Robust two-degree-of freedom control optimally balancing feedforward plant inversion and feedforward closed loop inversion

1st Leopoldo Jetto
Dep. of Information Engineering, UNIVPM, Ancona, Italy
l.ietto@univpm.it

2nd Valentina Orsini
Dep. of Information Engineering, UNIVPM, Ancona, Italy
vorsini@univpm.it

***Abstract*—This paper focuses on the problem of determining the most appropriate Two Degrees of Freedom (2DoF) control architecture, when the FeedForward (FF) action is the result of a stable model inversion procedure. The purpose is to define a control scheme with enhanced tracking performance even in the case of non minimum phase and/or non hyperbolic MIMO plants affected by polytopic uncertainty. The new proposed 2DoF control architecture is given by an optimal balance of the control actions produced by FeedForward Plant Inversion (FFPI) and FeedForward Closed Loop Inversion (FFCLI). This new architecture is referred to as FeedForward Optimally Balanced Inversion (FFOBI). Robustness with respect to polytopic uncertainty is obtained using a min-max optimization approach. Numerical results show that the FFOBI improves the tracking performances of both FFPI and FFCLI.**

Keywords: Output tracking, 2DoF control, model inversion, parametric uncertainty.

I. INTRODUCTION

The well established theoretical framework of model inversion [1]- [4] provides a powerful tool to define 2DoF control schemes based on a FeedForward (FF) inverting control action. This allows overcoming many limitations of classical 1DoF control schemes. For practical applications to output tracking problems, the most widely used 2DoF control architectures based on inverse control can be classified into two main categories: FeedForward Plant Inversion (FFPI) (see e.g. [5] - [6]), and FeedForward Closed Loop Inversion (FFCLI) (see e.g. [7]- [8]). In the FFPI methods the FF control is obtained through a plant model inversion procedure, in the FFCLI the FF input is obtained inverting the closed-loop system. The performances of the two architectures are discussed and compared in [9]- [10]. These papers evidence that the achievable tracking accuracy of both control schemes depend on the specific application and on the amount of uncertainty affecting the plant. As each of the two architectures has advantages over the other, it would be very useful to define a more general 2DoF control architecture optimally combining the FFPI and FFCLI control actions independently of the particular application. This problem can not be solved with the guidelines stated in the above papers because they refer to some specific applications and their qualitative nature makes it difficult any generalization. The problem considered in this paper is precisely to define a systematic quantitative method to obtain an FF control law such that: i) it is always given by an optimal combination of the FFPI and FFCLI control actions independently of the particular application, ii) it is robust with respect to polytopic uncertainties affecting the MIMO plant to be controlled. To the best authors knowledge, this is still an open question.

The solution proposed in this paper consists of an FFOBI control architecture where the two feedforward actions produced by FFPI and FFCLI are simultaneously acting and are optimally fused together minimizing a "worst case" quadratic cost functional of the predicted tracking error. The proposed approach can be framed in the context of pseudo-inversion [4], [11]. The results of these references are here extended to deal with a generalized robust inversion problem involving the simultaneous pseudo inversion of a polytopic plant and of the corresponding polytopic closed-loop system.

The design of the FFOBI architecture proposed in this paper can be summarized in the two following steps.

Step 1). Given a (possibly non-minimum phase and/or non hyperbolic) LTI polytopic plant Σ_p, an LTI dynamic output feedback controller is designed to guarantee the robust stability of the closed-loop system Σ_f. The controller is also endowed with an internal model of the steady-state component of the desired output $y_d(t)$ to be tracked.

Step 2). Σ_f is forced by two inputs $r_1(t)$ and $r_2(t)$ affinely depending on the outputs $\mathbf{s}_1(t)$ and $\mathbf{s}_2(t)$ of two feedforward input estimators IE_1 and IE_2 simultaneously operating according to the FFCLI and FFPI schemes respectively. The two signals $\mathbf{s}_1(t)$ and $\mathbf{s}_2(t)$ are searched in the linear space generated by B-spline basis functions of a fixed degree and are computed so that the corresponding $r_1(t)$ and $r_2(t)$ solve a "worst-case" optimization problem. Parametrizing $\mathbf{s}_1(t)$ and $\mathbf{s}_2(t)$ as B-splines involves significant advantages: B-spline functions are continuously differentiable universal approximators which admit a parsimonious parametric representation and belong to the convex hull defined by the relative control points [12]. These properties significantly reduce the number of parameters (the control points) with respect to which the quadratic cost functional is minimized. They also allow the minimization procedure to be formulated as a robust least square estimation problem where both the design matrix and observations are not exactly known due to plant uncertainty. The resulting optimal feedforward action is given by the optimal balance of the two contributions produced by FFPI and FFCLI control scheme because the estimated control points univocally define the corresponding B-splines and hence the corresponding $r_1(t)$ and $r_2(t)$. The weights of the two contributions are given by the 2-norms of the estimated B-splines. During the transient period, the optimal inputs are applied to Σ_f according to a Receding Horizon Control (RHC). This allows a continuous adaptation of $r_1(t)$ and $r_2(t)$, on the basis of a continuous upgrading of the state estimate.

The paper is organized in the following way. Some mathematical preliminaries are recalled in Section II. The FFOBI control architecture and problem statement are given in Section III. The solution of the problem is reported in Sections IV and V. A numerical example is shown in Section VI. Some concluding remarks are given in Section VII.

II. Mathematical Background

A. B-spline functions [12]

A scalar B-spline time function is defined as a linear combination of B-splines basis functions and control points:

$$s(t)=\sum_{i=1}^{\ell} c_i B_{i,d}(t), \qquad t\in[\hat{t}_1,\hat{t}_{\ell+d+1}]\subseteq \mathbb{R}, \tag{1}$$

where the c_i's are real numbers representing the control points of $s(t)$, the integer d is the degree of the B-spline, the $(\hat{t}_i)_{i=1}^{\ell+d+1}$ are the non decreasing knot points and the $B_{i,d}(t)$ are the B-spline basis functions which can be computed by the Cox-de Boor recursion formula, [12]. An equivalent representation of $s(t)$ in (1) is

$$s(t)=\mathbf{B}_d(t)\mathbf{c}, \qquad t\in[\hat{t}_1,\hat{t}_{\ell+d+1}]\subseteq \mathbb{R}, \tag{2}$$

where $\mathbf{c}\triangleq[c_1,\cdots,c_\ell]^T$ and $\mathbf{B}_d(t)\triangleq[B_{1,d}(t),\cdots,B_{\ell,d}(t)]$.

Convex hull property. Any value assumed by $s(t)$, $\forall t\in[\hat{t}_j,\hat{t}_{j+1}]$, $\; j>d$, lies in the convex hull of its $d+1$ control points $c_{j-d},\cdots,c_j$. $\triangle$

For a q-component vector $\mathbf{s}(t)=[s_1(t),\cdots,s_q(t)]^T$, a compact B-spline representation can be used

$$\mathbf{s}(t)=\bar{\mathbf{B}}_d(t)\bar{\mathbf{c}}, \quad t\in[\hat{t}_1,\hat{t}_{\ell+d+1}], \tag{3}$$

where $\bar{\mathbf{c}}\triangleq[\mathbf{c}_1{}^T,\cdots,\mathbf{c}_q{}^T]^T$ and $\bar{\mathbf{B}}_d(t)\triangleq \mathrm{diag}\,[\mathbf{B}_d(t),\cdots,\mathbf{B}_d(t)]$. Each $\mathbf{c}_i\triangleq[c_{i,1},\cdots,c_{i,\ell}]^T$, $i=1,\cdots,q$, is defined as in (2). The dimensions of $\bar{\mathbf{c}}$ are $(q\ell\times 1)$. The dimensions of the block diagonal matrix $\bar{\mathbf{B}}_d(t)$ are $(q\times q\ell)$.

Remark 1. From (2) it is apparent that, once the degree d and the knot points $\hat{t}_i$ have been fixed, the scalar B spline function $s(t)$, $t\in[\hat{t}_1,\hat{t}_{\ell+d+1}]$, is completely determined by the corresponding vector $\mathbf{c}$ of ℓ control points. $\triangle$

B. The robust least squares problem [13]

Given an overdetermined set of linear equations $Df\approx g$, with $D\in\mathbb{R}^{r\times m}$, $g\in\mathbb{R}^r$, subject to unknown but bounded errors: $\|\delta D\|_s\le\rho$ and $\|\delta g\|_s\le\xi$, the robust least squares estimate $\hat{f}\in\mathbb{R}^m$ is the value of f minimizing

$$\min_f \max_{\|\delta D\|_s\le\rho,\ \|\delta g\|_s\le\xi} \|(D+\delta D)f-(g+\delta g)\|, \tag{4}$$

where $\|\cdot\|_s$ denotes the spectral norm.

As shown in ([13], p. 206), problem (4) is equivalent to minimizing a sum of Euclidean norms

$$\min_f \ \|D\,f-g\|+\rho\|f\|+\xi \tag{5}$$

Possible constraints on f of the kind

$$\underline{f}\le f\le\bar{f} \tag{6}$$

can be taken into account by imposing all the scalar linear inequalities deriving from the above vector constraint.

III. The FFOBI control scheme and problem statement

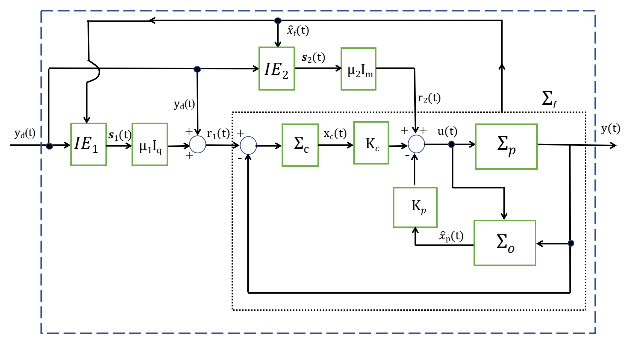

Fig. 1. The FFOBI control scheme

The new 2DoF control scheme proposed in this paper is shown in Fig. 1 where, without any loss of generality, a unitary feedback is assumed. The two blocks IE_1 and IE_2 are two feedforward input estimators operating according to the FFCLI and FFPI schemes respectively. The inputs of both feedforward filters are the desired output $y_d(t)$ to be tracked and the estimated state $\hat{x}_f(t)$ of Σ_f. The outputs of IE_1 and IE_2 are the two B-splines $\mathbf{s}_1(t)\in\mathbb{R}^q$ and $\mathbf{s}_2(t)\in\mathbb{R}^m$ respectively. The two scalars μ_1 and μ_2 are two binary variables. The FFOBI control scheme optimally combining the FFCLI and FFPI architectures corresponds to $\mu_1=\mu_2=1$. If $\mu_1=1$ and $\mu_2=0$, then $r_1(t)=y_d(t)+\mathbf{s}_1(t)$ and $r_2(t)=0$, so that the FFCLI is obtained, while $\mu_1=0$ and $\mu_2=1$ give $r_1(t)=y_d(t)$ and $r_2(t)=\mathbf{s}_2(t)$, so that the FFPI is obtained. For $\mu_1=\mu_2=0$, the 2DoF control scheme reduces to the usual 1DoF feedback control scheme with no feedforward action.

The block Σ_f is the feedback connection of a (possibly non-minimum phase and/or non hyperbolic) LTI polytopic plant Σ_p with an LTI robustly stabilizing dynamic output feedback controller. The plant $\Sigma_p\equiv(C_p,A_p(\alpha),B_p)$ is given by

$$\begin{aligned}\dot{x}_p(t) &= A_p(\alpha)x_p(t)+B_pu(t), u(t)\in\mathbb{R}^m, &(7)\\ y(t) &= C_px_p(t),\ y(t)\in\mathbb{R}^q, &(8)\end{aligned}$$

where: $x_p(t) \in \mathbb{R}^{n_p}$, $A_p(\alpha) \in \mathcal{A} \triangleq \mathrm{co}\{A_{p_i}, \cdots, A_{p_l}\} = \{A_p(\alpha) = \sum_{i=1}^{l} \alpha_i A_{p_i},\ \alpha \in \Lambda_l\}$.
The following assumptions on Σ_p are made:
A1): Σ_p is robustly stabilizable by an LTI dynamic output controller;
A2) let the steady state component $\tilde{y}_d(t)$ of the desired output trajectory $y_d(t)$ be generated as the free output response of an LTI unstable system Σ_y, then for no $\alpha \in \Lambda_l$, Σ_p has a transmission zero coinciding with an eigenvalue of the dynamical matrix of Σ_y.
The dynamic output feedback controller includes the internal model Σ_c of the steady state component of the desired output trajectory $y_d(t)$, whose state-space has the form $\dot{x}_c(t) = A_c x_c(t) + B_c(r_1(t) - y(t))$ for suitably defined A_c and B_c [14], and a full state observer Σ_o of the form

$$\dot{\hat{x}}_p(t) = \bar{A}_p \hat{x}_p(t) + B_p u(t) + L(y(t) - C_p\, \hat{x}_p(t)), \qquad (9)$$

where: $\bar{A}_p \triangleq (\sum_{i=1}^{l} A_{p_i})/l$ is the assumed nominal dynamical matrix of the plant.
The input $u(t) \in \mathbb{R}^m$ forcing the polytopic plant Σ_p is given by $u(t) = -K_p \hat{x}_p(t) + K_c x_c(t) + r_2(t)$.
The state space $(C_f, A_f(\alpha), B_f)$ of the closed loop system Σ_f with $x_f(t) \triangleq [x_p^T(t),\, x_c^T(t),\, \hat{x}_p^T(t)]^T \in \mathbb{R}^n$, $n \triangleq 2n_p + n_c$ and $r(t) \triangleq [r_1^T(t),\, r_2^T(t)]^T \in \mathbb{R}^{q+m}$ is

$$\begin{aligned} \dot{x}_f(t) &= \begin{bmatrix} A_p(\alpha) & B_p K_c & -B_p K_p \\ -B_c C_p & A_c & 0 \\ L C_p & B_p K_c & \bar{A}_p - L C_p - B_p K_p \end{bmatrix} x_f(t) \\ &+ \begin{bmatrix} 0 & B_p \\ B_c & 0 \\ 0 & B_p \end{bmatrix} r(t) \end{aligned} \qquad (10)$$

$$y(t) = \begin{bmatrix} C_p & 0 & 0 \end{bmatrix} x_f(t) \qquad (11)$$

where, analogously to $A_p(\alpha)$, also $A_f(\alpha) \in \mathcal{A}_f \triangleq \mathrm{co}\{A_{f_i}, \cdots, A_{f_l}\}$.
The existence of matrices K_c, K and L such that Σ_f is internally asymptotically stable $\forall A_f(\alpha) \in \mathcal{A}_f$, is guaranteed by assumption **A1**. These matrices can be computed by any existing robust stabilizing technique. This problem is not discussed here because the focus of the paper is on computing an optimally balanced feedforward control action $r(t)$. The presence of the internal model Σ_c and **A2** guarantees robust exact asymptotic tracking even with no feedforward action [14]. Hence, the problem of determining an optimally combined feedforward action can be limited to a sufficiently long but finite interval over which steady-state is practically attained. This drastically reduces the computational burden of the numerical procedure for the minimization of the cost functional (see Section V).

For this reason the following signals are partitioned in a transient and steady state component as follows:

$$y_d(t) = \begin{cases} y_{d,t}(t) & t \in [0, t_y) \triangleq T_y \\ \tilde{y}_d(t) & t \geq t_y \\ y_{d,t}(t_y^-) = \tilde{y}_d(t_y) \end{cases} \qquad (12)$$

where $y_{d,t}(t)$ and $\tilde{y}_d(t)$ are smooth functions denoting the transient and steady state components of $y_d(t)$, respectively, T_y is the time interval over which $y_{d,t}(t)$ is required to converge towards $\tilde{y}_d(t)$. According to assumption **A2**, the desired $\tilde{y}_d(t)$ is any function generated as the free output response of an unstable LTI system Σ_y.
Analogously:

$$\mathbf{s}_i(t) = \begin{cases} \mathbf{s}_{i,t}(t) & t \in [0, t_r) \triangleq T_r \\ \tilde{\mathbf{s}}_i(t) & t \geq t_r \\ \mathbf{s}_{i,t}(t_r^-) = \tilde{\mathbf{s}}_i(t_r) \end{cases} \quad i = 1, 2 \qquad (13)$$

$$r(t) = \begin{cases} r_t(t) \triangleq \begin{bmatrix} y_d(t) + \mu_1 \mathbf{s}_{1,t}(t) \\ \mu_2 \mathbf{s}_{2,t}(t) \end{bmatrix} & t \in [0, t_r) \triangleq T_r \\ \tilde{r}(t) \triangleq \begin{bmatrix} \tilde{y}_d(t) + \mu_1 \tilde{\mathbf{s}}_1(t) \\ \mu_2 \tilde{\mathbf{s}}_2(t) \end{bmatrix} & t \geq t_r \\ r_t(t_r^-) = \tilde{r}(t_r) \end{cases} \qquad (14)$$

The time instant t_r has to be sufficiently large to guarantee that $r_t(t_r^-) = \tilde{r}(t_r)$ (namely $\mathbf{s}_{i,t}(t_r^-) = \tilde{\mathbf{s}}_i(t_r)$, $i = 1, 2$) and the actual output $y(t)$ under the action of $r_t(t)$ has almost achieved the steady-state trajectory. A way to fix a lower bound on t_r is: $t_r \geq t_{st}$, where t_{st} is the settling time relative to the output of the nominal $\bar{\Sigma}_f \triangleq (C_f, \bar{A}_f, B_f)$ forced by $r(t) = [y_d(t)\ \ 0]^T$. Like $\bar{A}_p$, also $\bar{A}_f$ is chosen as the centroid of $\mathcal{A}_f$: $\bar{A}_f \triangleq \frac{\sum_{i=1}^{l} A_{f_i}}{l}$.
Definition. The optimal combination of FFPI and FFCLI is the one giving a minimum 2-norm predicted transient tracking error.

By the above definition, the problem of optimally balancing FFPI and FFCLI can be restated as the following equivalent Robust Almost Exact Output Tracking Problem.

(RAEOTP) Let $\Sigma_f \equiv (C_f, A_f(\alpha), B_f)$ be a robustly asymptotically stable closed described by (10), (11) with unknown initial state $x_f(0)$. Given a desired $y_d(t)$ defined as in (12), it is required to find a feedforward control input $r(t)$ defined as in (14) satisfying the following conditions $\forall A_f(\alpha) \in \mathcal{A}_f$:
Transient conditions: i) $r_t(t)$ is converging to $\tilde{r}(t_r)$ over T_r, ii) $r_t(t)$ is the result of an RHC strategy applied to the minimization of a suitably defined "worst case" quadratic cost functional of the predicted transient tracking error (see (15)-(17) of Section IV.B)
Steady-state condition: $\tilde{r}(t)$ yields a steady-state tracking error asymptotically converging to zero.
Boundedness condition: $r(t)$ is uniformly bounded for any uniformly bounded $y_d(t)$.

IV. COMPUTATION OF $r(t)$

In accordance with definition (14), this step is performed through a separate computation of the steady state $\tilde{r}(t)$ and transient $r_t(t)$ components of $r(t)$.

A. Computation of $\tilde{r}(t)$

As Σ_f is robustly asymptotically stable, then, by **A2**, the steady-state condition is automatically satisfied endowing the dynamic output feedback controller with the internal model of $\tilde{y}_d(t)$ [14]. Recalling the assumption of a unitary feedback, it is enough to choose $\tilde{s}_1(t) = \tilde{s}_2(t) = 0$, for $t \geq t_r$, which, by (14), implies $\tilde{r}(t) = [\tilde{y}_d^T(t)\ 0^T]^T$.

B. Computation of $r_t(t)$

In the following, the explicit dependence on α of the predicted output and tracking error will be omitted for simplicity of notation.

The robust optimization problem is numerically solved imposing to $r_{1,t}(t) = y_d(t) + \mu_1 \mathbf{s}_{1,t}(t)$ and $r_{2,t}(t) = \mu_2 \mathbf{s}_{2,t}(t)$ the structure deriving from the assumption of modeling the transient components of $\mathbf{s}_1(t)$ and $\mathbf{s}_2(t)$ respectively as B-spline functions given by (3). The parameter vector defining $r_t(t)$ is computed as the solution of the constrained optimization problem defined beneath.

Let $T'_r \triangleq [0, t_r + t_w)$ be partitioned as $T'_r = \bigcup_{k=0}^{n_r-1} T_k$, where $T_k \triangleq [t_k, t_k + t_w)$, $k = 0, 1, \cdots, n_r - 1$, with $t_k \triangleq k\Delta$ and $\Delta \triangleq \frac{t_r}{n_r}$ are disjoint sub-intervals such that: $t_0 = 0$, $t_{n_r} = n_r\Delta = t_r$. According to the RHC strategy, $t_w = w\Delta$, denotes the length of the moving window T_k, for a fixed $w \in \mathbb{Z}^+$. The transient $r_t(t)$ is determined from the minimization of the following "worst case" quadratic cost functional for any fixed $k = 0, 1, \cdots, n_r - 1$:

$$\max_{\alpha \in \Lambda_l} J_{k,\alpha} \triangleq \max_{\alpha \in \Lambda_l} \sum_{i=1}^{w-1} e^T(t_{k+i}|t_k) Q(t_k) e(t_{k+i}|t_k), \quad (15)$$

where

$$e(t_{k+i}|t_k) \triangleq y_d(t_{k+i}) - y(t_{k+i}|t_k), \quad (16)$$

with

$$\begin{aligned} y(t_{k+i}|t_k) &\triangleq C_f e^{A_f(\alpha)(t_{k+i}-t_k)} \hat{x}_f(t_k) \\ &+ \int_{t_k}^{t_{k+i}} C_f e^{A_f(\alpha)(t_{k+i}-\tau)} B_f r_t(\tau) d\tau, \quad (17) \end{aligned}$$

is the predicted tracking error at time t_{k+i} based on the state estimate $\hat{x}_f(t_k)$.

By definition of $r_t(t)$ and according to (3) one has

$$\begin{aligned} r_t(t) &= \begin{bmatrix} \mu_1 \bar{\mathbf{B}}_{d_1}(t) & 0 \\ 0 & \mu_2 \bar{\mathbf{B}}_{d_2}(t) \end{bmatrix} \begin{bmatrix} \bar{\mathbf{c}}_1 \\ \bar{\mathbf{c}}_2 \end{bmatrix} + \begin{bmatrix} y_d(t) \\ 0 \end{bmatrix} \\ &\triangleq \tilde{\mathbf{B}}(t)\tilde{\mathbf{c}} + \mathbf{y}_d(t) \quad (18) \end{aligned}$$

where the integer d_1 (d_2) indicates the degree of the scalar B spline functions composing $r_{t,1}(t) \in \mathbb{R}^q$ ($r_{t,2}(t) \in \mathbb{R}^m$). The dimensions of $\tilde{\mathbf{c}}$ are $(q\ell_1 + m\ell_2) \times 1$. The dimensions of the block diagonal matrix $\tilde{\mathbf{B}}(t)$ are $(q + m) \times (q\ell_1 + m\ell_2)$.

By (17),(18) and setting $\tilde{\mathbf{c}} = \tilde{\mathbf{c}}_k$, $e(t_{k+i}|t_k)$ results to be given by

$$\begin{aligned} e(t_{k+i}|t_k) &= y_d(t_{k+i}) - C_f e^{A_f(\alpha)(t_{k+i}-t_k)} \hat{x}_f(t_k) \\ &- \int_{t_k}^{t_{k+i}} C_f e^{A_f(\alpha)(t_{k+i}-\tau)} B_f \tilde{\mathbf{B}}(\tau) d\tau\, \tilde{\mathbf{c}}_k \\ &- \int_{t_k}^{t_{k+i}} C_f e^{A_f(\alpha)(t_{k+i}-\tau)} B_f \mathbf{y}_d(t) \quad (19) \end{aligned}$$

The input function $r_t(t), t \in T_r$, affinely dependent on $\mathbf{s}_{1,t}(t)$ and $\mathbf{s}_{2,t}(t)$, is robustly estimated minimizing the worst case error due to the parametric uncertainty. More precisely $r_t(t), t \in T_r$, is obtained solving the following sequence of n_r Min-Max Constrained Optimization Problems (MMCOP)

$$\text{MMCOP:} \quad \min_{\tilde{\mathbf{c}}_k} \max_{\alpha \in \Lambda_l} J_{k,\alpha}, \quad k = 0, \cdots, n_r - 1, \quad (20)$$

$$\text{subject to} \quad \tilde{\mathbf{c}}_{k,\min} \leq \tilde{\mathbf{c}}_k \leq \tilde{\mathbf{c}}_{k,\max}. \quad (21)$$

At $k = 0$ ($t_0 = 0$), the MMCOP is solved with reference to a $J_{0,\alpha}$, which, by (15), is defined over $[t_0, t_0 + t_w) = [0,\ t_w)$ and the corresponding minimizing $\tilde{\mathbf{c}}_0$ defines $r_t(t)$ over the same interval. According to the RHC strategy only the restriction of $r_t(t)$ to $T_0 = [t_0, t_1)$ is applied to Σ_f. Analogously, for $k = 1$, ($t_1 = \Delta$), the minimizing $\tilde{\mathbf{c}}_1$ gives $r_t(t)$, $t \in [t_1, t_1 + t_w)$ but only $r_t(t), t \in T_1 = [t_1, t_2)$ is applied. The iterative procedure stops at $k = n_r - 1$. The advantage of the above RHC policy is the possibility of exploiting an updated state estimate $\hat{x}_f(t_k)$ to improve the tracking error prediction. Over each moving window $[t_k, t_k + t_w)$, the constraints (21) on $\tilde{\mathbf{c}}_k$ are chosen so as to impose the convergence of $\mathbf{s}_{1,t}(t)$ and $\mathbf{s}_{2,t}(t)$ to the respective null steady state components within T_r according to (13). Consequently the convergence of $r_t(t)$ towards $\tilde{r}(t)$ is guaranteed. This assures the continuity of $r(t)$.

V. ROBUST ESTIMATION OF $r_t(t)$

This section shows how the MMCOP stated in Section IV can be reformulated as a robust least square problem. The starting point is to rewrite the closed loop dynamical matrix $A_f(\alpha)$ as $A_f(\alpha) \triangleq \bar{A}_f + \delta A_f(\alpha)$, $\alpha \in \Lambda_l$ where $\bar{A}_f$ is the nominal plant. Using the matrix identity $e^{(A+E)t} = e^{At} + \int_0^t e^{A(t-s)} E e^{(A+E)s} ds$ and replacing A and E with $\bar{A}_f$ and $\delta A_f(\alpha)$ respectively, one has

$$e^{(\bar{A}_f + \delta A_f(\alpha))t} = e^{\bar{A}_f t} + \int_0^t e^{\bar{A}_f(t-s)} \delta A_f(\alpha) e^{A_f(\alpha)s} ds \quad (22)$$

Then for any fixed $k = 0, 1, \cdots n_r - 1$, exploiting (22), the predicted $e(t_{k+i}|t_k)$ given by (19), can be rewritten as

$$\begin{aligned} e(t_{k+i}|t_k) &= (b(t_{k+i}|t_k) + \delta b_\alpha(t_{k+i}|t_k)) \\ &- (D(t_{k+i}|t_k) + \delta D_\alpha(t_{k+i}|t_k)) f_k \quad (23) \end{aligned}$$

where $f_k = \tilde{\mathbf{c}}_k$. The remaining terms (not explicitly reported for page limits) can be easily derived taking into account that only $\delta b_\alpha(t_{k+i}|t_k)$ and $\delta D_\alpha(t_{k+i}|t_k)$ are depending on α. Defining augmented vectors and matrices, equation (23) for $i = 1, \cdots, w - 1$ can be expressed in the compact form

$$\underline{e}_k(\alpha) = (\underline{b}_k + \delta \underline{b}_k(\alpha)) - (\underline{D}_k + \delta \underline{D}_k(\alpha)) f_k,$$

where $\underline{e}_k(\alpha) \triangleq [e^T(t_{k+1}|t_k), \cdots, e^T(t_{k+(w-1)}|t_k)]^T$. For each $k = 0, \cdots, n_r - 1$, functional $J_{k,\alpha}$ in (20) can be written as $J_{k,\alpha} = \underline{e}'_k(\alpha)^T \underline{e}'_k(\alpha)$, where $\underline{e}'_k(\alpha) \triangleq \underline{Q}_k^{1/2} \underline{e}_k(\alpha)$ and $\underline{Q}_k \triangleq \text{diag}[Q(t_k), \cdots, Q(t_k)]$. Also defining $\underline{b}'_k + \delta \underline{b}'_k(\alpha) \triangleq \underline{Q}_k^{1/2}(\underline{b}_k + \delta \underline{b}_k(\alpha))$ and $\underline{D}'_k + \delta \underline{D}'_k(\alpha) \triangleq \underline{Q}_k^{1/2}(\underline{D}_k + \delta \underline{D}_k(\alpha))$, it is evident that each MMCOP is equivalent to the constrained minimization of the squared 2-norm of the worst-case weighted residual $\underline{e}'_k(\alpha)$. Hence the sequence of the n_r MM-COP (20), (21) is equivalent to solve the following sequence of Constrained Robust Least Square Problems (CRLSP).

$$\min_{f_k} \max_{\|\delta \underline{D}'_k(\alpha)\|_s \le \rho_k \; \|\delta \underline{b}'_k(\alpha))\|_s \le \xi_k}$$
$$\|(\underline{D}'_k + \delta \underline{D}'_k(\alpha)) f_k - (\underline{b}'_k + \delta \underline{b}'_k(\alpha))\|, \tag{24}$$
$$\text{subject to } f_{k,\min} \le f_k \le f_{k,\max}, k = 0, 1, \cdots, n_r - 1, \tag{25}$$

where: (24) is of the kind (4) and (25) is of the kind (6).

Remark 2 The numerical calculation of ρ_k and ξ_k can be greatly simplified taking into account the following:

1. *As the term ξ of (5) is independent of f, it cannot be minimized. Hence it can be removed from the objective function. This, in turn, implies that in (24) only the upper bound ρ_k on $\|\delta \underline{D}'_k(\alpha)\|_s$ needs to be determined at each k.*
2. *The way the B-spline basis functions are defined by the Cox de Boor formula [12], implies that $\tilde{\mathbf{B}}(\tau) = \tilde{\mathbf{B}}(\tau + t_w)$, $\forall \tau \in [t_k, t_k + t_w)$, $k = 0, 1, \cdots, n_r - 1$ and hence it can be easily proven that $\rho_k \stackrel{\triangle}{=} \rho$, $\forall k = 0, 1, \cdots, n_r - 1$.*
3. *The calculation of ρ can be entirely executed off-line by performing a gridding on the parameter vector $\alpha \in \Lambda_l$.*

Theorem Under Assumptions A1 and A2, the feedforward input $r(t)$ of Σ_f resulting from the solution of the CRLSP (24), (25), and from the above RHC strategy, solves the RAEOTP. $\triangle$

The proof of Theorem is not reported for page limits.

VI. Numerical results

The example considered here is a more involved version of the tracking problem considered in [16].

The linearized model (26) is the unstable, non minimum phase near non hyperbolic system which represents the aircraft trimmed at a nominal 5° pitch attitude, with a mid-range weight, a mid-position center of gravity and operating in-ground effect at near sea level. The model is described by

$$\dot{x}_p(t) = A_p x_p(t) + B_p u(t), \; y(t) = C_p x_p(t) \tag{26}$$

where

$$x_p = \begin{pmatrix} U \\ W \\ Q \\ V \\ P \\ R \\ \theta \\ \chi \end{pmatrix} = \begin{pmatrix} \text{forward velocity} \\ \text{vertical velocity} \\ \text{pitch rate} \\ \text{lateral velocity} \\ \text{roll rate} \\ \text{yaw rate} \\ \text{pitch attitude} \\ \text{roll attitude} \end{pmatrix} \in \mathbb{R}^{n_p},$$
$$y = \begin{pmatrix} U & W & V & R \end{pmatrix}^T \in \mathbb{R}^q,$$
$$u = \begin{pmatrix} \delta_C \\ \delta_B \\ \delta_A \\ \delta_P \end{pmatrix} = \begin{pmatrix} \text{collective} \\ \text{longitudinal cyclic} \\ \text{lateral cyclic} \\ \text{tail rotor collective} \end{pmatrix} \in \mathbb{R}^m$$

The vectors $y(t)$, $u(t)$, represent the controlled and manipulated variables respectively. Like [16], the state vector $x_p(t) \in \mathbb{R}^n$, $n = 8$, is assumed to be measurable. The entries of A_p and of B_p (not reported for brevity) can be found in [16], matrix C_p directly follows by the way $y(t)$ is defined. With respect to [16], a polytopic dynamical matrix $A_p(\alpha)$ is here assumed. The dynamical matrix A_p reported in [16] is considered the nominal matrix $\bar{A}_p$ of $A_p(\alpha)$. The three elements $a_{i,i}$, $i = 2, 3, 5$ of $A_p(\alpha)$, have been here assumed to belong to the intervals $[\bar{a}_i - \varepsilon_{a_{i,i}}, \; \bar{a}_i + \varepsilon_{a_{i,i}}]$, $i = 2, 3, 5$, centered on the nominal value $\bar{a}_i$ where: $\bar{a}_2 = -0.39$, $\bar{a}_3 = -0.19$ and $\bar{a}_5 = -0.57$. The following uncertainty scenario is considered $\mathcal{S} = (\varepsilon_{a_{2,2}}, \varepsilon_{a_{3,3}}, \varepsilon_{a_{5,5}}) = (0.29, 0.09, 0.43)$.

For $\mathcal{S}$, the respective uncertain open loop plant $A_p(\alpha)$ belongs to the polytopic set $\mathcal{A} \stackrel{\triangle}{=} \{A_p(\alpha) = \sum_{i=1}^{l} \alpha_i A_{p_i}, \, \alpha \in \Lambda_l\}$ with $l = 8$.

As in [16], the forward velocity and the yaw rate are to be kept at zero $\forall t \in \mathbb{R}^+$, while the desired behavior of W and V is the smooth function converging to the constant value 0.2 in the interval $T_y = [0, t_y) = [0, 13)$, as shown in Fig. 2. Unlike [16], the unnecessity of a pre-actuation in the proposed approach allowed us to freely assign the desired profiles without requiring them to be null over an initial sufficiently long time interval.

The first step is the design of a robustly stabilizing feedback controller. As the state is assumed to be measurable, the observer Σ_o is not necessary. This directly implies that the control input $u(t)$ is given by $u(t) = -K_p x_p(t) + K_c x_c(t) + r_2(t)$. The gain matrices K_p and K_c defining a robustly stabilizing controller for $\mathcal{S}$ have been computed imposing the following eigenvalues $[-0.7 \pm 0.73i, \; -2 \pm 1.5i, \; -5, \; -5.5, \; -4.95, \; -3, \; -2.8, \; -2.24, \; -0.61, \; -0.33]$ to the nominal closed loop dynamical matrix $\bar{A}_f = \begin{bmatrix} \bar{A}_p & 0 \\ -B_c C_p & A_c \end{bmatrix} - \begin{bmatrix} B_p \\ 0 \end{bmatrix} \begin{bmatrix} K_p & -K_c \end{bmatrix}$.

For the given $\Sigma_p \equiv (C_p, A_p(\alpha), B_p)$, it is easy to verify that the system matrix $\begin{bmatrix} sI - A_p(\alpha) & B_p \\ -C_p & 0 \end{bmatrix}$ has rank $n + q = 12$ at $s = 0$, $\forall \alpha \in \Lambda_l$. This implies the fulfillment of $\mathbf{A}_2$ and, as a consequence, guarantees an exact steady-state tracking, provided endowing the stabilizing controller with the internal model Σ_c of the external reference [14]. As $\tilde{y}_d(t) \in \mathbb{R}^q$, $q = 4$, then the internal model Σ_c of constant signals is defined by $A_c = 0_{q \times q} = 0_{4 \times 4}$ and $B_c = I_{q \times q} = I_{4 \times 4}$.

The second step is to compute the input $r(t)$ solving the **RAEOT** problem. According to Section IV.A one has $\tilde{r}(t) = \begin{bmatrix} \tilde{y}_d^T(t) & \mathbf{0}_m^T \end{bmatrix}^T$, $\forall t \ge t_r$ where: $\mathbf{0}_\ell$ denotes the column vector of ℓ null elements and $t_r = 30 > t_s = 25$ is chosen. Both the transient components $s_{1,t}(t) \in \mathbb{R}^q$, and $s_{2,t}(t) \in \mathbb{R}^m$, defining $r_t(t) \stackrel{\triangle}{=} \begin{bmatrix} y_d(t) + \mu_1 s_{1,t}(t) \\ \mu_2 s_{2,t}(t) \end{bmatrix}$ have been modeled as two B spline functions vectors with $d_1 = d_2 = 1$ (order of the B-spline function) and $\ell_1 = \ell_2 = 4$ (number of control points defining the B-spline function over each moving window of length t_w). Fig. 2 evidences a desired $y_{d,t}(t)$, $t \in T_y$, given by two fast but smooth transitions between two set points. The time interval of each transition is $t_{tr} = 2$ and the value $t_w = t_{tr} = 2$ is chosen. By (18) it directly follows that $\tilde{\mathbf{c}} \stackrel{\triangle}{=} \begin{bmatrix} \bar{\mathbf{c}}_1 \\ \bar{\mathbf{c}}_2 \end{bmatrix} \in \mathbb{R}^{q\ell_1 + m\ell_2 = 32}$, $\bar{\mathbf{B}}_{d_1}(t) = \mathbf{B}_d(t)$ has dimensions

$1 \times q\ell_1$ and $\bar{\mathbf{B}}_{d_2}(t) = \mathbf{B}_d(t)$ has dimensions $1 \times m\ell_2$.

As $t_r = 30$ and $t_w = 2$ one has $T'_r = [0, 32)$ and choosing $n_r = 300$ one has $\Delta = 0.1$ and $w = 20$. The vector $f_k \triangleq \tilde{\mathbf{c}}_k$ defining $r_t(t)$ over each moving prediction horizon $[t_k, t_k + t_w) = [t_k, t_{k+w})$, $k = 0, \cdots, n_r - 1 = 299$, is iteratively estimated solving the sequence of n_r CRLSP (24),(25), using the software Yalmip [15]. All the weight matrices $Q(t_k)$ are set equal to the identity matrix. A large interval $[f_{k,\min}, f_{k,\max}]$ is initially chosen to allow $r_t(t)$ to freely vary at the beginning of the transition period $0 \le k \le k_y \triangleq \frac{t_y}{\Delta}$. For $k > k_y$, $f_{k,\min}$ and $f_{k,\max}$ (namely $s_{1,t}(t)$ and $s_{2,t}(t)$) converge to zero in such a way that $r_t(t) \to \tilde{r}(t_r)$ for $t \to t_r$. A general rule to fix the vectors $f_{k,\max}$, and $f_{k,\min}$, $k = 0, 1, \cdots, n_r - 1$, is: $f_{k,\max} = |f_{k,\min}| = \begin{cases} f\,\mathcal{I}_1 & 0 \le k \le k_y \\ f e^{-\beta(k-k_y)}\,\mathcal{I}_1 & k_y \le k \le n_r - 1 \end{cases}$ where $\mathcal{I}_1$ denotes a column vector of $(q\ell_1 + m\ell_2)$ elements equal to 1. In this case $f = 5$, $\beta = 0.4$ and $k_y = 130$ are set. With reference to $\mathcal{S}$, three simulations relative to FFCLI, FFPI and FFOBI control schemes have been performed starting from null initial conditions and choosing: $\alpha_5 = 1$ and $\alpha_l = 0$, $l \neq 5$. Tables I reports the 2-norm of the tracking error $e(t) \triangleq y_d(t) - y(t)$, the 2-norms of $\mathbf{s}_{1,t}(t)$ and $\mathbf{s}_{2,t}(t)$ over $T_r = [0, 30)$ and the value of ρ relative to $\mathcal{S}$. The same table shows that the FFOBI control scheme outperforms both FFCLI and FFPI because it provides the minimum 2-norm of the transient tracking error. This is produced by an optimal combination of FFCLI and FFPI, whose weights are given by the 2-norm of the B-splines $s_{1,t}(t)$ and $s_{2,t}(t)$ respectively. A measure of the improvement provided by the FFOBI has been calculated as the percentage of reduction of $\|e\|_2$ with respect to FFCLI and FFPI. These percentages, denoted by $P_{OBI/CLI}$ and $P_{OBI/PI}$, are 15% and 81% respectively. Figures 2 show the behavior of the controlled output $y(t) \triangleq [U(t)\ \ W(t)\ \ V(t)\ \ R(t)]^T$ produced by the FFOBI, FFCLI and FFPI schemes respectively.

TABLE I

$(\varepsilon_{a_{2,2}}, \varepsilon_{a_{3,3}}, \varepsilon_{a_{5,5}}) = (0.29, 0.09, 0.43)$, $\rho = 0.0685$			
	$\|e(t)\|_2$	$\|\mathbf{s}_{1,t}(t)\|_2$	$\|\mathbf{s}_{2,t}(t)\|_2$
FFOBI	0.6758	13.9686	7.2225
FFCLI	0.7950	16.2166	0
FFPI	3.6100	0	60.0232

VII. Conclusions

In the context of the model inversion based control, a new and more general 2DoF control architecture has been proposed here. It consists of an optimally weighted combination of FFPI and FFCLI control schemes. The numerical results confirmed that the best tracking performance is given by the FFOBI configuration.

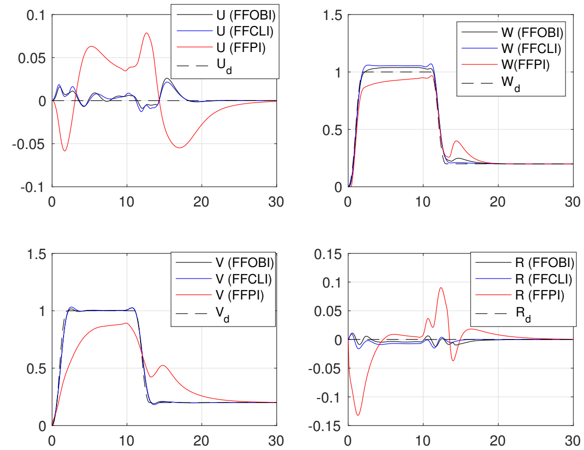

Fig. 2. The trajectories of $y_d(t) \triangleq [U_d(t)\ W_d(t)\ V_d(t)\ R_d(t)]^T$ (black dashed lines) and of the controlled output $y(t) \triangleq [U(t)\ W(t)\ V(t)\ R(t)]^T$ produced by FFOBI (black lines), FFCLI (blue lines) and FFPI (red lines) schemes respectively.

References

[1] Q. Zou, S. Devasia, "Preview based stable inversion for output tracking of linear systems", *Journal of Dynamic Systems Measurement and Control*, Vol. 121, pp. 625-630, 1999.

[2] Q. Zou, "Optimal preview-based stable-inversion for output tracking of nonminimum-phase linear systems", *Automatica*, Vol. 45, 2009, pp. 230-237.

[3] H. Perez, S. Devasia, "Optimal output-transition for linear systems", *Automatica*, Vol. 39, pp.181-192, 2003.

[4] L. Jetto, V. Orsini, R. Romagnoli, " A mixed numerical-analytical stable pseudo-inversion method aimed at attaining an almost exact tracking", *Intern. Journal of Robust and Nonlinear Control*, Vol. 25, pp. 809-823, 2015.

[5] C.Peng, C. Han, J.Zou, G. Zhang, "H_∞ Optimal inversion feedforward and robust feedback based 2DOF control approach for high speed precision positioning systems", *Journ. of Control Science and Engineering*, Vol. 2, pp. 1-11, 2016.

[6] R. Jeyasenthil, S.B. Choi, H. Purohit, D. Jung, "Robust position and disturbance rejection of an industrial plant emulator system using the feedforward-feedback control", *Mechatronics*, Vol. 57, pp. 29-38, 2019.

[7] A.Piazzi, A. Visioli, " Optimal inversion-based control for the set-point regulation of nonminimum-phase uncertain scalar systems", *IEEE Transactions on Automatic Control*, Vol.46, pp. 1654-1659, 2001.

[8] S. S. Aphale, S. Devasia, S.O.R. Moheimani, "High-bandwidth control of a piezoelectric nanopositioning stage in the presence of plant uncertainties", *Nanotechnology*, Vol. 19, pp. 125503-125511, 2008.

[9] B.P. Rigney, L.Y. Pao, D.A Lawrence, "Model inversion architectures for settle time applications with uncertainty", *45th IEEE Control Decis. Conf.*, San Diego (CA), 2006.

[10] J.A. Butterworth, L.Y. Pao, D.Y. Abramovitch, "A comparison of control architectures for atomic force microscopes", *Asian Journal of Control*, Vol. 11, pp.175-181, 2009.

[11] L. Jetto, V. Orsini, R. Romagnoli, " Optimal transient performance under output set-point reset", *Intern. Journal of Robust and Nonlinear Control*, Vol. 26, pp. 2788-2805, 2016.

[12] C. De Boor (1978). "*A practical guide to splines*", Springer Verlag, New York.

[13] M.S. Lobo, L. Vandenberghe, S. Boyd, H. Lébret, "Second-order cone programming", *Linear Algebra and its Applications*, Vol. 284, pp. 193-228, 1998.

[14] C.A. Desoer, Y.T. Wang, "Linear time invariant robust servomechanism problem: a self contained exposition", *Control and Dynamic Systems*C.T. Leondes (Ed.), Vol. 16, pp. 81-129,1980.

[15] J. Löfberg, *https://yalmip.github.io*

[16] S. Devasia, "Output Tracking with Nonhyperbolic and Near Nonhyperbolic Internal Dynamics: Helicopter Hover Control", *J. of Guidance, Control and Dynamics*, Vol 20, pp. 573-580, 1997.