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"ZERO" TEMPERATURE STOCHASTIC 3D ISING MODEL AND DIMER COVERING FLUCTUATIONS: A FIRST STEP TOWARDS INTERFACE MEAN CURVATURE MOTION

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ABSTRACT. We consider the Glauber dynamics for the Ising model with "+" boundary conditions, at zero temperature or at temperature which goes to zero with the system size (hence the quotation marks in the title). In dimension $d = 3$ we prove that an initial domain of linear size L of "−" spins disappears within a time τ_+ which is at most $L^2(\log L)^c$ and at least $L^2/(c \log L)$, for some $c > 0$. The proof of the upper bound proceeds via comparison with an auxiliary dynamics which mimics the motion by mean curvature that is expected to describe, on large time-scales, the evolution of the interface between "+" and "−" domains. The analysis of the auxiliary dynamics requires recent results on the fluctuations of the height function associated to dimer coverings of the infinite honeycomb lattice. Our result, apart from the spurious logarithmic factor, is the first rigorous confirmation of the expected behavior $\tau_+ \simeq const \times L^2$, conjectured on heuristic grounds [13, 7]. In dimension $d = 2$, τ_+ can be shown to be of order L^2 without logarithmic corrections: the upper bound was proven in [8] and here we provide the lower bound. For $d = 2$, we also prove that the spectral gap of the generator behaves like c/L for L large, as conjectured in [3].

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1. INTRODUCTION

A long standing open problem in the mathematical analysis of the stochastic Ising model [17] can be described as follows.

Consider the standard ± 1 spin Ising model at inverse temperature β and zero external magnetic field in a cubic box $\Lambda \subset \mathbb{Z}^d$, $d \geq 2$, of side L, with homogeneous, e.g. "plus", boundary conditions outside Λ (see Section 2 for the precise definition). Denote by π_Λ^+ the corresponding Gibbs measure and assume that β is larger (or much larger) than the critical value β_c where the "plus" and the "minus" phases start to coexist. On the spin configuration space Ω_Λ consider also the continuous time Markov chain, reversible w.r.t. the Gibbs measure π_Λ^+, in which each spin σ_x, $x \in \Lambda$, with rate one chooses a new value from ± 1 with probabilities given by the conditional Gibbs measure given the current values of the spins outside the site x. Such a chain is known in the literature as *Gibbs sampler* or *Glauber chain* and, because of reversibility, its unique stationary distribution coincides with π_Λ^+. Let μ_t^σ denote the distribution of the chain at time t when the initial configuration at time $t = 0$ is $\sigma \in \{-1, 1\}^\Lambda$ and let T_{mix} be the minimum time such that the variation distance between μ_t^σ and π_Λ^+ is smaller than, e.g., $1/(2e)$ for any σ. Because Λ is a finite set and the chain is ergodic, the Perron-Frobenius theorem guarantees that T_{mix} is finite and it is not too difficult to prove that $T_{\text{mix}} \leq e^{c\beta L^{d-1}}$ for some constant c (see e.g. [17]). Despite the fact that the above upper bound is known to be the correct order of growth of T_{mix} when the boundary conditions are absent, i.e. free (see e.g. [24, 17]), it is known that the presence of the homogeneous "plus" boundary conditions drastically modifies the whole process

of relaxation to equilibrium. In this case the conjectured correct growth of T_{mix} as a function of L is given by the Lifshitz's law $T_{\mathrm{mix}} = O(L^2)$ (see [13] and [7]).

Unfortunately, *any* polynomial upper bound on T_{mix} has escaped so far a rigorous analysis and the best known result, following a recent breakthrough [19], is confined to the two dimensional case $d = 2$ and it is of the form $T_{\mathrm{mix}} \leq \exp(c \log(L)^2)$ for any $\beta > \beta_c$ (see [15]).

The conjectured L^2 growth of T_{mix} is related to the shrinking of a "spherical" bubble of the "minus" phase under the influence of the "plus" boundary conditions. On a macroscopic scale one expects [7, 23] that the dynamics of the bubble follows a *motion by mean curvature* which, if true, implies immediately that in a time $O(L^2)$ the bubble disappears and equilibrium is achieved.

Some of the above questions and in particular the validity of the Lifshitz's law can be formulated (and its mathematical justification remains highly non-trivial) even at zero temperature ($\beta = +\infty$), a situation that was considered in great detail in [8] (see also [6], [23]). Similarly one can consider, as we do in the sequel, an almost zero temperature, i.e. $\beta = \beta(L)$ and increasing so fast with L that thermal fluctuations become irrelevant for the relaxation process. In the extreme case $\beta = +\infty$, the stationary distribution is concentrated on the "plus" configuration (i.e. all spins equal to $+1$) and the Glauber chain evolves towards it without ever increasing the spin energy. The mixing time T_{mix} becomes closely related to the hitting time of the "plus" configuration starting from all minuses and it is not too hard to prove, by induction on the dimension d (starting from the case $d = 2$, which requires non-trivial work [8]), an upper bound of the form $T_{\mathrm{mix}} \leq cL^d$. However, the inductive argument, which in $d = 3$ simply boils down to comparing the true evolution of the original cubic bubble of minuses with an auxiliary chain in which the L two-dimensional square layers of the bubble disappear one after the other starting e.g. from the top one, completely neglects the interesting cooperative effect in which the whole bubble, starting from the corners, is eroded by a "mean curvature effect" which enhances considerably its shrinking. In other words, the Lifshitz's law in $d = 3$ cannot be proved or even approached closely without considering in detail how a two-dimensional curved interface separating the pluses from the minuses evolves in time.

The first main contribution of the present work is a strategy to attack and solve the above problem at zero temperature for $d = 3$ (with logarithmic corrections) and for $d = 2$ (with different constants in the upper and lower bounds). We refer to Section 3 for the precise statements. In the challenging three dimensional case our method involves the use and adaptation to our specific situation of two different and beautiful sets of results concerning:

(i) the Gaussian Free Field-like equilibrium fluctuations of *random monotone surfaces* and their connection with *random dimer coverings* of the two-dimensional hexagonal lattice (see [5, 9, 10, 12, 22]);

(ii) the mixing time of a Glauber chain for monotone surfaces [25].

The key point of our approach is to prove that the evolution of the Glauber chain is dominated by that of another effective chain which follows a motion by mean curvature, only slowed down by a logarithmic factor (in the radius of the bubble). In turn, the evolution of the bounding chain is constructed by "peeling off" at time t a layer of logarithmic width from the bubble of radius R_t. The peeling process is realized by letting relax to equilibrium a mesoscopic spherical cap of radius $O(\sqrt{R_t} \times polylog(R_t))$ and height $O(polylog(R_t))$ (where $polylog(x)$ stands for a suitable polynomial of $\log x$) centered at each point of the surface of the bubble. Results (i) above are essential in order to prove the peeling effect while the results (ii) prove that the overall effect occurs on the correct time scale $O(R_t \times polylog(R_t))$. We strongly believe that our approach can be helpful in solving other related problems.

The second contribution, this time for the model in dimension $d = 2$, are upper and lower bounds, *linear* in L^{-1} and *uniform* in $\beta > c \log L$, c large enough, on the *spectral gap* (see (2.6)) of the Glauber chain. In [3] for $d = 2$ the upper bound was proved (apart from logarithmic corrections and with constants depending on β) for any $\beta > \beta_c$ and it was conjectured to be the correct behavior of the spectral gap (to be compared with the L^2 scaling of the mixing time). In our case the proof of the lower bound (the most interesting one) has an analytic flavor. We first unitarily transform the original Markov generator of the Glauber chain acting on $\ell^2(\Omega_\Lambda, \pi_\Lambda^+)$ into a new matrix acting on $\ell^2(\Omega_\Lambda)$ with the flat (i.e. counting) measure, whose off-diagonal elements corresponding to spin transitions which *do not conserve the energy* vanish very fast as $\beta \to \infty$. Such a property, for $\beta \geq c \log L$, allows us to write the new matrix into a block-form plus a remainder whose norm is very small with L. Since each block describes a suitably killed Glauber chain (killing occurs as soon as the spin energy decreases) the desired bound follows by a probabilistic analysis of the killing time for each block.

Contents

2. Model and preliminaries

Given $x, y \in \mathbb{Z}^d$, we write $x \sim y$ if x is a nearest neighbor of y. To each point $x \in \mathbb{Z}^d$ is assigned a spin σ_x which takes values in $\{-1, +1\}$. Given $\Lambda \subset \mathbb{Z}^d$, we let

$$\partial \Lambda := \{x \in \mathbb{Z}^d \setminus \Lambda : \exists y \in \Lambda \text{ such that } x \sim y\}$$

and $\Omega_\Lambda := \{-1, +1\}^\Lambda$ be the set of all possible configurations in Λ. The Gibbs measure in a finite domain Λ for a boundary condition $\eta \in \Omega_{\partial\Lambda}$ at inverse temperature $\beta > 0$ is given by

$$\pi_\Lambda^\eta(\sigma) = \frac{e^{-\beta H_\Lambda^\eta(\sigma)}}{Z_\Lambda^\eta} \tag{2.1}$$

with

$$H_\Lambda^\eta(\sigma) := -\sum_{\substack{x,y\in\Lambda \\ x\sim y}} \sigma_x\sigma_y - \sum_{\substack{x\in\Lambda, y\in\partial\Lambda \\ x\sim y}} \sigma_x\eta_y \tag{2.2}$$

and

$$Z_\Lambda^\eta := \sum_{\sigma\in\Omega_\Lambda} e^{-\beta H_\Lambda^\eta(\sigma)}. \tag{2.3}$$

When no danger of confusion arises, we will write just π for π_Λ^η. We will consider also the case $\beta = +\infty$, in which case π_Λ^η is taken to be the uniform measure over all configurations $\sigma \in \Omega_\Lambda$ which minimize H_Λ^η.

The dynamics we consider is the Glauber dynamics $\{\sigma^\xi(t)\}_{t\geq 0}$, where $\xi = \sigma^\xi(0)$ is the initial condition. To each $x \in \Lambda$ is associated an independent Poisson clock of rate 1. When the clock labeled x rings, say at time s, one replaces σ_x with a value sampled from the probability distribution $\pi_{x,\sigma^\xi(s)}(\cdot)$, where

$$\pi_{x,\sigma}(\cdot) := \pi(\cdot|\sigma_y, y \neq x). \tag{2.4}$$

We denote the law of $\sigma^\xi(t)$ for a given time t as μ_t^ξ. It is well known that $\{\sigma^\xi(t)\}_{t\geq 0}$ is a Markov process, reversible with respect to the equilibrium measure π.

We are interested in the order of magnitude of the "mixing time", defined for some $\epsilon \in (0, 1)$ as

$$\mathrm{T_{mix}}(\epsilon) = \inf\{t > 0 : \sup_{\sigma\in\Omega_\Lambda} \|\mu_t^\sigma - \pi\| \leq \epsilon\} \tag{2.5}$$

where

$$\|\mu - \nu\| = \frac{1}{2} \sum_{\sigma\in\Omega_\Lambda} |\mu(\sigma) - \nu(\sigma)|$$

denotes the total variation distance between two probability measures. When $\epsilon = 1/(2e)$, we will just write $\mathrm{T_{mix}}$ for $\mathrm{T_{mix}}(\epsilon)$. Another quantity we will focus on is the spectral gap,

$$\mathrm{gap} = \mathrm{gap}_\Lambda^\eta = \inf \frac{\pi_\Lambda^\eta(f(-\mathcal{L})f)}{\mathrm{Var}_{\pi_\Lambda^\eta}(f)} \tag{2.6}$$

where $\mathrm{Var}_{\pi_\Lambda^\eta}$ denotes the variance w.r.t. π_Λ^η, $\mathcal{L}$ is the generator of the dynamics,

$$\mathcal{L}f(\sigma) = \sum_{x\in\Lambda} (\pi_{x,\sigma}(f) - f(\sigma)) \tag{2.7}$$

(a more explicit expression for the generator is given in Section 9.1) and the infimum is taken over non-constant functions $f : \Omega_\Lambda \mapsto \mathbb{R}$. The inverse of the gap (respectively the mixing time) measures the speed of convergence to equilibrium in $L^2(d\pi)$ (resp. in total variation norm). Also, it is well known that

$$\sup_{\sigma\in\Omega_\Lambda} \|\mu_t^\sigma - \pi\| \leq e^{-\lfloor t/\mathrm{T_{mix}}\rfloor}, \tag{2.8}$$

which actually holds for any reversible Markov process.

2.1. **Preliminary results.**

Monotonicity and global coupling. For notational clarity, in this subsection we indicate explicitly the dependence on the boundary condition η in $\mu_{t,\eta}^{\xi}$, the law of the dynamics $\sigma_{\eta}^{\xi}(t)$ at time t, ξ being the initial condition. The Glauber dynamics enjoys the following well known monotonicity property. One can introduce a partial order in Ω_{Λ} by saying that $\sigma \leq \sigma'$ if $\sigma_x \leq \sigma_x'$ for every $x \in \Lambda$. Then, one has

$$\mu_{t,\eta}^{\xi} \preceq \mu_{t,\eta'}^{\xi'} \text{ if } \xi \leq \xi', \eta \leq \eta' \tag{2.9}$$

where $\preceq$ denotes stochastic domination (one writes $\mu \preceq \nu$ if $\mu(f) \leq \nu(f)$ for every increasing function f, i.e. $f(\sigma) \leq f(\sigma')$ whenever $\sigma \leq \sigma'$; an event will be called increasing if its characteristic function is increasing). In particular, letting $t \to \infty$ one obtains

$$\pi_{\Lambda}^{\eta} \preceq \pi_{\Lambda}^{\eta'}. \tag{2.10}$$

Also, it is possible to realize on the same probability space the trajectories of the Markov chain corresponding to distinct initial conditions ξ and/or distinct boundary conditions η, in such a way that, with probability one,

$$\sigma_{\eta}^{\xi}(t) \leq \sigma_{\eta'}^{\xi'}(t) \text{ for every } t \geq 0, \text{ if } \xi \leq \xi', \eta \leq \eta'. \tag{2.11}$$

This will be referred to as the "monotone global coupling": its law will be denoted by $\mathbb{P}$ and the corresponding expectation by $\mathbb{E}$.

   Throughout the paper we will apply several times the above monotonicity properties: for brevity, we will simply say "by monotonicity..." instead of referring explicitly to (2.9)-(2.11).

Comparing $\beta = +\infty$ and $\beta \geq C \log L$. In this section, it is understood that Λ is the cubic domain

$$\Lambda_L := \{-L, \ldots, L\}^d \tag{2.12}$$

of side $2L + 1$ and that $\eta_x = +$ for every $x \in \partial \Lambda_L$, so that we omit Λ and η from the notations.
   In this work, we consider the situation where β grows with L, and in particular

$$\beta \in (C \log L, \infty] \tag{2.13}$$

for a sufficiently large C.

   Note that, for $\beta = +\infty$, the equilibrium measure concentrates on the all "+" configuration: $\pi^{\infty}(\sigma) = \mathbf{1}_{\sigma \equiv +}$. Given $c > 0$, one can choose C large enough so that

$$\|\pi - \pi^{\infty}\| \leq \frac{1}{L^c} \tag{2.14}$$

for every β in the range (2.13) (this follows from easy Peierls estimates). Moreover, for every initial condition $\sigma \in \Omega$, one can find a coupling between the dynamics at $\beta = +\infty$ and $\beta > C \log L$ such that they coincide until time L^c with probability at least $1 - 1/L^c$ (just compare the transition rates). In particular, this implies that

$$\|\mu_t^{\sigma} - \mu_t^{\sigma,\infty}\| \leq \frac{1}{L^c} \text{ for every } t \leq L^c. \tag{2.15}$$

We will see later (cf. Remark 1) that, as an immediate consequence of Theorem 1, one actually has the estimate

$$\lim_{L \to \infty} \sup_{\sigma \in \Omega_{\Lambda_L}} \sup_{t \geq 0} \|\mu_t^{\sigma} - \mu_t^{\sigma,\infty}\| = 0. \tag{2.16}$$

We define the random time τ_+ as the first time when all spins are "+", starting from the all "−" configuration:

$$\tau_+ := \inf\{t > 0 : \sigma_x^-(t) = +1 \text{ for every } x \in \Lambda_L\}. \tag{2.17}$$

Observe that, under the global coupling, if $\beta = +\infty$, then $\sigma_x^\xi(t) = +1$ for every $t \geq \tau_+$ and for every initial condition ξ.

Notational conventions

- For notational clarity, quantities referring to the $\beta = +\infty$ dynamics will have a superscript ∞ (we will write for instance $\mu_t^{\xi,\infty}$ and T_{mix}^∞), while (for lightness of notation) we will not put a superscript β when $\beta < \infty$;
- our main focus will be on the case of the all "+" boundary condition ($\eta_x = +1$ for every $x \in \partial\Lambda$) and we will just write $\eta \equiv +$ in this case;
- we let $\mathbf{P}$ denote the law of the process $\{\sigma^-(t)\}_{t\geq 0}$ at $\beta = +\infty$, started from the "−" configuration, with boundary condition $\eta \equiv +$;
- given $\sigma \in \Omega_\Lambda$ and $x \in \Lambda$, we will denote by $\sigma^{(x)} \in \Omega_\Lambda$ the configuration obtained by flipping the spin at x to $-\sigma_x$.

Based on the observations (2.14) and (2.15), one can use the time τ_+ to estimate the mixing time for β large:

Lemma 1. *For every $\epsilon \in (0,1)$ one has*

$$T_{\text{mix}}^\infty(\epsilon) = \inf\{t > 0 : \mathbf{P}(\tau_+ > t) \leq \epsilon\}, \tag{2.18}$$

where we recall that $\mathbf{P}$ is the law of the process $\{\sigma^-(t)\}_{t\geq 0}$ for $\beta = +\infty$, with "+" boundary conditions.

Moreover, fix $\epsilon \in (0,1)$ and $\delta > 0$ and assume that, for some $c_0 > 0$, $T_{\text{mix}}^\infty(\epsilon) \leq L^{c_0}$ for L sufficiently large. If $\beta \in (C \log L, +\infty)$ with C sufficiently large, then for every L large one has

$$T_{\text{mix}}(\epsilon + \delta) \leq T_{\text{mix}}^\infty(\epsilon) \tag{2.19}$$

and

$$T_{\text{mix}}(\epsilon - \delta) \geq T_{\text{mix}}^\infty(\epsilon) \tag{2.20}$$

Proof of Lemma 1. Recall that $\pi^\infty(\sigma) = \mathbf{1}_{\sigma\equiv+}$ so that $\|\mu_t^{\xi,\infty} - \pi^\infty\| = 1 - \mu_t^{\xi,\infty}(\sigma \equiv +) \leq 1 - \mathbf{P}(\sigma^-(t) \equiv +)$ (where the inequality follows from monotonicity) and (2.18) is immediate.

To prove (2.19)-(2.20), it is sufficient to choose C sufficiently large so that (2.14) and (2.15) hold for $c = c_0$ and to take L sufficiently large so that $2/L^{c_0} < \delta$. □

3. RESULTS

3.1. Three-dimensional model.
Consider the 3D Ising model in the cubic box $\Lambda_L = \{-L, \ldots, L\}^3$ with boundary condition $\eta \equiv +$ and $\beta = +\infty$. The main result of this work is:

Theorem 1. *There exists a positive constant c such that for $L \geq 2$ one has*

$$\mathbf{P}\left[\frac{L^2}{c \log L} \leq \tau_+ \leq cL^2(\log L)^c\right] \geq 1 - \frac{c}{L}. \tag{3.1}$$

Remark 1.

(1) As we mentioned in the introduction, the previously known upper bound for τ_+ was of order L^3 [8, Th. 1.3], while (3.1) matches the heuristically expected behavior, except for the logarithmic factor. On the other hand, we are not aware of previously known lower bounds, except for the trivial estimate $\tau_+ \geq cL$;

(2) via Lemma 1, once we prove (3.1) we also have that

$$\frac{L^2}{c \log L} \leq \mathrm{T_{mix}} \leq c\, L^2 (\log L)^c \tag{3.2}$$

for $\beta \geq C \log L$ and C large;

(3) To prove (2.16), observe first of all that

$$\|\mu_t^\sigma - \mu_t^{\sigma,\infty}\| \leq \|\mu_t^\sigma - \pi\| + \|\mu_t^{\sigma,\infty} - \pi^\infty\| + \|\pi - \pi^\infty\|. \tag{3.3}$$

The first two terms are estimated through (2.8), taking say $t \geq L^3$ and using the upper bound (3.2) for the mixing time. The proof is concluded thanks to (2.14)-(2.15) (if C is chosen such that these bounds hold for $c = 3$).

3.2. Two-dimensional model.
For the 2D Ising model in the cubic box $\Lambda_L = \{-L, \ldots, L\}^2$ with boundary condition $\eta \equiv +$ and $\beta = +\infty$ we have

Theorem 2. *There exist positive constants c, γ such that*

$$\mathbf{P}\left(cL^2 \leq \tau_+ \leq (1/c)L^2\right) \geq 1 - e^{-\gamma L}. \tag{3.4}$$

for every $L \geq 1$.

Remark 2. Thanks to Lemma 1, this implies that $c_1 L^2 \leq \mathrm{T_{mix}} \leq c_2 L^2$ for $\beta \geq C \log L$. We mention that the bound $\mathbf{P}(\tau_+ \geq (1/c)L^2) \leq \exp(-\gamma L)$ was proven in [8, Th. 1.3]; as for the lower bound for τ_+, the authors of [8] proved the following weaker result: for every $\delta > 0$, $\mathbf{P}(\tau_+ \leq L^2/(\log L)^{1+\delta})$ tends to zero as $L \to \infty$. For a simplified version of the dynamics, the authors of [6] proved the sharp behavior $\tau_+ \sim const \times L^2$.

In two dimensions, always with "+" boundary conditions, we also have sharp bounds on the spectral gap:

Theorem 3. *There exist positive constants C, c such that for every $\beta \in [C \log L, +\infty)$, one has*

$$\frac{c}{L} \leq \mathrm{gap} \leq \frac{1}{cL}. \tag{3.5}$$

The reason why we require $\beta < \infty$ is that for $\beta = +\infty$ the equilibrium measure is concentrated on a single configuration (the all "+" configuration), and the spectral gap (2.6) is not well defined (all functions have zero variance with respect to π).

Remark 3. On the basis of the analysis of a one-dimensional toy model of evolution of a bubble of "−" phase inside the "+" phase, it was conjectured in [3, Sec. 7] that in dimension $d = 2$ the spectral gap behaves like $const/L$ for every $\beta > \beta_c$. Our result (3.5) is the first confirmation of this conjecture, in the $\beta \to \infty$ limit. Let us remark also that the authors of [3] showed, via the construction of a suitable test function, that $\mathrm{gap} \leq c_1(\beta)(\log L)^{c_2(\beta)}/L$ for every $\beta > \beta_c$. However, when β grows with L as in (2.13), the constants $c_{1,2}(\beta)$ possibly diverge. Therefore, even the upper bound in (3.5) is a qualitative improvement over known results.

The proof that $\mathrm{gap} \geq c/L$ for $\beta \geq C \log L$ extends easily to the three-dimensional model (cf. Section 9.1) but, as we discuss in a moment, there is no reason to believe that this is the correct behavior of the spectral gap in $d = 3$.

Two interesting questions which are left open by the above results are the following:

(1) What is the order of magnitude of the spectral gap in dimension $d = 3$? One expects it to be much larger than in dimension $d = 2$, possibly of order $(\log L)^{-c}$ for some $c > 0$, or even of order 1 [3, Sec. 7], [7];

(2) How to remove the logarithmic corrections in the upper and lower bounds for $\mathrm{T_{mix}}$ in dimension $d = 3$? A suggestive indication that indeed $\mathrm{T_{mix}} \geq cL^2$ comes from the following argument. If one defines the "entropy constant" (or modified Log-Sobolev constant) as

$$c_{ent} := \sup \frac{\pi(f \log f)}{\pi(\log f(-\mathcal{L})f)} \tag{3.6}$$

where the supremum is taken over positive functions f which verify $\pi(f) = 1$, it is possible to exhibit a test function which gives $c_{ent} \geq cL^2$ for some constant c which is positive, uniformly for $\beta \geq C \log L$ with C large. On the other hand, it has been conjectured [20] (and verified in various explicitly solvable examples) that for every reversible Markov process the mixing time is lower bounded by c_{ent} times some universal constant.

The organization of the paper is as follows. Theorem 1 is proven in Section 4 (upper bound on τ_+) and in Section 7 (lower bound). The upper bound requires precise estimates on the fluctuations of dimer coverings of the hexagonal (or honeycomb) lattice, see Section 5, and on the mixing time of a dynamics on plane partitions or monotone sets, proven in Section 6. As for the 2D model, Theorem 2 is proven in Section 8 and Theorem 3 in Section 9.

More notational conventions

- In the proof of the results, c, c', c'' etc. denote positive and finite constants (independent of L and β) which are not necessarily the same at each occurrence.
- If $u \in \mathbb{R}^d$, it will be understood that $u^{(a)}, a = 1, \ldots, d$ are its Cartesian coordinates.
- $d(\cdot, \cdot)$ will denote the Euclidean distance in $\mathbb{R}^d$.
- ∂B will denote the geometric boundary of a set $B \subset \mathbb{R}^d$, except when $B \subset \mathbb{Z}^d$, in which case $\partial B := \{x \in \mathbb{Z}^d \setminus B : \exists y \in B \text{ such that } d(x, y) = 1\}$.
- $\mathbb{N}$ denotes the set of non-zero integers $\{1, 2, \ldots\}$.

4. The mixing time in dimension $d = 3$ (upper bound)

Let us briefly explain the strategy of the proof. Consider the dynamics with initial condition "$-$" inside the discrete ball S_L of radius L and "$+$" outside. The basic step is to prove that, with overwhelming probability, after a time of order $L \times polylog(L)$, the set of minus spins is contained in S_{L-1}, i.e. the radius of the ball has shrinked by 1. Iterating L times this step one obtains the desired upper bound on τ_+. For this, we will need a local analysis of the dynamics of a spherical cap with basis disc radius $O(\sqrt{L} \times polylog(L))$ and height $O(polylog(L))$, when the rest of the spins in S_L are frozen to the initial value "$-$". On one hand, Theorem 4 below guarantees that the equilibration time of such a region is of order $L \times polylog(L)$. On the other hand, a detailed control of the equilibrium fluctuations (see Proposition 4) shows that, after such time, the spins inside the spherical cap belonging to the layer $S_L \setminus S_{L-1}$ have become "$+$" with overwhelming probability.

4.1. **A basic tool: dynamics of discrete monotone sets.**

Definition 1. *We say that $V \subseteq \mathbb{N}^3$ is a positive monotone set if $z \in V$ implies $y \in V$ whenever $y \in \mathbb{N}^3$ and $y^{(a)} \leq z^{(a)}, a = 1, 2, 3$. The collection of all positive monotone sets will be denoted by Σ^+, which is partially ordered with respect to inclusion. When $V \in \Sigma^+$ is a finite set, it is*

usually called a plane partition, *the two-dimensional generalization of an ordinary partition (or Young diagram).*

Given V^+, V_0, V^- in Σ^+ such that $V^- \subseteq V_0 \subseteq V^+$, we define a dynamics $\{V_t^{V_0}\}_{t \geq 0}$ on Σ^+ such that $V^- \subseteq V_t^{V_0} \subseteq V^+$ for all times, with initial condition $V_{t=0}^{V_0} = V_0$. Let $\Lambda := V^+ \setminus V^-$ and associate to each $z \in \Lambda$ an independent Poisson clock of rate one. When the clock labeled z rings at some time s:

- if both sets $V_{s,z,-} := V_s \setminus \{z\}$ and $V_{s,z,+} := V_s \cup \{z\}$ belong to Σ^+, then we replace V_s with $V_{s,z,a}$, with a chosen between "+" and "−" with equal probabilities $1/2$;
- otherwise, we keep V_s unchanged.

The link with the $\beta = +\infty$ dynamics of the Ising model is straightforward: Consider the Ising dynamics in the domain $\Lambda := V^+ \setminus V^-$, with initial condition

$$\xi_z = \begin{cases} - & \text{if} \quad z \in V_0 \cap \Lambda \\ + & \text{if} \quad z \in \Lambda \setminus V_0 \end{cases} \tag{4.1}$$

and with boundary conditions

$$\eta_z = \begin{cases} + & \text{if} \quad z \in \mathbb{N}^3 \setminus V^+ \\ - & \text{if} \quad z \in (\mathbb{Z}^3 \setminus \mathbb{N}^3) \cup V^- \end{cases} \tag{4.2}$$

and let $M_t := \{z \in \mathbb{N}^3 : \sigma_z^\xi(t) = -\}$. Then, M_t has the same law as $V_t^{V_0}$.

To avoid any risk of confusion we choose different notations for the Ising dynamics and the monotone set dynamics: we call $\nu_t^{V_0}$ the law of $V_t^{V_0}$ and $\rho := \rho_{V^\pm}$ its (reversible) invariant measure. Of course, $\rho_{V^\pm}$ is just the uniform measure over the positive monotone sets $V \in \Sigma^+$ such that $V^- \subseteq V \subseteq V^+$. The monotonicity properties of the Glauber dynamics discussed in Section 2.1 immediately give:

Proposition 1. *If $V^\pm, W^\pm \in \Sigma^+$ and $V^\pm \subseteq W^\pm$, then $\rho_{V^\pm} \preceq \rho_{W^\pm}$.*

The next key result quantifies the mixing time of the monotone set dynamics as a function of the shape of Λ:

Theorem 4. *Let*

$$D := \max_{y,z \in \Lambda} \left\{ d\left((z^{(1)}, z^{(2)}), (y^{(1)}, y^{(2)}) \right) \right\}$$

and

$$H := \max\{|z^{(3)} - y^{(3)}| : z, y \in \Lambda \ \text{and} \ y^{(a)} = z^{(a)}, a = 1, 2\}.$$

If $H \leq D$, the mixing time of the monotone set dynamics is $O(H^2 D^2 (\log D)^2)$.

We believe that the correct behavior is $O(D^2 \log D)$. The spurious factor H^2 is potentially dangerous, so we will take care to apply Theorem 4 only in situations where H is small, say of order of a power of $\log D$: this is a crucial step for the proof of the upper bound (3.2), which differs from the expected behavior $O(L^2)$ only by logarithmic corrections. We give the proof of Theorem 4 in Section 6.

4.2. **Upper bound on the mixing time.** Let

$$S_r := \mathbb{Z}^3 \cap B_r,$$

where

$$B_r := \{x \in \mathbb{R}^3 : d(x, 0) \leq r\}$$

is the ball of radius r centered at the origin. By monotonicity, the claim (3.1) follows if we prove the following result for the dynamics in S_{5L} with "+" boundary conditions, started from "−":

Proposition 2. *Consider the dynamics in S_{5L} with "+" boundary conditions on ∂S_{5L}. There exists $c > 0$ such that for every $L \geq 2$ and $t = c\, L^2 (\log L)^c$ one has*

$$\mathbf{P}\left(\exists x \in S_{5L} \setminus S_L \text{ such that } \sigma_x^-(t) = -\right) \leq c/L. \tag{4.3}$$

The reason why this result implies (3.1) is that the set $S_{5L} \setminus S_L$ contains a cube, call it Q, which is a translate of Λ_L and, by monotonicity, the marginal in Q of the evolution in S_{5L} is stochastically dominated by the evolution in Q started from "−", with "+" boundary conditions on ∂Q.

In order to get Proposition 2, we prove

Proposition 3. *Consider the dynamics in S_L with "+" boundary conditions and let $x \in S_L \setminus S_{L-1}$. There exists $c > 0$ such that for every $L \geq 2$*

$$\mathbf{P}\left(\sigma_x^-(t) = + \text{ for all times } t \in [c\, L(\log L)^c, L^3]\right) \geq 1 - c/L^4. \tag{4.4}$$

Proposition 2 then easily follows:

Proof of Proposition 2 (assuming Proposition 3). By the union bound, one then deduces that

$$\mathbf{P}\left(\exists x \in S_L \setminus S_{L-1} \text{ and } t \in [c\, L(\log L)^c, L^3] \text{ such that } \sigma_x^-(t) = -\right) \leq c/L^2. \tag{4.5}$$

For the dynamics in S_{5L} we prove, by induction on $i = 1, \dots, 4L$, that

$$\mathbf{P}(\mathcal{A}_i) := \mathbf{P}\left(\exists x \in S_{5L} \setminus S_{5L-i} \text{ and } t \in [5c\, iL(\log(5L))^c, L^3] \text{ such that } \sigma_x^-(t) = -\right) \leq \frac{c\, i}{L^2}, \tag{4.6}$$

which for $i = 4L$ implies (4.3). For $i = 1$, this follows from Eq. (4.5). If the claim holds for some $1 \leq i < 4L$, then

$$\mathbf{P}(\mathcal{A}_{i+1}) \leq \mathbf{P}(\mathcal{A}_i) + \mathbf{P}(\mathcal{A}_{i+1}|\mathcal{A}_i^{\mathsf{c}}) \leq \frac{c\, i}{L^2} + \mathbf{P}(\mathcal{A}_{i+1}|\mathcal{A}_i^{\mathsf{c}}) \tag{4.7}$$

where for any event A we let A^{c} denote its complement. Next, by monotonicity,

$$\mathbf{P}(\mathcal{A}_{i+1}|\mathcal{A}_i^{\mathsf{c}}) \leq \mathbf{P}\left(\exists x \in S_{5L-i} \setminus S_{5L-i-1}, t \in [5c\,(i+1)L(\log(5L))^c, L^3] : \hat{\sigma}_x(t) = -\right) \tag{4.8}$$

where $\hat{\sigma}^-(t)$ is the evolution in S_{5L-i} with "+" boundary conditions, which starts from "−" at time $t = 5c\, iL(\log(5L))^c$. Thanks to monotonicity we can restart from "−" the evolution at time $t = 5c\, iL(\log(5L))^c$ and we used the hypothesis $\mathcal{A}_i^{\mathsf{c}}$ to freeze all spins outside S_{5L-i} to the value "+" in the time interval $[5ciL(\log(5L))^c, L^3]$. Via a trivial time translation, the upper bound (4.5) (applied with L replaced by $5L - i$) shows that the right-hand side of (4.8) is smaller than c/L^2 (c being the same constant which appears in (4.7)) which, together with (4.7), completes the inductive proof.

$\square$

Proof of Proposition 3. Clearly, it is sufficient to prove the claim for L large enough. In the following, δ will denote a small positive universal constant (conditions on its smallness will be specified later). Given a point $x \in \mathbb{R}^3$, let $(r := \|x\|, \theta, \phi)$ be its spherical coordinates, where $\theta \in [0, \pi]$ is the polar angle and $\phi \in [0, 2\pi)$ is the azimuthal angle, with the convention that the half-plane $\{z \in \mathbb{R}^3 : z^{(1)} \geq 0, z^{(2)} = 0\}$ corresponds to $\phi = 0$ and that $\theta = 0$ identifies the positive $z^{(3)}$ axis. Note that the portion of ∂B_L contained in the octant $\mathcal{C}^+ := \{(\theta, \phi) \in (0, \pi/2)^2\}$ is a monotone surface, where:

Definition 2. *A smooth surface $\Gamma \subseteq \mathbb{R}^3$ is said to be monotone if for every $x \in \Gamma$ all three components of the normal vector $\mathbf{n}_x$ at x are non-zero and have the same sign.*

It is convenient to introduce a few geometric definitions:

Definition 3. *For a given $x \in S_L$, let*

(a) $\mathbb{L}$ be the infinite half-line which starts from the origin of $\mathbb{R}^3$ and goes through x

(b) x' be the intersection of $\mathbb{L}$ and ∂B_L

(c) Π be the plane perpendicular to $\mathbb{L}$ which meets $\mathbb{L}$ at distance $L - (\log L)^{3/2}$ from the origin (the exponent $3/2$ is somewhat arbitrary, and it could be replaced by $1 + \delta$ for any $\delta > 0$)

(d) Υ be the half-space not containing the origin and delimited by Π

(e) $\mathcal{M}$ be the spherical cap $B_L \cap \Upsilon$.

Also, decompose the boundary of $\mathcal{M}$ as $\partial \mathcal{M} = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \partial \mathcal{M} \cap \partial B_L$ is the "curved portion of the boundary" and $\Gamma_2 = \partial \mathcal{M} \cap \Pi$ is a disk.

Thanks to the discrete symmetries of the sphere S_L, it is clearly enough to prove (4.4) for points $x \in S_L \setminus S_{L-1}$ such that $\theta \in [0, \pi/2]$ and $\phi \in [0, \pi/4]$. Actually, we claim that it is enough to restrict to $\theta \in [0, \pi/2 - \delta]$ and $\phi \in [0, \pi/4]$, if $\delta > 0$ is small enough. Indeed, if we interchange the role of the first and third coordinates of $\mathbb{R}^3$, the points with $\phi \in [0, \pi/4]$ and θ close to $\pi/2$ are mapped into points with $\theta \leq \pi/4$ and ϕ small.

It is convenient to distinguish three cases (see Figure 1):
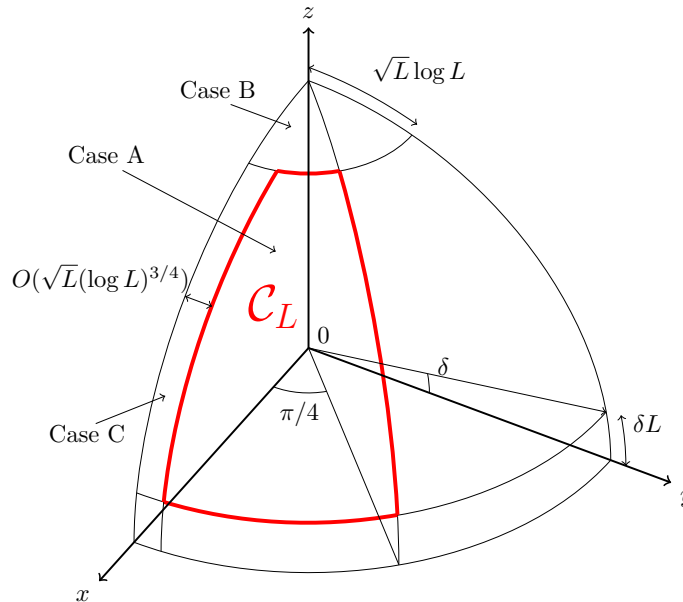
FIGURE 1. The monotone octant $\mathcal{C}^+$ of the surface of the sphere of radius L. By symmetry, we need to prove (4.4) only in the case where $\phi \leq \pi/4$ and x' (cf. Definition 3(b)) does not fall in the strip of width δL, adjacent to the (x, y) plane. Case **A** corresponds to x' in $\mathcal{C}_L$ (the subset of the surface of the sphere delimited by the thick line). Case **B** corresponds to x' at geodesic distance at most $\sqrt{L} \log L$ from the north pole. Finally, case **C** corresponds to x' in the strip of width $(1 + \delta)\sqrt{2L}(\log L)^{3/4}$ to the left of $\mathcal{C}_L$. For reasons of graphical clarity, proportions are not respected in the drawing.

Case A: $x \in S_L \setminus S_{L-1}$ is such that $(\theta, \phi) \in \mathcal{C}_L$ where

$$\mathcal{C}_L := ([0, \pi/2 - \delta] \times [0, \pi/4]) \cap \left\{ (\theta, \phi) : \theta \geq \frac{\log L}{\sqrt{L}}, \ \phi \sin(\theta) \geq \frac{(1+\delta)\sqrt{2}}{\sqrt{L}}(\log L)^{3/4} \right\}. \quad (4.9)$$

Remark that the lower bound on $\phi \sin(\theta)$ just means that, moving on ∂B_L along a line of constant θ, the distance between x' (cf. Definition 3(b)) and the set of points in ∂B_L where $\phi = 0$ is at least $(1+\delta)\sqrt{2L}(\log L)^{3/4}$.

Lemma 2. *Under the condition $(\theta, \phi) \in \mathcal{C}_L$, one has:*
(a) The disk Γ_2 has radius $\sqrt{2L}(\log L)^{3/4}(1 + o(1))$
(b) Γ_1 is contained in the monotone octant $\mathcal{C}^+ = (0, \pi/2)^2$.
As a consequence, both Γ_1 and Γ_2 are monotone surfaces, cf. Definition 2.

Proof of Lemma 2. Statement (a) and monotonicity of Γ_2 require just elementary geometric considerations. As for statement (b), it is easy to see that the condition $(\theta, \phi) \in \mathcal{C}_L$ implies that the geodesic distance of x' along ∂B_L from the boundary of $\mathcal{C}^+$ is larger than $(1+\delta)\sqrt{2L}(\log L)^{3/4}(1 + o(1))$. The fact that Γ_1 is contained (for L large enough) in $\mathcal{C}^+$ then just follows (thanks to the fact that δ is strictly positive) from statement (a). $\quad\square$

We can now continue the proof of Proposition 3 under the condition $(\theta, \phi) \in \mathcal{C}_L$. Let

$$\widehat{\mathcal{M}} := \mathcal{M} \cap \mathbb{Z}^3$$

be the collection of lattice sites contained in $\mathcal{M}$ (cf. Definition 3(e)). It is clear that the site $x \in S_L \setminus S_{L-1}$ under consideration belongs to $\widehat{\mathcal{M}}$ and that, thanks to Lemma 2(a),

$$D := \max_{y,z \in \widehat{\mathcal{M}}} \left\{ d\left((z^{(1)}, z^{(2)}), (y^{(1)}, y^{(2)}) \right) \right\} \leq 2\sqrt{2L}(\log L)^{3/4}(1 + o(1)). \quad (4.10)$$

By monotonicity, it is enough to prove (4.4) for a modified dynamics where only the spins $y \in \widehat{\mathcal{M}}$ evolve, starting from the "$-$" configuration, with boundary conditions given by $\eta_y = +$ for $y \notin S_L$ and $\eta_y = -$ for $y \in S_L \setminus \widehat{\mathcal{M}}$. For lightness of notation, we still call such dynamics $\sigma^-(t)$. Let

$$V^+ := S_L \cap \mathbb{N}^3, \quad V^- := V^+ \setminus \widehat{\mathcal{M}}, \quad (4.11)$$

and observe that $V^\pm$ are positive monotone sets in the sense of Definition 1, as guaranteed by the monotonicity of the surfaces Γ^1, Γ^2 (cf. Lemma 2). Also, call $M_t := \{z \in \mathbb{N}^3 : \sigma_z^-(t) = -\}$: thanks to the discussion in Section 4.1, we have that M_t is a positive monotone set at all times $t \geq 0$, and trivially $V^- \subseteq M_t \subseteq V^+$. To be coherent with the notations of Section 4.1, we have in the present case $V_0 = V^+$, $\Lambda = V^+ \setminus V^- = \widehat{\mathcal{M}}$; also, the law of M_t is just $\nu_t^{V_0}$, with invariant measure $\rho_{V^\pm}$. Finally note that, from the definition of the spherical cap $\mathcal{M}$, one has

$$H := \max\{|z^{(3)} - y^{(3)}| : z, y \in \widehat{\mathcal{M}} \text{ and } y^{(a)} = z^{(a)}, a = 1, 2\} \leq \frac{(\log L)^{3/2}}{\cos \theta} \leq c(\delta)(\log L)^{3/2} \quad (4.12)$$

(recall that we are working under the assumption that $\theta \leq \pi/2 - \delta$). Putting together Theorem 4 with the estimates (4.10), (4.12), we get that the mixing time of the dynamics in $\widehat{\mathcal{M}}$ is $O(L(\log L)^{13/2})$.

We have the following key equilibrium estimate:

Proposition 4. *Recall that $\rho_{V^\pm}$ is the uniform distribution over all positive monotone sets $V \in \Sigma^+$ such that $V^- \subseteq V \subseteq V^+$. There exists $c > 0$ such that for every $L \geq 2$ and whenever $(\theta, \phi) \in \mathcal{C}_L$,*

$$\rho_{V^\pm}\left[\exists z \in \widehat{\mathcal{M}} : d(z, V^-) \geq \frac{1}{4}(\log L)^{3/2} \ \text{ and } \ z \in V\right] \leq \frac{1}{c}\exp(-c(\log L)^{3/2}). \qquad (4.13)$$

In words, this result is saying that spins which are at distance of order $(\log L)^{3/2}$ away from the bottom of the spherical cap $\widehat{\mathcal{M}}$, where the "$-$" boundary conditions act, are "$+$" with overwhelming probability. It is crucial here that the estimate (4.13) (i.e. the value of c) does not depend on the spherical coordinates (θ, ϕ) of the point x under consideration, i.e. on the slope of the plane Π of Definition 3(c).

Roughly speaking, (4.13) follows from recent works on the height fluctuations of random monotone interfaces associated to dimer coverings of the hexagonal lattice [10, 9, 11], but it requires some work to really prove the precise statement we need. The proof of Proposition 4 is given in detail in Section 5.

Once we have Proposition 4, the proof of (4.4) proceeds via a standard argument which we simply sketch: Eq. (2.8), together with the fact that the mixing time of the dynamics in $\widehat{\mathcal{M}}$ is $O(L(\log L)^{13/2})$, implies that the variation distance between $\rho_{V^\pm}$ and the law of $\sigma^-(t)$ is smaller than $\exp(-c'(\log L)^{3/2})$ for all $t > c\,L(\log L)^8$. Since the dynamics undergoes $O(L^6)$ updates during the time interval $[c\,L(\log L)^8, L^3]$, via a union bound and the equilibrium estimate (4.13) one gets that the probability in (4.4) is lower bounded by

$$1 - L^{c''}\exp(-c(\log L)^{3/2}) \geq 1 - c\,L^{-4}$$

for L large, which is the desired bound.

Remark 4. It is important to notice that exactly the same proof gives, for some $c > 0$ and for all $L \geq 2$,

$$\inf_{\substack{x \in S_L \setminus S_{L-(3/4)(\log L)^{3/2}}: \\ (\theta, \phi) \in \mathcal{C}_L}} \mathbf{P}\left(\sigma_x^-(t) = + \text{ for all times } t \in [c\,L(\log L)^c, L^3]\right) \geq 1 - \frac{c}{L^4} \qquad (4.14)$$

(just look at the equilibrium estimate in Proposition 4).

Case B: $x \in S_L \setminus S_{L-1}$ is such that $\theta \leq L^{-1/2} \log L$.

Here, a rather rough argument suffices. Remark first of all that the vertical coordinate $x^{(3)}$ of x belongs to the interval $[L - (1/2)(\log L)^2(1 + o(1)), L]$. Call $\hat{S} := \{y \in S_L : y^{(3)} \geq x^{(3)}\}$. By monotonicity, to show (4.4) it is sufficient to prove that

$$\mathbf{P}(\tau_+ > cL(\log L)^c) \leq c/L^4 \qquad (4.15)$$

for a modified dynamics where only spins in $\hat{S}$ evolve, starting from the "$-$" configuration, with boundary conditions given by $\eta_z = +$ if $z \notin S_L$ and $\eta_z = -$ if $z \in S_L \setminus \hat{S}$. Note that $\hat{S}$ is a discrete spherical cap, and that the boundary conditions just described are "$-$" below its base, and "$+$" elsewhere. Of course, τ_+ in (4.15) is understood to be the first time when all spins in $\hat{S}$ are "$+$".

We note first of all that $\hat{S}$ is contained in a parallelepiped Q whose base is a square of side $\ell := 2\sqrt{L}\log L(1+o(1))$ and whose height is $h := (1/2)(\log L)^2(1+o(1))$. Next, by monotonicity we see that τ_+ is stochastically increased if we replace $\hat{S}$ with Q, with "$-$" boundary conditions below its base square, and "$+$" everywhere else. One can decompose Q into h horizontal squares

of side ℓ, stacked one on top of the other. If h were 1, we would simply have the evolution for $\beta = +\infty$ of the two-dimensional Ising model with "+" boundary conditions in a square of side ℓ (the "+" boundary conditions on the top face of Q would compensate exactly the "−" boundary conditions on the bottom face), and we know [8, Theorem 1.3 (a), case $d = 2$] that in this case one has $\mathbf{P}(\tau_+ \geq c\ell^2) \leq \exp(-\gamma\ell)$ for some positive γ, if c is large enough. Via a standard monotonicity argument (cf. [8, Proof of Theorem 1.3 (a), case $d > 2$] for details) this implies that, for the evolution in the parallelepiped Q, it is very unlikely that τ_+ exceeds $h \times c\ell^2$:

$$\mathbf{P}(\tau_+ > cL(\log L)^c) = \mathbf{P}\left[\tau_+ > c\ell^2 h(\log L)^{c-4}(1/4 + o(1))\right] \leq h\exp(-\gamma\ell) \tag{4.16}$$

if $c > 4$, which is actually stronger than the estimate (4.15) we wished to prove.

Case C: $x \in S_L \setminus S_{L-1}$ is such that

$$\theta \in \left[\frac{\log L}{\sqrt{L}}, \pi/2 - \delta\right] \quad \text{and} \quad \phi\sin\theta \in \left[0, \frac{(1+\delta)\sqrt{2}}{\sqrt{L}}(\log L)^{3/4}\right]. \tag{4.17}$$

Definition 4. *Let $\tilde{x}$ be the point of ∂B_L with angular coordinates*

$$(\theta, \phi + \phi') := \left(\theta, \phi + \frac{\gamma}{\sqrt{L}}\frac{(\log L)^{3/4}}{\sin\theta}\right) \tag{4.18}$$

where $\gamma > 0$ will be chosen later, and let $\bar{x}$ be a point of $\mathbb{Z}^3$ of minimal distance from $\tilde{x}$.

For every $y \in \mathbb{R}^3$, let $(\hat{r}(y), \hat{\theta}(y), \hat{\phi}(y))$ be the spherical coordinates of the vector $y + 3\bar{x}$, i.e., the spherical coordinates of y with respect to the point $-3\bar{x} \in \mathbb{Z}^3$.

Lemma 3. *We have, for the point $x \in S_L \setminus S_{L-1}$ under consideration,*

$$\hat{r}(x) = 4L - \frac{3\gamma^2}{8}(\log(4L))^{3/2}(1 + o(1)), \tag{4.19}$$

$$\theta(1 + o(1)) \leq \hat{\theta}(x) \leq \theta \tag{4.20}$$

$$\hat{\phi}(x)\sin\hat{\theta}(x) = \phi\sin\theta + \frac{3\gamma}{2\sqrt{4L}}(\log(4L))^{3/4}(1 + o(1)). \tag{4.21}$$

We omit the proof of Lemma 3, since it requires only tiresome but elementary computations: one first writes down the Cartesian coordinates of $\bar{x}$ and x, then one works out the spherical coordinates of $x + 3\bar{x}$; finally, the three statements are obtained by expanding these spherical coordinates for L large, using the fact that $\phi' \ll 1$.

We can now conclude the proof of (4.4), case **C**. First of all, remark that, thanks to (4.19), one has for L large enough and letting $S_{r,a} = \mathbb{Z}^3 \cap B_{r,a}$, with $B_{r,a}$ the ball of radius r centered at a,

$$x \in S_{4L,-3\bar{x}} \setminus S_{4L-(3/4)(\log(4L)^{3/2}),-3\bar{x}}$$

provided that

$$\frac{3\gamma^2}{8} < \frac{3}{4}. \tag{4.22}$$

Secondly, (4.21) implies that for L large enough

$$\hat{\phi}(x)\sin\hat{\theta}(x) \geq \frac{(1+\delta)\sqrt{2}}{\sqrt{4L}}(\log(4L))^{3/4}$$

provided that

$$\frac{3\gamma}{2} > (1 + \delta)\sqrt{2}. \tag{4.23}$$

Note that conditions (4.22) and (4.23) are compatible if δ is small enough, and choose a value for γ which satisfies both. Finally, (4.20) shows that $\hat{\theta} \leq \pi/2 - \delta$.

All in all, thanks to Eq. (4.14) in Remark 4 (applied with L replaced by $4L$, and modulo a trivial translation of the center of the sphere S_{4L} from 0 to $-3\bar{x}$), we have

$$\mathbf{P}\left(\sigma_x(t) = + \text{ for all times } t \in [c\,L(\log L)^c, L^3]\right) \geq 1 - c/(4L)^4 \tag{4.24}$$

with $\mathbf{P}$ the law of the dynamics in the sphere $S_{4L,-3\bar{x}}$ with "+" boundary conditions on $\partial S_{4L,-3\bar{x}}$, started from "−". Since $S_{4L,-3\bar{x}} \supset S_L$, by monotonicity (4.24) implies the same inequality when $\mathbf{P}$ is the law of the dynamics in S_L, always started from "−" and with boundary "+" conditions on ∂S_L. The desired estimate (4.4) is proven.

$\square$

5. Estimates on height fluctuations of monotone interfaces

Definition 5. *A subset $V \subseteq \mathbb{Z}^3$ is said to be monotone if $x \in V$ implies $y \in V$ whenever $y \in \mathbb{Z}^3$ and $y^{(a)} \leq x^{(a)}, a = 1, 2, 3$. The collection of all monotone sets is denoted by Σ.*

Remark that this definition differs from that of *positive monotone set* (cf. Definition 1) only in that it is not required that $V \subseteq \mathbb{N}^3$. Given a *positive* monotone set $V \in \Sigma^+$, in the following we will always implicitly identify it with the monotone set $V \cup (\mathbb{Z}^3 \setminus \mathbb{N}^3) \in \Sigma$.

Definition 6. *Given $V \in \Sigma$, we associate to it a vertical height function, which we denote v. The function $\{v_x\}_{x \in \mathbb{Z}^2}$ is defined as*

$$v_x := \max\{x^{(3)} : (x^{(1)}, x^{(2)}, x^{(3)}) \in V\} \text{ if } x = (x^{(1)}, x^{(2)}). \tag{5.1}$$

Observe that v_x takes values in $\mathbb{Z} \cup \{-\infty, +\infty\}$, and that

$$v_x \leq v_y \text{ if } y^{(a)} \leq x^{(a)}, a = 1, 2. \tag{5.2}$$

We will denote $\mathcal{V}$ the set of all possible functions $v : \mathbb{Z}^2 \mapsto \mathbb{Z} \cup \{-\infty, +\infty\}$ which satisfy (5.2). As discussed in Section 5.2, one can identify $\mathcal{V}$ with $\mathcal{D}_{\mathcal{H}} \times \mathbb{Z}$, where $\mathcal{D}_{\mathcal{H}}$ is the set of dimer coverings of the infinite honeycomb lattice $\mathcal{H}$.

5.1. **Proof of Proposition 4.** Recall the definition (4.11) of the monotone sets $V^{\pm}$ in Section 4, and note that the corresponding height functions $v^{\pm} \in \mathcal{V}$ coincide outside some domain $U \subseteq \mathbb{N}^2$ whose diameter is $O(\sqrt{L}(\log L)^{3/4})$ (cf. Lemma 2(a)). Note that U is just the projection on the plane (x, y) of the discrete spherical cap $\widehat{\mathcal{M}}$.

The estimate (4.13) is proven if we have, for some universal constant $c > 0$,

$$\rho_{v^-|_{U^c}}(A|\Gamma^1 \cap \Gamma^2) \leq \frac{1}{c} \exp(-c\,(\log L)^{3/2}) \tag{5.3}$$

for $L \geq 2$ where

$$A := \left\{\exists x \in U : v_x \geq v_x^- + \frac{(\log L)^{3/2}}{4\cos(\theta)}\right\} \tag{5.4}$$

$$\Gamma^1 := \{v_x^- \leq v_x \text{ for every } x \in U\} \tag{5.5}$$

$$\Gamma^2 := \{v_x^+ \geq v_x \text{ for every } x \in U\} \tag{5.6}$$

and $\rho_{\bar{v}|_{U^c}}(\cdot)$ (for some $\bar{v} \in \mathcal{V}$) denotes the uniform measure over the elements v of $\mathcal{V}$ such that $v_x = \bar{v}_x$ for $x \notin U$. (Since monotone sets and height functions are in one-to-one correspondence, we use the same notation ρ to denote the equilibrium uniform measure in both cases). Note that the event A is increasing with respect to the natural partial order in $\mathcal{V}$, where we say that $v \leq w$ (with $v, w \in \mathcal{V}$) if $v_x \leq w_x$ for every $x \in \mathbb{Z}^2$. To lighten notations, given $A \subseteq \mathbb{Z}^2$ we will write $v_A \leq w_A$ if $v_x \leq w_x$ for every $x \in A$.

We need the following monotonicity property, which is an immediate consequence of Proposition 1:

Lemma 4. *One has*

$$\rho_{w|_{U^c}}(\cdot|a_U \leq w_U \leq b_U) \preceq \rho_{w'|_{U^c}}(\cdot|a'_U \leq w_U \leq b'_U) \tag{5.7}$$

if $a, a', b, b', w, w' \in \mathcal{V}$ *are such that* $a_U \leq a'_U$, $b_U \leq b'_U$ *and* $w_{U^c} \leq w'_{U^c}$. *Moreover,* $\rho_{w|_{U^c}}(\cdot)$ *depends only on the value of w on ∂U.*

Applying Lemma 4, we get

$$\rho_{v^-|_{U^c}}(A|\Gamma^1 \cap \Gamma^2) \leq \rho_{v^-|_{U^c}}(A|\Gamma^1). \tag{5.8}$$

The key point is the following result, whose proof is given in Section 5.2:

Theorem 5. *There exists a probability measure ν on the elements $w \in \mathcal{V}$ which satisfies the following properties:*

$$\nu(\exists x \in \partial U : w_x < v_x^-) \leq \varepsilon_L \tag{5.9}$$

$$\nu(\rho_{w|_{U^c}}(A)) \leq \varepsilon_L \tag{5.10}$$

$$\nu(\rho_{w|_{U^c}}((\Gamma^1)^c)) \leq \varepsilon_L, \tag{5.11}$$

where

$$\varepsilon_L := (1/c)\exp(-c\,(\log L)^{3/2}) \tag{5.12}$$

and c is a universal positive constant.

We are now in a position to prove Proposition 4. We have

$$\nu(\rho_{w|_{U^c}}(A|\Gamma^1)) \geq \nu\left[\rho_{w|_{U^c}}(A|\Gamma^1)|\,v_{\partial U}^- \leq w_{\partial U}\right]\nu(v_{\partial U}^- \leq w_{\partial U}) \tag{5.13}$$

$$\geq (1 - \varepsilon_L)\rho_{v^-|_{U^c}}(A|\Gamma^1)$$

where we used (5.9) and Lemma 4. Therefore,

$$\begin{aligned}
\rho_{v^-|_{U^c}}(A|\Gamma^1) &\leq 2\nu(\rho_{w|_{U^c}}(A|\Gamma^1)) \tag{5.14}\\
&= 2\nu(\rho_{w|_{U^c}}(A|\Gamma^1); \rho_{w|_{U^c}}(\Gamma^1) \leq 1/2) + 2\nu(\rho_{w|_{U^c}}(A|\Gamma^1); \rho_{w|_{U^c}}(\Gamma^1) > 1/2)\\
&\leq 2\nu(\rho_{w|_{U^c}}(\Gamma^1) \leq 1/2) + 4\nu(\rho_{w|_{U^c}}(A))\\
&\leq 2\nu(\rho_{w|_{U^c}}((\Gamma^1)^c) \geq 1/2) + 4\varepsilon_L \leq 4\nu(\rho_{w|_{U^c}}((\Gamma^1)^c)) + 4\varepsilon_L \leq 8\varepsilon_L
\end{aligned}$$

where we used assumptions (5.10), (5.11), Markov's inequality and the obvious bound $\rho_{w|_{U^c}}(A|\Gamma^1) \leq \rho_{w|_{U^c}}(A)/\rho_{w|_{U^c}}(\Gamma^1)$. Recalling the definition of ε_L, this implies (4.13). $\qquad\square$

The idea behind Theorem 5 is that, while the height fluctuations of v under ρ for the *fixed boundary conditions* imposed by $v^-|_{U^c}$ are hard to control, they are instead easily described if the boundary conditions are sampled from the infinite measure ν described in Section 5.2. Such random boundary conditions are with high probability "higher" than the deterministic

ones (see (5.11)) and an application of monotonicity (cf. the steps in (5.13)-(5.14)) concludes the argument.

5.2. **Proof of Theorem 5: Dimers coverings and height functions.** We need first to recall some notions and results about the connection between monotone sets and dimer coverings of the infinite two-dimensional hexagonal lattice (cf. for instance [10, 9, 11, 12]).

For $x \in \mathbb{R}^3$, let $\pi_{111}(x)$ denote its orthogonal projection on the (111) plane $\{z \in \mathbb{R}^3 : z^{(1)} + z^{(2)} + z^{(3)} = 0\}$. Let also

$$\mathcal{T} = \cup_{z \in \mathbb{Z}^3} \pi_{111}(z) \tag{5.15}$$

and note that $\mathcal{T}$ is a two-dimensional triangular lattice, which we consider as a graph by putting an edge between any two nearest neighbors (whose mutual distance is $\sqrt{2/3}$). We also let the vector $\hat{e}_1$ (resp. $\hat{e}_2$ and $\hat{e}_3$) be the π_{111} projection of the vector which joins $(0,0,0)$ to $(1,0,0)$ (resp. to $(0,1,0)$ and to $(0,0,1)$): of course, the $\hat{e}_i$ have norm $\sqrt{2/3}$. See Figure 2.

Let $\mathcal{H}$ be the dual lattice of $\mathcal{T}$, again with edges between nearest neighbors. $\mathcal{H}$ is a hexagonal lattice which we decompose as $\mathcal{H} = \mathcal{H}_B \cup \mathcal{H}_W$, where both $\mathcal{H}_W$ and $\mathcal{H}_B$ are translates of the triangular lattice $\mathcal{T}$, such that all nearest neighbors of every vertex in $\mathcal{H}_W$ (resp. in $\mathcal{H}_B$) belong to $\mathcal{H}_B$ (resp. to $\mathcal{H}_W$). Vertices in $\mathcal{H}_W$ (resp. in $\mathcal{H}_B$) are said to be *white* (resp. *black*).

Conventions[1] We embed $\mathcal{T}$ and $\mathcal{H}$ in $\mathbb{R}^2$, with the convention that $\pi_{111}(0) \in \mathcal{T}$ is mapped to $(0,0)$, that $\hat{e}_3$ is mapped to the vertical vector $(0, \sqrt{2/3})$ and that $\hat{e}_1$ is obtained by $\hat{e}_3$ by a counter-clockwise rotation of $(2/3)\pi$. Also, given the two endpoints of the edge of $\mathcal{H}$ which crosses $\hat{e}_3$, we decide that the one which has negative horizontal coordinate (call it $w_{0,0}$) belongs to $\mathcal{H}_W$, while the other (call it $b_{0,0}$) belongs to $\mathcal{H}_B$. Observe that the vector $b_{0,0} - w_{0,0}$ is proportional to $\hat{e}_2 - \hat{e}_1$. We will label a site $v \in \mathcal{H}_W$ (resp. in $\mathcal{H}_B$) as $w_{x,y}$ if $v = w_{0,0} + x\hat{e}_1 + y\hat{e}_2$ (resp. as $b_{x,y}$ if $v = b_{0,0} + x\hat{e}_1 + y\hat{e}_2$). Edges of $\mathcal{H}$ which are perpendicular to $\hat{e}_1$ (resp. to $\hat{e}_2$ or $\hat{e}_3$) will be called edges *of type "a"* (resp. of type "b" or "c"). See Figure 2.

Given a monotone set $V \in \Sigma$, we define the height function $h := h(V) := \{h_x\}_{x \in \mathcal{T}}$ as follows:

$$h_x := \max_{z \in V}\{z^{(3)} : \pi_{111}(z) = x\} \in \mathbb{Z}. \tag{5.16}$$

We denote by $\mathcal{W}$ the set of all functions $h : \mathcal{T} \mapsto \mathbb{Z}$ such that there exists $V \in \Sigma$ with $h = h(V)$. It is easy to see that there is a one-to-one mapping between elements of Σ and elements of $\mathcal{V}$ (cf. Definition 6), and between elements of $\mathcal{V}$ and elements of $\mathcal{W}$. In other words, the functions $\{v_x\}_{x \in \mathbb{Z}^2}$ (cf. Definition 6) and $\{h_x\}_{x \in \mathcal{T}}$ are two equivalent ways to describe the height of the set V with respect to the horizontal plane.

A *dimer covering M* of the hexagonal lattice $\mathcal{H}$ is a subset of the edges of $\mathcal{H}$ covering each vertex of $\mathcal{H}$ exactly once. Note that each edge in M covers one black and one white vertex. To each height function (or monotone set) $h \in \mathcal{W}$ is uniquely associated a dimer covering, and conversely to a dimer covering one can associate uniquely a height function, provided that one fixes the height function at some arbitrary point $x \in \mathcal{T}$ (in other words, dimer coverings identify only gradients of the height function).

The construction of the dimer covering M given $h(V)$ goes as follows (see also Figure 3). Let e be an edge of $\mathcal{H}$, and let (x, y) be the edge of $\mathcal{T}$ which intersects e, with the convention that the vector $x - y$ is $+\hat{e}_i$ for some $i \in \{1, 2, 3\}$. If $x - y = \hat{e}_1$ or $x - y = \hat{e}_2$, then we put a dimer on the edge e if $h_x = h_y - 1$, and we do not put it if $h_x = h_y$ (it is easy to see that these are the

---

[1]The conventions we adopt on the orientation of the axes and on the labeling of the edges via the symbols a, b, c do not coincide with those of [10, 9], but this is simply an irrelevant matter of convention.
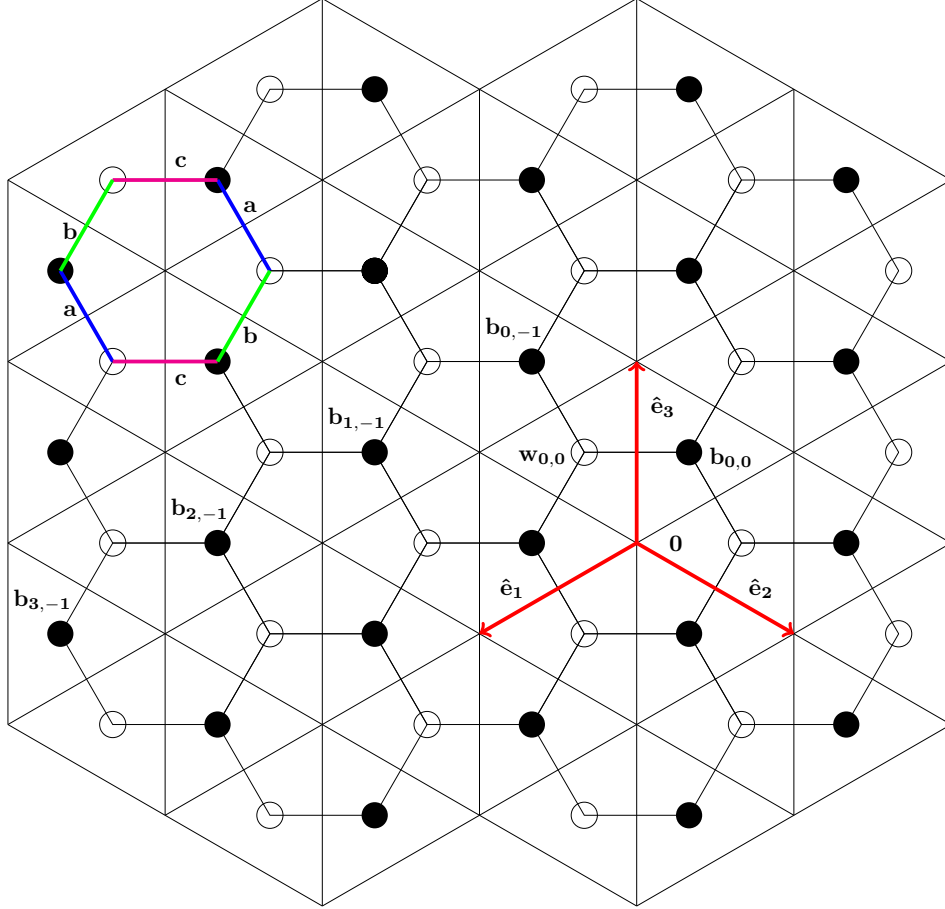
FIGURE 2. A portion of the triangular lattice $\mathcal{T}$ and of the hexagonal lattice $\mathcal{H}$.

only two possibilities since h is a monotone interface). If $x - y = \hat{e}_3$, then we put a dimer on e if $h_x = h_y$ and we do not put it if $h_x = h_y + 1$. Note that the asymmetry between the indices $1, 2, 3$ is due to the fact that we are computing heights with respect to the horizontal plane. A more symmetric choice (but less convenient for our purposes) would be to measure heights with respect to the (111) plane.

Conversely, the construction of h given a dimer covering M goes as follows. First we define a *flux* ω, i.e. a function on oriented edges e of $\mathcal{H}$, such that $\omega(e) = -\omega(-e)$. If e is oriented from the white to the black vertex, then:

- if e is of type "a" or of type "b", then $\omega(e) = 0$ if $e \notin M$ and $\omega(e) = 1$ otherwise
- if e is of type "c", then $\omega(e) = 0$ if $e \in M$ and $\omega(e) = -1$ otherwise.

Next, we fix some arbitrary value $h_{\bar{x}} \in \mathbb{Z}$ at some point $\bar{x} \in \mathcal{T}$. Finally, the difference $h_v - h_{\bar{x}}$ for $v \in \mathcal{T}$ is the total flux of ω which crosses, from right to left, a path ℓ of edges of $\mathcal{T}$, which
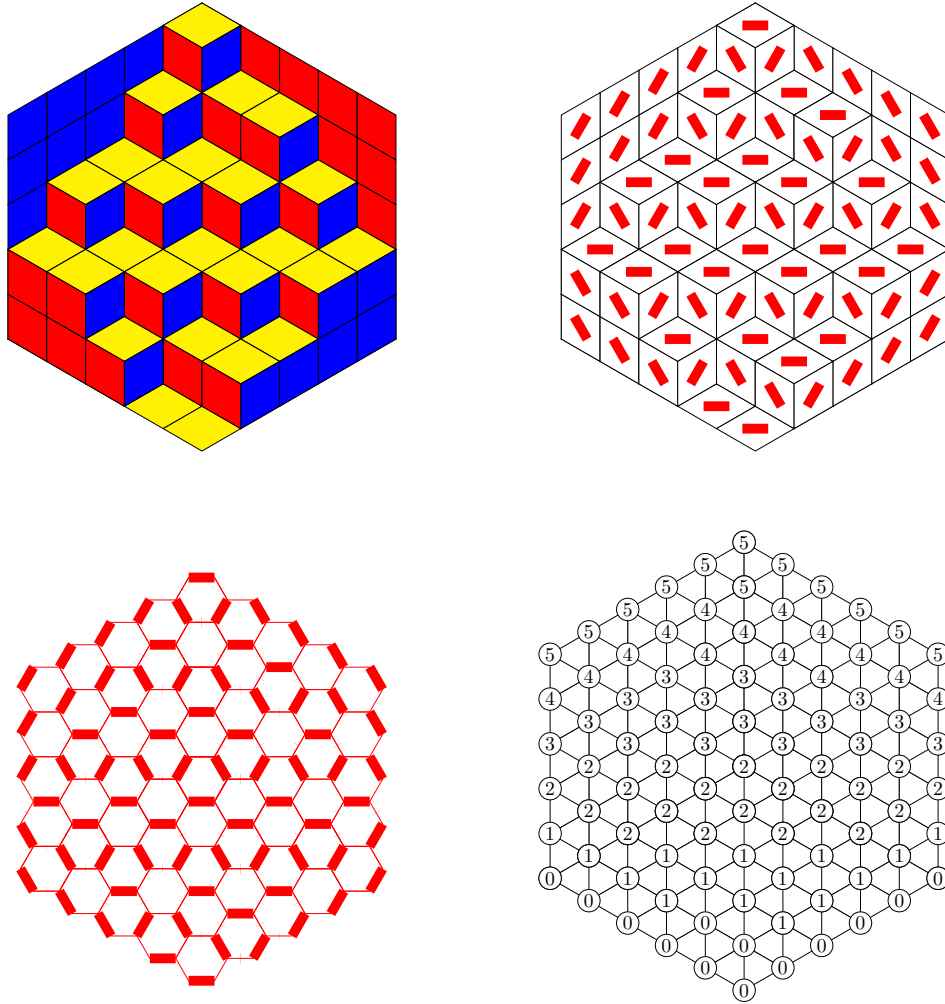
FIGURE 3. A graphically convenient way to visualize a monotone set V is to associate to every $x \in V$ a unit cube, centered at $x - (1/2, 1/2, 1/2)$. In this way, a monotone set, when seen from the 111 direction, appears as a tiling of a portion of the plane with three types of lozenges (top drawing on the left). If we mark a segment along the longer diagonal of each lozenge (top drawing on the right), we obtain a dimer covering of a portion of the hexagonal lattice (bottom, left). To each vertex of a lozenge, i.e. to every vertex in a triangular lattice, one associates its vertical height (bottom, right). One can check in the drawing that the dimer covering and the height function are linked by the construction explained in the text.

goes from $\bar{x}$ to v. The fact that $h_v - h_{\bar{x}}$ does not depend on the choice of the path ℓ is due to the fact that the flux ω has zero divergence, see [10, Sec. 2.2].

Given $\mathbf{p} = (p_a, p_b, p_c)$ with $p_a, p_b, p_c > 0$ and $p_a + p_b + p_c = 1$, take a triangle of perimeter 1 whose angles are $\theta_a := \pi p_a, \theta_b := \pi p_b, \theta_c := \pi p_c$ and let k_a (resp. k_b, k_c) be the length of the

side opposite to θ_a (resp. θ_b, θ_c). To every choice of $\mathbf{p}$ as above, one can associate a translation-invariant Gibbs measure $\mu_{\mathbf{p}}$ on dimer coverings of $\mathcal{H}$. Translation-invariance means that, if A is a set of edges of $\mathcal{H}$, then $\mu_{\mathbf{p}}(A \subseteq M) = \mu_{\mathbf{p}}(T(A) \subseteq M)$, where T is a translation which maps $\mathcal{H}$ into itself.

The precise statement is the following, whose different pieces were proved in [12, 22, 11]:

Theorem 6. *There exists a unique translation-invariant law $\mu_{\mathbf{p}}$ on dimer coverings, such that the probability that a given edge of type "a" (resp. of type "b", "c") belongs to M is p_a (resp. p_b, p_c) and such that, conditionally on the configuration M_{X^c} of the covering M outside a given domain $X \subseteq \mathcal{H}$, $\mu_{\mathbf{p}}$ is the uniform measure over all coverings M_X of X compatible with M_{X^c}, i.e., such that $M_X \cup M_{X^c}$ is a covering of $\mathcal{H}$ (we refer to this property as "DLR property").*

Explicitly, $\mu_{\mathbf{p}}$ is described as follows. Define the matrix $K := \{K(\mathrm{b}, \mathrm{w})\}_{\{\mathrm{b} \in \mathcal{H}_B, \mathrm{w} \in \mathcal{H}_W\}}$ as follows:

- *if b is not a nearest neighbor of w, then $K(\mathrm{b}, \mathrm{w}) = 0$;*
- *$K(\mathrm{b}, \mathrm{w}) = k_a$ (resp. k_b, k_c) if the edge (b, w) is of type "a" (resp. of type "b", "c").*

Define also the matrix $K^{-1} := K^{-1}(\mathrm{w}, \mathrm{b})_{\{\mathrm{w} \in \mathcal{H}_W, \mathrm{b} \in \mathcal{H}_B\}}$ as

$$K^{-1}(\mathrm{w}_{x,y}, \mathrm{b}_{x',y'}) = K^{-1}(\mathrm{w}_{0,0}, \mathrm{b}_{x'-x,y'-y}) \tag{5.17}$$

and

$$K^{-1}(\mathrm{w}_{0,0}, \mathrm{b}_{x,y}) = \frac{1}{(2\pi i)^2} \int_{\mathbb{T}} \frac{z^{-y} w^x}{k_c + k_a z + k_b w} \frac{dz}{z} \frac{dw}{w} \tag{5.18}$$

where the integral is taken over the two-dimensional torus $\mathbb{T} := \{(z, w) \in \mathbb{C}^2 : |z| = |w| = 1\}$. Then, given a set of edges $X = \{(\mathrm{w}_1, \mathrm{b}_1), \ldots, (\mathrm{w}_k, \mathrm{b}_k)\}$ in $\mathcal{H}$, one has

$$\mu_{\mathbf{p}}(X \subseteq M) = \left(\prod_{i=1}^{k} K(\mathrm{b}_i, \mathrm{w}_i)\right) \det\left(K^{-1}(\mathrm{w}_i, \mathrm{b}_j)\right)_{1 \le i, j \le k}. \tag{5.19}$$

Note that K is a weighted version of the adjacency matrix of $\mathcal{H}$. It is immediate to check that one has the relation $KK^{-1} = \mathbb{I}$, which justifies the notation K^{-1}. The infinite matrix K however does not admit a unique inverse, as discussed for instance in [9].

Remark 5. *If we fix deterministically the value $h_{\bar{x}}$ for some $\bar{x} \in \mathcal{T}$, then the function $\mathcal{T} \ni x \mapsto \mu_{\mathbf{p}}(h_x) - h_{\bar{x}}$ is linear in $x - \bar{x}$, thanks to translation invariance of $\mu_{\mathbf{p}}$. As a consequence, the set of points*

$$\{z \in \mathbb{R}^3 : \pi_{111}(z) \in \mathcal{T}, z^{(3)} = \mu_{\mathbf{p}}(h_{\pi_{111}(z)})\} \tag{5.20}$$

is contained in some plane $\Pi_{\mathbf{p}} \subseteq \mathbb{R}^3$ which of course contains the point $\bar{z} \in \mathbb{R}^3$ such that $\pi_{111}(\bar{z}) = \bar{x}$ and $\bar{z}^{(3)} = h_{\bar{x}}$. It is rather easy to check that the normal vector of the plane $\Pi_{\mathbf{p}}$ is parallel to $\mathbf{p}$. In particular, the plane $\Pi_{\mathbf{p}}$ is monotone in the sense of Definition 2.

Height fluctuations. Set for lightness of notation $\Delta = (1/4)(\log L)^{3/2}$.

Proposition 5. *Let $\bar{x} \in \mathcal{T}$ and fix $h_{\bar{x}}$ to some deterministic value. There exists $c > 0$ such that for every $L \ge 2$ one has*

$$\mu_{\mathbf{p}}\left(\exists x \in \mathcal{T} \text{ such that } d(x, \bar{x}) \le L \text{ and } |h_x - \mu_{\mathbf{p}}(h_x)| \ge \Delta/2\right) \le \frac{1}{c} \exp(-c(\log L)^{3/2}) \tag{5.21}$$

uniformly in $\mathbf{p}$ (where $d(\cdot, \cdot)$ denotes the Euclidean distance in $\mathcal{T}$).

Proof of Proposition 5. By translation invariance of $\mu_{\mathbf{p}}$, we can assume without loss of generality that $\bar{x} = 0$ and that we fixed $h_{\bar{x}} = 0$. If $d(x, 0) \leq L$, we can write $x = m\hat{e}_1 + n\hat{e}_2$ for some $m, n \in \{-L, -L+1, \ldots, L\}$. Therefore, via a union bound, to show (5.21) it is enough to prove that, for every $|n| \leq L$

$$\mu_{\mathbf{p}}(|h_{n\hat{e}_i} - \mu_{\mathbf{p}}(h_{n\hat{e}_i})| \geq \Delta/4) \leq \frac{1}{c} \exp(-c(\log L)^{3/2}) \tag{5.22}$$

for $i = 1, 2$. We consider for instance the case $n > 0$ and $i = 1$, the other cases being essentially identical. From the construction of the height function in Section 5.2, we know that

$$h_0 - h_{n\hat{e}_1} = |M \cap \{(\mathrm{b}_{1,0}, \mathrm{w}_{1,1}), \ldots (\mathrm{b}_{n,0}, \mathrm{w}_{n,1})\}| =: \mathcal{N}_n, \tag{5.23}$$

i.e., it is just the number of dimers in the set $\{(\mathrm{w}_{1,0}, \mathrm{b}_{1,1}), \ldots (\mathrm{w}_{n,0}, \mathrm{b}_{n,1})\}$. Thanks to the determinantal representation (5.19), one has [2] that $\mathcal{N}_n$ has the same law as a sum of n independent Bernoulli random variables $X_i, i \leq n$, whose parameters $q_i := P(X_i = 1)$ are the eigenvalues of the matrix $A := \{k_a K^{-1}(\mathrm{w}_{i,1}, \mathrm{b}_{j,0})\}_{1 \leq i,j \leq n}$.

In general, it is not easy to solve explicitly the double integral (5.18) which defines K^{-1}. However, there are special values of x, y for which $K^{-1}(\mathrm{w}_{0,0}, \mathrm{b}_{x,y})$ takes an easy form. In particular, one checks that[2], for $n \in \mathbb{Z} \setminus \{0\}$,

$$K^{-1}(\mathrm{w}_{0,0}, \mathrm{b}_{n,-1}) = (-1)^n \frac{\sin(n p_a \pi)}{\pi n k_a} \tag{5.24}$$

and that

$$K^{-1}(\mathrm{w}_{0,0}, \mathrm{b}_{0,-1}) = \frac{p_a}{k_a}. \tag{5.25}$$

As a consequence, the entries of the matrix A are given by

$$A_{i,j} = a_{i-j} := k_a K^{-1}(\mathrm{w}_{0,0}, \mathrm{b}_{j-i,-1}) = \begin{cases} (-1)^{j-i} \frac{\sin[(i-j)\pi p_a]}{\pi(i-j)} & i \neq j \\ p_a & i = j \end{cases} \tag{5.26}$$

and then

$$\mathrm{Var}(\mathcal{N}_n) = \sum_{i=1}^{n} q_i(1 - q_i) = Tr(A) - Tr(A^2) \tag{5.27}$$

$$= n a_0(1 - a_0) - a_1^2(2n - 2) - a_2^2(2n - 4) - \ldots - 2a_{n-1}^2. \tag{5.28}$$

We will show in a moment that

$$\mathrm{Var}(\mathcal{N}_n) \leq c \log n \tag{5.29}$$

for some $c < \infty$, uniformly in $\mathbf{p}$, which allows us to conclude the proof of (5.22): via the exponential Tchebyshev inequality,

$$\mu_{\mathbf{p}}(\mathcal{N}_n - \mu_{\mathbf{p}}(\mathcal{N}_n) \geq \Delta/4) = P\left(\sum_{i \leq n}(X_i - E(X_i)) \geq \Delta/4\right) \tag{5.30}$$

$$\leq e^{-\Delta/4} \prod_{i \leq n}\left(q_i e^{(1-q_i)} + (1 - q_i)e^{-q_i}\right). \tag{5.31}$$

---

[2] The formula (5.24) differs from the analogous one in [10, Sec. 6.3] by the factor $(-1)^n$, probably due to a typo there. In any case, the global sign is inessential for our computation, since (5.27) below depends only on the absolute value of K^{-1}.

Using the inequality $\exp(x) \leq 1 + x + x^2$ which holds for $-1 < x < 1$ and the fact that $n \leq L$, one deduces

$$
P(\mathcal{N}_n - \mu_{\mathbf{p}}(\mathcal{N}_n) \geq \Delta/4) \;\; \leq \;\; e^{-\Delta/4} \prod_{i \leq n} (1 + q_i(1 - q_i)) \tag{5.32}
$$

$$
\leq \;\; e^{-\Delta/4 + \mathrm{Var}(\mathcal{N}_n)} \leq e^{-(1/16)(\log L)^{3/2} + c \log L}. \tag{5.33}
$$

Analogously, one obtains the same upper bound for $P(\mathcal{N}_n - \mu_{\mathbf{p}}(\mathcal{N}_n) \leq -\Delta/4)$ and (5.22) is proven.

It remains only to prove the estimate (5.29). Essentially the proof can be found in [10, Sec. 6.3], where however uniformity with respect to $\mathbf{p}$ was not discussed, so we repeat quickly the necessary steps here. Observe first of all that

$$
|a_i| \leq \frac{c}{|i| + 1}, \quad i \in \mathbb{Z} \tag{5.34}
$$

uniformly in $\mathbf{p}$. Therefore,

$$
\mathrm{Var}(\mathcal{N}_n) \;\; = \;\; n \left[a_0(1 - a_0) - 2 \sum_{i=1}^{\infty} a_i^2 \right] + 2 \sum_{i=1}^{n-1} i\, a_i^2 + 2n \sum_{i=n}^{\infty} a_i^2 \tag{5.35}
$$

$$
\leq \;\; n \left[a_0(1 - a_0) - 2 \sum_{i=1}^{\infty} a_i^2 \right] + c \log n \tag{5.36}
$$

for some c independent of $\mathbf{p}$. Finally, one observes that the quantity in square brackets is identically equal to zero. To see this, let $f(x) = |x|(\pi - |x|)$ so that $a_0(1 - a_0) = f(\theta_a)/\pi^2$. One then observes that

$$
c_k := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{ikx} dx = \begin{cases} \frac{\pi^2}{6} & \text{if } k = 0 \\ -\frac{1 + (-1)^k}{k^2} & \text{if } k \in \mathbb{Z} \setminus \{0\} \end{cases} \tag{5.37}
$$

and then a straightforward computation shows that the Fourier identity $f(\theta_a) = \sum_{k \in \mathbb{Z}} c_k\, e^{-ik\theta_a}$ is equivalent to $a_0(1 - a_0) = 2 \sum_{i \in \mathbb{N}} a_i^2$. $\qquad \square$

Proof of Theorem 5. Recall the setting of Section 5.1. Let $u = (u^{(1)}, u^{(2)}) \in \partial U$, let $\bar{z} := (u^{(1)}, u^{(2)}, v_u^- + \lfloor \Delta/(2\cos(\theta)) \rfloor) \in \mathbb{Z}^3$, and $\bar{x} := \pi_{111}(\bar{z}) \in \mathcal{T}$. Choose $\mathbf{p}$ to be parallel to the normal to the plane Π of Definition 3(c) and consider the infinite volume distribution of height functions induced by $\mu_{\mathbf{p}}$, with the normalization $h_{\bar{x}} = v_u^- + \lfloor \Delta/(2\cos(\theta)) \rfloor$. Note that the plane $\Pi_{\mathbf{p}}$ defined in Remark 5 is obtained from Π by translating it for a distance $\lfloor \Delta/(2\cos(\theta)) \rfloor + O(1)$ in the positive vertical direction, so that $d(\Pi, \Pi_{\mathbf{p}}) = \Delta/2 + O(1)$.

For a given realization of the height function h, we let Y be the finite set of points

$$
Y(h) := \{z \in \mathbb{Z}^3 \text{ such that } d(\pi_{111}(z), \bar{x}) \leq L \text{ and } z^{(3)} = h_{\pi_{111}(z)}\} \subseteq \mathbb{Z}^3. \tag{5.38}
$$

Thanks to Proposition 5, one has that

$$
\mu_{\mathbf{p}}(\exists z \in Y(h) \text{ such that } d(z, \Pi_{\mathbf{p}}) \geq \Delta/2) \leq \frac{1}{c} \exp(-c(\log L)^{3/2}) \tag{5.39}
$$

with c independent of $\mathbf{p}$.

Recall that to a height function $h \in \mathcal{W}$ there corresponds a unique element $v = v(h) \in \mathcal{V}$ and let ν be the law on $\mathcal{V}$ induced by $\mu_{\mathbf{p}}$. Note that $v_{(u^{(1)}, u^{(2)})} = v_u^- + \lfloor \Delta/(2\cos(\theta)) \rfloor$ deterministically. We show now that ν satisfies the conditions (5.9)–(5.11). The point is that, as observed at the beginning of Section 5.1, the diameter of $U \cup \partial U$ is $o(L)$, so that the π_{111} projection of any

point $(x^{(1)}, x^{(2)}, v_{(x^{(1)}, x^{(2)})})$, with $(x^{(1)}, x^{(2)}) \in U \cup \partial U$, has distance from $\bar{x}$ smaller than L. As a consequence, given some $v = v(h) \in \mathcal{V}$,

$$B(v) := \{x \in \mathbb{Z}^3 \text{ such that } (x^{(1)}, x^{(2)}) \in U \cup \partial U \text{ and } x^{(3)} = v_{(x^{(1)}, x^{(2)})}\} \subseteq Y(h) \qquad (5.40)$$

and, from (5.39),

$$\nu\left(\exists z \in B(v) \text{ such that } d(z, \Pi_{\mathbf{p}}) \geq \Delta/2\right) \leq \frac{1}{c} \exp(-c(\log L)^{3/2}) \qquad (5.41)$$

which immediately implies condition (5.9), since the distance between $\Pi_{\mathbf{p}}$ and Π is Δ, and the graph of the function $U \cup \partial U \ni x \mapsto v_x^-$ is within distance $O(1)$ from the plane Π (recall that v^- is the vertical height function of the monotone set V^- defined in (4.11) so that, in U, v^- is just the lattice approximation of the plane Π). Conditions (5.10), (5.11) are also immediate from (5.41), once one realizes that $\nu(\rho_{v|U^c}(O)) = \nu(O)$ if O is an event which depends only on $\{v_x\}_{x \in U}$. This is because, thanks to the DLR property of $\mu_{\mathbf{p}}$ (cf. Theorem 6), the measure $\nu(\cdot|\{v_x\}_{x \in U^c})$ is the uniform measure over the elements of $\mathcal{V}$ which coincide with v outside U; in other words, $\nu(\cdot|\{v_x\}_{x \in U^c})$ is nothing but $\rho_{v|U^c}(\cdot)$. $\qquad \square$

6. ON THE MIXING TIME OF A DYNAMICS OF MONOTONE SETS

In this section we prove Theorem 4. Recall the notation from Section 4.1, in particular the definition of $\Lambda = V^+ \setminus V^-$, of D (the diameter of the horizontal projection of Λ) and of H (the maximal vertical distance between two points in Λ with the same horizontal projection). Without loss of generality, we can assume that Λ is contained in some parallelepiped $\Gamma \subset \mathbb{N}^3$ defined by

$$\Gamma = \{z \in \mathbb{N}^3 : 1 \leq z^{(1)} \leq a_1, 1 \leq z^{(2)} \leq a_2, \ h_1 \leq z^{(3)} \leq h_2\}, \qquad (6.1)$$

with $a_i \leq D$ and $h_2 = \max\{z^{(3)}, z \in \Lambda\}$, $h_1 = \min\{z^{(3)}, z \in \Lambda\}$. Recalling Definition 6, one can identify uniquely any monotone set $V \in \Sigma^+$, such that $V^- \subset V \subset V^+$, via a vertical height function $v_{(x,y)}$, indexed by pairs (x, y) of non-negative integers. Since the configuration of V outside Λ is fixed and coincides with V^+, what matters is the collection $v := \{v_{(x,y)}\}_{1 \leq x \leq a_1, 1 \leq y \leq a_2}$. This defines a *plane partition* in the box Γ (cf. also Definition 1), i.e. a collection of heights such that $v_{(x+1,y)} \leq v_{(x,y)}$, $v_{(x,y+1)} \leq v_{(x,y)}$. As discussed in Section 5, the bijective correspondence between v and V is given by

$$v_{(x,y)} = \max\{z : (x, y, z) \in V\}.$$

Each pair (x, y) can be identified with a unit square in the plane $z^{(3)} = 0$, in such a way that the center of this square is given by the point of $\mathbb{R}^3$ with coordinates $(x - \frac{1}{2}, y - \frac{1}{2}, 0)$. Thus, we interpret $v_{(x,y)}$ as the height of the column at (x, y). The minimal and maximal sets V^-, V^+ correspond to minimal and maximal column heights, denoted v^- and v^+ respectively (the same notation was used in Section 5.1). We define $\Omega = \Omega(v^{\pm})$ as the set of all plane partitions v in the box Γ such that $v^- \leq v \leq v^+$. Thus, Ω is in one-to-one correspondence with the set $\{V \in \Sigma^+$ such that $V^- \subset V \subset V^+\}$, and the measure $\rho = \rho_{V^{\pm}}$ of Section 4.1 becomes now the uniform probability measure on Ω (we still call it ρ).

6.1. **Column dynamics.** The first step in the proof of Theorem 4 consists in establishing a mixing time upper bound (Lemma 5 below) for a Markov chain that involves equilibration of full columns at each move. The key idea here borrows from Wilson's analysis [25] of the Luby-Randall-Sinclair Markov chain for lozenge tilings [16]. The second step (see Section 6.2) is to show that Lemma 5 implies the upper bound on the mixing time of the "single spin-flip"

dynamics $\nu_t^{V_0}$ which is under consideration in Theorem 4. A similar strategy was used in [18] in the simpler context of the $(1+1)$-dimensional SOS model.

A column (x, y) is said to be even/odd if $(x - y)$ is even/odd. Consider the continuous time Markov chain with state space Ω, where at each arrival time of a Poisson process with parameter 1 we flip a fair binary coin; if the coin is 0 (resp. 1) we update simultaneously all even (resp. odd) column heights $v_{(x,y)}$ with a sample from the distribution ρ conditioned on the current value of the height of the odd (resp. even) columns. Let $P_t(v, \cdot)$ denote the distribution at time t of such Markov process when the starting configuration is $v \in \Omega$. Note that the kernel $P_t = P_t(\cdot, \cdot)$ satisfies

$$P_t = e^{t\mathcal{G}}, \tag{6.2}$$

where the infinitesimal generator $\mathcal{G}$ acts on functions $f : \Omega \mapsto \mathbb{R}$ by

$$[\mathcal{G}f](v) = \frac{1}{2} \rho[\, f \,|\, \mathrm{odd}\,](v) + \frac{1}{2} \rho[\, f \,|\, \mathrm{even}\,](v) - f(v), \tag{6.3}$$

where we use the notation $\rho[\, f \,|\, \mathrm{odd}\,](v)$ (resp. $\rho[\, f \,|\, \mathrm{even}\,](v)$) for the expectation with respect to ρ conditioned on $\{v_{(x,y)}, \ (x, y) \ \mathrm{odd}\}$ (resp. $\{v_{(x,y)}, \ (x, y) \ \mathrm{even}\}$). Note that sampling from $\rho[\,\cdot\,|\,\mathrm{odd}\,]$ amounts to pick uniformly at random a new configuration of even column heights that is compatible with the current odd column heights (compatibility here means that the configuration of all column heights is then a plane partition v that satisfies $v^- \leq v \leq v^+$). It is important to remark that the probability measure $\rho[\,\cdot\,|\,\mathrm{odd}\,]$ is a product of single column probability measures, i.e. conditionally on $\{v_{(x,y)}, \ (x, y) \ \mathrm{odd}\}$, all even columns become independent. The same remarks apply to $\rho[\,\cdot\,|\,\mathrm{even}\,]$. Clearly, ρ is the reversible invariant distribution of our Markov chain.

Lemma 5. *If $H \leq D$, there exists a constant $c > 0$ such that*

$$\sup_{v \in \Omega} \| P_t(v, \cdot) - \rho \| \leq c\, D^4 \exp\left(-\frac{t}{c\, D^2} \right). \tag{6.4}$$

Proof of Lemma 5. We use the well known "lattice path representation" of a plane partition; see e.g. [16, 25]. Namely, any plane partition $v \in \Omega$ can be seen as a collection of $h_2 - h_1 + 1$ non-intersecting lattice paths $\phi^{(i)} = \phi^{(i)}(v)$, $i = h_1, \ldots, h_2$, each of length $2D$ (h_1, h_2 are the same integers that appear in (6.1)) which satisfy

$$\phi_0^{(j)} = \phi_{2D}^{(j)} = j, \quad \phi_{x+1}^{(j)} - \phi_x^{(j)} \in \{-1, +1\}, \quad \phi_x^{(j)} < \phi_x^{(j+1)} \tag{6.5}$$

for all $j = h_1, \ldots, h_2$, $x = 0, \ldots, 2D$. The polymer j describes the jth level set $\{z \in V \cap \Lambda : z^{(3)} = j\}$. For the precise construction of the paths, we refer to [25, Section 5], [16, Section 2.1] (see also Figure 4 for a graphical construction).

In the plane partition-to-lattice path mapping, inequalities are reversed, i.e. if $v_1 \leq v_2$ then $\phi(v_1) \geq \phi(v_2)$ (cf. Figure 4). We let $\phi^+ := (\phi^{+,(i)})_i$ and $\phi^- := (\phi^{-,(i)})_i$ denote the lattice paths corresponding to the maximal and minimal plane partitions v^+, v^-, so that $\phi^+ \leq \phi^-$. Then, the condition $v^- \leq v \leq v^+$ gives

$$\phi_x^{+,(j)} \leq \phi_x^{(j)} \leq \phi_x^{-,(j)}, \tag{6.6}$$

for all $j = h_1, \ldots, h_2$, $x = 0, \ldots, 2D$ or, more compactly, $\phi^+ \leq \phi \leq \phi^-$.

Let $\widetilde{\Omega}$ denote the set of configurations $\phi = (\phi_x^{(j)})_{x,j}$ of integer heights $\phi_x^{(j)} \in \mathbb{Z}$, $j = h_1, \ldots, h_2$ and $x = 0, \ldots, 2D$ satisfying the constraints (7.11), (6.6). The construction above establishes a one-to-one correspondence between the set Ω of plane partitions v satisfying $v^- \leq v \leq v^+$ and the set $\widetilde{\Omega}$. The image of the measure ρ is the uniform probability distribution on $\widetilde{\Omega}$ (which we
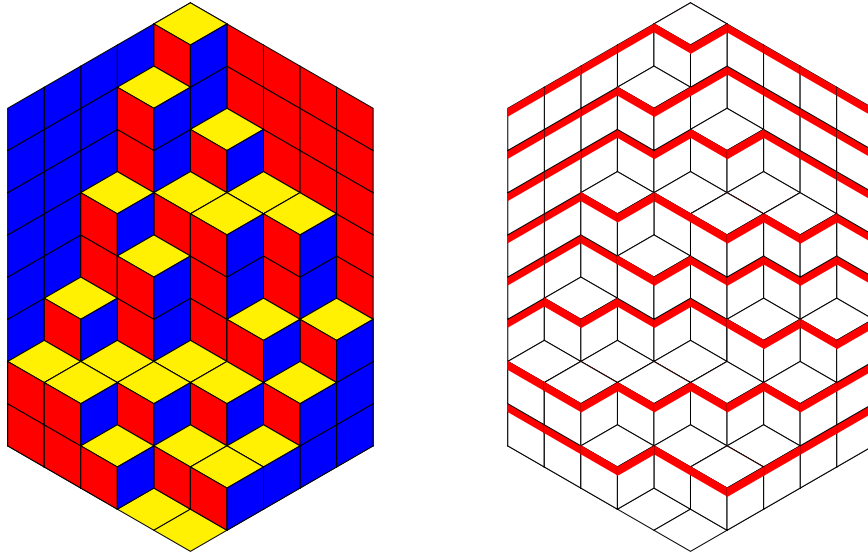
FIGURE 4. In the left drawing, the "column" or "plane partition" representation of a monotone set $V \in \Sigma^+$. In the right drawing, the corresponding "lattice path" representation. The lattice path $\phi^{(j)}$ is essentially the level set at height j of the column representation of V, seen from the (111) direction. It is clear that, whenever we *add* a cube to the plane partition on the left, one lattice path on the right will be *lowered*. This shows that $v_1 \leq v_2$ is reversed into $\phi(v_1) \geq \phi(v_2)$.

call again ρ with some abuse of language). Moreover, it is not hard to see that the image of the Markov process with "full column moves" under this map coincides with the continuous time Markov process with state space $\widetilde{\Omega}$ obtained as follows: at each arrival time of a Poisson process with parameter 1 we flip a fair binary coin; if the coin is 0 (resp. 1) we update simultaneously all $\phi_x^{(i)}$, $i = h_1, \ldots, h_2$, for x even (resp. odd) with a sample from the uniform distribution conditioned on the current values $\{\phi_x^{(i)}, \; i = h_1, \ldots, h_2, x \text{ odd}\}$ (resp. even). With a slight abuse of notation we call again $P_t(\cdot, \cdot)$ the kernel of the Markov process on lattice path configurations, and its invariant measure is of course the uniform measure ρ. Moreover, we write again (cf. (6.3))

$$\mathcal{G} = \frac{1}{2} \left(\rho[\,\cdot \,|\, \mathrm{odd}\,] - \mathbb{I} \right) + \frac{1}{2} \left(\rho[\,\cdot \,|\, \mathrm{even}\,] - \mathbb{I} \right),$$

for the generator of P_t, where $\mathbb{I}$ denotes the identity operator.

Next, we turn to Wilson's coupling argument. Define $\Phi : \widetilde{\Omega} \mapsto \mathbb{R}$ by

$$\Phi(\phi) = \sum_{x=1}^{2D-1} g(x) \sum_{j=h_1}^{h_2} \phi_x^{(j)}, \quad g(x) = \sin\left(\frac{\pi x}{2D}\right).$$

To compute the action of $\mathcal{G}$ on Φ, fix some even $0 < x < 2D$, and observe that if the constraints $\phi_x^{+,(j)} \leq \phi_x^{(j)} \leq \phi_x^{-,(j)}$ were absent, one would have

$$\rho\left[\sum_{j=h_1}^{h_2} \phi_x^{(j)} \,\Big|\, \mathrm{odd} \right] = \frac{1}{2} \sum_{j=h_1}^{h_2} (\phi_{x-1}^{(j)} + \phi_{x+1}^{(j)}). \tag{6.7}$$

Note that the sum is important in (6.7) since the identity has no reason to hold for a single j. Now, the constraint (6.6) is felt at x in the path j iff either $A_-(j,x) := \{\phi_x^{(j)} = \phi_x^{-,(j)} < \phi_{x-1}^{(j)} = \phi_{x+1}^{(j)}\}$ or $A_+(j,x) := \{\phi_x^{(j)} = \phi_x^{+,(j)} > \phi_{x-1}^{(j)} = \phi_{x+1}^{(j)}\}$. In the first case we have to compensate (6.7) with -1 since $\phi_x^{(j)}$ cannot move and $\phi_x^{(j)} = \frac{1}{2}(\phi_{x-1}^{(j)} + \phi_{x+1}^{(j)}) - 1$. In the second case we have to compensate (6.7) with $+1$ since $\phi_x^{(j)}$ cannot move and $\phi_x^{(j)} = \frac{1}{2}(\phi_{x-1}^{(j)} + \phi_{x+1}^{(j)}) + 1$. In conclusion,

$$\rho\Big[\sum_{j=h_1}^{h_2} \phi_x^{(j)} \,\big|\, \text{odd}\Big] = \frac{1}{2} \sum_{j=h_1}^{h_2} (\phi_{x-1}^{(j)} + \phi_{x+1}^{(j)}) + \sum_{j=h_1}^{h_2} (1_{A_+(j,x)}(\phi) - 1_{A_-(j,x)}(\phi)). \qquad (6.8)$$

Clearly, for any x even one has $\rho\Big[\sum_{j=h_1}^{h_2} \phi_x^{(j)} \,\big|\, \text{even}\Big] = \sum_{j=h_1}^{h_2} \phi_x^{(j)}$. Moreover, exactly the same equations hold when x is odd, provided we change the conditioning from odd to even. Therefore, using the notation $(\Delta\varphi)(x) = \frac{1}{2}(\varphi(x-1) + \varphi(x+1)) - \varphi(x)$ for the discrete Laplacian, (6.8) together with its analogue for odd x imply that

$$[\mathcal{G}\,\Phi](\phi) = \frac{1}{2} \sum_{x=1}^{2D-1} g(x) \sum_{j=h_1}^{h_2} (\Delta\phi^{(j)})(x) + \sum_{x=1}^{2D-1} g(x) \sum_{j=h_1}^{h_2} (1_{A_+(j,x)}(\phi) - 1_{A_-(j,x)}(\phi)).$$

Summing by parts and using $\Delta g = -\kappa_D\, g$, where $\kappa_D = 1 - \cos(\pi/(2D))$, one has

$$[\mathcal{G}\,\Phi](\phi) = -\frac{\kappa_D}{2}\,\Phi(\phi) + \sum_{x=1}^{2D-1} g(x) \sum_{j=h_1}^{h_2} (1_{A_+(j,x)}(\phi) - 1_{A_-(j,x)}(\phi)). \qquad (6.9)$$

Next, let $\phi^\xi(t)$ denote the state of the Markov process at time t with initial condition $\xi \in \widetilde{\Omega}$ at time 0. When $\xi = \phi^+$ (minimal state in terms of lattice paths) or $\xi = \phi^-$ (maximal state), we simply write $\phi^+(t)$ or $\phi^-(t)$. Define $u(t) = \mathbb{E}[\Phi(\phi^-(t)) - \Phi(\phi^+(t))] \geq 0$, where $\mathbb{E}$ denotes expectation with respect to the global monotone coupling of the lattice path Markov process with kernel P_t (cf. Section 2.1 and [25]). From (6.2) we infer

$$\frac{d}{dt}\,u(t) = -\frac{\kappa_D}{2}\,u(t) + \psi(t)\,, \qquad (6.10)$$

with

$$\psi(t) = \sum_{x=1}^{2D-1} g(x) \sum_{j=1}^{h} \mathbb{E}\left[(1_{A_-(j,x)}(\phi^+(t)) - 1_{A_-(j,x)}(\phi^-(t))) + (1_{A_+(j,x)}(\phi^-(t)) - 1_{A_+(j,x)}(\phi^+(t)))\right].$$

By monotonicity of the coupling it is immediate to see that $\psi(t) \leq 0$. Therefore (6.10) implies $u(t) \leq u(0)\,e^{-\frac{\kappa_D}{2}t}$. Note that $u(0)$ can be upper bounded by the volume enclosed between the minimal and maximal plane partition, i.e. $u(0) \leq |V^+ \setminus V^-| \leq D^2 H \leq D^3$ (recall that in Theorem 4 we are assuming $H \leq D$). It follows that $u(t) \leq D^3\,e^{-\frac{\kappa_D}{2}t}$.

To finish the proof it suffices to observe that by monotonicity $\Phi(\phi^-(t)) - \Phi(\phi^+(t)) \geq 0$ and $\phi^+(t) \neq \phi^-(t)$ iff $\Phi(\phi^-(t)) - \Phi(\phi^+(t)) \geq 2\sin(\pi/2D)$, so that by Markov's inequality

$$\mathbb{P}\left(\phi^+(t) \neq \phi^-(t)\right) \leq \frac{u(t)}{2\sin(\pi/2D)} \leq \frac{D^3}{2\sin(\pi/2D)}\,e^{-\frac{\kappa_D}{2}t}.$$

We can now bound the total variation distance by the probability of no coupling up to time t, and using monotonicity and the bound above this gives, for any initial state v

$$\|P_t(v,\cdot) - \rho\| \leq \frac{D^3}{2\sin(\pi/2D)}\,e^{-\frac{\kappa_D}{2}t},$$

which is easily seen to imply the desired estimate.

$\square$

6.2. **Proof of Theorem 4.** The proof of Theorem 4 uses the bound of Lemma 5 together with a comparison argument that allows us to translate the mixing time upper bound of the "column dynamics" to an upper bound for the mixing time of the dynamics $\nu_t^{V_0}$ defined in Section 4.1. In the plane partition language, the dynamics $V_t^{V_0}$ will be called $v_t^{v_0}$, with law $\nu_t^{v_0}$ and equilibrium distribution as usual denoted by ρ: it has initial condition $v_0 \in \Omega$ (v_0 being the plane partition corresponding to the monotone set V_0), to each column (x,y) is associated an independent Poisson clock with parameter 1 and the evolution proceeds by local updates. By this we mean that when the clock of the column (x,y) rings, one replaces $v_{(x,y)}$ by $\max\{v_{(x,y)} - 1, v_{(x+1,y)}, v_{(x,y+1)}, v_{(x,y)}^-\}$ (with probability $1/2$) or by $\min\{v_{(x,y)} + 1, v_{(x-1,y)}, v_{(x,y-1)}, v_{(x,y)}^+\}$ (with probability $1/2$). In other words, the chosen column performs a simple symmetric random walk step, except that jumps which violate the plane partition constraints $v_{(x,y)} \geq \max\{v_{(x+1,y)}, v_{(x,y+1)}\}$, $v_{(x,y)} \leq \min\{v_{(x-1,y)}, v_{(x,y-1)}\}$, or the overall constraint $v_{(x,y)}^- \leq v_{(x,y)} \leq v_{(x,y)}^+$, are rejected. We need to prove that the mixing time of this chain is $O(D^2 H^2 (\log D)^2)$.

We start with a simple observation that allows us to reduce to the case of maximal ($v_0 = v^+$) and minimal ($v_0 = v^-$) initial conditions.

Lemma 6. *For any $t > 0$ and any $v_0, v_0' \in \Omega$:*

$$\|\nu_t^{v_0} - \nu_t^{v_0'}\| \leq D^3 \|\nu_t^{v^+} - \nu_t^{v^-}\|.$$

Proof of Lemma 6. Let $\mathbb{P}$ denote a monotone coupling of $\nu_t^{v_0}, \nu_t^{v_0'}$. Then, using the fact that each column (x,y) has a minimal height $v_{(x,y)}^-$ and a maximal height $v_{(x,y)}^+$ such that $v_{(x,y)}^+ - v_{(x,y)}^- \leq H \leq D$ we have

$$\|\nu_t^{v_0} - \nu_t^{v_0'}\| \leq \mathbb{P}(v_t^{v_0} \neq v_t^{v_0'}) \leq \mathbb{P}(v_t^{v^+} \neq v_t^{v^-})$$

$$\leq \sum_{(x,y)} \sum_{h=v_{(x,y)}^-}^{v_{(x,y)}^+} [\mathbb{P}((v_t^{v^+})_{(x,y)} > h) - \mathbb{P}((v_t^{v^-})_{(x,y)} > h)] \leq D^3 \|\nu_t^{v^+} - \nu_t^{v^-}\|.$$

$\square$

Thanks to Lemma 6, to prove Theorem 4 it is sufficient to show that

$$\|\nu_t^{v^\pm} - \rho\| \leq \frac{1}{4eD^3}, \tag{6.11}$$

for some $t = O(D^2 H^2 (\log D)^2)$. Let us prove the statement for $\nu_t^{v^+}$, the argument for $\nu_t^{v^-}$ being identical. Consider i.i.d. Bernoulli($\frac{1}{2}$) random variables $\zeta = (\zeta_1, \zeta_2, \dots)$ and let $T_\ell = \ell T$, with $T = c_1 H^2 \log D$ and $\ell \in \mathbb{N} \cup \{0\}$, denote a partition of the time axis. Furthermore, consider i.i.d. Poisson processes with parameter 1 at each column (x,y), and let $\{S_i(x,y)\}_{i,(x,y)}$ denote the corresponding collection of arrival times. Call $\{\widetilde{S}_i(x,y)\}_{i,(x,y)}$ the collection of arrival times obtained from $\{S_i(x,y)\}_{i,(x,y)}$ by deleting (or "censoring") all arrivals $S_i(x,y)$ such that, for some $j \in \mathbb{N}$, $T_{j-1} \leq S_i(x,y) < T_j$ and (x,y) has the opposite parity as ζ_j (e.g. (x,y) is odd and $\zeta_j = 0$).

By construction, if we start from the configuration v^+ at time 0 and perform local updates using all the marks $\{S_i(x,y)\}_{i,(x,y)}$ up to time t we obtain the distribution $\nu_t^{v^+}$. Let us call $\widetilde{\nu}_{\zeta,t}^{v^+}$ the distribution of the "censored" dynamics obtained in the same way but only using the marks

$\{\widetilde{S}_i(x,y)\}_{i,(x,y)}$, for a fixed Bernoulli sequence ζ. From the "censoring inequality" of Peres and Winkler [21], [19, Th. 2.5] it follows that $\nu_t^{v^+}$ is stochastically dominated by $\widetilde{\nu}_{\zeta,t'}^{v^+}$, for any $t' \leq t$. Moreover, setting $\widetilde{\nu}_t^{v^+} := \mathbb{E}_\zeta \widetilde{\nu}_{\zeta,t}^{v^+}$, where $\mathbb{E}_\zeta$ denotes expectation over the random sequence ζ, by linearity of the expectation one sees that

$$\nu_t^{v^+} \preceq \widetilde{\nu}_{t'}^{v^+} \quad \text{for any } t' \leq t. \tag{6.12}$$

Let us fix $t_0 = c_2\, s\, T$, where $s = c_3\, D^2 \log D$ and c_2, c_3 are constants to be taken sufficiently large. Let $\{N(s), s \geq 0\}$ denote a Poisson process with parameter 1, and write $\mathbb{P}$ and $\mathbb{E}$ for the associated probability and expectation. Monotonicity implies that the event $A = \{v \in \Omega : \nu_{t_0}^{v^+}(v) > \rho(v)\}$ is increasing [19, Th. 2.5]. Therefore, the censoring inequality (6.12) gives

$$\begin{aligned}
\|\nu_{t_0}^{v^+} - \rho\| &= \nu_{t_0}^{v^+}(A) - \rho(A) \leq \mathbb{E}(\widetilde{\nu}_{N(s)T}^{v^+}(A)|N(s)T < t_0) - \rho(A) \tag{6.13}\\[2mm]
&\leq \frac{\mathbb{E}\widetilde{\nu}_{N(s)T}^{v^+}(A) - \rho(A)}{\mathbb{P}(N(s)T < t_0)} + \rho(A)\frac{\mathbb{P}(N(s)T \geq t_0)}{\mathbb{P}(N(s)T < t_0)}\\[2mm]
&\leq \frac{\|\mathbb{E}\widetilde{\nu}_{N(s)T}^{v^+} - \rho\|}{1 - \mathbb{P}(N(s)T \geq t_0)} + \frac{\mathbb{P}(N(s)T \geq t_0)}{\mathbb{P}(N(s)T < t_0)}.
\end{aligned}$$

Taking c_2 large in the definition of t_0 above and using standard estimates for the Poisson random variable we can make $\mathbb{P}(N(s)T \geq t_0)$ smaller than D^{-4}. Therefore, we see that thanks to Lemma 5 and (6.13), if c_3 in the definition of s is chosen large the claim (6.11) is a consequence of

$$\|\mathbb{E}\,\widetilde{\nu}_{N(s)T}^{v^+} - P_s(v^+, \cdot)\| \leq \frac{1}{5eD^3}, \tag{6.14}$$

where $P_s(\cdot, \cdot)$ is the kernel defined in Section 6.1, which involves full column equilibrations (cf. (6.2)). To prove (6.14) we need to compare column equilibration moves with local updates. Let us use the notation $\rho_0(v, \cdot)$ and $\rho_1(v, \cdot)$ for the probability kernels associated to $\rho[\cdot\,|\,\text{odd}\,](v)$ and $\rho[\cdot\,|\,\text{even}\,](v)$ respectively, see (6.3). That is, for any $f : \Omega \mapsto \mathbb{R}$ one has for instance $\rho[f\,|\,\text{odd}](v) = \sum_{v' \in \Omega} \rho_0(v, v')f(v')$. Define $P_{\zeta,n}^v = [\rho_{\zeta_1} \cdots \rho_{\zeta_n}](v, \cdot)$, and note that $Q_n^v := 2^{-n} \sum_{\zeta \in \{0,1\}^n} P_{\zeta,n}^v$ is nothing else but a discrete time version of the kernel $P_s(v, \cdot)$, i.e. $P_s(v, \cdot) = \mathbb{E}Q_{N(s)}^v$ where $\mathbb{E}$ and $N(\cdot)$ are as above. Since $s = c_3\, D^2 \log D$, excluding an event of probability $O(D^{-p})$ for some large $p > 0$ we can assume that $N(s) = n$ for some $n \leq 2s$. Recall that $\widetilde{\nu}_{\zeta,t}^{v^+}$ denotes the law $\widetilde{\nu}_t^{v^+}$ conditioned on the binary sequence ζ. The previous remarks imply that it will be sufficient to prove the upper bound

$$\sup_{\zeta \in \{0,1\}^n} \|\widetilde{\nu}_{\zeta,nT}^{v^+} - P_{\zeta,n}^{v^+}\| \leq \frac{1}{6eD^3}, \quad n \leq 2s. \tag{6.15}$$

Observe that, for the censored dynamics $\widetilde{\nu}_{\zeta,t}^{v^+}$, in the time interval $T_{j-1} \leq t < T_j$, all columns with the same parity as ζ_j are independently updated by local moves, i.e. on columns of that parity we have independent continuous-time simple symmetric random walks in segments (determined by the columns of opposite parity) of length bounded by H. It is standard that for some constant $c > 0$ the mixing time on each column is bounded by $c\,H^2$ and therefore after a time T we have the bound, for any $\zeta_1 \in \{0, 1\}$, uniformly in the starting configuration $v \in \Omega$:

$$\|\widetilde{\nu}_{\zeta,T}^{v}(\cdot) - \rho_{\zeta_1}(v, \cdot)\| \leq c\,D^2 \exp\left(-\frac{T}{c\,H^2}\right). \tag{6.16}$$

For general $n \in \mathbb{N}$, by recursive coupling of the distributions involved, using (6.16) at each step, one has

$$\|\widetilde{\nu}_{\zeta,nT}^{v^+}(\cdot) - [\rho_{\zeta_1} \cdots \rho_{\zeta_n}](v^+, \cdot)\| \leq 1 - \left(1 - c\, D^2 \exp\left(-\frac{T}{c\, H^2}\right)\right)^n. \tag{6.17}$$

Since $T = c_1\, H^2 \log D$ and $n \leq 2s = O(D^2 \log D)$, we see that the desired estimate (6.15) follows for a suitable choice of c_1.

$\square$

7. The mixing time in dimension $d = 3$ (lower bound)

Here we prove the bound

$$\mathbf{P}(\tau_+ \leq L^2/(c \log L)) \leq c/L \tag{7.1}$$

for a suitable constant c.

Let C_L be the cube $\{1, \ldots, L/2\}^3$ (we assume for definiteness that L is even) with boundary conditions $\eta_x = +$ if $x \in \partial C_L$ with $\min(x^{(1)}, x^{(2)}, x^{(3)}) = 0$ and $\eta_x = -$ otherwise. In other words, the boundary conditions η are "+" at the three faces of ∂C_L which meet at the origin of $\mathbb{Z}^3$, and "−" at the other three. Denote by $\{s^-(t)\}_t$ the zero-temperature Glauber evolution started from the "−" configuration (in order not to confuse it with the evolution $\sigma^-(t)$ in Λ_L) and by $\mathbf{P}_{C_L}^\eta$ its law. One has

Proposition 6. *There exists $c > 0$ such that*

$$\mathbf{P}_{C_L}^\eta \left(\text{there exists } t < \frac{L^2}{c \log L} \text{ such that } s_{(1,L/2,1)}^-(t) = +\right) \leq \frac{c}{L}. \tag{7.2}$$

Proof of Eq. (7.1), assuming Proposition 6. Since the set $\{x \in C_L : s_x^-(t) = +\}$ is a monotone set (cf. Definition 1) at all times, the event that $s_{(1,L/2,1)}^-(t) = -$ implies the event $s_y^-(t) = -$ for all $y \in C_L$ such that $y^{(2)} = L/2$. Therefore, (7.2) implies (using also symmetry among the three coordinate axes)

$$\mathbf{P}_{C_L}^\eta \left(\exists\, t < \frac{L^2}{c \log L}, x \in C_L \text{ with } \max(x^{(1)}, x^{(2)}, x^{(3)}) = L/2 \text{ such that } s_x^-(t) = +\right) \leq \frac{3c}{L}. \tag{7.3}$$

The cube $\Lambda_L = \{1, \ldots, L\}^3$ can be seen as the union of eight disjoint sub-cubes $C^{(i)}$ of side $L/2$, with $C^{(1)} = C_{L/2}$ while $C^{(i)}, i = 2, \ldots, 8$ are suitable translations of C_L. Let as usual $\mathbf{P}$ denote the law of the Glauber evolution inside Λ_L, with "+" b.c., started from "−" and let $\mathbf{P}'$ be law of the evolution, again started from "−", where the spin configuration inside each $C^{(i)}$ evolves independently for different i, with b.c. given by $\eta_x = +$ for $x \in \partial C^{(i)} \cap \partial \Lambda_L$ and $\eta_x = -$ for $x \in \partial C^{(i)} \cap \Lambda_L$. It is clear that, until the random time

$$t_1 := \inf\{t > 0 : \exists i \in \{1, \ldots, 8\}, x \in \partial C^{(i)} \cap \Lambda_L \text{ such that } \sigma_x^-(t) = +\}, \tag{7.4}$$

the two evolutions can be perfectly coupled, and that $t_1 < \tau_+$. On the other hand, thanks to (7.3) and to the symmetry among the various cubes $C^{(i)}$ (up to suitable translations of the origin and reflections of the coordinate axes) one sees that

$$\mathbf{P}\left(\tau_+ < \frac{L^2}{c \log L}\right) \leq \mathbf{P}\left(t_1 < \frac{L^2}{c \log L}\right) = \mathbf{P}'\left(t_1 < \frac{L^2}{c \log L}\right) \leq 8 \times \frac{3c}{L}. \tag{7.5}$$

$\square$

The main ingredient in the proof of Proposition 6 is a "column dynamics" $\tilde{s}^-(t)$ (analogous to the one used in Section 6.1) defined as follows. Start from the "all minus" configuration in C_L, $\tilde{s}^-(0) \equiv -$. Assign to each $x \in \{1, \ldots, L/2\}^2$ an i.i.d. Poisson clock of rate 2; when the clock

labeled $x = (x^{(1)}, x^{(2)})$ rings, we assign a new value to the collection of spins at sites $z \in C_L$ with $(z^{(1)}, z^{(2)}) = (x^{(1)}, x^{(2)})$, by sampling it from the equilibrium distribution conditioned on the present value of all the other spins. In other words, we set to equilibrium the column of horizontal coordinates $(x^{(1)}, x^{(2)})$, conditionally on the value of the neighboring columns. For every t we have the stochastic domination

$$s^-(t) \preceq \tilde{s}^-(t). \tag{7.6}$$

Indeed, for $k \in \mathbb{N}$ let $s^{-,k}(t)$ be the following dynamics. Set $s^{-,k}(0) \equiv -$ and, when the Poisson clock of the column labeled x rings, repeat k times the following procedure:

- flip a fair binary coin;
- if the coin gives "head", then make a heat bath update at each of the sites of the column x with vertical coordinate belonging to $2\mathbb{N}$, one by one, starting from the bottom site;
- if instead the coin gives "tail", then make a heat bath update at each of the sites of the column x with vertical coordinate belonging to $2\mathbb{N} + 1$, one by one, starting from the bottom site.

Here, "making a heat-bath update" at a site z means updating σ_z according to the equilibrium conditioned on the value of the spins outside z. It is immediate to realize that the process $\{s^{-,1}(t)\}_t$ has the same law as $\{s^-(t)\}_t$, and that the law of $\{s^{-,k}(t)\}_t$ converges to that of $\{\tilde{s}^-(t)\}_t$ for $k \to \infty$. Also, the stochastic domination $s^{-,k}(t) \preceq s^{-,k+1}(t)$ is an immediate consequence of the Peres-Winkler censoring inequality [21], [19, Th. 2.5].

Call $\tilde{\mathbf{P}}$ the law of the column dynamics $\{\tilde{s}^-(t)\}_t$. One has

Proposition 7. *Fix $c > 0$. For L sufficiently large and $t < L^2/(c \log L)$ one has*

$$\tilde{\mathbf{P}}(\tilde{s}^-_{(1,L/2,1)}(t) = +) \leq c' L^2 \, e^{-L^2/(64t)} \tag{7.7}$$

for some finite constant c'.

Proof of Proposition 6, given Proposition 7. Let

$$H := \int_0^{L^2/(c \log L)} \mathbf{1}_{\{s^-_{(1,L/2,1)}(t)=+\}} dt \tag{7.8}$$

so that, thanks to Proposition 7 and to the stochastic domination (7.6) one has

$$\mathbf{E}^\eta_{C_L}(H) \leq c' \, L^{4-(c/64)}. \tag{7.9}$$

Note that the desired bound (7.3) can be rewritten as $\mathbf{P}^\eta_{C_L}(H > 0) < 3c/L$. One has

$$\begin{aligned}
\mathbf{P}^\eta_{C_L}(H > 0) &= \mathbf{P}^\eta_{C_L}(H \geq L^{-c/128}) + \mathbf{P}^\eta_{C_L}(0 < H < L^{-c/128}) \\
&\leq c' L^{4-(c/128)} + \mathbf{P}^\eta_{C_L}(0 < H < L^{-c/128})
\end{aligned} \tag{7.10}$$

where in the first term we applied Markov's inequality. Choosing c sufficiently large, one can make both terms in the last expression smaller than c/L. Indeed, in order that $0 < H < L^{-c/128}$, there must be two times s, t with $0 < t - s < L^{-c/128}$ such that the Poisson clock associated to the site $(1, L/2, 1)$ rings both at times s and t. Via a simple union bound, and using the exponential form of the law of the intervals between two successive rings, one sees that the probability of such event is $O(L^{2-c/128})$.

$\square$

Proof of Proposition 7. Since the set $\{x \in C_L : \tilde{s}^-_x(t) = +\}$ is a monotone subset of C_L at all times, one can identify it (recall Section 6.1 and Fig. 4) with the set of $L/2$ paths

$\{\phi^{(j)}(t)\}_{j=1,\ldots,L/2}$ of length $L+1$, where the jth path is the collection $\{\phi_x^{(j)}(t)\}_{x=0,\ldots,L}$ and the following relations are satisfied:

$$\phi_0^{(j)}(t) = \phi_L^{(j)}(t) = j, \quad \phi_{x+1}^{(j)}(t) - \phi_x^{(j)}(t) \in \{-1, +1\}, \quad \phi_x^{(j)}(t) < \phi_x^{(j+1)}(t). \tag{7.11}$$

At time $t = 0$ one has $\{x \in C_L : \tilde{s}_x^-(0) = +\} = \emptyset$ and therefore $\phi_x^{(j)}(0) = j + x$ if $x \leq L/2$ and $\phi_x^{(j)}(0) = j + L - x$ if $x \geq L/2$. Defining

$$h_x(t) := \frac{2}{L} \sum_{j=1}^{L/2} \left[\phi_x^{(j)}(t) - j \right], \tag{7.12}$$

$u(t, x) := \tilde{\mathbf{E}}(h_x(t/2))$ and reasoning like in Section 6.1 (see also [25, Sec. 5]), one sees that $u(t, x)$ satisfies the discrete heat equation

$$\begin{cases} \frac{d}{dt} u(t,x) = (\Delta u)(t,x) := \frac{u(t,x+1)+u(t,x-1)-2u(t,x)}{2}, & x = 1, \ldots, L-1 \\ u(0,x) = x \, \mathbf{1}_{x \leq L/2} + (L - x) \, \mathbf{1}_{x > L/2} \end{cases} \tag{7.13}$$

with Dirichlet boundary conditions $u(t, 0) = u(t, L) = 0$. On the other hand, from the representation of monotone sets as collections of paths, one sees that

$$\begin{aligned}
\tilde{\mathbf{P}}(\tilde{s}_{(1,L/2,1)}^-(t) = +) &= \tilde{\mathbf{E}}\left(\mathbf{1}_{\{\phi_{L-1}^{(1)}(t)-1=-1\}} \right) = \tilde{\mathbf{E}}\left(\frac{1 - (\phi_{L-1}^{(1)}(t) - 1)}{2} \right) \tag{7.14} \\
&\leq \sum_{j=1}^{L/2} \tilde{\mathbf{E}}\left(\frac{1 - (\phi_{L-1}^{(j)}(t) - j)}{2} \right) = \frac{L}{2} \frac{1 - u(2t, L-1)}{2}.
\end{aligned}$$

Therefore, it is sufficient to prove the heat-equation estimate

$$1 - u(t, L-1) \leq c' L \, e^{-L^2/(32t)}, \tag{7.15}$$

uniformly for $t < 2L^2/(c \log L)$ and L large. While (7.15) can be obtained directly via Fourier analysis, we give a simple and more probabilistic argument. First one notes that, for $x = 1, \ldots, L$, the quantity $[1 + u(t,x) - u(t, x-1)]/2$ coincides with the probability that there is a particle at site x at time t, for a symmetric simple exclusion process on $\{1, \ldots, L\}$ with initial condition at time zero such that sites $x \leq L/2$ are occupied by a particle, while sites $\{L/2 + 1, \ldots, L\}$ are empty (each particle attempts with rate one to jump to one of its two neighboring sites with equal probability $1/2$ and the jump is rejected if either the site is already occupied or if it lies outside $\{1, \ldots, L\}$). In particular (recall that $u(t, L) = 0$) one has that $[1 - u(t, L-1)]/2$ is the probability that there is a particle at L at time t. By duality (cf. [14, Section II.3]), this equals the probability that a continuous-time simple random walk of rate 1 on $\{1, \ldots, L\}$, started from site L, is in $\{1, \ldots, L/2\}$ at time t. The bound (7.15) then follows from standard random walk estimates: if $P_x^{a,b}$ is the law of the continuous-time simple random walk X_t on $\{n \in \mathbb{N} : a \leq n \leq b\}$ started from x, one has

$$P_L^{1,L}\left(X_t \leq \frac{L}{2}\right) \leq P_L^{-\infty,L}\left(X_t \leq \frac{L}{2}\right) \leq P_{3L/4}^{-\infty,L}\left(X_t \leq \frac{L}{2}\right) \leq P_0^{-\infty,+\infty}\left(\exists s < t : |X_s| \geq \frac{L}{4}\right) \tag{7.16}$$

and the latter expression is easily seen (e.g. using the local central limit theorem) to be upper bounded by the r.h.s. of (7.15).

$\square$

8. Proof of Theorem 2

As discussed in Remark 2, we only have to prove that, for the $\beta = +\infty$ dynamics,

$$\mathbf{P}\left(\tau_+ < c_0 L^2\right) \leq \exp(-\gamma L) \tag{8.1}$$

for suitable positive constants c_0, γ. Also, thanks to monotonicity, it is enough to prove this fact for the dynamics $\sigma^-(t)$ in the domain

$$\tilde{\Lambda} := \Lambda_L \cap \{x \in \mathbb{Z}^2 : |x^{(1)}| + |x^{(2)}| \leq L + 1\}, \tag{8.2}$$

with "+" boundary conditions on $\partial\tilde{\Lambda}$. The advantage of looking at the dynamics in $\tilde{\Lambda}$ instead of Λ_L will be apparent in the proof of Theorem 7.

We need a certain number of geometric definitions:

Definition 7. *Given $\sigma \in \Omega_{\tilde{\Lambda}}$, let*

(a)

$$M(\sigma) := \{x : \sigma_x = -1\};$$

(b)

$$\Gamma(\sigma) := \cup_{x \in M(\sigma)} B_x,$$

where B_x is the unit square of side 1 centered at x, with sides parallel to the coordinate axes. The boundary of $\Gamma(\sigma)$ is denoted as $\partial\Gamma(\sigma)$ and its geometric length as $|\partial\Gamma(\sigma)|$;

(c) for $i = 1, 2$

$$u_{max}^{(i)}(\sigma) := \max\{x^{(i)} : x = (x^{(1)}, x^{(2)}) \in \partial\Gamma(\sigma)\}$$

and similarly

$$u_{min}^{(i)}(\sigma) := \min\{x^{(i)} : x = (x^{(1)}, x^{(2)}) \in \partial\Gamma(\sigma)\};$$

(d) $p(\sigma)$ be the configuration in $\Omega_{\tilde{\Lambda}}$ obtained by flipping every "−" spin in $\tilde{\Lambda}$ such that a strict majority of its nearest neighbors are "+", and repeating the same operation as long as such a site exists. We call $p(\cdot)$ the "majority transformation";

(e) $D := \{x \in \mathbb{Z}^2 : |x^{(1)}| + |x^{(2)}| \leq (9/10)L\} \subseteq \tilde{\Lambda}$;

(f) $\mathcal{G}$ be the subset of $\Omega_{\tilde{\Lambda}}$ defined by

$$\mathcal{G} := \left\{\sigma : \partial\Gamma(\sigma) \text{ is a simple curve (i.e. without loops)}, \right. \tag{8.3}$$

$$\left. |\partial\Gamma(\sigma)| = 2\sum_{i=1}^{2}(u_{max}^{(i)}(\sigma) - u_{min}^{(i)}(\sigma)), D \subseteq M(\sigma) \text{ and } p(\sigma) = \sigma\right\}. \tag{8.4}$$

Note that the constraint $|\partial\Gamma(\sigma)| = 2\sum_{i=1}^{2}(u_{max}^{(i)}(\sigma) - u_{min}^{(i)}(\sigma))$ in the definition of $\mathcal{G}$ is simply the requirement that σ minimizes the Hamiltonian $H_{\tilde{\Lambda}}^+(\cdot)$, given the values $\{u_{max}^{(i)}(\sigma), u_{min}^{(i)}(\sigma)\}_{i=1,2}$.

Next we introduce a modified dynamics $\{\tilde{\sigma}^\xi(t)\}_{t \geq 0}$ on $\tilde{\Lambda}$, with initial condition $\tilde{\sigma}^\xi(0) = \xi$, as follows. Let $\tau_D^\xi := \inf\{t > 0 : M(\tilde{\sigma}^\xi(t)) \not\supseteq (D \cup \partial D)\}$, and set $\tilde{\sigma}^\xi(t) := \tilde{\sigma}^\xi(\tau_D^\xi)$ for $t \geq \tau_D^\xi$. Whenever the Poisson clock labeled x rings at a time $t < \tau_D^\xi$, first refresh the current value of the spin at x according to the distribution $\pi(\cdot|\sigma_y = \tilde{\sigma}_y^\xi(t), y \neq x)$, and then apply the majority transformation $p(\cdot)$ of Definition 7(d) to the configuration thus obtained.

We call $\tilde{\mathcal{L}}$ the generator of such dynamics and $\tilde{\mu}_t^\xi$ its law at time t. When the initial condition is $\xi \equiv -$, it is immediate to see that, by the monotonicity of the usual Glauber dynamics, one has

$$\left\{M(\sigma^-(t)) \not\supseteq (D \cup \partial D)\right\} \Longrightarrow \left\{M(\tilde{\sigma}^-(t)) \not\supseteq (D \cup \partial D)\right\} \tag{8.5}$$

for every $t > 0$, so that (8.1) is proven if we show that

$$\tilde{\mathbf{P}}(\tau_D^- \leq c_0 L^2) \leq e^{-\gamma L} \qquad (8.6)$$

for some suitable c_0, where $\tilde{\mathbf{P}}$ is the law of the process $\{\tilde{\sigma}^-(t)\}_{t \geq 0}$.

The advantage of the modified dynamics $\tilde{\sigma}^-(t)$ with respect to the usual Glauber dynamics is that it belongs to the "good set" $\mathcal{G}$ for all times:

Theorem 7. *For every $t > 0$ one has that $\tilde{\sigma}^-(t) \in \mathcal{G}$. Moreover, there exists a deterministic positive constant ψ such that*

$$|M(\tilde{\sigma}^-(0))| - |M(\tilde{\sigma}^-(\tau_D^-))| \geq \psi L^2. \qquad (8.7)$$

Proof of Theorem 7. For this, we need some additional notions:

Definition 8. *Given $\sigma \in \mathcal{G}$ and $x \in \tilde{\Lambda}$, we say that x is a "flippable site" if two neighbors of x, at mutual distance $\sqrt{2}$, are "+" and the other two neighbors are "−". If x is flippable, recall that $\sigma^{(x)}$ is the configuration obtained by flipping σ_x to $-\sigma_x$. Then, for x flippable we say that*

- *x is a "mountain" if $x \in M(\sigma)$ and $\sigma^{(x)} \in \mathcal{G}$;*
- *x is a "vertex" if $x \in M(\sigma)$ and $p(\sigma^{(x)}) \neq \sigma^{(x)}$;*
- *x is a "valley" if $x \in \partial M(\sigma)$.*

Remark 6. Observe that, if x is a "vertex", then exactly one of its neighbors, call it y, is also a vertex. It is immediate to see that $M(p(\sigma^{(x)})) = M(\sigma) \setminus \{x, y\}$, cf. Figure 5.

We prove that $\tilde{\sigma}^-(t) \in \mathcal{G}$ for $t \leq \tau_D^-$ by induction: clearly the statement is true at time zero, and we show that if it is true until some time s then it remains true after the next update.

The first observation is that, if $\chi := \tilde{\sigma}^-(s) \in \mathcal{G}$, then nothing happens in the evolution until the Poisson clock of a *flippable site* x rings (the occurrence of sites with three neighbors of opposite sign, or sites with exactly two neighbors of opposite sign at mutual distance 2 is forbidden by the condition $\chi \in \mathcal{G}$). When such a ring happens, we have three possibilities:

(1) x is a valley. Then, with probability $1/2$ the configuration remains unchanged, and with probability $1/2$ we change χ to $\chi^{(x)}$, cf. Definition 8. We do not need to apply the transformation $p(\cdot)$, since one sees easily that $p(\chi^{(x)}) = \chi^{(x)}$. Also, it is easy to see that $|\Gamma(\chi)| = |\Gamma(\chi^{(x)})|$, and that $\Gamma(\chi^{(x)})$ remains a simple curve, see Figure 6.

(2) x is a vertex. With probability $1/2$ the configuration remains unchanged, and with probability $1/2$ we change χ to $\chi^{(x)}$; then, the application of the transformation $p(\cdot)$ has the effect of flipping also the vertex neighbor of x, cf. Remark 6. Altogether, Γ remains a simple curve; its length decreases by 2, but also does the sum $2 \sum_{i=1}^2 (u_{max}^{(i)}(\chi) - u_{min}^{(i)}(\chi))$, see Figure 6.

(3) x is a mountain. This case is more subtle, since it is not obvious apriori that $\Gamma(\chi^{(x)})$ is a simple curve. Just to fix ideas, assume that the "+" neighbors of x in χ are $x + (0,1), x + (1,0)$. If $\Gamma(\chi^{(x)})$ were not a simple curve, it would mean that $\chi_{x-(1,1)} = +$. Since $\chi \in \mathcal{G}$, one has that the set

$$Y(x) := \tilde{\Lambda} \cap (\{x - (1,1) - (i,j), i, j \geq 0\} \cup \{x + (i,j), i, j \geq 0, i+j > 0\}) \qquad (8.8)$$

belongs to $\tilde{\Lambda} \setminus M(\chi)$ (so in particular it has no intersection with D), otherwise the condition $|\partial \Gamma(\chi)| = 2 \sum_{i=1}^2 (u_{max}^{(i)}(\chi) - u_{min}^{(i)}(\chi))$ would be violated. However, by the definition of the domains $\tilde{\Lambda}$ and D, there exists no site $x \in \tilde{\Lambda}$ such that $Y(x) \cap D = \emptyset$. Indeed, for that to happen one would need that $x = (x^{(1)}, x^{(2)})$ with either $x^{(1)} \geq$
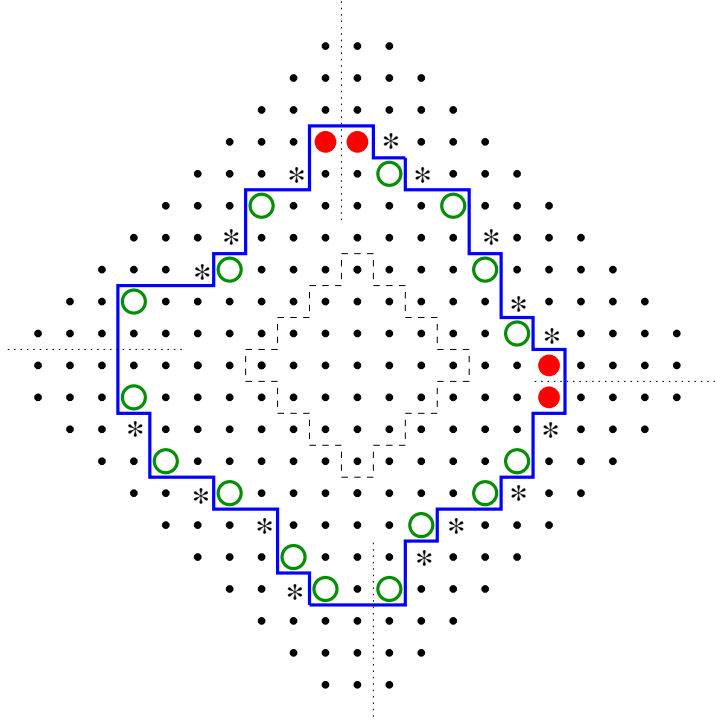
FIGURE 5. The domain $\tilde{\Lambda}$, the sub-domain D (enclosed by the dashed line; proportions are not respected) and the contour $\Gamma(\sigma)$ (thick line) of a configuration $\sigma \in \mathcal{G}$. Big full dots denote "vertex" sites (remark that they occur in nearest-neighboring pairs), empty dots denote "mountains" and $*$ denote "valleys". Vertices can occur only in the right-most or left-most column which intersects $M(\sigma)$, or in the highest or lowest row which intersects $M(\sigma)$, so there are plainly at most 8 vertices. The curve $\Gamma(\sigma)$ can be seen as the union of four monotone curves, which in the figure are delimited by the dotted lines. For each monotone curve, one has $m(\sigma) + w(\sigma) - v(\sigma) = 1$.

$(9/10)L$ and $x^{(2)} \leq -(9/10)L$ or $x^{(1)} \leq -(9/10)L$ and $x^{(2)} \geq (9/10)L$, which is clearly incompatible with $x \in \tilde{\Lambda}$, cf. (8.2). This shows that $\Gamma(\chi^{(x)})$ is a simple curve. Observe that it is for this issue that it was important to change the shape of the domain from Λ to $\tilde{\Lambda}$. Of course, the value $(9/10)$ in the definition of D could be changed to any number larger than $1/2$.

In all cases, the configuration belongs to $\mathcal{G}$ after the move, and the proof of $\tilde{\sigma}^-(t) \in \mathcal{G}$ for all $t \geq 0$ is complete.

To prove (8.7), assume that the last update before τ_D^- consisted in flipping from "$-$" to "$+$" a "mountain" site $x \in \partial D$ (if instead the move consisted in flipping from "$-$" to "$+$" a "vertex" site, the proof which follows would be very similar). Call χ the spin configuration just before the last update and, just to fix ideas, assume that the two "$-$" neighbors of x in χ are $x - (0,1)$ and $x - (1,0)$. Since x is a mountain and $\chi \in \mathcal{G}$, it is immediate to realize that the set

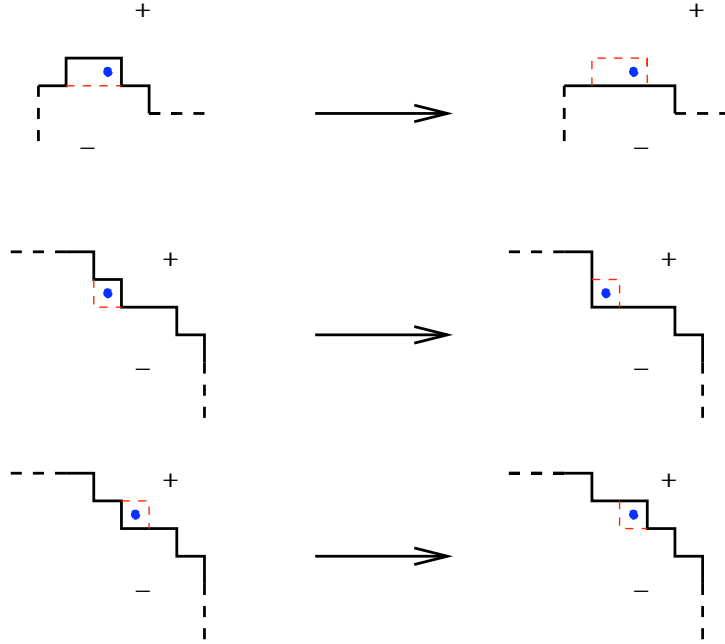$$J(x) := \tilde{\Lambda} \cap \{x + (i,j), i, j \geq 0, i + j > 0\} \tag{8.9}$$

FIGURE 6. In the top drawing, an update consisting in the flip of the "vertex" spin σ_x (x is marked by a dot) and of its neighboring vertex site from the value "−" to "+". In this case, $u_{max}^{(2)}$ decreases by 1 (while $u_{max}^{(1)}$ and $u_{min}^{(i)}, i = 1, 2$ are unchanged) and $\Gamma(\sigma)$ remains a simple curve, but its length decreases by 2. In the middle drawing, the effect of flipping from "−" to "+" a "mountain" σ_x. $\Gamma(\sigma)$ remains a simple curve and neither its length nor the values $\{u_{max}^{(i)}, u_{min}^{(i)}\}_{i=1,2}$ vary. Similarly, the third drawing shows the effect of flipping a "valley" spin. In the three cases, only a portion of $\partial\Gamma(\sigma)$ is drawn.

is a subset of $\tilde{\Lambda} \setminus M(\chi)$, otherwise the condition $|\partial\Gamma(\chi)| = 2\sum_{i=1}^{2}(u_{max}^{(i)}(\chi) - u_{min}^{(i)}(\chi))$ would be violated. Next, from the definition of the set D one sees that $J(x)$ has cardinality at least ψL^2, uniformly in $x \in \partial D$ (explicitly, one can take ψ to be slightly less than $\frac{1}{2}(1 - 9/10)^2$). The estimate (8.7) is then proven.

$\square$

The simplification of considering a dynamics which evolves in the set $\mathcal{G}$ is that for such configurations the numbers of valleys, mountains and vertices satisfy simple relations:

Lemma 7. *Given $\sigma \in \mathcal{G}$, let $v(\sigma)$ (resp. $m(\sigma), w(\sigma)$) be the number of valleys (resp. mountains, vertices) in $\tilde{\Lambda}$. Then, $m(\sigma) + w(\sigma) - v(\sigma) = 4$ and $w(\sigma) \leq 8$.*

The proof of Lemma 7 is best explained through a picture, and therefore we refer to the caption of Figure 5.

Now we can finish the proof of (8.1). Thanks to Theorem 7, we see that (8.6) follows if we have

$$\tilde{\mathbf{P}}(|M(\tilde{\sigma}^-(0))| - |M(\tilde{\sigma}^-(c_0 L^2))| \geq \psi L^2) \leq e^{-\gamma L}. \tag{8.10}$$

By the exponential Tchebyshev inequality one has for $\lambda > 0$

$$\tilde{\mathbf{P}}(|M(\tilde{\sigma}^-(0))| - |M(\tilde{\sigma}^-(c_0L^2))| \geq \psi L^2) \quad \leq \quad e^{-\lambda \psi L^2} \tilde{\mathbf{E}}\left[e^{\lambda(|M(\tilde{\sigma}^-(0))| - |M(\tilde{\sigma}^-(c_0L^2))|)}\right] \quad (8.11)$$

$$= \quad e^{-\lambda \psi L^2}\phi(c_0L^2) \quad (8.12)$$

where $\phi(t) := \tilde{\mathbf{E}}[f(\tilde{\sigma}^-(t))]$ and

$$f(\sigma) := e^{\lambda(|M(\tilde{\sigma}^-(0))| - |M(\sigma)|)}.$$

Note that one has $\phi(0) = 1$ and

$$\frac{d\phi(t)}{dt} = \tilde{\mu}_t^- \left(\tilde{\mathcal{L}}f\right), \quad (8.13)$$

where we recall that $\tilde{\mu}_t^-$ is the law of $\tilde{\sigma}^-(t)$ and $\tilde{\mathcal{L}}$ is its generator. By the very definition of the modified dynamics, if $M(\sigma) \not\supseteq (D \cup \partial D)$ then $\tilde{\mathcal{L}}(\sigma, \sigma')$ vanishes for every σ'. If instead $M(\sigma) \supseteq (D \cup \partial D)$, we know from Theorem 7 that we need only to consider the case $\sigma \in \mathcal{G}$. Therefore, for $M(\sigma) \supseteq (D \cup \partial D)$ one has (recall the discussion after Remark 6)

$$\tilde{\mathcal{L}}f(\sigma) = f(\sigma)\left[v(\sigma)\left(\frac{e^{-\lambda} - 1}{2}\right) + m(\sigma)\left(\frac{e^\lambda - 1}{2}\right) + w(\sigma)\left(\frac{e^{2\lambda} - 1}{2}\right)\right]. \quad (8.14)$$

Now we choose $\lambda = 1/L$. Since clearly there exists a constant c such that $v(\sigma), m(\sigma), w(\sigma) \leq cL$ for every $\sigma \in \mathcal{G}$, we have

$$\tilde{\mathcal{L}}f(\sigma) \leq f(\sigma)\left[\frac{\lambda}{2}(m(\sigma) + 2w(\sigma) - v(\sigma)) + \frac{c}{L}\right] \leq \frac{c'}{L}f(\sigma), \quad (8.15)$$

where we used Lemma 7. Plugging this inequality into (8.13), we find

$$\frac{d\phi(t)}{dt} \leq \frac{c'}{L}\phi(t) \quad (8.16)$$

which implies $\phi(c_0L^2) \leq e^{c' c_0 L}$ and, together with (8.11),

$$\tilde{\mathbf{P}}\left(|M(\tilde{\sigma}^-(0))| - |M(\tilde{\sigma}^-(c_0L^2))| \geq \psi L^2\right) \quad \leq \quad e^{-L(\psi - c' c_0)}. \quad (8.17)$$

Choosing $c_0 < \psi/c'$ we obtain (8.10) and therefore (8.1).

$\square$

9. Proof of Theorem 3

9.1. $\Omega(1/L)$ lower bound on the gap: a perturbative argument.
Here we prove the lower bound gap $\geq c/L$ for $\beta \geq C \log L$ with C large enough. The result is particularly interesting in $d = 2$, in view of the matching upper bound in Theorem 3, but as we mentioned the proof works also in $d = 3$. Actually, we give the proof in the three-dimensional case, which is slightly more complicated.

We introduce the matrix $U := \{U(\sigma, \sigma')\}_{\{\sigma, \sigma' \in \Omega_\Lambda\}}$, unitarily equivalent to the matrix $\mathcal{L}$ (the generator (2.7)), as

$$U(\sigma, \sigma') := \sqrt{\pi(\sigma)}\mathcal{L}(\sigma, \sigma')\frac{1}{\sqrt{\pi(\sigma')}}. \quad (9.1)$$

The spectrum of U coincides with the spectrum of $\mathcal{L}$, so that we have to prove that the smallest non-zero eigenvalue of $-U$ is lower bounded by c/L. Note that U is symmetric thanks to the reversibility condition $\pi(\sigma)\mathcal{L}(\sigma, \sigma') = \pi(\sigma')\mathcal{L}(\sigma', \sigma)$.

Remark 7. The transformation (9.1) is the analogue of *the inverse* of the "ground-state transformation" which maps a Schrödinger operator of the form $H = -\Delta + V$ (which acts on $\mathbb{L}^2(\mathbb{R}^d)$) into the operator

$$-\mathcal{L} = A^{-1}HA = -\Delta - \frac{2}{\psi_0}\nabla\psi_0 \cdot \nabla,$$

where $\psi_0(\cdot)$ is the ground state eigenfunction (we assume for definiteness that $H\psi_0 = 0$, i.e. the ground state energy is zero) and the unitary operator A acts as $(Af)(x) = \psi_0(x)f(x)$ for $f \in L^2(\mathbb{R}^d)$. Note that $\mathcal{L}$ is the generator of a diffusion with drift.

Looking at the definition of U and $\mathcal{L}$, one immediately realizes that

-

$$U(\sigma, \sigma) = \mathcal{L}(\sigma, \sigma) = -\sum_{\sigma' \neq \sigma} \mathcal{L}(\sigma, \sigma'). \tag{9.2}$$

- if $\sigma' = \sigma^{(x)}$ for some $x \in \Lambda$ and $H_\Lambda^+(\sigma) = H_\Lambda^+(\sigma')$, then

$$U(\sigma, \sigma') = \mathcal{L}(\sigma, \sigma') = \frac{1}{2};$$

- if $\sigma' = \sigma^{(x)}$ for some $x \in \Lambda$ and $|H_\Lambda^+(\sigma') - H_\Lambda^+(\sigma)| > 0$ (hence ≥ 4) then

$$\mathcal{L}(\sigma, \sigma') = \frac{1}{1 + \exp(-\beta(H_\Lambda^+(\sigma) - H_\Lambda^+(\sigma')))} \tag{9.3}$$

$$U(\sigma, \sigma') = \frac{\exp(-(\beta/2)(H_\Lambda^+(\sigma) - H_\Lambda^+(\sigma')))}{1 + \exp(-\beta(H_\Lambda^+(\sigma) - H_\Lambda^+(\sigma')))} \leq e^{-2\beta}; \tag{9.4}$$

- if $\sigma \neq \sigma'$ and there exists no x such that $\sigma' = \sigma^{(x)}$, then $\mathcal{L}(\sigma, \sigma') = U(\sigma, \sigma') = 0$.

If we write $U = U_\infty + R$ with $U_\infty = \lim_{\beta \to \infty} U$ we see that all the matrix elements of R are smaller than $1/L^5$ if β satisfies (2.13) with C large enough. Since each row of R has at most $|\Lambda| = O(L^3)$ non-zero elements, one sees easily that the spectral radius of R is $O(L^{-2})$.

We are therefore left with the task of proving that the smallest non-zero eigenvalue of $-U_\infty$ is larger than c/L. Thanks to formulas (9.2)-(9.4), we have that

$$U_\infty(\sigma, \sigma) = -\frac{1}{2}|\{x \in \Lambda : H_\Lambda^+(\sigma^{(x)}) = H_\Lambda^+(\sigma)\}| - |\{x \in \Lambda : H_\Lambda^+(\sigma^{(x)}) < H_\Lambda^+(\sigma)\}| \tag{9.5}$$

and, for $\sigma \neq \sigma'$,

$$U_\infty(\sigma, \sigma') = \begin{cases} \frac{1}{2} & \text{if} \quad \sigma' = \sigma^{(x)} \quad \text{and} \quad H_\Lambda^+(\sigma') = H_\Lambda^+(\sigma); \\ 0 & \text{otherwise.} \end{cases} \tag{9.6}$$

If we decompose Ω_Λ into a finite number M of equivalence classes $\mathcal{C}_i, 1 \leq i \leq M$, where two configurations belong to the same class iff they can be connected via a finite number of single spin-flips which do not change the energy $H_\Lambda^+(\cdot)$, then $U_\infty(\sigma, \sigma') = 0$ whenever $\sigma \in \mathcal{C}_i, \sigma' \in \mathcal{C}_j$ with $i \neq j$. In other words, one can write U_∞ in a block matrix form $U_\infty = \oplus_{i=1}^M U_\infty^{(i)}$ where $U_\infty^{(i)} = \{U_\infty(\sigma, \sigma')\}_{\{\sigma, \sigma' \in \mathcal{C}_i\}}$ and in particular, if $\mathcal{S}(U_\infty)$ denotes the spectrum of U_∞, one has $\mathcal{S}(U_\infty) = \cup_{i=1}^M \mathcal{S}(U_\infty^{(i)})$. It is clear that, if $\mathcal{C}_1 = \{+\}$ denotes the equivalence class whose unique element is the all "+" configuration, one has $\mathcal{S}(U_\infty^{(1)}) = \{0\}$ (cf. (9.5) and observe that any spin flip increases the energy if $\sigma \equiv +$). We need to prove that, for every $i > 1$, $\mathcal{S}(U_\infty^{(i)}) \subset (-\infty, -c/L)$ for some positive c.

Let us fix $i > 1$, let $\lambda = \lambda_i$ be the smallest eigenvalue of $-U_\infty^{(i)}$ and $g : \mathcal{C}_i \mapsto \mathbb{R}$ an associated eigenfunction. We want to show that

$$\lambda \geq \frac{c}{L} \tag{9.7}$$

with c independent of i. The key point is the following:

Lemma 8. *For every* $\eta, \eta' \in \mathcal{C}_i$ *and* $t > 0$*, one has*

$$e^{tU_\infty^{(i)}}(\eta, \eta') = \mathbf{P}^\eta\left(\sigma^\eta(t) = \eta'; \tau > t\right) \tag{9.8}$$

where $\mathbf{P}^\eta$ *denotes the law of the Ising evolution* $\{\sigma^\eta(t)\}_{t \geq 0}$ *in* Λ *at* $\beta = +\infty$ *with* $+$ *boundary condition, started from the configuration* η*, and* τ *is the random time*

$$\tau = \inf\{t > 0 : H_\Lambda^+(\sigma(t)) < H_\Lambda^+(\eta)\}. \tag{9.9}$$

Proof. Just check that the time derivatives of left- and right-hand side are the same. $\square$

Remark 8. Note that $U_\infty^{(i)}$ is just the generator of the "killed" Markov process which coincides with the Ising evolution except that it is killed when it exits the set $\mathcal{C}_i$, cf. (9.8).

Since the matrix $\{\exp(tU_\infty^{(i)})(\eta, \eta')\}_{\eta, \eta' \in \mathcal{C}_i}$ has strictly positive entries for $t > 0$ (this follows from (9.8) and the definition of the equivalence classes $\mathcal{C}_i$), the Perron-Frobenius theorem implies that the eigenvalue λ is non-degenerate and the eigenfunction g can be chosen strictly positive on $\mathcal{C}_i$. Normalizing g so that $\sum_{\sigma \in \mathcal{C}_i} g(\sigma)^2 = 1$, we have

$$e^{-\lambda t} = \sum_{\sigma, \sigma' \in \mathcal{C}_i} g(\sigma) e^{tU_\infty^{(i)}}(\sigma, \sigma') g(\sigma'). \tag{9.10}$$

The desired estimate (9.7) then follows from Lemma 8 by letting $t \to \infty$, once we prove the following result:

Theorem 8. *There exists* $a(L) < \infty$ *and* $c > 0$ *independent of* L *such that, for every* $i > 1$, $\eta, \eta' \in \mathcal{C}_i$ *and* $t > 0$ *one has*

$$\mathbf{P}^\eta\left(\sigma^\eta(t) = \eta'; \tau > t\right) \leq a(L) \exp\left(-c\frac{t}{L}\right). \tag{9.11}$$

Proof. Let V be the smallest parallelepiped which contains the set $\{x \in \Lambda : \eta_x = -\}$, and let V^+ be the rectangular layer of points of V with maximal vertical coordinate. Plainly, $\tau \leq \tau_V$ where

$$\tau_V = \inf\{t > 0 : \sigma_x^\eta(t) = + \text{ for every } x \in V^+\} : \tag{9.12}$$

this is because when the last "$-$" spin in V^+ flips to "$+$", the energy decreases by at least 8. Then,

$$\mathbf{P}^\eta\left(\sigma^\eta(t) = \eta'; \tau > t\right) \leq \mathbf{P}^\eta\left(\tau_V > t\right) \leq \mathbf{P}^{\eta_V}\left(\tau_V > t\right) \tag{9.13}$$

where η_V is the configuration where spins take value "$-$" in V and "$+$" in $\Lambda \setminus V$, and we used monotonicity of the dynamics (the event $\tau_V > t$ is decreasing, and $\eta_V \leq \eta$). Again by monotonicity, we can assume that V is the entire domain Λ, and that spins in $V \setminus V^+$ are frozen at the value "$-$" during the entire evolution: both these operations make τ_V stochastically larger. But, in this case, V^+ is just a $(2L + 1) \times (2L + 1)$ square, and the evolution in V^+ coincides with the evolution of the two-dimensional Ising model with $+$ boundary conditions

and $\beta = +\infty$ (the "+" boundary conditions on the upper face of V^+ and the "−" boundary conditions on the lower face compensate exactly). We have then

$$\mathbf{P}^{\eta_V}\left(\tau_V > t\right) \leq \mathbf{P}^{(d=2)}\left(\tau_+ > t\right) \tag{9.14}$$

where $\mathbf{P}^{(d=2)}$ is the law of the evolution of the two-dimensional Ising model in the $(2L+1) \times (2L+1)$ square with "+" boundary conditions started from the "−" configuration, and τ_+ was defined in (2.17) as the first time when all spins in the square are "+". Finally, one has

$$\mathbf{P}^{(d=2)}\left(\tau_+ > t\right) \leq \left[\mathbf{P}^{(d=2)}\left(\tau_+ \geq \frac{L^2}{c}\right)\right]^{\lfloor tc/L^2 \rfloor} \leq a(L)e^{-c\gamma\frac{t}{L}} \tag{9.15}$$

where in the first inequality we used monotonicity (if τ_+ has not been reached at time $i\,L^2/c, i = 1, \ldots, \lfloor tc/L^2 \rfloor - 1$, we restart the dynamics from the all "−" configuration) and in the second one the result (3.4) of Theorem 2. This ends the proof of Theorem 8. $\qquad\square$

Remark 9. The asymptotic behavior $\exp(-c\,t/L)$ can be understood by comparison with the symmetric simple exclusion process (SSEP), as already noticed in [8]. In fact, by monotonicity, the tail of the law of τ_+ is bounded from above by the tail of the hitting time τ_0 defined as follows. Consider the SSEP on $[-L, L]$ starting with all the negative sites occupied and all the non-negative ones empty and define τ_0 to be the first time at which the site "L" is occupied. In turn, using beautiful results by Liggett and by Arratia [1], the tail of τ_0 is controlled from above by the tail of the same hitting time but for the process in which the L particles evolve as independent random walks without the exclusion constraint. Finally the latter has a tail $\exp(-t/L)$ simply because the tail for a single random walk is $\exp(-t/L^2)$ and there are L of them.

9.2. **Upper bound on the gap.** We will prove here the upper bound gap $\leq 1/(cL)$ for the dynamics in the square box $\Lambda = \{-L, \ldots, L\}^2$. We use the variational characterization (2.6) of the spectral gap and we note that, if $\mathcal{Y}$ is some subset of the configuration space Ω_Λ, one has

$$\text{gap} \leq \frac{1}{\pi(\mathcal{Y}^c)} \inf_{f:f|_{\mathcal{Y}^c}=0} \frac{\pi(f(-\mathcal{L})f)}{\pi(f^2)}, \tag{9.16}$$

where the infimum is taken over functions which vanish on $\mathcal{Y}^c$: just observe that

$$\text{Var}_\pi(f) = \frac{1}{2}\int (f(\sigma) - f(\tau))^2 \pi(d\sigma)\pi(d\tau) \tag{9.17}$$

$$\geq \frac{1}{2}\int (f(\sigma) - f(\tau))^2 \pi(d\sigma)\pi(d\tau)(\mathbf{1}_{\tau\in\mathcal{Y}^c} + \mathbf{1}_{\sigma\in\mathcal{Y}^c}) = \pi(\mathcal{Y}^c)\pi(f^2). \tag{9.18}$$

The bound gap $\leq 1/(cL)$ is therefore proven if one exhibits a set $\mathcal{Y}$ such that $\pi(\mathcal{Y}^c) > 1/2$ and a function f which vanishes on $\mathcal{Y}^c$ and such that

$$\frac{\pi(f(-\mathcal{L})f)}{\pi(f^2)} \leq \frac{c}{L}. \tag{9.19}$$

For $\sigma \in \Omega_\Lambda$, we let $M(\sigma)$ and $\Gamma(\sigma)$ be as in Definition 7 and we let $Q \subset \mathbb{R}^2$ be the square $[-L - 1/2, L + 1/2]^2$. We consider the eight points of ∂Q which are at distance $L/2$ from a corner of Q and we call them $p_i, i = 1, \ldots, 8$, with the convention that we order them clockwise, starting from the right-most one on the north side of Q. Let us assume for definiteness that L is even.

Definition 9. *We let $\mathcal{Y}$ be the subset of configurations in Ω_Λ such that $\Gamma(\sigma)$ is a simple curve of length $4(2L+1)$ and such that $p_i \in \partial\Gamma(\sigma), i = 1, \ldots, 8$.*
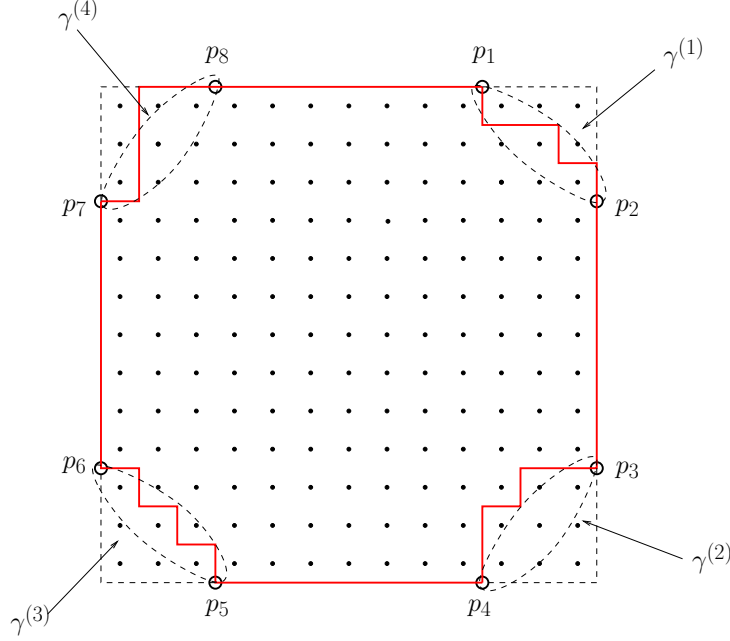
FIGURE 7. The domain Λ with $L = 6$ and a configuration $\sigma \in \mathcal{Y}$: only the boundary $\partial\Gamma(\sigma)$ is drawn (thick line). The four portions $\gamma^{(i)}, i = 1, \ldots, 4$ of $\partial\Gamma(\sigma)$ cannot intersect and, if suitably rotated, are just lattice paths. According to formula (9.21), one has in this example $\nabla^{(1)}_{-3} = \nabla^{(1)}_0 = \nabla^{(1)}_2 = -1$ and $\nabla^{(1)}_{-2} = \nabla^{(1)}_{-1} = \nabla^{(1)}_1 = +1$.

Note that $4(2L + 1)$ is just the length of ∂Q. Also, if $\sigma \in \mathcal{Y}$ then the following properties are immediately verified, cf. Figure 7 :

(1) the portion of $\partial\Gamma(\sigma)$ which connects p_{2i} to $p_{2i+1}, i = 1, 2, 3$ or p_8 to p_1 is a straight segment of length $L + 1$ included in ∂Q;

(2) the portion of $\partial\Gamma(\sigma)$ which connects p_{2i-1} to $p_{2i}, i = 1, \ldots, 4$ (call it $\gamma^{(i)}$) is a "lattice path" of length L (in the sense of Section 6). More precisely, if $\gamma^{(i)}$ is rotated counter-clockwise by an angle $\pi/4 + \pi/2(i - 1)$, then expanded by a factor $\sqrt{2}$ and finally is suitably translated, then it becomes the graph of a function $x \in [-L/2, L/2] \mapsto \phi^{(i)}_x :=$ $\phi^{(i)}_x(\sigma) \in [-L/2, L/2]$ such that

$$\phi^{(i)}_{\pm L/2} = 0 \tag{9.20}$$

$$\nabla^{(i)}_x := \phi^{(i)}_{x+1} - \phi^{(i)}_x \in \{-1, +1\} \text{ for } x \in \mathbb{Z} \cap [-L/2, L/2 - 1] \tag{9.21}$$

$$\phi^{(i)}_\cdot \text{ is linear in each interval } (x, x + 1), x \in \mathbb{Z} \cap [-L/2, L/2 - 1]. \tag{9.22}$$

The test function f in (9.19) is then

$$f(\sigma) := \mathbf{1}_{\{\sigma \in \mathcal{Y}\}} \prod_{i=1}^4 g(\phi^{(i)}(\sigma)), \tag{9.23}$$

where

$$g(\phi^{(i)}) := \prod_{x=-L/2}^{-1} \cos\left(\frac{\pi x}{L}\right)^{(1-\nabla_x^{(i)})/2} \prod_{x=0}^{(L/2)-1} \cos\left(\frac{\pi(x+1)}{L}\right)^{(1+\nabla_x^{(i)})/2} \tag{9.24}$$

with the convention that $0^0 = 1$. Note that f vanishes on $\mathcal{Y}^c$ and that

$$\pi(\mathcal{Y}^c) \geq \pi(\sigma_x = + \text{ for every } x \in \Lambda) = 1 + o(1) > 1/2$$

as required above.

Let ρ be the uniform measure over the set of lattice paths ϕ satisfying (9.20)-(9.22), and let $\hat{\mathcal{L}}$ be the generator

$$\hat{\mathcal{L}}g(\phi) = \sum_{x=-L/2+1}^{L/2-1} \left[\rho(g|\phi_{y,y\in\mathbb{Z},y\neq x}) - g(\phi)\right]. \tag{9.25}$$

Remark 10. Conditionally on $\sigma \in \mathcal{Y}$, the four lattice paths $\phi^{(i)}$ are independent and, since all their allowed configurations have the same length, $\pi(\cdot|\mathcal{Y})$ is just the product uniform measure $\rho^{\otimes 4}$, where each copy of ρ acts on a path $\phi^{(i)}, i = 1, \ldots, 4$. Indeed, the Ising Hamiltonian (2.2) equals

$$H_\Lambda^+(\sigma) = const + 2|\Gamma(\sigma)| \tag{9.26}$$

which does not depend on σ if $\sigma \in \mathcal{Y}$, cf. Definition 9.

Lemma 9. *If f is defined as in (9.23), one has*

$$\frac{\pi(f(-\mathcal{L})f)}{\pi(f^2)} \leq 4\frac{\rho(g(-\hat{\mathcal{L}})g)}{\rho(g^2)} + \frac{c}{L^2}. \tag{9.27}$$

Proof. Let us consider first the denominator of the left-hand side:

$$\pi(f^2) = \pi(\mathcal{Y})\pi\left(\prod_{i=1}^4 g^2(\phi_i)\,\middle|\,\mathcal{Y}\right) = \pi(\mathcal{Y})\left[\rho(g^2)\right]^4 \tag{9.28}$$

where we used the fact that $\pi(\cdot|\mathcal{Y}) = \rho^{\otimes 4}$, see Remark 10.

The numerator requires more work. From the definition (2.7) of the generator one has

$$\pi(f(-\mathcal{L})f) = \sum_{\sigma\in\mathcal{Y}} \pi(\sigma) \prod_{i=1}^4 g(\phi^{(i)}(\sigma)) \sum_{z\in\Lambda} (f(\sigma) - \pi_{z,\sigma}(f)) \tag{9.29}$$

where $\pi_{z,\sigma}(\cdot)$ is defined in (2.4). The first observation is that, if $\beta \geq C\log L$ with C large enough, then

$$|\pi_{z,\sigma}(f) - f(\sigma)| \leq \frac{1}{L^4}f(\sigma) \tag{9.30}$$

whenever $\sigma \in \mathcal{Y}$ and z is such that the configuration $\sigma^{(z)}$ satisfies $|\Gamma(\sigma^{(z)})| \geq |\Gamma(\sigma)| + 1$ (in particular, $\sigma^{(z)}$ does not belong to $\mathcal{Y}$). The reason is that, since $f(\sigma^{(z)}) = 0$, one has

$$\pi_{z,\sigma}(f) - f(\sigma) = -\pi_{z,\sigma}(-\sigma_z)f(\sigma), \tag{9.31}$$

where $\pi_{z,\sigma}(-\sigma_z)$ is the probability, under $\pi_{z,\sigma}$, that the spin at z equals $-\sigma_z$, and this probability is smaller than $\exp(-2\beta) \leq L^{-4}$ by the assumption that $|\Gamma(\sigma^{(z)})| \geq |\Gamma(\sigma)| + 1$ (cf. (9.26)).

Call $B(\sigma)$ the set of $z \in \Lambda$ such that $|\Gamma(\sigma^{(z)})| = |\Gamma(\sigma)|$. One has then

$$\pi(f(-\mathcal{L})f) \leq \sum_{\sigma \in \mathcal{Y}} \pi(\sigma) \prod_{i=1}^{4} g(\phi^{(i)}(\sigma)) \sum_{z \in B(\sigma)} (f(\sigma) - \pi_{z,\sigma}(f)) + \frac{c}{L^2} \pi(f^2) \qquad (9.32)$$

since, if $\sigma \in \mathcal{Y}$, there are no spin flips which decrease $|\Gamma(\sigma)|$ (i.e. the energy). Finally, recalling Remark 10 it is an easy task to check the identity

$$\sum_{\sigma \in \mathcal{Y}} \pi(\sigma) \prod_{i=1}^{4} g(\phi^{(i)}(\sigma)) \sum_{z \in B(\sigma)} (f(\sigma) - \pi_{z,\sigma}(f)) = 4 \pi(\mathcal{Y}) \left[\rho(g^2)\right]^3 \rho(g(-\hat{\mathcal{L}})g), \qquad (9.33)$$

the reason being that the flips which leave $|\Gamma(\sigma)|$ unchanged just correspond to the local updates in the generator $\hat{\mathcal{L}}$ of (9.25). The factor 4 is due to the symmetry among the lattice paths $\phi^{(i)}, i = 1, \ldots, 4$.   □

The proof of (9.19) is concluded once we prove

Theorem 9. *There exists $c > 0$ such that*

$$\frac{\rho(g(-\hat{\mathcal{L}})g)}{\rho(g^2)} \leq \frac{c}{L}. \qquad (9.34)$$

Proof. Using the form (9.25) of the generator and the fact that ρ is the uniform measure over lattice paths, one sees that

$$\rho(g(-\hat{\mathcal{L}})g) = \frac{1}{2} \sum_{x=-L/2+1}^{-1} \frac{[\cos(\pi x/L) - \cos(\pi(x-1)/L)]^2}{\cos^2(\pi x/L)} \rho\left(g^2 \mathbf{1}_{\{\nabla_{x-1}=+1, \nabla_x=-1\}}\right) \quad (9.35)$$

$$+ \frac{1}{2} \sum_{x=-L/2+1}^{-1} \frac{[\cos(\pi(x-1)/L) - \cos(\pi x/L)]^2}{\cos^2(\pi(x-1)/L)} \rho\left(g^2 \mathbf{1}_{\{\nabla_{x-1}=-1, \nabla_x=+1\}}\right)$$

where we recall that $\nabla_x = \phi_{x+1} - \phi_x$ and we used the symmetry $x \leftrightarrow -x$ to restrict the sum to $x < 0$. Clearly, one has

$$\sum_x [\cos(\pi x/L) - \cos(\pi(x-1)/L)]^2 \leq c/L. \qquad (9.36)$$

Moreover, by an "equivalence of ensembles" argument, one has

$$\frac{\rho\left(g^2 \mathbf{1}_{\{\nabla_{x-1}=+1, \nabla_x=-1\}}\right)}{\rho(g^2)} \leq c' \cos^2(\pi x/L) \qquad (9.37)$$

and a similar estimate for $\rho\left(g^2 \mathbf{1}_{\{\nabla_{x-1}=-1, \nabla_x=+1\}}\right)/\rho(g^2)$ where c' is finite uniformly in $x < -1$. The statement of the Theorem then immediately follows from Eqs. (9.35), (9.36) and (9.37).

The proof of (9.37) is a direct adaptation of the proof of [4, Prop. 3.8]. Loosely speaking, the idea is to observe that the law of $\{\nabla_x\}_{x=-L/2,\ldots,L/2-1}$ under $\rho(\cdot g^2)/\rho(g^2)$ is a product law of independent (but not identically distributed) random variables conditioned to the event $\sum_{x=-L/2}^{L/2-1} \nabla_x = 0$. Next, one shows that, when computing the average of a local function like $\mathbf{1}_{\{\nabla_{x-1}=+1, \nabla_x=-1\}}$, the conditioning can be eliminated, at the price of losing a multiplicative constant. Under the unconditioned product law, the expectation of $\mathbf{1}_{\{\nabla_{x-1}=+1, \nabla_x=-1\}}$ is easily computed and turns out to be proportional to the r.h.s. of (9.37).

□

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