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Abstract. We investigate the properties of the Ma–Trudinger–Wang nonlocal curvature tensor in the case of surfaces. In particular, we prove that a strict form of the Ma–Trudinger–Wang condition is stable under C^4 perturbation if the nonfocal domains are uniformly convex; and we present new examples of positively curved surfaces which do not satisfy the Ma–Trudinger–Wang condition. As a corollary of our results, optimal transport maps on a "sufficiently flat" ellipsoid are in general nonsmooth.

Contents

Introduction

The Ma–Trudinger–Wang (MTW) tensor is a nonlocal generalization of sectional curvature, involving fourth-order derivatives of the squared distance function [4, 7, 9, 10, 11, 18, 22, 25, 26, 27, 28, 29]. Various positivity or nonnegativity conditions on this tensor have been introduced and identified as a crucial tool in the regularity theory for the optimal transport in curved geometry [8, 10, 12, 19, 23, 24]; see [6] or [28, Chapter 12] for a presentation and survey. Besides, this tensor has led to results of a completely new kind concerning the geometry of the cut locus [10, 11, 24].

It is therefore natural to investigate the stability of the Ma–Trudinger–Wang conditions. But while the condition of, say, strictly positive sectional curvature is obviously stable under C^2 perturbation of the metric, it is not obvious at all that the condition of positive Ma–Trudinger–Wang curvature tensor is stable under C^4 perturbation, because this tensor is nonlocal. Partial results have already been obtained: roughly speaking, stability of the nonnegative curvature condition under Gromov–Hausdorff limits [29]; stability of the positive curvature condition under C^4 perturbation, away from the focal locus [4, 29]; stability of the positive curvature condition under C^4 perturbation of the round spheres [10, 12].

In the present paper we shall continue these investigations, sticking to the case of surfaces. Without making the problem trivial, this assumption does allow for more explicit calculations. The main results of this paper are:

(1) simplified analytic expressions for the MTW curvature tensor on surfaces (formulas (1.8) and (3.3)), and the discovery of a strict connection between the MTW tensor and the curvature of the tangent focal locus near the tangent cut locus (Proposition (3.1) and formula (3.4));

(2) the stability of the strict MTW condition under an assumption of uniform convexity (near the tangent cut locus) of nonfocal domains (Theorem 4.1);

(3) new counterexamples of positively curved surfaces which do not satisfy the MTW condition (Section 5).

We remark that a formula for the MTW curvature tensor on surfaces analogous to (1.8) has been found independently in [5].

Our stability results should be compared to those in [11]. In the latter paper we proved the stability around the round sphere $M = \mathbb{S}^n$; in the present paper, we only consider surfaces, but the assumption on M (convexity of nonfocal domains near the tangent cut locus) is much less restrictive. Moreover, the link we find between the MTW condition and the curvature of the tangent cut locus puts a new light on what the MTW condition geometrically means, and it explains why stability holds on the two-sphere (a result first proven in [10], see also [5]). It would be interesting to find a similar connection between the MTW condition and some convexity properties of the focal cut locus in higher dimension.

Let us further recall that on perturbations of $\mathbb{S}^n$, the stability of the MTW condition has strong geometric consequences, namely it implies the uniform convexity of all injectivity domains [10, 11].

As far as counterexamples are concerned, we shall see in particular that some ellipsoids do not satisfy the MTW condition. A striking consequence is that the smoothness of optimal transport on Riemannian manifolds may fail *even on ellipsoids*.

Finally, let us observe that to show the stability of the strict MTW condition near the tangent cut locus, we will need to prove a result on the focalization time at the tangent focal cut locus, which we believe being of independent interest: if $v \in T_xM$ is a focal velocity belonging to the tangent cut locus, then the tangent focal locus and the segment joining 0 to v are orthogonal (Proposition A.6). As a corollary of this fact, we obtain that the focal cut locus of any point $x \in M$ has zero Hausdorff dimension (Corollary A.8).

Notation:

Throughout all this paper (M, g) is a given C^∞ compact Riemannian manifold of dimension 2, equipped with its geodesic distance d, its exponential map $\exp : (x, v) \longmapsto \exp_x v$, and its Riemann curvature tensor Riem. We write $g(x) = g_x$, $g_x(v, w) = \langle v, w\rangle_x$. We further define

- $t_C(x, v)$: the *cut time* of (x, v):

$$t_C(x, v) = \max\Big\{t \geq 0;\ (\exp_x(sv))_{0 \leq s \leq t} \text{ is a minimizing geodesic}\Big\}.$$

- $t_F(x, v)$: the *focalization time* of (x, v):

$$t_F(x, v) = \inf\Big\{t \geq 0;\ \det(d_{tv} \exp_x) = 0\Big\}.$$

- $\mathrm{TCL}(x)$: the *tangent cut locus* of x:

$$\mathrm{TCL}(x) = \big\{t_C(x, v)v;\ v \in T_xM \setminus \{0\}\big\}.$$

- $\mathrm{cut}(x)$: the *cut locus* of x:

$$\mathrm{cut}(x) = \exp_x(\mathrm{TCL}(x)).$$

- $\mathrm{TFL}(x)$: the *tangent focal locus* of x:

$$\mathrm{TFL}(x) = \Big\{t_F(x, v)v;\ v \in T_xM \setminus \{0\}\Big\}.$$

- $\mathrm{TFCL}(x)$: the *tangent focal cut locus* of x:

$$\mathrm{TFCL}(x) = \mathrm{TFL}(x) \cap \mathrm{TCL}(x).$$

- $\mathrm{fcut}(x)$: the *focal cut locus* of x:

$$\mathrm{fcut}(x) = \exp_x\big(\mathrm{TFCL}(x)\big).$$

• $\mathrm{I}(x)$: the *injectivity domain* of the exponential map at x; so

$$\mathrm{I}(x) = \Big\{tv;\ 0 \le t < t_C(x,v),\ v \in T_xM\Big\}.$$

• $\mathrm{NF}(x)$: the *nonfocal domain* of the exponential map at x:

$$\mathrm{NF}(x) = \Big\{tv;\ 0 \le t < t_F(x,v),\ v \in T_xM\Big\}.$$

• $\exp^{-1}$: the *inverse of the exponential map*; by convention $\exp_x^{-1}(y)$ is the set of *minimizing* velocities v such that $\exp_x v = y$. In particular $\mathrm{TCL}(x) = \exp_x^{-1}(\mathrm{cut}(x))$, and $\mathrm{I}(x) = \exp_x^{-1}(M \setminus \mathrm{cut}(x))$.

Recall that $t_F \ge t_C$, or equivalently $\mathrm{I}(M) \subset \mathrm{NF}(M)$ [15, Corollary 3.77]. (The injectivity domain is included in the nonfocal domain.)

1. Two-dimensional Ma–Trudinger–Wang curvature

In this section we particularize to dimension two the general recipe for the computation of the Ma–Trudinger–Wang curvature, as given in [11, Section 2 and Paragraph 5.1]. W

1.1. **Jacobi fields and Hessian operator.** Let us fix a geodesic $(\overline{\gamma}(t))_{0\le t\le T}$ with $\overline{\gamma}(0) = x$, $\overline{\gamma}(1) = \overline{y}$, $\dot{\overline{\gamma}}(0) = \sigma$, $|\sigma| = 1$, and $T = t_F(\sigma) = t_F(x,\sigma)$. We choose a unit vector $\sigma^\perp$ orthogonal to σ, and identify tangent vectors at x to their coordinates in the g-orthonormal basis $(\sigma, \sigma^\perp)$. Thus, modulo identification, $T_xM = \mathbb{R}^2$, $\sigma = (1,0)$, $g_x = \mathrm{Id}_{\mathbb{R}^2}$.

Next, we let $(\gamma_\alpha(t))_{t\ge 0}$ be the geodesic starting at x with initial velocity $\sigma_\alpha = (\cos\alpha, \sin\alpha)$. We further define $\sigma_\alpha^\perp = (-\sin\alpha, \cos\alpha)$.

For any $\alpha \in [0, 2\pi]$ and $\tau \ge 0$ we let $k(\alpha,\tau)$ be the Gauss curvature of M at $\gamma_\alpha(\tau)$. This function determines two "fundamental solutions" f_0, f_1 given by

$$\begin{cases} \ddot{f}_i(\alpha,\tau) + k(\alpha,\tau)\, f_i(\alpha,\tau) = 0 & i = 0,1, \\ f_0(\alpha,0) = 0, \qquad \dot{f}_0(\alpha,0) = 1, \\ f_1(\alpha,0) = 1, \qquad \dot{f}_1(\alpha,0) = 0. \end{cases} \tag{1.1}$$

Here as in the sequel, dots stand for τ-derivatives, while we shall use primes for α-derivatives. We further let

$$\mathcal{F}(\alpha,\tau) = \frac{f_1(\alpha,\tau)}{f_0(\alpha,\tau)}.$$

Our goal is to express the MTW curvature in terms of f_0, $\mathcal{F}$, and their derivatives with respect to α. (In the case of the unit sphere, $f_1(\tau) = \cos\tau$, $f_0(\tau) = \sin\tau$, $\mathcal{F}(\tau) = \cot\tau$, and there is no α-dependence.) For this we only need to work with α close to 0.

For any α, we define an orthonormal basis by setting $e_1(\alpha, 0) = \sigma_\alpha$, $e_2(\alpha, 0) = \sigma_\alpha^\perp$, and from this we deduce $e_1(\alpha, \tau)$, $e_2(\alpha, \tau)$ by parallel transport along γ. Then we define fields J_0 and J_1 by their matrix in this local basis:

$$J_0(\alpha, \tau) = \begin{bmatrix} \tau & 0 \\ 0 & f_0(\alpha, \tau) \end{bmatrix} \qquad J_1(\alpha, \tau) = \begin{bmatrix} 1 & 0 \\ 0 & f_1(\alpha, \tau) \end{bmatrix}.$$

(Each J_i should be thought of as an array of two Jacobi fields.) Let $w = \tau\sigma_\alpha$ with $\tau < t_F(\sigma_\alpha)$, and let $S_{(x,w)}$ be the symmetric operator whose matrix, in the basis $(\sigma_\alpha, \sigma_\alpha^\perp)$, is

$$S(\alpha, \tau) = \tau J_0(\alpha, \tau)^{-1} J_1(\alpha, \tau) = \begin{bmatrix} 1 & 0 \\ 0 & \tau\mathcal{F}(\alpha, \tau) \end{bmatrix}.$$

Then $S_{(x,w)}$ is the *extended Hessian operator*, as defined in [11, Equation (2.6)]. (If $w \in \mathrm{I}(x)$ then $S_{(x,w)}$ coincides with $\nabla_x^2 d(\,\cdot\,, \exp_x w)^2/2$, see [28, Chapter 14, Third Appendix].)

1.2. **MTW tensor.** The Ma–Trudinger–Wang tensor is obtained basically by differentiating the Hessian operator twice.

Let

$$Q(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix},$$

so that for $v = \tau\,\sigma_\alpha$, the matrix of $S_{(x,v)}$ in the standard basis of $\mathbb{R}^2$ is $Q(-\alpha)S(\alpha,\tau)Q(\alpha)$. Equivalently,

$$\Big\langle S_{(x,\tau\,\sigma_\alpha)}\xi, \xi \Big\rangle = \Big\langle S(\alpha,\tau)\, Q(\alpha)\xi, Q(\alpha)\xi \Big\rangle. \tag{1.2}$$

Let now $v = (t, 0)$, $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$ (intrinsically, this means $\eta = \eta_1\,\sigma + \eta_2\,\sigma^\perp$), and $s \in \mathbb{R}$ small enough. Then $v + s\eta = \tau\sigma_\alpha$, where

$$\tau = |v + s\eta| = \sqrt{(t + s\eta_1)^2 + (s\eta_2)^2}, \qquad \alpha = \tan^{-1}\left(\frac{s\eta_2}{t + s\eta_1}\right). \tag{1.3}$$

We differentiate (1.2) twice with respect to s:

$$\frac{d}{ds}\Big\langle S_{(x,\tau\sigma_\alpha)}\xi,\xi\Big\rangle = \Big[\Big\langle\left(\frac{\partial S}{\partial\alpha}\right)Q\xi,Q\xi\Big\rangle + 2\Big\langle SQ\xi,\left(\frac{\partial Q}{\partial\alpha}\right)\xi\Big\rangle\Big]\left(\frac{\partial\alpha}{\partial s}\right) + \Big\langle\left(\frac{\partial S}{\partial\tau}\right)Q\xi,Q\xi\Big\rangle\left(\frac{\partial\tau}{\partial s}\right).$$

$$\begin{aligned}(1.4)\qquad \frac{d^2}{ds^2}\Big\langle S_{(x,\tau\sigma_\alpha)}\xi,\xi\Big\rangle = &\Big[\Big\langle\left(\frac{\partial^2 S}{\partial\alpha^2}\right)Q\xi,Q\xi\Big\rangle + 4\Big\langle\left(\frac{\partial S}{\partial\alpha}\right)\left(\frac{\partial Q}{\partial\alpha}\right)\xi,Q\xi\Big\rangle \\ &+2\Big\langle S\left(\frac{\partial Q}{\partial\alpha}\right)\xi,\left(\frac{\partial Q}{\partial\alpha}\right)\xi\Big\rangle + 2\Big\langle SQ\xi,\left(\frac{\partial^2 Q}{\partial\alpha^2}\right)\xi\Big\rangle\Big]\left(\frac{\partial\alpha}{\partial s}\right)^2 \\ &+\Big[2\Big\langle\left(\frac{\partial^2 S}{\partial\alpha\,\partial\tau}\right)Q\xi,Q\xi\Big\rangle + 4\Big\langle\left(\frac{\partial S}{\partial\tau}\right)Q\xi,\left(\frac{\partial Q}{\partial\alpha}\right)\xi\Big\rangle\Big]\left(\frac{\partial\tau}{\partial s}\right)\left(\frac{\partial\alpha}{\partial s}\right) \\ &+\Big\langle\left(\frac{\partial^2 S}{\partial\tau^2}\right)Q\xi,Q\xi\Big\rangle\left(\frac{\partial\tau}{\partial s}\right)^2 \\ &+\Big[\Big\langle\left(\frac{\partial S}{\partial\alpha}\right)Q\xi,Q\xi\Big\rangle + 2\Big\langle SQ\xi,\left(\frac{\partial Q}{\partial\alpha}\right)\xi\Big\rangle\Big]\left(\frac{\partial^2\alpha}{\partial s^2}\right) \\ &+\Big\langle\left(\frac{\partial S}{\partial\tau}\right)Q\xi,Q\xi\Big\rangle\left(\frac{\partial^2\tau}{\partial s^2}\right).\end{aligned}$$

For a function $f = f(\alpha,\tau)$, we will use a dot to designate a derivative with respect to τ ("time"), and a prime to designate a derivative with respect to α: $f' = \partial f/\partial\alpha$, $\dot f = \partial f/\partial t$, etc. By direct computation, at $s=0$ we have

$$S = \begin{bmatrix}1 & 0\\ 0 & t\mathcal{F}\end{bmatrix}\qquad \frac{\partial S}{\partial\alpha} = \begin{bmatrix}0 & 0\\ 0 & t\mathcal{F}'\end{bmatrix}\qquad \frac{\partial S}{\partial\tau} = \begin{bmatrix}0 & 0\\ 0 & \mathcal{F}+t\dot{\mathcal{F}}\end{bmatrix}$$

$$\frac{\partial^2 S}{\partial\alpha^2} = \begin{bmatrix}0 & 0\\ 0 & t\mathcal{F}''\end{bmatrix}\qquad \frac{\partial^2 S}{\partial\alpha\,\partial\tau} = \begin{bmatrix}0 & 0\\ 0 & \mathcal{F}'+t\dot{\mathcal{F}}'\end{bmatrix}\qquad \frac{\partial^2 S}{\partial\tau^2} = \begin{bmatrix}0 & 0\\ 0 & 2\dot{\mathcal{F}}+t\ddot{\mathcal{F}}\end{bmatrix},$$

$$Q = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\qquad \frac{\partial Q}{\partial\alpha} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\qquad \frac{\partial^2 Q}{\partial\alpha^2} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix},$$

$$\tau = t,\qquad \frac{\partial\tau}{\partial s} = \eta_1,\qquad \frac{\partial^2\tau}{\partial s^2} = \frac{\eta_2^2}{t},$$

$$\alpha = 0, \qquad \frac{\partial \alpha}{\partial s} = \frac{\eta_2}{t}, \qquad \frac{\partial^2 \alpha}{\partial s^2} = -\frac{2\,\eta_1\,\eta_2}{t^2}.$$

Plugging this back in (1.4), we obtain the following expression for the extended Ma–Trudinger–Wang tensor, as defined in [10] (see also [11, Definition 2.2]):

$$\begin{aligned}
(1.5) \qquad \frac{2}{3}\,\overline{\mathfrak{S}}_{(x,v)}(\xi,\eta) &= -\left.\frac{d^2}{ds^2}\right|_{s=0} \Big\langle S_{(x,\tau(s)\sigma_{\alpha(s)})}\xi, \xi \Big\rangle \\
&= \left[-(t\mathcal{F}'')\xi_2^2 + 4(t\mathcal{F}')\xi_1\xi_2 + 2(t\mathcal{F})\xi_2^2 + 2\xi_1^2 - 2\xi_2^2 - 2(t\mathcal{F})\xi_1^2\right] \frac{\eta_2^2}{t^2} \\
&\quad + \Big[-2(\mathcal{F}' + t\dot{\mathcal{F}}')\xi_2^2 + 4(\mathcal{F} + t\dot{\mathcal{F}})\xi_1\xi_2\Big] \frac{\eta_1\,\eta_2}{t} \\
&\quad - \Big(2\dot{\mathcal{F}} + t\ddot{\mathcal{F}}\Big)\,\xi_2^2\eta_1^2 \\
&\quad + \left[-(t\mathcal{F}')\xi_2^2 + 2(t\mathcal{F})\xi_1\xi_2 - 2\xi_1\xi_2\right] \left(-\frac{2\eta_1\eta_2}{t^2}\right) \\
&\quad - \Big(\mathcal{F} + t\dot{\mathcal{F}}\Big)\,\frac{\xi_2^2\eta_2^2}{t}.
\end{aligned}$$

Whenever $v \in \mathrm{I}(x)$, $\overline{\mathfrak{S}}_{(x,v)}$ coincides (modulo identification) with the "usual" MTW tensor, see [11] for more details.

At this stage we note that the identity

$$\frac{d}{d\tau}\big(f_1\dot{f}_0 - \dot{f}_1 f_0\big) = 0$$

implies

$$(1.6) \qquad f_1\dot{f}_0 - \dot{f}_1 f_0 = 1$$

(as the above identity holds at $\tau = 0$), or equivalently

$$(1.7) \qquad \dot{\mathcal{F}} = -\frac{1}{f_0^2}.$$

It follows

$$\ddot{\mathcal{F}} = \frac{2\dot{f}_0}{f_0^3}, \qquad \dot{\mathcal{F}}' = \frac{2f_0'}{f_0^3}.$$

Plugging this in (1.5) yields

$$
\begin{aligned}
(1.8)\quad \frac{2}{3}\,\overline{\mathfrak{S}}_{(x,v)}(\xi,\eta) = &\\
&\left(\frac{2}{t^2}-\frac{2\mathcal{F}}{t}\right)\xi_1^2\eta_2^2+\left(\frac{4}{t^2}-\frac{4}{f_0^2}\right)\xi_1\xi_2\eta_1\eta_2+\frac{4\mathcal{F}'}{t}\,\xi_1\xi_2\eta_2^2\\
&+\left(\frac{2}{f_0^2}-\frac{2t\dot{f}_0}{f_0^3}\right)\xi_2^2\eta_1^2-\frac{4f_0'}{f_0^3}\,\xi_2^2\eta_1\eta_2+\left(-\frac{2}{t^2}+\frac{\mathcal{F}}{t}-\frac{\mathcal{F}''}{t}+\frac{1}{f_0^2}\right)\xi_2^2\eta_2^2.
\end{aligned}
$$

Now, if we take $\eta=\xi^\perp=(-\xi_2,\xi_1)$ we get

$$
(1.9)\quad \frac{2}{3}\,\overline{\mathfrak{S}}_{(x,v)}(\xi,\xi^\perp) \;=\; A(t)\,\xi_1^4\;+\;B(t)\,\xi_1^3\xi_2\;+\;C(t)\,\xi_1^2\xi_2^2\;+\;D(t)\,\xi_1\xi_2^3\;+\;E(t)\,\xi_2^4,
$$

where

$$
(1.10)\qquad \begin{cases}
A(t)=\dfrac{2}{t^2}-\dfrac{2\mathcal{F}}{t}\\[2ex]
B(t)=\dfrac{4\mathcal{F}'}{t}\\[2ex]
C(t)=\dfrac{5}{f_0^2}-\dfrac{6}{t^2}+\dfrac{\mathcal{F}}{t}-\dfrac{\mathcal{F}''}{t}\\[2ex]
D(t)=\dfrac{4f_0'}{f_0^3}\\[2ex]
E(t)=\dfrac{2}{f_0^2}-\dfrac{2t\dot{f}_0}{f_0^3}.
\end{cases}
$$

1.3. **MTW conditions.** We say that the *MTW condition* (in short **(MTW)**) holds if (1.9) is non-negative for all ξ, for any (x,v) in the injectivity domain $\mathrm{I}(M)$. The strict form of the Ma–Trudinger–Wang condition amounts to saying that (1.9) is positive whenever $(\xi_1,\xi_2)\neq(0,0)$,

(a) either for any choice of (x,v) in the injectivity domain $\mathrm{I}(M)$;

(b) or for any choice of (x,v) in the nonfocal domain $\mathrm{NF}(M)$.

In case (a), we say that the *strict MTW condition* (denoted $(\mathbf{MTW}^+)$) holds. In case (b), we say that the *extended strict MTW condition* (denoted $(\overline{\mathbf{MTW}}^+)$) holds.

A useful quantitative form of this inequality is

$$\forall\, \xi, \eta \in T_x M, \qquad \overline{\mathfrak{S}}_{(x,v)}(\xi, \eta) \geq K\, |\xi|^2\, |\eta|^2 - C\, \langle \xi, \eta \rangle^2,$$

where K, C are positive constants. If this holds true for all $(x, v) \in \mathrm{I}(M)$ (resp. for all $(x, v) \in \mathrm{NF}(M)$), we say that M satisfies the condition $(\mathbf{MTW}(K,\, C))$ (resp. $(\overline{\mathbf{MTW}}(K,\, C))$).

All these conditions may or may not be satisfied by M. They are anyway strictly stronger than the condition of positive Gauss curvature. Examples and counterexamples are discussed at the end of this paper.

2. Curvature of tangent focal locus

It is near the tangent focal cut locus that the study of the MTW condition becomes tricky. We shall see that there, the curvature of the tangent focal locus plays a crucial role. In this section we compute this curvature.

2.1. **Local behavior of** TFL. Let us define the function $\alpha \mapsto \rho(\alpha) = t_F(\sigma_\alpha) = t_F(x, \sigma_\alpha)$, so that the tangent focal locus of M at x is given by the equation $\{\rho = \rho(\alpha)\}$ in polar coordinates. The function ρ is the first nonzero solution of the implicit equation

$$f_0(\alpha, \rho(\alpha)) = 0. \tag{2.1}$$

The identity (1.6) implies that $\dot{f}_0$ does not vanish in a neighborhood of $\{f_0 = 0\}$, so by the implicit function theorem ρ is a smooth function of α. (As before, for a function $f = f(\alpha, \tau)$ we write $f' = \partial f/\partial\alpha$, $\dot{f} = \partial f/\partial\tau$.)

Differentiating (2.1) with respect to α and using (1.6) again yields

$$\rho'(\alpha) = -\frac{f_0'}{\dot{f}_0}(\alpha, \rho(\alpha)) = -(f_0'\, f_1)(\alpha, \rho(\alpha)). \tag{2.2}$$

A second differentiation, combined with (2.2), yields

$$\rho'' = -\frac{f_0'' + 2\rho' \dot{f}_0' + (\rho')^2 \ddot{f}_0}{\dot{f}_0} = -f_1 f_0'' + 2 f_1^2 f_0' \dot{f}_0', \tag{2.3}$$

where the right-hand side is evaluated at $(\alpha, \rho(\alpha))$. (The term $\ddot{f}_0$ has disappeared because $\ddot{f}_0 = -k f_0 = 0$.)

Now we can apply classical formulas to compute the signed curvature $\kappa(\alpha)$ of $\mathrm{TFL}(x)$ at $(\alpha, \rho(\alpha))$: with $v = \rho(\alpha)\sigma_\alpha$,

$$\kappa(\alpha) = \frac{\det \begin{bmatrix} v_1' & v_1'' \\ v_2' & v_2'' \end{bmatrix}}{\big((v_1')^2 + (v_2')^2\big)^{3/2}} = \frac{\rho^2 + 2(\rho')^2 - \rho\,\rho''}{(\rho^2 + (\rho')^2)^{3/2}}$$

$$= \frac{\rho^2 + 2(f_0' f_1)^2 + \rho f_0'' f_1 - 2\rho f_1 \dot f_1 (f_0')^2 + 2\rho f_1' f_0'}{\big[\rho^2 + (f_0' f_1)^2\big]^{3/2}}, \tag{2.4}$$

evaluated at $(\alpha, \rho(\alpha))$. (In the last formula we have used again $f_1 \dot f_0 = 1$ and $f_1 \dot f_0' = \dot f_1 f_0' - f_1' \dot f_0$, both deduced from (1.6).)

So the nonfocal domain is convex (resp. uniformly convex) around $\overline{v} = t\sigma_{\overline{\alpha}}$ if and only (2.4) is nonnegative (resp. positive) for any α in a neighborhood of $\overline{\alpha}$.

2.2. **Local behavior of** TFCL. Now we particularize the preceding computation by considering a velocity which is not only focal, but also a cut velocity. So let again $\overline{v} = t\sigma_{\overline{\alpha}} \in \mathrm{TFCL}$, and let $\overline{\rho} = t_C(\overline{v}) = t_F(\overline{v}) = \rho(\overline{\alpha})$. By Proposition A.6 in the Appendix,

$$\rho'(\overline{\alpha}) = 0. \tag{2.5}$$

By (1.6) we have $f_1 \neq 0$ at $(\overline{\alpha}, \overline{\rho})$, so (2.5) is equivalent to

$$f_0'(\overline{\alpha}, \overline{\rho}) = 0. \tag{2.6}$$

Then the expressions obtained in Subsection 2.1 simplify as follows:

$$\rho''(\overline{\alpha}) = -f_1 f_0'', \qquad \kappa(\overline{\alpha}) = \frac{1}{\overline{\rho}}\left(1 + \frac{f_1 f_0''}{\overline{\rho}}\right), \tag{2.7}$$

where f_1 and f_0'' are evaluated at $(\overline{\alpha}, \overline{\rho})$.

3. MTW conditions at the edge

From the expression of the MTW tensor given in (1.8), one can easily see that $\overline{\mathfrak{S}}_{(x,v)}$ varies smoothly with respect the metric, as long as f_0 is bounded away from 0, i.e. as long as $v \in \mathrm{NF}(x)$ is far away from $\mathrm{TFL}(x)$. Hence, when one is interested in the stability of the Ma–Trudinger–Wang condition inside $\mathrm{I}(M)$, the critical part is to understand this condition near the "natural boundary" of its domain of validity, which is not the tangent cut locus, but rather the tangent *focal* cut locus.

The main theme of this section is that there is "almost" equivalence of the three following conditions:

(a) the MTW tensor is "nonnegative" near TFCL

(b) the MTW tensor is "bounded below" near TFCL

(c) TFCL is "locally convex".

Condition (c) really means that for any x, the nonfocal domain $\mathrm{NF}(x)$ is convex in the neighborhood of any focal cut velocity. Since the description of the tangent cut locus is easy away from focalization, condition (c) allows to prove a rather strong geometric property, not known for general manifolds: injectivity domains are *semiconvex*, i.e. smooth deformations of convex sets (see [13]). (Compare with the open problem stated in [16, Problem 3.4].)

We do not know how far one can push the equivalence between (a), (b) and (c). For the moment we shall only establish certain partial implications between variants of these conditions.

Proposition 3.1. *Let* $\overline{v} = \overline{\rho}\sigma_{\overline{\alpha}} \in \mathrm{TFCL}(x)$ *and let* $\overline{\kappa}$ *be the signed curvature of* $\mathrm{TFL}(x)$ *at* $\overline{v}$*. Then:*

(i) if $\overline{\kappa} > 0$*, there are* $\delta, K, C > 0$*, depending only on upper bounds on* $|\dot{f}_i|$*,* $|f_i'|$*,* $|f_i''|$ *(*$i = 0, 1$*), and on a lower bound on* $\overline{\kappa}$ *and on the injectivity radius of* M*, such that for all* $\alpha \in (\overline{\alpha} - \delta, \overline{\alpha} + \delta)$ *and* $t \in (t_F(\sigma_\alpha) - \delta, t_F(\sigma_\alpha))$*,* $v = t\sigma_\alpha$*,*

$$\forall\, (\xi, \eta) \in T_xM \times T_xM, \qquad \overline{\mathfrak{S}}_{(x,v)}(\xi, \eta) \geq K \left(\xi_1^2 + \frac{\xi_2^2}{f_0^2}\right) |\eta|^2 \;-\; C\,\langle \xi, \eta\rangle^2. \tag{3.1}$$

(ii) if $\overline{\kappa} < 0$*, then*

$$\liminf \left\{ \overline{\mathfrak{S}}_{(x,t\sigma_\alpha)}(\xi, \xi^\perp);\ t \to t_F(\sigma_\alpha)^-,\ \alpha \to \overline{\alpha},\ \xi \in T_xM,\ |\xi| = 1 \right\} = -\infty. \tag{3.2}$$

The above proposition says the following remarkable thing: whenever $\mathrm{TFL}(x)$ is uniformly convex near a point $\overline{v} \in \mathrm{TFCL}(x)$, then $(\overline{\mathbf{MTW}}(K,\ C))$ holds in a neighborhood of $\overline{v}$ in T_xM. As already observed in [10, 11, 24], this analytic property allows to deduce strong geometric consequences on the injectivity domains.

Remark 3.2. As can be easily seen from the proof of point (i), the constant K actually depends only on a lower bound on $\overline{\kappa}$ and on the injectivity radius of M, provided δ is chosen sufficiently small. Moreover, as can be immediately seen from the proof, the assumption that $\bar{v} \in \mathrm{TFCL}(x)$ is needed only to ensure that $\left|f_0'(\alpha, t_F(\alpha))\right|$ is sufficiently small in a neighborhood of $(\overline{\alpha}, t_F(\overline{\alpha}))$. Hence all the results in Proposition 3.1 hold true for any $\bar{v} = t_F(\overline{\alpha})\sigma_{\overline{\alpha}} \in \mathrm{TFL}(x)$ such that $\left|f_0'(\overline{\alpha}, t_F(\overline{\alpha}))\right| \leq \eta$, for some universal constant $\eta > 0$.

Proof of Proposition 3.1. We start by rewriting (1.8). After repeated use of (1.6), we obtain

$$\begin{aligned}
(3.3)\qquad \frac{2}{3}\,\overline{\mathfrak{S}}_{(x,v)}(\xi,\eta) = & -\frac{2f_1}{tf_0}\left[\left(\frac{f_0'}{f_0}-\frac{f_1'}{f_1}\right)\xi_2\,\eta_2 + \frac{t\,\dot f_0}{f_0}\,\xi_2\,\eta_1 + \xi_1\,\eta_2\right]^2 \\
& + \frac{2}{t^2}\,\xi_1^2\,\eta_2^2 \\
& + 4\left(\frac{1}{t^2}+\frac{\dot f_1}{f_0}\right)\xi_1\,\xi_2\,\eta_1\,\eta_2 \\
& + \left(\frac{2}{f_0^2}+2t\,\frac{\dot f_0\,\dot f_1}{f_0^2}\right)\xi_2^2\,\eta_1^2 \\
& + \frac{4}{f_0^2}\left(f_0'\,\dot f_1 - \dot f_0\,f_1'\right)\xi_2^2\,\eta_1\,\eta_2 \\
& + \left[-\frac{2}{t^2}+\frac{f_1}{tf_0}-\frac{1}{t}\left(\frac{f_1}{f_0}\right)'' + \frac{1}{f_0^2}+\frac{2f_1}{tf_0}\left(\frac{f_0'}{f_0}-\frac{f_1'}{f_1}\right)^2\right]\xi_2^2\,\eta_2^2.
\end{aligned}$$

Note that the coefficient $-2f_1/(tf_0)$ is positive near the edge (near focalization). (This follows from (1.6), observing that $\dot f_0 < 0$ near the edge.) Let us examine the behavior of the various coefficients:

- $f_0 \to 0$ as we approach the focal locus.
- Since $f_0'(\overline{\alpha}, t_F(\overline{\alpha})) = 0$, we may choose δ small enough that we can impose $|f_0'| \le \phi$, with ϕ arbitrarily small.
- In the coefficient of $\xi_2^2\,\eta_2^2$ the highest order terms $\pm 2f_1(f_0')^2/(tf_0^3)$ cancel each other; in the end this coefficient is

$$\gamma = \frac{1}{f_0^2}\left[1+\frac{f_1\,f_0''}{t} - 2\,\frac{f_1'\,f_0'}{t}\right] + O\left(\frac{1}{f_0}\right).$$

By (2.7), if the curvature $\overline{\kappa}$ of TFL(x) at $(\overline{\alpha}, t_F(\overline{\alpha}))$ is nonzero and if ϕ is small enough, then for δ sufficiently small

$$\gamma = \frac{1}{f_0^2}\,t\,\overline{\kappa}\,(1+\omega),$$

where $\omega = \omega(\alpha,t)$ satisfies $|\omega| \le 1/4$.

• Then we choose f_0 very small with respect to the other parameters. In the end

$$
\begin{aligned}
\frac{2}{3}\,\overline{\mathfrak{S}}_{(x,v)}(\xi,\eta) = &-\frac{2f_1}{t f_0}\,\left[\left(\frac{f_0'}{f_0}-\frac{f_1'}{f_1}\right)\xi_2\,\eta_2 + \frac{t\,\dot f_0}{f_0}\,\xi_2\,\eta_1 + \xi_1\,\eta_2\right]^2 \\
&+\frac{2}{t^2}\,\xi_1^2\,\eta_2^2 + \frac{t\,\overline{\kappa}}{f_0^2}\,(1+\omega)\,\xi_2^2\,\eta_2^2 \\
&+O\left(\frac{1}{f_0}\right)\xi_1\,\xi_2\,\eta_1\,\eta_2 + O\left(\frac{1}{f_0^2}\right)\xi_2^2\,\eta_1^2 + O\left(\frac{1}{f_0^2}\right)\xi_2^2\,\eta_1\,\eta_2.
\end{aligned}
\tag{3.4}
$$

Next, we write

$$
\left|O\left(\frac{1}{f_0^2}\right)\xi_2^2\,\eta_1\,\eta_2\right| \le \frac{\lambda}{f_0^2}\,\xi_2^2\,\eta_2^2 + O\left(\frac{1}{f_0^2}\right)\xi_2^2\,\eta_1^2,
$$

$$
\left|O\left(\frac{1}{f_0}\right)\xi_1\,\xi_2\,\eta_1\,\eta_2\right| \le \lambda\,\xi_1^2\,\eta_2^2 + O\left(\frac{1}{f_0^2}\right)\xi_2^2\,\eta_1^2,
$$

where $\lambda > 0$ is arbitrarily small but fixed. We conclude that for f_0 and δ small enough,

$$
\begin{aligned}
\frac{2}{3}\,\overline{\mathfrak{S}}_{(x,v)}(\xi,\eta) = &-\frac{2f_1}{t f_0}\,\left[\left(\frac{f_0'}{f_0}-\frac{f_1'}{f_1}\right)\xi_2\,\eta_2 + \frac{t\,\dot f_0}{f_0}\,\xi_2\,\eta_1 + \xi_1\,\eta_2\right]^2 \\
&+\frac{2}{t^2}\,(1+\omega)\,\xi_1^2\,\eta_2^2 + \frac{t\,\overline{\kappa}}{f_0^2}\,(1+\omega)\,\xi_2^2\,\eta_2^2 + O\left(\frac{1}{f_0^2}\right)\xi_2^2\,\eta_1^2,
\end{aligned}
\tag{3.5}
$$

where, say, $|\omega| \le 1/2$.

After these preparations we can prove Proposition 3.1.

(I) First we assume $\overline{\kappa} < 0$ and we wish to prove instability. From (3.5),

$$
\begin{aligned}
\frac{2}{3}\,\overline{\mathfrak{S}}_{(x,v)}(\xi,\xi^\perp) = &-\frac{2f_1}{t f_0}\,\left[\left(\frac{f_0'}{f_0}-\frac{f_1'}{f_1}\right)\xi_1\,\xi_2 - \frac{t\,\dot f_0}{f_0}\,\xi_2^2 + \xi_1^2\right]^2 \\
&+\frac{2}{t^2}\,(1+\omega)\,\xi_1^4 + \frac{t\,\overline{\kappa}}{f_0^2}\,(1+\omega)\,\xi_1^2\,\xi_2^2 + O\left(\frac{1}{f_0^2}\right)\xi_2^4,
\end{aligned}
\tag{3.6}
$$

From the definition of t_F we have $f_0(\alpha, t_F(\alpha)) \equiv 0$, so that we deduce

$$
0 = \frac{\partial}{\partial\alpha}\big[f_0(\alpha, t_F(\alpha))\big] = f_0'(\alpha, t_F(\alpha)) + \dot f_0(\alpha, t_F(\alpha))\,t_F'(\alpha),
$$

and since $t_F'(\overline{\alpha}) = 0$ this implies

$$0 = \frac{\partial}{\partial \alpha}\Big|_{\alpha=\overline{\alpha}} \big[f_0'(\alpha, t_F(\alpha))\big] + \dot{f}_0(\overline{\alpha}, t_F(\overline{\alpha}))\, t_F''(\overline{\alpha}),$$

whence, recalling (2.7),

$$\begin{aligned}\frac{\partial}{\partial \alpha}\Big|_{\alpha=\overline{\alpha}} \big[f_0'(\alpha, t_F(\alpha))\big] &= -\dot{f}_0(\overline{\alpha}, t_F(\overline{\alpha}))\, t_F''(\overline{\alpha}) \\ &= (\dot{f}_0\, f_1\, f_0'')(\overline{\alpha}, t_F(\overline{\alpha})) \\ &= \dot{f}_0(\overline{\alpha}, t_F(\overline{\alpha})\, t_F(\overline{\alpha})\, \big(\overline{\kappa}\, t_F(\overline{\alpha}) - 1\big).\end{aligned}$$

The latter quantity is positive since $\dot{f}_0(\overline{\alpha}, t_F(\overline{\alpha})) < 0$ and $\overline{\kappa} < 0$ by assumption. Recalling that $f_0'(\overline{\alpha}, t_F(\overline{\alpha})) = 0$, we conclude that $f_0'(\alpha, t_F(\alpha))$ is a small positive number for $\alpha > \overline{\alpha}$, $\alpha \simeq \overline{\alpha}$. We fix such an α.

Then for $t < t_F(\alpha)$, $t \simeq t_F(\alpha)$, we have $f_0'(\alpha, t) \neq 0$, $\dot{f}_0(\alpha, t) \neq 0$, and we can choose a unit vector $\xi = \xi(\alpha, t)$ such that

$$\frac{\xi_2}{\xi_1} = \frac{\left(\frac{f_1'}{f_1} - \frac{f_0'}{f_0}\right) + \sqrt{\left(\frac{f_1'}{f_1} - \frac{f_0'}{f_0}\right)^2 + 4t\frac{\dot{f}_0}{f_0}}}{-2t\frac{\dot{f}_0}{f_0}} \;\simeq\; -\frac{f_0(\alpha, t)}{f_0'(\alpha, t_F(\alpha))} \qquad \text{as } t \to t_F(\alpha).$$

Then the first term in the right-hand side of (3.6) vanishes, and as $t \to t_F(\alpha)^-$ we have

$$\frac{2}{3}\,\overline{\mathfrak{S}}_{(x,t\sigma_\alpha)}(\xi, \xi^\perp) = \frac{2}{t_F(\overline{\alpha})^2}\,(1+\omega) + \frac{t_F(\overline{\alpha})\,\overline{\kappa}}{f_0'(\alpha, t_F(\alpha))^2}\,(1+\omega) + O(f_0^2).$$

This expression takes arbitrarily large negative values as $\alpha \to \overline{\alpha}$, which proves (3.2).

(II) Next we assume $\overline{\kappa} > 0$ and we wish to prove stability. The first three terms in the right-hand side of (3.5) have the "right" sign, so the issue is to control the remaining one. We introduce a small constant $\varepsilon > 0$, and distinguish three cases:

• If $|\eta_1| \leq \varepsilon|\eta_2|$, then the "dangerous" term is obviously controlled by the term involving $\overline{\kappa}$, as soon as ε is small enough. Moreover $|\eta|^2 \leq 2\,\eta_2^2$, and so

$$\overline{\mathfrak{S}}_{(x,v)}(\xi, \eta) \geq \frac{2}{t^2}\,(1+\omega)\,\xi_1^2\,\eta_2^2 + \frac{t\,\overline{\kappa}}{f_0^2}\,(1+\omega)\,\xi_2^2\,\eta_2^2 \geq K\,\left(\xi_1^2 + \frac{\xi_2^2}{f_0^2}\right)\,|\eta|^2,$$

for $K > 0$ sufficiently small.

• If $|\eta_2| \leq \varepsilon^{-1}\,|\eta_1|$ and

$$\left| t\frac{\dot f_0}{f_0}\,\xi_2\,\eta_1 \right| \geq \varepsilon^{-1}\,|\xi_1\,\eta_2| \quad \text{or} \quad \left| t\frac{\dot f_0}{f_0}\,\xi_2\,\eta_1 \right| \leq \varepsilon\,|\xi_1\,\eta_2|,$$

then choosing δ small enough we have

$$\left| \left(\frac{f_0'}{f_0} - \frac{f_1'}{f_1} \right) \xi_2\,\eta_2 \right| \leq \frac{1}{4} \left| \frac{t\,\dot f_0}{f_0}\,\xi_2\,\eta_1 \right|,$$

and so

$$-\frac{f_1}{t f_0} \left[\left(\frac{f_0'}{f_0} - \frac{f_1'}{f_1} \right) \xi_2\,\eta_2 + \frac{t\dot f_0}{f_0}\,\xi_2\,\eta_1 + \xi_1\,\eta_2 \right]^2 \geq -\frac{f_1}{2t} \left(\frac{t^2\,\dot f_0^2}{f_0^3}\,\xi_2^2\,\eta_1^2 + \frac{\xi_1^2\,\eta_2^2}{f_0} \right),$$

which easily dominates the "dangerous" term for ε small enough. Hence (since $|\eta|^2 \leq 2\,\varepsilon^{-1}\eta_1^2$)

$$\begin{aligned} \overline{\mathfrak{S}}_{(x,v)}(\xi,\eta) &\geq -\frac{f_1}{2t} \left(\frac{t^2\,\dot f_0^2}{f_0^3}\,\xi_2^2\,\eta_1^2 + \frac{\xi_1^2\,\eta_2^2}{f_0} \right) + \frac{2}{t^2}\,(1+\omega)\,\xi_1^2\,\eta_2^2 + \frac{t\,\overline{\kappa}}{f_0^2}\,(1+\omega)\,\xi_2^2\,\eta_2^2 \\ &\geq c\left(\xi_1^2\,\eta_2^2 + \xi_2^2\,\eta_1^2 + \xi_2^2\,\eta_2^2\right) + c\frac{\xi_2^2}{f_0^2}|\eta|^2, \end{aligned}$$

for some small constant $c > 0$. Thanks to the inequality $a^2 \leq 2\,b^2 + 2\,(a+b)^2$, we deduce that

$$\xi_1^2\,\eta_1^2 \leq 2\,\xi_2^2\,\eta_2^2 + 2\left|\langle \xi,\eta\rangle\right|^2,$$

which easily implies $\overline{\mathfrak{S}}_{(x,v)}(\xi,\eta) \geq K\left(|\xi|^2 + f_0^{-2}\xi_2^2\right)|\eta|^2 - C\langle\xi,\eta\rangle^2$ for some positive constants K, C.

• If $|\eta_2| \leq \varepsilon^{-1}|\eta_1|$ and

$$\varepsilon\,|\xi_1\,\eta_2| \leq \left| t\frac{\dot f_0}{f_0}\,\xi_2\,\eta_1 \right| \leq \varepsilon^{-1}\,|\xi_1\,\eta_2|,$$

then $\xi_2^2\,\eta_1^2 \leq O(f_0^2\,\varepsilon^{-1})\,\xi_1^2\,\eta_2^2$ (note that $\dot f_0$ is strictly negative near $f_0 = 0$), so for $|\xi| = |\eta| = 1$ we have

$$\overline{\mathfrak{S}}_{(x,v)}(\xi,\eta) \geq -\frac{C_0}{\varepsilon}, \tag{3.7}$$

for some constant $C_0 \geq 0$.

On the other hand, $|\xi_2| = O(f_0\,\varepsilon^{-1}|\xi_1|\,|\eta_2|/|\eta_1|) = O(f_0\,\varepsilon^{-2})$ is bounded above by $\varepsilon/8$ for f_0 small enough, and we get

$$\big|\langle \xi,\eta\rangle\big| \geq |\xi_1\,\eta_1| - |\xi_2\,\eta_2| \geq \frac{\varepsilon}{4}.$$

Also $\xi_2^2/f_0^2 = O(1)$. Combining this with (3.5) and (3.7), for δ and f_0 small enough we do have

$$\overline{\mathfrak{S}}_{(x,v)}(\xi,\eta) \geq K\,|\xi|^2\,|\eta|^2 + \frac{\xi_2^2}{f_0^2}\,|\eta|^2 - C\,\langle \xi,\eta\rangle^2$$

for $K > 0$ small, and for some large enough constant C. This completes the proof of Proposition 3.1. □

4. Stability

If the condition $(\mathbf{MTW^+})$ is unstable, this can only be near TFCL. The main result of this section shows that a geometric condition on the focal locus near this "dangerous set" will prevent the instability.

For any $x \in M$, let $\underline{\kappa}(x) = \inf_\alpha \kappa(\alpha)$, where $\kappa(\alpha)$ is the signed curvature of $\mathrm{TFL}(x)$ at $(\alpha,\rho(\alpha))$. (See Subsection 2.1.) Then we define $\underline{\kappa}(M) = \inf_{x\in M} \underline{\kappa}(x)$.

Theorem 4.1 (Stability of $(\mathbf{MTW^+})$ on surfaces with convex nonfocal domain)**.** *Let (M,g) be a compact Riemannian surface satisfying* $(\mathbf{MTW^+})$. *If $\underline{\kappa}(M) > 0$ then any C^4 perturbation of g satisfies* $(\mathbf{MTW}(K,\,C))$ *for some constants $K, C > 0$. This applies in particular to g itself.*

In other words, there is $\delta > 0$ such that for any other metric $\widetilde{g}$ on M, if $\|g-\widetilde{g}\|_{C^4} < \delta$ then $(M,\widetilde{g})$ satisfies $(\mathbf{MTW}(K,\,C))$. Here the C^4 norm is measured by means of local charts on M.

Remark 4.2. Theorem 4.1 was proven in [10] in the particular case when M is the sphere $\mathbb{S}^2$. In that case however, there is a stronger statement according to which the *extended* condition $(\overline{\mathbf{MTW}}(K,\,C))$ survives perturbation.

Proof of Theorem 4.1. Let $\mathcal{G}$ be the set of Riemannian metrics on M, equipped with the C^4 topology. First we note that $\mathrm{TFL}(M)$ is continuous on $\mathcal{G}$ (the C^2 topology would be sufficient for that); and according to formula (2.4), the curvature of $\mathrm{TFL}(x)$ at v is a continuous function of x, v and also of $\widetilde{g} \in \mathcal{G}$.

Next, $\mathrm{TCL}(M)$ is continuous on $\mathcal{G}$ (also here the C^2 topology would be sufficient.) In particular, the injectivity radius of M is continuous on $\mathcal{G}$; and $\mathrm{TFCL}(M,\widetilde{g})$ remains within an open neighborhood of $\mathrm{TFCL}(M,g)$. (This set can shrink drastically, as the perturbation of the sphere shows.) So if $\|g - \widetilde{g}\|_{C^4}$ is small enough, we still have $\underline{\kappa}(M, \widetilde{g}) > 0$.

For each $(x, v) \in \mathrm{TFCL}(M, g)$, the curvature of $\mathrm{TFL}(x)$ at v is positive; so Proposition 3.1 (i) implies the existence of a neighborhood $U_{(x,v)}$ of (x, v) in TM, and a neighborhood $O_{(x,v)}$ of g in $\mathcal{G}$ such that for any $(y, w) \in U_{(x,v)}$ and any $\widetilde{g} \in O_{(x,v)}$,

$$\forall\, \xi, \eta \in T_yM \setminus \{0\}, \qquad \overline{\mathfrak{S}}^{\widetilde{g}}_{(y,w)}(\xi, \eta) \geq K\, |\xi|_y^2\, |\eta|_y^2 - C\, \langle \xi, \eta \rangle_y^2.$$

By compactness, we can find an open neighborhood U of $\mathrm{TFCL}(M, g)$ and a neighborhood O of g in $\mathcal{G}$ on which this inequality holds.

Outside of U, (1.8) shows that $\overline{\mathfrak{S}}^{\widetilde{g}}_{(y,w)}(\xi, \eta)$ is a uniformly continuous function of $(y, w) \in \mathrm{I}(M, \widetilde{g})$, $\widetilde{g} \in \mathcal{G}$, and ξ, η in the unitary tangent bundle. Then one can conclude by the same compactness argument as in [11, Section 6]. □

Example 4.3. Consider an ellipsoid E which is not too far from the sphere. If it satisfies a strict MTW condition and has uniformly convex nonfocal domains, then any C^4 perturbation of E will also satisfy the MTW condition. The point is that the MTW condition may be checked numerically on E, since geodesics and focal loci are given by known analytic expressions.

5. New counterexamples

Following [28, Chapter 12], let us agree that a manifold is *regular* if it satisfies the Ma–Trudinger–Wang condition and has convex injectivity domains. Examples of regular manifolds appear in [10, 11, 23, 24, 19]. Regularity of the manifold is a necessary condition for the regularity theory of optimal transport [28, Chapter 12].

In the class of positively curved surfaces, counterexamples were constructed in [17] and [24] (in the last paper, this is essentially a cone touching a paraboloid). Here we shall construct new counterexamples, also with positive curvature.

5.1. Surfaces of revolutions.

In this subsection, we give simple formulas for the MTW curvature along certain well-chosen geodesics of surfaces with revolution symmetry.

Let N and S respectively stand for the North and South Poles on $\mathbb{S}^2$. We parameterize $\mathbb{S}^2 \setminus \{N, S\}$ by polar coordinates (θ, r) from N, and define the Riemannian metric

$$g = m(r)^2\, d\theta^2 + dr^2,$$

where m is a positive smooth function. In the sequel, we shall identify points of $\mathbb{S}^2$ with their coordinates and denote by $m^{(k)}$ the k-th derivative of m.

Only two of the Christoffel symbols of g are nonzero:

$$\Gamma^r_{\theta\theta} = -m(r)\, m^{(1)}(r) \quad \text{and} \quad \Gamma^\theta_{r\theta} = \frac{m^{(1)}(r)}{m(r)};$$

so the equation for geodesics is

$$\begin{cases} \ddot{\theta} + 2\Gamma^\theta_{r\theta}\dot{r}\dot{\theta} &= 0 \\ \ddot{r} + \Gamma^r_{\theta\theta}\big(\dot{\theta}\big)^2 &= 0. \end{cases}$$

Further, the Gauss curvature of g at a point (θ, r) is equal to

$$k(r) = -\frac{m^{(2)}(r)}{m(r)}. \tag{5.1}$$

We assume that $k > 0$, so that m is strictly concave. We define $\overline{r}$ as the unique value r such that $m^{(1)}(r) = 0$, and we assume that $m^{(3)}(\overline{r}) = 0$, so that $k^{(1)}(\overline{r}) = 0$. We write $\overline{k} = k(\overline{r})$.

Let $\overline{\gamma} : \mathbb{R}_+ \to \mathbb{S}^2$ be the unit-speed geodesic starting at $\overline{\theta} = 0$ in the θ-direction:

$$\overline{\gamma}(t) = (t, \overline{r}).$$

We shall study variations of $\overline{\gamma}$. First of all, since the curvature is constant along $\overline{\gamma}$, the functions f_0, f_1 introduced in (1.1) are given by

$$f_0(t) = \frac{\sin(\sqrt{\overline{k}}t)}{\sqrt{\overline{k}}}, \qquad f_1(t) = \cos(\sqrt{\overline{k}}t). \tag{5.2}$$

Next, let γ_α be the geodesic starting at $\overline{p} = \overline{\gamma}(0)$ with velocity $\sigma_\alpha = (\cos\alpha, \sin\alpha)$. From (5.2) we deduce

$$\left.\frac{\partial \gamma_\alpha}{\partial \alpha}\right|_{\alpha=0} = \left(0, \frac{\sin(\sqrt{\overline{k}}t)}{\sqrt{\overline{k}}}\right).$$

Then

$$\left.\frac{\partial k}{\partial \alpha}\right|_{\alpha=0} = 0 \tag{5.3}$$

$$\begin{aligned} \left.\frac{\partial^2 k}{\partial \alpha^2}\right|_{\alpha=0} &= \left\langle (\nabla^2 k)\frac{\partial \gamma_\alpha}{\partial \alpha}, \frac{\partial \gamma_\alpha}{\partial \alpha}\right\rangle \\ &= \big(m^{(2)}(\overline{r})^2 - m^{(4)}(\overline{r})\, m(\overline{r})\big)\,\frac{\sin^2(\sqrt{\overline{k}}t)}{\overline{k}}. \end{aligned} \tag{5.4}$$

Differentiating the Jacobi equation, we obtain

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial f_i}{\partial \alpha}\right) + \overline{k}\left(\frac{\partial f_i}{\partial \alpha}\right) = 0$$

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 f_i}{\partial \alpha^2}\right) + \overline{k}\left(\frac{\partial^2 f_i}{\partial \alpha^2}\right) = -\left(\frac{\partial^2 k}{\partial \alpha^2}\right) f_i.$$

We deduce

$$\frac{\partial f_i(t)}{\partial \alpha} = 0$$

$$\frac{\partial^2 f_i(t)}{\partial \alpha^2} = \left(\int_0^t \mathcal{K}(s)\, f_1(s)\, f_i(s)\, ds\right) f_0(t) - \left(\int_0^t \mathcal{K}(s)\, f_0(s)\, f_i(s)\, ds\right) f_1(t),$$

where the function $\mathcal{K}$ is defined as

$$\mathcal{K}(s) := \left(m^{(4)}(\bar{r})\, m(\bar{r}) - m^{(2)}(\bar{r})^2\right) \frac{\sin^2(\sqrt{\bar{k}}s)}{\bar{k}}. \tag{5.5}$$

Let us assume

$$m(\bar{r}) = 1, \tag{5.6}$$

so that the matrix of the metric at $(\overline{\theta}, \overline{r})$ in the basis $(\partial_\theta, \partial_r)$ is the identity. Then we can apply (1.10) to compute the MTW curvature along $\overline{\gamma}$. (Recall that $\mathcal{F} = \frac{f_1}{f_0}$.) The expression simplifies when

$$t = \frac{\pi}{2\sqrt{\bar{k}}} =: \bar{t},$$

since then $\dot{f}_0(\bar{t}) = f_1(\bar{t}) = 0$. Hence

$$\frac{2}{3}\,\overline{\mathfrak{S}}_{(\bar{p},\bar{t}\sigma_0)}(\xi, \xi^\perp) = \overline{A}\,\xi_1^4 + \overline{C}\,\xi_1^2\xi_2^2 + \overline{E}\,\xi_2^4, \tag{5.7}$$

where

$$\begin{cases} \overline{A} = \dfrac{2}{\bar{t}^2} - \dfrac{2\mathcal{F}}{\bar{t}} = \dfrac{2}{\bar{t}^2} \\[2ex] \overline{C} = \dfrac{5}{f_0^2} - \dfrac{6}{\bar{t}^2} + \dfrac{f_1}{\bar{t} f_0} - \dfrac{\mathcal{F}''}{\bar{t}} = 5\,\bar{k} - \dfrac{6}{\bar{t}^2} - \dfrac{\mathcal{F}''}{\bar{t}} \\[2ex] \overline{E} = \dfrac{2}{f_0^2} - \dfrac{2\bar{t}\dot{f}_0}{f_0^3} = 2\,\bar{k}, \end{cases} \tag{5.8}$$

with

$$(5.9)\qquad \mathcal{F}''(\bar{t}) = \frac{f_1''(\bar{t})}{f_0(\bar{t})} = \frac{1}{\bar{k}}\Big(m^{(4)}(\bar{r})m(\bar{r}) - m^{(2)}(\bar{r})^2\Big)\int_0^{\bar{t}} \sin^2\big(\sqrt{\bar{k}}s\big)\,\cos^2\big(\sqrt{\bar{k}}s\big)\,ds.$$

This expression is well suited to the construction of counterexamples: for that it is sufficient to devise the function m in such a way that $\mathcal{F}''$ is very large near focalization.

In practice, it is convenient to consider the metric as induced by a graph. Let $F : [-a,a] \to \mathbb{R}_+$ be a smooth function satisfying $F(-a) = F(a) = 0$, $F > 0$ on $(-a,a)$; then we may "rotate" the graph $(z = F(x))$ along the x-axis, thus sweeping a two-dimensional surface which is isometric to $(\mathbb{S}^2, g)$ with

$$g = m_F(r)^2\,d\theta^2 + dr^2,$$

where m_F is determined by the identity

$$\forall\, x \in [-a,a], \qquad m_F(r(x)) = F(x), \qquad r(x) = \int_{-a}^{x} \sqrt{1 + F^{(1)}(u)^2}\,du.$$

We assume $F(0) = 1$, $F^{(1)}(0) = F^{(3)}(0) = 0$, and define $\bar{r} = \int_{-a}^{0} \sqrt{1 + (F^{(1)})^2}$. Then

$$(5.10)\qquad \begin{cases} m_F(\bar{r}) = 1 \\ m_F^{(1)}(\bar{r}) = 0 \\ m_F^{(2)}(\bar{r}) = F^{(2)}(0) \\ m_F^{(3)}(\bar{r}) = 0 \\ m_F^{(4)}(\bar{r}) = F^{(4)}(0) - 4F^{(2)}(0)^3, \end{cases}$$

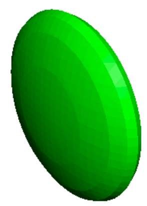
and we can apply (5.8).

5.2. **Ellipsoids of revolution.** Let $\big(E_\varepsilon\big)$ be an ellipsoid of revolution (of parameter $\varepsilon > 0$) given in $\mathbb{R}^3$ by the equation

$$(5.11)\qquad \frac{x^2}{\varepsilon^2} + y^2 + z^2 = 1.$$

In the formalism of the previous subsection, this is the surface defined by rotating the graph of the function $F_\varepsilon : [-\varepsilon,\varepsilon] \to \mathbb{R}_+$ defined by

$$(5.12)\qquad F_\varepsilon(x) = \sqrt{1 - \Big(\frac{x}{\varepsilon}\Big)^2}.$$



Ellipsoid $\big(E_\varepsilon\big)$ with $\varepsilon = 0.29$

Then

$$F_\varepsilon(0) = 1, \quad F_\varepsilon^{(1)}(0) = 0, \quad F_\varepsilon^{(2)}(0) = -\frac{1}{\varepsilon^2}, \quad F_\varepsilon^{(3)}(0) = 0, \quad F_\varepsilon^{(4)}(0) = -\frac{3}{\varepsilon^4}. \tag{5.13}$$

So all the computations in the preceding subsection apply with $m = m_{F_\varepsilon}$. The curvature along $\overline{\gamma}$ is $\overline{k} = 1/\varepsilon^2$, the focalization time along that geodesic is $t = \pi\varepsilon$, and we can compute the various terms in (5.7) for $\overline{t} = \pi\varepsilon/2$:

$$\int_0^{\frac{\pi\varepsilon}{2}} \sin^2(s/\varepsilon)\,\cos^2(s/\varepsilon)\,ds = \frac{\pi\,\varepsilon}{16},$$

$$m^{(4)}(\overline{r})\,m(\overline{r}) - m^{(2)}(\overline{r})^2 = \frac{4}{\varepsilon^4}\left(\frac{1}{\varepsilon^2} - 1\right),$$

$$\begin{cases} \overline{A} = \dfrac{8}{\pi^2\varepsilon^2} \\[2ex] \overline{C} = \dfrac{5}{\varepsilon^2} - \dfrac{24}{\pi^2\varepsilon^2} - \dfrac{1}{2\,\varepsilon^4} + \dfrac{1}{2\,\varepsilon^2} \\[2ex] \overline{E} = \dfrac{2}{\varepsilon^2}. \end{cases} \tag{5.14}$$

Therefore the MTW condition is violated as soon as $-\overline{C} > 2\sqrt{\overline{A}\,\overline{E}}$, or equivalently

$$\frac{1}{2\,\varepsilon^4} - \frac{1}{2\,\varepsilon^2} - \frac{5}{\varepsilon^2} + \frac{24}{\pi^2\varepsilon^2} > \frac{8}{\pi\,\varepsilon^2}.$$

This is equivalent to

$$1 - \Big(11 - \frac{48}{\pi^2} + \frac{16}{\pi}\Big)\varepsilon^2 > 0,$$

which in turn holds for

$$\varepsilon < \frac{1}{\sqrt{11 - \frac{48}{\pi^2} + \frac{16}{\pi}}} \sim 0.2984.$$

Thanks to a classical result of Klingenberg on even-dimensional Riemannian manifolds [20], we know that the injectivity and the conjugate radius coincide. Since along our geodesic the curvature is maximal (this can be easily checked by an explicit computation), we easily deduce that $t_C(\dot{\overline{\gamma}}(0)) = t_F(\dot{\overline{\gamma}}(0))$; in particular $\overline{t} < t_C(\dot{\overline{\gamma}}(0))$. Hence, invoking for instance [28, Theorem 12.44], we deduce an extremely strong negative result as regards the smoothness of optimal transport:

Corollary 5.1. *If E_ε is the ellipsoid of revolution defined by* (5.11) *with $\varepsilon \le 0.29$, then there are C^∞ positive probability densities f, g on E_ε such that the solution T of the optimal transport between $\mu(dx) = f(x)\,\mathrm{vol}\,(dx)$ and $\nu(dy) = g(y)\,\mathrm{vol}\,(dy)$, with transport cost d^2, is discontinuous.*

5.3. **Another counterexample to regularity.** The previous subsection has shown that the MTW condition does not like variations of curvature. In this subsection we shall present another illustration of this phenomenon, by considering two half-balls joined by a cylinder. Set

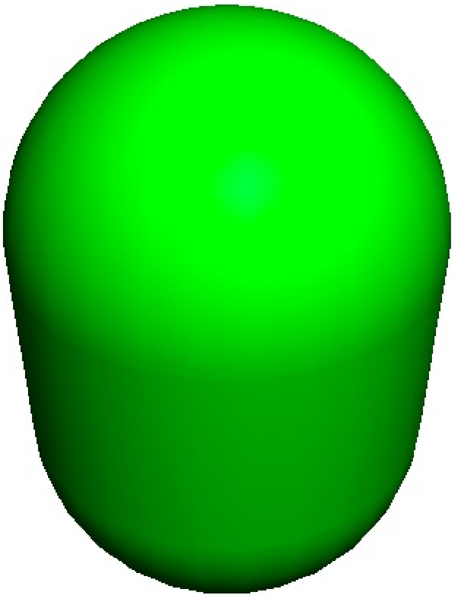
$$\mathcal{C} = \big\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, z \in [-1,0]\big\}\,,$$

$$\mathcal{S}^+ = \big\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, z \ge 0\big\}$$

$$\text{and} \quad \mathcal{S}^- = \big\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + (z+1)^2 = 1, z \le -1\big\}\,.$$

Let us denote by M the cylinder with boundary defined by

$$M = \mathcal{C} \cup \mathcal{S}^+ \cup \mathcal{S}^-.$$



The nonsmooth surface M

This submanifold of $\mathbb{R}^3$ is not C^∞, but it is sufficiently smooth to define an exponential mapping and the concept of regular costs. We denote by d the geodesic distance on M and consider as usual the cost $c = d^2$.

We set

$$A = (0, -1, -1), \quad \text{and} \quad B = (0, -1, 0).$$

If $v = (v_1, 0, v_3)$ is a unit vector in $T_A M$ with $v_3 \neq 0$, then the geodesic γ starting from A with initial speed v is given by

$$\gamma(t) = \left(\cos\left(at - \frac{\pi}{2}\right), \sin\left(at - \frac{\pi}{2}\right), bt - 1\right) \quad \text{if} \quad t \in \left[0, \frac{1}{v_3}\right],$$

$$\text{and} \quad \gamma(t) = \cos\left(t - \frac{1}{v_3}\right)\left(\cos\left(-\frac{\pi}{2} + \frac{v_1}{v_3}\right), \sin\left(-\frac{\pi}{2} + \frac{v_1}{v_3}\right), 0\right) + \sin\left(t - \frac{1}{v_3}\right)\left(-v_1 \sin\left(-\frac{\pi}{2} + \frac{v_1}{v_3}\right), v_1 \cos\left(-\frac{\pi}{2} + \frac{v_1}{v_3}\right), v_3\right)$$

$$\text{for} \quad t \geq \frac{1}{v_3} \quad \text{small enough.}$$

In the sequel, given a unit vector $v = (v_1, 0, v_3) \in T_A M$ and $l > 0$, we denote by $B(v, l)$ the end-point of the geodesic starting from A with initial speed v and of length l. We set

$$\eta = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad V(s) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + s\eta, \quad v(s) = \frac{1}{l(s)} \begin{pmatrix} s \\ 0 \\ 1+s \end{pmatrix}, \quad l(s) = \sqrt{1 + 2s^2 + 2s}.$$

Given $s > 0$, we set

$$B^- = B(v(-s), l(-s)) \quad \text{and} \quad B^+(s) = B(v(s), l(s)).$$

Then we have

$$B^- = \left(\cos\left(v_1(-s)\, l(-s) - \frac{\pi}{2}\right), \sin\left(v_1(-s)\, l(-s) - \frac{\pi}{2}\right), v_3(-s)\, l(-s) - 1\right)$$

$$\text{and} \quad B^+ = \left(\cos\left(l(s) - \frac{1}{v_3(s)}\right)\sin\left(\frac{v_1(s)}{v_3(s)}\right) + v_1(s)\sin\left(l(s) - \frac{1}{v_3(s)}\right)\cos\left(\frac{v_1(s)}{v_3(s)}\right), \right.$$
$$- \cos\left(l(s) - \frac{1}{v_3(s)}\right)\cos\left(\frac{v_1(s)}{v_3(s)}\right) + v_1(s)\sin\left(l(s) - \frac{1}{v_3(s)}\right)\sin\left(\frac{v_1(s)}{v_3(s)}\right),$$
$$\left. \sin\left(l(s) - \frac{1}{v_3(s)}\right) v_3(s)\right).$$

By construction,

$$c(A, B) = 1,$$
$$c(A, B^-) = l(-s)^2 = 1 + 2s^2 - 2s,$$
$$c(A, B^+) = l(s)^2 = 1 + 2s^2 + 2s.$$

Let $X \in M$ be a point given by (θ, z) in cylindrical coordinates, that is

$$X = (\cos(\theta), \sin(\theta), z)\,.$$

Then we have

$$c(X,B) = \Big(\theta + \frac{\pi}{2}\Big)^2 + z^2,$$

$$c(X,B^-) = \Big(\theta + \frac{\pi}{2} - v_1(-s)\, l(-s)\Big)^2 + \Big(z + 1 - v_3(-s)\, l(-s)\Big)^2,$$

Therefore,

$$\Delta = c(A,B) - c(X,B) = 1 - \Big(\theta + \frac{\pi}{2}\Big)^2 - z^2,$$

and

$$\begin{aligned}
\Delta^- &= c(A,B^-) - c(X,B^-) \\
&= l(-s)^2 - \Big(\theta + \frac{\pi}{2} - v_1(-s)\, l(-s)\Big)^2 - \Big(z + 1 - v_3(-s)\, l(-s)\Big)^2 \\
&= \left[1 - \Big(\theta + \frac{\pi}{2}\Big)^2 - z^2\right] + 2s^2 - 2s - v_1(-s)^2\, l(-s)^2 + 2\Big(\theta + \frac{\pi}{2}\Big) v_1(-s)\, l(-s) \\
&\quad - \Big(1 - v_3(-s)\, l(-s)\Big)^2 - 2z\Big(1 - v_3(-s)\, l(-s)\Big) \\
&= \Delta + 2\Big(\theta + \frac{\pi}{2}\Big) v_1(-s)\, l(-s) + 2(1+z)\Big(v_3(-s)\, l(-s) - 1\Big).
\end{aligned}$$

We compute easily

$$\begin{aligned}
l(s) &= 1 + s + s^3 - 7s^4 + 114s^5 + o(s^5) \\
v_1(s) &= s - s^2 + s^3 - 2s^4 + 10s^5 + o(s^5) \\
v_3(s) &= 1 - s^3 + 8s^4 - 8s^5 + o(s^5) \\
v_1(s)/v_3(s) &= s - s^2 + s^3 - s^4 + s^5 + o(s^5) \\
\cos(v_1(s)/v_3(s)) &= 1 - s^2/2 + s^3 - (35/24)s^4 + 11s^5/6 + o(s^5) \\
\sin(v_1(s)/v_3(s)) &= s - s^2 + (5/6)s^3 - s^4/2 + 61s^5/120 + o(s^5) \\
l(s) - 1/v_3(s) &= s + s^4 + 106s^5 + o(s^5) \\
\cos(l(s) - 1/v_3(s)) &= 1 - s^2/2 + s^4/24 - s^5 + o(s^5) \\
\sin(l(s) - 1/v_3(s)) &= s - s^3/6 + s^4 + \Big(106 + \frac{1}{120}\Big)s^5 + o(s^5)
\end{aligned}$$

Let $\alpha(s) > 0$ be such that the point

$$X_\alpha = \Big(\cos\Big(\alpha - \frac{\pi}{2}\Big), \cos\Big(\alpha - \frac{\pi}{2}\Big), -1 - \alpha\Big)$$

has the same first two coordinates as B^+. In this way, we have

$$\cos\left(\alpha - \frac{\pi}{2}\right)$$
$$= \frac{\cos\left(l(s) - \frac{1}{v_3(s)}\right)\sin\left(\frac{v_1(s)}{v_3(s)}\right) + v_1(s)\sin\left(l(s) - \frac{1}{v_3(s)}\right)\cos\left(\frac{v_1(s)}{v_3(s)}\right)}{\sqrt{\cos^2\left(l(s) - \frac{1}{v_3(s)}\right) + v_1(s)^2\sin^2\left(l(s) - \frac{1}{v_3(s)}\right)}}.$$

Also

(5.15) $$c(X_\alpha, B^+) = \left(\left(1+\alpha\right) + \arcsin\left(v_3(s)\sin\left(l(s) - \frac{1}{v_3(s)}\right)\right)\right)^2.$$

Since

$$\alpha(s) = s + s^4/3 + 23s^5/60 + o(s^5),$$

(5.15) implies

$$c(X_\alpha, B^+) = 1 + 4s + 4s^2 + 2s^4/3 + \left(\frac{21}{10} + 228\right)s^5 + o(s^5),$$

and

$$\begin{aligned} \Delta^+ &= c(A, B^+) - c(X_\alpha, B^+) \\ &= -2s - 2s^2 - 2s^4/3 - \left(\frac{21}{10} + 228\right)s^5 + o(s^5). \end{aligned}$$

We check that

$$\Delta = -2s - 2s^2 - 2s^4/3 - \left(\frac{5}{6} + \frac{4}{3}\right)s^5 + o(s^5) > \Delta^+,$$

and

$$\Delta^- = \Delta.$$

Given s small enough and fixed, we can approximate $X_{\alpha(s)}$ by X such that

$$c(A,B) - c(X,B) > \max\left\{c(A,B^-) - c(X,B^-), c(A,B^+) - c(X,B^+)\right\}.$$

This inequality shows that the cost c is not regular [28, Chapter 12]. Now let us regularize M into a smooth surface M'; this can be done in such a way that the Gauss curvature of M' takes values in $[0,1]$. Then by a classical result of Klingenberg [21] we have $t_C \geq \pi$ throughout the unit tangent bundle of M'. Since $d(A,B) = 1 < \pi$, we can include A and X into a region Ω, and B, B^+, B^- in an open set Λ, such that for any $x \in \Omega$, the convex hull of $\log_x \Lambda$ stays away from $\mathrm{TCL}(x)$; and for any $y \in \Lambda$, the convex hull of $\log_y \Omega$ stays away from $\mathrm{TCL}(y)$. For small values of the regularization parameter, the squared distance is not regular $\Omega \times \Lambda \subset M' \times M'$, otherwise we could pass to the limit (as in [29]) to deduce the same property for the limit M. Then we can apply the method of Loeper [22] [28, Proof of Theorem 12.36] to show that M' does not satisfy the MTW condition.

Appendix: On the Riemannian cut locus of surfaces

A.1. **Generalities.** Let M be a smooth, compact, connected n-dimensional Riemannian manifold equipped with a Riemannian metric $\langle \cdot, \cdot \rangle$ and a geodesic distance d. We recall that $d : M \times M \to \mathbb{R}$ is defined by

$$d(x,y)^2 = \inf \left\{ \int_0^1 |\dot\gamma(t)|^2 \, dt \mid \gamma \in \mathrm{Lip}([0,1];M), \gamma(0) = x, \gamma(1) = y \right\}.$$

Let $x \in M$ be fixed, we denote by $d_x = d(x, \cdot)$ the distance to the point x. The function d_x is locally semiconcave on $M \setminus \{x\}$. We denote by Σ_x the singular set of d_x^2 (or equivalently of d_x in $M \setminus \{x\}$), that is

$$\Sigma_x = \big\{ y \in M \ ; \ d_x^2 \text{ not differentiable at } y \big\}.$$

Since the function d_x is locally semiconcave, thanks to Rademacher's Theorem, it is differentiable almost everywhere. For every $y \in M$, we call *limiting gradient* of d_x at y the subset of T_yM defined as

$$D^* d_x(y) = \Big\{ w \in T_yM \ ; \ \exists w_k \to w \text{ s.t. } w_k = \nabla d_x(y_k), y_k \to y \Big\}.$$

For every $y \in M \setminus \{x\}$, there is a one-to-one correspondence between $D^*d_x(y)$ and the set of minimizing geodesics joining x to y: for every $w \in D^*d_x(y)$, there is a minimizing geodesic $\gamma : [0,1] \to \mathbb{R}$ joining x to y such that $\dot\gamma(1) = d(x,y)w$. Then we have

$$\Sigma_x = \Big\{ y \in M; \ \exists v \neq v' \in \mathrm{TCL}(x) \text{ s.t. } \exp_x(v) = \exp_x(v') = y \Big\}.$$

We denote by J_x the set

$$J_x = \big\{ \exp_x(v); \ v \in \mathrm{TCL}(x) \text{ s.t. } d_v \exp_x \text{ is singular} \big\} = \mathrm{fcut}(x).$$

We notice that if $y \in \Sigma_x$ is such that the set $D^*d_x(y)$ is infinite, then it belongs to J_x. The following result holds [1]:

Proposition A.2. *If M is a compact Riemannian manifold, then for any $x \in M$ we have*

$$\mathrm{cut}(x) = \overline{\Sigma}_x = \Sigma_x \cup J_x.$$

In the sequel, we shall say that a point $y \in \mathrm{cut}(x)$ (or by abuse of language that the tangent vector $v = \exp_x^{-1}(y)$ is purely focal) if y does not belong to Σ_x, that is if the function d_x is differentiable at y. We denote by J_x^0 the set of purely focal points. In particular, we have

$$\mathrm{cut}(x) = \overline{\Sigma}_x = \Sigma_x \cup J_x^0.$$

A.2. **Cut loci on surfaces.** Let M be a smooth, compact, connected Riemannian surface equipped with a Riemannian metric $\langle \cdot, \cdot \rangle$ and $x \in M$ be fixed. For every $y \in M$, we call generalized gradient of d_x at y, the convex subset of T_yM defined by

$$\partial d_x(y) = \mathrm{conv}\left(D^* d_x(y)\right).$$

The set $\partial d_x(y)$ being convex, it has dimension 0, 1 or 2. In fact, given $y \neq x$, the function d_x is differentiable at y if and only $\partial d_x(y)$ has dimension 0. We set for $i = 1, 2$,

$$\Sigma_x^i = \Big\{ y \in M \setminus \{x\} \ ; \ \dim(\partial d_x(y)) = i \Big\}.$$

Proposition A.3. *The set Σ_x^2 is discrete, and Σ_x^1 is countably 1-rectifiable.*

We stress that Σ_x^2 is not necessarily a closed set, as it may have accumulation points which do not belong to Σ_x^2. The following proposition on the propagation of singularities will be useful:

Proposition A.4. *Let $y \in \Sigma_x, p_0 \in \partial d_x(y) \setminus D^* d_x(y)$, and $q \in T_yM \setminus \{0\}$ such that*

$$\langle q, p - p_0 \rangle_y \geq 0, \qquad \forall\, p \in \partial d_x(y).$$

Then there exists a Lipschitz arc $y : [0, \varepsilon] \to M$ such that $\dot{y}(0) = q$ and

$$y(s) \in \Sigma_x, \qquad \forall\, s \in [0, \varepsilon].$$

The above two results can be found in [1].

A.3. **On focal velocities in dimension** 2. We assume from now on that $\langle \cdot, \cdot \rangle$ is a given Riemannian metric on a smooth compact surface M. Let $x \in M$ be fixed, we denote by S_x^1 the unit sphere in T_xM, that is

$$S_x^1 = \{ v \in T_xM; \ |v|_x = 1 \}.$$

We define the focal function (at x) and the cut function (at x) by

$$t_F = t_F(x, \cdot) : v \in S_x^1 \longmapsto t_F(x, v) \quad \text{ and } \quad t_C = t_C(x, \cdot) : v \in S_x^1 \longmapsto t_C(x, v)$$

The function t_F is smooth on S_x^1 (its domain), while t_C is Lipschitz. For every pair $v \neq v' \in S_x^1$ with v close to v', we denote by $I(v, v')$ the shortest of the two curves in S_x^1 which joins v to v'.

Lemma A.5. *There is $\epsilon > 0$ such that for every pair $v \neq v' \in S^1_x$ satisfying $|v' - v|_x < \epsilon$ and $\exp_x(t_C(v)v) = \exp_x(t_C(v')v')$, there is $\bar{v} \in I(v, v')$ such that either $t_C(\bar{v})\bar{v}$ is purely focal (that is $D^* d_x(\exp_x(t_C(\bar{v})\bar{v})$ is a singleton), or $\bar{v}$ is focal and $t'_F(\bar{v}) = 0$.*

Proof of Lemma A.5. There is $\varepsilon > 0$ such that for any $v \neq v' \in S^1_x$ with $|v' - v|_x < \varepsilon$, if we denote by γ, γ' the two minimizing geodesics with constant speed $t_C(v) = t_C(v')$ joining x to the point

$$y = \exp_x(t_C(v)v) = \exp_x(t_C(v')v'),$$

then the set $C = \gamma([0,1]) \cup \gamma'([0,1])$ separates M into two connected components. Moreover, we can also assume that for each velocity w in the small interval $I(v, v')$, any point $\exp_x(tw)$ with $t \in (0, t_C(w))$ belongs to the smallest component $\mathcal{O}$. Let $v, v' \in S^1_x$ with $|v' - v|_x < \varepsilon$ be fixed. Denote by $\bar{v}$ a speed in $I(v, v')$ such that

$$t_C(w) \geq t_C(\bar{v}) \qquad \forall\, w \in I(v, v').$$

We claim that either $\bar{v}$ is purely focal, or that $\bar{v}$ is focal and satisfies $t'_F(\bar{v}) = 0$. Indeed, assume that $\bar{v}$ is not purely focal. Set $y = \exp_x(t_C(\bar{v})\bar{v})$, and note that y belongs to $\mathcal{O} \cap \Sigma_x$. By Proposition A.4, there is no $p_0 \in \partial d_x(y) \setminus D^* d_x(y)$ and $q \neq 0$ such that

$$\langle q, p - p_0 \rangle \geq 0, \qquad \forall\, p \in \partial d_x(y),$$

and such that the Lipschitz curve given by Proposition A.4 makes the function d_x strictly decreasing. This means that there is necessarily a non-constant curve $w : [0,1] \mapsto D^* d_x(y)$ such that $w(1) = \bar{w}$, where $\bar{w}$ is the velocity at time $t_C(\bar{v})$ of the minimizing geodesic starting at x with speed $\bar{v}$. For every t close to 1, denote by v_t the speed in S^1_x such that $\exp_x(t_C(v_t)v_t) = \exp_x(t_C(\bar{v}))$ and such that the speed of the minimizing geodesic starting at x with speed v_t has the velocity $w(t)$ at time $s = t_C(\bar{v})$. Any v_t is focal and satisfies $t_F(v_t) = t_F(\bar{v}) = t_C(\bar{v})$. This shows that $\bar{v}$ is focal and that $t'_F(\bar{v})$ has to be zero. □

To our knowledge, the following result and its corollary are new.

Proposition A.6. *Let $v \in S^1_x$ be such that $t_C(v) = t_F(v)$. Then $t'_F(v) = 0$.*

For the proof we shall use the following lemma from [1]. Here, $\cdot$ will denote the Euclidean scalar product in $\mathbb{R}^n$.

Lemma A.7. *Let Ω be an open subset of $\mathbb{R}^n$ and $F : \Omega \to \mathbb{R}^n$ a map of class C^2. Let $\bar{z} \in \Omega$ be such that $DF(\bar{z})$ has rank $n-1$. Set $\bar{y} = F(\bar{z})$, let θ be a generator of*

Ker $DF(\bar{z})$ *and* $\bar{w}$ *be a nonzero vector orthogonal to Im* $DF(\bar{z})$*. Suppose that*

$$\frac{\partial^2 F}{\partial \theta^2}(\bar{z}) \cdot \bar{w} > 0.$$

Then there exist $\rho, \sigma > 0$ *such that the equation*

$$F(z) = \bar{y} + sw, \quad z \in B(\bar{z}, \rho),$$

has no solution if $-\sigma < s < 0$.

Proof of Proposition A.6. Without loss of generality, we may assume that the metric $\langle \cdot, \cdot \rangle$ in T_xM is given by the Euclidean metric. In this way, we can see a speed $v \in S_x^1$ as an angle on $\mathbb{S}^1$. Define $F : \mathbb{R}^2 \to \mathbb{S}^2$ by

$$F(\theta, r) = \exp_x(r \cos\theta, r \sin\theta).$$

(Up to a change of coordinates, we may indeed assume that F is valued in $\mathbb{R}^2$.) By construction of the focal function t_F, for all θ we have

$$\frac{\partial F}{\partial \theta}(\theta, t_F(\theta)) = 0,$$

therefore

$$\frac{\partial^2 F}{\partial \theta^2}(\theta, t_F(\theta)) + t_F'(\theta)\,\frac{\partial^2 F}{\partial \theta\, \partial r}(\theta, t_F(\theta)) = 0$$

Since $\frac{\partial^2 F}{\partial \theta \partial r}(\theta, t_F(\theta))$ never vanishes, we have to show that

$$\frac{\partial^2 F}{\partial \theta^2}(\theta, t_F(\theta)) = 0,$$

for every θ such that $t_C(\theta) = t_F(\theta)$. Let $\bar{\theta}$ be fixed such that $t_C(\bar{\theta}) = t_F(\bar{\theta})$. Argue by contradiction and assume that

$$\frac{\partial^2 F}{\partial \theta^2}(\bar{\theta}, t_F(\bar{\theta})) \neq 0,$$

Set $\bar{z} = (\bar{\theta}, t_C(\bar{\theta}))$ and $\bar{y} = F(\bar{z})$. Two cases may appear.

Case 1: $\bar{y} \in J_x^0$

By Lemma A.7, there is a small ball $B(\bar{z}, \rho)$ such that the equation

$$F(z) = \bar{y} + sw$$

has no solution for small negative s. Therefore, for each $s = -1/k$ with $k \in \mathbb{N}$, there is a minimizing geodesic $\gamma_k : [0, 1] \to M$ joining x to the point $\bar{y} + s_k w$ whose initial speed v_k does not belong to $B(\bar{z}, \rho)$. Taking the limit as $k \to +\infty$, we obtain a minimizing geodesic $\bar{\gamma} : [0,1] \to M$ joining x to $\bar{y}$ with initial speed $\bar{v} \notin B(\bar{z}, \rho)$. This contradicts the fact that $\bar{y}$ is purely focal.

Case 2: $\bar{y} \in J_x \setminus J_x^0$
Denote by V the set of $\theta \in \mathbb{S}^1$ such that $F(\theta, t_C(\theta)) = \bar{y}$. Two cases may appear.

Subcase 2.1: $\bar{\theta}$ is isolated in V.
Thus the minimizing geodesic $\bar{\gamma} : [0,1] \to M$ joining x to $\bar{z}$ is isolated among the set of minimizing geodesics joining x to $\bar{y}$. Therefore, we can modify the Riemannian metric outside a neighborhood of $\bar{\gamma}([0,1])$ in such a way that $\bar{\gamma}$ becomes the unique minimizing geodesic between x and $\bar{y}$. Since the modification of the metric has been done far from $\bar{\gamma}$, the function t_F and the new focal function $\widehat{t_F}$ coincide in a neighborhood of $\bar{\theta}$ which is purely conjugate. By Case 1, we obtain that $t'_F(\bar{\theta}) = 0$.

Subcase 2.2: $\bar{\theta}$ is not isolated in V.
Thus there is a sequence $\{\theta_k\}$ in V converging to $\bar{\theta}$. By Lemma A.5, this yields a sequence $\{\theta'_k\}$ converging to $\bar{\theta}$ such that each θ'_k is either purely focal or such that $t'_F(\theta'_k) = 0$. In any case, thanks to Case 1 above and the continuity of t'_F, one has $t'_F(\bar{\theta}) = 0$. □

We deduce as a consequence of Proposition A.6 an improvement of the classical result $\mathcal{H}^1\left(J_x^0\right) = 0$, as follows:

Corollary A.8. *The set J_x has zero Hausdorff dimension.*

Proof. Consider the smooth map $\Psi : v \in S_x^1 \mapsto \exp(t_F(v)v)$. Any $v \in \mathrm{TFCL}(x)$ is a critical point of Ψ. We conclude by Sard's Theorem. □

Remark A.9. In fact, a generalization of Corollary A.8 holds in any dimension: the set J_x has Hausdorff dimension at most $n-2$; see [12].

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