Received: 30 March 2021 | Accepted: 7 April 2021

DOI: 10.1002/asmb.2622

RESEARCH ARTICLE

WILEY

A micro-to-macro approach to returns, volumes and waiting times

Guglielmo D'Amico[1] | **Filippo Petroni**[2]

[1]Department of Pharmacy, Università 'G. d'Annunzio' di Chieti-Pescara, Italy

[2]Department of Management, Università Politecnica delle Marche, Ancona, Italy

Correspondence
Filippo Petroni, Department of Management, Università Politecnica delle Marche, Ancona, Italy.
Email: f.petroni@univpm.it

Abstract

Modelling stock prices has been a research topic for many decades and it is still an open question. Different approaches have been used in the literature, the majority of which can be classified within the so-called econometric framework and sometimes also referred to as the macro-to-micro approach. Another strand of literature relies on the modelling of directly observable quantities, the so-called micro-to-macro approach. Based on this second line of research, we propose a new multivariate stochastic process to model simultaneously price returns, trading volumes and the time interval between changes in trades, price and volume. The proposed model is based on a generalization of semi-Markov chain models and copulas and is motivated by empirical evidence that the three mentioned variables are correlated and long-range autocorrelated. Utilizing Monte Carlo simulations, we compared our model with real data from the Italian stock market and show that it can reproduce many empirical pieces of evidence. The proposed model can be used in the field of portfolio optimization, development of risk measure and volatility forecasting.

KEYWORDS

copula function, high frequency finance, multivariate semi-Markov chain

1 | INTRODUCTION

In financial markets, high frequency data and modelling have acquired a dominant role due to relevant information brought by intraday observations. Nowadays, increasingly sophisticated models and ideas can be advanced and tested on real market data based on the huge amount of information that can be stored and processed on modern computers.

A large part of the effort in market microstructure studies has been produced to understand, mimic and predict basic empirical regularities observed in the most important financial variables. Special attention has been dedicated to the relation between financial volumes and returns. The majority of the works in this area can be classified within the so-called econometric framework, sometimes also referred to as the macro-to-micro approach. The cornerstone of this approach is to consider the observed price to be a collateral effect of an unobservable volatility process to which a noise process transformation is applied, see, for example Reference 1. This line of research has flourished during the last two decades and considerable attention has been dedicated to the problem of irregular spacing in time of observations when dealing with high frequency financial data; the seminal work by Engle[2] and the recent review by Bhogal and Thekke Variyam[3] can provide a wide overview of the subject. Rapidly, econometricians turned the attention on multivariate models of logarithmic price returns, volumes and duration (waiting times), the latter to be intended as the time interval between changes in trades, price and volume of stocks, see for example References 4-7

Another strand of literature relies on the modelling of directly observable quantities, the so-called micro-to-macro approach that is philosophically in contrast with the econometric approach. This framework has a long tradition that has its roots in the paper by Bachelier[8] and also embraces lattice-based models including the popular binomial and trinomial models (see, e.g. References 9 and 10). The micro-to-macro approach has undergone a revival in recent years mainly due to the work of econophysicists that introduced the continuous time random walks (CTRW) apparatus in the modelling of financial returns, see References 11-13. Furthermore, it is known that CTRW can be used to reproduce ordinary diffusions, Levy flights, fractional Brownian motion and ambivalent processes, see Reference 14. The evolution equation of CTRW was formulated and it was shown that it can catch non-Markovian effects. Sometimes the non-Markovian behaviour of stocks has been accommodated considering a latent Markov process acting as a switching process as done in Reference 15. In any case, a viable solution to the non-Markovian problem is given by semi-Markov based models. Semi-Markov processes are the equivalent of CTRW having a nonindependent space–time dynamic. They appeared in the 1950s in the probability field due to the independent contributions by Levy[16] and Smith.[17] They have been successfully investigated and applied in connection with a very wide range of problems including reliability theory, queuing, stochastic systems and DNA analysis, see, for example References 18, 19, 20,21 and 22,23.

Financial modelling is not an exception and has assisted to the progressive abandonment of the Markovian property in favour of the semi-Markov one. Examples come from credit rating (e.g. References 24-26), high-frequency finance,[27-29] financial time series[30,31] and pricing problems.[32-34]

However, only recently it has been recognized that also semi-Markov processes (CTRW included) are not able to reproduce accurately the statistical properties of high-frequency financial data and a more general solution has been advanced in a series of papers where the concept of weighted-indexed semi-Markov chains (WISMC) has appeared, see References 35 and 36. The WISMC model represents a generalization of ordinary semi-Markov processes and revealed to be particularly useful to reproduce long-term dependence in the stock returns among other stylized facts of statistical finance. WISMC models were extended using different strategies to multivariate settings and applied to measure the risk of financial portfolios, see References 37 and 38. In the meantime, they were also successfully applied to the modelling of financial volumes by D'Amico et al.[39]

So far, models proposed in the literature of the micro-to-macro approach have not yet been able to describe a unifying framework where returns, volumes and waiting times are jointly modelled in such a way to reproduce known empirical regularities they possess. The contribution of this article is to present a modelling framework where these three variables are managed contemporaneously in a quite satisfactory and flexible way. Furthermore, the model is nonparametric and then does not rely on any particular assumption on the probability density functions of the main variables. In particular, we first conduct detailed explorative data analysis with the aim of a better understanding of the empirical relationships among the considered financial variables. The data analyzed are from four Italian stocks for the period August 2015 to August 2017 observed at a 1-min frequency. The analysis has shown many empirical pieces of evidence such as the autocorrelation of the absolute value of price and volume returns, the dependence between volume, returns and trading time interval, the nongaussianity of price and volume returns. The WISMC model is presented for the first time with a general state space in a discrete-time framework. It is considered as a model for the log-returns and also for the log-volume returns but with different kernels. To achieve the objective of a multivariate model of price-volumes-waiting times (triplet process), specific data-driven assumptions are advanced and the dependence structure between the price and the volume returns, conditional on the waiting times, is embodied by using a copula function on the joint distribution of the modulus of the returns and the modulus of the volume returns that exhibit a general dependence structure. The dynamic of the multivariate model is characterized by the determination of the kernel of the triplet process. In general, this allows the computation of any financial statistics that can be written as a function of the kernel of the triplet process. The model is used to compute linear and nonlinear measures of dependence, joint first passage time distribution (fptd) function of price and volumes and shows the ability to reproduce probability density functions of both variables as well as cross and auto-correlation functions.

The article is organized as follows. Section 2 provides a statistical analysis of financial data with a particular focus on the relationships among price returns, volume returns and waiting times. Section 3 sets out the marginal models of price and volumes and the multivariate extension using copula functions. In this section, the kernel of the triplet process is studied under some assumptions that are justified by the data. The section presents also the computation of some financial functions of broad interest. Section 4 illustrates the result of the application to real data and demonstrates the accuracy

TABLE 1 Stocks used in the application and their symbols

Code	Name
TIT	Telecom
ISP	Intesa San Paolo
TEN	Tenaris
F	FCA group

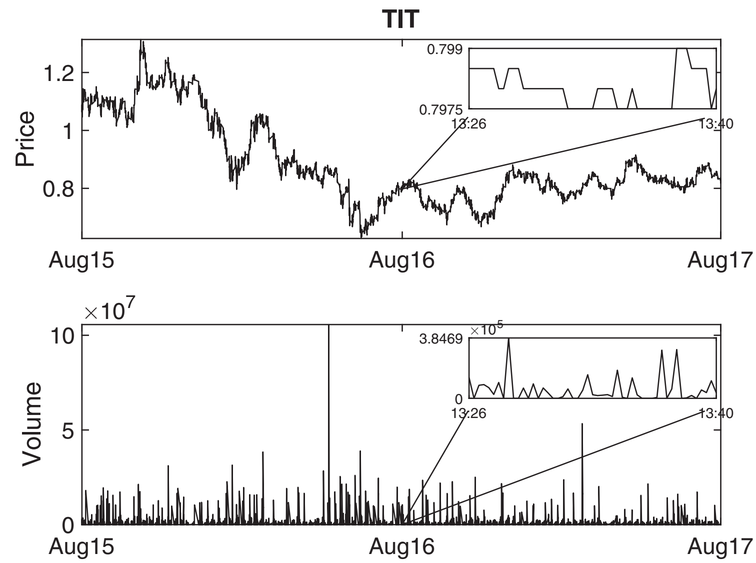

FIGURE 1 Time series of prices and volumes for stock TIT

of the model in reproducing the main empirical regularities observed in financial markets. Section 5 summarizes our contribution and results. All proofs are deferred to the Appendix.

2 | DATA ANALYSIS

Empirical research on price changes, financial volumes and waiting times has identified some characteristics often called the stylized facts.[40-46] In this section we conduct an explorative data analysis with the aim of a better understanding of empirical relations among the considered financial variables. This analysis will inspire the main assumptions under which our model is going to be built on in the next section.

The data used are quotes of Italian stocks for the period August 2015 to August 2017 (2 full years) with 1 min frequency. Every minute, the last price and the cumulated volume (number of transactions) is recorded. For each stock the database is composed of about $2.6 * 10^5$ volumes and prices. The list of stocks analyzed and their symbols are reported in Table 1. From now onward we will use only the codes in the table to identify each stock.

The analyzed stocks are chosen to represent different market sectors. According to the Global Industry Classification Standard, **F** is in the industrial sector, **ISP** is one of the largest banks in Italy (financial sector), **TIT** is in the telecommunication and **TEN** in the energy sector. In Figure 1 we show an example, for the stock **TIT**, of the time series of price $S(t)$ and trading volumes $V(t)$ in the analyzed period.

As a first step we analyze the time series and look for the most important statistical features. From prices we build a time series of the price returns defined as $r(t) = \log(S(t)/S(t-1))$ and from trading volumes we define the log variation (from now onward volume returns) as $v(t) = \log(V(t)/V(t-1))$ where t is the time variable in 1-min frequency. To be sure to use only variation in 1-min period we exclude from the analyses the variation of both variables from the closing of the stock market at day d to the reopening in the next trading day $d+1$ (we remind that the stock market is open from 9 am to 17:30 pm in weeks day). Descriptive statistics are summarized in the Appendix.

As far as the probability density function are concerned, we performed a Jarque–Bera test that rejected the Gaussian hypothesis for both $r(t)$ and $v(t)$ at 1% significance level.

One of the most important statistical feature of both time series is that their absolute values are long range correlated, see Figure 2. For space reason we show the plots only for two of the analyzed stocks. We also found that there is zero

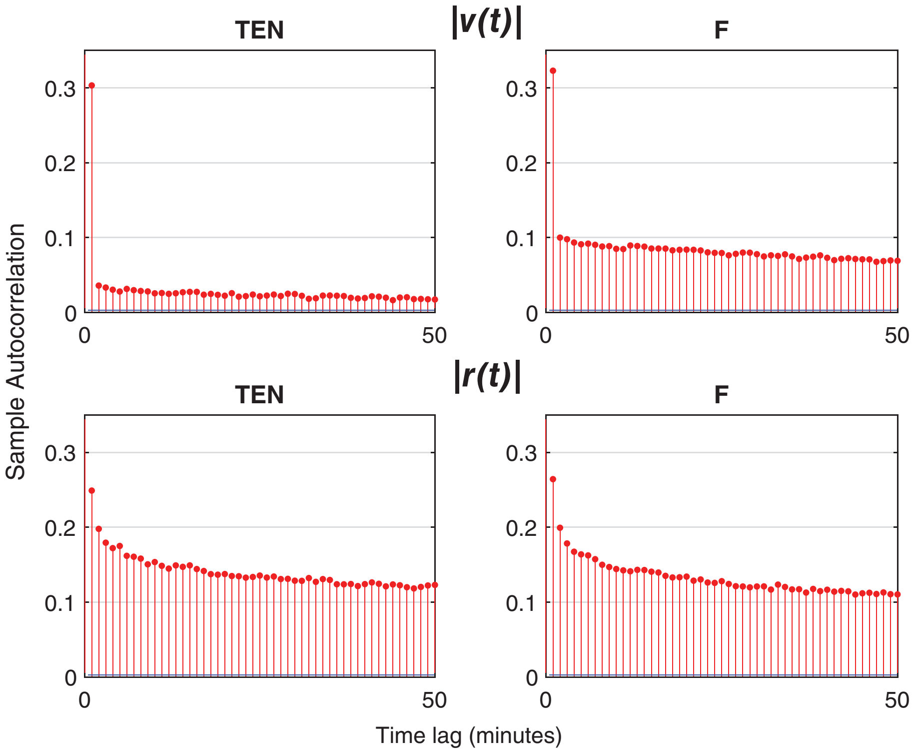

FIGURE 2 Sample autocorrelation of $|v(t)|$ (upper panels) and $|r(t)|$ (lower panels)

TABLE 2 Cross-correlation between price and volume returns

	$\rho(r(t), v(t))$	p-value	$\rho(\lvert r(t)\rvert, v(t))$	p-value	$\rho(r(t), \lvert v(t)\rvert)$	p-value	$\rho(\lvert r(t)\rvert, \lvert v(t)\rvert)$	p-value
TIT	0.0089	0	0.086	0	−0.0032	0.023	0.020	0
ISP	−0.010	0	0.086	0	−0.0036	0.0095	−0.019	0
TEN	−0.0091	0	0.086	0	0.00055	0.69	0.040	0
F	−0.012	0	0.11	0	−0.0041	0.0037	−0.027	0

correlation between $r(t)$ and $v(t)$ while a nonzero correlation between $|r(t)|$ and $v(t)$ is present, see Table 2. In this table we show all possible combinations of correlation between $r(t)$ and $v(t)$ and their absolute values, we also show the p-values which gives significance of nonzero correlation.

Given these properties a good model should be able to take all of them into account. Another property that we found quite interesting and that should be included into a model is the following: for both time series ($r(t)$ and $v(t)$) we found that there is a dependence with the waiting time which is defined as the time it takes for returns to change their values. This result is confirmed from contingency tables where we can see the dependence between $r(t)$, $v(t)$ and the waiting time T. The contingency tables, shown in Table 3, have been obtained by a discretization of both $r(t)$ and $v(t)$ into five states.

It can be easily noticed, from Table 3, that there is a dependence of the number of transition from T (for $T = 1$ there are much more transitions than for $T = 2$ or $T = 3$). More specifically, in brackets we give the number of transitions that we would expect for independent processes where the probability of finding a given number of transition is simply given by the product of the frequencies of having each variable at that given state. From Table 3 it is obvious that the independent hypothesis does not hold for both processes. We obtained similar results for all other stocks which, for reasons of space, are not shown here. From the above tables and from all results obtained in this section, we can say that a good real world model of price and volumes should take into account all the aforementioned stylized facts that we can summarize in a list:

- distributions of price returns and volume returns are not Gaussian;

TABLE 3 Contingency table for F

	$r(t)$				
	$-\infty:-0.13\%$	$-0.13\%:-0.05\%$	$-0.05\%:0.05\%$	$0.05\%:0.13\%$	$0.13\%:+\infty$
T $0:2$	24,647 (23386,7)	45,762 (43588,8)	43,459 (50186,2)	46,381 (44216,1)	23,984 (22855,1)
$2:4$	696 (1436,3)	1487 (2677,1)	6805 (3082,3)	1553 (2715,6)	774 (1403,7)
$4:+\infty$	45 (565,0)	70 (1053,1)	4217 (1212,5)	66 (1068,2)	53 (552,2)
	$v(t)$				
	$-\infty:-4$	$-4:-1.3$	$-1.3:1.3$	$1.3:4$	$4:+\infty$
T $0:2$	3784 (3071,1)	58,063 (47168,3)	36,466 (60051,0)	59,221 (48145,5)	4787 (3885,2)
$2:4$	0 (289,7)	54 (4449,2)	15,157 (5664,3)	100 (4541,3)	0 (366,5)
$4:+\infty$	0 (423,2)	0 (6499,5)	22,367 (8274,7)	0 (6634,2)	0 (535,4)

Note: In the first table we tested the dependence between $r(t)$ and T, while in the second table we tested the dependence between $v(t)$ and T. The numbers in brackets are obtained under the independence hypotheses. The χ^2 test rejects the null hypothesis of independence.

- the absolute values of price returns are long range correlated;
- the absolute values of volume returns are long range correlated;
- price returns and volume returns are uncorrelated while a nonzero correlation between $|r(t)|$ and $v(t)$ is present;
- $r(t)$ and the waiting times influence each other;
- $v(t)$ and the waiting times influence each other.

The majority of them was already known and extensively documented in the financial literature, here they have been confirmed in our dataset. A very interesting summary and financial implications of those empirical regularities are discussed in Reference 47 and in the references therein. The empirical evidences in the list are the cornerstones on which is built the model we are going to present in next section.

3 | MATHEMATICAL MODEL

In this section we first present the WISMC model that is used as a marginal model for both the price and volume returns. Successively, we extend the model in a multivariate setting by considering a dependence structure between price, volumes and waiting times (durations) using a copula function.

3.1 | Weighted-indexed semi-Markov chains

Here, we introduce discrete-time WISMC model with Borel phase space in relation to the financial problem to which we are interested in. Note that, in previous papers,[35,36] time was considered as a continuous variable but an appropriate discretization was made when the application was performed.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F} := (\mathcal{F}_n)_{n\in\mathbb{N}}$ where all upcoming random variables are defined.

Let $S(t)$ be the price of a financial asset at time $t \in \mathbb{N}$. The time varying log return, defined as $\log(S(t)/S(t-1))$, is usually the main variable object of investigation in financial literature. As commented in the previous section, at the short-time scales considered in high-frequency finance, this variable changes values only in correspondence of an increasing sequence of times $\{T_n^J\}_{n\in\mathbb{N}}$, the so-called jump-times of the asset price process.

In correspondence of the times $\{T_n^J\}_{n\in\mathbb{N}}$, the logarithmic return process assumes different values denoted by $\{J_n\}_{n\in\mathbb{N}}$ and along any waiting time $X_n := T_{n+1}^J - T_n^J$ it does not change value and remains constant. Thus, J_n is the value of the logarithmic change in price at its nth transition.

Let assume that at current time, say $t_0 = 0$, we dispose of a set of past data consisting of two vectors of observations collecting the last $m+1$ visited states of the log-return process and corresponding transitions times, respectively, that is

$$\mathbf{J}_{-m}^{0} = (J_{-m}, J_{-m+1}, \dots, J_0),$$
$$\mathbf{T}_{-m}^{0} = (T_{-m}^{J}, T_{-m+1}^{J}, \dots, T_0^{J}).$$

Consider also an index process:

$$I_n^J(\lambda) := \sum_{r=0}^{m+n-1} \sum_{a=T_{n-1-r}^J}^{T_{n-r}^J-1} f^{\lambda}(J_{n-1-r}, T_n^J, a) + f^{\lambda}(J_n, T_n^J, T_n^J), \tag{1}$$

where $f^{\lambda} : \mathbb{R} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is a bounded function.

The process $I_n^J(\lambda)$ can be interpreted as an accumulated reward process with the function f^{λ} as a measure of the weighted rate of reward per unit time. The parameter λ is a memory parameter that should be calibrated on the data. A calibration procedure is discussed in the application (Section 4). It should also be remarked that the index process considered in this article is slightly more general than those considered in previous research articles because we added the term $f^{\lambda}(J_n, T_n^J, T_n^J)$ that add to the index process also the score deriving from observing the current log-return state J_n at present time T_n^J.

Introduce the counting process $N^J(t) := \max\{n \in \mathbb{N} : T_n^J \le t\}$, and let us now introduce the notion of weighted-indexed semi-Markov chains.

Definition 1. The process $Z^J(t) := J_{N^J(t)}$ is said to be a weighted-indexed semi-Markov chain with phase-space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ if $\forall i, x, j \in \mathbb{R}$ and $\forall t \in \mathbb{N}$ there exists a function $\mathbf{q}^J = q^J(i, x; j, t)$, called the indexed semi-Markov kernel, such that $\forall n \in \mathbb{N}$ the following equality holds true:

$$\begin{aligned} &\mathbb{P}[J_{n+1} \le j, T_{n+1}^J - T_n^J = t | \sigma(J_h, T_h^J, I_h^J(\lambda), h \le n), J_n = i, I_n^J(\lambda) = x] \\ &= \mathbb{P}[J_{n+1} \le j, T_{n+1}^J - T_n^J = t | J_n = i, I_n^J(\lambda) = x] =: q^J(i, x; j, t). \end{aligned} \tag{2}$$

Remark 1. Relation (2) asserts that the knowledge of the values of the variables $J_n, I_n^J(\lambda)$ is sufficient to give the conditional distribution of the couple $J_{n+1}, T_{n+1}^J - T_n^J$ whatever the values of the past variables might be. Therefore, to assess the probability of the next value of the log-return process and of the time in which the process is going to change state, we need only the knowledge of the last state of the log-return and the last value of the index process.

Remark 2. The function $Q^J(i, x; j, t) := \sum_{s \le t} q^J(i, x; j, s)$ satisfies the following properties:

a) $Q^J(i, x; j, \cdot)$ is a nondecreasing discrete real function such that

$$Q^J(i, x; j, 0) = 0.$$

b) $p^J(\cdot, x; \cdot) := Q^J(\cdot, x; \cdot, \infty)$ is a Markov transition probability function from $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ to itself.

Remark 3. If the indexed semi-Markov kernel is constant in x, that is fixed the triple (i, j, t) for all $y \ne x$

$$q^J(i, x; j, t) \ne q^J(i, y; j, t),$$

then, it degenerates in a semi-Markov kernel and the WISMC model becomes equivalent to classical semi-Markov chain model, see, for example References 48 and 49.

The triplet $\{J_n, T_n^J, I_n^J(\lambda)\}$ describes the system in correspondence of any jump time T_n^J. However, it is also important to describe the system in correspondence of any time t, which can be a jump time ($t = T_n^J$) or not ($t \ne T_n^J$). The random process $Z^J(t) := J_{N^J(t)}$ introduced in definition (1) marks the log-return at any time t, while the backward recurrence time process $B^J(t) := t - T_{N^J(t)}^J$ denotes the time elapsed since the last transition. In our model this information is not sufficient to completely characterize the status of the system because we need to know also the value of the index process. To this end we extended the definition of the index process allowing to consider any time $t \in \mathbb{N}$ as follows:

$$I^J(\lambda;t) = \sum_{r=0}^{m+N^J(t)-1+\theta} \sum_{a=T^J_{N^J(t)+\theta-1-r}}^{(t\wedge T^J_{N^J(t)+\theta-r})-1} f^\lambda(J_{N^J(t)+\theta-1-r}, t, a) + f^\lambda(J_{N^J(t)}, t, t), \tag{3}$$

where $\theta = 1_{\{t>T^J_{N^J(t)}\}}$. If $t = T^J_n$, then $I^J(\lambda;t) = I^J_n(\lambda)$.

The following definition and result, which reduces the complexity of the model, is important for practical application of the WISMC model.

Definition 2 (Shift operator). Let $(i,t)^n_{-m} = \{(i_\alpha, t_\alpha), \alpha = -m, \dots, n\}$ be a sequence of states and transition times, that is $i_\alpha \in \mathbb{R}, t_\alpha \in \mathbb{Z}, t_\alpha < t_{\alpha+1}$.

Let denote by $\Theta^n_{-m} = \{(i_\alpha, t_\alpha), \alpha = -m, \dots, 0, \dots, n, i_\alpha \in \mathbb{R}, t_\alpha \in \mathbb{Z}\}$, then we define the shift operator

$$\circ : \Theta^{n+1}_{-m} \to \Theta^n_{-m-1}$$

defined by

$$\circ((i,t)^{n+1}_{-m}) = (s,k)^n_{-m-1},$$

where $s_\alpha = i_{\alpha+1}, k_\alpha = t_{\alpha+1} - t_{n+1}, \quad \alpha = -m-1, \dots, 0, \dots, n.$

From an intuitive point of view, the shift operator when applied to a trajectory $(i,t)^{n+1}_{-m}$ gives back a new trajectory where the sequence of visited states is the same as in the input trajectory with the difference that transition times are translated of t_{n+1} time units backward and the number of transitions is set one unit backward.

The following assumption concerning the score function f^λ will be needed in the rest of the article:

A1: $\forall i \in \mathbb{R}, t \in \mathbb{N}, a \in \mathbb{N}, f^\lambda(i,t,a) = f^\lambda(i, t-a)$.

Lemma 1. *For a WISMC with score function f^λ that satisfies assumption* **A1***, for fixed arbitrary state j and time t and $(i,t)^{n+1}_{-m} \in \Theta^{n+1}_{-m}$, we have:*

$$\mathbb{P}[J_{n+2} \le j, T^J_{n+2} - T^J_{n+1} = t | (J, T^J)^{n+1}_{-m} = (i,t)^{n+1}_{-m}] = \mathbb{P}[J_{n+1} \le j, T^J_{n+1} - T^J_n = t | (J, T^J)^n_{-m-1} = \circ((i,t)^{n+1}_{-m})]. \tag{4}$$

Proof. See the Appendix. ■

The result presented in Lemma 1 focuses on a class of score functions leading probability (4) to be independent of n. Accordingly, the WISMC inherits a homogeneity property that is particularly useful for the applications of the model. Throughout this article, we are going to consider homogeneous WISMC.

In this research we consider also financial volume as one important variable worthwhile to be investigated. The WISMC model was also applied to the modelling of financial volumes in a recent article by D'Amico et al.[39] and revealed to be able to reproduce several statistical properties of volumes at high-frequency scales. In order to be able to distinguish between the WISMC model for returns and that for volumes we introduce an additional notation for the volume model. Precisely, if $V(t)$ is the volume of a financial asset at time $t \in \mathbb{N}$, the time varying log volume is defined as $\log(V(t)/V(t-1))$. This variable at short-time scales changes values in correspondence of an increasing sequence of times $\{T^V_n\}_{n\in\mathbb{N}}$, the so-called jump-times of the asset volume process. In correspondence of the times $\{T^V_n\}_{n\in\mathbb{N}}$, the logarithmic volume process assumes different values denoted by $\{J^V_n\}_{n\in\mathbb{N}}$. We introduce the index process for the volume by replacing in formula (1) the variables J_n, T^J_n and λ by V_n, T^V_n and γ, respectively. The semi-Markov kernel for the volume process will be denotes by $\mathbf{q}^V = q^V(i,x;j,t)$ and the WISMC process for the volume variable is defined by $Z^V(t) := V_{N^V(t)}$ being $N^V(t) := \max\{n \in \mathbb{N} : T^V_n \le t\}$.

3.2 | The multivariate model

In this section we extend the WISMC model into a multivariate setting in such a way that it is able to describe jointly the time evolution of the three considered variables: log-returns, log-volumes and waiting times. The extension is done advancing a series of assumption that allow us to merge the WISMC kernel of the log-return process and that of the log-volumes in a new kernel that is completely characterized in this section.

The first step in the joint modelization of returns, volumes and durations is to synchronize the time events of the returns and volumes. In order to do it let us start from the two sequences

$$(J_n, T_n^J)_{n\in\mathbb{N}}, \qquad (V_n, T_n^V)_{n\in\mathbb{N}}. \tag{5}$$

They mark the values and points in time where log-returns and log-volumes change states, respectively. First, we define a new sequence of transition times:

$$\{\tilde{T}_n\} = \{T_n^J\} \cup \{T_n^V\}, \quad \text{with} \quad \tilde{T}_0 = T_0^J = T_0^V = 0. \tag{6}$$

Relation (6) means that we consider the union between the sets of transition times of returns and volumes and the obtained ordered sequence of times is denoted with the symbol $\{\tilde{T}_n\}_{n\in\mathbb{N}}$. Intuitively, the time $\tilde{T}_1$ is the first time when a change in the returns or in the volumes occurred, $\tilde{T}_2$ the second point in time when a second change of state of whichever of the two processes J_n and V_n occurred, and so on. The related interarrival times are denoted by

$$\tilde{X}_n = \tilde{T}_{n+1} - \tilde{T}_n.$$

Furthermore we define the corresponding values of the returns and volumes for each time of the random sequence $\tilde{T}_n$ according to the following relations:

$$\begin{aligned}\tilde{J}_n = J_s, &\quad \text{if } s = \max\{h \in \mathbb{N} : T_h^J \le \tilde{T}_n\},\\ \tilde{V}_n = V_s, &\quad \text{if } s = \max\{h \in \mathbb{N} : T_h^V \le \tilde{T}_n\},\end{aligned}$$

Thus, we ended up with three variables $(\tilde{J}_n, \tilde{V}_n, \tilde{T}_n)$ that denote the synchronized sequences of log-returns, log-volumes and transition times. In order to advance a joint model for this three-variate process we need to advance some specific properties concerning their interdependence and dynamics.

Suppose the following conditional independence relation, namely assumption **A2**, holds true:

$$\begin{aligned}&\mathbb{P}[\tilde{J}_{n+1} \le j, \tilde{V}_{n+1} \le a, \tilde{X}_n = t | \sigma(\tilde{J}_h, \tilde{V}_h, \tilde{T}_h, h \le n), \tilde{J}_n = i, \tilde{V}_n = v, \tilde{I}_n^J = x,\\ &\quad \tilde{I}_n^V = w, \tilde{T}_n = s, \tilde{T}_n - T_{N^J(s)}^J = b^J, \tilde{T}_n - T_{N^V(s)}^V = b^V]\\ &\quad = \mathbb{P}[\tilde{J}_{n+1} \le j, \tilde{V}_{n+1} \le a, \tilde{X}_n = t | \tilde{J}_n = i, \tilde{V}_n = v, \tilde{I}_n^J = x, \tilde{I}_n^V = w, \tilde{T}_n = s,\\ &\tilde{T}_n - T_{N^J(s)}^J = b^J, \tilde{T}_n - T_{N^V(s)}^V = b^V].\end{aligned} \tag{7}$$

Assumption **A2** considers a quasi-Markovian-type hypothesis which asserts that the knowledge of the last values of synchronized variables $(\tilde{J}_n = i, \tilde{V}_n = v, \tilde{T}_n = s)$ together with corresponding values of the index processes $(\tilde{I}_n^J = x, \tilde{I}_n^V = w)$ and of the time elapsed from last transition of both log-return and log-volume $(\tilde{T}_n - T_{N^J(s)}^J = b^J, \tilde{T}_n - T_{N^V(s)}^V = b^V)$ suffices to give the conditional distribution of the triplet $(\tilde{J}_{n+1}, \tilde{V}_{n+1}, \tilde{X}_n)$ whatever the values of the past variables might be.

The following information sets are introduced for notational convenience:

$$\begin{aligned}\mathcal{A}_{n,s}^J &:= \{\tilde{J}_n = i, \tilde{I}_n^J = x, \tilde{T}_n = s, \tilde{T}_n - T_{N^J(s)}^J = b^J\},\\ \mathcal{A}_{n,s}^V &:= \{\tilde{V}_n = v, \tilde{I}_n^V = w, \tilde{T}_n = s, \tilde{T}_n - T_{N^V(s)}^V = b^V\},\\ \mathcal{A}_{n,s}^{JV} &:= \mathcal{A}_{n,s}^J \bigcup \mathcal{A}_{n,s}^V,\\ \mathcal{A}_{n,s}^{JVT} &:= \mathcal{A}_{n,s}^{JV} \bigcup \{\tilde{X}_n = t\}.\end{aligned}$$

Probability (7) is so important to merit a formal definition:

Definition 3. Let $(\tilde{J}_n, \tilde{V}_n, \tilde{T}_n)$ be the synchronized triplet process of log-return, log-volume and transition times. The function

$$\mathbf{q}^{JV} = q^{JV}(\mathcal{A}^{JV}_{n,s}; j, a, t),$$

with $i, v, x, w, j, a \in \mathbb{R}$ and $b^J, b^V, t \in \mathbb{N}$ defined by

$$q^{JV}(\mathcal{A}^{JV}_{n,s}; j, a, t) := \mathbb{P}[\tilde{J}_{n+1} \le j, \tilde{V}_{n+1} \le a, \tilde{X}_n = t | \mathcal{A}^{JV}_{n,s}]. \tag{8}$$

is called the kernel of the triplet process.

The kernel of the triplet process, can be factorized into the product of the conditional joint distribution of log-return and log-volumes multiplied by the conditional distribution of interarrival times, that is

$$\mathbb{P}[\tilde{J}_{n+1} \le j, \tilde{V}_{n+1} \le a | \mathcal{A}^{JVT}_{n,s}] \cdot \mathbb{P}[\tilde{X}_n = t | \mathcal{A}^{JV}_{n,s}]. \tag{9}$$

Our next main task is to give a representation of this kernel in such a way that the dynamic of the joint process $(\tilde{J}_n, \tilde{V}_n, \tilde{T}_n)$ could be completely characterized. We shall now consider reasonable data-driven assumptions that permits this computation.

Assumption **A3**: synchronized waiting time distribution.

We assume that the probability distribution of interarrival time is independent on current time s, this avoid the use of time nonhomogeneous probabilistic structures of the process. Moreover we assume that the waiting time distribution does not explicitly depends on the time elapsed by log-return and log-volume into their current states but includes past information depending on the index processes of returns and volumes. In formula

$$\mathbb{P}[\tilde{X}_n = t | \mathcal{A}^{JV}_{n,s}] = \mathbb{P}[\tilde{X}_n = t | \tilde{J}_n = i, \tilde{V}_n = v, \tilde{I}^J_n = x, \tilde{I}^V_n = w]. \tag{10}$$

Assumption **A3** implies that the distributional properties of the waiting-times in our model can differ according to price and volume movements ($\tilde{J}_n$ and $\tilde{V}_n$ values) as well as with their past behaviour measured by the index processes $\tilde{I}^J_n$ and $\tilde{I}^V_n$.

Denote the conditional probability of $\tilde{X}_n$ by

$$\tilde{H}_{i,v}(x, w; t) := \mathbb{P}[\tilde{X}_n \le t | \tilde{J}_n = i, \tilde{V}_n = v, \tilde{I}^J_n = x, \tilde{I}^V_n = w], \tag{11}$$

and the corresponding probability mass function by

$$\mathbb{P}[\tilde{X}_n = t | \tilde{J}_n = i, \tilde{V}_n = v, \tilde{T}_n = s, \tilde{I}^J_n = x, \tilde{I}^V_n = w] = \tilde{H}_{i,v}(x, w; t) - \tilde{H}_{i,v}(x, w; t-1) =: \tilde{h}_{i,v}(x, w; t). \tag{12}$$

The independence of the cdf of waiting times on the number of transitions n, and on the time of last transition s is done in order to avoid unnecessary complications that would have made the model inhomogeneous in time.

The knowledge of the kernel of the triplet process needs also the specification of the conditional joint probability distribution of log-returns and log-volumes. In this respect we propose the following

Assumption **A4**: the conditional joint distribution of modulus of log-returns and log-volumes is given by

$$\mathbb{P}[|\tilde{J}_{n+1}| \le j, |\tilde{V}_{n+1}| \le a \mid \mathcal{A}^{JVT}_{n,s}] = C\left(F^{|J|}(i, x, t + b^J; j), F^{|V|}(v, w, t + b^V; a)\right), \tag{13}$$

where C is a Copula-function and the marginal distributions $F^{|J|}(i, x, t + b^J; j)$ and $F^{|V|}(v, w, t + b^V; a)$ are given by

$$\begin{aligned} F^{|J|} &= \mathbb{P}[|J_{N^J(s)+1}| \le j \mid T_{N^J(s)+1} - s = t, J_{N^J(s)} = i, I^J_{N^J(s)} = x, s - T^J_{N^J(s)} = b^J], \\ F^{|V|} &= \mathbb{P}[|V_{N^V(s)+1}| \le a \mid T_{N^V(s)+1} - s = t, V_{N^V(s)} = i, I^V_{N^V(s)} = x, s - T^V_{N^V(s)} = b^V]. \end{aligned}$$

Assumption **A4** is motivated by the data analysis executed in Section 2, specifically in Table 2 we have shown that the two processes are dependent on each other. Essentially this assumption allows us to consider a dependence structure between the modulus of the log-returns and log-volumes that is managed through the use of any copula function. The copula maps the two marginal distributions $F^{|J|}$ and $F^{|V|}$ into a joint probability distribution function. The quantity $F^{|J|}$ expresses the probability to get the modulus of log-return less or equal to j conditionally on the last value of the variable, corresponding index process, waiting time length and duration in the last visited states. The same interpretation can be given to the quantity $F^{|V|}$ with the only exception that it is related to the modulus of log-volume process.

The $F^{|J|}$ can be evaluated as follows:

$$\begin{aligned}
F^{|J|}(i,x,t+b^J;j) &= \mathbb{P}[|J_{N^J(s)+1}| \le j \mid T_{N^J(s)+1} - s = t, J_{N^J(s)} = i, I^J_{N^J(s)} = x, s - T^J_{N^J(s)} = b^J] \\
&= \mathbb{P}[|J_{N^J(s)+1}| \le j \mid T_{N^J(s)+1} - T_{N^J(s)} + T_{N^J(s)} - s = t, J_{N^J(s)} = i, \\
I^J_{N^J(s)} = x, T^J_{N^J(s)} = s - b^J] &= \mathbb{P}[|J_{N^J(s)+1}| \le j \mid T_{N^J(s)+1} - T_{N^J(s)} = t + b^J, J_{N^J(s)} = i, I^J_{N^J(s)} = x] \\
&= \mathbb{P}[|J_{n+1}| \le j \mid T_{n+1} - T_n = t + b^J, J_n = i, I^J_n = x] \\
&= \frac{\mathbb{P}[|J_{n+1}| \le j, T_{n+1} - T_n = t + b^J \mid J_n = i, I^J_n = x]}{\mathbb{P}[T_{n+1} - T_n = t + b^J \mid J_n = i, I^J_n = x]} \\
&= \frac{q^J(i,x;j,t+b^J) - q^J(i,x;-j,t+b^J)}{H^J_i(x;t+b^J) - H^J_i(x;t+b^J-1)}.
\end{aligned}$$

Similar computations gives

$$F^{|V|}(v,w,t+b^V;a) = \frac{q^V(v,w;a,t+b^V) - q^V(v,w;-a,t+b^V)}{H^V_v(w;t+b^V) - H^V_v(w;t+b^V-1)}.$$

By means of assumptions **A3** and **A4** we can get information on the joint distribution of modulus of log-returns and modulus of log-volumes. Nonetheless, it is our interest to recover information on the exact values (with signs) of these variables. This is motivated by the empirical observation that although $\{|\tilde{J}_n|\}$ and $\{|\tilde{V}_n|\}$ are significantly correlated, $\{\tilde{J}_n\}$ and $\{\tilde{V}_n\}$ are not.

To be able to reach this objective we advance a final assumptions:

A5: For each $n \in \mathbb{N}$, $\tilde{J}_n$ and $\tilde{V}_n$ satisfy the following relations:

$$\begin{aligned}
\tilde{J}_n &= |\tilde{J}_n| \cdot \eta^J_n; \\
\tilde{V}_n &= |\tilde{V}_n| \cdot \eta^V_n;
\end{aligned}$$

where η^J_n and η^V_n are two sequences of i.i.d. random variables with pmf

$$\eta^J_n \sim \begin{cases} +1 & \text{with probability } p^J \\ -1 & \text{with probability } 1 - p^J \end{cases}$$

$$\eta^V_n \sim \begin{cases} +1 & \text{with probability } p^V \\ -1 & \text{with probability } 1 - p^V \end{cases}$$

This assumptions allows us to get the value of the variables starting from the knowledge of their modulus. Indeed, the variables η^J_n and η^V_n provides the sign of the size of the variation. Obviously, the parameters p^J and p^V need to be estimated on the data.

The next theorem characterizes the kernel of the triplet process.

Theorem 1. *Under assumptions* **A1–A5**, $\forall s \in \mathbb{N}$ *the kernel of the triplet process* $(\tilde{J}_n, \tilde{V}_n, \tilde{T}_n)$, *defined in formula (8), is given by:*

(i) for $j \ge 0, a \ge 0$

$$\begin{aligned}
q^{JV}(\mathcal{A}^{JV}_{n,s}; j, a, t) = \tilde{h}_{i,v}(x,w;t) \cdot [1 + p^J p^V (1 - F^{|J|}(i,x,t+b^J;j) \\
- F^{|V|}(v,w,t+b^V;a) + C(F^{|J|}(i,x,t+b^J;j), F^{|V|}(v,w,t+b^V;a))) \\
- p^V(1 - F^{|V|}(v,w,t+b^V;a)) - p^J(1 - F^{|J|}(i,x,t+b^J;j))],
\end{aligned} \tag{14}$$

(ii) for $j < 0, a < 0$

$$\begin{aligned}
q^{JV}(\mathcal{A}^{JV}_{n,s}; j, a, t) = \tilde{h}_{i,v}(x,w;t) \cdot (1 - p^J)(1 - p^V)[1 - F^{|J|}(i,x,t+b^J;-j) \\
- F^{|V|}(v,w,t+b^V;-a) + C(F^{|J|}(i,x,t+b^J;-j), F^{|V|}(v,w,t+b^V;-a))],
\end{aligned} \tag{15}$$

(iii) for $j<0, a>0$

$$q^{JV}(\mathcal{A}^{JV}_{n,s};j,a,t) = \tilde{h}_{i,v}(x,w;t)\cdot[(1-p^J)\cdot[F^{|V|}(v,w,t+b^V;a) - C(F^{|J|}(i,x,t+b^J;-j),F^{|V|}(v,w,t+b^V;a))] + (1-p^J)(1-p^V)\cdot[1-F^{|J|}(i,x,t+b^J;-j) - F^{|V|}(v,w,t+b^V;a) + C(F^{|J|}(i,x,t+b^J;-j),F^{|V|}(v,w,t+b^V;a))]], \tag{16}$$

(iv) for $j>0, a<0$

$$q^{JV}(\mathcal{A}^{JV}_{n,s};j,a,t) = \tilde{h}_{i,v}(x,w;t)\cdot[(1-p^V)\cdot[F^{|J|}(i,x,t+b^J;j) - C(F^{|J|}(i,x,t+b^J;j),F^{|V|}(v,w,t+b^V;-a))] + (1-p^J)(1-p^V) \cdot[1-F^{|J|}(i,x,t+b^J;j) - F^{|V|}(v,w,t+b^V;-a) + C(F^{|J|}(i,x,t+b^J;j),F^{|V|}(v,w,t+b^V;-a))]]. \tag{17}$$

Proof. See the Appendix. ■

3.3 | Financial functions

In this subsection we show how it is possible to compute financial functions of specific interest using the characterization of the kernel of the triplet process given in Theorem 1. Results are confined to marginal distributions of log-returns and log-volumes, correlation structures and joint first passage time distributions. In general given the kernel, it is possible to compute any type of functional of the kernel.

3.3.1 | The one-step marginal distributions of returns and volumes

The first question to which we are interested in is the determination of the marginal distributions of log-returns and log-volumes. Since the dependence structure has been introduced on the modulus of these variables the marginal distributions we are looking for do not coincide with those used in the copula, that is with $F^{|J|}$ and $F^{|V|}$.

Let us consider the problem of finding the marginal distribution of the return process. Let us proceed by integration of the volume variable and summation on the duration one, this gives $\sum_{t\geq 0} q^{JV}(\mathcal{A}^{JV}_{n,s};j,\infty,t)$. Thus, for $j>0$, using the kernel representation (14) and the fact that $F^{|V|}(v,w,|\infty|,t+b^V)=1$ and that $C(F^{|J|}(i,x,t+b^J;j),F^{|V|}(v,w,t+b^V;\infty)) = F^{|J|}(i,x,t+b^J;j)$ we obtain the following sequence of equalities:

$$\begin{aligned}\sum_{t\geq 0} q^{JV}(\mathcal{A}^{JV}_{n,s};j,\infty,t) &= \sum_{t\geq 0}\tilde{h}_{i,v}(x,w;t)\cdot[1+p^Jp^V(1-F^{|J|}(i,x,t+b^J;j)-1+F^{|J|}(i,x,t+b^J;j)) \\ &\quad - p^V(1-1)) - p^J(1-F^{|J|}(i,x,t+b^J;j))] \\ &= \sum_{t\geq 0}\tilde{h}_{i,v}(x,w;t)\cdot[1-p^J(1-F^{|J|}(i,x,t+b^J;j))] \\ &= 1-p^J\cdot\left(1-\sum_{t\geq 0}\tilde{h}_{i,v}(x,w;t)\cdot F^{|J|}(i,x,t+b^J;j)\right).\end{aligned}$$

This marginal distribution expresses the probability to observe with next transition, executed at any future time t, a return not greater than j. Symmetric arguments prove that the marginal distribution of volumes results in

$$1-p^V\cdot\left(1-\sum_{t\geq 0}\tilde{h}_{i,v}(x,w;t)\cdot F^{|V|}(v,w,t+b^V;a)\right).$$

3.3.2 | Dependence measures

The kernel of the triplet process (8) completely describes the dependence structure between returns and volumes and waiting times. Nevertheless, it is relevant to measure it using classical indicators of linear and nonlinear dependence.

The most widely studied measure of linear dependence is the correlation coefficient. Let $\rho_{\mathcal{A}^{JV}_{n,s}}(|\tilde{J}_{n+1}|, |\tilde{V}_{n+1}|)$ be the correlation coefficient between the modulus of returns and the modulus of volumes at next transition unconditionally on the time when the next transition will happen, that is

$$\rho_{\mathcal{A}^{JV}_{n,s}}(|\tilde{J}_{n+1}|, |\tilde{V}_{n+1}|) = \frac{Cov_{\mathcal{A}^{JV}_{n,s}}(|\tilde{J}_{n+1}|, |\tilde{V}_{n+1}|)}{\sigma_{\mathcal{A}^{J}_{n,s}}(|\tilde{J}_{n+1}|) \cdot \sigma_{\mathcal{A}^{V}_{n,s}}(|\tilde{V}_{n+1}|)}. \tag{18}$$

Using the formula discussed above, we can calculate the correlation coefficient by using the joint probability density function of $(|\tilde{J}_{n+1}|, |\tilde{V}_{n+1}|)$ conditional on the information set $\mathcal{A}^{JV}_{n,s}$. For every $j, a \geq 0$, one gets:

$$\begin{aligned} F_{(|\tilde{J}_{n+1}|,|\tilde{V}_{n+1}|)}(j,a) &:= \mathbb{P}[|\tilde{J}_{n+1}| \leq j, |\tilde{V}_{n+1}| \leq a \mid \mathcal{A}^{JV}_{n,s}] \\ &= \sum_{t\geq 0} \mathbb{P}[|\tilde{J}_{n+1}| \leq j, |\tilde{V}_{n+1}| \leq a \mid \mathcal{A}^{JVT}_{n,s}] \cdot \mathbb{P}[\tilde{X}_n = t \mid \mathcal{A}^{JV}_{n,s}] \\ &= \sum_{t\geq 0} \tilde{h}_{i,v}(x, w; t) \cdot C(F^{|J|}(i, x, t + b^J; j), F^{|V|}(v, w, t + b^V; a)). \end{aligned} \tag{19}$$

Consequently the density can be obtained by derivation of the cumulative distribution function, that is

$$\begin{aligned} f_{(|\tilde{J}_{n+1}|,|\tilde{V}_{n+1}|)}(j,a) &= \frac{\partial^2}{\partial j \partial a} F_{(|\tilde{J}_{n+1}|,|\tilde{V}_{n+1}|)}(j,a) \\ &= \sum_{t\geq 0} \tilde{h}_{i,v}(x, w; t) \cdot \frac{\partial^2}{\partial j \partial a} C(F^{|J|}(i, x, t + b^J; j), F^{|V|}(v, w, t + b^V; a)). \end{aligned} \tag{20}$$

Accordingly we get

$$Cov_{\mathcal{A}^{JV}_{n,s}}(|\tilde{J}_{n+1}|, |\tilde{V}_{n+1}|) = \int_0^\infty \int_0^\infty j \cdot a \cdot f_{(|\tilde{J}_{n+1}|,|\tilde{V}_{n+1}|)}(j,a) dj da - \int_0^\infty j f_{|\tilde{J}_{n+1}|}(j) dj \cdot \int_0^\infty a f_{|\tilde{V}_{n+1}|}(a) da. \tag{21}$$

This allows the recovering of $\rho_{\mathcal{A}^{JV}_{n,s}}(|\tilde{J}_{n+1}|, |\tilde{V}_{n+1}|)$ once the standard deviations $\sigma_{\mathcal{A}^{J}_{n,s}}(|\tilde{J}_{n+1}|)$ and $\sigma_{\mathcal{A}^{V}_{n,s}}(|\tilde{V}_{n+1}|)$ are known. They can be obtained by using the univariate densities $f_{|\tilde{J}_{n+1}|}(\cdot)$ and $f_{|\tilde{V}_{n+1}|}(\cdot)$ that in turn can be obtained by integration of the joint density.

It is also interesting to compute the covariance function between the modulus of log-returns and the log-volumes at next transition, that is

$$\begin{aligned} Cov_{\mathcal{A}^{JV}_{n,s}}(|\tilde{J}_{n+1}|, \tilde{V}_{n+1}) &= \mathbb{E}_{\mathcal{A}^{JV}_{n,s}}[|\tilde{J}_{n+1}| \cdot \tilde{V}_{n+1}] - \mathbb{E}_{\mathcal{A}^{J}_{n,s}}[|\tilde{J}_{n+1}|] \cdot \mathbb{E}_{\mathcal{A}^{V}_{n,s}}[\tilde{V}_{n+1}] \\ &= \mathbb{E}_{\mathcal{A}^{JV}_{n,s}}[|\tilde{J}_{n+1}| \cdot \eta^V_{n+1} \cdot |\tilde{V}_{n+1}|] - \mathbb{E}_{\mathcal{A}^{J}_{n,s}}[|\tilde{J}_{n+1}|] \cdot \mathbb{E}_{\mathcal{A}^{V}_{n,s}}[\eta^V_{n+1} \cdot |\tilde{V}_{n+1}|] \\ &= \mathbb{E}_{\mathcal{A}^{V}_{n,s}}[\eta^V_{n+1}] \cdot (Cov_{\mathcal{A}^{JV}_{n,s}}(|\tilde{J}_{n+1}|, |\tilde{V}_{n+1}|)). \end{aligned} \tag{22}$$

Note that if $\mathbb{E}_{\mathcal{A}^{V}_{n,s}}[\eta^V_{n+1}] = 0$, then the modulus of log-returns and log-volumes are uncorrelated at next transition.

One may also be interested in providing nonlinear measures of dependence between random variables. Mutual information, which goes back to Reference 50, possesses relevant properties that imposed it as a suitable measure of nonlinear dependence, see, for example Reference 51. It is simple to express the mutual information within our model:

$$MI_{\mathcal{A}^{JV}_{n,s}}(|\tilde{J}_{n+1}|, |\tilde{V}_{n+1}|) = \int_0^\infty \int_0^\infty f_{(|\tilde{J}_{n+1}|,|\tilde{V}_{n+1}|)}(j,a) \log \frac{f_{(|\tilde{J}_{n+1}|,|\tilde{V}_{n+1}|)}(j,a)}{f_{|\tilde{J}_{n+1}|}(j) \cdot f_{|\tilde{V}_{n+1}|}(a)} dj da, \tag{23}$$

where the densities are given in formula (20).

3.3.3 | First passage time distributions

The first passage time distribution (fptd) has attracted a lot of attention in finance. It has been considered for different assumptions about the stochastic processes that describes the asset behaviour. It has been investigated for log-returns when described by Ornstein–Uhlenbeck processes (see, for example Reference 52) and more recently for generalized semi-Markov models in References 35,36,53. We shall now derive the fptd for our multivariate model.

Let $\tilde{M}_t^J(\tau)$ be the accumulation factor of the return process in the multivariate model from time t to $t+\tau$. Formally, it is defined as follows:

$$\tilde{M}_t^J(\tau) = e^{\sum_{r=0}^{\tau-1} \tilde{Z}^J(t+r)}.$$

A similar definition applies for the volume process, that is

$$\tilde{M}_t^V(\tau) = e^{\sum_{r=0}^{\tau-1} \tilde{Z}^V(t+r)}.$$

For $\rho \in \mathbb{R}_+$ and $\psi \in \mathbb{R}_+$, denote the joint first passage time by

$$\Gamma_{(\rho;\psi)} := \min\{\tau \geq 0 : \{\tilde{M}_0^J(\tau) \geq \rho\} \cup \{\tilde{M}_0^V(\tau) \geq \psi\}\}. \tag{24}$$

Thus, $\Gamma_{(\rho;\psi)}$ is the first time when at least one accumulation factor exceeds its own thresholds. Denote the corresponding conditional survival function by

$$R_{(\rho;\psi)}((i,v,t)_{-m}^0, u; t) = \mathbb{P}[\Gamma_{(\rho;\psi)} > t | (\tilde{J}, \tilde{V}, \tilde{T})_{-m}^0 = (i,v,t)_{-m}^0, \tilde{B}(u) = u],$$

where $\tilde{B}(u) = u - \tilde{T}_{\tilde{N}(u)}$.

The definition of the shift operator given in Definition 2 can be easily extended to triplet sequences $(i,v,t)_{-m}^n$.

Definition 4. Let $(i,v,t)_{-m}^n = \{(i_\alpha, v_\alpha, t_\alpha), \alpha = -m, \dots, n\}$ be a sequence of returns, volumes and corresponding transition times. Let denote by $\Phi_{-m}^n = \{(i_\alpha, v_\alpha, t_\alpha), \alpha = -m, \dots, 0, \dots, n, i_\alpha \in \mathbb{R}, v_\alpha \in \mathbb{R}, t_\alpha \in \mathbb{Z}, t_\alpha < t_{\alpha+1}\}$, then we define the shift operator

$$\circ : \Phi_{-m}^{n+1} \to \Phi_{-m-1}^n$$

defined by

$$\circ((i,v,t)_{-m}^{n+1}) = (s,y,k)_{-m-1}^n,$$

where $s_\alpha = i_{\alpha+1}, y_\alpha = v_{\alpha+1}, k_\alpha = t_{\alpha+1} - t_{n+1}, \quad \alpha = -m-1, \dots, 0, \dots, n.$

We prove a theorem which provides an equation for the joint fptd.

Theorem 2. *Let f^λ and g^γ be the score functions of the return and volume index processes, respectively. For $i_0 \geq 0$ and $v_0 \geq 0$, it results that*

$$\begin{aligned} R_{(\rho;\psi)}((i,v,t)_{-m}^0, u; t) &= 1_{\{e^{i_0 t} < \rho\}} 1_{\{e^{v_0 t} < \psi\}} \frac{1 - \tilde{H}_{i_0,v_0}(\alpha_0, \beta_0; t)}{1 - \tilde{H}_{i_0,v_0}(\alpha_0, \beta_0; u)} \\ &+ \sum_{t_1=u+1}^{t} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} di_1 dv_1 \frac{1_{\{e^{i_0 t_1} < \rho\}} 1_{\{e^{v_0 t_1} < \psi\}}}{1 - \tilde{H}_{i_0,v_0}(\alpha_0, \beta_0; u)} \\ &\cdot \frac{\partial^2 q^{JV}(i_0, v_0, \alpha_0, \beta_0; i_1, v_1, t_1)}{\partial i_1 \partial v_1} \cdot R_{\left(\frac{\rho}{e^{i_0 t_1}}; \frac{\psi}{e^{v_0 t_1}}\right)}(\circ((i,v,t)_{-m}^1), 0; t - t_1), \end{aligned} \tag{25}$$

where

$$\begin{aligned} \alpha_0 &= \sum_{r=0}^{m-1} \sum_{a=t_{-r-1}}^{t_{-r}-1} f^\lambda(i_{-r-1}, -a) + f^\lambda(i_0, 0), \\ \beta_0 &= \sum_{r=0}^{m-1} \sum_{a=t_{-r-1}}^{t_{-r}-1} g^\gamma(v_{-r-1}, -a) + g^\gamma(v_0, 0). \end{aligned} \tag{26}$$

For $i_0 < 0$ replace $e^{i_0 t_1}$ with e^{i_0} everywhere in formula (25).
For $v_0 < 0$ replace $e^{v_0 t_1}$ with e^{v_0} everywhere in formula (25).

Proof. See the Appendix. ∎

4 | APPLICATION TO REAL HIGH FREQUENCY DATA

To verify the validity of the model described above, we applied it to the database introduced in Section 2. Following References 39,53 we use, as definition of the function f^{λ} in (1), an exponentially weighted moving average (EWMA) of the squares of J_n which has the following expression:

$$f^{\lambda}(J_{n-1-k}, T_n^J, a) = \frac{\lambda^{T_n^J - a} J_{n-1-k}^2}{\sum_{k=0}^{m+n-1} \sum_{a=T_{n-1-k}^J}^{T_{n-k}^J - 1} \lambda^{T_n^J - a} + 1} = \frac{\lambda^{T_n^J - a} J_{n-1-k}^2}{\sum_{a=T_{-m}^J}^{T_n^J} \lambda^a}. \tag{27}$$

A similar choice is done for the volume return process leading to the choice of

$$g^{\gamma}(V_{n-1-k}, T_n^V, a) = \frac{\gamma^{T_n^V - a} V_{n-1-k}^2}{\sum_{k=0}^{m+n-1} \sum_{a=T_{n-1-k}^V}^{T_{n-k}^V - 1} \gamma^{T_n^V - a} + 1} = \frac{\gamma^{T_n^V - a} V_{n-1-k}^2}{\sum_{a=T_{-m}^V}^{T_n^V} \gamma^a}. \tag{28}$$

We remark that the choice for the functional form of f^{λ} is not obtained through any optimization procedure. One can probably find functional forms that perform better according to some performance measure. Our choice is motivated by its simplicity and the fact that it gives very good results. Moreover, it is justified by the empirical evidence that price and volumes returns dynamics do depend on volatility regime.

Next step in the application is finding the optimal parameters to be used in the model. We followed the same procedure described in Reference 38 and summarized in the following subsection.

4.1 | Parameters optimization

We describe here the whole procedure to set and optimize the parameters used in the univariate models.

1. The first step to set the WISMC model is, by using the descriptive statistics of the dataset, to fix a number of states s and a value for the weight parameter λ;
2. Build the trajectory (J_n, T_n^J) implied by the choice of s and λ;
3. Estimate the weighted-indexed semi-Markov kernel $\mathbf{q}^J$ applying the empirical estimators to the trajectory obtained at previous step;
4. Perform Monte Carlo simulation to build synthetic time series;
5. Estimate the autocorrelation function (ACF) for the synthetic time series $\Sigma(\tau; s, \lambda)$. Note that this ACF depends on the number of states s and on the value of the weight parameter λ;
6. Compare the real ACF, $\Sigma(\tau)$, with the synthetic one, $\Sigma(\tau; s, \lambda)$, by computing the mean absolute percentage error (MAPE) between them. The MAPE depends on the number of states and on the value of the weight parameter, then it is denoted by $MAPE(s, \lambda)$;
7. Change the number of states and the parameter λ, restart from point 2 and repeat all points;

At the end of the whole process, choose the number of states s^* and parameter λ^* that best represent the dataset by minimizing the $MAPE(s, \lambda)$, that is

$$(s^*, \lambda^*) = \underset{(s,\lambda)}{\operatorname{argmin}}\{MAPE(s, \lambda)\}.$$

Notice that the algorithm can stop whenever the increase in the number of states does not decrease the MAPE more than a given threshold ϵ.

This procedure should be repeated for all stocks in the portfolio and also for the variable $v(t)$. Once all the parameters for the two univariate models are optimized use a copula to build the bivariate model.

TABLE 4 Parameters used in the application to real data

	$r(t)$			$v(t)$		
	s	λ	$MAPE(\%)$	s	γ	$MAPE(\%)$
TIT	5	0.97	7.7	5	0.97	10.2
ISP	5	0.98	4.6	5	0.98	6.1
TEN	5	0.97	6.2	7	0.97	9.2
F	5	0.97	3.7	9	0.97	5.5

TABLE 5 Cross-correlation between price and volume returns for simulated data

	$\rho(r(t), v(t))$	p-value	$\rho(\lvert r(t)\rvert, v(t))$	p-value	$\rho(r(t), \lvert v(t))\rvert$	p-value	$\rho(\lvert r(t)\rvert, \lvert v(t))\rvert$	p-value
TIT	0.0001	0	0.091	0	−0.0042	0.36	0.0039	0
ISP	−0.00390	0	0.083	0	−0.0025	0.053	−0.0086	0
TEN	−0.0046	0	0.081	0	0.0011	0.46	0.022	0
F	−0.0075	0	0.12	0	−0.0029	0.025	−0.0092	0

4.2 | Results

Here we show some results obtained and a comparison with real data. Using the optimization procedure described above we found the optimal parameters which are summarized in Table 4 for the four stocks

The dependence between the two real processes $v(t)$ and $r(t)$ is kept in the model by using a copula function. We tested different copulas like Gaussian, t-student, Gumbel and Clayton finding almost no differences in the results. This is mainly due to the fact that 1-min price returns are almost discrete and varies in a small range, then, in this dataset there is no tail effect. To keep the application as simple as possible we decided to use a Gaussian copula that has only one parameter. We simulated, using the estimated kernels and a Gaussian copula, the joint process $|r(t)|$ and $v(t)$ and obtained $r(t)$ by using the relation described in assumption **A5**. The results are trajectories with the same time length of real data for both variables $v(t)$ and $r(t)$.

In Table 5 we show the cross-correlation between the synthetic $v(t)$ and $r(t)$ and their absolute values, for all combinations, as done in Table 2. The table shows that there is a good agreement with what was found for real data.

The model is also used to compare the fptd of the joint process $(v(t), r(t))$. From the synthetic variables ($r(t)$ and $v(t)$) we build the variables price and volumes in the following way: at each discrete state of $r(t)$ ($v(t)$) is associated a range of variability of the continuous real $r(t)$ ($v(t)$), inside this range a continuous value is chosen by extracting a random number form a uniform distribution and then inverting the empirical distribution of real continuous price (volume) return. Once $r(t)$ ($v(t)$) are transformed back into continuous values, prices $S(t)$ (volume $V(t)$) are obtained by $S(t) = S_0 \times e^{\sum_{k=1}^{k=t} r(k)}$ ($V(t) = V_0 \times e^{\sum_{k=1}^{k=t} v(k)}$). Synthetic prices and volumes are then used to build the distribution of time at which there is the first cross of given thresholds.

To verify if the price fptd depends on volume values we estimated the fptd as a function of the value of initial condition on the discretized volume returns. In this way, for each initial $v(t)$ value, we obtain a different fptd. Moreover, we verified if the proposed model also keeps this dependence structure. In Figure 3 we show the results and comparison with real data (for two of the given stocks). We fixed a price increment threshold at 0.5% and we see that it takes 10–15 min to reach this price change.

From the figure we can see that there is, indeed, a dependence on the initial conditions on volume. Furthermore, the dependence lasts up to 15 min and after this time all distributions converge to similar values. The behaviour depends on the different stocks but almost all of them show faster achievements of threshold when the initial volume return is in the third state. Figure 3 also shows that, although with some differences, the model has the same behaviour of real data, there is a dependence on the initial conditions of volume return and it is very similar to real data. Finally, in Figure 4 we show the joint fptd that represent the first time that both $S(t)$ and $V(t)$ cross a given threshold. Again, the price increment threshold is set at 0.5% while the volume increment threshold is 100. Overall we can say that the model is able to capture all statistical features of real data keeping all the dependencies between price, volumes and waiting times. Furthermore, we found very good agreement between real data and model also for the first passage time distribution.

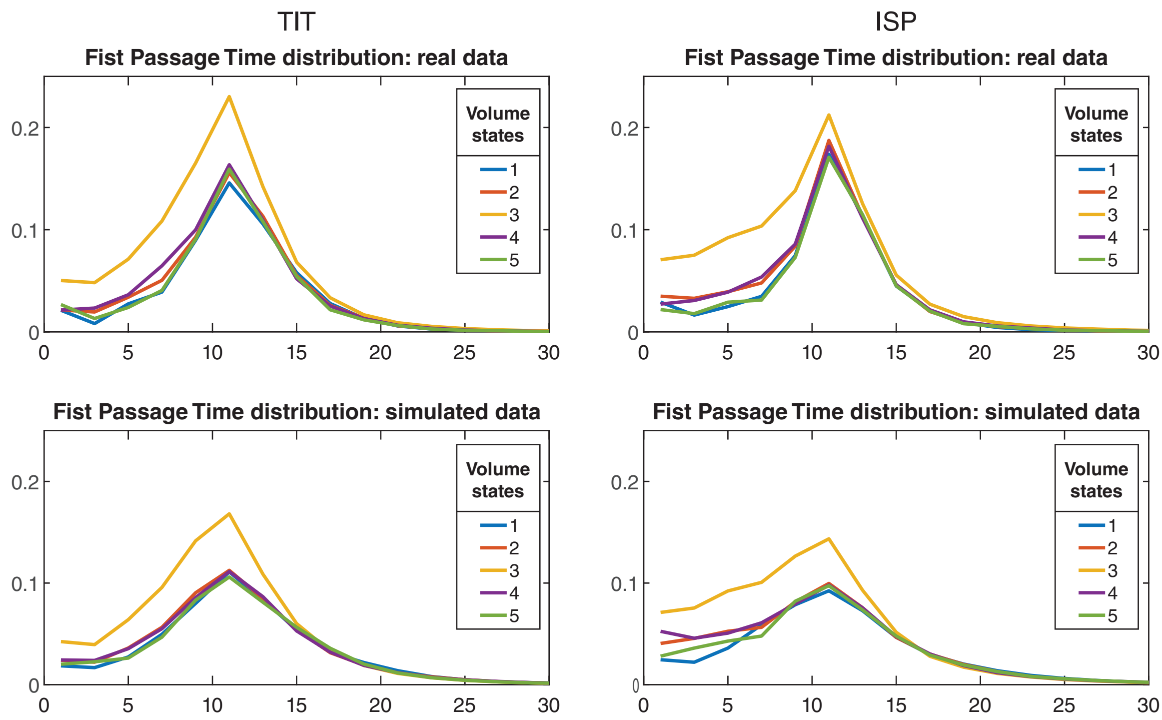

FIGURE 3 First passage time distribution of real data compared with synthetic data

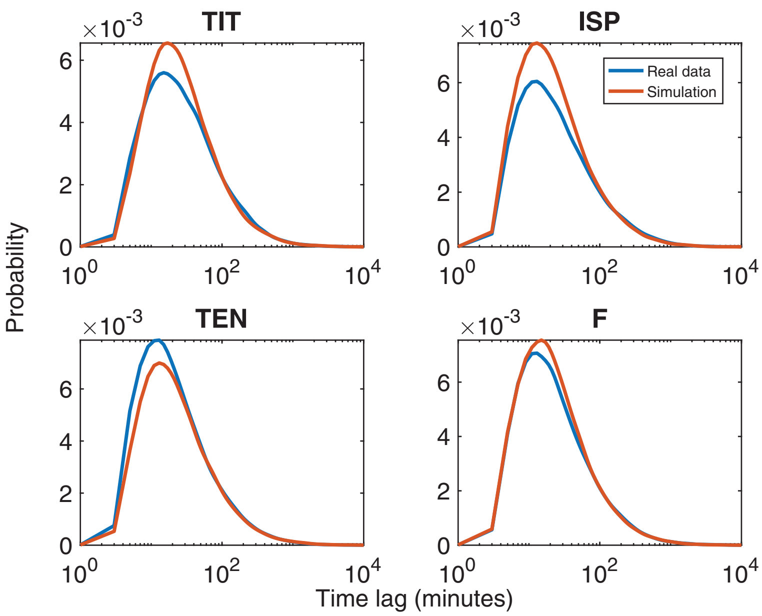

FIGURE 4 First passage time distribution of real data compared with synthetic data

5 | CONCLUSIONS

In this work we have advanced a new stochastic model, based on weighted-indexed semi-Markov chain, for modelling price, volumes and waiting times in high frequency finance. After showing all the empirical evidences that support the use of a multivariate model, we defined the probabilistic structure of the model and give a detailed mathematical implementation. Furthermore, mathematical expressions for covariances and first passage time distributions are given. In the last part we show, by using Monte Carlo simulations, that the model has the same statistical features of real data. In fact, the model is able to reproduce the autocorrelation functions, the dependence between price and volume and the first passage time distributions. The parameter optimization is based on a performance measure that relies on mean square error between autocorrelation function, further studies could be conducted to verify if there exist better measures. Furthermore, the model could be used in portfolio optimization, development of risk measure and volatility forecasting.

ORCID

Filippo Petroni https://orcid.org/0000-0001-8467-8481

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How to cite this article: D'Amico G, Petroni F. A micro-to-macro approach to returns, volumes and waiting times. *Appl Stochastic Models Bus Ind*. 2021;37:767–789. https://doi.org/10.1002/asmb.2622

APPENDIX

A.1 Descriptive statistics

In Table A1 we summarize the descriptive statistics of price returns $r(t)$, while in Table A2 we show the descriptive statistics of volume returns $v(t)$.

TABLE A1 Descriptive statistics of the price returns $r(t)$

Stock	Mean	Median	SD	Skewness	Kurtosis
TIT	$-2 * 10^{-6}$	0	$4.7 * 10^{-4}$	$0.18 * 10^{-2}$	2.5
ISP	$2.5 * 10^{-6}$	0	$5.4 * 10^{-4}$	$-0.25 * 10^{-2}$	2.2
TEN	$-4.2 * 10^{-7}$	0	$4.8 * 10^{-4}$	$0.24 * 10^{-3}$	2.8
F	$4.7 * 10^{-6}$	0	$5.5 * 10^{-4}$	$-0.88 * 10^{-2}$	2.3

TABLE A2 Descriptive statistics of volume returns $v(t)$

Stock	Mean	Median	SD	Skewness	Kurtosis
TIT	$-2.9 * 10^{-2}$	−0.06	1.58	$6.6 * 10^{-2}$	3.94
ISP	$-2.2 * 10^{-2}$	−0.05	1.53	$9.2 * 10^{-2}$	3.96
TEN	$-2.9 * 10^{-2}$	−0.05	1.31	$5.9 * 10^{-3}$	4.08
F	$-2.3 * 10^{-2}$	−0.06	1.51	$8.7 * 10^{-2}$	3.88

A.2 Proofs

Proof of Lemma 1. Given $(J,T)_{-m}^{n+1} = (i,t)_{-m}^{n+1}$, we can compute the value of the index process at the $(n+1)$th transition through formula (1) and assumption **A1**:

$$\begin{aligned} I_{n+1}^{J}(\lambda) &= \sum_{r=0}^{m+n+1-1} \sum_{a=T_{n+1-1-r}}^{T_{n+1-r}-1} f^{\lambda}(J_{n+1-1-r}, T_{n+1}-a) + f^{\lambda}(J_{n+1}, T_{n+1}-T_{n+1}) \\ &= \sum_{r=0}^{m+nt} \sum_{a=t_{n-r}}^{_{n+1-r}-1} f^{\lambda}(i_{n-r}, t_{n+1}-a) + f^{\lambda}(i_{n+1}, 0). \end{aligned} \tag{A1}$$

For simplicity of notation, denote by x this value, that is $I_{n+1}^{J}(\lambda) = x$. Thus,

$$\begin{aligned} &\mathbb{P}[J_{n+2} \le j, T_{n+2} - T_{n+1} = t | (J,T)_{-m}^{n+1} = (i,t)_{-m}^{n+1}] = \\ &\mathbb{P}[J_{n+2} \le j, T_{n+2} - T_{n+1} = t | J_{n+1} = i_{n+1}, I_{n+1}^{J}(\lambda) = x] = q(i_{n+1}, x; j, t). \end{aligned} \tag{A2}$$

Let us consider now the probability $\mathbb{P}[J_{n+1} \le j, T_{n+1} - T_n = t | (J,T)_{-m-1}^{n} = \circ((i,t)_{-m}^{n+1})]$ and apply the definition of the shift operator to have:

$$\circ((i,t)_{-m}^{n+1}) = (s,k)_{-m-1}^{n}, \tag{A3}$$

and in turn

$$\begin{aligned} &\mathbb{P}[J_{n+1} \le j, T_{n+1} - T_n = t | (J,T)_{-m-1}^{n} = \circ((i,t)_{-m}^{n+1})] \\ &\quad = \mathbb{P}[J_{n+1} \le j, T_{n+1} - T_n = t | (J,T)_{-m-1}^{n} = (s,k)_{-m-1}^{n})] \\ &\quad = \mathbb{P}[J_{n+1} \le j, T_{n+1} - T_n = t | J_n = s_n, I_n^{J}(\lambda) = b], \end{aligned} \tag{A4}$$

where

$$b = \sum_{r=0}^{m+n} \sum_{a=k_{n-1-r}}^{k_{n-r}-1} f^{\lambda}(s_{n-1-r}, k_n - a) + f^{\lambda}(s_n, 0). \tag{A5}$$

Since $s_{n-1-r} = i_{n-r}$ and $k_{n-1-r} = t_{n-r} - t_{n+1}$ it follows that

$$b = \sum_{r=0}^{m+nt} \sum_{a=t_{n-r}-t_{n+1}}^{_{n-r+1}-t_{n+1}-1} f^{\lambda}(i_{n-r}, -a) + f^{\lambda}(i_{n+1}, 0).$$

A change of variable $y = a + t_{n+1}$ gives

$$b = \sum_{r=0}^{m+nt} \sum_{y=t_{n-r}}^{_{n-r+1}-1} f^{\lambda}(i_{n-r}, t_{n+1} - y) + f^{\lambda}(i_{n+1}, 0) = x.$$

Accordingly, we get

$$\begin{aligned} &\mathbb{P}[J_{n+1} \le j, T_{n+1} - T_n = t | (J,T)_{-m-1}^{n} = \circ((i,t)_{-m}^{n+1})] \\ &\quad = \mathbb{P}[J_{n+1} \le j, T_{n+1} - T_n = t | J_n = i_{n+1}, I_n^{J}(\lambda) = x] = q(i_{n+1}, x; j, t), \end{aligned}$$

which completes the proof. ∎

Proof of Theorem 1. The kernel of the triplet process has been represented in formula (9) as follows: $q^{JV}(\mathcal{A}_{n,s}^{JV}; j, a, t) = \mathbb{P}[\tilde{J}_{n+1} \le j, \tilde{V}_{n+1} \le a | \mathcal{A}_{n,s}^{JVT}] \cdot \mathbb{P}[\tilde{X}_n = t | \mathcal{A}_{n,s}^{JV}]$, and from assumption **A3** we get $\tilde{h}_{i,v}(x, w; t) = \mathbb{P}[\tilde{X}_n = t | \mathcal{A}_{n,s}^{JV}]$.

Thus, it remains to evaluate the conditional probability of the joint distribution of log-return and log-volume. Let consider the case when $j \ge 0$ and $a \ge 0$ and introduce the notation $F^{|J|}(j)$ and $F^{|V|}(a)$ to denote in a compact form the marginal distributions of the copula.

Let us consider the following representation:

$$
\begin{aligned}
\mathbb{P}[\tilde{J}_{n+1} \leq j, \tilde{V}_{n+1} \leq a | \mathcal{A}_{n,s}^{JVT}] &= \mathbb{P}[|\tilde{J}_{n+1}| \leq j, |\tilde{V}_{n+1}| \leq a | \mathcal{A}_{n,s}^{JVT}] \\
&+ \mathbb{P}[|\tilde{J}_{n+1}| > j, \eta_{n+1}^{J} = -1, |\tilde{V}_{n+1}| \leq a | \mathcal{A}_{n,s}^{JVT}] \\
&+ \mathbb{P}[|\tilde{J}_{n+1}| > j, \eta_{n+1}^{J} = -1, |\tilde{V}_{n+1}| > a, \eta_{n+1}^{V} = -1 | \mathcal{A}_{n,s}^{JVT}] \\
&+ \mathbb{P}[|\tilde{J}_{n+1}| \leq j, |\tilde{V}_{n+1}| > a, \eta_{n+1}^{V} = -1 | \mathcal{A}_{n,s}^{JVT}].
\end{aligned} \tag{A6}
$$

Let us proceed to the computation of each one of the four addenda in (A6). From assumption **A4** we know that

$$
\mathbb{P}[|\tilde{J}_{n+1}| \leq j, |\tilde{V}_{n+1}| \leq a | \mathcal{A}_{n,s}^{JVT}] = C(F^{|J|}(j), F^{|V|}(a)). \tag{A7}
$$

Next consider $\mathbb{P}[|\tilde{J}_{n+1}| > j, \eta_{n+1}^{J} = -1, |\tilde{V}_{n+1}| > a, \eta_{n+1}^{V} = -1 | \mathcal{A}_{n,s}^{JVT}]$. From assumption **A5** this probability is equal to

$$
\begin{aligned}
&\mathbb{P}[\eta_{n+1}^{J} = -1] \cdot \mathbb{P}[\eta_{n+1}^{V} = -1] \cdot \mathbb{P}[|\tilde{J}_{n+1}| > j, |\tilde{V}_{n+1}| > a | \mathcal{A}_{n,s}^{JVT}] \\
&\quad = (1 - p^{J})(1 - p^{V})\{1 - \mathbb{P}[|\tilde{J}_{n+1}| \leq j, |\tilde{V}_{n+1}| \leq a | \mathcal{A}_{n,s}^{JVT}] \\
&\qquad - \mathbb{P}[|\tilde{J}_{n+1}| > j, |\tilde{V}_{n+1}| \leq a | \mathcal{A}_{n,s}^{JVT}] - \mathbb{P}[|\tilde{J}_{n+1}| \leq j, |\tilde{V}_{n+1}| > a | \mathcal{A}_{n,s}^{JVT}]\} \\
&\quad = (1 - p^{J})(1 - p^{V})\{1 - C(F^{|J|}(j), F^{|V|}(a)) - (\mathbb{P}[|\tilde{J}_{n+1}| \leq j | \mathcal{A}_{n,s}^{JVT}] \\
&\qquad - \mathbb{P}[|\tilde{J}_{n+1}| \leq j, |\tilde{V}_{n+1}| \leq a | \mathcal{A}_{n,s}^{JVT}]) - (\mathbb{P}[|\tilde{V}_{n+1}| \leq a | \mathcal{A}_{n,s}^{JVT}] - \mathbb{P}[|\tilde{J}_{n+1}| \leq j, |\tilde{V}_{n+1}| \leq a | \mathcal{A}_{n,s}^{JVT}])\} \\
&\quad = (1 - p^{J})(1 - p^{V})\{1 - C(F^{|J|}(j), F^{|V|}(a)) - F^{|J|}(j) + C(F^{|J|}(j), F^{|V|}(a)) - F^{|V|}(a) + C(F^{|J|}(j), F^{|V|}(a))\} \\
&\quad = (1 - p^{J})(1 - p^{V})[1 - F^{|J|}(j) - F^{|V|}(j) + C(F^{|J|}(j), F^{|V|}(a))].
\end{aligned} \tag{A8}
$$

Then, proceed to compute $\mathbb{P}[|\tilde{J}_{n+1}| > j, \eta_{n+1}^{J} = -1, |\tilde{V}_{n+1}| \leq a | \mathcal{A}_{n,s}^{JVT}]$. Apply again assumptions **A4** and **A5** to get

$$
\begin{aligned}
&\mathbb{P}[\eta_{n+1}^{J} = -1] \cdot \mathbb{P}[|\tilde{J}_{n+1}| > j, |\tilde{V}_{n+1}| \leq a | \mathcal{A}_{n,s}^{JVT}] \\
&\quad = (1 - p^{J})[\mathbb{P}[|\tilde{V}_{n+1}| \leq a | \mathcal{A}_{n,s}^{JVT}] - \mathbb{P}[|\tilde{J}_{n+1}| \leq j, |\tilde{V}_{n+1}| \leq a | \mathcal{A}_{n,s}^{JVT}]] \\
&\quad = (1 - p^{J})[F^{|V|}(a) - C(F^{|J|}(j), F^{|V|}(a))].
\end{aligned} \tag{A9}
$$

Analogous computations allow to get

$$
\mathbb{P}[|\tilde{J}_{n+1}| \leq j, \eta_{n+1}^{V} = -1, |\tilde{V}_{n+1}| > a | \mathcal{A}_{n,s}^{JVT}] = (1 - p^{V})[F^{|J|}(j) - C(F^{|J|}(j), F^{|V|}(a))]. \tag{A10}
$$

A substitution of (A7), (A8), (A9) and (A10) into (A6) and some algebraic manipulations produces

$$
\begin{aligned}
\mathbb{P}[\tilde{J}_{n+1} \leq j, \tilde{V}_{n+1} \leq a | \mathcal{A}_{n,s}^{JVT}] &= 1 - p^{J} p^{V} \cdot (1 - F^{|J|}(j) - F^{|V|}(a) + C(F^{|J|}(j), F^{|V|}(a))) \\
&\quad - p^{V}(1 - F^{|V|}(a)) - p^{J}(1 - F^{|J|}(j)).
\end{aligned} \tag{A11}
$$

A multiplication of (A11) by $\tilde{h}_{i,v}(x, w; t)$ concludes the proof for the case (i). The remaining cases (ii)–(iv) can be accomplished by similar arguments. ∎

Proof of Theorem 2.

$$
\begin{aligned}
R_{(\rho;\psi)}((i, v, t)_{-m}^{0}, u; t) &= \mathbb{P}[\Gamma_{(\rho;\psi)} > t | (\tilde{J}, \tilde{V}, \tilde{T})_{-m}^{0} = (i, v, t)_{-m}^{0}, \tilde{B}(u) = u] \\
&= \mathbb{P}[\Gamma_{(\rho;\psi)} > t, \tilde{T}_{1} > t | (\tilde{J}, \tilde{V}, \tilde{T})_{-m}^{0} = (i, v, t)_{-m}^{0}, \tilde{B}(u) = u]
\end{aligned} \tag{A12}
$$

$$
+ \mathbb{P}[\Gamma_{(\rho;\psi)} > t, \tilde{T}_{1} \leq t | (\tilde{J}, \tilde{V}, \tilde{T})_{-m}^{0} = (i, v, t)_{-m}^{0}, \tilde{B}(u) = u]. \tag{A13}
$$

By the definition of conditional probability (A12) can be written as

$$
\mathbb{P}[\Gamma_{(\rho;\psi)} > t | \tilde{T}_{1} > t, (\tilde{J}, \tilde{V}, \tilde{T})_{-m}^{0} = (i, v, t)_{-m}^{0}, \tilde{B}(u) = u] \times \mathbb{P}[\tilde{T}_{1} > t | (\tilde{J}, \tilde{V}, \tilde{T})_{-m}^{0} = (i, v, t)_{-m}^{0}, \tilde{B}(u) = u]. \tag{A14}
$$

Note that by definition $\tilde{X}_0 := \tilde{T}_1 - \tilde{T}_0$ but since $\tilde{T}_0 = t_0 = 0$, we can replace $\tilde{T}_1$ with the corresponding sojourn time $\tilde{X}_0$. Also note that the event $\{\tilde{B}(u) = u\}$ is equivalent to the event $\{\tilde{T}_{N(u)} = 0, \tilde{T}_{N(u)+1} > u\}$. The latter equality between events means that at least one between returns and volumes did last transition at time $t_0 = 0$ and the other process made its last transition at some time before. Let b^J and b^V generically denote the times since last transition of the backward recurrence time processes, that is

$$\tilde{T}_0 - T_0^J = b^J, \quad \tilde{T}_0 - T_0^V = b^V.$$

Besides, note that the information set $(\tilde{J}, \tilde{V}, \tilde{T})_{-m}^0 = (i, v, t)_{-m}^0$ generates a value of the index process of returns equal to

$$\tilde{I}_0^V = \sum_{r=0}^{m-1} \sum_{a=t_{-r-1}}^{t_{-r}-1} f^{\lambda_J}(i_{-r-1}, -a) + f^{\lambda_J}(i_0, 0) =: \alpha_0, \tag{A15}$$

and of the index process of volumes equal to

$$\tilde{I}_0^V = \sum_{r=0}^{m-1} \sum_{a=t_{-r-1}}^{t_{-r}-1} g^{\lambda_V}(v_{-r-1}, -a) + g^{\lambda_V}(v_0, 0) =: \beta_0. \tag{A16}$$

Thus, in virtue of assumption **A2**, the probability (A14) becomes equal to

$$\begin{aligned} &\mathbb{P}[\Gamma_{(\rho;\psi)} > t | \tilde{X}_0 > t, \tilde{J}_0 = i_0, \tilde{V}_0 = v_0, \tilde{I}_0^J = \alpha_0, \tilde{I}_0^V = \beta_0, \tilde{T}_1 > u, \tilde{T}_0 - T_0^J = b^J, \tilde{T}_0 - T_0^V = b^V] \\ &\times \mathbb{P}[\tilde{X}_0 > t | (\tilde{J}, \tilde{V}, \tilde{T})_{-m}^0 = (i, v, t)_{-m}^0, \tilde{B}(u) = u]. \end{aligned} \tag{A17}$$

Nevertheless, according to assumption **A3**, we have

$$\begin{aligned} &\mathbb{P}[\tilde{X}_0 > t | (\tilde{J}, \tilde{V}, \tilde{T})_{-m}^0 = (i, v, t)_{-m}^0, \tilde{B}(u) = u] \\ &= \mathbb{P}[\tilde{X}_0 > t | \tilde{X}_0 > u, \tilde{J}_0 = i_0, \tilde{V}_0 = v_0, \tilde{I}_0^J = \alpha_0, \tilde{I}_0^V = \beta_0] \\ &= \frac{\mathbb{P}[\tilde{X}_0 > t | \tilde{J}_0 = i_0, \tilde{V}_0 = v_0, \tilde{I}_0^J = \alpha_0, \tilde{I}_0^V = \beta_0]}{\mathbb{P}[\tilde{X}_0 > u | \tilde{J}_0 = i_0, \tilde{V}_0 = v_0, \tilde{I}_0^J = \alpha_0, \tilde{I}_0^V = \beta_0]} = \frac{1 - \tilde{H}_{i_0,v_0}(\alpha_0, \beta_0; t)}{1 - \tilde{H}_{i_0,v_0}(\alpha_0, \beta_0; u)}. \end{aligned} \tag{A18}$$

By the definition of joint first passage time we have that

$$\begin{aligned} &\mathbb{P}[\Gamma_{(\rho;\psi)} > t | \tilde{X}_0 > t, \tilde{J}_0 = i_0, \tilde{V}_0 = v_0, \tilde{I}_0^J = \alpha_0, \tilde{I}_0^V = \beta_0, \tilde{T}_1 > u, \tilde{T}_0 - T_0^J = b^J, \tilde{T}_0 - T_0^V = b^V] \\ &\qquad = \mathbb{P}[\min\{\tau \geq 0 : \{\tilde{M}_0^J(\tau) \geq \rho\} \cup \{\tilde{M}_0^V(\tau) \geq \psi\}\} | \mathcal{A}_{0,0}^{JV}, \tilde{T}_1 > u], \end{aligned} \tag{A19}$$

where

$$\mathcal{A}_{0,0}^{JV} = \{\tilde{J}_0 = i_0, \tilde{V}_0 = v_0, \tilde{I}_0^J = \alpha_0, \tilde{I}_0^V = \beta_0, \tilde{T}_0 = 0, \tilde{T}_0 - T_0^J = b^J, \tilde{T}_0 - T_0^V = b^V\}.$$

It is clear that since $i_0 \geq 0$, $v_0 \geq 0$ and $\tilde{T}_1 > t$, the processes $\tilde{M}_0^J(\tau)$ and $\tilde{M}_0^V(\tau)$ are both increasing with respect to the variable τ. Accordingly,

$$\max_{\tau \in \{0,1,\ldots,t\}} \{\tilde{M}_0^J(\tau)\} = \tilde{M}_0^J(t) = e^{\sum_{r=0}^{t-1} \tilde{Z}^J(r)} = e^{i_0 t},$$

and analogously $\max_{\tau \in \{0,1,\ldots,t\}} \{\tilde{M}_0^V(\tau)\} = e^{v_0 t}$. Thus, formula (A19) becomes

$$\mathbb{P}[e^{i_0 t} < \rho, e^{v_0 t} < \psi | \mathcal{A}_{0,0}^{JV}, \tilde{T}_1 > u] = 1_{\{e^{i_0 t} < \rho\}} 1_{\{e^{v_0 t} < \psi\}}. \tag{A20}$$

A substitution of (A20) and (A18) in (A17) gives:

$$\mathbb{P}[\Gamma_{(\rho;\psi)} > t | (\tilde{J}, \tilde{V}, \tilde{T})_{-m}^0 = (i, v, t)_{-m}^0, \tilde{B}(u) = u] = 1_{\{e^{i_0 t} < \rho\}} 1_{\{e^{v_0 t} < \psi\}} \frac{1 - \tilde{H}_{i_0,v_0}(\alpha_0, \beta_0; t)}{1 - \tilde{H}_{i_0,v_0}(\alpha_0, \beta_0; u)}. \tag{A21}$$

It remains to compute probability (A13). The law of total probability and the definition of conditional probability give the following chain of equality:

$$
\begin{aligned}
&\mathbb{P}[\Gamma_{(\rho;\psi)} > t, \tilde{T}_1 \leq t | (\tilde{J}, \tilde{V}, \tilde{T})_{-m}^{0} = (i, v, t)_{-m}^{0}, \tilde{B}(u) = u] = \sum_{t_1=1}^{t} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbb{P}[\Gamma_{(\rho;\psi)} > t, \\
&\quad \tilde{T}_1 = t_1, \tilde{J}_1 \in (i_1, i_1 + di_1), \tilde{V}_1 \in (v_1, v_1 + dv_1) | (\tilde{J}, \tilde{V}, \tilde{T})_{-m}^{0} = (i, v, t)_{-m}^{0}, \tilde{B}(u) = u] \\
&\quad = \sum_{t_1=1}^{t} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbb{P}[\Gamma_{(\rho;\psi)} > t | \tilde{T}_1 = t_1, \tilde{J}_1 = i_1, \tilde{V}_1 = v_1, (\tilde{J}, \tilde{V}, \tilde{T})_{-m}^{0} = (i, v, t)_{-m}^{0}, \tilde{B}(u) = u] \\
&\quad \times \mathbb{P}[\tilde{T}_1 = t_1, \tilde{J}_1 \in (i_1, i_1 + di_1), \tilde{V}_1 \in (v_1, v_1 + dv_1) | (\tilde{J}, \tilde{V}, \tilde{T})_{-m}^{0} = (i, v, t)_{-m}^{0}, \tilde{B}(u) = u].
\end{aligned}
$$

Let start to compute the following probability:

$$
\begin{aligned}
&\mathbb{P}[\tilde{T}_1 = t_1, \tilde{J}_1 \in (i_1, i_1 + di_1), \tilde{V}_1 \in (v_1, v_1 + dv_1) | (\tilde{J}, \tilde{V}, \tilde{T})_{-m}^{0} = (i, v, t)_{-m}^{0}, \tilde{B}(u) = u] \\
&\quad = \mathbb{P}[\tilde{T}_1 = t_1, \tilde{J}_1 \in (i_1, i_1 + di_1), \tilde{V}_1 \in (v_1, v_1 + dv_1) | \mathcal{A}_{0,0}^{JV}, \tilde{T}_1 > u] \\
&\quad = \frac{\mathbb{P}[u < \tilde{T}_1 = t_1, \tilde{J}_1 \in (i_1, i_1 + di_1), \tilde{V}_1 \in (v_1, v_1 + dv_1) | \mathcal{A}_{0,0}^{JV}]}{\mathbb{P}[\tilde{T}_1 > u | \mathcal{A}_{0,0}^{JV}]} \\
&\quad = \frac{1_{\{t_1 > u\}} \frac{\partial^2 q^{JV}(i_0, v_0, \alpha_0, \beta_0; i_1, v_1, t_1)}{\partial i_1 \partial v_1} di_1 dv_1}{1 - \tilde{H}_{i_0, v_0}(\alpha_0, \beta_0; u)}.
\end{aligned} \tag{A22}
$$

It remains to compute

$$
\begin{aligned}
&\mathbb{P}[\Gamma_{(\rho;\psi)} > t | \tilde{T}_1 = t_1, \tilde{J}_1 = i_1, \tilde{V}_1 = v_1, (\tilde{J}, \tilde{V}, \tilde{T})_{-m}^{0} = (i, v, t)_{-m}^{0}, \tilde{B}(u) = u] \\
&\quad = \mathbb{P}[\max_{s \in \{0,1,\dots,t\}} \{\tilde{M}_0^{J}(s)\} < \rho, \max_{s \in \{0,1,\dots,t\}} \{\tilde{M}_0^{V}(s)\} < \psi | (\tilde{J}, \tilde{V}, \tilde{T})_{-m}^{1} = (i, v, t)_{-m}^{1}, \tilde{B}(u) = u].
\end{aligned} \tag{A23}
$$

Now observe that since $\tilde{T}_1 = t_1$ we have that

$$
\max_{s \in \{0,1,\dots,t\}} \{\tilde{M}_0^{J}(s)\} = \max\{\max_{s \in \{0,1,\dots,t_1\}} \{\tilde{M}_0^{J}(s)\}, \max_{s \in \{1,\dots,t-t_1\}} \{\tilde{M}_0^{J}(t_1 + s)\}\},
$$

and due to the fact that $i_0 \geq 0$ it results that $\max_{s \in \{0,1,\dots,t_1\}} \{\tilde{M}_0^{J}(s)\} = \tilde{M}_0^{J}(t_1) = e^{t_1 i_0}$. Accordingly we can deduce that

$$
\max_{s \in \{0,1,\dots,t\}} \{\tilde{M}_0^{J}(s)\} = \max\{e^{t_1 i_0}, \max_{s \in \{1,\dots,t-t_1\}} \{e^{t_1 i_0} e^{\sum_{r=0}^{s-1} \tilde{Z}^{J}(t_1 + r)}\}\}.
$$

Similarly we have

$$
\max_{s \in \{0,1,\dots,t\}} \{\tilde{M}_0^{V}(s)\} = \max\{e^{t_1 v_0}, \max_{s \in \{1,\dots,t-t_1\}} \{e^{t_1 v_0} e^{\sum_{r=0}^{s-1} \tilde{Z}^{V}(t_1 + r)}\}\}.
$$

Thus by substitution, the probability (A23) becomes

$$
\begin{aligned}
&= \mathbb{P}[\max\{e^{t_1 i_0}, \max_{s \in \{1,\dots,t-t_1\}} \{e^{t_1 i_0} e^{\sum_{r=0}^{s-1} \tilde{Z}^{J}(t_1 + r)}\}\} < \rho, \\
&\quad \times \max\{e^{t_1 v_0}, \max_{s \in \{1,\dots,t-t_1\}} \{e^{t_1 v_0} e^{\sum_{r=0}^{s-1} \tilde{Z}^{V}(t_1 + r)}\}\} < \psi | (\tilde{J}, \tilde{V}, \tilde{T})_{-m}^{1} = (i, v, t)_{-m}^{1}, \tilde{B}(u) = u]. \\
&= 1_{\{e^{t_1 i_0} < \rho\}} 1_{\{e^{t_1 v_0} < \psi\}} \\
&\quad \cdot \mathbb{P}[\max_{s \in \{1,\dots,t-t_1\}} \{e^{t_1 i_0} e^{\sum_{r=0}^{s-1} \tilde{Z}^{J}(t_1 + r)}\} < \rho, \max_{s \in \{1,\dots,t-t_1\}} \{e^{t_1 v_0} e^{\sum_{r=0}^{s-1} \tilde{Z}^{V}(t_1 + r)}\} < \psi \\
&\quad \times | (\tilde{J}, \tilde{V}, \tilde{T})_{-m}^{1} = (i, v, t)_{-m}^{1}, \tilde{B}(u) = u].
\end{aligned}
$$

$$= 1_{\{e^{t_1 i_0}<\rho\}}1_{\{e^{t_1 v_0}<\psi\}}\cdot\mathbb{P}[\max_{s\in\{1,\dots,t-t_1\}}\{e^{\sum_{r=0}^{s-1}\tilde{Z}^J(t_1+r)}\}<\frac{\rho}{e^{t_1 i_0}},$$
$$\times\max_{s\in\{1,\dots,t-t_1\}}\{e^{\sum_{r=0}^{s-1}\tilde{Z}^V(t_1+r)}\}<\frac{\psi}{e^{t_1 v_0}}|(\tilde{J},\tilde{V},\tilde{T})_{-m}^{1}=(i,v,t)_{-m}^{1},\tilde{B}(u)=u].$$
$$= 1_{\{e^{t_1 i_0}<\rho\}}1_{\{e^{t_1 v_0}<\psi\}}\cdot\mathbb{P}[\Gamma_{(\frac{\rho}{e^{t_1 i_0}};\frac{\psi}{e^{t_1 v_0}})}>t-t_1|(\tilde{J},\tilde{V},\tilde{T})_{-m}^{1}=(i,v,t)_{-m}^{1},\tilde{B}(u)=u].$$

The latter probability, making use of Definition 4 can be expressed as

$$1_{\{e^{t_1 i_0}<\rho\}}1_{\{e^{t_1 v_0}<\psi\}}\cdot\mathbb{P}[\Gamma_{(\frac{\rho}{e^{t_1 i_0}};\frac{\psi}{e^{t_1 v_0}})}>t-t_1|(\tilde{J},\tilde{V},\tilde{T})_{-m-1}^{0}=\circ(i,v,t)_{-m}^{1},\tilde{B}(u)=u]$$
$$= 1_{\{e^{t_1 i_0}<\rho\}}1_{\{e^{t_1 v_0}<\psi\}}\cdot R_{(\frac{\rho}{e^{i_0 t_1}};\frac{\psi}{e^{v_0 t_1}})}(\circ((i,v,t)_{-m}^{1}),0;t-t_1). \tag{A24}$$

A substitution of (A24) into (A23) and then of the obtained quantity in (A13) together with (A22) concludes the proof. ∎