Generalization of Okamoto's equation to arbitrary 2x2 Schlesinger systems

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Abstract

The 2×2 Schlesinger system for the case of four regular singularities is equivalent to the Painlevé VI equation. The Painlevé VI equation can in turn be rewritten in the symmetric form of Okamoto's equation; the dependent variable in Okamoto's form of the PVI equation is the (slightly transformed) logarithmic derivative of the Jimbo-Miwa tau-function of the Schlesinger system. The goal of this note is twofold. First, we find a symmetric uniform formulation of an arbitrary Schlesinger system with regular singularities in terms of appropriately defined Virasoro generators. Second, we find analogues of Okamoto's equation for the case of the 2×2 Schlesinger system with an arbitrary number of poles. A new set of scalar equations for the logarithmic derivatives of the Jimbo-Miwa tau-function is derived in terms of generators of the Virasoro algebra; these generators are expressed in terms of derivatives with respect to singularities of the Schlesinger system.

1   Introduction

The Schlesinger system is the following non-autonomous system of differential equations for N unknown matrices $A_j \in \mathfrak{sl}(M)$ depending on N variables $\{\lambda_j\}$:

$$\frac{\partial A_j}{\partial \lambda_i} = \frac{[A_j, A_i]}{\lambda_j - \lambda_i} \,, \quad i \neq j \,, \qquad \frac{\partial A_j}{\partial \lambda_j} = -\sum_{i \neq j} \frac{[A_j, A_i]}{\lambda_j - \lambda_i} \,. \tag{1.1}$$

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The system (1.1) determines isomonodromic deformations of a solution of matrix ODE with meromorphic coefficients

$$\frac{\partial \Psi}{\partial \lambda} = A(\lambda)\, \Psi \equiv \sum_{j=1}^{N} \frac{A_j}{\lambda - \lambda_j}\, \Psi \ . \tag{1.2}$$

The solution of this system normalized at a fixed point λ_0 by $\Psi(\lambda_0) = I$ solves a matrix Riemann-Hilbert problem with some monodromy matrices around the singularities λ_j.

The Schlesinger equations were discovered almost 100 years ago [1]; however, they continue to play a key role in many areas of mathematical physics: the theory of random matrices, integrable systems, theory of Frobenius manifolds, etc. . The system (1.1) is a non-autonomous hamiltonian system with respect to the Poisson bracket

$$\{A_j^a,\ A_k^b\} = \delta_{jk}\, f^{ab}{}_c\, A_j^c \ , \tag{1.3}$$

where $f^{ab}{}_c$ are structure constants of $\mathfrak{sl}(M)$; δ_{jk} is the Kronecker symbol. Obviously, the traces $\mathrm{tr} A_j^n$ are integrals of the Schlesinger system for any value of n. The commuting Hamiltonians defining evolution with respect to the times λ_j are given by

$$H_j = \frac{1}{4\pi \mathrm{i}} \oint_{\lambda_i} \mathrm{tr} A^2(\lambda)\, d\lambda \equiv \frac{1}{2} \sum_{k \neq j} \frac{\mathrm{tr} A_j A_k}{\lambda_k - \lambda_j} \ . \tag{1.4}$$

The generating function $\tau_{\mathrm{JM}}(\{\lambda_j\})$ of the hamiltonians H_j, defined by

$$\frac{\partial}{\partial \lambda_j} \log \tau_{\mathrm{JM}} = H_j \ , \tag{1.5}$$

was introduced by Jimbo, Miwa and their co-authors [2, 3]; it is called the τ-function of the Schlesinger system. The τ-function plays a key role in the theory of the Schlesinger equations; in particular, the divisor of zeros of the τ-function coincides with the divisor of singularities of the solution of the Schlesinger system; on the same divisor the underlying Riemann-Hilbert problem looses its solvability.

In the simplest non-trivial case when the matrix dimension equals $M = 2$ and the number of singularities equals $N = 4$, the Schlesinger system can equivalently be rewritten as a single scalar differential equation of order two — the Painlevé VI equation

$$\begin{aligned}
\frac{d^2 y}{dt^2} &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} \\
&\quad + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right) \ ,
\end{aligned} \tag{1.6}$$

where t is the cross-ratio of the four singularities $\lambda_1, \ldots, \lambda_4$, and y is the position of a zero of the upper right corner element of the matrix $\sum_{k=1}^{4} \frac{A_k}{\lambda - \lambda_k}$. Let us denote the eigenvalues of the matrices A_j by $\alpha_j/2$ and $-\alpha_j/2$. Then the constants α, β, γ and δ from the Painlevé VI equation (1.6) are related to the constants α_j as follows:

$$\alpha = \frac{(\alpha_1 - 1)^2}{2} \ , \quad \beta = -\frac{\alpha_2^2}{2} \ , \quad \gamma = \frac{\alpha_3^2}{2} \ , \quad \delta = \frac{1}{2} - \frac{\alpha_4^2}{2} \ . \tag{1.7}$$

It was further observed by Okamoto [4, 5], that the Painlevé VI equation (and, therefore, the original 2×2 Schlesinger system with four singularities) can be rewritten alternatively in a simple form in terms of the so-called auxiliary hamiltonian function $h(t)$. To define this function we need to introduce first four constants b_j, which are expressed in terms of the eigenvalues of the matrices A_j as follows:

$$b_1 = \frac{1}{2}(\alpha_2 + \alpha_3) , \qquad b_2 = \frac{1}{2}(\alpha_2 - \alpha_3) ,$$

$$b_3 = \frac{1}{2}(\alpha_4 + \alpha_1) , \qquad b_4 = \frac{1}{2}(\alpha_4 - \alpha_1) . \tag{1.8}$$

The auxiliary hamiltonian function $h(t)$ is defined in terms of solution y of equation (1.6) and the constants b_j as follows:

$$
\begin{aligned}
h \;=\;& y(y-1)(y-t)\left(\frac{dy}{dt}\right)^2 \\
& - \left\{(b_1 + b_2)(y-1)(y-t) + (b_1 - b_2)y(y-t) + (b_3 + b_4)y(y-1)\right\}\frac{dy}{dt} \\
& + \left\{\frac{1}{4}(2b_1 + b_3 + b_4)^2 - \frac{1}{4}(b_3 - b_4)^2\right\}(y-t) + \sigma_2'[b]t - \frac{1}{2}\sigma_2[b] ,
\end{aligned}
\tag{1.9}
$$

where

$$\sigma_2'[b] := b_1 b_3 + b_1 b_4 + b_3 b_4 , \qquad \sigma_2[b] := \sum_{\substack{j,k=1 \\ j<k}}^{4} b_j b_k . \tag{1.10}$$

In terms of the function h, the Painlevé equation (1.6) can be represented in a remarkably symmetric form as follows:

$$\frac{dh}{dt}\left[t(1-t)\frac{d^2h}{dt^2}\right]^2 + \left[\frac{dh}{dt}\left\{2h - (2t-1)\frac{dh}{dt}\right\} + b_1 b_2 b_3 b_4\right]^2 - \prod_{k=1}^{4}\left(\frac{dh}{dt} + b_k^2\right) \;=\; 0 . \tag{1.11}$$

Okamoto's form (1.11) of the Painlevé VI equation turned out to be extremely fruitful for establishing the hidden symmetries of the equation (the so-called Okamoto symmetries). These symmetries look very simple in terms of the auxiliary hamiltonian function h, but are highly non-trivial on the level of the solution y of the Painlevé VI equation, the corresponding monodromy group and the solution of the associated fuchsian system [6, 7].

The goal of this paper is twofold. First, we show how to rewrite the Schlesinger system in an arbitrary matrix dimension in a symmetric universal form. Second, we use this symmetric form to find natural analogues of the Okamoto equation (1.11) for 2×2 Schlesinger systems with an arbitrary number of simple poles. Our approach is similar to the approach used by J. Harnad to derive analogues of the Okamoto equation for Schlesinger systems corresponding to higher order poles (non-fuchsian systems) [8].

Namely, introducing the following differential operators (which satisfy the commutation relations of the Virasoro algebra):

$$L_m := \sum_{j=1}^{N} \lambda_j^{m+1}\frac{\partial}{\partial \lambda_j} , \qquad m = -1, 0, 1, \dots ,$$

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and the following dependent variables:

$$\mathcal{B}_n := \sum_j \lambda_j^n A_j \equiv \text{res}|_{\lambda=\infty}\{\lambda^n A(\lambda)\}, \qquad n = 0, 1, \ldots,$$

one can show that the Schlesinger system (1.1) implies

$$L_m \mathcal{B}_n = \sum_{k=1}^{n-1} [\mathcal{B}_k, \mathcal{B}_{m+n-k}] + n\mathcal{B}_{m+n}, \qquad (1.12)$$

for all $n \geq 0$ and $m \geq -1$. The infinite set of equations (1.12) is of course dependent for any given N. To derive the original Schlesinger system (1.1) from (1.12) it is sufficient to take the set of equations (1.12) for $n \leq N$ and $m \leq N$. The advantage of the system (1.12) is in its universality: its form is independent of the number of the poles; the positions of the poles enter only the definition of the differential operators L_m.

Consider now the case of 2×2 matrices. To formulate the analog of the Okamoto equation for the case of an arbitrary number of poles we introduce the following "hamiltonians":

$$\widehat{\mathcal{H}}_m := -\tfrac{1}{4} \sum_{k=0}^{m} \text{tr}\, \mathcal{B}_k \, \mathcal{B}_{m-k},$$

which can be viewed as symmetrised analogues of the Hamiltonians (1.4); they coincide with $L_m \log \tau_{JM}$ up to an elementary transformation. The simplest equation satisfied by $\widehat{\mathcal{H}}_m$ in the case of 2×2 system is given by

$$\begin{aligned}
\tfrac{1}{8} \left(L_2 L_2 \widehat{\mathcal{H}}_2 + 2L_3 \widehat{\mathcal{H}}_3 - 5L_4 \widehat{\mathcal{H}}_2 - 2\widehat{\mathcal{H}}_6 \right)^2 &= (L_3 \widehat{\mathcal{H}}_3 - L_4 \widehat{\mathcal{H}}_2 - \widehat{\mathcal{H}}_6) \left((\widehat{\mathcal{H}}_3)^2 + 4\widehat{\mathcal{H}}_2 (L_2 \widehat{\mathcal{H}}_2 - \widehat{\mathcal{H}}_4) \right) \\
&\quad - (L_3 \widehat{\mathcal{H}}_2 - \widehat{\mathcal{H}}_5) \left(\widehat{\mathcal{H}}_2 L_3 \widehat{\mathcal{H}}_2 + \widehat{\mathcal{H}}_3 L_2 \widehat{\mathcal{H}}_2 - \widehat{\mathcal{H}}_2 \widehat{\mathcal{H}}_5 \right) \\
&\quad + (L_2 \widehat{\mathcal{H}}_2 - \widehat{\mathcal{H}}_4)(L_2 \widehat{\mathcal{H}}_2)^2, \qquad (1.13)
\end{aligned}$$

as we shall show in the following.

Since the $\widehat{\mathcal{H}}_m$ themselves are combinations of the first order derivatives of the tau-function, this equation is of the third order; it also has cubic non-linearity. In the case $N = 4$ the equation (1.13) boils down to the standard Okamoto equation (1.11).

The paper is organized as follows. In Section 2 we derive the symmetrised form of the Schlesinger system. In Section 3 we derive the generalized Okamoto equations. In Section 4 we show how the usual Okamoto equation is obtained from the generalized equation (1.13) in the case $N = 4$. In section 5 we discuss some open problems.

2 Symmetrisation of Schlesinger system in terms of Virasoro generators

2.1 Variation of a Riemann surface by vector fields on a contour

The "times" λ_j of the Schlesinger system can be viewed as coordinates on the space of genus zero Riemann surfaces with N punctures (we do not speak about the moduli space since we

do not identify two configurations of punctures related by a Möbius transformation). The vectors $\partial/\partial\lambda_j$ span the tangent space to the space of N-punctured spheres. However, there exist many different ways to parametrize the tangent space to the space of Riemann surfaces of given genus with a fixed number of punctures.

One example is the variation of a Riemann surface by vector fields on a chosen closed contour l enclosing a disc D (see [9]). To vary an unpunctured Riemann surface $\mathcal{L}$ of genus g by a vector field $\mathbf{v}$ on the contour l one cuts the disc D and attaches it back with infinitesimal shift of the boundary along the vector field $\mathbf{v}$. The complex structure of the Riemann surface also changes infinitesimally (or remains the same). The infinite-dimensional space V of vector fields on l can be represented as direct sum of three subspaces, V_{in} (which consists of vector fields which can be analytically continued inside of the disc D), V_{out} (which consists of vector fields which can be analytically continued outside of the disc D) and V_0 (which consists of vector fields whose analytical continuation is impossible neither inside nor outside of the disc D). The moduli of Riemann surface $\mathcal{L}$ are not changed by vector fields from V_{in} and V_{out}; the dimension of V_0 turns out to be equal to $3g-3$ i.e. to dimension of the moduli space, and the vector fields from V_0 do change the complex structure of the Riemann surface.

If the Riemann surface $\mathcal{L}$ has punctures (which are assumed to lie outside of D), the vector fields from V_{out} should not only admit analytical continuation to the exterior of D, but also vanish at all the punctures.

To apply this general scheme in our present framework we choose the contour l to be a circle around $\lambda = \infty$ (such that all points λ_j lie "outside" of this circle). The standard basis in the space of vector fields on l is given by $v_m := \lambda^{m+1} d/d\lambda$, $m \in \mathbb{Z}$; the vectors v_m satisfy standard commutation relations $[v_m, v_n] = (n-m)v_{m+n}$. How to relate variation of positions of the punctures λ_j to variation along vector fields v_m? Notice that the fields v_m with $m \leq -3$ can be holomorphically continued in a neighbourhood of $\lambda = \infty$, and, therefore, do not vary the singularities λ_m; they span the vector space V_{in} in our case. The vector fields v_m with $m = -2, 0, 1, \ldots$ can not be analytically continued in the neighbourhood of ∞. Being analytically continued in the neighbourhood of 0, these fields do not vanish at λ_m; however, there exists an infinite-dimensional space (this is V_{out} in our case) of linear combinations of these v_m's which vanish at all points λ_m. The space V_0 can be chosen in different ways. For example, V_0 can be chosen to be spanned by all linear combinations of $v_{-1}, \ldots, v_{N-2}$.

However, our goal will be to describe all Schlesinger equations (independently of the number of singularities) within one setting. Therefore, we shall vary λ_j by all v_m with $m = -1, 0, 1, 2, \ldots$ in spite of the fact that for any given N all these vectors can be expressed as linear combinations of N vectors $\partial/\partial\lambda_j$.

Namely, the action of the vector field v_m on $\{\lambda_j\}$ is given by the following linear combination of the tangent vectors $\partial/\partial\lambda_m$:

$$L_m := \sum_{j=1}^{N} \lambda_j^{m+1} \frac{\partial}{\partial\lambda_j} \ .$$

L_m and v_m coincide as tangent vector to the space of N-punctured spheres. The vectors L_m also satisfy commutation relations of Virasoro algebra:

$$[L_m, \ L_n] = (n-m)L_{m+n} \ . \tag{2.1}$$

2.2 Schlesinger system in terms of Virasoro generators

To symmetrise the Schlesinger equations we also introduce the symmetric dependent variables:

$$\mathcal{B}_m := \sum_j \lambda_j^m A_j \equiv \mathrm{res}|_{\lambda=\infty}\{\lambda^m A(\lambda)\} \ . \tag{2.2}$$

The new variable

$$\mathcal{B}_0 = \sum_j A_j \ , \tag{2.3}$$

plays a distinguished role: it vanishes on-shell (i.e. on solutions of the Schlesinger system); however, off-shell it plays the role of a generator (with respect to the Poisson bracket (1.3)) of constant gauge transformations (i.e. constant simultaneous similarity transformations of all matrices A_j).

   To describe the dynamics under the action of the differential operators L_m we introduce the symmetrised Hamiltonians $\mathcal{H}_m$:

$$\mathcal{H}_m := -\frac{1}{2}\mathrm{res}|_{\lambda=\infty}\mathrm{tr}A^2(\lambda) \equiv \sum_j \lambda_j^{m+1} H_j \ . \tag{2.4}$$

These Hamiltonians can be expressed in terms of the variables $\mathcal{B}_k$ as follows:

$$\mathcal{H}_m = \widehat{\mathcal{H}}_m + \tfrac{1}{4}(m{+}1)\sum_j \lambda_j^m C_j \ , \tag{2.5}$$

where the modified Hamiltonians $\widehat{\mathcal{H}}$ are given by

$$\widehat{\mathcal{H}}_m := -\tfrac{1}{4}\sum_{k=0}^{m} \mathrm{tr}\,\mathcal{B}_k\,\mathcal{B}_{m-k} \ , \tag{2.6}$$

and $C_j := \mathrm{tr}A_j^2 = \tfrac{1}{2}\alpha_j^2$. In particular, $\widehat{\mathcal{H}}_{-1} = \widehat{\mathcal{H}}_0 = \widehat{\mathcal{H}}_1 = 0$ (taking into account that $\mathcal{B}_0 = 0$), such that the first three symmetrised Hamiltonians take the form

$$\mathcal{H}_{-1} = 0 \ , \qquad \mathcal{H}_0 = \tfrac{1}{4}\sum_j C_j \ , \qquad \mathcal{H}_1 = \tfrac{1}{2}\sum_j \lambda_j C_j \ . \tag{2.7}$$

   In terms of the Virasoro generators L_m the equations (1.5) for the Jimbo-Miwa τ-function τ_{JM} look as follows:

$$L_m\,(\log \tau_{\mathrm{JM}}) = \mathcal{H}_m \ . \tag{2.8}$$

It is convenient to introduce also a modified τ-function, invariant under Möbius transformations:

Lemma 1 *The modified τ-function $\tilde{\tau}$ defined by*

$$\tilde{\tau} \quad \equiv \quad \tau_{\mathrm{JM}} \prod_{i<j}(\lambda_j - \lambda_i)^{-\frac{1}{2(N-2)}(C_i+C_j)+\frac{2}{(N-1)(N-2)}\mathcal{H}_0} \ , \tag{2.9}$$

is annihilated by the first three Virasoro generators:

$$L_{-1}\,\tilde{\tau} = L_0\,\tilde{\tau} = L_1\,\tilde{\tau} = 0 \ . \tag{2.10}$$

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Proof: by straightforward computation.

$\square$

In terms of the new variables (2.2), the Schlesinger system (1.1) takes a very compact form:

Theorem 1 *The differential operators L_m act on the symmetrised variables $\mathcal{B}_n$ as follows:*

$$L_m\mathcal{B}_n = \sum_{k=1}^{n-1} [\mathcal{B}_k, \mathcal{B}_{m+n-k}] + n\mathcal{B}_{m+n} \ , \tag{2.11}$$

for $m = -1, 0, 1, 2, \ldots$, $n = 1, 2, \ldots$.

Proof. Using the Schlesinger equations (1.1), we have

$$L_m\mathcal{B}_n \equiv \sum_{i=1}^{N} \lambda_i^{m+1} \frac{\partial}{\partial\lambda_i} \left\{ \sum_{j=1}^{N} A_l \right\} = \sum_{i\neq j} \lambda_i^{m+1} \frac{\lambda_i^n - \lambda_j^n}{\lambda_i - \lambda_j} [A_j, \ A_i] + n \sum_{i=1}^{N} \lambda_i^{m+n} A_i \ .$$

Expanding

$$\frac{\lambda_i^n - \lambda_j^n}{\lambda_i - \lambda_j} = \lambda_j^{n-1} + \lambda_j^{n-2}\lambda_i + \ldots + \lambda_j\lambda_i^{n-2} + \lambda_i^{n-1} \ ,$$

we further rewrite this expression for $L_m\mathcal{B}_n$ as

$$[\mathcal{B}_{n-1}, \mathcal{B}_{m+1}] + [\mathcal{B}_{n-2}, \mathcal{B}_{m+2}] + \ldots + [\mathcal{B}_1, \mathcal{B}_{m+n+1}] + n\mathcal{B}_{m+n} \ ,$$

which coincides with the right hand side of (2.11).

$\square$

Remark 1 *The system (2.11) can be equivalently rewritten as follows:*

$$L_m\mathcal{B}_n = \sum_{k=1}^{m} [\mathcal{B}_k, \mathcal{B}_{m+n-k}] + n\mathcal{B}_{m+n} \ . \tag{2.12}$$

i.e. the right-hand side of (2.11) does not change if the upper limit $n-1$ is substituted by m.

The system of equations (2.11), or (2.12) is the symmetric form of the Schlesinger system. Using (2.12) we can express the commutators $[\mathcal{B}_m, \mathcal{B}_n]$ as follows:

$$[\mathcal{B}_m, \mathcal{B}_n] = L_m\mathcal{B}_n - L_{m-1}\mathcal{B}_{n+1} + \mathcal{B}_{m+n} \ . \tag{2.13}$$

Acting on the modified hamiltonians $\widehat{\mathcal{H}}_n$ by the operators L_m, we get the following equation:

$$L_m\widehat{\mathcal{H}}_n = -\tfrac{1}{2} \sum_{k=1}^{n-1} k \operatorname{tr}\mathcal{B}_{m+k}\mathcal{B}_{n-k} \ . \tag{2.14}$$

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In particular, we have

$$L_m \widehat{\mathcal{H}}_n - L_n \widehat{\mathcal{H}}_m = (n - m) \widehat{\mathcal{H}}_{m+n} . \qquad (2.15)$$

The same equation holds for the Hamiltonians $\mathcal{H}_n$ as a corollary of the integrability of equations (2.8).

The Poisson bracket (1.3) induces the following Poisson bracket between variables $\mathcal{B}_n$, $n = 0, 1, 2, \ldots$:

$$\{\mathcal{B}_n^a, \, \mathcal{B}_m^b\} = f^{ab}{}_c \mathcal{B}_{m+n}^c . \qquad (2.16)$$

Then equations (2.11) can then be written in the following form:

$$L_m \mathcal{B}_n = \{\mathcal{H}_m, \, \mathcal{B}_n\} + n \mathcal{B}_{m+n} . \qquad (2.17)$$

We note that formally the second term can be absorbed into the symplectic action $\{\mathcal{H}_m, \, \mathcal{B}_n\}$ upon extending the affine Poisson structure (2.16) by the standard central extension.

3 Generalized Okamoto equations

Here we shall use the symmetric form (2.11) of the Schlesinger equations to derive an analog of Okamoto's equation (1.11) for an arbitrary 2×2 Schlesinger system. In fact, one can write down a whole family of scalar differential equations for the tau-function in terms of the Virasoro generators L_m. In the next theorem we prove two equations of this kind.

Theorem 2 *The τ-function τ_{JM} (1.5) of an arbitrary 2×2 Schlesinger system satisfies the following two differential equations:*

- *The third order equation with cubic non-linearity:*

$$
\begin{aligned}
\tfrac{1}{8} \left(L_2 L_2 \widehat{\mathcal{H}}_2 + 2 L_3 \widehat{\mathcal{H}}_3 - 5 L_4 \widehat{\mathcal{H}}_2 - 2 \widehat{\mathcal{H}}_6 \right)^2 \;=\; & (L_3 \widehat{\mathcal{H}}_3 - L_4 \widehat{\mathcal{H}}_2 - \widehat{\mathcal{H}}_6) \left((\widehat{\mathcal{H}}_3)^2 + 4 \widehat{\mathcal{H}}_2 (L_2 \widehat{\mathcal{H}}_2 - \widehat{\mathcal{H}}_4) \right) \\
& - (L_3 \widehat{\mathcal{H}}_2 - \widehat{\mathcal{H}}_5) \left(\widehat{\mathcal{H}}_2 L_3 \widehat{\mathcal{H}}_2 + \widehat{\mathcal{H}}_3 \, L_2 \widehat{\mathcal{H}}_2 - \widehat{\mathcal{H}}_2 \widehat{\mathcal{H}}_5 \right) \\
& + (L_2 \widehat{\mathcal{H}}_2 - \widehat{\mathcal{H}}_4)(L_2 \widehat{\mathcal{H}}_2)^2 . \qquad (3.1)
\end{aligned}
$$

- *The fourth order equation with quadratic non-linearity:*

$$
\begin{aligned}
L_2 L_2 L_2 \widehat{\mathcal{H}}_2 \;=\; & 8 \widehat{\mathcal{H}}_8 - 9 L_4 \widehat{\mathcal{H}}_4 + 10 L_5 \widehat{\mathcal{H}}_3 - 4 L_3 L_3 \widehat{\mathcal{H}}_2 + 10 L_4 L_2 \widehat{\mathcal{H}}_2 \qquad (3.2) \\
& - 4 \left(L_2 \widehat{\mathcal{H}}_2 (2 \widehat{\mathcal{H}}_4 - 3 L_2 \widehat{\mathcal{H}}_2) - \widehat{\mathcal{H}}_3 (\widehat{\mathcal{H}}_5 - 2 L_3 \widehat{\mathcal{H}}_2) \right) \\
& - 8 \widehat{\mathcal{H}}_2 \left(2 \widehat{\mathcal{H}}_6 - 3 L_3 \widehat{\mathcal{H}}_3 + 4 L_4 \widehat{\mathcal{H}}_2 \right) ,
\end{aligned}
$$

where according to (2.5), (2.8)

$$\widehat{\mathcal{H}}_m = L_m \log \tau_{JM} - \frac{m+1}{4} \sum_{j=1}^{N} \lambda_j^m C_j .$$

Proof. Inverting the system of equations (2.14), we can express $\operatorname{tr}\mathcal{B}_m\mathcal{B}_n$ in terms of the Hamiltonians $\widehat{\mathcal{H}}_n$ as follows:

$$\operatorname{tr}\mathcal{B}_m\mathcal{B}_n = 4L_m\widehat{\mathcal{H}}_n - 2\left(L_{m-1}\widehat{\mathcal{H}}_{n+1} + L_{m+1}\widehat{\mathcal{H}}_{n-1}\right) . \tag{3.3}$$

¿From the Schlesinger system (2.11), we furthermore get

$$L_k\operatorname{tr}(\mathcal{B}_m\mathcal{B}_n) = \sum_{j=1}^{k}\left(\operatorname{tr}(\mathcal{B}_m[\mathcal{B}_j,\mathcal{B}_{n+k-j}]) + \operatorname{tr}(\mathcal{B}_n[\mathcal{B}_j,\mathcal{B}_{m+k-j}])\right)$$
$$+ n\operatorname{tr}\mathcal{B}_m\mathcal{B}_{k+n} + m\operatorname{tr}\mathcal{B}_n\mathcal{B}_{k+m} . \tag{3.4}$$

Inverting this relation, we obtain for $k < m < n$

$$\operatorname{tr}\left(\mathcal{B}_k[\mathcal{B}_m,\mathcal{B}_n]\right) = \sum_{j=m}^{n-1}\left(L_k\operatorname{tr}\mathcal{B}_j\mathcal{B}_{m+n-j} - \tfrac{1}{2}L_{k-1}\operatorname{tr}\mathcal{B}_{j+1}\mathcal{B}_{m+n-j} - \tfrac{1}{2}L_{k+1}\operatorname{tr}\mathcal{B}_j\mathcal{B}_{m+n-1-j}\right)$$
$$+ \operatorname{tr}\mathcal{B}_n\mathcal{B}_{k+m} - \operatorname{tr}\mathcal{B}_m\mathcal{B}_{k+n} . \tag{3.5}$$

Combining this equation with (3.3) we can thus express also $\operatorname{tr}(\mathcal{B}_k[\mathcal{B}_m,\mathcal{B}_n])$ entirely in terms of the action of the operators L_m on the Hamiltonians $\widehat{\mathcal{H}}_n$, which can further be simplified upon using the commutation relations (2.1) and (2.15). This leads to the closed expression

$$\begin{aligned}\operatorname{tr}\left(\mathcal{B}_k[\mathcal{B}_m,\mathcal{B}_n]\right) = \ & 2\left(L_{n-1}L_{m+1} - L_{n+1}L_{m-1}\right)\widehat{\mathcal{H}}_k\\ & + 2\left(L_{n+1}L_m - L_nL_{m+1}\right)\widehat{\mathcal{H}}_{k-1}\\ & + 2\left(L_nL_{m-1} - L_{n-1}L_m\right)\widehat{\mathcal{H}}_{k+1}\\ & - 4L_{m+n}\widehat{\mathcal{H}}_k + 2L_{m+n+1}\widehat{\mathcal{H}}_{k-1} - 2L_{m+n-1}\widehat{\mathcal{H}}_{k+1}\\ & - 4L_{k+m}\widehat{\mathcal{H}}_n + 2L_{k+m-1}\widehat{\mathcal{H}}_{n+1} + 2L_{k+m+1}\widehat{\mathcal{H}}_{n-1}\\ & + 4L_{k+n}\widehat{\mathcal{H}}_m - 2L_{k+n-1}\widehat{\mathcal{H}}_{m+1} - 2L_{k+n+1}\widehat{\mathcal{H}}_{m-1} . \end{aligned} \tag{3.6}$$

In particular, for the lowest values of k, m, n we obtain

$$\operatorname{tr}(\mathcal{B}_1[\mathcal{B}_2,\mathcal{B}_3]) = -2L_2L_2\widehat{\mathcal{H}}_2 - 4L_3\widehat{\mathcal{H}}_3 + 10L_4\widehat{\mathcal{H}}_2 + 4\widehat{\mathcal{H}}_6 . \tag{3.7}$$

To derive from these relations the desired result, we make use of the following algebraic identity

$$\operatorname{tr}\left(M_1[M_2,M_3]\right)\operatorname{tr}\left(M_4[M_5,M_6]\right) = -2\left(\operatorname{tr}(M_1M_4)\operatorname{tr}(M_2M_5)\operatorname{tr}(M_3M_6) + \dots\right), \tag{3.8}$$

valid for an arbitrary set of six matrices $M_j \in \mathfrak{sl}(2)$, where the dots on the right-hand side denote complete antisymmetrisation of the expression with respect to the indices $1,2,3$. In terms of the structure constants of $\mathfrak{sl}(2)$, this identity reads

$$f_{abc}f^{def} = 6\,\delta^{[d}_{[a}\delta^e_b\delta^{f]}_{c]} , \tag{3.9}$$

where adjoint indices $a, b, \dots$ are raised and lowered with the Cartan-Killing form. Setting in (3.8) $M_1 = M_4 = \mathcal{B}_1$, $M_2 = M_5 = \mathcal{B}_2$ and $M_3 = M_6 = \mathcal{B}_3$, and using (3.3), (3.7), we arrive (after some calculation) at (3.1).

9

Equation (3.2) descends from another algebraic identity

$$\text{tr}\left([M_1, M_2][M_3, M_4]\right) \;=\; -2\left(\text{tr}(M_1 M_3)\,\text{tr}(M_2 M_4) - \text{tr}(M_2 M_3)\,\text{tr}(M_1 M_4)\right), \quad (3.10)$$

valid for any four $\mathfrak{sl}(2)$-valued matrices M_j. In terms of the structure constants of $\mathfrak{sl}(2)$, this identity reads

$$f_{abf}\, f^{cdf} \;=\; 2\, \delta^{[c}_{[a} \delta^{d]}_{b]}\,, \quad (3.11)$$

and is obtained by contraction from (3.9). We consider the action of L_2 on (3.7) which yields

$$3\text{tr}(\mathcal{B}_1[\mathcal{B}_2, \mathcal{B}_5]) - 2\text{tr}(\mathcal{B}_1[\mathcal{B}_3, \mathcal{B}_4]) \;=\; \text{tr}[\mathcal{B}_1, \mathcal{B}_3][\mathcal{B}_1, \mathcal{B}_3] - \text{tr}[\mathcal{B}_1, \mathcal{B}_2][\mathcal{B}_1, \mathcal{B}_4] - \text{tr}[\mathcal{B}_1, \mathcal{B}_2][\mathcal{B}_2, \mathcal{B}_3]$$
$$- 2L_2 L_2 L_2 \widehat{\mathcal{H}}_2 - 4L_2 L_3 \widehat{\mathcal{H}}_3 + 10 L_2 L_4 \widehat{\mathcal{H}}_2 + 4L_2 \widehat{\mathcal{H}}_6\,.$$

The l.h.s. of this equation can be reduced by (3.5) while the first terms on the r.h.s. are reduced by means of the algebraic relations (3.10) together with (3.3). As a result we obtain equation (3.2).

$$\square$$

4 Four simple poles: reproducing the Okamoto equation

As remarked above, the explicit form of the differential equations (3.1), (3.2) for the τ-function τ_{JM} is obtained upon expressing the modified Hamiltonians $\widehat{\mathcal{H}}_m$ in terms of τ_{JM} by virtue of (2.5), (2.8). As an illustration, we will work out these equations for the Schlesinger system with four singularities and show that they reproduce precisely Okamoto's equation (1.11). For $N = 4$, the modified τ-function $\tilde{\tau}$ from (2.9) depends only on the cross-ratio

$$t \;=\; \frac{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_3)}\,. \quad (4.1)$$

We furthermore define the auxiliary function

$$S(t) \;=\; 2t(1+t)\frac{d}{dt}\log \tilde{\tau}(t)\,. \quad (4.2)$$

Then equation (3.1) in terms of S after lengthy but straightforward calculation gives rise to the second order differential equation

$$3\left(t(1-t)\,S''\right)^2 \;=\; 3(C_2 - C_3)(C_1 - C_4)\,(S - tS') - 3(C_1 - C_3)(C_2 - C_4)\,S'$$
$$+ 12\,(S - tS')S'^2 + 12(S - tS')^2 S'$$
$$+ 2\sigma_1[C]\left((S - tS')^2 + (S - tS')S' + S'^2\right)$$
$$- \tfrac{1}{18}\sigma_1[C]^3 + \tfrac{1}{2}\sigma_1[C]\,\sigma_2[C] - 3\sigma_3[C]\,. \quad (4.3)$$

where $\sigma_i[C]$ are the elementary symmetric polynomials of the C_i's

$$\sigma_1[C] = C_1 + C_2 + C_3 + C_4\,, \qquad \sigma_2[C] = \sum_{j<k} C_j C_k\,, \qquad \sigma_3[C] = \sum_{j<k<l} C_j C_k C_l\,.$$

Finally, it is straightforward to verify that with $h(t) \equiv S(t) - \frac{1}{12}(1 - 2t)\,\sigma_1[C]$, equation (4.3) is equivalent to Okamoto's equation (1.11).

In turn, equation (3.2) leads to the following quadratic, third order differential equation in the function S:

$$
\begin{aligned}
6t(1 - t)\left((1 - 2t)\,S'' + t(1 - t)\,S^{(3)}\right) \;=\; & 3\Big((C_1 - C_3)\,(C_2 - C_4) - (C_2 - C_3)\,(C_1 - C_4)\,t\Big) \\
& -12\,S^2 + 2(1 - 2t)\,S\,\left(\sigma_1[C] - 12\,S'\right) \\
& +4\sigma_1[C]\,(1 - t + t^2)\,S' + 36t(1 - t)\,S'^2 \;.
\end{aligned}
$$
(4.4)

Indeed, this equation can also be obtained by straightforward differentiation of (4.3) with respect to t. In terms of the function h equation (4.4) takes the following form

$$
\begin{aligned}
(1 - t)t\left(6h'^2 - (1 - 2t)h'' - (1 - t)t\,h^{(3)}\right) \;=\; & 4h^2 + 8(1 - 2t)\,hh' - 2(b_1^2 + b_2^2 + b_3^2 + b_4^2)\,h' \\
& -\prod_{i<j} b_i^2 b_j^2 + 2(1 - 2t)\,b_1 b_2 b_3 b_4 \;,
\end{aligned}
$$
(4.5)

equivalently obtained by derivative of Okamoto's equation (1.11).

5   Discussion and Outlook

We have shown in this paper that the symmetric form (2.11), (2.12) of the Schlesinger system gives rise to a straightforward algorithm that allows to translate the algebraic $\mathfrak{sl}(2)$ identities (3.9), (3.11) into differential equations for the τ-function of the Schlesinger system. In the simplest case of four singularities, the resulting equations reproduce the known Okamoto equation (1.11). In the case of more singularities, the same equations (3.1), (3.2) give rise to a number of non-trivial differential equations to be satisfied by the τ-function.

Apart from this direct extension of Okamoto's equation, the link between the algebraic structure of $\mathfrak{sl}(2)$ and the Schlesinger system's τ-function gives rise to further generalizations. Note, that in the proof of Theorem 2, with equation (3.6) we have already given the analogue of (3.7) to arbitrary values of k, m, n. Combining this equation with the identity (3.8) thus gives rise to an entire hierarchy of third order equations that generalize (3.1). Likewise, the construction leading to the fourth order equation (3.2) can be generalized straightforwardly upon applying (3.10) to other Virasoro descendants of the cubic equation.

As an illustration, we give the first three equations of the hierarchy generalizing (3.2):

$$
\begin{aligned}
L_3 L_2 L_2 \widehat{\mathcal{H}}_2 \;=\; & 6\widehat{\mathcal{H}}_9 - 6L_5\widehat{\mathcal{H}}_4 + 10L_6\widehat{\mathcal{H}}_3 - 5L_7\widehat{\mathcal{H}}_2 - L_3 L_3 \widehat{\mathcal{H}}_3 + L_4 L_3 \widehat{\mathcal{H}}_2 + 6L_5 L_2 \widehat{\mathcal{H}}_2 \\
& + 8\widehat{\mathcal{H}}_2(2L_4\widehat{\mathcal{H}}_3 - 3L_5\widehat{\mathcal{H}}_2 - \widehat{\mathcal{H}}_7) + 4\widehat{\mathcal{H}}_3(L_3\widehat{\mathcal{H}}_3 - L_4\widehat{\mathcal{H}}_2) - 8\widehat{\mathcal{H}}_4 L_3 \widehat{\mathcal{H}}_2 + 12 L_2 \widehat{\mathcal{H}}_2 L_3 \widehat{\mathcal{H}}_2 \;,
\end{aligned}
$$

$$
\begin{aligned}
L_4 L_2 L_2 \widehat{\mathcal{H}}_2 \;=\; & 2\widehat{\mathcal{H}}_{10} + 3L_5\widehat{\mathcal{H}}_5 - 8L_6\widehat{\mathcal{H}}_4 + 9L_7\widehat{\mathcal{H}}_3 - 9L_8\widehat{\mathcal{H}}_2 - L_4 L_3 \widehat{\mathcal{H}}_3 \\
& + 4L_4 L_4 \widehat{\mathcal{H}}_2 - 3L_5 L_3 \widehat{\mathcal{H}}_2 + 6L_6 L_2 \widehat{\mathcal{H}}_2 + 8\widehat{\mathcal{H}}_2(L_4\widehat{\mathcal{H}}_4 + 3L_5\widehat{\mathcal{H}}_3 - 3L_6\widehat{\mathcal{H}}_2) \\
& -4\widehat{\mathcal{H}}_3(L_4\widehat{\mathcal{H}}_3 - 3L_5\widehat{\mathcal{H}}_2) + 8\widehat{\mathcal{H}}_4 L_4 \widehat{\mathcal{H}}_2 - 12 L_2 \widehat{\mathcal{H}}_2 L_4 \widehat{\mathcal{H}}_2 \;,
\end{aligned}
$$

11

$$
\begin{aligned}
L_3 L_3 L_2 \widehat{\mathcal{H}}_2 \;=\;& 8\widehat{\mathcal{H}}_{10} - 6L_5\widehat{\mathcal{H}}_5 + 4L_6\widehat{\mathcal{H}}_4 + 6L_7\widehat{\mathcal{H}}_3 - 6L_8\widehat{\mathcal{H}}_2 - L_4L_3\widehat{\mathcal{H}}_3 - 2L_4L_4\widehat{\mathcal{H}}_2 + 6L_5L_3\widehat{\mathcal{H}}_2 \\
&+ 4L_6L_2\widehat{\mathcal{H}}_2 - 4(\widehat{\mathcal{H}}_5 - 2L_3\widehat{\mathcal{H}}_2)L_3\widehat{\mathcal{H}}_2 + 4L_2\widehat{\mathcal{H}}_2 L_3\widehat{\mathcal{H}}_3 - 8\widehat{\mathcal{H}}_4 L_4\widehat{\mathcal{H}}_2 \\
&+ 4\widehat{\mathcal{H}}_3(L_4\widehat{\mathcal{H}}_3 - 3L_5\widehat{\mathcal{H}}_2) - 8\widehat{\mathcal{H}}_2\left(2\widehat{\mathcal{H}}_8 - 2L_4\widehat{\mathcal{H}}_4 + L_5\widehat{\mathcal{H}}_3 + 2L_6\widehat{\mathcal{H}}_2\right) .
\end{aligned}
\tag{5.1}
$$

Obviously, these equations are not all independent, but related by the action of the lowest Virasoro generators $L_{\pm 1}$, using that

$$
L_1(L_2 L_2 L_2 \widehat{\mathcal{H}}_2) \;=\; 4\,(L_3 L_2 L_2 \widehat{\mathcal{H}}_2) + \ldots \; ,
\tag{5.2}
$$

etc., since $\widehat{\mathcal{H}}_1 = 0$. The number and structure of the independent equations in this hierarchy is thus organized by the structure of representations of the Virasoro algebra. For the case of $N = 4$ singularities, the explicit form of all the equations of the hierarchy reduces to equivalent forms of (1.11) and (4.5). With growing number of simple poles, the number of independent differential equations induced by the hierarchy increases.

Therefore, we arrive to a natural question: which set of derived equations for the tau-function is equivalent to the original Schlesinger system? We stress that all differential equations for the tau-function are PDE with respect to the variables $\lambda_1, \ldots, \lambda_N$. However, if one gets a sufficiently high number of independent equations, one can actually come to a set of ODE's for the tau-function. This situation resembles the situation with the original form of the Schlesinger system (1.1): if one ignores the second set of equations in (1.1), one gets a system of PDE's for the residues A_j; only upon adding the equations for $\partial A_j/\partial \lambda_j$ one gets a system of ODE's with respect to each λ_j (the flows with respect to different λ_j commute).

Let us finally note that the construction we have presented in order to derive the differential equations (3.1), (3.2) suggests a number of interesting further generalizations that deserve further study.

- At the origin of our derivation have figured the algebraic $\mathfrak{sl}(2)$ identities (3.9), (3.11) that we have translated into differential equations. Similar identities exist also for higher rank groups (e.g. $M > 2$, or the Schlesinger system for orthogonal, symplectic, and exceptional groups) where the number of independent tensors may be larger. It would be highly interesting to understand if equations analogous to (3.1), (3.2) can be derived from such higher rank algebraic identities. As those identities will be built from a larger number of invariant tensors (structure constants, etc.), the corresponding differential equations would be of higher order in derivatives.

- Is it possible to combine our present construction applicable to Schlesinger systems with simple poles only with construction of [8] which requires the presence of higher order poles? What would be the full set of equations for the tau-function with respect to the full set of deformation parameters in presence of higher order poles?

- The Schlesinger system (1.1) has also been constructed for various higher genus Riemann surfaces [10, 11, 12, 13]. It would be interesting to first of all find the proper generalization of the symmetric form (2.11), (2.12) of the Schlesinger system to higher genus surfaces which in turn should allow to derive by an analogous construction the non-trivial differential equations satisfied by the associated τ-function. We conjecture that in some sense the form (2.11) should be universal: it should remain the same, although the definition of the Virasoro generators L_m and the variables $\mathcal{B}_m$ may change.

- As we have mentioned above, the extra term $n\mathcal{B}_{m+n}$ in the Hamiltonian dynamics of the symmetrised Schlesinger system (2.17) can be absorbed into the symplectic action upon replacing the standard affine Lie-Poisson bracket (2.16) by its centrally extended version. However, this central extension is not seen in any of the finite-N Schlesinger systems. This seems to suggest that the system (2.11) should be considered not just as a symmetric form of the usual Schlesinger system with finite number of poles, but as a "universal" Schlesinger system which involves an infinite set of independent variables $\mathcal{B}_n$. Presumably, this full system involves the generators L_n and coefficients $\mathcal{B}_n$ not only for positive, but also for negative n.

  In this setting, the centrally extended version of the bracket (2.16) should appear naturally. The most interesting problem would be to find the geometric origin of such a generalized system; a possible candidate could be the isomonodromic deformations on higher genus curves.

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