Distances on Lozenge Tilings

Olivier Bodini, Éric Rémila, Thomas Fernique

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O. Bodini[1], T. Fernique[2], and E. Remila[3]

[1] LIP6 UMR 7606 CNRS - Université Paris 6,
4 place Jussieu 75005 Paris - France,
olivier.bodini@lip6.fr
[2] LIF UMR 6166 CNRS - Université de la Méditerranée,
39 rue Joliot-Curie - F-13453 Marseille Cedex13 - France,
thomas.fernique@lif.univ-mrs.fr
[3] LIP UMR 5668 CNRS-ENS, Université de Lyon 1,
46 allée d'Italie 69364 Lyon Cedex 7 - France,
eric.remila@ens-lyon.fr

Abstract. In this paper, a structural property of the set of lozenge tilings of a $2n$-gon is highlighted. We introduce a simple combinatorial value called *Hamming-distance*, which is a lower bound for the flip-distance (i.e. the number of necessary local transformations involving three lozenges) between two given tilings.
It is here proven that, for $n \leq 4$, the flip-distance between two tilings is equal to the Hamming-distance. Conversely, for $n \geq 6$, We show that there is some deficient pairs of tilings for which the flip connection needs more flips than the combinatorial lower bound indicates.

1 Introduction

Lozenge tilings are now a classical model, used by physicists as a model for quasicrystals [10], since the discovery of the famous Penrose tilings, with a pentagonal symmetry. We are especially interested on tilings of finite $2n$-gons. If such a tiling contains three rhombic tiles which pairwise share an edge, then a new tiling of the same $2n$-gon can be obtained by just changing the position of those three tiles. This operation is called a *flip*. The *tiling space* of a fixed $2n$-gon is the graph whose vertices are tilings of and two tilings being linked by an edge if they differ by a single flip.

The combinatorial properties of tiling spaces are not trivial for $n \geq 4$. The connectivity of these spaces has been proved in [11], nevertheless in the case of infinite tilings as Penrose tiling this property is not true even if we admit an infinite number of flips [4,5]. The main argument is the following: each tiling is linked to a special fixed tiling $\mathcal{T}_0$. Therefore, this proof does not give a precise result for the *flip-distance* between any pair $(\mathcal{T}, \mathcal{T}')$ of tilings in the tiling space. Moreover, by elementary geometrical considerations (de Bruijn lines), a lower bound for the flip-distance, called the *Hamming-distance*, is canonically obtained.

In this paper, we precisely investigate the quality of this lower bound. We first prove that, for octagonal tilings (i.e. $n = 4$), the Hamming-distance is actually the flip-distance (section 4). This is an extension of the same result about the previously known hexagonal tilings [12]. The lack of a distributive lattice structure on the tiling space for the octagonal tilings makes the proof more difficult and the result more surprising. Conversely, there exists a (very small in average) difference when $n \geq 6$. Indeed, there are a few number of pairs of tilings for which the Hamming-distance is strickly lower than the flip-distance (section 5). The case $n = 5$ needs a tedious case-study. We only indicates, in this paper, some hints about the way of proving it (section 6). The exhaustive proof is a work in progress.

Regarding related results, it has be proven in [9] that there exists a fixed special tiling $\mathcal{T}_0$ in a unitary $2n$-gon such that the flip-distance and the Hamming-distance between a variable tiling $\mathcal{T}$ and $\mathcal{T}_0$ are always equal. But this situation cannot be extended to any arbitrary pairs of tilings of general $(m_1, \ldots, m_n)$-$2n$-gons (i.e. with m_i pseudo-lines of type i).

2 $2n$-gons, tilings and de Bruijn lines

Let $V = (v_1, \ldots, v_n)$ be a n-uple of vectors of $\mathbb{R}^2$ (with no pair of colinear vectors) and $(m_1, \ldots, m_n)$ be a n-uple of positive integers ($n > 1$). The $(m_1, \ldots, m_n)$-$2n$-*gon* is the subset of $\mathbb{R}^2$:

$$Z(V, M) = \{v | v = \sum_{k=1}^{n} \lambda_k \ v_k, \ \lambda_k \in [-m_k, m_k]\}$$

The $2n$-gon is *regular* if, for $0 \leq k < n$, we have $v_{k+1} = (\cos(\pi k/n), \sin(\pi k/n))$. We only work on regular zonotopes. When $m_1 = \ldots = m_n = 1$, the regular $2n$-gon is *unitary*.

For $1 \leq i < j \leq n$, we denote by T_{ij} the *lozenge prototile*: $T_{ij} = \{\lambda \boldsymbol{v}_i + \mu \boldsymbol{v}_j, -1 \leq \lambda, \mu \leq 1\}$. A *lozenge tilings* of $2n$-gon is a set of translated copies of lozenge prototiles with pairwise disjoint interiors whose union is the $2n$-gon. Let $\mathcal{T}$ be a lozenge tiling, the *vertices* (resp. *edges*) of $\mathcal{T}$ are the vertices (resp. edges) of the tiles which belong to $\mathcal{T}$.

The combinatorial structure of tilings of a $(m_1, \ldots, m_n)$-$2n$-gon depends only on the n-uple $(m_1, \ldots, m_n)$ but not on the vectors v_i of the $2n$-gon. This important property is not true in dimension 3 or higher [8], and induces that the choice to study tilings of regular $2n$-gons is not a restriction 'in fine' for the 2D-problem.

For each integer $k \in \{1, ..., n\}$, and each tiling $\mathcal{T}$ of $Z(V, M)$, a *k-located height function* (classical height functions have been introduced by Thurston [13] $h_{T,k}$ is a function from vertices to $\mathbb{Z}$ such that, for any pair (x, x') of vertices of T with $x' = x + 2v_i$ and $[x, x']$ being an edge of T,

- $h_{T,k}(x') = h_{T,k}(x) + 1$ if $i = k$,
- $h_{T,k}(x') = h_{T,k}(x)$ otherwise.

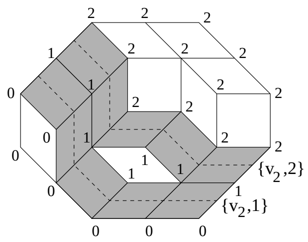

Fig. 1. The 2-located height function and two de Bruijn lines.

We use the normalized k-located height function such that, for each vertex x, $h_{T,k}(x) \geq 0$ and there exists a vertex x_0 with $h_{T,k}(x_0) = 0$. The existence of height function and uniqueness of normalized height functions is well known [3].

Following [7], we define the *de Bruijn line* $\mathcal{S}_{i,j}$ of T as the set of tiles whose normalized i-located function is $j-1$ on one edge, and j on the opposite one. See Figure 1.

A de Bruijn line $\mathcal{S}_{i,j}$ is said of *type* i. It is interesting to note that two distinct de Bruijn lines of the same type do not intersect while two de Bruijn lines of different types share a single tile. Conversely, each tile is the intersection of two de Bruijn lines of different types.

The de Bruijn line $\mathcal{S}_{i,j}$ disconnects $\mathcal{T}$ into two parts:

– $\triangle(\mathcal{S}_{i,j})$ formed by tiles for which the i-located function is at least j on any vertex.
– $\bigtriangledown(\mathcal{S}_{i,j})$ formed by tiles for which the i-located function is at most $j-1$ on any vertex.

Three de Bruijn lines of pairwise different types i, j, k define a tiled region of the zonotope, called *pseudo-triangle* of type ijk. A *minimal* (for inclusion) pseudo-triangle is reduced to three tiles, each of them being the intersection of a pair of de Bruijn lines.

3 Hamming-distance and flip-distance between two tilings

We introduce in this section the notions of flip-distance and Hamming-distance between two tilings.

3.1 Flip-distance

Two tiles are *adjacent* if they share an edge. Assume that three tiles of a tiling $\mathcal{T}$ are pairwise adjacent (i. e. form a minimal pseudo-triangle). In this case, one can replace in a unique way these three tiles by three other tiles of the same

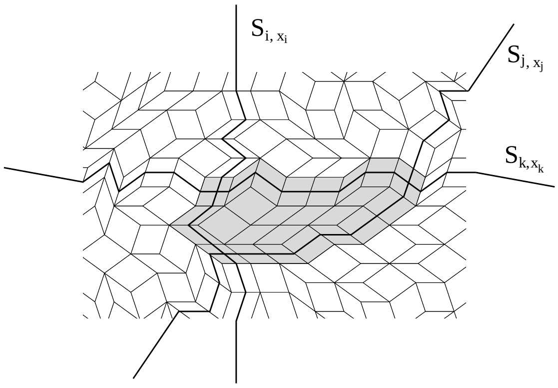

Fig. 2. In gray, a pseudo-triangle

type, to obtain another tiling $\mathcal{T}'$ of the same $2n$-gon. This operation is called a *flip* (see. Fig.3). The *tiling space* of $Z(V, M)$ is the undirected graph whose vertices are tilings of $Z(V, M)$, and two tilings are linked by an edge if they differ by a flip.

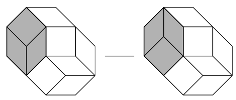

Fig. 3. Two neighbor tilings. One can pass from one to the other one by a single flip.

Definition 1. *The* flip-distance *between two tilings $\mathcal{T}_1$ and $\mathcal{T}_2$ of $Z(V, M)$, denoted by $d_F(\mathcal{T}_1, \mathcal{T}_2)$, is the length of the shortest path relying $\mathcal{T}_1$ with $\mathcal{T}_2$ in the tiling space. It is a finite value, since the tiling space is connected [11].*

The figure 4 illustrates the topology of such a graph.

3.2 Hamming-distance

For every triple of de Bruijn lines $(\mathcal{S}_{i,\alpha_i}, \mathcal{S}_{j,\alpha_j}, \mathcal{S}_{k,\alpha_k})$, such that $i < j < k$, we state:

- $\mathbb{T}(\mathcal{S}_{i,\alpha_i}, \mathcal{S}_{j,\alpha_j}, \mathcal{S}_{k,\alpha_k}) = +$ if the tile $\mathcal{S}_{i,\alpha_i} \cap \mathcal{S}_{j,\alpha_j}$ belongs to $\triangle(\mathcal{S}_{k,\alpha_k})$,
- $\mathbb{T}(\mathcal{S}_{i,\alpha_i}, \mathcal{S}_{j,\alpha_j}, \mathcal{S}_{k,\alpha_k}) = -$ otherwise.

In this way, each tiling induces a one dimensionnal array $\mathbb{T}$ containing $+$ or $-$ as entries and indexed on the set $\mathcal{L}_{\mathbf{m}}$ of all possible triples. It can be easily proved that the array $\mathbb{T}$ totally characterises the tiling.

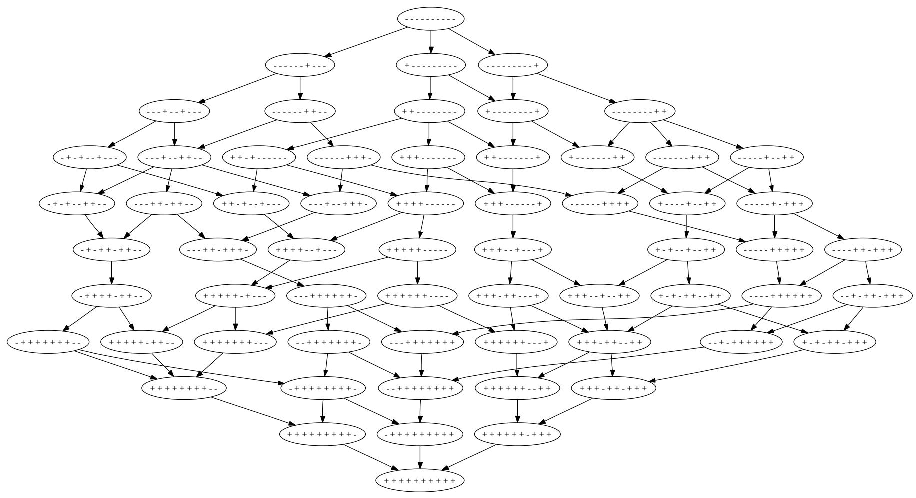

Fig. 4. The tiling space of the $(1,1,1,1,1)$-10-gon.

Nevertheless, there exist some $\mathbf{m}$-uples of $\{-,+\}^{\mathbf{m}}$ that do not correspond to any tiling. A characterization of $\mathbf{m}$-uples induced by tilings has been given by Chavanon-Rémila [6]. It uses a set of "local" conditions, in a sense that each of them involving a finite number (two or four) values of the array.

Let $\mathcal{T}$ and $\mathcal{T}'$ be two tilings whose associated arrays are respectively $\mathbb{T}$ and $\mathbb{T}'$. The triple $(\mathcal{S}_{i,\alpha_i}, \mathcal{S}_{j,\alpha_j}, \mathcal{S}_{k,\alpha_k})$ (or, by extension, the pseudo-triangle of corresponding de Bruijn lines) is *inverted* when

$$\mathbb{T}(\mathcal{S}_{i,\alpha_i}, \mathcal{S}_{j,\alpha_j}, \mathcal{S}_{k,\alpha_k}) \neq \mathbb{T}'(\mathcal{S}_{i,\alpha_i}, \mathcal{S}_{j,\alpha_j}, \mathcal{S}_{k,\alpha_k}).$$

Definition 2. *The* Hamming-distance *between $\mathcal{T}$ and $\mathcal{T}'$, denoted by $d_H(\mathcal{T},\mathcal{T}')$, is the number of inverted triples. This is exactly the classical Hamming-distance between arrays $\mathbb{T}$ and $\mathbb{T}'$.*

Proposition 1. *For every two tilings $\mathcal{T}$ and $\mathcal{T}'$ of a $2n$-gon, the following inequality holds:*

$$d_H(\mathcal{T},\mathcal{T}') \leq d_F(\mathcal{T},\mathcal{T}')$$

Proof. Each inverted triangle need to be reversed. But a flip can only turn upside down a unique triangle.

4 Distance comparison: our results

The goal of this paper is to compare flip-distance and Hamming-distance. More precisely, we are interested in the existence and the ratio of deficient configurations. The deficience are naturally defined as follows:

Definition 3. *The pair* $(\mathcal{T}_1, \mathcal{T}_2)$ *of tilings is* deficient *if its flip-distance is strictly greater than its Hamming-distance.*

Proposition 2 ([12]). *For hexagons (i. e.* $n = 3$*), the Hamming-distance between two lozenge tilings is equal to the flip-distance between them.*

This result, which also holds for any polygon, is strongly related to the structure of distributive lattice of the space of lozenge tilings, (for $n = 3$). It also can be interpreted in terms of "stepped surface" [1]: the Hamming-distance is exactly the volume of the solid inbetween the stepped surfaces defined by the tilings. It is possible to crush this gap by deforming the stepped surfaces decreasing the current volume. But for the general $2n$-gons one cannot use the same type of arguments, since the lattice structure and the volume interpretation are both lost.

4.1 Distance equality for $n = 4$

Distances between the tilings of the unitary 8-gon. It is well-known that there exists only 8 tilings of the unitary 8-gon centered in Ω. These 8 tilings are isometrically equivalent. They can be obtain of acting the dihedral group of order 16 (which is isomorph to the group of isometry that preserves the octagon) from one of them. More precisely, the flip which is in general a local move can be seen here as the global map $s \circ \rho$ or $s \circ \rho^{-1}$ where ρ is the rotation of angle $2\pi/2n$ centered in Ω and s is the central symmetry of center Ω. The tiling space of these tilings is a cycle (of length 8) which is also the orbit of a tiling under $s \circ \rho$. We can easily remark that the Hamming-distance between two unitary 8-gon tilings is equal to their flip-distance. Up to isometry, there only exists 4 types of pair of tilings which corresponds to important configurations related to the next lemma 1.

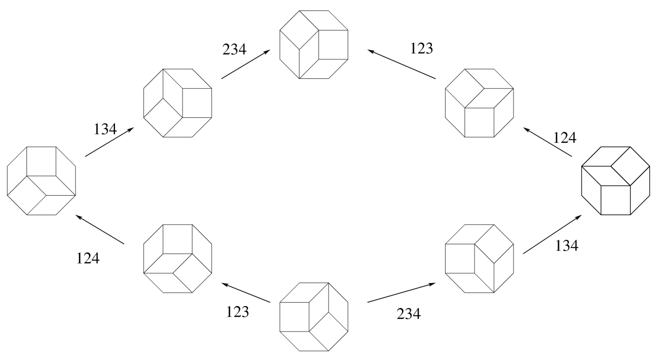

Fig. 5. The tiling space of the unitary octagon.

Distances between general 8-gon tilings. In fact, the distance equality is also true for general 8-gons. But, the proof is not straightforward. Let $(\mathcal{T}_1, \mathcal{T}_2)$ be a pair of tilings, we need to introduce another array $\mathbb{B}$ as follows:

– $\mathbb{B}(\mathcal{S}_{i,\alpha_i}, \mathcal{S}_{j,\alpha_j}, \mathcal{S}_{k,\alpha_k}) = 1$ if the pseudo-triangle $(\mathcal{S}_{i,\alpha_i}, \mathcal{S}_{j,\alpha_j}, \mathcal{S}_{k,\alpha_k})$ is inverted (i.e. $\mathbb{T}(\mathcal{S}_{i,\alpha_i}, \mathcal{S}_{j,\alpha_j}, \mathcal{S}_{k,\alpha_k}) \neq \mathbb{T}'(\mathcal{S}_{i,\alpha_i}, \mathcal{S}_{j,\alpha_j}, \mathcal{S}_{k,\alpha_k})$.),
– $\mathbb{B}(\mathcal{S}_{i,\alpha_i}, \mathcal{S}_{j,\alpha_j}, \mathcal{S}_{k,\alpha_k}) = 0$ otherwise.

Some consistence conditions (lemma 1,2) in $\mathbb{B}$ are similar (and local) to those appearing in the characterization of $\mathbb{T}$.

Lemma 1. *Let $(\mathcal{T}_1, \mathcal{T}_2)$ be a pair of tilings of the $(m_1, \ldots, m_n)$-$2n$-gon and let us consider four de Bruijn lines, namely S_1, S_2, S_3, S_4, and consider that their relative positions in $\mathcal{T}_1$ are:*

1. *such that S_3 and S_4 have the same type and S_4 cuts the pseudo-triangle (S_1, S_2, S_3) (see figure 6a). Then, we have:*

$$\mathbb{B}(S_1, S_2, S_3) = 1 \Rightarrow \mathbb{B}(S_1, S_2, S_4) = 1$$

and

$$\mathbb{B}(S_1, S_2, S_4) = 0 \Rightarrow \mathbb{B}(S_1, S_2, S_3) = 0.$$

In other words, $(\mathbb{B}(S_1, S_2, S_3), \mathbb{B}(S_1, S_2, S_4))$ belongs to $\{(0,0),(0,1),(1,1)\}$.

2. *such that the pseudo-triangle (S_2, S_3, S_4) is included in the pseudo-triangle (S_1, S_2, S_3) as described in the figure 6b. Then, the following sequence:*

$$(\mathbb{B}(S_1, S_3, S_4), \mathbb{B}(S_1, S_2, S_4), \mathbb{B}(S_1, S_2, S_3), \mathbb{B}(S_2, S_3, S_4))$$

is monotonic. In other words, it belongs to the set

$$\{(0,0,0,0),(0,0,0,1),(0,0,1,1),(0,1,1,1),(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)\}.$$

Proof. This can easily be done with the following arguments :

1. We have to keep the order between the de Bruijn lines of a same type in the both tilings.
2. This can be done from the study of the 8 tilings of the unitary octagon.

Lemma 2. *Let $(\mathcal{T}_1, \mathcal{T}_2)$ be a pair of tilings of the $(m_1, \ldots, m_n)$-$2n$-gon, if the pseudo-triangle (S_i, S_j, S_k) of $\mathcal{T}_1$ of type ijk is inverted and it is only cut by de Bruijn lines of type i,j or k then every sub-pseudo-triangle (S'_i, S'_j, S'_k) included in (S_i, S_j, S_k) is inverted. In particular, (S_i, S_j, S_k) contains an inverted minimal sub-pseudo-triangle.*

Proof. First, by lemma 1, an easy induction on the number of de Bruijn lines proves that every sub-pseudo-triangle (S'_i, S'_j, S'_k) of (S_i, S_j, S_k) is inverted. In the same way, it is clear that the configuration always contains a minimal pseudo-triangle.

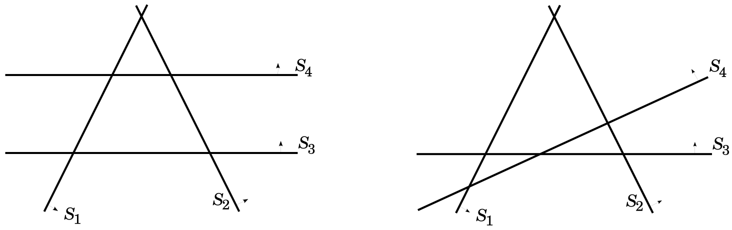

Fig. 6. The configurations for the lemma 1.

Theorem 1. *Let $(\mathcal{T}_1, \mathcal{T}_2)$ be a pair of tilings of the $(m_1, \ldots, m_4)$-8-gon, then the Hamming-distance between them is equal to their flip-distance.*

Proof. We first prove that for every pair of distinct tilings $(\mathcal{T}_1, \mathcal{T}_2)$ of the $(m_1, \ldots, m_4)$-8-gon, $\mathcal{T}_1$ contains an inverted minimal pseudo-triangle (a closer flip is feasible on it).

Considering all these previous lemmas, an induction on the number m_4 of de Bruijn lines of type 4 can be done.

For initialization, if $m_4 = 0$, the 8-gon is actually a hexagon for which the result is previously known. Suppose that $m_4 = 1$ and when we remove the de Bruijn line S of type 4, $\mathcal{T}_1 \backslash S$ are identical to $\mathcal{T}_2 \backslash S$. In this case, the positions of the de Bruijn line S in $\mathcal{T}_1$ and $\mathcal{T}_2$ mark the boundary of a stepped surface U in $\mathcal{T}_1 \backslash S$ which is a hexagonal tiling (see figure 7).

The flip-distance and the Hamming-distance between $\mathcal{T}_1$ and $\mathcal{T}_2$ are exactly the number of tiles in U. Indeed, one easily proves that, in any case, a flip can be done, corresponding to a local transformation on the position of the de Bruijn line S in $\mathcal{T}_2$ (i.e. surrounding a unique tile by the other side).

Now, for the other cases, we can remove a de Bruijn line $S = \mathcal{S}_{4,\alpha}$, of type 4, in both tilings $\mathcal{T}_1$, $\mathcal{T}_2$ in such a way that $\mathcal{T}_1 \backslash S$ are distinct to $\mathcal{T}_2 \backslash S$ (this is always possible when $m_4 > 1$). Henceforward, we are going to only work on the tiling $\mathcal{T}_1$. By hypothesis of induction, there exists an inverted minimal pseudo-triangle (S_a, S_b, S_c), in the tiling $\mathcal{T}_1$ obtained by removing of S and sticking the two remaining parts of the initial tiling.

After this, let us replace the removed de Bruijn line S. The only tricky case arises when S cuts the pseudo-triangle (S_a, S_b, S_c) and the types of de Bruijn lines S_a, S_b and S_c are respectively 1, 2 and 3. Moreover, the minimal sub-pseudo-triangle of (S_a, S_b, S_c) (which is (S_a, S_b, S) or (S_b, S_c, S)) is not inverted (in any other configuration, the existence of an inverted minimal pseudo-triangle is trivial).

We can consider without loss of generality that $\mathbb{T}(S_a, S_b, S_c) = +$ and that (S_a, S_b, S) is the non-inverted minimal sub-pseudo-triangle of (S_a, S_b, S_c). The 3 other cases are in fact isometrically equivalent.

Since (S_a, S_b, S) is not inverted, we have by lemma 1.2 applying on the pseudo-lines $\{S_a, S_b, S_c, S\}$ that the pseudo-triangle (S_a, S_c, S) is inverted. If this pseudo-triangle is minimal, then the result ensues. Otherwise, (S_a, S_c, S)

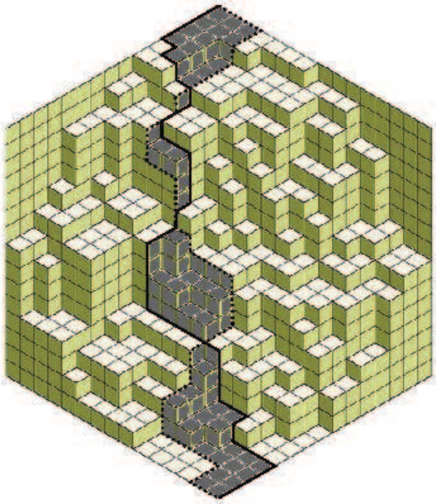

Fig. 7. The bold (resp. dotted) line indicates the position of the de Bruijn line S in $\mathcal{T}_1$ (resp. in $\mathcal{T}_2$). The stepped surface U is in dark gray inbetween the two de Bruijn lines.

can only be cut by de Bruijn lines of type 1 or 2, because of the minimality of the pseudo-triangle (S_a, S_b, S_c) in $\mathcal{T}_1 \backslash S$. Let $\mathcal{S}_{1,j_1}$ (resp. $\mathcal{S}_{2,j_2}$) be (if there exists) the de Bruijn line of type 1 (resp. type 2), with j_1 minimal such that $\mathcal{S}_{1,j_1}$ cuts (S_a, S_c, S) (resp. with j_2 minimal such that $\mathcal{S}_{2,j_2}$ cuts (S_a, S_c, S)) (see fig.8). The pseudo-triangle $(\mathcal{S}_{1,j_1}, S_c, S)$ (resp. $(\mathcal{S}_{2,j_2}, S_c, S)$) is inverted. Indeed this follows of applying lemma 1.1 to the pseudo-triangles (S_a, S_c, S) (resp. (S_b, S_c, S)) which are both inverted. If one of them is minimal, we can conclude.

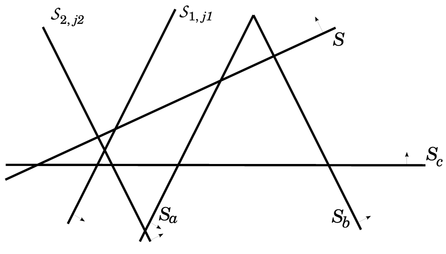

Fig. 8. A configuration involving the de Bruijn lines $\mathcal{S}_{1,j_1}$ and $\mathcal{S}_{2,j_2}$ such that their intersection is in (S_a, S_c, S).

Otherwise, the tile $\mathcal{S}_{1,j_1} \cap \mathcal{S}_{2,j_2}$ belongs to $\triangle(S_c) \cap \bigtriangledown(S)$ and necessarily at least one of the pseudo-triangles $(\mathcal{S}_{1,j_1}, \mathcal{S}_{2,j_2}, S_c)$ and $(\mathcal{S}_{1,j_1}, \mathcal{S}_{2,j_2}, S)$ is inverted (by applying lemma 1.2 on $\{S_c, S, \mathcal{S}_{1,j_1}, \mathcal{S}_{2,j_2}\}$ with $(\mathcal{S}_{1,j_1}, S_c, S)$ inverted). But $(\mathcal{S}_{1,j_1}, \mathcal{S}_{2,j_2}, S_c)$ (resp. $(\mathcal{S}_{1,j_1}, \mathcal{S}_{2,j_2}, S)$) can only be cut by de Bruijn lines of type 1 (resp. type 2). Because of the minimality of j_1 and j_2, it includes by lemma 2 an inverted minimal sub-pseudo-triangle of type 123 (resp. type 124). Thus, we always have a inverted minimal pseudo-triangle in $\mathcal{T}_1$.

Now, we can prove the theorem. Indeed, by absurd, let $(\mathcal{T}_1, \mathcal{T}_2)$ be a deficient pair of tiling of a $(m_1, \ldots, m_4)$-8-gon. Moreover, we suppose that it is flip-distance minimum belong the deficient pairs. So, $\mathcal{T}_1$ does not contain any inverted minimal pseudo-triangle. Indeed, an inverted minimum pseudo-triangle corresponds to a flippable position which brings closer $\mathcal{T}_1$ from $\mathcal{T}_2$. But this is in contradiction with what we proved above.

Corollary 1. *A* $(m_1, \ldots, m_4)$-8-*gon tiling (resp.* $4 \to 2$ *tiling of the plan) is uniquely determined by the value of* $\mathbb{T}$ *for its minimal pseudo-triangles.*

Proof. Consider $\mathcal{T}_1$ and $\mathcal{T}_2$ be a pair of tiling of a $(m_1, \ldots, m_4)$-8-gon with the same set M of minimal pseudo-triangle. If $\mathbb{T}_1(v) = \mathbb{T}_2(v)$ for every $v \in M$, then $\mathcal{T}_1$ has no inverted minimal pseudo-triangle. So, by theorem 1, $\mathcal{T}_1 = \mathcal{T}_2$. For the $4 \to 2$ tilings, it suffices to take the limit when (m_1, m_2, m_3, m_4) tends to ∞.

5 Counterexamples for $n = 6$

There exist deficient pairs of tilings of the unitary 12-gon, which yields that this is also true for every $2n$-gon, for $n \geq 6$. It is possible, using a computer, to find every deficient pair for the unitary 12-gon : up to isometry there exists only two deficient pairs of tilings. For these two pairs the Hamming-distance is 16 but the flip-distance is 18. From the first pair $(\mathcal{T}_1, \mathcal{T}_2)$ which has no symmetry (Fig. 9), one obtains, by symmetry 12 different deficient pairs of tilings. The second one (Fig.10) induces 4 different deficient pairs of tilings. These 16 pairs of tilings are exactly the deficient pairs of tilings among the $(908)^2$ possible pairs of tilings (there are 908 different tilings in the unitary 12-gon). The expected flip-distance between two tilings is not 10 (i. e. the expected Hamming-distance, as it would be if there were no deficient pair) but around 10,000 07. This is somewhat surprising.

6 Distances between general 10-gon tilings and Conclusion

This section is dedicated to explain the remained objectives of this work. In particular, we give below the main ideas of a currently hypothetic proof for the following conjecture:

Conjecture 1. The hamming-distance between two tilings of any $(m_1, \ldots, m_5)$-10-gon is equal to their flip-distance.

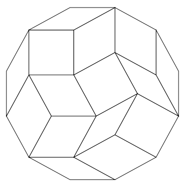

Fig. 9. The first deficient pair of tilings of the unitary 12-gon.

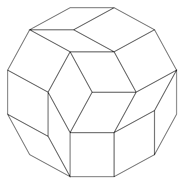

Fig. 10. The second deficient pair of tilings of the unitary 12-gon.

At present time, we have not already proven all the huge number of cases that occurs in the process of such a proof. But we keep hope alive to be able to achieve this tedious case-study. The following proved lemmas are the cornerstone of our plan.

Lemma 3 (The harp lemma). *Let $P = (S_1, S_2, S_3)$ be an inverted pseudo-triangle of type ijk. Let us consider that P is cut only by de Bruijn lines $S_4, ..., S_p$ of type l and m different with i, j and k. Then, the configuration formed by $S_1, ..., S_p$ contains an inverted minimal subtriangle $(S_\alpha, S_\beta, S_\gamma)$.*

Lemma 4 (10-cycle lemma). *Suppose that there exits a deficient pair of tilings $(\mathcal{T}_1, \mathcal{T}_2)$ for a $(m_1, \ldots, m_5)$-10-gon, and consider the smallest m_5 possible. If we delete a de Bruijn line S of type 5, we obtain a pair of tilings which (are equal or) contains an inverted minimal pseudo-triangle P of type ijk (say 123). Then, when we undelete the de Bruijn line S, we have that P is cut by S and an inverted pseudo-triangle P_1 of type 135 is created. The pseudo-triangle P_1 also needs to be cut by a de bruijn line which is of type 4. This creates an inverted pseudo-triangle of type 345, and so on. The complete process gives the 10-cycle described below (fig.11).*

But in the 10-cycle lemma, it seems to be impossible to start from a pseudo-triangle T of type ijk and after a 10-cycle to return to the same pseudo-triangle T. In each already studied case, we always reach a new pseudo-traingle T' of type ijk. So, we strongly conjecture that the property that the Hamming-distance between two tilings is equal to the flip-distance is also true for 10-gons too, but the proof is a huge case-study that has not been precisely checked yet. It is a work in progress.

This study is the first step of a more general research. Indeed, the combinatorics of the set parallelogram tiling of general n-dimensional zonotopes is always

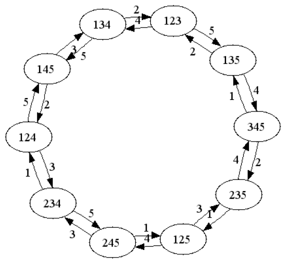

Fig. 11. The 10-cycle.

very misleading, even though recent works [6] [8] have proved that the tiling space is connected in a large case of zonotopes [2]. But, the problem is always opened for $6 \to 3$ zonotopes (i. e. constructed with 6 vectors of $\mathbb{R}^3$) with icosaedral symmetry. The answer could have interesting application in quasicrystal theory and statistical mechanics. It is a persective for the future.

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