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A model of influence with a continuum of actions*

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Abstract. In the paper, we generalize a two-action (yes-no) model of influence to a framework in which every player has a continuum of actions and he has to choose one of them. We assume the set of actions to be an interval. Each player has an inclination to choose one of the actions. Due to influence among players, the final decision of a player, i.e., his choice of one action, may be different from his original inclination. In particular, a coalition of players with the same inclination may influence another player with different inclination, and as a result of this influence, the decision of the player is closer to the inclination of the influencing coalition than his inclination was. We introduce and study a measure of such a positive influence of a coalition on a player. Several unanimous influence functions in this generalized framework are considered. Moreover, we investigate other tools for analyzing influence, like the concept of a follower of a given coalition, its particular case - a perfect follower, and the kernel of an influence function. We study properties of these concepts. Also the set of fixed points under a given influence function is analyzed. Furthermore, we study linear influence functions. We also introduce a measure of a negative influence of a coalition on a player.

JEL Classification: C7, D7

Keywords: action, decision, influence index, unanimous influence function, follower of a coalition, kernel, fixed point, linear influence function

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1 Introduction

1.1 Short overview of the related literature

In the voting literature, usually binary voting is assumed, for instance, casting 'yes' or 'no' vote on a particular proposal or voting for one of two available candidates. However, one may find several works that extend the number of options in voting situations. For instance, voting systems with abstention (as a third option together with 'yes' and 'no' votes) are studied e.g. in Braham and Steffen (2002, [6]), Felsenthal and Machover (1997, 1998, 2001, [8–10]), Fishburn (1973, [11]). Works on voting systems with several levels of approval in the input and output can be found in Freixas (2005a, 2005b, [12, 13]), Freixas and Zwicker (2003, [14]). In particular, standard simple games and games with abstention can be easily defined in such an extended framework. Also in Hsiao and Raghavan (1993,

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[21]) multi-choice games are studied, and in Bolger (1986, 1993, 2000, 2002, [2–5]) games with n players and r alternatives are analyzed.

Considering a continuum of alternatives or choices in models of game theory and social choice theory is interesting both from a theoretical and an application point of view, since many real life situations can be modeled in the framework with a continuum of actions or options. However, not that many works so far, in particular in the voting literature, concern models with a continuum of alternatives. One of such examples is presented in Chang and Stauber (2006, [7]), where the authors extend a costly voting model with binary voting to a model in which a choice has to be made from a continuum of alternatives. The agents have to decide whether or not to participate in a collective decision-making process that determines the level of a public good. There are costs associated with the act of voting and the implementation of the voting outcome. In the social choice literature, models with a continuum of alternatives are studied, e.g., in Abdou (1988, [1]), Huang (2004, 2009, [24, 25]).

One of the concepts naturally related to voting situations is the concept of influence. In the original two-action model of influence introduced in Hoede and Bakker (1982, [20]) and later studied, e.g., in Grabisch and Rusinowska (2009b, [16]), each player (also referred to as agent or actor) has to make an acceptance-rejection decision, and he has an *inclination* to vote either 'yes' or 'no'. The inclination of a player is defined as a decision the player would make if he were to decide completely on his own. However, an agent may experience *influence* of other players, and as a consequence of such an influence, his *decision* may be different from his original inclination. Such an influence between players is defined in a broad sense and may cover many possible situations: forcing by players that are higher in a hierarchical structure, following authorities, learning and observing the behavior of others, etc. The transformation of the inclinations into the decisions is represented by an *influence function*. In Grabisch and Rusinowska (2009b, [16]), we define and investigate weighted influence indices that measure influence between players in the yes-no model. We consider several special cases of the weighted influence indices, like the possibility influence index and the certainty influence index, and we analyze two kinds of influence, i.e., a direct influence and an opposite influence. Under the direct influence of a coalition on a player, while the inclination of the player is different from the inclination of the coalition, his decision is the same as the inclination of the coalition in question. The opposite influence of a coalition on a player means that the inclinations of the player and the coalition coincide, but the player's vote is different from this inclination. In Grabisch and Rusinowska (2009a, [15]) we compare our yes-no model of influence with the framework of command games introduced by Hu and Shapley (2003a, 2003b, [22, 23]). One of the central concepts of the command games is the concept of command function. In Grabisch and Rusinowska (2010a, [17]) we study, e.g., the relation between influence function and follower function, and the relation between command games and influence functions. In Grabisch and Rusinowska (2010b, [18]) we enlarge the set of possible yes-no actions of the influence model to a multi-action framework in which every player has a totally ordered set of possible actions, the same for each player, and he has an inclination to choose a particular action. We investigate the generalized influence indices and other tools related to the influence in the multi-choice model. First, we analyze a positive influence of a coalition on a player, which measures how much a decision of the player with inclination different from the inclination of the given coalition is closer to the inclination of the coalition than his inclination was. A direct influence in the yes-no model is a particular case of the positive influence. We also investigate a negative influence of a coalition on a player, which measures how much a decision of the player with inclination different from extreme (farthest) actions of the coalition, comes 'closer' to the extreme action. An opposite influence in the yes-no model is a particular case of the negative influence.

1.2 Aims and a summary of the present paper

The present paper is a continuation of our research on the influence indices defined for the yes-no model and for the multi-action framework of influence. We consider now a generalization of the influence model in a social network and assume that each player has a *continuum* of actions to choose, and he has an inclination to choose one of them. The aim of the present paper is therefore to define the influence concepts for the continuum case and to verify if, and if so how, our previous results obtained in the multi-choice framework change when we switch to the continuum case. Similar as in our former investigations on influence, we are interested in situations where a decision of a player (the final choice of one action) is different from his (preliminary) inclination. To be more precise, we define a *positive influence index* of a coalition on a player for the continuum case. This index measures how much the player chooses an action which is closer to the inclination of the coalition than his inclination was. We also define a *negative influence index* for the framework with a continuum of actions. The negative influence of a coalition on a player means that while the inclination of the player is different from the extreme action(s) of the coalition (that is, the action(s) lying as far as possible from the inclination of the coalition), the player's decision comes 'closer' to such an extreme action and goes farther from the inclination of the coalition. Furthermore, we investigate several useful tools for studying influence. In particular, we consider the concept of *followers*, where by follower of a given coalition of players we mean the agent whose decision is always closer to the inclination of the coalition in question than his inclination was. A particular case of follower is the *perfect follower* of a coalition, that is, a player who always decides according to the inclination of the coalition in question. We also define several *unanimous influence functions* for the continuum case, like the majority function, the guru function, the identity function, and the mass psychology function, and we study their properties. Under such influence functions, unanimously inclined players always decide according to their common inclination. If this holds only for the extreme actions ($\mathbf{0}$ and $\mathbf{1}$ for the model with the set of actions $[0, 1]$), such an influence function is called *unanimous on the boundaries*. We compare the results on followers and the influence functions defined for the model with a continuum of actions with the results on followers and the analogous influence functions introduced in the multi-choice model of influence. Moreover, we look at the set of *fixed points* under a given influence function, i.e., the set of the inclination vectors that coincide with the decision vectors resulted from these inclinations. Also *linear influence functions*, i.e., functions that can be written as a matrix, are studied. For a function unanimous on the boundary $\mathbf{1}$ (for the set of actions $[0, 1]$), we find necessary and sufficient conditions for the existence of a positive influence of a coalition on a player, and we calculate the value of the influence index.

The structure of the paper is the following. In Section 2, we define the positive influence index for the model with a continuum of actions. In order to illustrate clearly the concepts introduced in the paper, in Sections 3.1 and 3.2 we present examples of a two-agent social network and an analogous example of a three-player network, respectively. Next, other tools for studying the influence are described. The concepts of follower and perfect follower of a coalition, the kernel, and the concept of purely influential function are studied in Section 4. In Section 5, we investigate several unanimous influence functions and study their properties. Sections 6 and 7 concern fixed points of an influence function and linear influence functions, respectively. Section 8 is devoted to the concept of negative influence. In Section 9, we present some concluding remarks.

2 The positive influence index in the generalized framework

We consider a social network with the set of players (agents, actors) denoted by $N = \{1, ..., n\}$. Each player has a continuum of actions to choose, i.e., we assume that the set of actions is an interval $[a, b]$, where $a, b \in \mathbb{R}$ and $a < b$. Each player has an *inclination* to choose one of the actions, i.e., by the inclination of a player we mean the particular action from $[a, b]$ the player wants to choose. Let i_m denote the inclination of player $m \in N$, i.e., $i_m \in [a, b]$. Let $I = [a, b]^n$ denote the set of all n-inclination vectors $\mathbf{i} = (i_1, ..., i_n) \in [a, b]^n$.

It is assumed that players may influence each other, and due to the influence in the network, the final decision of a player may be different from his original inclination. In other words, each inclination vector $\mathbf{i} \in I$ is transformed into a *decision vector* $B(\mathbf{i})$, where $B : I \to I, \mathbf{i} \mapsto B(\mathbf{i})$ is the *influence function*. The decision vector $B(\mathbf{i})$ is an n-vector consisting of the decisions made by all the players, i.e., the actions chosen by the players. Our only requirement on the function B is that it should be continuous, up to a countable number of points. The set of all such influence functions with N will be denoted by $\mathcal{B}$.

A coalition $\emptyset \neq S \subset N$ of players is assumed to be able to influence an outside agent $j \notin S$ *only* if all members of that coalition have the same inclination. In this case we say that the coalition is *unanimous*. Such n-inclination vectors where the coalition S is unanimous form the set

$$I_S := \{\mathbf{i} \in [a, b]^n : \ \forall m, p \in S \ [i_m = i_p]\}. \tag{1}$$

The cardinality of S will be denoted by s. We use the notation $i_S \in [a, b]$ for the inclination of the members of the unanimous coalition S. This suggests to consider the whole coalition S as a single "macro"-player, which we denote by $[S]$. Therefore, there is bijective correspondence between our n-dimensional vector $\mathbf{i} \in I_S$ and the $(n - s + 1)$-dimensional vector $\mathbf{i}^{[S]}$, defined by

$$i_j^{[S]} := i_j, \text{ if } j \notin S, \text{ and } i_{[S]}^{[S]} := i_S. \tag{2}$$

Let for each $\emptyset \neq S \subset N$ and $j \in N \setminus S$

$$I_{S \to j}^{[S]} := \{\mathbf{i}^{[S]} \in [a, b]^{n-s+1} : \ i_j \neq i_S\}. \tag{3}$$

$I_{S \to j}^{[S]}$ denotes the set of all $(n - s + 1)$-inclination vectors of *potential positive influence* of S on j. These are all the $(n - s + 1)$-inclination vectors for which the inclination of player j is different from the inclination of coalition S.

Next, for each $\emptyset \neq S \subset N$, $j \in N \setminus S$, and $B \in \mathcal{B}$, we define the set of all $(n-s+1)$-inclination vectors of *(observed) positive influence of S on j under given B* as

$$I^{*[S]}_{S \to j}(B) = \{\mathbf{i}^{[S]} \in I^{[S]}_{S \to j} : \ |(B(\mathbf{i}))_j - i_S| < |i_j - i_S|\}, \tag{4}$$

where it is understood that the vectors $\mathbf{i}^{[S]}$ and $\mathbf{i}$ in the above formula correspond to each other by (2). The set $I^{*[S]}_{S \to j}(B)$ consists therefore of all the $(n-s+1)$-inclination vectors of potential positive influence of S on j such that the difference between the inclinations of j and of S is greater than the difference between the decision of player j and the inclination of S. Consequently, we identify the cases where player j chooses an action which is closer to the inclination of coalition S than his inclination was.

Definition 1 *Given $B \in \mathcal{B}$, for each $\emptyset \neq S \subset N$, $j \in N \setminus S$, the positive influence index of coalition S on player j under the influence function B is defined as*

$$D(B, S \to j) := \frac{\int_{I^{*[S]}_{S \to j}(B)} [|i_j - i_S| - |(B(\mathbf{i}))_j - i_S|] \ d\mathbf{i}^{[S]}}{\int_{I^{[S]}_{S \to j}} |i_j - i_S| \ d\mathbf{i}^{[S]}}. \tag{5}$$

Note that due to the continuity assumptions on B and its boundedness, it is Riemann-integrable, and therefore $D(B, S \to j)$ exists, and $D(B, S \to j) \in [0, 1]$. Moreover, it corresponds to the possibility positive influence index defined in the model of influence in which every player has a finite totally ordered set of possible actions; see Grabisch and Rusinowska (2010b, [18]).

Lemma 1 *If $I = [a, b]^n$, then for each $\emptyset \neq S \subset N$ and $j \in N \setminus S$*

$$\int_{I^{[S]}_{S \to j}} |i_j - i_S| \ d\mathbf{i}^{[S]} = \frac{(b-a)^{n-s+2}}{3}. \tag{6}$$

Proof:

$$\begin{aligned}
\int_{I^{[S]}_{S \to j}} |i_j - i_S| \ d\mathbf{i}^{[S]} &= 2 \int_a^b \dots \int_a^b \int_a^b \int_a^{i_j} (i_j - i_S) \ di_S \ di_j \ di_{k_1} \dots di_{k_{n-s-1}} \\
&= 2 \int_a^b \dots \int_a^b \int_a^b \left[i_j \cdot i_S - \frac{i_S^2}{2} \right]_{i_S = a}^{i_S = i_j} di_j \ di_{k_1} \dots di_{k_{n-s-1}} \\
&= \frac{1}{3} \int_a^b \dots \int_a^b (b-a)^3 \ di_{k_1} \dots di_{k_{n-s-1}} = \frac{(b-a)^{n-s+2}}{3}
\end{aligned}$$

■

3 Examples

3.1 The two-player examples

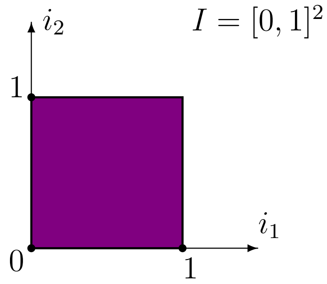
In order to illustrate the concepts introduced in Section 2, we present a very simple 2-player example, i.e., $N = \{1, 2\}$. We skip the upper index $[S]$, because for $n = 2$ and all $\emptyset \neq S \subset N$, we have $N = [S] \cup (N \setminus S)$.

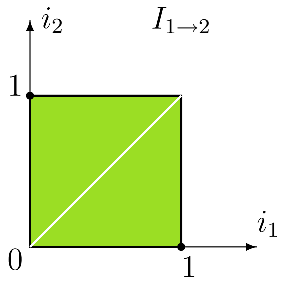
Suppose that a two-member committee has to evaluate a scientific project and decide about the amount of funding for the project. Each of the two referees has to write his report and propose an amount of funding, i.e., to choose a percent (from 0% till 100 %) of the grant demanded by the project coordinator to be assigned to the project. Before preparing the reports, the referees discuss with each other the project. Each referee knows the inclination (opinion) of the another one, and such a pre-decision discussion is a good moment for exercising the influence between the referees.

We can assume that each player has to choose an action from the interval $[0, 1]$, where each action means a fraction of the demanded grant. In particular, the actions 0, 0.5 and 1 denote assigning no grant to the project, giving half of the requested grant, and giving the whole demanded amount, respectively. The set of inclination vectors is then equal to $I = [0, 1]^2$. We consider the positive influence of player 1 on player 2, and therefore

$$I_{1\to 2} = \{(i_1, i_2) \in [0, 1]^2 : i_2 \neq i_1\}.$$

These sets are presented in the figures below. The set of all inclination vectors I consists of all the points (i_1, i_2) of the square, while the set $I_{1\to 2}$ consists of all the points of the square without the diagonal $i_1 = i_2$.





First, let us assume that the influence function B is defined as follows:

$$B(i_1, i_2) = \left(i_1, \frac{i_1 + i_2}{2}\right) \text{ for each } (i_1, i_2) \in [0, 1]^2, \tag{7}$$

which means that player 1 always follows his own inclination, while the decision of player 2 is a middle point between the inclinations of the two players. Consequently, if their inclinations are not equal, then the decision of player 2 is closer to the inclination of player 1 than his inclination was. By virtue of (6), we have:

$$\int_{I_{1\to 2}} |i_2 - i_1| \; d\mathbf{i} = \frac{1}{3}.$$

$$I^*_{1\to 2}(B) = \{\mathbf{i} \in I_{1\to 2} : \left|\frac{i_1 + i_2}{2} - i_1\right| < |i_2 - i_1|\} = I_{1\to 2}.$$

If we calculate the positive influence index $D(B, 1 \to 2)$ of player 1 on player 2 under the influence function B, then from (5) and (7) we have

$$\int_{I^*_{1\to 2}(B)} [|i_2 - i_1| - |(B(\mathbf{i}))_2 - i_1|] \; d\mathbf{i} = \frac{1}{2} \cdot \int_{I_{1\to 2}} |i_2 - i_1| \; d\mathbf{i}$$

and therefore

$$D(B, 1 \to 2) = \frac{1}{2}.$$

Let us consider now a negative influence and assume that the decision of player 2 is as extreme with respect to the inclination of player 1 as possible, that is,

$$(B'(i_1, i_2))_1 = i_1, \quad (B'(i_1, i_2))_2 = \begin{cases} 0 & \text{if } i_1 \geq \frac{1}{2} \\ 1 & \text{if } 0 \leq i_1 < \frac{1}{2} \end{cases} \quad \text{for each } (i_1, i_2) \in [0,1]^2. \tag{8}$$

In this case, $I^*_{1\to 2}(B') = \emptyset$, and therefore $D(B', 1 \to 2) = 0$.

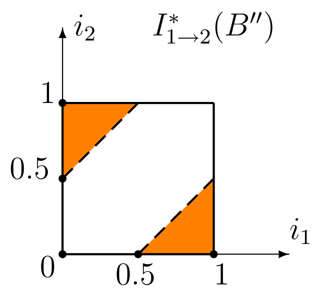
Another influence function which we study is defined as follows:

$$(B''(i_1, i_2))_1 = i_1, \quad (B''(i_1, i_2))_2 = \begin{cases} i_2 & \text{if } |i_2 - i_1| \leq \frac{1}{2} \\ i_1 + \frac{1}{2} & \text{if } i_2 > i_1 + \frac{1}{2} \\ i_1 - \frac{1}{2} & \text{if } i_2 < i_1 - \frac{1}{2} \end{cases} \quad \text{for each } (i_1, i_2) \in [0,1]^2. \tag{9}$$

According to (9), if the difference between the inclinations of players 1 and 2 is not greater than $\frac{1}{2}$, then player 2 decides according to his own inclination, otherwise the decision of player 2 is of an exact distance $\frac{1}{2}$ from the inclination of player 1. It is a kind of a partial influence meaning that player 2 likes to decide according to his own inclination if his inclination is not that far from the inclination of player 1, and he 'adjusts' his inclination if there is a rather serious difference between the inclinations of both players. We have

$$\begin{aligned} I^*_{1\to 2}(B'') &= \{(i_1, i_2) \in I_{1\to 2} : \ |(B''(\mathbf{i}))_2 - i_1| < |i_2 - i_1|\} \\ &= \left\{(i_1, i_2) \in [0,1]^2 : \ i_2 > i_1 + \frac{1}{2}\right\} \cup \left\{(i_1, i_2) \in [0,1]^2 : \ i_2 < i_1 - \frac{1}{2}\right\}. \end{aligned}$$

The figure below presents the set $I^*_{1\to 2}(B'')$ with the influence function B'' defined in (9), and it consists of all the points (i_1, i_2) of the two triangles without the lines $i_2 = i_1 + \frac{1}{2}$ and $i_2 = i_1 - \frac{1}{2}$.



Let us calculate the positive influence index $D(B'', 1 \to 2)$ of player 1 on player 2 under the influence function B''.

$$\int_{I^*_{1\to 2}(B'')} [|i_2 - i_1| - |(B''(\mathbf{i}))_2 - i_1|] \ d\mathbf{i} =$$

$$= \int_0^{\frac{1}{2}} \int_{i_1+\frac{1}{2}}^{1} (i_2 - i_1 - \frac{1}{2})\ di_2\ di_1 + \int_{\frac{1}{2}}^{1} \int_0^{i_1-\frac{1}{2}} (i_1 - i_2 - \frac{1}{2})\ di_2\ di_1 = \frac{1}{24}.$$

Hence,

$$D(B'', 1 \to 2) = \frac{1}{8}.$$

Finally, we analyze the following family of influence functions:

$$B^x(i_1, i_2) = (i_1, x) \text{ for each } (i_1, i_2) \in [0,1]^2, \tag{10}$$

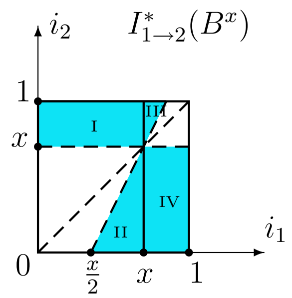
where $x \in [0,1]$. This means that the first player always decides according to his own inclination, and the second player always chooses the action x. There can be several reasons for choosing always the same action x. For instance, a certain fixed percent of funding is coherent with a policy of the given foundation, or the experience shows that a certain percent usually represents well real needs to cover expenses of an average project within the given framework, etc. In this case, we cannot really speak of the influence of player 1 on player 2, but rather of a kind of a 'luck' rate of player 1, meaning that player 2 makes a decision which is closer to the inclination of player 1 than his inclination was. Nevertheless, we like to calculate the influence index for this example, because such a 'constant decision' is not that rare in real life situations. Moreover, we like to show which decision of a referee makes the other referee 'most lucky'. In this example, we have

$$I^*_{1\to 2}(B^x) = \{\mathbf{i} \in I_{1\to 2} : |x - i_1| < |i_2 - i_1|\}.$$

We can distinguish four possible cases when solving the inequality $|x - i_1| < |i_2 - i_1|$ in the set $I^*_{1\to 2}(B^x)$:

- Case I: If $i_1 \leq x$ and $i_2 > i_1$, then $i_2 > x$;
- Case II: If $i_1 \leq x$ and $i_2 < i_1$, then $i_2 < 2i_1 - x$;
- Case III: If $i_1 > x$ and $i_2 > i_1$, then $i_2 > 2i_1 - x$;
- Case IV: If $i_1 > x$ and $i_2 < i_1$, then $i_2 < x$.

The figure below presents the set $I^*_{1\to 2}(B^x)$ with the influence function B^x defined in (10). This set $I^*_{1\to 2}(B^x)$ consists of four subsets given by the cases listed above.



The subsets of $I^*_{1\to 2}(B^x)$ corresponding to the four cases are the following:

$$I^I_{1\to 2}(B^x) = \{(i_1, i_2) \in [0,1]^2 : \ [0 \leq i_1 \leq x \ \wedge \ x < i_2 \geq 1]\}$$

$$I^{II}_{1\to 2}(B^x) = \left\{(i_1, i_2) \in [0,1]^2 : \ \left[\frac{x}{2} < i_1 \leq x \ \wedge \ 0 \leq i_2 < 2i_1 - x\right]\right\}$$

$$I^{III}_{1\to 2}(B^x) = \left\{(i_1,i_2)\in[0,1]^2 :\ \left[x < i_1 < \frac{1+x}{2}\ \wedge\ 2i_1 - x < i_2 \le 1\right]\right\}$$
$$I^{IV}_{1\to 2}(B^x) = \left\{(i_1,i_2)\in[0,1]^2 :\ [x < i_1 \le 1\ \wedge\ 0 \le i_2 < x]\right\}.$$

Let us calculate the positive influence index $D(B^x, 1\to 2)$ of player 1 on player 2 under the influence function B^x.

$$\begin{aligned} D(B^x, 1\to 2) &= \frac{\int_{I^*_{1\to 2}(B^x)} [|i_2 - i_1| - |(B^x(\mathbf{i}))_2 - i_1|]\ d\mathbf{i}}{\int_{I_{1\to 2}} |i_2 - i_1|\ d\mathbf{i}} \\ &= 3 \sum_{L\in\{I,II,III,IV\}} \int_{I^L_{1\to 2}(B^x)} [|i_2 - i_1| - |x - i_1|]\ d\mathbf{i} \end{aligned}$$

$$\begin{aligned} \int_{I^I_{1\to 2}(B^x)} [|i_2 - i_1| - |x - i_1|]\ d\mathbf{i} &= \int_0^x \int_x^1 (i_2 - x)\ di_2\ di_1 = \int_0^x \left[\frac{i_2^2}{2} - x\cdot i_2\right]_{i_2=x}^{i_2=1} di_1 \\ &= \int_0^x \left(\frac{1}{2} - x + \frac{1}{2}x^2\right)\ di_1 = \frac{1}{2}x(x-1)^2 \end{aligned}$$

$$\begin{aligned} \int_{I^{II}_{1\to 2}(B^x)} [|i_2 - i_1| - |x - i_1|]\ d\mathbf{i} &= \int_{\frac{x}{2}}^x \int_0^{2i_1 - x} (2i_1 - i_2 - x)\ di_2\ di_1 \\ &= \int_{\frac{x}{2}}^x \left[2i_1\cdot i_2 - \frac{i_2^2}{2} - x\cdot i_2\right]_{i_2=0}^{i_2=2i_1-x} di_1 \\ &= 2\int_{\frac{x}{2}}^x \left(i_1 - \frac{x}{2}\right)^2\ di_1 = \frac{1}{12}x^3 \end{aligned}$$

$$\begin{aligned} \int_{I^{III}_{1\to 2}(B^x)} [|i_2 - i_1| - |x - i_1|]\ d\mathbf{i} &= \int_x^{\frac{1+x}{2}} \int_{2i_1 - x}^1 (i_2 - 2i_1 + x)\ di_2\ di_1 \\ &= \int_x^{\frac{1+x}{2}} \left[\frac{i_2^2}{2} - 2i_1\cdot i_2 + x\cdot i_2\right]_{i_2=2i_1-x}^{i_2=1} di_1 = \frac{1}{12}(1-x)^3 \end{aligned}$$

$$\begin{aligned} \int_{I^{IV}_{1\to 2}(B^x)} [|i_2 - i_1| - |x - i_1|]\ d\mathbf{i} &= \int_x^1 \int_0^x (x - i_2)\ di_2\ di_1 \\ &= \int_x^1 \left(x^2 - \frac{x^2}{2}\right)\ di_1 = \frac{1}{2}x^2(1-x). \end{aligned}$$

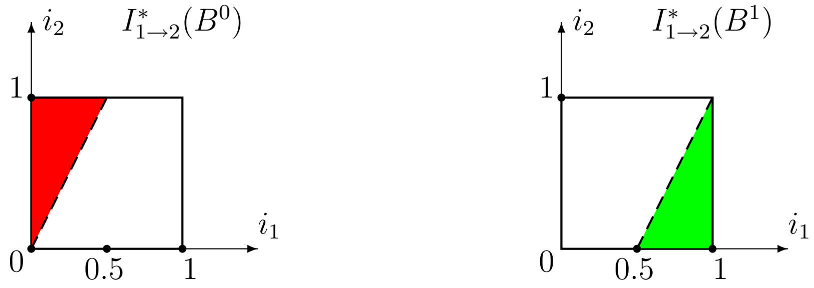
Hence,

$$\begin{aligned} D(B^x, 1\to 2) &= 3\left[\frac{1}{2}x(x-1)^2 + \frac{1}{12}x^3 + \frac{1}{12}(1-x)^3 + \frac{1}{2}x^2(1-x)\right] \\ &= \frac{1}{4}(1 + 3x - 3x^2). \end{aligned}$$

Note that, for instance, for $x = 0$ the analysis is reduced to the case III, while for $x = 1$ we have the case II, and

$$D(B^0, 1 \to 2) = D(B^1, 1 \to 2) = \frac{1}{4}.$$

The figures below present $I^*_{1\to 2}(B^0)$ and $I^*_{1\to 2}(B^1)$.



Note that

$$\frac{1}{4} \le D(B^x, 1 \to 2) \le \frac{7}{16} \text{ for all } x \in [0, 1]$$

and

$$\arg\max_{x\in[0,1]} D(B^x, 1 \to 2) = \frac{1}{2}$$

which means that the positive influence index of player 1 on player 2 is maximal if player 2 proposes to assign 50% of the requested budget to the evaluated project, no matter what the inclinations of both referees are.

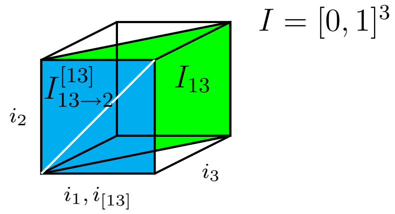
3.2 The three-player example

Let us consider a three-player example which is similar to the 2-player example presented in Section 3.1, except that the committee evaluating a scientific project consists now of 3 referees, that is, $N = \{1, 2, 3\}$. Suppose $S = \{1, 3\}$, $j = 2$, and we measure the influence of coalition S on player j. We omit braces for sets, i.e., in particular, $\{1, 3\}$ will be written 13. We have $I = [0, 1]^3$, that is, $\mathbf{i} = (i_1, i_2, i_3)$, $\mathbf{i}^{[S]} = (i_{[13]}, i_2)$. Moreover,

$$I_{13} = \{(i_1, i_2, i_3) \in [0, 1]^3 : \ i_1 = i_3\}$$

$$I^{[13]}_{13\to 2} = \{(i_{13}, i_2) \in [0, 1]^2 : \ i_2 \neq i_{13}\}$$

The following picture shows the respective sets of influence vectors: I, I_{13}, and $I^{[13]}_{13\to 2}$.



We define the influence function $\widetilde{B} : [0, 1]^3 \to [0, 1]^3$ as follows:

$$\widetilde{B}(i_1, i_2, i_3) = \left(i_1, \frac{i_1 + 2i_2 + i_3}{4}, i_2\right) \text{ for each } (i_1, i_2, i_3) \in [0, 1]^3. \tag{11}$$

Note that the influence function $B : [0,1]^2 \to [0,1]^2$ defined by (7) in Section 3.1 is the corresponding influence function for $\widetilde{B}$, i.e.,

$$B(i_{13}, i_2) = \left(i_{13}, \frac{i_{13} + i_2}{2}\right) \text{ for each } (i_{13}, i_2) \in [0,1]^2.$$

Moreover,

$$I^{*[13]}_{13\to 2}(\widetilde{B}) = \left\{(i_{13}, i_2) \in I^{[13]}_{13\to 2} : \left|\frac{i_{13} + i_2}{2} - i_{13}\right| < |i_2 - i_{13}|\right\} = I^{[13]}_{13\to 2}.$$

We get therefore

$$D(\widetilde{B}, 13 \to 2) = D(B, [13] \to 2) = \frac{1}{2}.$$

It is easy to show that such a relation holds for a general case, i.e., there is the equivalence between the influence index of a coalition on a player in a social network and the analogous influence index in a corresponding network in which the coalition in question is treated as one player.

4 The set of followers and the purely influential functions

Similar as in Grabisch and Rusinowska (2010b, [18]), we define the concepts of follower and perfect follower of a given coalition. A *follower* of a given coalition of agents is a player whose decision is never farther from the inclination of the coalition in question than his inclination was. A player who always decides according to the inclination of the coalition in question is called a *perfect follower* of that coalition.

Definition 2 *Let $\emptyset \neq S \subseteq N$ and $B \in \mathcal{B}$. The* follower function $F_B : 2^N \to 2^N$ *is defined as follows*

$$F_B(S) := \{j \in N : \forall \mathbf{i} \in I_S \; [[i_j \neq i_S \;\Rightarrow\; |(B(\mathbf{i}))_j - i_S| < |i_j - i_S|] \wedge [i_j = i_S \;\Rightarrow\; (B(\mathbf{i}))_j = i_S]]\} \quad (12)$$

where $F_B(S)$ is the set of followers of S under B, and $F_B(\emptyset) := \emptyset$.
The perfect follower function $F^{per}_B(S) : 2^N \to 2^N$ *is given by*

$$F^{per}_B(S) := \{j \in N : \forall \mathbf{i} \in I_S \; [(B(\mathbf{i}))_j = i_S]\}, \quad (13)$$

where $F^{per}_B(S)$ is the set of perfect followers of S under B.

Of course, each perfect follower is also a follower, i.e., for each $B \in \mathcal{B}$ and $S \subseteq N$,

$$F^{per}_B(S) \subseteq F_B(S).$$

The next proposition studies the properties of these functions. The analogous result for an influence model in which players have a finite totally ordered set of actions is given in Grabisch and Rusinowska (2010b, [18]). For the proofs, see [18].

Proposition 1 *Let $B \in \mathcal{B}$. Then the following holds:*

(i) Whenever $S \cap T = \emptyset$, $F_B(S) \cap F_B(T) = \emptyset$.

(ii) F_B *is an isotone function* ($S \subseteq S'$ *implies* $F_B(S) \subseteq F_B(S')$).
Consequently, if $F_B(N) = \emptyset$, *then* $F_B \equiv \emptyset$.

(iii) *For each* $j \in F_B^{per}(S) \setminus S$, $D(B, S \to j) = 1$.

Assume F_B is not identically the empty set. Similarly as in the multi-action model, we define the *kernel* $\mathcal{K}(B)$ of B

$$\mathcal{K}(B) := \{S \in 2^N : F_B(S) \neq \emptyset, \text{ and } S' \subset S \Rightarrow F_B(S') = \emptyset\}.$$

which is the set of "true" influential coalitions, and *a purely influential function* which, in the framework of influence with a continuum of actions, has similar properties as the purely influential function for the model with a finite totally ordered set of actions (see Grabisch and Rusinowska, 2010b, [18]).

Definition 3 *Let* S, T *be two disjoint nonempty subsets of* N. *The influence function* $B \in \mathcal{B}$ *is said to be a* purely influential function of S upon T *if it satisfies for all* $\mathbf{i} \in I_S$:

$$(B(\mathbf{i}))_j = \begin{cases} i_S & \text{if } j \in T \\ i_j & \text{otherwise.} \end{cases} \tag{14}$$

The set of such functions is denoted by $\mathcal{B}_{S \to T}$. *The* canonical purely influential function of S upon T *satisfies in addition* $B(\mathbf{i}) = \mathbf{i}$ *for each* $\mathbf{i} \in I \setminus I_S$. *This function is denoted by* $B_{S \to T}$.

Proposition 2 *Let* S, T *be two disjoint nonempty subsets of* N. *Then the following holds:*

(i) *For all* $B \in \mathcal{B}_{S \to T}$, $F_B(S) \supseteq S \cup T$, *and* $F_B^{per}(S) = S \cup T$.

(ii) *For each* $B \in \mathcal{B}_{S \to T}$ *and* $j \in N \setminus S$,

$$D(B, S \to j) = \begin{cases} 1 & \text{if } j \in T \\ 0 & \text{if } j \in N \setminus (S \cup T). \end{cases} \tag{15}$$

Example 1 For the example presented in Section 3.1, we have

$$F_B(1) = F_B(N) = N, \; F_B(2) = \emptyset, \; F_B^{per}(1) = F_B^{per}(N) = \{1\}, \; \mathcal{K}(B) = \{1\}.$$

Moreover, for each $\hat{B} \in \{B', B'', B^x\}$, and for each $x \in [0, 1]$, we have

$$F_{\hat{B}}(S) = F_{\hat{B}}^{per}(S) \text{ for each } S \subseteq N, \; F_{\hat{B}}(N) = F_{\hat{B}}(1) = \{1\}, \; F_{\hat{B}}(2) = \emptyset, \text{ and } \mathcal{K}(\hat{B}) = \{1\}.$$

5 The unanimous influence functions

When we describe phenomena of a positive influence in terms of a corresponding influence function, it is usually sufficient to restrict the analysis to unanimous functions.

Definition 4 *The influence function* $B \in \mathcal{B}$ *is* unanimous *if* $B(x, x, ..., x) = (x, x, ..., x)$ *for each* $x \in [a, b]$, *and* unanimous on the boundaries *if this holds only for* $x = a$ *and* $x = b$.

Obviously, influence functions depicting a negative (opposite) influence should not satisfy the unanimous properties. In Section 3.1, the influence functions B and B'' defined in (7) and (9), respectively, are unanimous, as well as the function $\widetilde{B}$ presented in Section 3.2 (see (11)).

In this section, we present several other examples of influence functions that are unanimous. These influence functions which have been introduced in Grabisch and Rusinowska (2009b, 2010b, [16, 18]), are now generalized for the influence model with a continuum of actions. We investigate the properties of these functions and compare them with our results on the analogous functions in the model with a finite totally ordered set of actions. In particular, for the influence functions analyzed, we determine the follower functions, the kernels, and the influence indices for some particular cases. Basic properties of the influence functions defined for the continuum case appear to be the same as the properties of the analogous influence functions defined for the model with a finite totally ordered set of actions. The proofs of propositions presented below (Propositions 3, 4, 5, and 6) are analogous to the proofs of the corresponding results in the model with a totally ordered set of actions; see Grabisch and Rusinowska (2010b, [18]).

The first influence function we like to mention is the *majority function*. If the players decide according to this function, then in case a majority of players has an inclination x, all players decide for x, and if not, then each player decides according to his own inclination.

Definition 5 (The majority function)
Let $n \geq t > \lfloor \frac{n}{2} \rfloor$, and introduce for any $\mathbf{i} \in I$ and $x \in [a, b]$, the set

$$i^x := \{k \in N : \; i_k = x\}. \tag{16}$$

The majority influence function $\mathsf{Maj}^{[t]} \in \mathcal{B}$ is defined by

$$\left(\mathsf{Maj}^{[t]}(\mathbf{i})\right)_j := \begin{cases} x, & \text{if } \exists x \in [a,b] \; [|i^x| \geq t] \\ i_j, & \text{otherwise} \end{cases}, \quad \forall \mathbf{i} \in I, \quad \forall j \in N. \tag{17}$$

One can show that:

Proposition 3 *Let $n \geq t > \lfloor \frac{n}{2} \rfloor$ and consider the majority function $\mathsf{Maj}^{[t]}$. Then the following holds:*

(i) For each $\emptyset \neq S \subset N$ such that $s \geq t$, and for each $j \in N \setminus S$, $D(\mathsf{Maj}^{[t]}, S \to j) = 1$. If $t = n$, then for each $\emptyset \neq S \subset N$ and $j \in N \setminus S$, $D(\mathsf{Maj}^{[t]}, S \to j) = 0$.

(ii) For each $S \subseteq N$,

$$F_{\mathsf{Maj}^{[t]}}(S) = \begin{cases} N, & \text{if } s \geq t \\ S, & \text{if } n - t < s < t \\ \emptyset, & \text{if } s \leq n - t. \end{cases} \tag{18}$$

(iii) The kernel is $\mathcal{K}(\mathsf{Maj}^{[t]}) = \{S \subseteq N : \; |S| = n - t + 1\}$.

Another influence function analyzed in this paper is the *guru function* which simply means that when a guru exists, every player always follows the guru.

Definition 6 (The guru function)
Let $\widetilde{k} \in N$ be a particular player called the guru. The guru influence function $\mathsf{Gur}^{[\widetilde{k}]} \in \mathcal{B}$ *is defined by*

$$(\mathsf{Gur}^{[\widetilde{k}]}(\mathbf{i}))_j = i_{\widetilde{k}}, \quad \forall \mathbf{i} \in I, \quad \forall j \in N. \tag{19}$$

The following holds:

Proposition 4 *Let $\widetilde{k} \in N$ and consider the guru influence function $\mathsf{Gur}^{[\widetilde{k}]}$. Then the following holds:*

(i) For each $\emptyset \neq S \subseteq N$ such that $\widetilde{k} \in S$, and for each $j \in N \setminus S$, $D(\mathsf{Gur}^{[\widetilde{k}]}, S \to j) = 1$.
(ii) For each $S \subseteq N$,

$$F_{\mathsf{Gur}^{[\widetilde{k}]}}(S) = \begin{cases} N, & \text{if } \widetilde{k} \in S \\ \emptyset, & \text{if } \widetilde{k} \notin S. \end{cases} \tag{20}$$

(iii) The kernel is $\mathcal{K}(\mathsf{Gur}^{[\widetilde{k}]}) = \{\widetilde{k}\}$.
(iv) $\mathsf{Gur}^{[\widetilde{k}]}$ is the unique purely influential function of $\widetilde{k}$ upon $N \setminus \widetilde{k}$, i.e., $\mathcal{B}_{\widetilde{k} \to N \setminus \widetilde{k}} = \{\mathsf{Gur}^{[\widetilde{k}]}\}$.

Also the *identity function*, according to which every player always follows himself, is a particular influence function where no influence is visible.

Definition 7 (The identity function)
The identity function $\mathsf{Id} \in \mathcal{B}$ is defined by

$$\mathsf{Id}(\mathbf{i}) = \mathbf{i}, \quad \forall \mathbf{i} \in I. \tag{21}$$

It is not surprising that the identity function has the following properties:

Proposition 5 *Let us consider the identity function Id. Then the following holds:*

(i) For each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$, $D(\mathsf{Id}, S \to j) = 0$.
(ii) For each $S \subseteq N$, $F_{\mathsf{Id}}(S) = S$.
(iii) The kernel is $\mathcal{K}(\mathsf{Id}) = \{\{k\}, k \in N\}$.

Finally, we like to mention the *mass psychology function*. According to this influence function, if there is a sufficiently high number of players with inclination x, none of these players will decide differently than x, and they will possibly attract other players to choose action x.

Definition 8 (The mass psychology function)
Let $t \in (0, n]$ and $x \in [a, b]$. Functions $B \in \mathcal{B}$ satisfying for each $\mathbf{i} \in I$

$$\text{if } |i^x| \geq t, \text{ then } (B(\mathbf{i}))^x \supseteq i^x \tag{22}$$

are called mass psychology influence functions. We denote by $\mathcal{B}^{[x,t]}$ the set of such influence functions.

One can prove that:

Proposition 6 *Let $t \in (0, n]$ and $x \in [a, b]$ be fixed, and consider any influence function B in $\mathcal{B}^{[x,t]}$. Then the following holds:*

(i) There exists $B \in \mathcal{B}^{[x,t]}$ such that for each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$, $D(B, S \to j) = 0$.

(ii) For each $\emptyset \neq S \subseteq N$ such that $s > n - t$, $t > 1$, and $j \in N \setminus S$, there exists $B \in \mathcal{B}^{[x,t]}$ such that $D(B, S \to j) = 1$.

(iii) For each $S \subseteq N$, $F_B(S) \subseteq S$ if $s \leq n - t$. Moreover, there exists $B \in \mathcal{B}^{[x,t]}$ such that for each $S \subseteq N$, $F_B(S) = S$.

6 Fixed points of an influence function

A central notion in studying influence functions is the notion of fixed point, that is, those inclination vectors which, despite the influence between players, remain invariant.

Definition 9 *An inclination vector $\mathbf{i} \in I$ is a* fixed point *under $B \in \mathcal{B}$ if $B(\mathbf{i}) = \mathbf{i}$. The set of fixed points under B will be denoted by $I^{FP}(B)$.*

Fixed points are of fundamental interest in the study of the convergence of the influence process, when several steps of influence are considered. Indeed, we may consider that there are several steps of discussion before every voter decides, and in each step of discussion, when every player reveals his opinion, influence occurs. Assuming that the same mechanism of influence depicted by the influence function B acts in each step, the successive vectors of the opinion of the voters, starting from the original inclination $\mathbf{i}$, are: $\mathbf{i}$, $B(\mathbf{i})$, $B(B(\mathbf{i})) = B^2(\mathbf{i})$, $B^3(\mathbf{i})$, etc. Ideally, the discussion should end when convergence to a final influence function $B^\infty = \lim_{k\to\infty} B^k$ is attained, if convergence occurs at all. Clearly, if $\mathbf{i}$ is a fixed point, then

$$\mathbf{i} = B(\mathbf{i}) = B^2(\mathbf{i}) = \cdots = B^\infty(\mathbf{i}).$$

Conversely, if $\lim_{k\to\infty} B^k(\mathbf{i})$ exists, then $\lim_{k\to\infty} B^k(\mathbf{i})$ is a fixed point of B. Consequently, if this limit exists for every $\mathbf{i} \in I$, then $B^\infty(I) = I^{FP}(B)$. Therefore, knowing the set of fixed points of B is of primary importance.

In this section, we give general results on this issue, borrowing the main results of fixed point theory (see, e.g., Granas and Dugundji (2003) [19]). Moreover, we determine the set of fixed points for some of the examples given in this paper. A deeper study can be done only for specific families of influence functions, but this is a topic of further research.

A first remark is that if B is unanimous for some $x \in [a, b]$, that is, $B(x, x, \ldots, x) = (x, x, \ldots, x)$, then $(x, x, \ldots, x)$ is a fixed point. Hence for a unanimous function B, $I^{FP}(B)$ contains the diagonal of $[a, b]^n$. In fact, by the Brouwer theorem, if B is a continuous function, the set $I^{FP}(B)$ is nonempty and compact.

Existence and uniqueness of fixed points can be proved using the Banach theorem: if the function B is contracting, that is, if $d(B(\mathbf{i}), B(\mathbf{i}')) < d(\mathbf{i}, \mathbf{i}')$ holds for any distinct vectors $\mathbf{i}, \mathbf{i}' \in I$ and a given metric distance d, then there exists a unique fixed point. However, remark that influence functions being unanimous for at least two vectors $\mathbf{i}, \mathbf{i}'$ cannot be contracting, and since unanimity is a rather natural property for influence functions (at least for a positive influence), this result is of little interest.

We have the following simple result. Let $I_{S\to j}$ denote the set of all n-inclination vectors of potential positive influence of S on j, i.e.,

$$I_{S\to j} := \{\mathbf{i} \in I_S : i_j \neq i_S\} \tag{23}$$

and let $I^*_{S\to j}(B)$ be the set of all n-inclination vectors of positive influence of S on j under B, i.e.,

$$I^*_{S\to j}(B) := \{\mathbf{i} \in I_S : [i_j \neq i_S \ \wedge \ |(B(\mathbf{i}))_j - i_S| < |i_j - i_S|]\}. \tag{24}$$

Proposition 7 *For any influence function B,*

$$I^{FP}(B) \subseteq I \setminus \bigcup_{S,j} I^*_{S\to j}(B). \tag{25}$$

Proof: Suppose that the inclusion does hold, i.e., there is $\mathbf{i} \in I$ such that $\mathbf{i} \in I^{FP}(B)$ and $\mathbf{i} \in \bigcup_{S,j} I^*_{S\to j}(B)$. Hence, $B(\mathbf{i}) = \mathbf{i}$ and there is S and j such that $\mathbf{i} \in I^*_{S\to j}(B)$. This means that $i_j \neq i_S$ and $|(B(\mathbf{i}))_j - i_S| < |i_j - i_S|$, while $(B(\mathbf{i}))_j = i_j$, a contradiction. ■

Note that equality does not hold in general (see Example 3, function B'), but it may hold for some cases (see Examples 2 and 3, function B'').

We end this section by investigating the set of fixed points for several examples of influence functions introduced above.

Example 2 For the influence function B defined by (7) in Section 3.1, the set of fixed points is

$$I^{FP}(B) = \{(i_1, i_2) \in [0,1]^2 : i_1 = i_2\}.$$

Note that, for this example

$$I^{FP}(B) = I \setminus \left(I^*_{1\to 2}(B) \cup I^*_{2\to 1}(B)\right).$$

For the function $\widetilde{B}$ defined by (11) in the three-player example presented in Section 3.2, we have

$$I^{FP}(\widetilde{B}) = \{(i_1, i_2, i_3) \in [0,1]^3 : i_1 = i_2 = i_3\}.$$

Example 3 Let us calculate the sets of fixed points for the remaining functions of Section 3.1, i.e., the functions B', B'', and B^x defined by (8), (9), and (10), respectively. We have

$$I^{FP}(B') = \left\{(i_1, 0) : i_1 \geq \frac{1}{2}\right\} \cup \left\{(i_1, 1) : 0 \leq i_1 < \frac{1}{2}\right\}$$

$$I^{FP}(B'') = \left\{(i_1, i_2) \in [0,1]^2 : i_1 - \frac{1}{2} \leq i_2 \leq i_1 + \frac{1}{2}\right\}$$

$$I^{FP}(B^x) = \{(i_1, x) : 0 \leq i_1 \leq 1\}.$$

Note that

$$I^{FP}(B'') = I \setminus \left(I^*_{1\to 2}(B'') \cup I^*_{2\to 1}(B'')\right).$$

However, we have $I^*_{1\to 2}(B') = I^*_{2\to 1}(B') = \emptyset$, therefore, $I^{FP}(B') \neq I$.

Example 4 For the unanimous functions presented in Section 5, the sets of fixed points are equal to

$$I^{FP}(\mathsf{Id}) = I, \quad I^{FP}(\mathsf{Gur}^{[\widetilde{k}]}) = \{\mathbf{i} \in I : \forall j, k \in N \ [i_j = i_k]\}$$

$$I^{FP}(\mathsf{Maj}^{[t]}) = \{\mathbf{i} \in I : \forall j, k \in N \ [i_j = i_k]\} \cup \{\mathbf{i} \in I : \forall x \in [a,b] \ [|i^x| < t]\}.$$

7 Linear influence functions

An important particular case is when an influence function B is *linear*, that is, it can be written as a matrix, and then we write $\mathbf{Bi}$ instead of $B(\mathbf{i})$. We assume throughout this section that $I = [0,1]^n$.

Example 5 The influence function B defined by (7) in Section 3.1 is linear. For this function, the corresponding matrix is equal to

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Also the function $\widetilde{B}$ defined by (11) in a three-player example presented in Section 3.2 is linear, and

$$\widetilde{\mathbf{B}} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{bmatrix}.$$

The guru influence function $\mathsf{Gur}^{[\widetilde{k}]}$ is also linear, with matrix

$$\mathbf{Gur}^{[\widetilde{k}]} = \begin{bmatrix} 0 & \cdots 1 \cdots & 0 \\ \cdots & \cdots \vdots \cdots & \cdots \\ 0 & \cdots 1 \cdots & 0 \end{bmatrix}$$

i.e., $\mathbf{Gur}^{[\widetilde{k}]}$ has the form $b_{j\widetilde{k}} = 1$ and $b_{jk} = 0$ for $k \neq \widetilde{k}$, $j \in N$. Obviously, the identity function Id corresponds to the identity matrix.

Let us denote by $\mathbf{1}, \mathbf{0}$ the n-dimensional vectors having all components equal to 1 (respectively, 0). A first remark is that any linear influence function is unanimous on the value 0, since $\mathbf{B0} = \mathbf{0}$. Saying that $\mathbf{B}$ is unanimous on the value 1 means that $\mathbf{B1} = \mathbf{1}$, which implies $\sum_{k=1}^n b_{jk} = 1$ for each row $\mathbf{b}_j$. Another noteworthy fact is that the entries of any influence matrix $\mathbf{B}$ shoud be nonnegative, for, if it would exist some negative entry, the resulting decision vector $\mathbf{Bi}$ might have negative components, which is impossible since $I = [0,1]^n$. These observations are summarized in the next lemma.

Lemma 2 *Let* $\mathbf{B} = [b_{ij}]_{i,j=1,\ldots,n}$ *be the matrix of some influence function.*

(i) $b_{ij} \geq 0$ *for* $i,j = 1,\ldots,n$.
(ii) $\mathbf{B}$ *is unanimous on the value 0.*
(iii) $\mathbf{B}$ *is unanimous on the value 1 if and only if* $\mathbf{B}$ *is unanimous if and only if* $\sum_{k=1}^n b_{jk} = 1$ *for each row* $\mathbf{b}_j$.

An important consequence is that the set of unanimous linear influence functions on $[0,1]$ corresponds to the set of stochastic matrices. It is well known that the set of stochastic matrices is closed under matrix multiplication, and if $\mathbf{B}$ is a stochastic matrix then its inverse, whenever it exists, is also a stochastic matrix (and therefore a linear unanimous influence function).

By definition, $\mathbf{i}$ is a fixed point of B if and only if $\mathbf{i}$ is a solution of the linear system $(\mathbf{B} - \mathbf{I})\mathbf{i} = \mathbf{0}$, where $\mathbf{I}$ is the identity matrix (equivalently, if $\mathbf{i}$ is an eigenvector associated with the eigenvalue 1). If $\mathbf{B}$ is a stochastic matrix, the problem of finding fixed points corresponds in Markov chain theory to finding limit probability vectors on states, up to the fact that $\mathbf{i}$ need not be a probability vector.

A remarkable feature of linear functions is that they map polyhedra to polyhedra. Since I is a polyhedron, it follows that all $\mathbf{B}^k(I)$, $k = 1, 2, \ldots$ are polyhedra. Moreover, computing $\mathbf{B}^k(I)$ amounts to computing only $\mathbf{B}^k(\mathbf{i})$ for all vertices of I since we have the property

$$B^k(I) = \text{conv}(B^k(\text{ext}(I))),$$

where $\text{ext}(\cdot)$ and $\text{conv}(\cdot)$ denote the vertices of a polyhedron and the convex closure of a set.

We turn now to more specific results.

Proposition 8 *Let* $\mathbf{B}$ *be unanimous on the boundary* $\mathbf{1}$. *Take* $S \subseteq N$, $j \notin S$. S *has a positive influence on* j *for every* $\mathbf{i} \in I_{S \to j}$ *if and only if* $b_{jj} < 1$ *and* $b_{jk} = 0$ *for all* $k \in N \setminus (S \cup j)$. *Moreover, in this case*

$$D(\mathbf{B}, S \to j) = 1 - b_{jj}. \tag{26}$$

Proof: Let $\mathbf{B}$ be arbitrary. There is positive influence of S on j for $\mathbf{i} \in I_{S \to j}$ if

$$|(\mathbf{Bi})_j - i_S| < |i_j - i_S|. \tag{27}$$

By Lemma 2 we find

$$\begin{aligned}
(\mathbf{Bi})_j - i_S &= i_S \sum_{k \in S} b_{jk} + \sum_{k \notin S} b_{jk} i_k - i_S \\
&= i_S \big(- \sum_{k \notin S} b_{jk} \big) + \sum_{k \notin S} b_{jk} i_k \\
&= b_{jj}(i_j - i_S) + \sum_{\substack{k \notin S \\ k \neq j}} b_{jk}(i_k - i_S).
\end{aligned}$$

We have to find conditions on $\mathbf{B}$ such that for all $\mathbf{i} \in I_{S \to j}$, (27) holds, i.e.

$$|b_{jj}(i_j - i_S) + \sum_{\substack{k \notin S \\ k \neq j}} b_{jk}(i_k - i_S)| < |i_j - i_S|. \tag{28}$$

We have

$$\Big| b_{jj}(i_j - i_S) + \sum_{\substack{k \notin S \\ k \neq j}} b_{jk}(i_k - i_S) \Big| \leq |b_{jj}(i_j - i_S)| + \Big| \sum_{\substack{k \notin S \\ k \neq j}} b_{jk}(i_k - i_S) \Big|.$$

Moreover, $\sum_{\substack{k \notin S \\ k \neq j}} b_{jk}(i_k - i_S) \leq \sum_{\substack{k \notin S \\ k \neq j}} b_{jk}$ since $-1 \leq i_k - i_S \leq 1$ for all $k \neq j, k \notin S$, and equality can be attained by choosing $\mathbf{i}$ with $i_k - i_S = 1$. Therefore, for every $\mathbf{i} \in I_{S \to j}$

$$0 \leq \Big| b_{jj}(i_j - i_S) + \sum_{\substack{k \notin S \\ k \neq j}} b_{jk}(i_k - i_S) \Big| \leq |b_{jj}(i_j - i_S)| + \sum_{\substack{k \notin S \\ k \neq j}} b_{jk}.$$

The right hand expression and $|i_j - i_S|$ being continuous functions of i_j on $[0,1]$, we have by (28)

$$0 \leq \lim_{i_j \to i_S} \Big(|b_{jj}(i_j - i_S)| + \sum_{\substack{k \notin S \\ k \neq j}} b_{jk} \Big) \leq \lim_{i_j \to i_S} |i_j - i_S|$$

which becomes

$$0 \leq \sum_{\substack{k \notin S \\ k \neq j}} b_{jk} \leq 0$$

and yields $b_{jk} = 0$ for every $k \in N \setminus (S \cup j)$. Substituting into (28) yields

$$|b_{jj}(i_j - i_S)| < |i_j - i_S|$$

implying $b_{jj} < 1$.

Let us take $\mathbf{B}$ satisfying the above conditions. Then

$$(\mathbf{Bi})_j - i_S = b_{jj} i_j + \sum_{k \in S} b_{jk} i_S - i_S = b_{jj}(i_j - i_S).$$

Note that for such a $\mathbf{B}$ we have $I^*_{S \to j}(B) = I_{S \to j}$. Therefore

$$\begin{aligned} D(\mathbf{B}, S \to j) &= \frac{\int_{I^{*[S]}_{S \to j}(B)} \big(|i_j - i_S| - |(\mathbf{Bi})_j - i_S| \big) d\mathbf{i}^{[S]}}{\int_{I^{[S]}_{S \to j}} |i_j - i_S| d\mathbf{i}^{[S]}} \\ &= \frac{\int_{I^{[S]}_{S \to j}} |i_j - i_S| \big(1 - b_{jj}\big) d\mathbf{i}^{[S]}}{\int_{I^{[S]}_{S \to j}} |i_j - i_S| d\mathbf{i}^{[S]}} \\ &= 1 - b_{jj}. \end{aligned}$$

■

Remark 1 *Note that:*

(i) The expression

$$(\mathbf{Bi})_j - i_S = b_{jj}(i_j - i_S)$$

shows that:

- *If $b_{jj} = 0$, then $(\mathbf{Bi})_j = i_S$, i.e., the influence is full and j is a perfect follower of S.*
- *If $b_{jj} > 0$, then $sgn\big((\mathbf{Bi})_j - i_S\big) = sgn(i_j - i_S)$, which means that due to influence, the decision of j is always going closer to the inclination of S, i.e., is in between the original inclination i_j and the inclination of S. Therefore, j is a follower of S.*
- *If b_{jj} were negative, then $sgn\big((\mathbf{Bi})_j - i_S\big) = -sgn(i_j - i_S)$, which means that the decision of j would go beyond the inclination of S (we may call this a* switch*), i.e., the original inclination of S would be in between the original inclination of j and his decision. Since $b_{jj} < 0$ is not possible, linear unanimuous influence functions cannot represent positive influence with a switch.*

(ii) The condition that $\mathbf{B}$ has to fulfill for the influence of S on j concerns only row j of the matrix. Therefore, it is very easy to build/customize a matrix $\mathbf{B}$ such that for some players the influencing coalition is specified (see also Corollary 1 below).

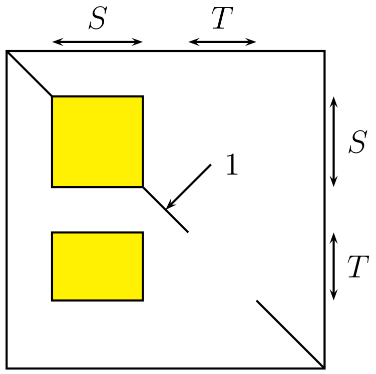
The following is an immediate consequence of Proposition 8.

Corollary 1 *Let S, T be two disjoint coalitions. A matrix $\mathbf{B}$ of the form*

$$\begin{aligned} & b_{jj} = 1 \textit{ and } b_{jk} = 0 \textit{ for } k \neq j, \quad j \notin T \cup S \\ & b_{jk} = 0 \textit{ for } k \notin S, \quad j \in T \\ & b_{jk} = 0 \textit{ for } k \notin S, \quad j \in S \end{aligned}$$

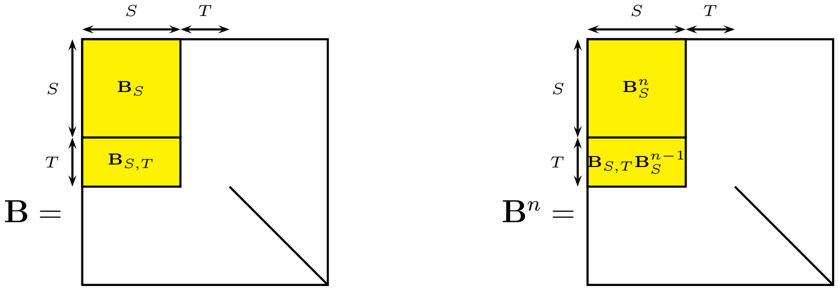
is a purely influential function of S upon T (see figure below: blank area means 0, elements of each row in the shaded area sum up to 1).

Note that the converse is false, because for such functions nothing is specified when $\mathbf{i}$ is outside I_S (in particular they need not be linear). With the above matrix, the behavior of the function outside I_S is specified. In particular, the canonical purely influential function of S upon T is *not* of this form, because outside I_S, it should be the identity matrix.



It is easy to compute $\mathbf{B}^n$ when $\mathbf{B}$ is a purely influential function of S upon T. First we order the players so that $S = \{1, \ldots, s\}$ and $T = \{s+1, \ldots, s+t\}$. Let us call $\mathbf{B}_S$ the square submatrix corresponding to rows and columns from 1 to s, and $\mathbf{B}_{ST}$ the submatrix corresponding to rows from $s+1$ to $s+t$ and columns from 1 to s (see figure below, left). We may denote therefore the matrix by $\mathbf{B} = (\mathbf{B}_S, \mathbf{B}_{ST})$. Note that matrices $\mathbf{B}_S, \mathbf{B}_{ST}$ are arbitrary, provided the sum of coefficients on each row is equal to 1. It is not difficult to see that $\mathbf{B}^n$ is still a purely influential function of S upon T with the following structure (see figure, right):

$$\mathbf{B}^n = (\mathbf{B}_S^n, \mathbf{B}_{S,T}\mathbf{B}_S^{n-1}).$$



8 The negative influence index in the generalized framework

8.1 Definition of the index

The concept of negative influence for the model with a continuum of actions is naturally related to the negative influence defined in the model with a totally ordered set of actions. Under the negative influence in the multi-action framework, the inclination of the player is different from the extreme action(s) of the coalition (that is, the action(s) placed farthest from the inclination of the coalition), and the player's decision comes 'closer' to such an extreme action. We follow here a similar approach.

Let us formalize the concepts of negative influence. For each action $x \in [a, b]$, $M(x)$ is the bound of $[a, b]$ which is farthest from x, i.e.:

$$M(x) = \begin{cases} b, & \text{if } x \in [a, \frac{a+b}{2}[\\ a, & \text{if } x \in]\frac{a+b}{2}, b] \\ a \text{ or } b & \text{if } x = \frac{a+b}{2}. \end{cases}$$

In the third case, a or b can be chosen, without any influence on the subsequent results. For each $S \subseteq N$ and $j \in N \setminus S$, the set of all $(n-s+1)$-inclination vectors of *potential negative influence of S on j* is defined as

$$\widetilde{I}_{S \to j}^{[S]} := \{\mathbf{i}^{[S]} \in [a, b]^{n-s+1} : i_j \neq M(i_S)\}. \tag{29}$$

This means that in any inclination vector of potential negative influence of S on j, the inclination of player j is not the extreme action of coalition S.

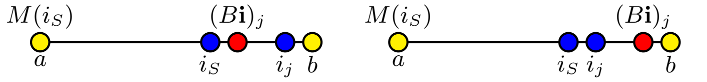
The *set of all inclination vectors of negative influence of S on j under given B* is defined as

$$I_{S \to j}^{neg[S]}(B) := \left\{\mathbf{i}^{[S]} \in \widetilde{I}_{S \to j}^{[S]} : \Big[|i_j - i_S| < |(B(\mathbf{i}))_j - i_S| \ \wedge \ |i_j - M(i_S)| > |(B(\mathbf{i}))_j - M(i_S)| \Big] \right\} \tag{30}$$

$I_{S \to j}^{neg[S]}(B)$ is therefore the set of all inclination vectors in $[a, b]^{n-s+1}$ such that the inclination of player j is not the extreme action of coalition S, and it satisfies two conditions:

- The first condition of $I_{S \to j}^{neg[S]}(B)$ means that the decision of player j goes farther from the inclination of coalition S than his inclination was.
- The second condition says that the decision of player j comes closer to the extreme action of coalition S than his inclination was. Note that in the special case $i_S = \frac{a+b}{2}$, this second condition should be modified as follows: $|i_j - \epsilon| > |(B(\mathbf{i}))_j - \epsilon'|$, where ϵ, ϵ' are respectively the bounds closer to i_j and closer to $(B(\mathbf{i}))_j$. However, this is unimportant since when computing integrals this distinction will disappear.

Note that we need both conditions to define the negative influence, since none of these conditions implies the another one. The two pictures below show the situations in which one of the conditions is not satisfied:



Definition 10 *Given $B \in \mathcal{B}$, for each $S \subseteq N$, $j \in N \setminus S$, the negative influence index of coalition S on player j is defined as*

$$D^{neg}(B, S \to j) := \frac{\int_{I^{neg[S]}_{S\to j}(B)} [|i_j - M(i_S)| - |(B(\mathbf{i}))_j - M(i_S)|] \; d\mathbf{i}^{[S]}}{\int_{\widetilde{I}^{[S]}_{S\to j}} |i_j - M(i_S)| \; d\mathbf{i}^{[S]}}. \tag{31}$$

The negative influence index defined in (31) is related to the possibility negative influence index for the multi-action model; see Grabisch and Rusinowska (2010b, [18], Definition 2). Similarly as for the positive influence, we can calculate the integral in the denominator of definition (31).

Lemma 3 *If $I = [a, b]^n$, then for each $\emptyset \neq S \subset N$ and $j \in N \setminus S$*

$$\int_{\widetilde{I}^{[S]}_{S\to j}} |i_j - M(i_S)| \; d\mathbf{i}^{[S]} = \frac{(b-a)^{n-s+2}}{2}. \tag{32}$$

Proof:

$$\int_{\widetilde{I}^{[S]}_{S\to j}} |i_j - M(i_S)| \; d\mathbf{i}^{[S]} = \int_a^b ... \int_a^b \int_a^b \int_a^{\frac{a+b}{2}} (b - i_j) \; di_S \; di_j \; di_{k_1} \; ... \; di_{k_{n-s-1}} +$$

$$+ \int_a^b ... \int_a^b \int_a^b \int_{\frac{a+b}{2}}^b (i_j - a) \; di_S \; di_j \; di_{k_1} \; ... \; di_{k_{n-s-1}} =$$

$$= \int_a^b ... \int_a^b \int_a^b (b - i_j) \left(\frac{a+b}{2} - a\right) \; di_j \; di_{k_1} \; ... \; di_{k_{n-s-1}} +$$

$$+ \int_a^b ... \int_a^b \int_a^b (i_j - a) \left(b - \frac{a+b}{2}\right) \; di_j \; di_{k_1} \; ... \; di_{k_{n-s-1}} =$$

$$= \int_a^b ... \int_a^b \left(\frac{b-a}{2}\right) \left[b \cdot i_j - \frac{i_j^2}{2}\right]_{i_j=a}^{i_j=b} \; di_{k_1} \; ... \; di_{k_{n-s-1}} +$$

$$+ \int_a^b ... \int_a^b \left(\frac{b-a}{2}\right) \left[\frac{i_j^2}{2} - a \cdot i_j\right]_{i_j=a}^{i_j=b} \; di_{k_1} \; ... \; di_{k_{n-s-1}} = \frac{(b-a)^{n-s+2}}{2}$$

■

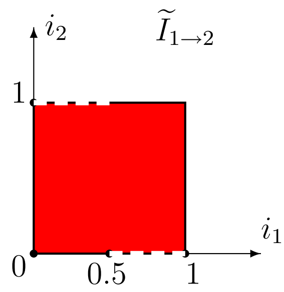
8.2 Examples of negative influence

Let us calculate the negative influence indices for some examples for which we have already calculated the positive influence indices.

Example 6 We consider Example 3.1 and the negative influence of player 1 on player 2. The set of potential negative influence is

$$\widetilde{I}_{1\to 2} = \left\{(i_1, i_2) \in [0, 1]^2 : i_2 \neq M(i_1)\right\}.$$

The set $\widetilde{I}_{1\to 2}$ is presented on the figure below. It consists of all the points of the square without two intervals: one with $i_1 \in \left[\frac{1}{2}, 1\right]$ and $i_2 = 0$, and one with $i_1 \in \left[0, \frac{1}{2}\right]$ and $i_2 = 1$.



By virtue of (32), we have

$$\int_{\widetilde{I}_{1\to 2}} |i_2 - M(i_1)| \ d\mathbf{i} = \frac{1}{2}.$$

For the influence function B defined by (7), we have $I^{neg}_{1\to 2}(B) = \emptyset$, because while there are inclination vectors (i_1, i_2) that satisfy the second condition, no inclination vector satisfies the first condition of $I^{neg}_{1\to 2}(B)$. Hence, $D^{neg}(B, 1 \to 2) = 0$.

Similarly for the influence function B'' defined by (9), $I^{neg}_{1\to 2}(B'') = \emptyset$ and $D^{neg}(B'', 1 \to 2) = 0$.

Let us consider now the influence function B' defined by (8). Note that $I^{neg}_{1\to 2}(B') = \widetilde{I}_{1\to 2}$, and $(B'(\mathbf{i}))_2 = M(i_1)$ for each $\mathbf{i} \in \widetilde{I}_{1\to 2}$. Hence,

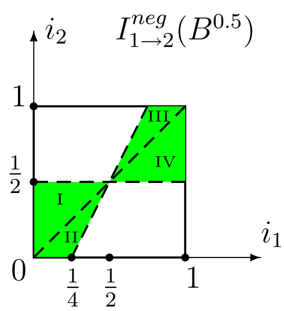
$$\int_{I^{neg}_{1\to 2}(B')} [|i_2 - M(i_1)| - |(B'(\mathbf{i}))_2 - M(i_1)|] \ d\mathbf{i} = \int_{\widetilde{I}_{1\to 2}} |i_2 - M(i_1)| \ d\mathbf{i} = \frac{1}{2}$$

and $D^{neg}(B', 1 \to 2) = 1$.

For the influence function B^x defined in (10), the negative influence indices will have also a positive value, although smaller than 1. Let us calculate the negative influence of player 1 on player 2 for $x = \frac{1}{2}$, that is, for the parameter leading to the maximal positive influence index equal to $\frac{7}{16}$.

$$I^{neg}_{1\to 2}(B^{0.5}) = \left\{ \mathbf{i} \in \widetilde{I}_{1\to 2} : \left[|i_2 - i_1| < \left| \frac{1}{2} - i_1 \right| \ \wedge \ |i_2 - M(i_1)| > \left| \frac{1}{2} - M(i_1) \right| \right] \right\}$$

The figure below presents the set $I^{neg}_{1\to 2}(B^{0.5})$ which consists of four subsets.



The subsets of $I^{neg}_{1\to 2}(B^{0.5})$ are the following:

$$I^{negI}_{1\to 2}(B^{0.5}) = \left\{ (i_1, i_2) \in [0,1]^2 : \ \left[0 \le i_1 \le \frac{1}{2} \ \wedge \ i_1 \le i_2 < \frac{1}{2} \right] \right\}$$

$$I^{negII}_{1\to 2}(B^{0.5}) = \left\{ (i_1, i_2) \in [0,1]^2 : \ \left[0 \le i_1 \le \frac{1}{2} \ \wedge \ 2i_1 - \frac{1}{2} < i_2 < i_1 \right] \right\}$$

$$I^{negIII}_{1\to 2}(B^{0.5}) = \left\{(i_1,i_2)\in[0,1]^2: \ \left[\frac{1}{2} < i_1 \le 1 \ \wedge \ i_1 \le i_2 < 2i_1 - \frac{1}{2}\right]\right\}$$

$$I^{negIV}_{1\to 2}(B^{0.5}) = \left\{(i_1,i_2)\in[0,1]^2: \ \left[\frac{1}{2} < i_1 \le 1 \ \wedge \ \frac{1}{2} < i_2 < i_1\right]\right\}.$$

We calculate the negative influence index $D(B^{0.5}, 1\to 2)$ of player 1 on player 2 under the influence function $B^{0.5}$.

$$\begin{aligned}
D^{neg}(B^{0.5}, 1\to 2) &= \frac{\int_{I^{neg}_{1\to 2}(B^{0.5})}\left[|i_2 - M(i_1)| - \left|\frac{1}{2} - M(i_1)\right|\right] \ d\mathbf{i}}{\int_{\widetilde{I}_{1\to 2}} |i_2 - M(i_1)| \ d\mathbf{i}} \\
&= 2\sum_{L\in\{I,II,III,IV\}} \int_{I^{negL}_{1\to 2}(B^{0.5})} \left[|i_2 - M(i_1)| - \left|\frac{1}{2} - M(i_1)\right|\right] \ d\mathbf{i}
\end{aligned}$$

$$\begin{aligned}
\int_{I^{negI}_{1\to 2}(B^{0.5})} \left[|i_2 - M(i_1)| - \left|\frac{1}{2} - M(i_1)\right|\right] \ d\mathbf{i} &= \int_0^{\frac{1}{2}}\int_{i_1}^{\frac{1}{2}} \left(\frac{1}{2} - i_2\right) \ di_2 \ di_1 = \frac{1}{48} \\
\int_{I^{negII}_{1\to 2}(B^{0.5})} \left[|i_2 - M(i_1)| - \left|\frac{1}{2} - M(i_1)\right|\right] \ d\mathbf{i} &= \int_0^{\frac{1}{2}}\int_{2i_1-\frac{1}{2}}^{i_1} \left(\frac{1}{2} - i_2\right) \ di_2 \ di_1 = \frac{1}{16} \\
\int_{I^{negIII}_{1\to 2}(B^{0.5})} \left[|i_2 - M(i_1)| - \left|\frac{1}{2} - M(i_1)\right|\right] \ d\mathbf{i} &= \int_{\frac{1}{2}}^{1}\int_{i_1}^{2i_1-\frac{1}{2}} \left(i_2 - \frac{1}{2}\right) \ di_2 \ di_1 = \frac{1}{16} \\
\int_{I^{negIV}_{1\to 2}(B^{0.5})} \left[|i_2 - M(i_1)| - \left|\frac{1}{2} - M(i_1)\right|\right] \ d\mathbf{i} &= \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{i_1} \left(i_2 - \frac{1}{2}\right) \ di_2 \ di_1 = \frac{1}{24}
\end{aligned}$$

Hence,

$$D^{neg}(B^{0.5}, 1\to 2) = 2\left[\frac{1}{48} + \frac{1}{16} + \frac{1}{16} + \frac{1}{24}\right] = \frac{3}{8} < \frac{7}{16} = D(B^{0.5}, 1\to 2).$$

We conclude that under the influence function $B^{0.5}$, the negative influence of player 1 on player 2 is slightly smaller than the positive influence.

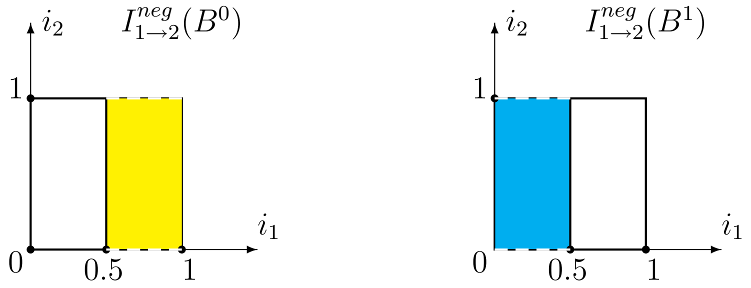
Let us calculate the negative influence of player 1 on player 2 for $x = 0$ and for $x = 1$, that is, for the parameters leading to the minimal positive influence index equal to $\frac{1}{4}$.

The sets of $I^{neg}_{1\to 2}(B^0)$ and $I^{neg}_{1\to 2}(B^1)$ are the following:

$$I^{neg}_{1\to 2}(B^{0}) = \left\{(i_1,i_2)\in[0,1]^2: \ \left[\frac{1}{2} \le i_1 \le 1 \ \wedge \ 0 < i_2 < 1\right]\right\}$$

$$I^{neg}_{1\to 2}(B^{1}) = \left\{(i_1,i_2)\in[0,1]^2: \ \left[0 \le i_1 \le \frac{1}{2} \ \wedge \ 0 < i_2 < 1\right]\right\}$$

These sets are presented in the figures below.



We have

$$\int_{I^{neg}_{1\to 2}(B^0)} [|i_2 - M(i_1)| - M(i_1)] \; d\mathbf{i} = \int_{\frac{1}{2}}^{1} \int_{0}^{1} i_2 \; di_2 \; di_1 = \frac{1}{4}$$

$$\int_{I^{neg}_{1\to 2}(B^1)} [|i_2 - M(i_1)| - |1 - M(i_1)|] \; d\mathbf{i} = \int_{0}^{\frac{1}{2}} \int_{0}^{1} (1 - i_2) \; di_2 \; di_1 = \frac{1}{4}.$$

Hence,

$$D^{neg}(B^0, 1 \to 2) = D^{neg}(B^1, 1 \to 2) = \frac{1}{2} > \frac{1}{4} = D(B^0, 1 \to 2) = D(B^1, 1 \to 2).$$

We conclude that under the influence functions B^0 and B^1, the negative influence of player 1 on player 2 is greater than the positive influence.

Example 7 For the three-player example presented in Section 3.2 and the influence function $\widetilde{B}$ defined by (11), similarly as for the function given in (7), we have $I^{neg[13]}_{13\to 2}(\widetilde{B}) = \emptyset$ and $D^{neg}(\widetilde{B}, 13 \to 2) = 0$.

Since the analysis of the negative influence is analogous to the study of the positive influence, in this paper we focus mainly on the positive influence. Consequently, most of the influence functions considered in this paper do not represent any negative influence. We can show that the negative influence index is equal to zero also for the set of perfect followers and purely influential functions defined in Section 4, as well as for the unanimous influence functions analyzed in Section 5.

Proposition 9 *$I^{neg}_{S\to j}(B) = \emptyset$, and consequently $D^{neg}(B, S \to j) = 0$, where $B \in \mathcal{B}$, $\emptyset \neq S \subset N$ and $j \in N \setminus S$ are the following:*

(i) for each $j \in F^{per}_B(S) \setminus S$, where $F^{per}_B(S)$ is the set of perfect followers of S under B;
(ii) for each $B \in \mathcal{B}_{S\to T}$ (purely influential function of S upon T);
(iii) for each majority influence function $\mathsf{Maj}^{[t]}$, where $n \geq t > \lfloor \frac{n}{2} \rfloor$, and each S such that $s \geq t$;
(iv) for each guru influence function $\mathsf{Gur}^{[\widetilde{k}]}$, and each S such that $\widetilde{k} \in S$;
(v) for the identity function Id.

We omit the proofs here, since they are analogous to the proofs of the corresponding results on the negative influence in the model with a totally ordered set of actions; see Grabisch and Rusinowska (2010b, [18]).

9 Concluding remarks

In this paper, we have aimed at defining and investigating the influence model with a continuum of actions and at comparing the results of such a model with the ones obtained in the model with an ordered set of possible actions (and its particular case, the yes-no model). For the continuum case, we have defined and studied the positive and negative influence indices of a coalition on a player, the unanimous influence functions, the purely influential function, the set of followers and perfect followers, the kernel of an influence function, the set of fixed points under a given influence function, and the linear influence functions. The main difference between the two models lies naturally in the definitions of the influence indices. While in the previous model the influence indices have been defined by sums of some expressions over the particular sets, in the continuum case sums are replaced by integrals.

We like to stress that apart from comparing the results of the continuum case with the ones obtained in the model with an ordered set of possible actions, in the present paper we have also aimed at studying two concepts which we did not consider before, i.e., the fixed points under an influence function, and the linear influence functions. Due to the representation of the linear functions by matrices, such influence functions are very convenient and worth studying. In particular, for the linear functions, being unanimous on the boundaries is equivalent to being unanimous, and is also equivalent to the fact that the matrix is stochastic. According to one of our results, a quick look at a matrix representing an unanimous linear function shows us immediately if there is a positive influence of a coalition on a player and also what the value of the influence index is.

To the best of our knowledge, the influence model with a continuum of actions has not been investigated before. Since in real-life situations people frequently have to make a choice of one option from among a continuum of options, we believe that our influence model with a continuum of actions is of importance and contributes significantly to the voting literature.

As already mentioned, within our project on influence in a social network, we have investigated the influence indices and other tools for measuring the influence in the yes-no model, in the multi-choice framework, and in the model with a continuum of actions, and we have compared the yes-no model of influence with command games. The next important research on the influence issues which we are going to conduct will be to introduce dynamic aspects into the model. More precisely, we would like to investigate a model in which the mutual influence does not stop necessarily after one step but may iterate. In particular, we intend to study the behavior of the series of influence functions and to look for convergent conditions for such series.

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