Symmetric spaces of higher rank do not admit differentiable compactifications

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SYMMETRIC SPACES OF HIGHER RANK DO NOT ADMIT DIFFERENTIABLE COMPACTIFICATIONS

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ABSTRACT. Any nonpositively curved symmetric space admits a topological compactification, namely the Hadamard compactification. For rank 1 spaces, this topological compactification can be endowed with a differentiable structure such that the action of the isometry group is differentiable. Moreover, the restriction of the action on the boundary leads to a flat model for some geometry (conformal, CR or quaternionic CR depending of the space). One can ask whether such a differentiable compactification exists for higher rank spaces, hopefully leading to some knew geometry to explore. In this paper we answer negatively.

1. INTRODUCTION

Let M be a symmetric space of nonpositive curvature, G its group of isometries and G_0 the identity component in G.

As a Riemannian manifold, M is a Hadamard space and is diffeomorphic to an open ball. Its *Hadamard compactification* (or *geodesic compactification*) is a topological gluing of M and its *Hadamard boundary* $M(\infty)$ such that $\overline{M} = M \cup M(\infty)$ is a closed ball. The group G acts continuously on $\overline{M}$.

When M is of rank one, that is to say when it is negatively curved, this topological compactification admits "nice" models, carrying an invariant differentiable structure: in these models, the action of G is differentiable on $\overline{M}$. Moreover, the restriction of this action to the boundary is a flat model for some geometry. The boundary spheres of the real, complex and quaternionic hyperbolic spaces wield the standard conformal, CR and quaternionic CR structures respectively. Concerning the octonionic hyperbolic plane, the corresponding geometry has not been studied yet, as far as we know.

It is natural to ask whether such a differentiable compactification exists when M is of higher rank. One could expect such a model to give birth to a new, luckily interesting, geometry.

In this paper, we give a negative answer to this question, and show that the obstruction comes from the spherical building at infinity. This combinatorial structure is trivial only in the Euclidean and rank one spaces. Thus there is an alternative: a symmetric space of nonpositive curvature admits either an interesting building at infinity or a differentiable compactification, not both.

1.1. **Differentiable compactifications.** Our goal is to extend the differentiable structure of M to the manifold with boundary $\overline{M}$, so that we do not lose symmetry in the process. This leads to the following definition.

Definition 1. *A* differentiable Hadamard compactification *of M is a differentiable (C^1) structure $\mathcal{D}$ on $\overline{M}$ compatible with the differentiable structure of M and such that the action of G is C^1.*

We can define a C^r Hadamard compactification *in the same way, where r can be finite, ∞ or ω, C^ω meaning real analytic. When no precision is given, differentiable means C^1.*

By a C^r action, we mean that the map $G \times M \to M$ is C^r. It implies that G acts by C^r diffeomorphisms and that the map $G \to \mathrm{Diff}(M)$ is continuous in the C^r topology.

This condition can be greatly relaxed thanks to the Bochner and Montgomery theorem: if G acts continuously by C^r diffeomorphisms, then its action is in fact C^r [2].

For the sake of brevity we will often write "differentiable compactification" instead of "differentiable Hadamard compactification". However, a symmetric space admits other topological compactifications than the Hadamard one, e.g. Martin, Satake and Furstenberg compactifications. It would be interesting to extend our study to these, but the Hadamard compactification seems to be of utmost importance for our question. First, it is very natural, defined directly by the geometry of the space for a large class of Riemannian manifolds. Second, there is as far as we know little hope to get a manifold with boundary from the other compactifications. Either the infinity does not have the right dimension (e.g. the Poisson boundary) or the most natural differentiable structure is that of a manifold with boundary *and corners* (e.g. the maximal Satake compactification). A detailed account on all classical compactifications can be found in [3] and [6].

1.2. **Existence of differentiable compactifications.** Let us now discuss the existence of differentiable compactifications for the three types of nonpositively curved symmetric spaces.

Symmetric spaces of rank 1. It is well known that the real hyperbolic space $\mathbb{H}^n$ admits a differentiable Hadamard compactification, given for example by the closure of Klein's ball: the central projection of the hyperboloid $Q = -1$ (where Q is the canonical Lorentzian metric on $\mathbb{R}^{n+1}$) gives an embedding of $\mathbb{H}^n$ into $\mathbb{RP}^n$ where the group $\mathrm{SO}_0(1,n)$ of isometries of $\mathbb{H}^n$ acts analytically. This construction can be generalized to all symmetric spaces of nonpositive curvature and rank 1.

It is worth noticing that $\mathbb{H}^n$ admits other differentiable Hadamard compactifications. For example, the action of $\mathrm{SO}_0(1,n)$ on Poincaré's ball extends analytically to the closed ball and the resulting action is not C^1 conjugate to the previous one (this can be seen by looking at asymptotic geodesics: they are tangent one to another in the closure of Poincaré's ball, not in Klein's ball.) Details are given in [8], where it is shown that $\mathbb{H}^n$ admits an infinite number of nonconjugate analytic compactifications in the sense of definition 1.

Euclidean spaces. If M is a Euclidean space, once again it admits a differentiable Hadamard compactification we briefly describe. Identify $\mathbb{R}^n$ with the affine hyperplane $\{x_0 = 1\}$ of $\mathbb{R}^{n+1}$ where n is the dimension of M. The projection of center 0 of M on the open upper unit half-sphere is a diffeomorphism. Pushing forward by this map we get an action of G (the affine group) on the open upper half-sphere whose continuous prolongation to the closed half-sphere is real analytic. This action is a real analytic Hadamard compactification of $M = \mathbb{R}^n$.

Symmetric spaces of higher rank. The main result of this paper is the following.

Theorem 1. *No noneuclidean symmetric space of rank $k \geqslant 2$ admits a differentiable Hadamard compactification.*

Structure of the paper. From now on, M is supposed to be a symmetric space of rank $k \geqslant 2$.

We shall start with a simple remark about the natural projection of a fiber S_xM of the unit tangent bundle of M on $M(\infty)$.

In the second section we prove that $\mathbb{H}^2 \times \mathbb{R}$ admits no differentiable Hadamard compactification.

Next we generalize this fact to every product $F \times \mathbb{R}^{k-1}$ where $k \geqslant 2$ and F is a symmetric space of noncompact type of rank 1.

Finally, we prove Theorem 1.

Note that the different parts are more or less independent: the proof of Theorem 1 does not make use of preceding results. However some arguments of Section 4 will be useful, and Section 3 gives a good insight of the general phenomenon on the simplest case.

2. Apartments and the visual projection

2.1. **Apartments.** We give some basic vocabulary about the building structure of $M(\infty)$. More details can be found in [1], Appendix 5. Our main reference for the building structure of a symmetric space is [5]. For details about general buildings, see [4].

Let A be a maximal flat (*i.e.* a totally geodesic submanifold isometric to the Euclidean space of maximal dimension) of M, $\overline{A}$ its closure in $\overline{M}$ and $A(\infty) = \overline{A} \cap M(\infty)$ its boundary. $A(\infty)$ is called an *apartment* of $M(\infty)$. It is a topological submanifold.

Every point of $M(\infty)$ belongs to at least one apartment. A point is said to be *regular* if it belongs to exactly one apartment, otherwise it is said to be *singular*.

Let x be a point of $M(\infty)$. We denote by $a(x)$ the set of all apartments containing x. If x is singular, it is said to have *index* 1 if $a(x)$ is minimal with respect to inclusion among sets $a(y)$ of singular y's.

The connected component of x in the set of points y such that $a(x) = a(y)$ is a *facet*. Facets are topological submanifolds. If x is regular, we call its facet a *Weyl chamber* or simply a *chamber*; if x is singular of index 1, we call its facet a *panel*.

The dimension of every apartment is $k-1$ (where k is the rank of M). Chambers have dimension $k-1$, panels have dimension $k-2$.

Two facets are *adjacent* if their closures intersect. If they are adjacent and of different dimensions, one is contained in the closure of the other.

The facets form a simplicial complex on $M(\infty)$ if M is of noncompact type. If M has a Euclidean factor, some of the cells are spheres rather than simplicies.

This complex has the incidence structure of a spherical thick building, which means:

(1) each apartment is a spherical Coxeter complex (see [4] for details),
(2) for any two facets, there is an apartment containing both of them,
(3) there exists at least three chambers adjacent to any given panel,
(4) if there are two apartments A, A' containing two facets F and F', then there is an isomorphism $A \mapsto A'$ fixing F and F' pointwise.

The group G acts by isomorphisms on this building: it preserves the adjacency relation and sends facets onto facets of the same dimension.

2.2. **Non smoothness of the visual projection.** Let x be a point of M. A unit vector v tangent to M at x defines a geodesic ray γ_v, hence a point $\gamma_v(\infty)$ of the Hadamard boundary $M(\infty)$. The map

$$\pi_x : \begin{array}{rcl} S_xM & \to & M(\infty) \\ v & \mapsto & \gamma_v(\infty) \end{array}$$

is called the *visual projection* from the point x.

For all x, the visual projection from x is a homeomorphism. It seems reasonable to expect the visual projections to be diffeomorphisms for a "good" differentiable Hadamard compactification. However, it cannot be.

Proposition 1. *If M is a nonpositively curved symmetric space of higher rank, there is no differentiable structure on $M(\infty)$ such that all apartments are submanifolds.*

Proof. Let γ be some geodesic ray in M that is singular of index 1. As the spherical building at infinity of M is thick, there are at least three (in fact, an infinite number of) chambers C_1, C_2, C_3 adjacent to P.

For each pair C_i, C_j $(i \neq j)$ there is a flat A_{ij} such that $C_i \subseteq A_{ij}(\infty)$ and $C_j \subseteq A_{ij}(\infty)$. But $A_{ij}(\infty)$ is an embedded submanifold of $M(\infty)$, thus C_i and C_j have opposite tangent half spaces E_i, E_j at $\gamma(\infty)$. See figure 1.

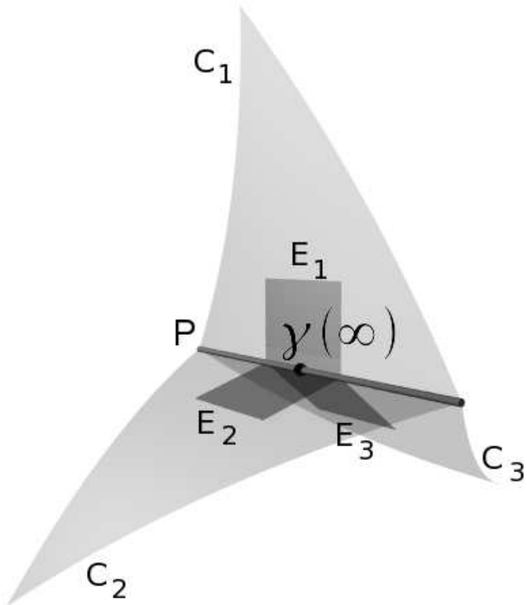

Figure 1. Three chambers meeting at a panel.

Thus we get three half subspaces E_1, E_2, E_3 of $T_{\gamma(\infty)}M(\infty)$ such that $E_1 = -E_2$, $E_1 = -E_3$, $E_2 = -E_3$, a contradiction. □

Corollary 1. *There is no differentiable structure on $M(\infty)$ such that π_x is a diffeomorphism for all $x \in M$.*

Proof. Suppose there is such a differentiable structure.

Let A be a maximal flat of M, x be a point of A.

Then S_xA is an embedded submanifold of S_xM and π_x is a diffeomorphism. Thus $A(\infty) = \pi_x(S_xA)$ is a submanifold of $M(\infty)$. A contradiction with Proposition 1. □

3. Study of $\mathbb{H}^2 \times \mathbb{R}$

We summarize briefly the building structure of $\mathbb{H}^2 \times \mathbb{R}$.

The singular geodesics are those of the form $\{x\} \times \mathbb{R}$ where x is a point of $\mathbb{H}^2$; they are parallel (asymptotic at both ends) to one another. The maximal flats are the products $\gamma \times \mathbb{R}$ where γ is a geodesic of $\mathbb{H}^2$ (see figure 2).

The boundary of $\mathbb{H}^2 \times \mathbb{R}$ is a 2-sphere partitionned into two points and a family of nonintersecting curves joining them. The points are the end points of every singular geodesic, therefore panels of the building. The curves are the Weyl chambers, there is one of them for each point in the boundary of $\mathbb{H}^2$ (see figure 3). The union of any two of them and of the two panels is an apartment.

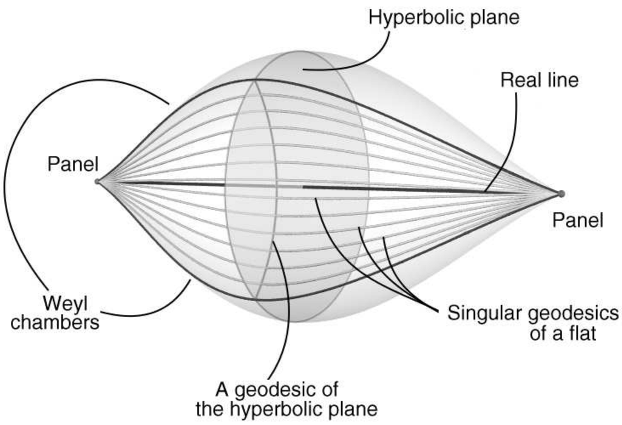

FIGURE 2. A family of parallel singular geodesics lying on the same flat.

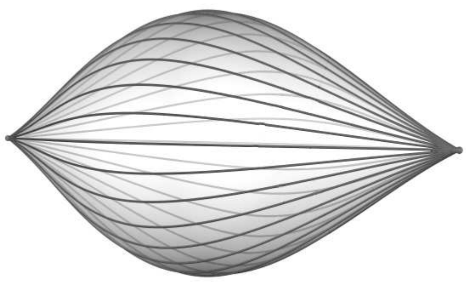

FIGURE 3. Here we show the Weyl chambers in the boundary of $\mathbb{H}^2 \times \mathbb{R}$. Any choice of two of them corresponds to a geodesic of $\mathbb{H}^2$, a flat of $\mathbb{H}^2 \times \mathbb{R}$ and an apartment.

Proposition 2. *The space* $\mathbb{H}^2 \times \mathbb{R}$ *admits no differentiable Hadamard compactification.*

Proof. Suppose $\mathcal{D}$ is a differentiable Hadamard compactification of $M = \mathbb{H}^2 \times \mathbb{R}$ and $\overline{M}$ is endowed with $\mathcal{D}$.

Let $\gamma = \{x\} \times \mathbb{R}^+$ be any singular geodesic ray of unit speed. We denote by $\gamma(\infty)$ the point of $M(\infty)$ defined by γ.

Since $\gamma(\infty)$ is fixed by all orientation-preserving isometries of $\mathbb{H}^2$, the derivatives of these isometries give a linear representation of $\mathrm{PSL}_2(\mathbb{R})$ on $T_{\gamma(\infty)}\overline{M}$. This representation is reducible as $T_{\gamma(\infty)}M(\infty)$ is an invariant subspace. Let ρ be the induced representation. As a representation of a simple Lie group ρ is trivial or faithful.

Let s_x be the geodesic symmetry of $\mathbb{H}^2$ around x. We identify s_x with the isometry $s_x \times \mathrm{Id}$ of $\mathbb{H}^2 \times \mathbb{R}$. For every time $t \in \mathbb{R}$, $ds_x(\gamma(t))$ has eigenvalues 1, -1, -1. By continuity, $ds_x(\gamma(\infty))$ must have the same eigenvalues, thus the restriction to $T_{\gamma(\infty)}M(\infty)$ of $ds_x(\gamma(\infty))$ is $-\mathrm{Id}$, that is to say $\rho(s_x) = -\mathrm{Id}$.

Let y be any point of $\mathbb{H}^2$ different from x. Then $\rho(s_y) = -\mathrm{Id}$ too. Thus $\rho(s_x s_y) = \mathrm{Id}$, but $s_x s_y$ is a non-trivial hyperbolic transformation (it is a translation along the geodesic containing x and y). Thus ρ is neither faithful nor trivial, a contradiction. □

4. Product of a Euclidean space by a rank 1 space

We now generalize Proposition 2 to the case when $M = F \times \mathbb{R}^{k-1}$ is the product of an Euclidean space by a symmetric space F of rank 1.

However, as we will need it later, we prove something stronger.

Definition 2. *A* weak differentiable Hadamard compactification *of M is defined as a differentiable Hadamard compactification where we replace G by its identity component G_0.*

We prove that M admits no weak differentiable Hadamard compactification. Thus, in order to generalize the argument used in the proof of Proposition 2 we need the geodesic symmetries to belong to G_0. Of course this is false if F is of odd dimension, and we shall use another argument in this case.

Proposition 3. *Let F be a rank 1 symmetric space of noncompact type. If $\dim F$ is even, then the geodesic symmetries belong to G_0. If $\dim F$ is odd, then F is a real hyperbolic space $\mathbb{H}^{2m+1}$.*

Proof. From the classification of symmetric spaces (see for example [7], Chapter IX) we know that the rank 1 symmetric spaces of noncompact type are: the real hyperbolic spaces, the complex hyperbolic spaces, the quaternionic hyperbolic spaces and an exceptionnal space, the octonionic hyperbolic plane (therefore the last assertion is clear). The identity components of their isometry groups are respectively $\mathrm{SO}_0(1,n)$, $\mathrm{SU}(1,n)$, $\mathrm{Sp}(1,n)$ and $F_{4(-20)}$, which are simple Lie groups.

We shall use the following criterion. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of the Lie algebra of G_0. Then the geodesic symmetries are in G_0 if and only if $\mathfrak{k}$ contains a maximal abelian algebra of $\mathfrak{g}$ (see [7], Chapter IX 3).

It is know sufficient to compare the ranks of $\mathfrak{g}$ and $\mathfrak{k}$ (most of them can be found in [9], Appendix C):

- in the real case, $\mathfrak{g} = \mathfrak{so}(n, 1)$ is of rank $\lfloor \frac{1}{2}(n+1) \rfloor$ and $\mathfrak{k} = \mathfrak{so}(p)$, of rank $\lfloor n/2 \rfloor$. These two ranks coincide exactly when n is even,
- in the complex case, $\mathfrak{g} = \mathfrak{su}(n, 1)$ and $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(n) \times \mathfrak{u}(1))$ are both of rank n,
- in the quaternionic case, $\mathfrak{g} = \mathfrak{sp}(n, 1)$ and $\mathfrak{k} = \mathfrak{sp}(n) \times \mathfrak{sp}(1)$ are both of rank $n + 1$,
- in the octonionic case, $\mathfrak{g} = \mathfrak{f}_{4(-20)}$ and $\mathfrak{k} = \mathfrak{so}(9)$ are both of rank 4. I wish to thank Fokko du Cloux, Jérôme Germoni and Bruno Sévennec for explaning this case to me.

□

We can now prove the following.

Proposition 4. *If $M = F \times \mathbb{R}^{k-1}$ where F is a rank 1 symmetric space of noncompact type, then M admits no weak differentiable Hadamard compactification. In particular, M admits no differentiable Hadamard compactification.*

Proof. Suppose there is such a differentiable structure on $\overline{M}$.

We identify an isometry g of F with the isometry $g \times \mathrm{Id}$ of M. We denote the component of identity of the group of isometries of F by G_0^F and consider it a subgroup of G_0.

We first suppose that F is of even dimension.

Let s_x and s_y be the geodesic symmetries around two different points x and y of F. Let d be a geodesic of $\mathbb{R}^{k-1}$. Then $\gamma_1 = \{x\} \times d$ and $\gamma_2 = \{y\} \times d$ are asymptotic and $z = \gamma_1(+\infty) = \gamma_2(+\infty)$ is a singular point of index 1 of $M(\infty)$.

Differentiation $g \mapsto dg(z)$ gives us a linear representation of G_0^F in $T_z\overline{M}$. This representation is reducible since $T_zM(\infty)$ is an invariant subspace. Let ρ be the representation induced on $T_zM(\infty)$.

We now decompose the representation ρ. For all $t \in \mathbb{R}$, the eigenvalues of $ds_x(\gamma_1(t))$ are 1 with multiplicity $k-1$ and -1 with multiplicity $\dim F$. Thus, $\rho(s_x)$ must have eigenvalues 1 with multiplicity $k - 2$ and -1 with multiplicity $\dim F$ (an eigendirection transverse to the boundary must have a nonnegative eigenvalue).

We shall decompose ρ using the following lemma.

Lemma 1. *The panel P of $\gamma_1(\infty)$ is a differentiable submanifold of $M(\infty)$.*

Proof. The panel P is pointwise fixed by all isometries of F. Thus it is pointwise fixed by s_x. In a local chart, it is defined by $p \in P \Rightarrow s_x(p) - p = 0$. Since $s_x - \mathrm{Id}$ has rank $\dim F$ and $\dim M(\infty) = \dim F + k - 2$, the inverse function theorem implies that P is contained in a differentiable submanifold of $M(\infty)$ of dimension $k-2$ (namely the set of fixed points of s_x). But, as a panel in the boundary of a rank k symmetric space, it is an open topological manifold of dimension $k-2$. Thus P is a differentiable submanifold of dimension $k-2$ of $M(\infty)$. □

The tangent space T_zP is an invariant subspace of ρ. Thus, this representation splits in two parts : $\rho = \rho_0 \oplus \rho_1$ where ρ_0 is the trivial representation of dimension $k-2$ and ρ_1 is a representation of dimension $\dim F$.

Now we have a representation ρ_1 (which, as G_0^F is simple, must be faithful or trivial) with $\rho_1(s_x) = -\mathrm{Id}$. So ρ_1 cannot be trivial. But $ds_y(\gamma_2(t))$ has the same eigenvalues as $ds_x(\gamma_1(t))$, and thus $\rho_1(s_y) = -\mathrm{Id}$ too. Now we have $\rho_1(s_x s_y) = \mathrm{Id}$ with $s_x s_y$ a hyperbolic transformation, so ρ_1 cannot be faithful, a contradiction.

Suppose now that F is of odd dimension.

We have $F = \mathbb{H}^{2m+1}$, the real hyperbolic space. The geodesic symmetries are not in G_0 (their determinant is -1) and we shall use Proposition 1.

Let A be a maximal flat. Then A is the product of a geodesic γ of F by $\mathbb{R}^{k-1}$. Let $r \in G_0^F$ be the rotation of angle π around γ in F; A is the set of fixed points of r, and $A(\infty)$ is the set of fixed points of r in $M(\infty)$. Since r is an involution, $A(\infty)$ is a submanifold of $M(\infty)$, a contradiction to Proposition 1.

We give an alternative proof for the odd dimension case, less elegant but more useful for the proof of Theorem 1. We can define the representation ρ_1 like in the even dimension case. Then ρ_1 is a representation of dimension $\dim F$ of G_0^F. Since, for all $x \in M$ fixed by r, $dr(x)$ has eigenvalues -1 with multiplicity $2m$ and 1 with multiplicity k, $\rho_1(r)$ has eigenvalues -1 with multiplicity $2m$ and 1 with multiplicity 1. Thus $\rho_1(r) \neq \mathrm{Id}$, hence ρ_1 is not trivial.

But $G_0^F = \mathrm{SO}_0(2m+1, 1)$ admits no non trivial representation of dimension less than $2m+2$, a contradiction. □

5. Proof of Theorem 1

We shall now prove Theorem 1 with the same ideas that we used for the previous propositions.

Let M be a noneuclidean symmetric space of nonpositive curvature of dimension n and rank $k > 1$. As before, G is the group of all isometries of M, G_0 is the identity component of G and $\mathfrak{g}$ is the Lie algebra of G.

Suppose that there exists a differentiable Hadamard compactification $\mathcal{D}$ of M.

We denote by α the action of G on $\overline{M}$. We also denote by α the corresponding action of $\mathfrak{g}$.

The first step is to find in M an embedded product $F \times \mathbb{R}^{k-1}$.

Let γ be a singular geodesic of index 1. Let F_γ be the union of geodesics parallel to γ (recall that parallel means that they are both positively and negatively asymptotic). Then, F_γ is a totally geodesic submanifold of M isometric to a product $F \times \mathbb{R}^{k-1}$ where F is a symmetric space of rank 1 (see [5] Section 2.11).

Let $\overline{F}_\gamma$ be the closure of F_γ in $\overline{M}$ and $F_\gamma(\infty) = \overline{F}_\gamma \cap M(\infty)$.

It would be interesting to prove that $\overline{F}_\gamma$ is a submanifold of $\overline{M}$, since we could directly use Proposition 4 to get a contradiction, but a weaker statement (namely Lemma 7) will be sufficient.

Up to a change of parametrization we can write

$$\gamma(t) = (p, (t, 0, \dots, 0))$$

where $p \in F$, and $F^t = F \times \{(t, 0 \dots, 0)\}$ is identified with its embedding into M.

Since F, identified with F^0, is a totally geodesic submanifold of M, the Lie algebra of the group G^F of isometries of F is a subalgebra of $\mathfrak{g}$ and the identity component G_0^F of G^F is a subgroup of G_0 (that's why we needed the stronger statement in Proposition 4). Thus taking derivatives gives us a representation ρ of G_0^F on $T_{\gamma(\infty)}\overline{M}$.

Let $\mathfrak{k}^t \oplus \mathfrak{p}^t$ be the Cartan decomposition of $\mathfrak{g}$ at $\gamma(t)$. Since F_γ is a totally geodesic submanifold we have a further decomposition $\mathfrak{p}^t = \mathfrak{p}^t_F \oplus \mathfrak{p}^t_{eucl} \oplus \mathfrak{p}^t_0$ where the terms are pairwise orthogonal (with respect to the Killing form), $\mathfrak{p}^t_F$ is mapped by α onto T_pF^t, $\mathfrak{p}^t_{eucl}$ is mapped onto $T_p\mathbb{R}^{k-1}$ and $\mathfrak{p}^t_0$ is mapped onto $(T_pF_\gamma)^\perp$. We define $\mathfrak{p}^t_\gamma = \mathfrak{p}^t_F \oplus \mathfrak{p}^t_{eucl}$; α maps $\mathfrak{p}^t_\gamma$ onto $T_{\gamma(t)}F_\gamma$. Moreover, $\mathfrak{p}^t_\gamma$ is the set of all Killing fields in $\mathfrak{p}$ commuting with that Killing field $X \in \mathfrak{p}^t$ such that $\alpha(X)$ is the unit tangent vector of γ.

We shall split ρ in three parts in correspondence with the splitting $\mathfrak{p}^t = \mathfrak{p}^t_F \oplus \mathfrak{p}^t_{eucl} \oplus \mathfrak{p}^t_0$. To achieve this, we use the following stability result.

Lemma 2. *Let K be any compact group and $(\mu_t)_{t\in\mathbb{R}}$ be a continuous family of linear representations of K on some finite-dimensional real vector space V. Then for all pairs (t_1, t_2) of real numbers, the representations μ_{t_1} and μ_{t_2} are conjugate.*

Proof. As K is compact and V is finite-dimensional, the conjugacy class of a representation μ_t is determined by its character. More precisely, the multiplicity in μ_t of some irreducible representation ν is given by the scalar product of the characters of μ_t and ν, hence is a continuous map. This multiplicity is an integer and is thus constant. □

Lemma 3. *The tangent subspaces $T_{\gamma(t)}F^t$, $T_{\gamma(t)}\mathbb{R}^{k-1}$ and $(T_{\gamma(t)}F_\gamma)^\perp$ admit limits when $t \to \infty$, denoted respectively by V_F, V_{eucl} and V_0. Moreover one has $T_{\gamma(\infty)}\overline{M} = V_F \oplus V_{eucl} \oplus V_0$.*

Proof. Let K_0^F be the isotropy group of p in F. Taking derivatives gives linear representations ρ^t of K_0^F in $T_{\gamma(t)}M$ for all $t \in \mathbb{R}$. Then $T_{\gamma(t)}F^t$, $T_{\gamma(t)}\mathbb{R}^{k-1}$ and $(T_{\gamma(t)}F_\gamma)^\perp$ are invariant spaces of ρ_t.

By continuity, the restriction ρ^∞ of ρ to K_0^F splits into three parts and the conclusion holds. □

From Lemma 2 we deduce that the action of ρ^∞ is conjugate with that of ρ^0. Since ρ^t acts trivially on $T_{\gamma(t)}\mathbb{R}^{k-1}$, ρ^∞ acts trivially on V_{eucl}.

We shall now prove that V_F is an invariant subspace for ρ.

Lemma 4. *Let $\mathcal{O}$ be the orbit of $\gamma(\infty)$ under the action of G_0. Then $V_0 = T_{\gamma(\infty)}\mathcal{O}$ and, for all t, the restriction of $\alpha_{\gamma(\infty)}$ to $\mathfrak{p}_0^t$ is one-to-one and onto V_0.*

Proof. As an orbit, $\mathcal{O}$ is a submanifold of $M(\infty)$, invariant under the action of G_0, thus $T_{\gamma(\infty)}\mathcal{O}$ is an invariant space of ρ.

The definition of $\mathcal{O}$ shows that α sends $\mathfrak{p}_0^t$ on $T_{\gamma(\infty)}\mathcal{O}$. We want to prove that this map is one-to-one and onto.

Let H be an element of $\mathfrak{p}_0^t$. By definition, $\alpha(\exp(H))(\gamma)$ is not parallel to γ. If $\alpha(H)_{\gamma(\infty)} = 0$, then $\alpha(H)_{\gamma(-\infty)} \neq 0$. But after conjugacy by the geodesic symmetry at $\gamma(t)$ we find $\alpha(-H)_{\gamma(-\infty)} = 0$, a contradiction. Thus the restriction of $\alpha_{\gamma(\infty)}$ to $\mathfrak{p}_0^t$ is one-to-one.

Since $\dim \mathfrak{p}_0^t = n - \dim F_\gamma = (n-1) - (\dim F_\gamma - 1) = \dim \mathcal{O}$, it is onto and $V_0 = T_{\gamma(\infty)}\mathcal{O}$. □

Lemma 5. *The subspace V_F contains no subspace where ρ is trivial.*

Let X be that vector of $\mathfrak{p}_{eucl}^0$ such that $\gamma'(t) = \alpha(X)_{\gamma(t)}$. The linear operator $d_{\gamma(\infty)}\alpha(\exp X) - \mathrm{Id}$ acting on V_0 is of maximal rank.

Proof. The linear action ρ^∞ of K_0^F on $T_{\gamma(\infty)}M(\infty)$ is transitive on V_F and ρ^∞ is the restriction of ρ to K_0^F thus V_F contains no trivial part.

To prove the second part of the lemma, we use the root space decomposition $\mathfrak{g} = \mathfrak{g}_0 + \sum \mathfrak{g}_\lambda$ given by some maximal flat containing γ. We have

$$\mathfrak{p}_0^0 \subseteq \sum_{\lambda(X)\neq 0} \mathfrak{g}_\lambda$$

and $Ad(\exp X) = e^{\lambda(X)} I$ on $\mathfrak{g}_\lambda$.

Since α is onto from $\mathfrak{p}_0^0$ to V_0, it is onto from $\sum_{\lambda(X)\neq 0} \mathfrak{g}_\lambda$ to V_0.

For all $H \in \mathfrak{g}_\lambda$, we have

$$d_{\gamma(\infty)}\alpha(\exp X)(\alpha(H)) = \alpha(Ad(\exp(X))(H)) = e^{\lambda(X)}\alpha(H),$$

and thus $d_{\gamma(\infty)}\alpha(\exp X) - \mathrm{Id}$ is nondegenerate on V_0. □

Lemma 6. *The panel P of $\gamma(\infty)$ is a submanifold of $M(\infty)$ and its tangent space at $\gamma(\infty)$ equals $V_{eucl} \cap T_{\gamma(\infty)}M(\infty)$.*

Proof. The panel P is contained in the set of the points of $M(\infty)$ left fixed by the actions of G and of $\exp(X)$. Written in local coordinates, this gives us an infinite system of equations. By Lemma 5 we know that this system is of rank at least $d = \dim V_0 + \dim V_F$ at $\gamma(\infty)$. We can extract a subsystem of d equations that is of maximal rank at $\gamma(\infty)$. The inverse function theorem implies that this subsystem defines a submanifold of $M(\infty)$ of dimension $k-2 = \dim V_{eucl} \cap T_{\gamma(\infty)}M(\infty)$ and containing P. But P is topologically a manifold of dimension $k-2$ and thus must be a differentiable submanifold of $M(\infty)$.

Since K_0^F acts trivially on P, its tangent space must be $V_{eucl} \cap T_{\gamma(\infty)}M(\infty)$. □

Since G acts trivially on P, it must preserve its tangent space. We are now ready to prove the following.

Lemma 7. *The subspace V_F is invariant by ρ.*

Proof. From previous lemmas we know that V_0 and $V_{eucl} \cap T_{\gamma(\infty)}M(\infty)$ are invariant subspaces for ρ. Since ρ is totally reducible, there exists some subspace V' invariant by ρ such that one has the following decomposition $T_{\gamma(\infty)}M(\infty) = V' \oplus V_{eucl} \cap T_{\gamma(\infty)}M(\infty) \oplus V_0$. But V' must be invariant by ρ^∞, and thus $V' = V_F$ and V_F is invariant by ρ. □

Denote by ρ_1 the representation of G_0^F induced by ρ on V_F.

Since the restriction of ρ_1 to K_0^F is the limit of the restriction of ρ^t to $T_{\gamma(t)}F^t$, we can now use the same arguments as in the proof of Proposition 4.

If F is of even dimension, ρ_1 is neither trivial nor faithful, a contradiction.

If F is of odd dimension, ρ_1 is a nontrivial representation of dimension $\dim F$ of G_0^F, a contradiction.

Theorem 1 is proved. Note that we actually get something stronger: there exists no *weak* differentiable Hadamard compactification of M; the obstructions to differentiability appear in the identity component of G.

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