On the Representation of Stream Functions in Denotational Domains

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We investigate the representation of functions on streams in some denotational domains. As expected, a total continuous stream function can always be represented by a Scott-continuous function, and moreover by a strongly stable map in the corresponding Hypercoherence.

It seems however difficult to represent an arbitrary stream function by a monotone map on Scott domains such that the stream function is continuous if and only if its representant is Scott-continuous. The difficulty is that the set of Scott-approximants of an open subset of a not (topologically) compact set of streams may not be Scott-open. We show that this problem does not occur in the compact case.

1 Introduction

We investigate the representation of functions on streams in Scott domains for the corresponding PCF types [16].

As expected, it is easy to define from an *arbitrary* $f : A^\omega \to B^\omega$, a map f^* on the corresponding domains which is always Scott-continuous. Such an f^* always under-approximates f, and represents f if and only if f is continuous. As also expected, it is not difficult to adapt the definition scheme of f^* in order to get a Strongly Stable map in the corresponding Hypercoherences [3, 4].

The main question discussed in this paper is: how to "uniformly" define a (monotone) representant f^* for an arbitrary f, such that f^* is Scott-continuous if f is continuous. This question is motivated by the use of stream functions to model infinite computations in frameworks related to software verification such as Büchi automata (see e.g. [19, 8]), or in game based frameworks related to descriptive set theory [17]. These frameworks

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consider more than continuous functions or open sets of streams. It seems interesting to understand how Curry-Howard based interpretations can give an account of these phenomena. Since Scott-topology on type 1 maps is not far from the usual product topology on streams, we do not expect to give an account of these behaviours in usual Scott domains. However, as a first step it seems important to understand how a stream function f can, *first*, be represented by a monotone map f^* on the corresponding domain independently from the continuity of f, *and then*, to relate the continuity of f with the Scott-continuity of f^*.

The main results of this paper suggest that this is not as easy as it seems. The difficulty is that in general, the set of Scott-approximants (in the cpo $[\![A^\omega]\!]$ of Scott-continuous maps from the flat cpo $\omega_\perp$ to the flat cpo $A_\perp$) of an open subset of A^ω form a Scott-open subset of $[\![A^\omega]\!]$ only when A^ω is compact. This contrasts with the converse situation, where the set of streams $\alpha \in A^\omega$ whose lift $\alpha_\perp$ belong to a Scott-open subset of $[\![A^\omega]\!]$ is always an open subset of A^ω.

Technically, we relax the straightforward definition of f^* from a monotone map of Scott-finite elements (that we call *finitary* in this paper), and do not require that the result on a arbitrary element of a cpo is uniquely determined by the finitary elements below it. Such a relaxation seems necessary in order to prevent f^* from being automatically Scott-continuous.

Our result on Scott-topology makes this difficult (if not impossible) to work in the general case. We show that this works well when A^ω is topologically compact (*i.e.* when the alphabet A is finite). This corresponds to the kind of streams dealt-with in automata theory [19, 8].

Section 2 gives technical preliminaries and formulates the main question discussed in the paper. In Sect. 3, we gather (more or less well-known) material on the representation of continuous stream functions by Scott-continuous maps. Our main contributions are in Sect.4. Finally, we give in Sect. 5 an account in Hypercoherences of some of the material of Sect. 3. We conclude in Sect. 6, where we also briefly discuss other possible choices for the representation of streams in denotational domains.

2 Preliminaries

2.1 Sequences

Let ω be the set of natural numbers. An infinite sequence (or stream) over a set A is map $\alpha : \omega \to A$. Streams are the carrier of the final coalgebra for the functor $(-) \times A$ in **Set** [10]. We write A^ω for the set of streams over A and let $\alpha, \beta, \gamma, \dots$ range over streams. A finite sequence (or word) of length n over A is a map $p : \{0, \dots, n-1\} \to A$. We write $A^{<\omega}$ for the set of words over A and let $p, q, \dots$ range over words. Given a stream α and $n \in \omega$, the word $\alpha \restriction n$ is the restriction of α of domain $\{0, \dots, n-1\}$.

Seen as relations, streams and words over A are subsets of $\omega \times A$. Hence $p \subseteq \alpha$ means that $p \in A^{<\omega}$ is a finite prefix (and can be though of as a finite approximation) of $\alpha \in A^\omega$. More generally, given a partial map $s : \omega \rightharpoonup A$ (i.e. a subset of $\omega \times A$ satisfying the following *coherence* property[1]: if $(n,a)(n,a') \in s$ then $a = a'$), we let $[s]$ be the set of streams approximated by s:

$$[s] := \{\alpha \in A^\omega \mid s \subseteq \alpha\} \ .$$

2.2 Type 2 Continuity

Let A and B be at most countable. It is generally accepted that a "computable" total map f from A^ω to B^ω must only need to read a finite part of $\alpha \in A^\omega$ in order to produce a finite part of its answer in B^ω. This can be formalized by stating that for any input stream $\alpha \in A^\omega$ and any output index $m \in \omega$, there is an input index $n \in \omega$ such that for any $\beta \in A^\omega$ approximated by $\alpha \restriction n$, we have $f(\beta)(m) = f(\alpha)(m)$. This precisely means that f is a continuous map from A^ω to B^ω, where A, B are discrete and A^ω, B^ω are equipped with the corresponding product topology. It is convenient to reformulate the continuity of f as: for all $(m,b) \in \omega \times B$ and all $\alpha \in f^{-1}[\{(m,b)\}]$, there is some $p \in A^{<\omega}$ such that

$$\alpha \in [p] \subseteq f^{-1}[\{(m,b)\}] \ , \tag{1}$$

where f^{-1} is the inverse image of f:

$$f^{-1}S := \{\alpha \mid f(\alpha) \in S\} \qquad (S \subseteq B^\omega) \ .$$

2.3 Topology

2.3.1 Basic Definitions.

A *topological space* is a set X together with a collection $\Omega(X)$ of subsets of X such that $\Omega(X)$ contains both X and the empty set $\emptyset$, and is moreover preserved by finite intersections and arbitrary unions. A subset of X is *closed* if its complement w.r.t. X is open.

As advocated (and made precise in the relevant context) in [18] (see also [5]), the closure under arbitrary unions and finite intersections of opens suggests that "belonging to a given open set" may be through of as a kind of finite observation. Any set X can be endowed with the *discrete topology* $\Omega(X) = \mathcal{P}(X)$, meaning that any point of X is observable. In other words, the discrete topology is the finer possible topology on X since it distinguishes all its elements. At the other extreme, the coarsest topology on X contains only $\emptyset$ and X. More generally, given two topologies $\Omega(X)$ and $\Omega'(X)$ on X, we say that $\Omega(X)$ is *coarser* than $\Omega'(X)$ if $\Omega(X) \subseteq \Omega'(X)$.

Given topological spaces $(X, \Omega(X))$ and $(Y, \Omega(Y))$, a map $f : X \to Y$ is *continuous* if the inverse image by f of any open subset of $(Y, \Omega(Y))$ is open in $(X, \Omega(X))$. If $(X_i)_{i \in I}$ is a family of topological spaces, the set $\Pi_{i \in I} X_i$ of maps $f : i \in I \mapsto f(i) \in X_i$ can be equipped with the *product topology*, which is the coarsest topology such that all projections $\pi_i : f \in \Pi_{i \in I} X_i \mapsto f(i) \in X_i$ are continuous.

A *basis* of a topological space $(X, \Omega(X))$ is a collection of sets $\mathcal{B}$ such that every open set of X is the union of elements of $\mathcal{B}$ (by convention $\emptyset \in \Omega(X)$ is the empty union).

[1] They correspond exactly to cliques of the coherence space [7] $\omega \multimap A$ where ω and A are discrete.

A *pre-basis* of $(X, \Omega(X))$ is a collection of sets whose finite intersections form a basis of $(X, \Omega(X))$. Note that every basis is a pre-basis. Since inverse images preserve unions and intersections, in order to show that $f : (X, \Omega(X)) \to (Y, \Omega(Y))$ is continuous, it suffices to check that the inverse images by f of pre-basic opens of $(Y, \Omega(Y))$ are open in $(X, \Omega(X))$.

2.3.2 A Topology on Streams.

As usual (see e.g. [11, 19]), we equip $A^\omega (= \Pi_\omega A)$ with the product topology issued from the discrete topology on A. With this topology, a subset U of A^ω is open if and only if for any $\alpha \in U$, there is $n \in \omega$ such that $[\alpha \restriction n] \subseteq U$ (see e.g. [11]). This means that only a finite part of α must have been read in order to observe that α belongs to U.

We consider two basis for A^ω:

- the set of all sets of the form $[p]$ with $p \in A^{<\omega}$,
- the set of all sets of the form $[s]$ with s a finite partial map from ω to A.

The set of all sets of the form $[\{(n, a)\}]$ with $(n, a) \in \omega \times A$ is a pre-basis of these two basis.

The two above basis provide two equivalent formulations of the continuity of a stream function $f : A^\omega \to B^\omega$: besides (1) one can equivalently state that for any $(m, b) \in \omega \times B$ and any $\alpha \in f^{-1}[\{(m, b)\}]$ there is some finite $s : \omega \rightharpoonup A$ such that

$$\alpha \in [s] \subseteq f^{-1}[\{(m, b)\}] \ . \tag{2}$$

2.4 Domains

We refer to [1] for a more comprehensive treatment. Most of the material we use can also be found in [16].

2.4.1 Basic Definitions.

We write $D \to_{\mathrm{po}} E$ for the set of monotone functions from the partial order $(D, \sqsubseteq_D)$ to the partial order $(E, \sqsubseteq_E)$. A subset Δ of a partial order $(D, \sqsubseteq)$ is *directed* if it is non empty and for all $d, d' \in \Delta$, there is some $e \in \Delta$ such that $d, d' \sqsubseteq e$.

A *complete partial order* (cpo) is a partial order $(D, \sqsubseteq)$ which has a least element $\bot$ and such that all directed $\Delta \subseteq D$ have a least upper bound $\bigsqcup \Delta$.

A monotone map of cpos $g : D \to_{\mathrm{po}} E$ is *Scott-continuous* if g commutes with lubs of directed sets: $\bigsqcup \{g(d) \mid d \in \Delta\} = g(\bigsqcup \Delta)$ for all directed $\Delta \subseteq D$. The set $D \to_{\mathrm{cpo}} E$ is itself a cpo w.r.t. the pointwise ordering: $g \sqsubseteq h$ if $g(d) \sqsubseteq_E h(d)$ for all $d \in D$.

The *discrete* (or flat) cpo $(X_\bot, \sqsubseteq_{X_\bot})$ associated to a set X is defined as $X_\bot := X + \{\bot\}$ and $d \sqsubseteq_{X_\bot} e$ if $d = \bot$ or $d = e$. Note that a directed set $\Delta \subseteq X_\bot$ is either a singleton or of the form $\{\bot, x\}$ with $x \in X$. It follows that any monotone map of flat cpos is Scott-continuous.

An element e of a cpo $(D, \sqsubseteq)$ is *finitary*[2] if for all directed $\Delta \subseteq D$ such that $e \sqsubseteq \bigsqcup \Delta$, there is some $d \in \Delta$ such that $e \sqsubseteq d$. A cpo $(D, \sqsubseteq)$ is *algebraic* if for any $d \in D$, the set $\{e \mid e \text{ is finitary and } \sqsubseteq d\}$ is directed and its limit is d.

2.4.2 A Domain for Streams.

We write $[\![A^\omega]\!]$ for $\omega_\perp \to_{\text{po}} A_\perp = \omega_\perp \to_{\text{cpo}} A_\perp$, where $\omega_\perp$ and $A_\perp$ are discrete. If s is a partial map from ω to A, then we let $s_\perp$ be the corresponding element of $[\![A^\omega]\!]$:

$$s_\perp \;:\; x \in \omega_\perp \;\mapsto\; \begin{cases} s(x) & \text{if } x \in \text{dom}(s), \\ \perp & \text{otherwise.} \end{cases}$$

We say that $d \in [\![A^\omega]\!]$ is a *Scott-approximant* of $\alpha \in A^\omega$ if $d \sqsubseteq \alpha_\perp$, and write $[d]_\uparrow$ for the set of streams which are Scott-approximated by d. Note that this set is never empty if d is strict ($d(\perp) \neq \perp$) and also that $d \sqsubseteq d'$ implies $[d']_\uparrow \subseteq [d]_\uparrow$. Moreover, given $s : \omega \rightharpoonup A$, we have $[s] = [s_\perp]_\uparrow$.

If $g : D \to E$ is a map of cpos, we let $\text{dom}(g) := \{d \in D \mid g(d) \neq \perp_E\}$.

Lemma 2.1 *For any $d \in [\![A^\omega]\!]$,*

(i) $[d]_\uparrow$ is closed in A^ω,

(ii) and moreover open if $\text{dom}(d)$ is finite.

Proof. (i) If $d(\perp) \neq \perp$, then $d \sqsubseteq \alpha_\perp$ for no $\alpha \in A^\omega$. Hence $[d]_\uparrow$ is empty and thus open.

Assume that $d(\perp) \neq \perp$ and let $\alpha \in A^\omega$ such that $\alpha \notin [d]_\uparrow$. Hence $d \not\sqsubseteq \alpha_\perp$. Since $\alpha_\perp(\perp) = \perp = d(\perp)$, there is some $n \in \omega$ such that $d(n) \in A$ and $d(n) \neq \alpha(n)$. It follows that $d \not\sqsubseteq \beta_\perp$ for all $\beta \in A^\omega$ such that $\beta{\restriction}(n+1) = \alpha{\restriction}(n+1)$. Hence $A^\omega \setminus [d]_\uparrow$ is open and $[d]_\uparrow$ is closed.

(ii) Since $\text{dom}(d)$ is finite let $m = \max(\text{dom}(d)) \in \omega$. If $\alpha_\perp \in [d]_\uparrow$, then for all β such that $\beta{\restriction}(m+1) = \alpha{\restriction}(m+1)$ we have $d \sqsubseteq \beta_\perp$, hence $\beta \in [d]_\uparrow$. □

□

It is well-known (see e.g. [16]) that $[\![A^\omega]\!]$ is algebraic, and moreover that if s is a finite partial map from ω to A, then $s_\perp$ is finitary in $[\![A^\omega]\!]$. Hence, the $[d]_\uparrow$ with $d \in [\![A^\omega]\!]$ and $\text{dom}(d)$ finite form a basis A^ω. Since all elements of $A_\perp$ are finitary, the constant step function $\perp \Rightarrow a$ of value $a \in A$ is finitary (see e.g. [16]). These step functions $\perp \Rightarrow a$ with $a \in A$ are the only possible non strict elements of $[\![A^\omega]\!]$.

[2] In contrast with e.g. [1], we do not call them "compact" in order to avoid confusion with the topological notion of compactness, to be used in Sect 4.2.

2.5 Representation

Given $f : A^\omega \to B^\omega$ and $f^* : [\![A^\omega]\!] \to_{\mathrm{po}} [\![B^\omega]\!]$, we say that f^* *represents* f if $f^*(\alpha_\perp) = f(\alpha)_\perp$ for all $\alpha \in A^\omega$.

The main question of this paper is: given $f : A^\omega \to B^\omega$, is there some f^* that represents f, and, if f^* represents f, how do their respective continuity properties relate. Trying to be a bit more precise, we could say that we are looking for mappings

$$(-)^* : (A^\omega \to B^\omega) \to ([\![A^\omega]\!] \to_{\mathrm{po}} [\![B^\omega]\!])$$

such that f^* represents f and such that f^* is Scott-continuous iff f is continuous. We are looking for schemes $(-)^*$ which are as much as possible independent from the continuity of f. We shall however not to try to axiomatize them.

3 Representation of Continuous Stream Functions

In this section, we review basic material for the representation of $f : A^\omega \to B^\omega$ by some Scott-continuous $f^* : [\![A^\omega]\!] \to_{\mathrm{cpo}} [\![B^\omega]\!]$. The main (and expected) points are that the set of streams approximated by a given Scott-open subset of $[\![A^\omega]\!]$ is open, and therefore that f is continuous if it is represented by a Scott-continuous f^*.

3.0.1 Scott Topology.

Given a cpo $(D, \sqsubseteq)$, a subset $U \subseteq D$ is *Scott-open* if it is upward closed ($(d \in U \wedge d \sqsubseteq e)$ implies $e \in U$), and if moreover for any directed $\Delta \subseteq D$ such that $\bigsqcup \Delta \in U$, there is some $d \in \Delta$ such that $d \in U$. This means that if some limit $\bigsqcup \Delta$ belongs to U, then an approximant $d \in \Delta$ of $\bigsqcup \Delta$ already belongs to U. It is well-known (see e.g. [1]) that a map of cpos is continuous for the Scott topology if and only if it is Scott-continuous (*i.e.* a monotone map of cpos which commutes with directed limits). Moreover, if $(D, \sqsubseteq)$ is algebraic, then the collection of the sets $\uparrow d := \{e \mid d \sqsubseteq e\}$ with d finitary is a basis for the Scott topology.

3.1 From Scott-Continuity to Continuity

Lemma 3.1 *If $U \subseteq [\![A^\omega]\!]$ is Scott-open, then $\{\alpha \in A^\omega \mid \alpha_\perp \in U\}$ is open.*

Proof. Let $\alpha \in A^\omega$ such that $\alpha_\perp \in U$. Consider the set S_α of all $p_\perp$ such that $p \in A^{<\omega}$ approximates α. Then S_α is directed and has limit $\alpha_\perp$. Since U is Scott-open, it follows that there is some finite $p \subseteq \alpha$ such that $p_\perp \in U$. Now, if $\beta \in [p]$, then $p_\perp \sqsubseteq \beta_\perp$, hence $\beta_\perp \in U$ since U is Scott-open. It follows that $\alpha \in [p] \subseteq \{\gamma \in A^\omega \mid \gamma_\perp \in U\}$. □ □

Lemma 3.1 implies the expected fact that if f is represented by a Scott-continuous f^*, then f is continuous.

Lemma 3.2 *If f^* represents f and f^* is Scott-continuous then f is continuous.*

Proof. Let $(m, b) \in \omega \times B$. We show that $f^{-1}[\{(m, b)\}]$ is open in A^ω. Since f^* represents f, for all $\alpha \in A^\omega$ we have $\alpha \in f^{-1}[\{(m, b)\}]$ iff $\alpha_\perp \in (f^*)^{-1}\uparrow\{(m, b)\}_\perp$. But $(f^*)^{-1}\uparrow\{(m, b)\}_\perp$ is Scott-open thanks to the Scott-continuity of f^*, and it follows from Lem. 3.1 that $f^{-1}[\{(m, b)\}]$ is open in A^ω. □ □

3.2 Traces

It is well-known that a Scott-continuous map $g : D \to_{\text{cpo}} E$ on algebraic cpos $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$ is completely determined by its values on finitary elements. More precisely, $g : D \to_{\text{po}} E$ is Scott-continuous if and only if for all $d \in D$ we have:

$$g(d) \quad = \quad \bigsqcup\{g(d') \mid d' \text{ is finitary and } \sqsubseteq d\} \ .$$

Hence, in order to define $f^* : [\![A^\omega]\!] \to_{\text{cpo}} [\![B^\omega]\!]$ from a continuous $f : A^\omega \to B^\omega$, it suffices to specify f^* on the finitary elements of $[\![A^\omega]\!]$. A possibility to analyse this is to look at how f^* is generated from a "trace"[3]. We say that $\tau \subseteq [\![A^\omega]\!] \times (\omega \times B)$ is a *trace* of $f^* : [\![A^\omega]\!] \to_{\text{po}} [\![B^\omega]\!]$ if for all $d \in [\![A^\omega]\!]$,

$$f^*(d) \quad = \quad \bigsqcup\{\{(m, b)\}_\perp \mid \exists e,\ e \sqsubseteq d \text{ and } (e, (m, b)) \in \tau\} \ . \tag{3}$$

We say that τ is *finitary* if it contains only finitary $e \in [\![A^\omega]\!]$. This gives a necessary and sufficient condition for f^* to be Scott-continuous:

Proposition 3.3 *$f^* : [\![A^\omega]\!] \to_{\text{po}} [\![B^\omega]\!]$ is Scott-continuous if and only if it has a finitary trace.*

Proof. The "if" direction is trivial, and the "only if" direction follows from the algebraicity of $[\![A^\omega]\!]$. □ □

Remark 3.4 *Note that (3) is possible for a given $d \in [\![A^\omega]\!]$ only when the set*

$$\{\{(m, b)\}_\perp \mid \exists e,\ e \sqsubseteq d \textit{ and } (e, (m, b)) \in \tau\}$$

is bounded ($[\![A^\omega]\!]$ is bounded complete *[1, 16]). In this case, (3) is equivalent to*

$$f^*(d) \quad = \quad \{(m, b) \mid \exists e,\ e \sqsubseteq d \textit{ and } (e, (m, b)) \in \tau\}_\perp \ .$$

Given $f : A^\omega \to B^\omega$, the two equivalent definitions of continuity (1) and (2) of Sect. 2.3 lead to two possible finitary traces to define an f^*:

$$\begin{aligned} \tau_f \quad &:= \quad \{(s_\perp, (m, b)) \mid s : \omega \rightharpoonup A \text{ finite and } [s] \subseteq f^{-1}[\{(m, b)\}]\} \\ \tilde{\tau}_f \quad &:= \quad \{(p_\perp, (m, b)) \mid p \in A^{<\omega} \text{ and } [p] \subseteq f^{-1}[\{(m, b)\}]\} \end{aligned} \tag{4}$$

In order to ensure that τ_f and $\tilde{\tau}_f$ can indeed be traces of some $f^* : [\![A^\omega]\!] \to [\![B^\omega]\!]$, we can check that with $\tau = \tau_f$ and $\tau = \tilde{\tau}_f$, the set

$$\{(m, b) \mid \exists e,\ e \sqsubseteq d \text{ and } (e, (m, b)) \in \tau\}$$

[3] By lack of a better name, we depart from the usual terminology (see e.g. [1]) since we do not require any minimality in the trace. This will be regained with Hypercoherences in Sect. 5.

defines a partial function from ω to B. This follows from a more general result (recall that $[s] = [s_\bot]_\uparrow$ and that $d \sqsubseteq d'$ implies $[d']_\uparrow \subseteq [d]_\uparrow$):

Proposition 3.5 *Given $f : A^\omega \to B^\omega$ and $d \in [\![A^\omega]\!]$, the set*

$$\{(m, b) \in \omega \times B \mid [d]_\uparrow \subseteq f^{-1}[\{(m, b)\}]\}$$

defines a partial map from ω to B.

Proof. If $[d]_\uparrow \subseteq f^{-1}[\{(m, b)\}]$ and $[d]_\uparrow \subseteq f^{-1}[\{(m, b')\}]$, then for all $\alpha \in [d]_\uparrow$, we have $(m, b), (m, b') \in f(\alpha)$, hence $b = b'$. □ □

The two traces τ_f and $\tilde{\tau}_f$ are finitary and hence they both generate a Scott-continuous f^*. According to Lem. 3.2, such an f^* can not represent a non continuous f. However, f^* always under-approximates f.

Lemma 3.6 *Let $f : A^\omega \to B^\omega$ and let $f^* : [\![A^\omega]\!] \to_{\text{cpo}} [\![B^\omega]\!]$ have trace τ_f or $\tilde{\tau}_f$. Then for all $\alpha \in A^\omega$, we have $f^*(\alpha_\bot) \sqsubseteq_{[\![B^\omega]\!]} f(\alpha)_\bot$.*

Actually, we can say more: f^* represents f whenever f is continuous.

Proposition 3.7 *Let $f : A^\omega \to B^\omega$ and let $f^* : [\![A^\omega]\!] \to_{\text{cpo}} [\![B^\omega]\!]$ have trace τ_f or $\tilde{\tau}_f$. Then f^* represents f if and only if f is continuous.*

Proof. Lemma 3.2 implies that f is continuous whenever it is represented by f^*. We show the converse for $\tilde{\tau}_f$. Assume that $f : A^\omega \to B^\omega$ is continuous and let $\alpha \in A^\omega$. We only have to show that $f(\alpha)_\bot \sqsubseteq_{[\![B^\omega]\!]} f^*(\alpha_\bot)$. We have $f(\alpha)_\bot(\bot) = \bot \sqsubseteq_{B_\bot} f^*(\alpha_\bot)(\bot)$ by definition. Let $(m, b) \in f(\alpha)_\bot$. By continuity of f, there is some $p \in A^{<\omega}$ such that $\alpha \in [p]$ (*i.e.* $p_\bot \sqsubseteq \alpha_\bot$) and $[p] \subseteq f^{-1}[\{(m, b)\}]$, hence $(m, b) \in f^*(\alpha_\bot)$. □ □

4 Representation of Arbitrary Stream Functions

The previous section suggests that systematically building f^* from a finitary trace may not be the right approach to define f^* from f such that f^* always represents f and f^* is Scott-continuous if and only if f is continuous.

We try to relax the finitary condition on traces, because it fixes the Scott-continuity of f^* inside its definition. In this section, given $f : A^\omega \to B^\omega$, we define $f^* : [\![A^\omega]\!] \to_{\text{po}} [\![B^\omega]\!]$ by:

$$f^*(d) \quad := \quad \{(m, b) \in \omega \times B \mid [d]_\uparrow \subseteq f^{-1}[\{(m, b)\}]\}_\bot \tag{5}$$

That f^* is indeed a function follows from Prop. 3.5. Its monotony is obvious. The set

$$\{(d, (m, b)) \mid [d]_\uparrow \subseteq f^{-1}[\{(m, b)\}]\}$$

is a trace for f^*. This trace is *sufficient* in the sense that for all $(m, b) \in \omega \times B$ and all finitary $d \in [\![A^\omega]\!]$, if $[d]_\uparrow \subseteq f^{-1}[\{(m, b)\}]$ then $(m, b) \in f^*(d)$.

The scheme (5) satisfies our first requirement on f^*.

Lemma 4.1 *Given any $f : A^\omega \to B^\omega$, the f^* defined from f as in (5) is monotone and represents f.*

Proof. The monotonicity follows from the fact that $d \sqsubseteq d'$ implies $[d']_\uparrow \subseteq [d]_\uparrow$.

Let $\alpha \in A^\omega$. We have $[\alpha_\bot]_\uparrow = \{\alpha\}$, and therefore $(m, b) \in f(\alpha)$ implies $(m, b) \in f^*(\alpha_\bot)$. It follows that $f(\alpha)_\bot \sqsubseteq f^*(\alpha_\bot)$. Conversely, if $(m, b) \in f^*(\alpha_\bot)$, then we have $[\alpha_\bot]_\uparrow \subseteq f^{-1}[\{(m, b)\}]$, hence $(m, b) \in f(\alpha)$. It follows that $f^*(\alpha_\bot) \sqsubseteq f(\alpha)_\bot$. □ □

However, we show in Sect. 4.1 that the continuity of f does not imply the Scott-continuity of f^*. This does not happen when A^ω is (topologically) compact: as shown in Sect. 4.2, if A^ω is compact then the continuity of f implies the Scott-continuity of f^*. We moreover show that in this case, f^* is Scott-continuous whenever it has a sufficient trace and represents a continuous f.

4.1 Continuity Does Not Entail Scott-Continuity

We now show that (5) can induce a non Scott-continuous $f^* : [\![A^\omega]\!] \to_{\mathrm{po}} [\![B^\omega]\!]$ from a continuous $f : A^\omega \to B^\omega$. The surprising point is that this comes from a failure to the converse of Lem. 3.1, namely that when A is infinite, an open $U \subseteq A^\omega$ can be such that the set of its Scott-approximants $\{d \in [\![A^\omega]\!] \mid [d]_\uparrow \subseteq U\}$ is not Scott-open. It will be shown in Sect. 4.2, this is not possible when A is finite. We work here with the Baire space $\mathcal{N} := \omega^\omega$.

Lemma 4.2 *There is an open $U \subseteq \mathcal{N}$ such that $\{d \in [\![A^\omega]\!] \mid [d]_\uparrow \subseteq U\}$ is not Scott-open.*

Proof. Consider the open set $U \subseteq \mathcal{N}$ made of the $\alpha \in \mathcal{N}$ such that

$$\alpha(\alpha(0) + 1) = 0 \ .$$

Note that U is the union of the basic open sets $[\{(0, i), (i + 1, 0)\}]$ for $i \in \omega$. Consider now the directed family $(d_i)_{i \in \omega} \in [\![\mathcal{N}]\!]$ with

$$d_i(a) := \begin{cases} 0 & \text{if } a \in \{1, \dots, i + 1\}, \\ \bot & \text{otherwise.} \end{cases}$$

The set $\{d \in [\![A^\omega]\!] \mid [d]_\uparrow \subseteq U\} \subseteq [\![\mathcal{N}]\!]$ is not Scott-open since

(i) $[\bigsqcup\{d_i \mid i \in \omega\}]_\uparrow \subseteq U$,

(ii) for all i, $[d_i]_\uparrow$ is not contained in U.

Proofs of the claims (i) and (ii).

(i) Let $\alpha \in [\bigsqcup\{d_i \mid i \in \omega\}]_\uparrow$. For all $i \in \omega$ we have $d_i \sqsubseteq \alpha$, hence $\alpha(i + 1) = 0$. It follows that $\alpha(\alpha(0) + 1) = 0$, so that $\alpha \in U$.

(ii) Let $i \in \omega$ and define α_i as

$$\alpha_i = \begin{cases} 0 \mapsto i+1 & \\ k \mapsto 0 & (1 \leq k \leq i+1) \\ k \mapsto 1 & (k \geq i+2) \end{cases}$$

We have $\alpha_i \in [d_i]_\uparrow$ since $\alpha(k) = 0$ for all $1 \leq k \leq i+1$. But $\alpha_i \notin U$ since $\alpha_i(\alpha_i(0)+1) = \alpha_i(i+2) = 1$. □

□ □

Using Lem. 4.2, we can show that (5) gives a non Scott-continuous representation of a suitable continuous f. It suffices for f to test whether its argument belongs to the open set U of Lem. 4.2.

Lemma 4.3 *There is a continuous $f : \mathcal{N} \to \mathcal{N}$ such that $f^* : [\![\mathcal{N}]\!] \to_{\mathrm{po}} [\![\mathcal{N}]\!]$ defined from (5) is not Scott-continuous.*

Proof. Consider the open set U of Lem. 4.2 and define $f : \mathcal{N} \to \mathcal{N}$ as

$$f(\alpha) = \begin{cases} 1^\omega & \text{if } \alpha \in U \\ 0^\omega & \text{otherwise} \end{cases}$$

Define f^* from f as in (5). We claim that:

(i) f is continuous,

(ii) f^* is not Scott-continuous: $f^*(\bigsqcup\{d_i \mid i \in \omega\}) \not\sqsubseteq \bigsqcup\{f^*(d_i) \mid i \in \omega\}$, where $(d_i)_{i\in\omega}$ is defined in the proof of Lem. 4.2.

Proofs of the claims.

(i) We only have to show that $\mathcal{N} \setminus U$ is open. But $\alpha \notin U$ iff $\alpha(\alpha(0)+1) \neq 0$ iff $\alpha \in [\{(0,i),(i+1,k)\}]$ for some $i \in \omega$ and $k \geq 1$.

(ii) Since $[\bigsqcup\{d_i \mid i \in \omega\}]_\uparrow \subseteq U$, for all $\alpha \in [\bigsqcup_{i\in\omega} d_i]_\uparrow$ we have $f(\alpha) = 1^\omega$, hence $f^*(\bigsqcup_{i\in\omega} d_i) = (1^\omega)_\perp = \{(n,1) \mid n \in \omega\}_\perp$. However, by Claim (ii) of the proof of Lem. 4.2, for all $i \in \omega$ there is some α in $[d_i]_\uparrow$ such that $f(\alpha) = 0^\omega$. It follows that for all $n \in \omega$, we have $(n,1) \notin f(\alpha)_\perp \sqsupseteq f^*(d_i)$, hence $(n,1) \notin \bigsqcup_{i\in\omega} f^*(d_i)$. □

□ □

Remark 4.4 *The point is the definition of f^*. In the case of Lem. 4.3, since f is definable in Gödel's system T, it is representable by a (strongly) stable function.*

We see no other way in this situation than to use some (to our opinion) more contrived definition of f^*, such as:

$$f^*(\alpha_\perp) := f(\alpha)_\perp\,, \qquad f^*(\perp \Rightarrow a) := f(a^\omega)_\perp\,,$$
$$f^*(d) := \{(m,b) \mid \exists p \in A^{<\omega},\ p_\perp \sqsubseteq d \text{ and } [p] \subseteq f^{-1}[\{(m,b)\}]\}_\perp \quad \text{otherwise.}$$

4.2 The Compact Case

In this section we show that the problem raised by Lem. 4.2 does not appear when A^ω is a compact topological space, which exactly means that A is finite. We thus avoid the counter-example of Lem. 4.3. This implies that f^* is Scott-continuous whenever it has a sufficient trace and represents a continuous f.

4.2.1 Topological Compactness.

A subset set C of a topological space $(X, \Omega(X))$ is *compact* if for any family $(A_i)_{i \in I}$ of open sets such that $C \subseteq \bigcup_{i \in I} A_i$, there is a *finite* $I_0 \subseteq I$ such that $C \subseteq \bigcup_{i \in I_0} A_i$. We say that $(X, \Omega(X))$ is compact when X is compact in $(X, \Omega(X))$. It is well-known (see e.g. [11]) that a closed subset of a compact set is compact and also that A^ω is compact if and only if A is finite.

4.2.2 The Set of Scott-Approximants of an Open is Scott-Open.

We now show that the topological compactness of A^ω implies that $\{d \mid [d]_\uparrow \subseteq U\}$ is Scott-open in $[\![A^\omega]\!]$ whenever U is open in A^ω. The upward closure is trivial, and as suggested by Lem. 4.2, the point is the "finite" observation of limits.

Lemma 4.5 *Assume that A is finite. Let Δ be a directed subset of $[\![A^\omega]\!]$ and U be an open subset of A^ω. If $[\bigsqcup \Delta]_\uparrow \subseteq U$, then there is some $d \in \Delta$ such that $[d]_\uparrow \subseteq U$.*

Proof. The set $[\bigsqcup \Delta]_\uparrow$ is closed by Lem. 2.1, and hence compact since A^ω is compact. So, we may assume that there are $s_1, \dots, s_n$ finite partial from ω to A such that $[\bigsqcup \Delta]_\uparrow \subseteq [s_1] \cup \dots \cup [s_n] \subseteq U$ and $[\bigsqcup \Delta]_\uparrow \cap [s_i] \neq \emptyset$ for all $1 \leq i \leq n$. Since the set $S := s_1 \cup \dots \cup s_n$ is finite and Δ is directed, there is some $d \in \Delta$ such that $S \cap (\bigsqcup \Delta) \subseteq d$ (note that $S \subseteq \omega \times A \subseteq \omega_\bot \times A_\bot$ and that $\bigsqcup \Delta \subseteq \omega_\bot \times A_\bot$ since $\bigsqcup \Delta \in [\![A^\omega]\!]$).

We will show that $[d]_\uparrow \subseteq [s_1] \cup \dots \cup [s_n]$. We proceed by contradiction, by showing that from any $\alpha \in [d]_\uparrow \setminus ([s_1] \cup \dots \cup [s_n])$ we can build an $\alpha' \in [\bigsqcup \Delta]_\uparrow \setminus ([s_1] \cup \dots \cup [s_n])$, contradicting the assumption on the s_i's.

First, note that if $(m, a) \in S \setminus \bigsqcup \Delta$, then no (m, b) with $b \neq a$ belongs to $\bigsqcup \Delta$, since (m, a) belongs to some s_i while $[s_i] \cap [\bigsqcup \Delta]_\uparrow$ is not empty by assumption.

Consider some $\alpha \in [d]_\uparrow$. Since $\bigsqcup \Delta \in [\![A^\omega]\!]$, for all $m \in \mathrm{dom}(\bigsqcup \Delta) \setminus \mathrm{dom}(d)$, there is a unique b such that $(m, b) \in \bigsqcup \Delta$. We can thus define a sequence α' as α where all the (m, a) with $m \in \mathrm{dom}(\bigsqcup \Delta) \setminus \mathrm{dom}(d)$ have been replaced by (m, b) with $(m, b) \in \bigsqcup \Delta$. We therefore have $\alpha' \in [\bigsqcup \Delta]_\uparrow$.

Assume now that $\alpha \notin [s_1] \cup \dots \cup [s_n]$. Hence, for all $1 \leq i \leq n$ there is some $(m, a) \in s_i \setminus \alpha$. Since $S \cap (\bigsqcup \Delta) \subseteq \alpha$ by assumption, none of these m is in the domain of $\bigsqcup \Delta$, hence $\alpha'(m) = \alpha(m)$. It follows that $\alpha' \notin [s_1] \cup \dots \cup [s_n]$. □ □

Corollary 4.6 *If A is finite and $U \subseteq A^\omega$ is open, then $\{d \in [\![A^\omega]\!] \mid [d]_\uparrow \subseteq U\}$ is Scott-open.*

4.2.3 Representation.

Lemma 4.5 implies that f^* is Scott-continuous whenever it has a sufficient trace and represents a continuous $f : A^\omega \to B^\omega$ with A finite. Recall that if f^* is defined from f as in (5), then f^* represents f and has a sufficient trace.

Lemma 4.7 *Let A be finite set and let $f : A^\omega \to B^\omega$ be represented by $f^* : [\![A^\omega]\!] \to_{\mathrm{po}} [\![B^\omega]\!]$. Assume that f^* has a sufficient trace:*

- *for all $(m, b) \in \omega \times B$ and all finitary $d \in [\![A^\omega]\!]$, if $[d]_\uparrow \subseteq f^{-1}[\{(m, b)\}]$ then $(m, b) \in f^*(d)$.*

If f is continuous then f^* is Scott-continuous.

Note that if f^* represents f then $f^*(\alpha_\perp)$ is strict by strictness of $f(\alpha)_\perp$. Recall that $[d]_\uparrow$ is empty if and only if $d = \perp \Rightarrow a$ for some $a \in A$, and in this case d is finitary.

Proof. Let Δ be a directed subset of $[\![A^\omega]\!]$. Since f^* is monotone, it suffices to check that $f^*(\bigsqcup \Delta) \sqsubseteq_{[\![A^\omega]\!]} \bigsqcup f^*(\Delta)$. This is trivial if $\bigsqcup \Delta$ is finitary and in particular if $[\bigsqcup \Delta]_\uparrow$ is empty.

We have to show that $f^*(\bigsqcup \Delta)(e) \sqsubseteq_{A_\perp} \bigsqcup f^*(\Delta)(e)$ for all $e \in \omega_\perp$.

Consider first the case of $e = \perp$. If $f^*(\bigsqcup \Delta)(\perp) = \perp$ then we are done. Otherwise, $f^*(\bigsqcup \Delta)(\perp) = b \in B$. Since $[\bigsqcup \Delta]_\uparrow$ is assumed to be non-empty, let $\alpha \in [\bigsqcup \Delta]_\uparrow$. By monotonicity of f^*, we have $f^*(\alpha_\perp)(\perp) = b$, which contradicts the strictness of $f^*(\alpha_\perp)$.

Consider now the case of $e = m \in \omega$. If $f^*(\bigsqcup \Delta)(m) = \perp$ then we are done. So let $b \in B$ such that $(m, b) \in f^*(\bigsqcup \Delta)$. We show that there is some $d \in \Delta$ such that $(m, b) \in f^*(d)$. Since $[\bigsqcup \Delta]_\uparrow$ is assumed to be non-empty, let $\alpha \in [\bigsqcup \Delta]_\uparrow$. By monotonicity of f^*, we have $(m, b) \in f^*(\alpha_\perp)$ hence $(m, b) \in f(\alpha)$ since f^* represents f. It follows that $[\bigsqcup \Delta]_\uparrow \subseteq f^{-1}[\{(m, b)\}]$. But $f^{-1}[\{(m, b)\}] \subseteq A^\omega$ is open by continuity of f. Since A is finite, by Lem. 4.5 there is $d \in \Delta$ such that $[d]_\uparrow \subseteq f^{-1}[\{(m, b)\}]$. If d is finitary, then we are done by assumption on f^*. Otherwise, by algebraicity of $[\![A^\omega]\!]$ and since $[d]_\uparrow \subseteq f^{-1}[\{(m, b)\}]$, by Lem. 4.5 again there is a finitary $d' \sqsubseteq d$ such that $[d']_\uparrow \subseteq f^{-1}[\{(m, b)\}]$, and we are done by assumption on f^*. □ □

5 An Account in Hypercoherences

We give an account of our preceding results in Ehrhard's *Hypercoherences* [3]. Our motivation is to check that a trace similar to the "sequential" trace $\tilde{\tau}_f$ of (4) is indeed "sequential". We are thus looking at *Strongly Stable* maps of Hypercoherences. Maps of Hypercoherences are maps of the underlying *qualitative domains* [6]. At higher type, these maps work on representation of elements by some coding (traces) and not on the "extensional" elements themselves. We will thus represent f by a map $\tilde{f}$ of the corresponding qualitative domains.

5.0.4 Basic Definitions.

We gather here some basic definitions on Hypercoherences. See e.g. [3, 4] for details. We need some notations. Given two sets A and B, we say that A is a *multisection* of B (written $A \lhd B$) if

$$\forall a \in A,\ \exists b \in B,\ a \in b \qquad \text{and} \qquad \forall b \in B,\ \exists a \in A,\ a \in b\ .$$

Moreover, given $S \subseteq X \times Y$, we let S_1 and S_2, the two projections of S, be:

$$S_1 := \{x \in X \mid \exists y \in Y,\ (x, y) \in S\} \quad \text{and} \quad S_2 := \{y \in Y \mid \exists x \in X,\ (x, y) \in S\}$$

An *Hypercoherence* is a set X together with a collection $\Gamma(X)$ of non-empty finite subsets of X such that $\{x\} \in \Gamma(X)$ for all $x \in X$. The set X is called the *web* of the Hypercoherence $(X, \Gamma(X))$.

Let $(X, \Gamma(X))$ be an Hypercoherence. The set of states $\mathrm{qD}(X)$ of its associated *qualitative domain* is be the set of all $u \subseteq X$ such that all non-empty finite subset of u belong to $\Gamma(X)$. The corresponding *coherence* $\mathcal{C}(X)$ is the set of all non-empty finite subset S of $\mathrm{qD}(X)$ such that all multisections of S belong to $\Gamma(X)$. Note that $(X, \mathrm{qD}(X))$ is a cpo whose finitary elements are exactly the finite $u \in \mathrm{qD}(X)$. Write $\mathrm{qD}_{\mathrm{fin}}(X)$ for the set of finite $u \in \mathrm{qD}(X)$.

Given Hypercoherences X and Y, a *Strongly Stable function* $g : X \to_{\mathrm{FS}} Y$ is a Scott-continuous function from $(\mathrm{qD}(X), \subseteq)$ to $(\mathrm{qD}(Y), \subseteq)$ such that for all $S \in \mathcal{C}(X)$, we have $g(S) \in \mathcal{C}(Y)$ and $\bigcap g(S) = g(\bigcap S)$. The Hypercoherence associated to $X \to_{\mathrm{FS}} Y$ has web $\mathrm{qD}_{\mathrm{fin}}(X) \times Y$ and $\Gamma(X \to_{\mathrm{FS}} Y)$ is the set of all non-empty and finite subsets u of $\mathrm{qD}_{\mathrm{fin}}(X) \times Y$ such that if $u_1 \in \mathcal{C}(X)$, then $u_2 \in \Gamma(Y)$ and if moreover u_2 is a singleton, then u_1 is a Singleton.

To each $g : X \to_{\mathrm{FS}} Y$, we can associate its *trace* $\mathsf{tr}(g) \subseteq \mathrm{qD}_{\mathrm{fin}}(X) \times Y$:

$$\mathsf{tr}(g) \quad := \quad \{(u, y) \mid y \in g(u) \text{ and } u \text{ is minimal}\}\ ,$$

and g can be recovered from its trace as

$$g(u) \quad = \quad \{y \mid \exists u' \subseteq u,\ (u', y) \in \mathsf{tr}(g)\}\ .$$

By Proposition 19 of [3], $\mathrm{qD}(X \to_{\mathrm{FS}} Y)$ is exactly the set of traces of strongly stable maps from X to Y.

5.0.5 Representation.

Given a set A, its discrete Hypercoherence $\Gamma(A)$ is the set of singletons $\{a\}$ with $a \in A$.

Proposition 5.1 *If X is a discrete Hypercoherence, then*

(i) $\mathrm{qD}(X) = \{\{x\} \mid x \in X\} \cup \{\emptyset\}$,

(ii) $\mathcal{C}(X) = \{\{\{x\}\} \mid x \in X\} \cup \{S \subseteq^*_{\mathrm{fin}} \mathrm{qD}(X) \mid \emptyset \in S\}$.

Proof. (i) First, we have $\emptyset \in \mathrm{qD}(X)$ since the empty set has no non-empty subset. Furthermore, if $x \in X$, then the only non-empty subset of $\{x\}$ is $\{x\}$, and $\{x\} \in \Gamma(X)$ since X is discrete.

Conversely, assume that $s \in \mathrm{qD}(X)$ with distinct $x, y \in s$. Then $\{x, y\}$ is a finite non-empty subset of s which is not in $\Gamma(X)$.

(ii) If $x \in X$ then $\{\{x\}\} \subseteq^*_{\mathrm{fin}} \mathrm{qD}(X)$. Moreover, $u \lhd \{\{x\}\}$ implies $u = \{x\}$, so that $u \in \Gamma(X)$. On the other hand, if $S \subseteq^*_{\mathrm{fin}} \mathrm{qD}(X)$ contains the empty set, then S has no multisection and we are done.

Conversely, if $S \in \mathcal{C}(X)$ does not contain the empty set, then S is of the form $\{\{x_1\}, \ldots, \{x_n\}\}$. Since $\{x_1, \ldots, x_n\}$ is a multisection of S, we must have $\{x_1, \ldots, x_n\} \in \Gamma(X)$, but this is possible only when $n = 1$. □

□

Proposition 5.1.(i) tells us that given a set X equipped with the discrete Hypercoherence, we can identify $(\mathrm{qD}(X), \subseteq)$ with the discrete cpo $X_\perp$.

The Hypercoherence associated to A^ω is the Hypercoherence of $\omega \to_{\mathrm{FS}} A$, where ω, A are discrete and the function space is the strongly stable one.

It is possible to show that $\omega \to_{\mathrm{FS}} A$ and $[\![A^\omega]\!]$ are isomorphic w.r.t. the pointwise order (which is different from the *stable* order, corresponding to $(\mathrm{qD}(\omega \to_{\mathrm{FS}} A), \subseteq)$). We do not need this generality since it is easy to directly represent any $s : \omega \rightharpoonup A$ by its trace: let

$$s_{\mathsf{tr}} \quad := \quad \{(\{n\}, a) \mid (n, a) \in s\} \ .$$

Note that this encompasses the cases of $p \in A^{<\omega}$ and $\alpha \in A^\omega$.

Proposition 5.2 *If $s : \omega \rightharpoonup A$, then $s_{\mathsf{tr}} \in \mathrm{qD}(\omega \to_{\mathrm{FS}} A)$.*

Proof. Let $S \subseteq (s_{\mathsf{tr}})_{\mathrm{fin}}$ be non-empty and such that $S_1 \in \mathcal{C}(\omega)$. Since $\emptyset \notin S_1$, we have $S_1 = \{\{n\}\}$, hence $S_2 = \{a\} \in \Gamma(A)$. Moreover, both S_1 and S_2 are singletons. □ □

Given $f : A^\omega \to B^\omega$, define $\tilde{f} : \mathrm{qD}(\omega \to_{\mathrm{FS}} A) \to \mathrm{qD}(\omega \to_{\mathrm{FS}} B)$ by

$$\tilde{f}(d) \quad := \quad \{(\{m\}, b) \mid \exists p \in A^{<\omega},\ p_{\mathsf{tr}} \subseteq d \text{ and } [p] \subseteq f^{-1}[\{(m, b)\}]\}$$

We say that $\tilde{f}$ represents f if $f(\alpha)_{\mathsf{tr}} = \tilde{f}(\alpha_{\mathsf{tr}})$ for all $\alpha \in A^\omega$.

We have for $\tilde{f}$ similar properties as for the f^*'s of Sect. 3.

Lemma 5.3 *$\tilde{f}$ is Scott-continuous and moreover represents f if and only if f is continuous.*

Proof. Monotonicity follows from the shape of the definition. Scott-continuity follows from the fact that $p_{\mathsf{tr}} \in \mathrm{qD}(\omega \to_{\mathrm{FS}} A)$ is finite. That $\tilde{f}$ represents f if and only if f is continuous directly follows from the definition of $\tilde{f}$. □

We now show that $\tilde{f}$ is strongly stable. Thanks to Ehrhard's collapse theorem [4], this means that $\tilde{f}$ can be "extensionally" represented by a sequential algorithm in the sense of [2].

Lemma 5.4 *$\tilde{f}$ is strongly stable.*

Given sets A and B, we write $A \sqsubseteq B$ when

$$\forall a \in A\ \exists b \in B\ a \subseteq b \qquad \text{and} \qquad \forall b \in B\ \exists a \in A\ a \subseteq b\ .$$

It is shown in [3] that if $S \in \mathcal{C}(A)$ for some Hypercoherence A, and S' is a non-empty finite subset of $\mathrm{qD}(A)$ such that $S' \sqsubseteq S$, then $S' \in \mathcal{C}(A)$.

Proof. We show that $\tilde{f}$ preserves coherences $S \in \mathcal{C}(\omega \to_{\mathrm{FS}} A)$ and their intersections.

- We first show that if $S \in \mathcal{C}(\omega \to_{\mathrm{FS}} A)$ then $\tilde{f}(S) \in \mathcal{C}(\omega \to_{\mathrm{FS}} B)$.

  Let $S \in \mathcal{C}(\omega \to_{\mathrm{FS}} A)$. Then S is a finite non-empty subset of $\mathrm{qD}(\omega \to_{\mathrm{FS}} A)$. Hence S is of the form $\{d_1, \dots, d_n\}$. Let

  $$M \quad := \quad \{(x_1, b_1), \dots, (x_m, b_m)\} \quad \subseteq \quad \mathrm{qD}(\omega) \times A$$

  be a finite multisection of $\{\tilde{f}(d_1), \dots, \tilde{f}(d_n)\}$ such that $M_1 \in \mathcal{C}(\omega)$.

  Assume first that $\emptyset \in M_1$. Hence we have $x_j = \emptyset$ for some j, and it follows that $(\emptyset, b_j) \in \tilde{f}(d_i)$ for some i. But this is impossible by definition of $\tilde{f}$.

  It thus follows from Prop. 5.1.(ii) that $M_1 = \{\{k\}\}$ for some $k \in \omega$, hence that M is of the form $\{(\{k\}, b_1), \dots, (\{k\}, b_m)\}$.

  We show that $b_1 = \dots = b_m$.

  By assumption, for all $1 \leq j \leq m$, there is some $1 \leq i \leq n$ such that $(\{k\}, b_j) \in \tilde{f}(d_i)$, hence a minimal $p_j \in A^{<\omega}$ such that $(p_j)_{\mathsf{tr}} \subseteq d_i$ and $[p_j] \subseteq f^{-1}[(k, b_j)]$. We claim that $p_1 = \dots = p_m$.

  - Assume that we do not have $p_1 = \dots = p_m$. Hence, there is a least l such that there are $1 \leq j_0, j_1 \leq m$ and $a \in A$ such that $(l, a) \in p_{j_0} \setminus p_{j_1}$.

    Assume that $l \in \mathrm{dom}(p_j)$ for all j. Hence $\{(\{l\}, p_1(l)), \dots, (\{l\}, p_m(l))\}$ is a finite multisection of $\{d_1, \dots, d_n\}$, so that $\{(\{l\}, p_1(l)), \dots, (\{l\}, p_m(l))\}$ belongs to $\Gamma(\omega \to_{\mathrm{FS}} A)$. It follows that $p_1(l) = \dots = p_m(l)$, but this contradicts the assumption that $p_{j_0}(l) \neq p_{j_1}(l)$ for some j_0, j_1.

    It therefore must be the case that $l \in \mathrm{dom}(p_{j_0}) \setminus \mathrm{dom}(p_{j_1})$ for some j_0, j_1. But by assumption on l, for all $l' < l$ we have $l' \in \mathrm{dom}(p_1) \cap \dots \cap \mathrm{dom}(p_m)$ and $p_1(l') = \dots = p_m(l')$. Hence p_{j_1} is a prefix of all the p_j's, and therefore $(p_{j_1})_{\mathsf{tr}}$ is contained in all the d_i's. By monotonicity of $\tilde{f}$ (Lem. 5.3), we get $(\{k\}, b_{j_1}) \in \tilde{f}(d_i)$ for all $1 \leq i \leq n$. This implies $b_j = b_{j_1}$ for all $1 \leq j \leq m$. But then the minimality of the p_j's would imply that $p_{j_0} = p_{j_1}$, contradicting $l \in \mathrm{dom}(p_{j_0}) \setminus \mathrm{dom}(p_{j_1})$ (note that in this case, we already have $b_1 = \dots = b_m$, which was our primarily goal). □

  We therefore have $p_1 = \dots = p_m$, which implies $b_1 = \dots = b_m$.

- We now show that if $S \in \mathcal{C}(\omega \to_{\mathrm{FS}} A)$, then $\bigcap \tilde{f}(S) = \tilde{f}(\bigcap S)$.

  The inclusion ($\supseteq$) follows from the monotonicity of $\tilde{f}$. For the other direction, let $S = \{d_1, \ldots, d_n\} \in \mathcal{C}(\omega \to_{\mathrm{FS}} A)$ and $(\{m\}, b) \in \bigcap \tilde{f}(S)$. Hence, for all $1 \leq i \leq n$ there is a minimal $p_i \in A^{<\omega}$ such that $(p_i)_{\mathsf{tr}} \subseteq d_i$ and $(m, b) \in f[p_i]$. But $\{(p_1)_{\mathsf{tr}}, \ldots, (p_n)_{\mathsf{tr}}\} \sqsubseteq \{d_1, \ldots, d_n\}$. Reasoning similarly as above, we get that $p_1 = \cdots = p_n$ $(=: p)$, hence $p_{\mathsf{tr}} \subseteq \bigcap S$ and $(\{m\}, b) \in \tilde{f}(\bigcap S)$. □

□

6 Concluding Remarks

6.0.6 Other Possible Domains for Streams.

A similar study could have been carried out in domains for finite and infinite lists. There are different possibilities, depending on the strictness conditions imposed on the data type. Typical HASKELL lists are fully lazy, and their domain (which contains streams by completeness) contains also approximations with "holes" such as, say, $[\bot, \mathtt{1}]$. Hence most of the material of Sect. 3 and Sect. 4, and in particular the negative results of Sect. 4.1 should apply to this framework. Another possibility is to use *head strict* lazy lists, as in e.g. [15]. In this case, the only possible Scott-approximants are finite lists of non-bottom elements. The negative results of Sect. 4.1 should not apply to this case, but the positive results of Sect. 4.2 should also not apply. Note these both kinds of finite/infinite lists are available in the (strict, call-by-value) language OCAML [12] thanks to the `Lazy` module, as well as in the (lazy) language HASKELL [13] thanks to strictness '`!`' type annotation.

6.0.7 Conclusion.

We have studied the representation of stream functions in Scott domains for the corresponding PCF types. As expected, these domains are well suited for the representation of continuous functions. This is moreover adapted to the framework of Ehrhard's Hypercoherences.

However, our results seem to indicate that it is difficult to represent a stream function by a monotone map on the correspond domain, and to require that the representant is Scott-continuous if and only if the represented function is continuous. This kind of modularity fails in the general case for reasons that are not clear to us. Fortunately, things work well in the compact case, which corresponds to one of our motivations.

It is nevertheless unclear how to use this kind of domains to represent backtracking mechanisms used to approximate non-continuous functions (see e.g. [17]). Further work will concern more concrete and flexible framework such as Hyland-Ong games [9], or game presentations of sequential data structures such as [14].

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