Rotational Mobility Analysis of the 3-RFR Class of Spherical Parallel Robots

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Abstract. Spherical parallel manipulators (SPMs) are used to orient a tool in the space with three degrees of freedom exploiting the strengths of a multi-limb architecture. On the other hand, the performance of parallel kinematics machines (PKMs) is often affected by the occurrence of different kinds of singular configurations. The paper aims at characterizing a class of SPMs for which all singularities come to coincide and a single expression is able to describe all the singular configurations of the machines. The study is focused on a class of SPMs with 3-RFR topology (Revolute-Planar-Revolute pairs for each of the three limbs) addressing the mobility and singularity analysis by means of polynomial decomposition and screw theory. The neatness of the equations that are worked out, expressed in a robust formulation based on rotation invariants, allows a straightforward planning of singularity free tasks and simplifies the synthesis of dexterous machines.

Keywords: Mobility Analysis; Polynomial Decomposition; Singularity; Spherical Parallel Machines.

1 Introduction

Spherical parallel manipulators (SPMs) are a specific class of parallel kinematics machines (PKMs) which are able to generate a three-dof spherical motion; they are sometimes also called three-dof orientational parallel manipulators or rotational parallel manipulators [1]. SPMs can be divided into two families: *overconstrained* or *hyperstatic* SPMs and *non-overconstrained* or *isostatic* SPMs. The main advantage of isostatic architectures is that they do not need the strict dimensional and geometric tolerances of overconstrained machines during manufacturing and assembly; thus they can work even when precise geometrical conditions are not accurately met, if at the expense of parasitic motions. Moreover, modular solutions characterized by three identical legs (symmetrical PKMs) are usually preferred for economic reasons. The present work is focused on the 3-RFR class of tripod SPMs which have an isostatic and symmetrical architecture. This class of machines has already been studied by many researchers, with notable contributions by Karouia and Hervé [2] that first studied the mobility of this class of SPM by means of Lie algebra, Kong and Gosselin [3] that used screw theory for the synthesis of parallel wrists and Di Gregorio, who developed the kinematic relations of a number of SPMs [4], [5], [6], [7].

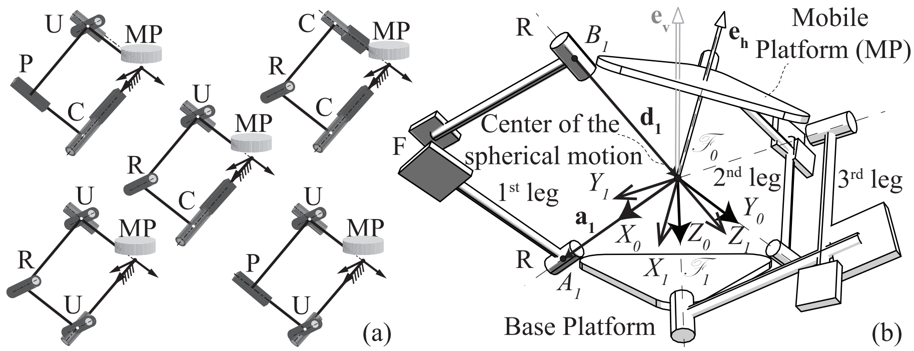

Fig. 1. The 3-RFR class of SPMs: (a) different leg topologies; (b) kinematic model.

After a first description of this class of manipulators, the paper is divided into two sections: the analysis of mobility and a comprehensive study on singular configurations, which are addressed by means of powerful geometrical tools such as polynomial decomposition and screw theory. The latter has been used to analyze the instantaneous mobility of the machines while the global mobility has been addressed by using algebraic geometry as done by Kong in [8]. In this work, the pose of the mobile platform (MP) is described by means of the Study parameters, which allow a polynomial representation of legs constraint equations. The subset of the Study parameters used to represent rotations are the same Euler-Rodrigues Parameters (ERPs) which are related to the invariants of the rotational matrix [9]. They allow a robust representation of the problem and an effective manipulation of constraints equations, therefore they have been used for pose and trajectory optimization by the same A.'s in [10,11].

The synthesis of PKMs is usually driven by the goal of maximizing stiffness and accuracy: Section 4 shows how the actuated joints can be chosen so as to obtain the coincidence of the singularity loci, therefore simplifying the avoidance of singular poses. Wrist configurations that cause singularities are identified and interpreted by means of *screw theory* [12]. This theory is based on the concept of the instant screw axis, represented in Plücker coordinates by a six-dimensional array: it is composed by the unit vector defining the direction of the axis and by its moment about the origin [13]. The present study has been developed according to the approach of Conconi and Carricato [14], by which the linear dependence of three screw moments is sufficient to identify all singularities, i.e., unactuated and actuated kinematic chain singularities and passive constraint singularities.

2 The 3-RFR Class of Spherical Parallel Manipulators

Three-legged isostatic SPMs include the relevant class of 3-RFR parallel mechanisms, see Fig. 1(b): their limbs are attached to the fixed and mobile platforms by means of two revolute (R) pairs, whose axes intersect at the center of the spherical motion; in between three lower pairs are equivalent to a planar joint (F) whose plane contains the axes of the revolute joints and the center of the motion. Corinaldi et al. [15] showed that this architecture can yield a large orientational workspace characterized by good dexterity around the home configuration, i.e., when the reference frame $\mathscr{F}_1$ attached to the MP has the same orientation as the reference frame $\mathscr{F}_0$ fixed to the ground. The planar joint in the middle of each limb can be indifferently substituted by any of the PRR, RPR, PPR, PRP and RRR topologies, provided that some rules are observed: the directions of the prismatic joints must be parallel to the plane of motion, while the axes of the revolute pairs must be perpendicular to such plane, as shown in Fig. 1(a). Finally, such pairs may be merged in a cylindrical (C) joint when the axes of revolute and prismatic pairs are coaxial; in the same way two orthogonal revolute pairs are equivalent to a universal (U) joint. The following set of leg topologies arise: (RP)R(RR) = CRU, (RR)P(RR) = UPU, (RP)P(RR) = CPU, (RP)R(PR) = CRC and (RR)R(RR) = URU.

3 Global Mobility Verification

The pose of the MP is expressed by means of the Study parameters [16]. This notation was introduced by Study who used a superabundant set of eight parameters in the seven-dimensional projective space to map the Euclidean space through a one to one representation $\chi : \mathbf{T} \in SE(3) \mapsto \mathbf{z} \in \mathbb{P}^7$. The eight parameters y_i and r_i for $i = 0,1,2,3$ are related through a quadric polynomial equation, namely the Study's quadric, and through a metric equation which ensures that the transformations actually represent rigid body motions:

$$\begin{aligned} &\sigma_1 : r_0 y_0 + \mathbf{r}^T \mathbf{y}, \quad \sigma_1 = 0 \\ &\sigma_2 : r_0^2 + \mathbf{r}^T \mathbf{r} - 1, \ \ \sigma_2 = 0 \end{aligned} \tag{1}$$

having collected the parameters in the vectors $\mathbf{r} = [r_1\ r_2\ r_3]^T$ and $\mathbf{y} = [y_1\ y_2\ y_3]^T$ for terseness. By using the homogeneous notation, the transformation $\mathbf{T}$ between the mobile frame $\mathscr{F}_1$ and the fixed frame $\mathscr{F}_0$ of Fig. 2(a) can be written as usual in terms of the rotation matrix $\mathbf{Q}$, and the position vector $\mathbf{p}$ between their origins:

$$\begin{aligned} &\mathbf{Q} = (r_0^2 - \mathbf{r}^T\mathbf{r})\mathbf{1} + 2\mathbf{r}\mathbf{r}^T + 2r_0\mathbf{R}, \quad &&\mathbf{R} = \mathrm{CPM}(\mathbf{r}) \\ &\mathbf{p} = -2(r_0\mathbf{y} + y_0\mathbf{r} + \mathbf{Y}\mathbf{r}), &&\mathbf{Y} = \mathrm{CPM}(\mathbf{y}) \end{aligned} \tag{2}$$

where CPM($\mathbf{r}$) stands for the cross-product matrix operator on the vector $\mathbf{r}$[1]. Besides the Study's quadric equations (1), additional algebraic relations are written to describe the constraints imposed by joints' arrangement on the MP; in this way the mobility of the platform within its workspace is fully characterized, as done for the 3-URU PKM by Carbonari et al. in [17]. These relations, which are specific of each legs' architecture, are introduced in the following lines by making reference to the sketch in Fig. 2(a). The i^{th} limb, for $i = 1,2,3$, is built so as to constrain the axes of the first and the last revolute joint on the plane π_i. Equivalently, the line ℓ_i, identified by the axis of such joint, has an intersection point with all the lines that lie on the plane π_i, for example the axis of the first revolute pair, here called m_i. The two revolute joints are connected through a planar joint that allows them a relative motion parallel to the plane π_i. This joint constraints

[1] That is, $\mathrm{CPM}(\mathbf{r}) = \partial(\mathbf{r} \times \mathbf{v})/\partial\mathbf{v},\ \forall\mathbf{v},\ \mathbf{r} \in \mathbb{R}^3$

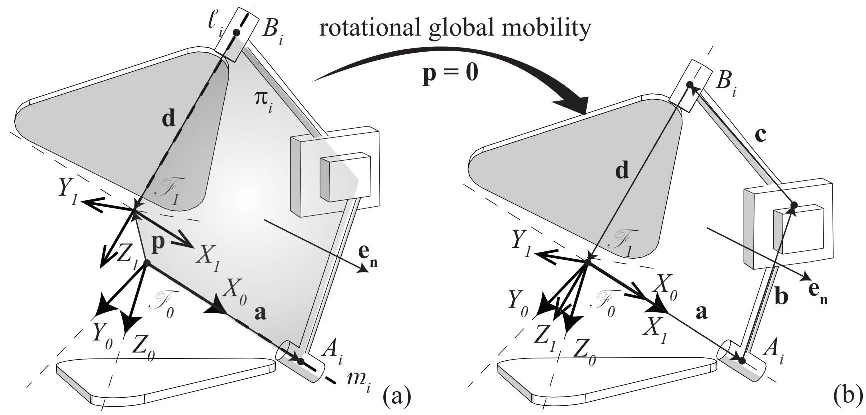

Fig. 2. (a) Kinematics of the i^{th} leg for a generic assembly of the PKM class and (b) the building architecture that grants the rotational behavior.

the two axes to lie on the same plane for all wrist configurations. The two lines can be parametrized through their direction plus a passing point, as

$$\ell_i : \alpha_i \hat{\mathbf{d}}_i + \mathbf{p}, \;\; m_i : \beta_i \hat{\mathbf{a}}_i \tag{3}$$

where $\alpha_i, \beta_i \in \mathbb{R}$ and the hat on the vectors indicates that they are unit vectors. The legs' constraints can be expressed by

$$\ell_i = m_i \;\rightarrow\; \alpha_i \hat{\mathbf{d}}_i + \mathbf{p} = \beta_i \hat{\mathbf{a}}_i \;\Rightarrow\; \left[[\mathbf{Q}]_0[\hat{\mathbf{d}}_i]_1 \;\; -[\hat{\mathbf{a}}_i]_0 \;\; [\mathbf{p}]_0\right]\left[\alpha_i \; \beta_i \; 1\right]^T = \mathbf{0} \tag{4}$$

where in the second part of Eq.4 the three column vectors composing the matrix are expressed in the ground frame $\mathscr{F}_0$. A non-trivial solution of this homogeneous equation can be found by letting the determinant of the respective matrix vanish. Substituting $\mathbf{Q}$ and $\mathbf{p}$ relationships Eq.(2) into Eq.4, three polynomials in the Study parameters are obtained, one for each leg. After simplification [18], the three constraints can be expressed by the varieties of the following polynomials

$$\begin{aligned} g_1 &: r_0y_2 + r_1y_3 + r_2y_0 + r_3y_1, \;\; g_1 = 0 \\ g_2 &: r_0y_3 + r_1y_2 + r_2y_1 + r_3y_0, \;\; g_2 = 0 \\ g_3 &: r_0y_1 + r_1y_0 + r_2y_3 + r_3y_2, \;\; g_3 = 0 \end{aligned} \tag{5}$$

The vanishing set of polynomials g_1, g_2, and g_3, together with Study quadric equations (1) are not yet sufficient to fully describe the kinematics of the 3-RFR class of SPMs, because a set of actuation dependent equations must still be written. Nevertheless, such relations provide the searched information about the mobility of the MP [19]. In fact, when the four Study parameters y_0, y_1, y_2, and y_3 are null, the general transformation matrix $\mathbf{T}$ describes a pure rotation since the translation vector $\mathbf{p}$ vanishes. Under these conditions, the $\mathbf{Q}$ matrix fully describes the motion of the MP by means of the four scalar quantities in r_0 and $\mathbf{r}$, that correspond to the Euler-Rodrigues parametrization of rotations [13]. It can be easily verified that such condition satisfies both constraints equations (5) and the Study quadric relation $\sigma_1 = 0$, thus demonstrating that a pure rotational motion has been obtained by the 3-RFR SPM.

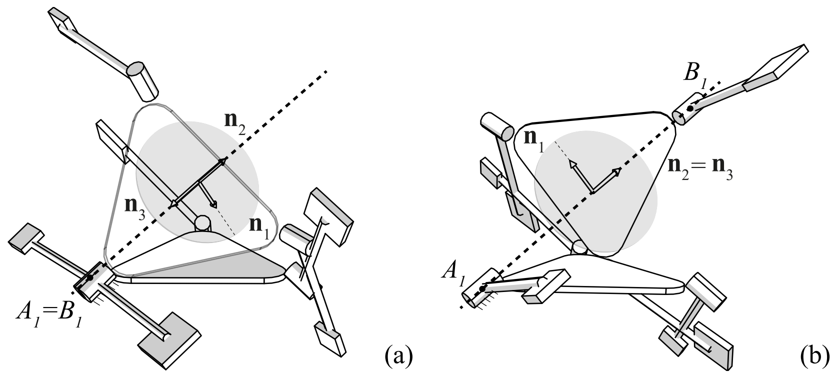

Fig. 3. Singular robot postures when a leg singularity occurs on the first limb.

4 Singularity Analysis

Differently from serial robots, where all joints are actuated, parallel robots are composed by many passive joints, which makes their analysis more complex and increases the number of singular configurations. Therefore the identification of these configurations and their parametric representation in terms of ERPs is useful for the singularity avoidance during path-planning and for a possible maximization of their dexterity during kinematic synthesis. The following section develops such kind of analysis through geometrical inspection with the aim of selecting the most suitable joints to be actuated in order to obtain a simple-to-use singularity relationship.

Unactuated Kinematic Chain Singularities Since the MP is subject to a pure rotation, the point B_i of Fig. 2(b) moves on the surface of a sphere; moreover, legs topology constraints the points A_i and B_i to have a relative planar motion. Keeping in mind these two facts, it is possible to identify the singular configurations of the legs with a geometric inspection of the mechanism. At this point no joint is actuated, thus kineto-static properties only depend on the kind, number, and mutual disposition of the joints composing the kinematic chain. When a subset of screws of a leg becomes linearly dependent a *leg singularity* occurs. In particular, for the case under study, two singular postures for each leg are identified that appear when the axes of two the revolute joints of the same leg become coaxial: Fig. 3(a) shows the posture in which the revolute pairs are coincident while in Fig. 3(b) they lie on the opposite sides of the sphere surface. Moreover, through an appropriate arrangement of the R and P joints which compose the planar pair, such singular configurations lead also to the linear dependence of the planar sub-chain pairs. In these configurations, the leg generates a new mobility constraint, i.e., it is unable to rotate around an axis perpendicular to the plane generated by the two revolute axes and the rotation axis of the planar pair. The relationships in terms of the ERPs can be obtained for $i = 1,2,3$ from

$$\hat{\mathbf{d}}_i^T \hat{\mathbf{a}}_i = 1 \quad \Rightarrow \quad \left([\mathbf{Q}]_0 \, [\mathbf{d}_i]_1\right)^T [\mathbf{a}_i]_0 = 1 \tag{6}$$

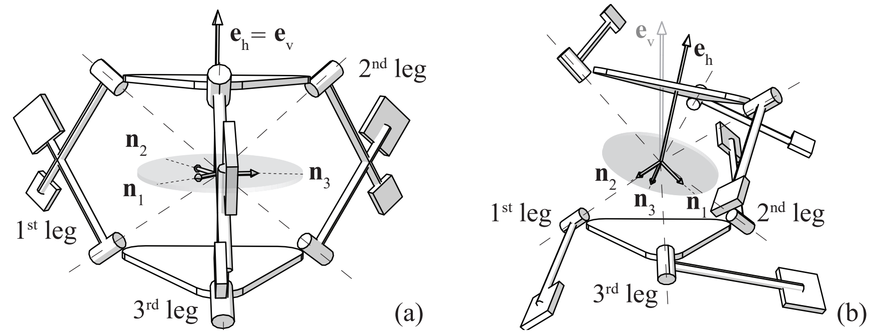

Fig. 4. Passive-constraint singularity postures for two different pointing directions.

Passive-Constraint Singularities A *passive-constraint singularity* (PCS) occurs when the dimension of the MP-motion space increases with respect to its global mobility value. Each leg develops on the MP a constraining force orthogonal to the plane of the leg and passing through the origin so that the MP motion is a three-dimensional rotation around the origin. When these forces become linearly dependent, e.g. they are parallel to the same plane, the dimension of the motion space increases from three to four and the mechanism is at a PCS. Unlike the previous case, there is at least a PCS posture for each pointing direction of MP, i.e. given a pointing vector, it is always possible to find a singular posture by rotating the MP around this axis. Two examples are shown in Fig. 4: case (a) shows a PCS posture for the vertical direction while case (b) sketches a different pointing direction. In both cases, the constraints imposed by each leg on the MP, i.e. the three forces perpendicular to the leg planes (shown in the figures as applied vectors in the center of spherical motion), become linearly dependent: in fact, the three vectors lie on the same plane and the twist system of MP gains a translational degree of mobility perpendicular to the force plane.

Actuated Kinematic Chain Singularities The parallel kinematic chain turns into a machine once the actuated joints are specified. The total number of actuators is obviously equal to the dimension of the global mobility space, i.e. three; moreover the motors should be placed at the base platform, i.e. the ground, for practical considerations. A first layout of the motors is obtained through the actuation of the revolute joints at the base so that their action on the MP consists of a force perpendicular to the leg plane.

A better alternative solution is obtained by actuating the prismatic joints closest to the ground that compose the planar joint of each leg, once they have been brought back to the base platform. This choice brings many advantages; in fact, this time the wrench acting on the platform is given by a force that lies in the leg plane, whose direction and application point depend of the planar joint sub-chain. Anyhow, this force causes a moment on the MP that is perpendicular to the leg plane, i.e, it is in the same direction of the constraint force imposed by each leg on the MP. In this way, the condition of linear dependence of the wrench constraints becomes coincident with the linear dependence

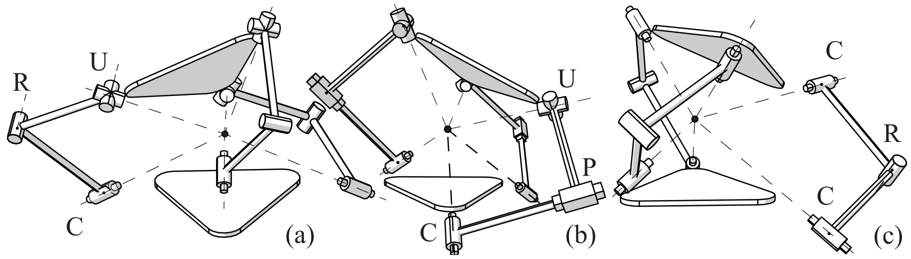

Fig. 5. Kinematic sketches of the (a) 3-CRU, (b) 3-CPU, (c) 3-CRC architectures.

of the actuation wrenches. Furthermore, it should be noted that, if these moment vectors are computed as cross product between the axis of the revolute joint $\mathbf{a}_i$ and the vector $\mathbf{d}_i$, then their magnitudes vanish when the two revolute joints into the leg become coaxial, i.e. when a leg singularity occurs. Due to these reasons, the study of these three moments is sufficient to fully describe the singularities assumed by the SPMs with linear actuation fixed to the base. Within the class of 3-RFR spherical manipulators of Fig. 1, only three of them can have the prismatic joint fixed to the base, i.e., 3-CRU, 3-CPU and 3-CRC, whose architecture is illustrated in Fig. 5. As a matter of fact, the R and P joints of the first cylindrical joint can be switched without affecting the MP mobility because of the coaxiality of the direction of the P joint with the axis of the R joint. If the moments of the actuation forces are denoted by $\mathbf{n}_i$, for $i = 1,2,3$, then the singular configurations of the 3-RFR wrists with prismatic actuated joints are given by:

$$\mathbf{n}_1 \times \mathbf{n}_2 \cdot \mathbf{n}_3 = 0, \quad \mathbf{n}_i = -\mathbf{d}_i \times \hat{\mathbf{a}}_i \Rightarrow [\mathbf{n}_1]_0 = d[\mathbf{a}_i]_0 \times [\mathbf{Q}]_0[\mathbf{d}_i]_1 \tag{7}$$

where d is the constant magnitude of the vector $\mathbf{d}$. Writing the rotation matrix in terms of the ERPs using Eq. 2, it is obtained

$$\begin{aligned}\mathbf{n}_1 \times \mathbf{n}_2 \cdot \mathbf{n}_3 = d^3&(r_1 + r_2 + r_3 - r_0)(-r_1 - r_2 + r_3 - r_0)\\ &\cdot(r_1 - r_2 - r_3 - r_0)(-r_1 + r_2 - r_3 - r_0)\end{aligned} \tag{8}$$

that vanishes when one of its factors become zero, indicating the robot poses for which a singular configuration is attained by the robot.

5 Conclusions

The paper has characterized the rotational mobility of the 3-RFR class of SPMs and has shown that it is beneficial to actuate the prismatic joint of each leg, once it has been brought back to the frame. In this way, wrench constraints become linearly dependent only when the actuation wrenches do and thus the study of wrist singularities can be performed by simply studying their moments. Furthermore, such moments represent the row vectors of the direct Jacobian matrix of the 3-CPU that map the angular velocity vector of the MP into the actuated joint rates. This means that for this particular robot a single 3×3 Jacobian matrix is able to describe all the possible singularities.

This feature can be exploited for posture optimization, e.g. when the 3-CPU robot is functionally redundant with respect to the task to be performed [15]. During path planning, when singularities must be avoided, the simple structure of Eq. (8) may represent a beneficial aspect for the considered class of manipulators. The interested reader can find a mapping of singularities in the Cartesian space of pointing directions in reference [20].

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