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ASYMPTOTICS OF THE VISIBILITY FUNCTION IN THE BOOLEAN MODEL

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The aim of this paper is to give a precise estimate on the tail probability of the visibility function in a germ-grain model: this function is defined as the length of the longest ray starting at the origin that does not intersect an obstacle in a Boolean model. We proceed in two or more dimensions using coverage techniques. Moreover, convergence results involving a type I extreme value distribution are shown in the two particular cases of small obstacles or a large obstacle-free region.

1. Presentation of the model and results. In [19] G. Pólya introduced the question of the visibility in a forest in a discrete lattice case as well as in a random case. He first treated the problem of a person standing at the origin of the regular square lattice of $\mathbb{R}^2$, when identical trees (discs with constant radius R) are situated at the other points of the lattice. In this framework he showed that in order to see at a distance r the radius R should be (asymptotically when r is large) taken as $1/r$. More recently V. Janković gave in [12] an elegant proof of a detailed version of this result. The random case studied by G. Pólya was the one of the visibility in one direction: we are here interested in the global solution to this problem considering all directions simultaneously. The spherical contact distribution which can be seen as the infimum of the visibility over all directions has been intensively used

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for a geometric description of random media (see e.g. [17, 20, 11, 15, 6, 1]). In comparison, the total visibility, *i.e.* the supremum of the same function, has been rarely studied in the litterature. The work of reference is due to S. Zacks and is strongly motivated by military applications ([26], see also his work with M. Yadin [25]). However his interest was mainly focused on the probability that given points could be seen and not on the total visibility. Very recently, the visibility problem has been investigated in the hyperbolic disc by I. Benjamini et alt. [2]. In particular, the authors show the existence of a critical intensity for the almost sure visibility at infinity. In this connection, one of the consequences of our work will be that with probability one, we can see only at a finite distance in the Euclidean space $\mathbb{R}^d$ (see Proposition 4.1 and also Remark 4.1 concerning the possibility to see at infinity).

In this paper, one of our goals is to present new distributional properties of the total visibility in order to develop a future use of this indicator for the study of porous media and more particularly in forestry. Potential applications concern the optimization of directional logging of trees, the measurement of competition level between growing trees in forest dynamics or even an estimation of the light transmission through the canopy of a tree. In such context, the total visibility seems to have an important role to play even though it is understood that in some particular cases, another quantity of interest could be the mean of the visibility in all directions.

The model is the following: consider a Boolean model (see [16, 24]) with random almost surely diameter-bounded *convex* grain K with law μ based on a Poisson Point Process $\mathbf{X}$ with intensity measure the Lebesgue measure on $\mathbb{R}^d$, $d \geq 2$. Define $\mathfrak{O}$ the *occupied phase* of this model,

$$\mathfrak{O} = \bigcup_{x \in \mathbf{X}} (x \oplus K_x),$$

where $(K_x)_{x \in \mathbf{X}}$ are independent identically distributed copies of K, independent of $\mathbf{X}$. We condition this model by the event $O \notin \mathfrak{O}$ where O is the origin of $\mathbb{R}^d$. In particular, it has a positive probability equal to $\exp(-\mathbb{E}[\mathrm{Leb}_d(K)])$. We then define the *visibility* in the following way:

Definition 1.1. *Let* $\mathbf{u}$ *be a unit vector in* $\mathbb{R}^d$*, the visibility in direction* $\mathbf{u}$ *is defined as*

$$V(\mathbf{u}) = \inf\{r > 0 \ : \ r\mathbf{u} \in \mathfrak{O}\},$$

the total visibility is defined as

$$\mathfrak{V} = \sup_{\|\mathbf{u}\|=1} V(\mathbf{u}).$$

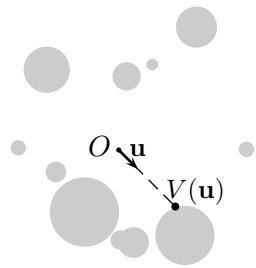

FIG 1. *Directional visibility*

For a convex body K and $\mathbf{u} \in \mathbb{S}^{d-1}$, we define the width of K in the direction v orthogonal to $\mathbf{u}$ as

$$W_u(K) = \sup_{x,y \in K} \langle y - x, \mathbf{v} \rangle.$$

The mean width of K is denoted by $\mathbf{W}(K)$ and satisfies

$$\mathbf{W}(K) = \frac{1}{d\omega_d} \int W_u(K) \, \mathrm{d}\sigma_d(u),$$

where σ_d is the uniform measure on $\mathbb{S}^{d-1}$ and ω_d is the d-dimensional Lebesgue measure of the unit-ball in $\mathbb{R}^d$.

It is well known that the law of the directional visibility is exponential, indeed

LEMMA 1.1. *For each unit direction* $\mathbf{u}$ *in* $\mathbb{R}^d$ *one has*

$$\mathbb{P}\left(V(\mathbf{u}) > r\right) = \exp(-r\mathbb{E}[\mathbf{W}(K)]).$$

The aim of this paper is to give precise estimates on the tail probability for the visibility function in all directions: in section 2 we present the general method of coverage processes used throughout this paper. In section 3 we give sharp upper and lower bounds in the same exponential order $\exp(-\text{Const}.r)$ in dimension two and in the two cases of circular obstacles and of more general rotation-invariant random obstacles. In higher dimensions results on coverage processes are more sparse, thus the bounds presented in section 4 for dimension $d \geq 3$ are rougher. Section 5 is devoted to two similar convergence results for the asymptotics of the visibility with small obstacles, and when the spherical contact length is conditionned to be large. Both results state a convergence in law towards a Gumbel distribution, are valid for any dimension $d \geq 2$ and are based on an extension of a result of [13].

The present work has been first announced in a small note [5].

2. Random coverage of the circle and the sphere and visibility. The visibility up to length r may be blocked only by those obstacles that intersect $B_d(0,r)$, the number N_r of those obstacles is Poisson distributed with parameter

$$g_d(r,K) = \mathbb{E}[\mathrm{Leb}_d((B_d(0,r) \oplus \check{K}) \setminus \check{K})],$$

where $\check{K} = \{-x \,:\, x \in K\}$, and those obstacles K_i, $i \in \{1,\dots,N_r\}$, are independent and identically distributed with the same law. Each of them projects a *shadow* of solid angle S_i on the sphere $r\mathbb{S}^{d-1} = \partial B_d(0,r)$, which is a random cap C_i:

$$C_i = \{(r,\mathbf{u}) \in \mathbb{R}_+ \times \mathbb{S}^{d-1} \;:\; \exists s \le r \text{ s.t. } (r,\mathbf{u}) \in K_i\}, \tag{2.1}$$

and the solid angle S_i is equal to

$$S_i = \{\mathbf{u} \in \mathbb{S}^{d-1} \;:\; (r,\mathbf{u}) \in C_i\}. \tag{2.2}$$

We have thus the following ansatz:

Ansatz 2.1. *The visibility $\mathfrak{V}$ is greater than r if and only if the solid angles S_i, $i \in \{1,\dots,N_r\}$ do not cover the sphere $\mathbb{S}^{d-1}$.*

This equivalence links coverage properties of the sphere with our initial problem: this problem of random coverings has been quite intensively studied in the literature in the two-dimensional case, see for instance [7, 14, 8] in the context of Dvoretzky covering, and [22, 21, 23, 25] for a more general approach. The properties of coverings in higher dimensions are less known, let us cite the works of S. Janson ([13] and other papers) dealing with some asymptotic properties for coverage processes with small caps.

Let us remark that each obstacle K_i is distributed according to the law

$$g_d(r,K)^{-1}\mathbf{1}_{(x\oplus K)\cap B_d(0,r)\neq\emptyset}\,\mathrm{d}x\mathrm{d}\mu(K).$$

We denote by $\tilde{\nu}_r$ the distribution of the associated solid angle S_i. The probability measure $\tilde{\nu}_r$ is naturally invariant under rotations. We call $P(\tilde{\nu}_r,n)$ the probability that n independent solid angles distributed as $\tilde{\nu}_r$ cover the sphere. With Ansatz 2.1 the following result becomes straightforward:

Proposition 2.1. *For every $r > 0$,*

$$\mathbb{P}\left(\mathfrak{V} \ge r\right) = \exp\left(-g_d(r,K)\right) \sum_{n\ge 0} \frac{(g_d(r,K))^n}{n!}(1 - P\left(\tilde{\nu}_r,n\right)). \tag{2.3}$$

When in dimension two, for every probability measure ν on $[0,1]$, $P(\nu,n)$ denotes the covering probability of the circle with perimeter one by n i.i.d. isotropic arcs of ν-distributed length. We shall denote by ν_r the probability law of the random arc shadowed by an obstacle intersecting the disc $B_2(0,r)$. When $K = B_2(0,R)$ for a constant radius R, one has for instance the following values for the characteristic data of proposition 2.1:

$$g_2(r,K) = \pi(2rR + r^2), \tag{2.4}$$

$$\begin{aligned}\frac{\mathrm{d}\nu_r}{\mathrm{d}u}(u) &= \frac{\pi r}{rR + \frac{r^2}{2}}\,\mathbf{1}_{\left[0,\frac{1}{\pi}\arctan\left(\frac{R}{r}\right)\right]}(u) \times \\ &\quad \left(r\sin(2\pi u) + \frac{\sin(\pi u)(R^2 + r^2\cos(2\pi u))}{\sqrt{R^2 - r^2\sin^2(\pi u)}}\right) \\ &\quad + \frac{\pi R^2}{rR + \frac{r^2}{2}}\,\mathbf{1}_{\left[\frac{1}{\pi}\arctan\left(\frac{R}{r}\right),\frac{1}{2}\right]}(u)\,\frac{\cos(\pi u)}{\sin^3(\pi u)}.\end{aligned} \tag{2.5}$$

Remark 2.1 (The two-dimensional case). *Let us recall that the covering probability $P(\nu,n)$ has been computed in dimesion two by Stevens in the case of arcs with deterministic lengths [23] and Siegel & Holst in the general case [22]: for every probability measure ν on $[0,1]$ and $n \in \mathbb{N}^*$,*

$$P(\nu,n) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\int\left[\prod_{i=1}^{k}F_\nu(u_i)\right]\left[\sum_{i=1}^{k}\int_0^{u_i}F_\nu(t)\,\mathrm{d}t\right]^{n-k}\mathrm{d}\tilde{\lambda}_k(u) \tag{2.6}$$

where F_ν is the cumulative distribution function of ν and $\tilde{\lambda}_k$, $k \in \mathbb{N}^*$, is the normalised uniform measure on the simplex $\{(x_1,\cdots,x_k) \in [0,1]^k; x_1 + \cdots + x_k = 1\}$.

We can consequently substitute $P(\nu,n)$ by its expression (2.6) to get an explicit formula for $\mathbb{P}(\mathfrak{V} \geq r)$, $r > 0$. Nevertheless, it seems more or less intractable for doing some asymptotic estimations.

3. Sharp asymptotics in dimension two. In dimension two it becomes possible to give sharp estimates for the tail probability, for instance we have from lemma 1.1 the following lower bound:

$$\mathbb{P}(\mathfrak{V} > r) \geq \mathbb{P}(V_{\mathbf{u}} > r) = \exp(-r\mathbb{E}[\mathbf{W}(K)]), \tag{3.1}$$

in this section we shall give two sharper lower bounds and an upper bound:

- When obstacles are fixed discs with constant radius R, for each $\mu \in (0, 2/R)$ there exists a function ε_μ converging to 0 as $r \to \infty$ such that:

$$\mathbb{P}(\mathfrak{V} \geq r) \geq \mu r\exp(-2Rr)\left(1 + \varepsilon_\mu(r)\right), \tag{3.2}$$

for r large enough;

- When obstacles are discs with random radius R, we have for r large enough

$$\begin{aligned} \mathbb{P}(\mathfrak{V} \geq r) &\leq 2(\pi(2r\mathbb{E}[R]+r^2)+2) \times \\ &\quad \exp\left(-\pi(2r\mathbb{E}[R]+r^2)m_r\right), \end{aligned} \tag{3.3}$$

$$\begin{aligned} \mathbb{P}(\mathfrak{V} \geq r) &\geq 2\pi m_r(2r\mathbb{E}[R]+r^2)\exp\left(-\pi m_r(2r\mathbb{E}[R]+r^2)\right) + \\ &\quad \exp\left(-2\pi m_r(2r\mathbb{E}[R]+r^2)\right), \end{aligned} \tag{3.4}$$

where m_r is the mean of law ν_r, satisfying

$$m_r \sim \frac{2\mathbb{E}[R]}{\pi r}. \tag{3.5}$$

¿From (3.3), (3.4) and (3.5) we obtain directly the following theorem:

THEOREM 3.1. *For the Boolean model in dimension two with random discs, the asymptotics of the visibility is given by*

$$\lim_{r\to+\infty} \frac{1}{r} \log \mathbb{P}(\mathfrak{V} \geq r) = -2\mathbb{E}[R].$$

The estimate (3.2) is obtained without the use of the previous section, by more direct considerations based on the estimation of the visibility in a finite number of directions. Details of the proof are postponed to the appendix 1.

In the following lines, we shall prove estimates (3.3) and (3.4), as well as a generalisation of theorem 3.1 for generall convex shapes. Most of the arguments rely on comparison results for covering probabilities of the circle.

3.1. *Lower bound via comparison of coverage probabilities, estimate (3.3).* In this subsection and the next we shall follow the main steps of a previous work [4] that dealt with the circumscribed radius of the typical Poisson-Voronoi cell: a comparison result on covering probabilities states that if two probability measures (the laws of the arcs) are comparable in some sense, then the coverage probabilities are also comparable. Up to now the ordering induced by the concentration around the mean has been the main criterion for comparing covering probabilities [21, 4] but the convex ordering (which is implied by the previous ordering) is in fact enough to deduce the required inequalities.

For the special case of random discs, the computation of the parameters of equation (2.3) is done in the following way. Let us denote by μ the law of the radius of the random discs of the Boolean model. The law of the normalised

lengths of the shadowed arcs on the circle is denoted by ν_r, this law is the image of the couple (R, U) with law $\mu \otimes \mathcal{U}(0,1)$ by the map $(R, U) \mapsto \ell$:

$$\ell = \begin{cases} \dfrac{1}{\pi} \arcsin \dfrac{1}{\sqrt{1 + \frac{r}{R}\left(2 + \frac{r}{R}\right) U}} & \text{if } 0 \leq U \leq \dfrac{r}{r+2R}; \\ \dfrac{1}{\pi} \arccos \dfrac{\frac{r}{R} + \left(2 + \frac{r}{R}\right) U}{2\sqrt{1 + \frac{r}{R}\left(2 + \frac{r}{R}\right) U}} & \text{if } \dfrac{r}{r+2R} \leq U \leq 1. \end{cases} \tag{3.6}$$

As r tends to infinity, we obtain $\ell \simeq (1/\pi) \arcsin(R/(r\sqrt{U}))$ so that the asymptotics of the expectation becomes $m_r := \mathbb{E}[\ell] \simeq 2\mathbb{E}[R]/(\pi r)$.

We shall use here the convex domination of measures, let us first remark the following: for any probability measure ν on $[0, 1/2]$ with mean m and any convex function $f : [0, 1/2] \longrightarrow \mathbb{R}$, we have

$$\int f \, \mathrm{d}\nu \leq (1 - 2m) f(0) + 2m f(1/2), \tag{3.7}$$

which means that $\nu <_{cv} \left[(1 - 2m)\delta_0 + (2m)\delta_{1/2}\right]$ (where $<_{cv}$ denotes the usual convex order [18]).

It is a consequence of the proof of theorem 13 in [4] that the convex order implies the order of the covering probabilities: if μ_1 and μ_2 are two probability measures on $[0, 1/2]$ such that $\mu_1 <_{cv} \mu_2$ then $P(\mu_1, n) \leq P(\mu_2, n)$. Inserting the inequality $P(\nu_r, n) \leq P((1 - 2m_r) + 2m_r\delta_{1/2}, n)$ in (2.3) for every $n \in \mathbb{N}$, we get that

$$\begin{aligned} \mathbb{P}(\mathfrak{V} \geq r) \quad &\geq \quad \exp\left(-\pi(2r\mathbb{E}[R] + r^2)\right) \times \\ &\sum_{n \geq 0} \frac{(\pi(2r\mathbb{E}[R] + r^2))^n}{n!} (1 - P\left((1 - 2m_r)\delta_0 + 2m_r\delta_{1/2}, n\right)). \end{aligned} \tag{3.8}$$

It follows from ([4], Corollary 1) that

$$1 - P\left((1 - 2m_r)\delta_0 + 2m_r\delta_{1/2}, n\right) = 2nm_r(1 - m_r)^{n-1} + (1 - 2m_r)^n.$$

Inserting that result in (3.8), we obtain that

$$\begin{aligned} \mathbb{P}(\mathfrak{V} \geq r) \quad &\geq \quad 2\pi m_r(2r\mathbb{E}[R] + r^2) \exp\left(-\pi m_r(2r\mathbb{E}[R] + r^2)\right) + \\ &\quad \exp\left(-2\pi m_r(2r\mathbb{E}[R] + r^2)\right). \end{aligned} \tag{3.9}$$

3.2. *Upper bound via comparison of coverage probabilities, estimate (3.4).* By Jensen's inequality, we have that $\nu_r >_{cv} \delta_{m_r}$. Consequently, the same argument as for the lower bound shows that $P(\nu_r, n) \geq P(\delta_{m_r}, n)$ for every $n \in \mathbb{N}$. Inserting this inequality in (2.3), we have

(3.10)
$$\mathbb{P}\left(\mathfrak{V} \geq r\right) \leq \exp\left(-\pi(2r\mathbb{E}[R] + r^2)\right) \sum_{n\geq 0} \frac{(\pi(2r\mathbb{E}[R] + r^2))^n}{n!}(1 - P\left(\delta_{m_r}, n\right)).$$

For the estimation of $(1 - P\left(\delta_a, n\right))$, Shepp obtained a basic inequality [21] which holds for $a \in [0, 1/4]$ and $n \in \mathbb{N}^*$ and is easier to use than Steven's explicit formula:

$$1 - P\left(\delta_a, n\right) \leq \frac{2(1-a)^{2n}}{\int_0^a (1-a-t)^n \,\mathrm{d}t + (\frac{1}{4} - a)(1-2a)^n}. \tag{3.11}$$

A straightforward consequence of (3.11) is that for every $a \in [0, 1/4]$ and $n \in \mathbb{N}$, we have

$$1 - P\left(\delta_a, n\right) \leq 2(n+1)(1-a)^{n-1}. \tag{3.12}$$

In particular, for r sufficiently large, the mean m_r is in the interval $[0, 1/4]$ so the equality (2.3) combined with (3.12) leads us to

$$\mathbb{P}\left(\mathfrak{V} \geq r\right) \leq 2(\pi(2r\mathbb{E}[R] + r^2) + 2)\exp(-\pi(2r\mathbb{E}[R] + r^2)m_r).$$

It remains to use the estimation on the mean m_r to get that

$$\limsup_{r\to+\infty} \frac{1}{r} \log \mathbb{P}\left(\mathfrak{V} \geq r\right) \leq -2\mathbb{E}[R]. \tag{3.13}$$

(3.9) and (3.13) now complete the proof of Theorem 3.1 for random discs.

3.3. *The case of general convex shapes.* Let K be a random convex body of $\mathbb{R}^2$ containing the origin, which is supposed to be invariant under any rotation and is such that its diameter is bounded almost surely by a constant $D > 0$. For instance, $\mathcal{K}$ can be the image of a deterministic convex body by a uniform random rotation. By the rotation-invariance of K, we have $\mathbb{E}[W_u(K)] = \mathbb{E}[\mathbf{W}(K)]$.

THEOREM 3.2. *For the Boolean model with random rotation-invariant grains distributed as K, the asymptotics of the visibility is given by*

$$\lim_{r\to+\infty} \frac{1}{r} \log \mathbb{P}\left(\mathfrak{V} \geq r\right) = -\mathbb{E}[\mathbf{W}(K)].$$

Proof. The lower bound is obtained by lemma 1.1. It remains to show that

$$\limsup_{r\to+\infty} \frac{1}{r} \log \mathbb{P}\left(\mathfrak{V} \geq r\right) \leq -\mathbb{E}[\mathbf{W}(K)].$$

In order to do it, we need an intermediary geometric lemma whose proof is postponed to the appendix 2:

LEMMA 3.1. *Let L be a convex body of diameter bounded by D containing the origin. We define $\Psi(r\mathbf{u}+L)$ as the angle of vision of $(r\mathbf{u}+L)$ from O. Then*

$$\lim_{r\to+\infty} r\Psi(r\mathbf{u}+L) = W_u(L),$$

the limit being uniform over all unit-vectors $\mathbf{u}$ and all such convex bodies L.

As in the proof of Proposition 2.1, the event $\{\mathfrak{V} \geq r\}$, $r > 0$, can be seen as the uncovering of the circle $C(0,r) = r\mathbb{S}^1$ by the 'shadows' produced by the obstacles $(x \oplus K_x)$ such that $(x \oplus K_x) \cap B_2(O,r) \neq \emptyset$.

Let $\varepsilon > 0$. By Lemma 3.1, let us fix $r_\varepsilon > 0$ such that for every x such that $\|x\| > r_\varepsilon$ and every convex body L (with a diameter bounded by D), we have

$$\|x\|\Psi(x+L) \geq (W_{x/\|x\|}(L) - \varepsilon). \tag{3.14}$$

Then for $r > r_\varepsilon + D$, the probability of uncovering the circle $C(0,r)$ is greater if we only keep the shadows produced by the obstacles $(x \oplus K_x)$ such that $r_\varepsilon < \|x\| < r - D$. In that case, such a shadow is a random rotation-invariant arc on the circle $C(O,r)$ whose normalised length is $(2\pi)^{-1}\Psi(x \oplus K_x)$. Let us denote by η_r the mean of $(2\pi)^{-1}\Psi(Z \oplus K_Z)$ when Z is uniformly distributed in $B_2(r-D) \setminus B_2(r_\varepsilon)$ and K_Z is independent from Z and distributed as K. Following the method already used to obtain the upper-bound (3.10), we have

$$\mathbb{P}\left(\mathfrak{V} \geq r\right) \leq e^{-\pi((r-D)^2 - r_\varepsilon^2)} \sum_{n=0}^{+\infty} \frac{\pi((r-D)^2 - r_\varepsilon^2)}{n!}(1 - P(\delta_{\eta_r}, n)).$$

Forecasting that η_r will be small enough, we may apply the inequality (3.12) in order to obtain that

$$\mathbb{P}\left(\mathfrak{V} \geq r\right) \leq 2(\pi((r-D)^2 - r_\varepsilon^2) + 2)\exp(-\pi((r-D)^2 - r_\varepsilon^2)\eta_r). \tag{3.15}$$

Let us now estimate the mean η_r: using (3.14), we get

$$\begin{aligned}
\eta_r &= \frac{1}{2\pi} E_K \left[\int_0^{2\pi} \int_{r_\varepsilon}^{r-D} \Psi(\rho u_\theta \oplus K) \rho \frac{\mathrm{d}\rho}{\pi((r-D)^2 - r_\varepsilon^2)} \, \mathrm{d}\theta \right] \\
&\geq \frac{1}{2\pi} E_K \left[\int_0^{2\pi} (W_\theta(K) - \varepsilon) \, \mathrm{d}\theta \right] \cdot \int_{r_\varepsilon}^{r-D} \frac{\mathrm{d}\rho}{\pi((r-D)^2 - r_\varepsilon^2)} \\
&= (\mathbb{E}[\mathbf{W}(K)] - \varepsilon) \int_{r_\varepsilon}^{r-D} \frac{\mathrm{d}\rho}{\pi((r-D)^2 - r_\varepsilon^2)} \\
&\underset{r \to +\infty}{\sim} \frac{\mathbb{E}[\mathbf{W}(K)] - \varepsilon}{\pi r}.
\end{aligned}$$

Inserting this last result in (3.15), we obtain that

$$\limsup_{r \to +\infty} \frac{1}{r} \log \mathbb{P}(\mathfrak{V} \geq r) \leq -(\mathbb{E}[\mathbf{W}(K)] - \varepsilon).$$

When ε goes to 0, we get the required result. □

4. Rough estimates in dimension greater than three. The problem of maximal visibility in a Boolean model is investigated in $\mathbb{R}^d$ with deterministic radii $R_x = R$, $x \in \mathbf{X}$. The obstacles are balls of deterministic radius R. The same connection between the distribution of $\mathfrak{V}$ and the non-covering of the sphere by random circular caps occurs. What prevents us from obtaining the analogue of Theorem 3.2 is that the calculation of the probability to cover the sphere with caps of random radii is not known. We have to restrict ourselves to coverings of the sphere with caps of deterministic radii. This explains that the following result is weaker than Theorem 3.2:

PROPOSITION 4.1. *In dimension $d \geq 3$, we have*

$$\liminf_{r \to +\infty} \frac{1}{r} \log \mathbb{P}(\mathfrak{V} \geq r) \geq -\omega_{d-1} R^{d-1}$$

and

$$\limsup_{r \to +\infty} \frac{1}{r} \log \mathbb{P}(\mathfrak{V} \geq r) \leq -\frac{1}{d} \omega_{d-1} R^{d-1}.$$

Proof. As in the two-dimensional case, the lower-bound is obtained by considering the visibility in a fixed direction.

Let us focus on the upper-bound: the maximal visibility is larger than $r > 0$ if and only if the shadows produced by the obstacles on the sphere centred at the origin and of radius r do not cover that sphere. The concerned

balls are those such that their centres are at distance $\rho \in [R, R+r]$ from the origin. Since we look for an upper-bound of a probability of non-covering of the sphere by random circular caps, we can take less and smaller caps. For sake of simplicity, we only keep the shadows produced by the balls with a centre at distance $\rho \in [R, r]$.

For such a ball, it comes from (3.6) that the angular radius of its shadow on the sphere is at least $\arcsin\left(\frac{R}{r}\right)$, which is bigger than $\frac{R}{r}$.

In conclusion, the probability that the maximal visibility is greater than r is lesser than the probability of non-covering of the unit-sphere by a Poissonian number (of mean $\omega_d(r^d - R^d)$) of circular caps of angular radius $\frac{R}{r}$. Upper-bounds for covering probabilities of the unit-sphere have been provided by Gilbert [9] in dimension three, Hall [10] in any dimension when the unit-sphere is replaced by the unit-cube and more recently by Bürgisser, Cucker & Lotz [3]. A very minor consequence of Theorem 1.1 of this last work is the following: let $\widetilde{P}(f, n)$ denote the probability to cover the unit-sphere with n random circular caps which are independent, with uniformly-distributed centres and with a fractional area of f. Then

$$\lim_{n\to+\infty} \frac{1}{n} \log(1 - \widetilde{P}(f, n)) = \log(1 - f). \tag{4.1}$$

In particular, the fractional area occupied by a circular cap of angular radius R/r is

$$f_r = \frac{(d-1)\omega_{d-1}}{d\omega_d} \int_0^{R/r} \sin^{d-2}(\theta)\, \mathrm{d}\theta \underset{r\to+\infty}{\sim} \frac{\omega_{d-1} R^{d-1}}{d\omega_d r^{d-1}}.$$

Consequently, we have

$$\mathbb{P}\left(\mathfrak{V} \geq r\right) \leq e^{-\omega_d(r^d - R^d)} \sum_{n=0}^{+\infty} \frac{(\omega_d(r^d - R^d))^n}{n!} (1 - \widetilde{P}(f_r, n))$$

and a direct application of (4.1) provides that

$$\log \mathbb{P}\left(\mathfrak{V} \geq r\right) \underset{r\to+\infty}{\lesssim} -\omega_d r^d f_r \underset{r\to+\infty}{\sim} -\frac{\omega_{d-1}}{d} R^{d-1} r.$$

This completes the proof of Proposition 4.1. □

Remark 4.1. *Proposition 4.1 implies that the total visibility $\mathfrak{V}$ is finite almost surely. Nevertheless, when the intensity measure of the underlying Poisson point process is of the form (in spherical coordinates) $r^{\alpha-1}\mathrm{d}r\mathrm{d}\sigma_d(u)$, $\alpha \in \mathbb{R}$, it can be shown in the same way that the visibility at infinity exists with positive probability as soon as $\alpha < 1$.*

5. Small or distant obstacles: convergence towards the law of extreme values. When the size of the covering objects becomes smaller and the number of objects grows at the same time accordingly, S. Janson [13] showed in very general setting that a particular scaling yields a convergence towards the Gumbel law. We shall use this kind of result in two contexts below:

- the asymptotics of the visibility when the obstacles are small (or equivalently when the intensity of the centres is small);
- the study of the visibility when there exists a large region around the origin with no obstacle at all.

5.1. *Small obstacles.* In this subsection, the radius R of the obstacles will be deterministic but no longer constant. For sake of clarity, we will denote by $\mathfrak{V}_R$ the visibility when the obstacles are discs of radius $R > 0$. We aim at giving the asymptotic behaviour of the visibility when the size of the obstacles goes to 0. Let us define the quantity

$$\xi_R = \omega_{d-1} R^{d-1} \mathfrak{V}_R + d(d-1)\log(R) - 2(d-1)\log(-\log(R)) - K_d \tag{5.1}$$

where

$$K_d = \log\left(\frac{d^{2(d-1)}(d-1)^{3(d-1)-1}\Gamma\left(\frac{d}{2}-\frac{1}{2}\right)^{2d-2}}{(d-1)!\pi^{\frac{(d-1)^2+1}{2}}2^{2d-3}\Gamma\left(\frac{d}{2}\right)^{d-2}}\right).$$

THEOREM 5.1. *When R goes to 0, the quantity ξ_R (provided by (5.1)) converges in distribution to the extreme value distribution, i.e. for every $u \in \mathbb{R}$,*

$$\lim_{R\to 0} \mathbb{P}\left(\xi_R \le u\right) = \exp\left(-e^{-u}\right).$$

Proof. The proof relies essentially on the application of a result due to Janson (Lemma 8.1. in [13]) about random coverings of a compact Riemannian manifold by small geodesic balls.

As before, we exploit the connection between the cumulative distribution function of $\mathfrak{V}_R$ and the probability of covering the sphere with circular caps:

$$\begin{aligned}\mathbb{P}\left(\xi_R \le u\right) &= \mathbb{P}\left(\mathfrak{V}_R \le -\frac{d(d-1)}{\omega_{d-1}}\frac{\log(R)}{R^{d-1}} + \frac{2(d-1)}{\omega_{d-1}}\frac{\log(-\log(R))}{R^{d-1}} + \right.\\ &\qquad \left.\frac{K_d+u}{\omega_{d-1}}\frac{1}{R^{d-1}}\right)\\ &= \mathbb{P}(\text{the sphere of radius } f(R) \text{ is covered}\\ &\qquad\qquad \text{by circular caps coming from the obstacles})\end{aligned} \tag{5.2}$$

where

$$f(R) = -\frac{d(d-1)}{\omega_{d-1}}\frac{\log(R)}{R^{d-1}} + \frac{2(d-1)}{\omega_{d-1}}\frac{\log(-\log(R))}{R^{d-1}} + \frac{K_d+u}{\omega_{d-1}}\frac{1}{R^{d-1}}.$$

Let us focus on this covering probability: the concerned obstacles are those such that their centres x are at distance $\rho \in (R, R+f(R))$. Their number is Poisson distributed, of mean $\omega_d((R+f(R))^d - R^d)$. The set of all $x/\|x\|$, where $x \in \mathbf{X} \cap [B(O, R+f(R)) \setminus B(O,R)]$, is a homogeneous Poisson point process on the unit-sphere of intensity $((f(R)+R)^d - R^d)/d$.

The induced shadow of each of theses obstacles is a geodesic ball on the unit-sphere of angular radius equal to $\arcsin(R/\rho)$ if $\rho \in (R, \sqrt{R^2+f(R)^2})$ and equal to $\arccos((f(R)^2+\rho^2-R^2)/2f(R)\rho)$ for $\rho \in [\sqrt{R^2+f(R)^2}, R+r)$ (see (3.6)). In particular, it can be verified that the normalized geodesic radius Θ_R of this circular cap satisfies that

$$\frac{1}{a_R}\Theta_R \overset{D}{\to} \mathbf{1}_{[1,+\infty)}(u)\frac{d}{u^{d+1}}\,\mathrm{d}u. \tag{5.3}$$

where $a_R = \frac{R}{f(R)} \underset{R\to 0}{\to} 0$. Consequently, the required covering probability in (5.2) is the probability that the unit-sphere is covered by a Boolean model on the sphere of intensity $\lambda_R = \frac{(f(R)+R)^d - R^d}{d}$, such that the geodesic balls have i.i.d. radii distributed as Θ_R (with Θ_R satisfying the convergence (5.3)).

It only remains to verify that all the hypotheses of Lemma 8.1. in [13] are satisfied ($\mathbb{S}^{d-1}$ being a $(d-1)$-dimensional Riemannian manifold):

- the only notable difference is that we should not have $\frac{1}{a_R}\Theta_R$ converging in distribution but have it of fixed distribution for any $R > 0$. Nevertheless, the proof of Lemma 8.1. in [13] relies essentially on convergence results [(7.15), (7.20), *ib.*] which also work in this context without any changes;
- the moments of order $((d-1)+\varepsilon)$ of the limit distribution obtained in (5.3) are finite for every $\varepsilon \in (0,1)$. Moreover, the moment of order $(d-1)$ is d and the moment of order $(d-2)$ is $d/2$;
- the constants b and α defined in ([13], Lemma 8.1) can be calculated:

$$b = \frac{\omega_{d-1}}{d\omega_d}\int_1^{+\infty}\frac{d}{u^2}\,\mathrm{d}u = \frac{\omega_{d-1}}{\omega_d}$$

and

$$\alpha = \frac{1}{d!}\left(\frac{\sqrt{\pi}\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(\frac{d+1}{2}\right)}\right)^{d-1}\cdot\frac{\left(\frac{d}{2}\right)^{d-1}}{d^{d-2}};$$

- The convergence (8.1) in [13] is satisfied:

$$\lim_{R\to 0}\Big\{ba_R^{d-1}d\omega_d\lambda_R+\log(ba_R^{d-1})+ \\ (d-1)\log(-\log(ba_R^{d-1}))-\log(\alpha)\Big\}=u.$$

Consequently, the proof of Theorem 5.1 is complete.

□

REMARK 5.1. *The same type of method and result should also occur in dimension two when the discs are replaced by rotation-invariant i.i.d convex bodies.*

REMARK 5.2. *In any dimension, the result could be extended to radii of the form $R_x=\varepsilon U_x$, $x\in\mathbf{X}$, where ε goes to 0 and the U_x are i.i.d. bounded random variables.*

5.2. *Conditioning by a large clearing.* We define here S the clearing radius as

$$S=\sup\{r>0; B_2(O,r)\subset\mathbb{R}^2\setminus\mathfrak{O}\}.$$

The distribution of S is called the spherical contact distribution. This section aims at estimating the distribution of the maximal visibility $\mathfrak{V}$ conditionally on S. In particular we show that when S is large, $\mathfrak{V}$ is asymptotically equivalent to S (see Theorem 5.2) and we estimate precisely the difference $(\mathfrak{V}-S)$ via an extreme value result (see Theorem 5.3).

A first estimation based on techniques similar to the proofs of (3.3) and (3.4) provides the following result:

THEOREM 5.2. *For every $\alpha\in(0,1)$, we have*

$$\mathbb{P}\left(\mathfrak{V}\geq r+r^\alpha|S=r\right)\leq\mathbb{P}\left(\mathfrak{V}\geq r+r^\alpha|S\geq r\right)=O\left(e^{-2E(R)r^\alpha}\right).$$

Proof. Let us fix $r>0$. Conditionally on $\{S\geq r\}$, the process of couples (x,R_x) is a Poisson point process on $\mathbb{R}^2\times\mathbb{R}_+$ of intensity measure $\mathbf{1}_{\|x\|-R>r}dx\otimes\mu$.

As in Proposition 2.1, $\mathfrak{V}$ is greater than $r+u$, $u>0$, if and only if the circle $C(O,r+u)$ is not totally hidden by the 'shadows' of the obstacles. Moreover, the discs $B_2(x,R_x)$ which produce a non-empty shadow are those which satisfy $r<\|x\|-R_x<(r+u)$. The formula for the length of the shadow depends on whether $\|x\|\leq\sqrt{(r+u)^2+R_x^2}$ or not (see equalities

(3.6)). Consequently, if we only consider the shadows produced by the discs $B_2(x, R_x)$ such that

$$(r + R_x < \|x\| < \sqrt{(r+u)^2 + R_x^2}) \quad \text{and} \quad u > \sqrt{r^2 + 2rR_x} - r,$$

then the probability of not covering the circle is greater.
When $u > \sqrt{r^2 + 2rR^*} - r$, the number of such discs is Poissonian, of mean $\pi(u^2 + 2ru - 2rE(R))$. Moreover, for these discs, the length of the shadow decreases with $\|x\|$ and is minimal when $\|x\| = \sqrt{(r+u)^2 + R_x^2}$, equal to $L_{\min} = (1/\pi)\arcsin(R_x/((r+u)^2 + R_x^2))$ (see (3.6)). In the sequel, we denote by $\nu_{r,u}$ the distribution of $L_{\min}$ and $m_{r,u}$ its mean.

In conclusion, we have proved the following inequality: for every $u > \sqrt{r^2 + 2rR^*} - r$,

$$\begin{aligned}
&\mathbb{P}\left(\mathfrak{V} \geq r + u | L \geq r\right) \\
&\leq e^{-\pi(u^2+2ru-2rE(R))} \sum_{n=0}^{+\infty} \frac{(\pi(u^2 + 2ru - 2rE(R)))^n}{n!} (1 - P(\nu_{r,u}, n)) \\
&\leq e^{-\pi(u^2+2ru-2rE(R))} \sum_{n=0}^{+\infty} \frac{(\pi(u^2 + 2ru - 2rE(R)))^n}{n!} (1 - P(\delta_{m_{r,u}}, n)). \qquad (5.4)
\end{aligned}$$

In particular, when $u = r^\alpha$, $0 < \alpha < 1$, we have

$$m_{r,r^\alpha} \underset{r\to+\infty}{\sim} \frac{1}{\pi}\frac{E(R)}{r} \quad \text{and } \pi(u^2 + 2ru - 2rE(R)) \underset{r\to+\infty}{\sim} 2\pi r^{1+\alpha}$$

Using the inequality (3.12) and inserting the two previous estimates in (5.4), we obtain the required result, i. e.

$$\mathbb{P}\left(\mathfrak{V} \geq r + r^\alpha | S \geq r\right) = O\left(e^{-2E(R)r^\alpha}\right).$$

Finally, it remains to study the distribution of $\mathfrak{V}$ conditionally on $S = r$. We remark that conditionally on $\{S = r\}$, the process is the same as in the case of the conditioning on $\{S \geq r\}$ with a supplementary random disc $B_2(Y, R_Y)$ such that R_Y is μ-distributed and conditionally on R_Y, Y is uniformly distributed on the circle $C(O, r + R_Y)$. Since there is one more obstacle, the maximal visibility must be lesser than in the case of the conditioning on $\{S \geq r\}$. □

Theorem 5.2 implies that the difference $(\mathfrak{V} - S)$ is negligible in front of S but we can get a far more precise three-terms development in the following way: for every $r > 0$, we denote by $\mathfrak{V}_r$ a random variable distributed as $\mathfrak{V}$

when the Boolean model is conditioned on $\{S \geq r\}$, *i.e.* on not having any grain at distance lesser than r from the origin. Let us define the quantity

$$\psi_r = \omega_{d-1} E_\mu(R^{d-1})(\mathfrak{V}_r - r) - (d-1)\log(r) - (d-1)\log(\log(r)) - K'_d \tag{5.5}$$

where

$$K'_d = \log\left(\frac{1}{(d-1)!}\left(\frac{\sqrt{\pi}\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\right)^{d-2}\frac{E_\mu(R^{d-2})^{d-1}}{E_\mu(R^{d-1})^{d-2}}\right)$$
$$+(d-1)\log(d-1) - \log\left(\frac{\omega_{d-1}E_\mu(R^{d-1})}{d\omega_d}\right).$$

For every $t \in \mathbb{R}$, we have

$$\begin{aligned}\mathbb{P}(\psi_r \leq t) &= \mathbb{P}\Big(\omega_{d-1}E_\mu(R^{d-1})(\mathfrak{V} - r) - \\ &\qquad (d-1)\log(r) - (d-1)\log(\log(r)) - K \leq t | S \geq r\Big).\end{aligned}$$

THEOREM 5.3. *When r goes to ∞, the quantity ψ_r (provided by (5.5)) converges in distribution to the extreme value distribution, i.e. for every $t \in \mathbb{R}$,*

$$\lim_{r\to+\infty} \mathbb{P}(\psi_r \leq t) = \exp\left(-e^{-t}\right).$$

Proof. As previously in the case of small obstacles, the proof relies essentially on the application of a result due to Janson (Lemma 8.1. in [13]) about random coverings of a compact Riemannian manifold by small geodesic balls. Let us consider the quantity

$$f(r) = \frac{d-1}{\omega_{d-1}E_\mu(R^{d-1})}\log(r) + \frac{d-1}{\omega_{d-1}E_\mu(R^{d-1})}\log(\log(r)) + \frac{K+t}{\omega_{d-1}E_\mu(R^{d-1})}.$$

such that $(\psi_r \leq t) \iff (\mathfrak{V}_r - r \leq f(r))$. The connection with a covering probability is the following:

$$\begin{aligned}\mathbb{P}(\psi_r \leq u) &= \mathbb{P}(\text{the sphere of radius } (r + f(r)) \text{ is covered} \\ &\qquad \text{by circular caps coming from the obstacles}).\end{aligned}$$

It remains to investigate asymptotics of this covering probability: the concerned obstacles are those such that their centers x are at distance $\rho \in (r+R, r+f(r)+R)$. Their number is Poisson distributed, of mean $\omega_d E_\mu[(R+f(r)+r)^d - (R+r)^d]$.

The induced shadow of each of theses obstacles is a geodesic ball on the unit-sphere of angular radius equal to:

- $\arcsin(R/\rho)$ if $\rho \in (R+r, \sqrt{R^2+(f(r)+r)^2})$,
- $\arccos(((f(r)+r)^2+\rho^2-R^2)/2(f(r)+r)\rho)$ if $\rho \geq \sqrt{R^2+(f(r)+r)^2}$ and $\rho < R+r+f(r))$ (see (3.6)).

In particular, it can be verified that the normalized geodesic radius Θ_r of this circular cap satisfies that $r\Theta_r \xrightarrow{D} \mu$. In the rest of the proof, we will use the quantity $a_r = 1/r$ in order to be as close as possible to the notations of Janson's lemma.

Consequently, the required covering probability is the probability that the unit-sphere is covered by a Boolean model on the sphere of intensity

$$\lambda_r = \frac{1}{d} E_\mu[(R+f(r)+r)^d - (R+r)^d] \underset{r\to+\infty}{\sim} r^{d-1} f(r),$$

such that the geodesic balls have i.i.d. radii distributed as Θ_r.

As in the proof of Theorem 5.1, we verify that all the hypotheses of Lemma 8.1. in [13] are satisfied:

- the moments of order $((d-1)+\varepsilon)$ of the limit distribution μ are finite for every $\epsilon > 0$.
- the constants b and α defined in ([13], Lemma 8.1) can be calculated:

$$b = \frac{\omega_{d-1} E_\mu(R^{d-1})}{d\omega_d}$$

and

$$\alpha = \frac{1}{(d-1)!}\left(\frac{\sqrt{\pi}\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\right)^{d-2} \cdot \frac{E_\mu(R^{d-2})^{d-1}}{E_\mu(R^{d-1})^{d-2}};$$

- The convergence (8.1) in [13] is satisfied:

$$\lim_{r\to+\infty} \Big\{ b a_r^{d-1} d\omega_d \lambda_r + \log(b a_r^{d-1}) + \\ (d-1)\log(-\log(b a_r^{d-1})) - \log(\alpha) \Big\} = t.$$

Consequently, the proof of Theorem 5.1 is complete. □

Appendix 1: proof of the lower bound via direct computation, estimate (3.2). It is quite reasonable to try to obtain directly a lower bound on the tail probability, let us explain the sketch of the proof: the visibility is greater than r if and only if there exists a direction in which one can see farther, so that if one discretises the circle $\partial B_2(0,r)$, one could argue that there exists one of those directions such that the visibility is greater

than r, the number of those directions is $2/\pi r$, hence the order $r\ \exp(-2rR)$. We shall make this statement more rigorous below.

Let us take $\zeta \in (0, 2/R)$, we define N_r as the integer part of ζs and $\theta_r = 2\pi/(\zeta r)$, and define the points $A_{k,r} = (r, k\theta_r)$ for $k \in \{0, \dots, N_r - 1\}$. We see easily that if we define

$$G_{k,r} = (B_2(0,R) \oplus [0, A_{k,r}]) \setminus B_2(0,R),$$

then for $r > R$ and $k_1 \neq k_2$ one has

$$G_{k_1,r} \cap G_{k_2,r} \subset B_2(0,r).$$

The sets $G_{k,r}$ are called 'fingers', we denote by $\rho_r = R/\sin(\theta_r/2)$ the maximal norm of a point belonging to the intersection of two fingers, we shall denote by $E_{k,r}$ the intersection $G_{k,r} \cap B_2(0,\rho_r)$ and $F_{k,r} = G_{k,r} \setminus E_{k,r}$.

We will assume from now on that r is large enough. If at least one of those points $A_{k,r}$ is not shadowed by the discs intersecting $B_2(0,r)$, the visibility $\mathfrak{V}$ is greater than r: hence the probability of this event is greater than the probability that one of the 'fingers' $G_{k,r}$ in figure 2 does not contain a point of $\mathbf{X}$.

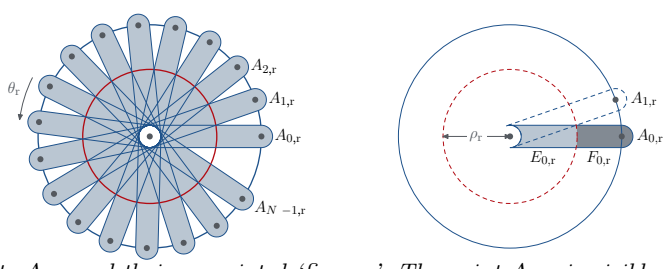

FIG 2. *The points $A_{i,r}$ and their associated 'fingers'. The point $A_{k,r}$ is visible if and only if no point of the process $\mathbf{X}$ belongs to $G_{k,r} = E_{k,r} \cup F_{k,r}$.*

We have

$$\mathrm{Leb}_2\left(F_{k,r}\right) = 2R(1-\kappa)s + \mathrm{O}(1),$$

where $\kappa = R\zeta\pi^{-1}$, so that $1 - \kappa > 0$, and $\mathrm{O}(1)$ means a bounded function. Let us define the following events

$$\begin{aligned} V_{k,r} &= \left\{\mathbf{X} \cap G_{k,r} = \varnothing\right\}, \\ Z_{k,r} &= \left\{\mathbf{X} \cap E_{k,r} = \varnothing\right\}, \\ W_{k,r} &= \left\{\mathbf{X} \cap F_{k,r} = \varnothing\right\}. \end{aligned}$$

We have

$$V_{k,r} = Z_{k,r} \cap W_{k,r},$$

and we want to evaluate $\mathbb{P}\left(\bigcup_{k=0}^{N_r-1} V_{k,r}\right)$, this is equal thanks to Poincaré's formula to

(5.6)

$$\mathbb{P}\left(\bigcup_{k=0}^{N_r-1} V_{k,r}\right) = \sum_{k=0}^{N_r-1} \mathbb{P}\left(V_{k,r}\right) + \sum_{\ell=2}^{N_r} (-1)^{\ell-1} \sum_{k_1<\cdots<k_\ell} \mathbb{P}\left(V_{k_1,r} \cap \cdots \cap V_{k_\ell,r}\right).$$

We shall prove that the dominating term in this expansion is the first one: it rewrites as

$$\begin{aligned} \sum_{k=0}^{N_r-1} \mathbb{P}\left(V_{k,r}\right) &= \sum_{k=0}^{N_r-1} \exp\left(-\operatorname{Leb}_2\left(E_{k,r} \cup F_{k,r}\right)\right) \\ &= N_r \exp\left(-2Rr\right). \end{aligned}$$

Let us consider $\ell \geq 2$ and $0 \leq k_1 < \cdots < k_\ell < N_r$, we have:

$$\mathbb{P}\left(V_{k_1,r} \cap \cdots \cap V_{k_\ell,r}\right) = \mathbb{P}\left(\bigcap_{i=1}^{\ell} Z_{k_i,r} \cap \bigcap_{i=1}^{\ell} W_{k_i,r}\right),$$

where the events $\bigcap_{i=1}^{\ell} Z_{k_i,r}$, $W_{k_1,r}$, $\ldots$, $W_{k_\ell,r}$ are independent as the sets $\bigcup_{i=1}^{\ell} E_{k_i,r}$ and $F_{k_1,r}, \ldots, F_{k_\ell,r}$ are disjoint, hence:

$$\mathbb{P}\left(V_{k_1,r} \cap \cdots \cap V_{k_\ell,r}\right) = \exp\left(-\operatorname{Leb}_2\left(\bigcup_{i=1}^{\ell} E_{k_i,r}\right)\right) \exp\left(-\ell \operatorname{Leb}_2\left(F_{0,r}\right)\right).$$

To estimate this probability, let us introduce the triangle T_r which is the greatest triangle included in $E_{0,r} \setminus \left(\cup_{i=1}^{N_r-1} E_{k,r}\right)$ (see 3). We easily get:

$$\operatorname{Leb}_2\left(T_r\right) = 2R\,\frac{\kappa}{4} s + \mathrm{O}(1).$$

For each k, $E_{k,r}$ contains a triangle that is isometric to T_r and disjoint from all others $E_{k',r}$, hence we have:

$$\begin{aligned} \operatorname{Leb}_2\left(\bigcup_{i=1}^{\ell} E_{k_i,r}\right) &\geq (\ell-1) \operatorname{Leb}_2\left(T_r\right) + \operatorname{Leb}_2\left(E_{0,r}\right), \\ &\geq \ell \operatorname{Leb}_2\left(T_r\right) + \operatorname{Leb}_2\left(E_{0,r}\right) - \operatorname{Leb}_2\left(T_r\right). \end{aligned}$$

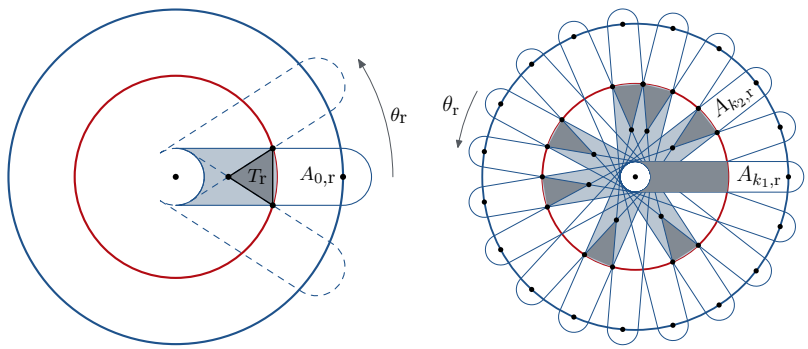

FIG 3. *Left: definition of T_r. Right: the set $E_{k_1,r} \cup \cdots \cup E_{k_\ell,r}$ in light grey and in dark grey the subset with area* $\mathrm{Leb}_2(E_{0,r}) + (\ell-1)\,\mathrm{Leb}_2(T_r)$.

Thus for all choice of $\ell \geq 2$ and $0 \leq k_1 < k_2 < \cdots < k_\ell < N_r$ we have:

$$\exp\left(-\mathrm{Leb}_2\left(B_2(0,\rho_r)\right)\right)\exp\left(-\ell\,\mathrm{Leb}_2\left(F_{0,r}\right)\right) \leq \mathbb{P}\left(\cap_{i=1}^{\ell} V_{k_i,r}\right) \leq$$
$$\exp\left(-\mathrm{Leb}_2\left(E_{0,r}\right)+\mathrm{Leb}_2\left(T_r\right)\right)\exp\left(-\ell(\mathrm{Leb}_2\left(F_{0,r}\right)+\mathrm{Leb}_2\left(T_r\right))\right).$$

The number of such terms is $\binom{N_r}{\ell}$, hence the sum S_ℓ of all those terms satisfies

$$|S_\ell| \leq \exp\left(\mathrm{Leb}_2\left(T_r\right)-\mathrm{Leb}_2\left(E_{0,r}\right)\right)\binom{N_r}{\ell}\exp\left(-\ell(\mathrm{Leb}_2\left(F_{0,r}\right)+\mathrm{Leb}_2\left(T_r\right))\right)$$

and the residual term $S = \sum_{\ell=2}^{N_r}(-1)^{\ell-1}S_\ell$ is bounded from above by:

$$\begin{aligned}
|S| \;&\leq\; \exp\left(\mathrm{Leb}_2\left(T_r\right)-\mathrm{Leb}_2\left(E_{0,r}\right)\right)\times \\
&\quad \sum_{\ell=2}^{N_r}\binom{N_r}{\ell}\exp\left(-\ell(\mathrm{Leb}_2\left(F_{0,r}\right)+\mathrm{Leb}_2\left(T_r\right))\right), \\
&\leq\; \exp\left(\mathrm{Leb}_2\left(T_r\right)-\mathrm{Leb}_2\left(E_{0,r}\right)\right)\times \\
&\quad \Big(\left(1+\exp\left(-(\mathrm{Leb}_2\left(F_{0,r}\right)+\mathrm{Leb}_2\left(T_r\right))\right)\right)^{N_r} \\
&\quad -1-N_r\exp\left(-(\mathrm{Leb}_2\left(F_{0,r}\right)+\mathrm{Leb}_2\left(T_r\right))\right)\Big), \\
&\leq\; \frac{N_r^2}{2}\exp\left(\mathrm{Leb}_2\left(T_r\right)-\mathrm{Leb}_2\left(E_{0,r}\right)\right)\times \\
&\quad \exp\left(-2(\mathrm{Leb}_2\left(F_{0,r}\right)+\mathrm{Leb}_2\left(T_r\right))\right)(1+\mathrm{O}(1)).
\end{aligned} \tag{5.7}$$

Using the asymptotic expansions of $F_{0,r}$, $E_{0,r}$ and T_r and the unpper bound

$N_r \leq \zeta r$, we obtain:

$$\begin{aligned} |S| &\leq \frac{1}{2}\exp\left(-\frac{3}{2}R\kappa s + \mathrm{O}(1)\right)\zeta^2 r^2 \times \\ &\quad \exp\left(-2R\left(2-\frac{3}{2}\kappa\right)r + \mathrm{O}(1)\right)(1+\mathrm{O}(1)) \\ &\leq C\zeta^2 r^2 \exp\left(-2Rr\left(1+\left(1-\frac{3}{4}\kappa\right)\right)\right). \end{aligned}$$

hence $S = \mathrm{o}(\zeta r \exp(-2Rr))$, which completes the proof of estimate 3.2.

Remark 5.3. *In the proof above, we could have taken only one triangle to obtain that the sum S is negligible with respect to the first term $\zeta s \exp(-2Rr)$, however the accuracy of the development would have been less interesting. Discs with bounded random radius $R \in [R_\star, R^\star]$ can also be treated this way, at a cost of a loss on the accuracy because of a non-optimal size of the fingers.*

Appendix 2: proof of Lemma 3.1. For sake of simplicity, we call $x = r\mathbf{u}$. Let us denote by y (resp. y') a point in the intersection of $(r\mathbf{u}+L)$ with its tangent line emanating from O and situated on the left-hand side (resp. on the right-hand side) of the half-line $(O+\mathbb{R}_+\mathbf{u})$. We define z (resp. z') as the orthogonal projection of y (resp. y') on $(O+\mathbb{R}_+\mathbf{u})$ and α (resp. α') as the angle between $(O+\mathbb{R}_+\mathbf{u})$ and $(O+\mathbb{R}_+y)$ (resp. $(O+\mathbb{R}_+y')$). Then we have

$$\Psi(r\mathbf{u}+L) = \alpha + \alpha' = \arctan\left(\frac{||y-z||}{||z||}\right) + \arctan\left(\frac{||y'-z'||}{||z'||}\right). \tag{5.8}$$

Let us now describe $W_u(L)$: there exist two points w and w' (w being on the left-hand side of $(O+\mathbb{R}_+\mathbf{u})$) such that

$$W_u(L) = \mathrm{dist}(w, O+\mathbb{R}\mathbf{u}) + \mathrm{dist}(w', O+\mathbb{R}\mathbf{u}) \tag{5.9}$$

where $\mathrm{dist}(\cdot, O+\mathbb{R}\mathbf{u})$ is the Euclidean distance to the line $(O+\mathbb{R}\mathbf{u})$. Comparing (5.8) with (5.9), we observe that we only have to prove that

$$\lim_{r\to+\infty} r\alpha = \lim_{r\to+\infty} r\arctan\left(\frac{||y-z||}{||z||}\right) = \mathrm{dist}(w, O+\mathbb{R}\mathbf{u}) \tag{5.10}$$

and

$$\lim_{r\to+\infty} r\alpha' = \lim_{r\to+\infty} r\arctan\left(\frac{||y'-z'||}{||z'||}\right) = \mathrm{dist}(w', O+\mathbb{R}\mathbf{u}), \tag{5.11}$$

the limits being uniform as required. Let us now concentrate on the first limit (the second can be proved in the same way):

since $||y-z|| \leq ||y-x|| \leq D$ and $||z-x|| \leq ||y-x|| \leq D$, we have for every $r \geq D$

$$\tan(\alpha) = \frac{||y-z||}{||z||} \leq \frac{||y-z||}{r-D} \leq \frac{D}{r-D}. \tag{5.12}$$

Moreover,

$$0 \leq \mathrm{dist}(w, O+\mathbb{R}\mathbf{u}) - ||y-z|| = ||y-w|| \sin(\beta) \leq D \sin(\beta)$$

where β is the angle between $(O+\mathbb{R}_+\mathbf{u})$ and the line from y to w. Since y is a contact point of a support line of L and w is in L, this angle β must necessarily be lesser than α. Consequently, we get by a direct use of (5.12) that

$$0 \leq \mathrm{dist}(w, O+\mathbb{R}\mathbf{u}) - ||y-z|| \leq D\sin(\alpha) \leq D\tan(\alpha) \leq \frac{D^2}{r-D}. \tag{5.13}$$

Inserting this last estimate in the first equality of (5.12), we have

$$r \arctan\left(\frac{\mathrm{dist}(w, O+\mathbb{R}\mathbf{u}) - \frac{D^2}{r-D}}{r+D}\right) \leq r\alpha \leq r \arctan\left(\frac{\mathrm{dist}(w, O+\mathbb{R}\mathbf{u})}{r-D}\right),$$

which provides the required convergence result (5.10) with a uniformity in $\mathbf{u}$ and in L.

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