given in (2.12) converges weakly to $\overline{\boldsymbol{A}}(t) := \overline{\boldsymbol{E}}(t) - \overline{\boldsymbol{R}}(t)$. First, we show that $\overline{A}_{ij}(t)$ is strictly increasing for any $i, j \geq 1$. Let $t_1, t_2 \geq 0$ with $0 \leq t_1 < t_2$. Assume that there exists a subinterval in $[t_1, t_2]$ such that the total queue length at node i is full. Without loss of generality, assume that there exists $\tau \in [t_1, t_2]$ such that $\overline{Q}_i(s) = K_i$ for any $s \in [\tau, t_2]$. First, assume that $\tau > t_1$, then we have that

$$\overline{A}_{ij}(t_2) - \overline{A}_{ij}(t_1) = \overline{E}_{ij}(t_2) - \overline{R}_{ij}(t_2) - \overline{E}_{ij}(t_1) + \overline{R}_{ij}(t_1)$$
$$\geq \overline{E}_{ij}(t_2) - \overline{R}_{ij}(t_2) - \overline{E}_{ij}(t_1) \geq \overline{E}_{ij}(\tau) - \overline{E}(t_1) > 0.$$

If $\tau = t_1$, then by (4.3), (4.4), and the fact that $\overline{Q}_{ij}(t_2) = \overline{Q}_{ij}(t_1)$, we obtain

$$\overline{A}_{ij}(t_2) - \overline{A}_{ij}(t_1) = \overline{D}_{ij}(t_2) - \overline{D}_{ij}(t_1) = \int_{t_1}^{t_2}\delta_{ij}(s)ds,$$

where $\delta_{ij}(s) = \lim_{\epsilon\to 0}\frac{\overline{Q}_{ij}(s) - \overline{\mathcal{Q}}_{ij}(s)([\epsilon,\infty))}{\epsilon}$. Furthermore, by the proof of Proposition 7.1, we have that $\delta_{ij}(s) = \delta_{ij}(t_1) > 0$ for $s \in [t_1, t_2]$, and hence $\overline{A}_{ij}(t_2) - \overline{A}_{ij}(t_1) > 0$. Now, consider a type j EV l at node i. Observe that by the constraints $p_{ij}(\cdot) \leq c^{\max}_j$, we have that $\frac{B_{ijl}}{p_{ij}(\cdot)} \wedge D_{ijl} \geq \frac{B_{ijl}}{c^{\max}_j} \wedge D_{ijl}$. That is, EV l will stay in the network at least $\frac{B_{ijl}}{c^{\max}_j} \wedge D_{ijl}$ time units after its arrival. Hence, the stochastic process $Z_{ij}(\cdot)$ is bounded from below by the queue length $Q^{\inf}_{ij}(\cdot)$ of the infinite-server queue with arrival process $A_{ij}(\cdot)$, $Q^{\inf}_{ij}(0) = 0$, and i.i.d. service requirements $\left\{\frac{B_{ijl}}{c^{\max}_j} \wedge D_{ijl}, l \in \mathbb{N}\right\}$. Recalling that $\overline{A}_{ij}(\cdot)$ is strictly increasing by Remerova et al. (2014, lemma 3.14), there exists $C'_\delta > 0$ such that, for any $\Delta > \delta$,

$$\mathbb{P}^n(\inf_{\delta\le t\le\Delta}\min_{i,j}\overline{Z}^n_{ij}(t) \geq C_\delta{}') \geq \mathbb{P}^n(\inf_{\delta\le t\le\Delta}\min_{i,j}\overline{Q}^{n,\inf}_{ij}(t) \geq C_\delta{}') \to 1,$$

as $n \to \infty$. Now, using again the Portmanteau theorem, we have that

$$\mathbb{P}^n(\phi_{[\delta,\Delta]}(\overline{\boldsymbol{Z}}) \geq C_\delta{}') \geq \limsup_{n\to\infty}\mathbb{P}^n(\phi_{[\delta,\Delta]}(\overline{\boldsymbol{Z}}^n) \geq C_\delta{}') = 1. \quad \square$$

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8.2.1. Fluid Limits Are Fluid Model Solutions. In the sequel, we focus on proving that any fluid limit satisfies the fluid model equations given in Definition 4.1. Let $(\overline{\boldsymbol{\mathcal{Q}}}(\cdot),\overline{\boldsymbol{Q}}(\cdot),\overline{\boldsymbol{\mathcal{Z}}}(\cdot),\overline{\boldsymbol{Z}}(\cdot),\overline{\boldsymbol{R}}(\cdot))$ be a fluid limit along a subsequence, which with an abuse of notation, we denote again by $(\overline{\boldsymbol{\mathcal{Q}}}^n(\cdot),\overline{\boldsymbol{Q}}^n(\cdot),\overline{\boldsymbol{\mathcal{Z}}}^n(\cdot),\overline{\boldsymbol{Z}}^n(\cdot),\overline{\boldsymbol{R}}^n(\cdot))$. Recall that $\mathcal{C} := \{[x,\infty), x \in \mathbb{R}_+\}$ and $\mathcal{C}' := \{[x,\infty)\times[y,\infty), x,y \in \mathbb{R}_+\}$. Proposition 8.1 and Gromoll et al. (2008, lemma 6.2) imply that for any $\mathcal{A} \in \mathcal{C}$ and $\mathcal{A}' \in \mathcal{C}'$, almost surely $\overline{\mathcal{Q}}_{ij}(t)(\partial\mathcal{A}) = 0$ and $\overline{\mathcal{Z}}_{ij}(t)(\partial\mathcal{A}') = 0$ for $t \geq 0$ and $i,j \geq 1$. Hence, we can restrict $\mathcal{C}$ and $\mathcal{C}'$ to the following restricted classes: $\mathcal{C}_+ := \{[x,\infty), x>0\}$ and $\mathcal{C}'_+ := \{[x,\infty)\times[y,\infty), x\wedge y > 0\}$. In addition, we fix $T>0$ and we work in the time interval $[0,T]$.

The total number of type j EVs at node i can be written as follows:

$$\mathcal{Q}^n_{ij}(t)(A) = \mathcal{Q}^n_{ij}(0)(A+t) + \sum_{l=1}^{E^n_{ij}(t)} \mathbb{1}_A(D_{ijl} - (t-\zeta_{ijl}))\mathbb{1}_{\{Q^n_i(\zeta^-_{ijl})<K_i\}}.$$

Furthermore, the previous expression can be rewritten as

$$\mathcal{Q}^n_{ij}(t)(A) = \mathcal{Q}^n_{ij}(0)(A+t) + \sum_{l=1}^{A^n_{ij}(t)} \mathbb{1}_A(D_{ijl} - (t-\xi_{ijl})),$$

where ξ_{ijl} represents the time of the lth accepted EV and $A^n_{ij}(\cdot)$ represents the number of accepted type j EVs at node i. In the same way, the number of uncharged type j EVs at node i is given by

$$\mathcal{Z}^n_{ij}(t)(A') = \mathcal{Z}^n_{ij}(0)(A' + (S^n_{ij}(\boldsymbol{Z}^n,0,t),t)) + \sum_{l=1}^{A^n_{ij}(t)} \mathbb{1}_{A'}(B_{ijl} - S^n_{ij}(\boldsymbol{Z}^n,\xi_{ijl},t), D_{ijl} - (t-\xi_{ijl})).$$

In these expressions, we relabel the parking times and the charging requirements accordingly, where with abuse of notation we denote them by the same letters. Now, we can follow the strategy in Remerova et al. (2014, section 7.6). Consider a partition $0 < t_0 < \cdots < t_N = t$ and take a nonincreasing function $y(\cdot)$ in $[t_0,t]$ such that

$$\sup_{t_0\leq s\leq t} |S_{ij}(\overline{\boldsymbol{Z}}^n,s,t) - y(s)| \leq \delta,$$

for some $\delta > 0$. For $\xi_{ijl} \in (t_r,t_{r+1}]$, the following inequalities hold:

$$D_{ijl} - (t-t_r) \leq D_{ijl} - (t-\xi_{ijl}) \leq D_{ijl} - (t-t_{r+1}),$$
$$B_{ijl} - (y(t_r)+\delta) \leq B_{ijl} - S^n_{ij}(\boldsymbol{Z}^n,\xi_{ijl},t) \leq B_{ijl} - (y(t_{r+1})+\delta).$$

Now, define the following quantities:

$$\Gamma^{n,1}_{ij}(t) := \sum_{r=0}^{N-1}(\overline{A}_{ij}(t_{r+1}) - \overline{A}_{ij}(t_r))F_{D_{ij}}(A+(t-t_r)) - \tilde{X}^n,$$

$$\Gamma^{n,2}_{ij}(t) := \sum_{r=0}^{N-1}(\overline{A}_{ij}(t_{r+1}) - \overline{A}_{ij}(t_r))F_{D_{ij}}(A+(t-t_{r+1})) + \tilde{X}^n + \overline{A}_{ij}(t_0) + \tilde{X}^n,$$

$$\Gamma^{n,3}_{ij}(t) := \sum_{r=0}^{N-1}(\overline{A}_{ij}(t_{r+1}) - \overline{A}_{ij}(t_r))F_{ij}(A' + (y(t_r)+\delta, t-t_r)) - X^n,$$

$$\Gamma^{n,4}_{ij}(t) := \sum_{r=0}^{N-1}(\overline{A}_{ij}(t_{r+1}) - \overline{A}_{ij}(t_r))F_{ij}(A' + (y(t_{r+1})+\delta, t-t_{r+1})) + X^n + \overline{A}_{ij}(t_0) + X^n,$$

where

$$\tilde{X}^n := \sup_{A\in\mathcal{C}}\sup_{0\leq s\leq t\leq T}\left\|\overline{\mathfrak{L}}^{n,Q}(s,t)(A) - (\overline{\boldsymbol{A}}(t) - \overline{\boldsymbol{A}}(s))\circ \boldsymbol{F_D}(A)\right\|,$$

with

$$\overline{\mathfrak{L}}^{n,Q}_{ij}(s,t)(A) = \frac{1}{n}\sum_{l=1}^{A^n_{ij}(nt)}\delta_{D_{ijl}}(A) - \frac{1}{n}\sum_{l=1}^{A^n_{ij}(ns)}\delta_{D_{ijl}}(A),$$

and

$$X^n := \sup_{A' \in \mathcal{C}'} \sup_{0 \le s \le t \le T} \left\| \overline{\mathfrak{L}}^n(s,t)(A') - (\overline{\boldsymbol{A}}(t) - \overline{\boldsymbol{A}}(s)) \circ \boldsymbol{F}(A') \right\|,$$

with

$$\overline{\mathfrak{L}}_{ij}^n(s,t)(A') = \frac{1}{n} \sum_{l=1}^{A_{ij}^n(nt)} \delta_{(B_{ijl}, D_{ijl})}(A') - \frac{1}{n} \sum_{l=1}^{A_{ij}^n(ns)} \delta_{(B_{ijl}, D_{ijl})}(A').$$

Then, the following bounds hold:

$$\Gamma_{ij}^{n,1}(t) \le \overline{\mathcal{Q}}_{ij}^n(t)(A) - \overline{\mathcal{Q}}_{ij}^n(0)(A+t) \le \Gamma_{ij}^{n,2}(t)$$

and

$$\Gamma_{ij}^{n,3}(t) \le \overline{\mathcal{Z}}_{ij}^n(t)(A') - \overline{\mathcal{Z}}_{ij}^n(0)(A' + (S_{ij}^n(\boldsymbol{Z}^n, 0, t), t)) \le \Gamma_{ij}^{n,4}(t).$$

By Gromoll et al. (2008, lemma 5.1), we have that

$$\tilde{X}^n \xrightarrow{d} 0 \text{ and } X^n \xrightarrow{d} 0,$$

as $n \to \infty$. By Skorokhod's representation theorem (Billingsley 1999), we can assume that all random elements are defined on a common probability space. Furthermore, by the dominated convergence theorem (Rudin 1987), we have that $S_{ij}(\overline{\boldsymbol{Z}}^n, s, t) \to S_{ij}(\overline{\boldsymbol{Z}}, s, t)$, for $s \in [t_0, t]$ as $n \to \infty$. Moreover, the function $S_{ij}(\overline{\boldsymbol{Z}}, s, t)$ is continuous, and $S_{ij}(\overline{\boldsymbol{Z}}^n, s, t)$ is monotone in s. Hence, we have that

$$\sup_{t_0 \le s \le t} |S_{ij}(\overline{\boldsymbol{Z}}^n, s, t) - S_{ij}(\overline{\boldsymbol{Z}}, s, t)| \to 0.$$

Now, the convergence follows by adapting the conclusion of the proof of Remerova et al. (2014, theorem 5, section 7.6).

In the remainder, we show that the fluid limit also satisfies the additional relations in Definition 4.1. Observe that by (2.13) and the definition of the Riemann-Stieltjes integral, we have that

$$\overline{R}_i^n(t) := \sum_{j=1}^J \overline{R}_{ij}^n(t) = \int_0^t 1_{\{\overline{Q}_i^n(s^-) = K_i\}} d \sum_{j=1}^J \overline{E}_{ij}^n(s).$$

Now, define

$$\overline{H}_i^n(t) := \int_0^t \sum_{j=1}^J \lambda_{ij}(s) \, \mathbb{1}_{\{\overline{Q}_i^n(s^-) = K_i\}} ds,$$

and notice that

$$\overline{R}_i^n(t) - \overline{H}_i^n(t) = \int_0^t 1_{\{\overline{Q}_i^n(s^-) = K_i\}} d \sum_{j=1}^J (\overline{E}_{ij}^n(s) - \int_0^s \lambda_{ij}(u) du).$$

By our assumptions for the arrival processes, we obtain that $\overline{R}_i^n(\cdot) - \overline{H}_i^n(\cdot) \xrightarrow{d} 0$ as $n \to \infty$, and hence $\overline{H}_i^n(\cdot) \xrightarrow{d} \overline{R}_i(\cdot)$. Now, by (2.13), the number of rejected type j EVs at node i can be written as follows:

$$\overline{R}_{ij}^n(t) = \int_0^t 1_{\{\overline{Q}_i^n(s^-) = K_i\}} d(\overline{E}_{ij}^n(s) - \int_0^s \lambda_{ij}(u) du) + \int_0^t \frac{\lambda_{ij}(s)}{\sum_{h=1}^J \lambda_{ih}(s)} d\overline{H}_i^n(s).$$

Using the assumption of the external arrival process and the fact that $\overline{H}_i^n(\cdot) \xrightarrow{d} \overline{R}_i(\cdot)$, we derive that $\overline{R}_{ij}^n(\cdot) \xrightarrow{d} \overline{R}_{ij}(\cdot)$ and

$$\overline{R}_{ij}(t) = \int_0^t \frac{\lambda_{ij}(s)}{\sum_{h=1}^J \lambda_{ih}(s)} d\overline{R}_i(s).$$

We proved that any subsequential limit $(\overline{\boldsymbol{\mathcal{Q}}}(\cdot), \overline{\boldsymbol{Q}}(\cdot), \overline{\boldsymbol{\mathcal{Z}}}(\cdot), \overline{\boldsymbol{Z}}(\cdot), \overline{\boldsymbol{R}}(\cdot))$ satisfies the fluid model equations given in Definition 4.1, and hence the proof of Theorem 5.1 is completed.

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