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# Entanglement entropy of highly degenerate states and fractal dimensions

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We consider the bi-partite entanglement entropy of ground states of extended quantum systems with a large degeneracy due to a spontaneously broken global Lie group symmetry. In general, a ground state is a linear combination of basis elements of a representation space, and for extended systems, these basis elements form a manifold. For instance, the spins of a spin-1/2 representation, pointing in various directions, form a sphere. We show that for subsystems with a large number $m$ of local degrees of freedom, the entanglement entropy diverges logarithmically as $\frac{d}{2}\log m$, where $d$ is the fractal dimension of the sub-variety of the manifold of basis elements occurring in the linear combination. We interpret this result by seeing $d$ as the (not necessarily integer) number of zero-energy Goldstone bosons describing the ground state. We suggest that this result holds in general for largely degenerate ground states, with potential applications to quenched disorder.

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## INTRODUCTION AND DISCUSSION

The entanglement entropy is a measure of entanglement between two complementary sets of observables in a quantum system [1]. It is defined as the von Neumann entropy [2] of the reduced density matrix of the state $|\Psi\rangle$ with respect to a tensor factor of the Hilbert space $\mathcal{H}$:

$$S = -\mathrm{Tr}_{\mathcal{A}}(\rho_A \log \rho_A) \quad \text{with} \quad \rho_A = \mathrm{Tr}_{\mathcal{B}}|\Psi\rangle\langle\Psi|, \qquad (1)$$

and $\mathcal{H} = \mathcal{A}\otimes\mathcal{B}$. A related measure is obtained from the Rényi entropy [3], $S_n = \frac{1}{1-n}\log \mathrm{Tr}_{\mathcal{A}}(\rho_A^n)$; clearly, $S = S_1$.

In extended quantum systems near to critical points, the entanglement entropy has turned out to reveal fundamental properties of ground states. An important result is the so-called area law. Consider a quantum system of dimensionality $D \geq 2$ with correlation length $\xi$, and a subsystem $\mathcal{A}$ composed of the local degrees of freedom on a $D$-dimensional region $A$ of linear extension $\ell$. It turns out that the entanglement entropy between the subsystem and the rest diverges as $\xi$ and $\ell$ increase, the ratio $r = \ell/\xi$ being fixed, with a power law $\ell^{D-1}$ [4, 5]. But this area law is modified in the cases where $D = 1$. There, the divergence is logarithmic: $\frac{qc}{6}\log(\ell)$ where $q$ is the number of points separating $A$ from the rest and $c$ is the central charge of the critical theory [6, 7]. Interestingly, the number $c$ comes out, which essentially measures the number of degrees of freedom that are carried over from the microscopic theory to the macroscopic universal theory (described by a quantum field theory). Further, for $D = 1$ again, subtracting this divergence, the rest is a finite quantity which depends on $r$, which saturates to a finite value at $r = \infty$, and which tends to this value in an exponential way that is solely determined by the spectrum of masses of the corresponding perturbation of the critical point (hence independent of the scattering matrix) [8, 9]. The spectrum of asymptotic particles characterises the low-energy degrees of freedom of the universal theory. Moreover, in systems with a boundary, the boundary degeneracy appears also by a natural substraction [7, 10]. This degeneracy characterises the number of degrees of freedom carried by the boundary. These results point to the observation that if the entanglement entropy diverges logarithmically at large subsystem size $\ell$, then the way it diverges is controlled by some basic counting of universal degrees of freedom.

All these results were established for non-degenerate ground states, or ground states with small, finite degeneracies. A question arises as to the entanglement entropy for highly-degenerate ground states (states lying in a high-dimensional lowest-energy subspace). Although it is not *a priori* clear how to characterise such states in complete generality, important examples are those where a global symmetry group of an extended system is dynamically broken. For clarity, let us discuss the particular example of the Heisenberg ferromagnet: an $N$-site lattice $\mathtt{L}$ (e.g. a $D$-dimensional hypercubic lattice of linear size $N^{1/D}$) where local degrees of freedom are spin-1/2 modules, with Hamiltonian:

$$H = J \sum_{(i,j)\,\in\,\text{edges of }\mathtt{L}} \vec{\sigma}_i \cdot \vec{\sigma}_j, \quad J < 0 \qquad (2)$$

($\vec{\sigma}_i$ is a vector of Pauli matrices acting on site $i$). This model has an $SU(2)$ global symmetry, and the states $|\Psi_{\vec{v}}\rangle = \otimes_{i\in\mathtt{L}}|\psi_{\vec{v}}\rangle_i$, where all spins point in the same direction $\vec{v}$ (that is $\vec{\sigma}_i\cdot\vec{v}|\psi_{\vec{v}}\rangle_i = |\psi_{\vec{v}}\rangle_i$ where $\vec{v}$ is a 3-dimensional unit vector), span the lowest-energy subspace. In the usual description, the ground state is a choice of an arbitrary direction $\vec{v}_0$ (chosen for instance by a small external magnetic field). Such a state is not invariant under $SU(2)$ transformations, hence the symmetry is dynamically broken. The Hilbert space $\mathcal{H}_{\vec{v}_0}$ of excitations in the thermodynamic limit $N\to\infty$ is taken with respect to such a choice of ground state, and excludes all ground states and excited states associated to

other directions, $\mathcal{H}_{\vec{v}}$ for $\vec{v} \neq \vec{v}_0$.

But linear combinations of $|\Psi_{\vec{v}}\rangle$s also give lowest-energy states. The lowest-energy subspace is the $N+1$-dimensional subspace forming a spin-$N/2$ representation, and a basis can be taken as $|\Psi_{\vec{v}_k}\rangle$ for an appropriate set of $N+1$ different points $\vec{v}_k$ on the unit sphere. This basis is appropriate for discussing the infinite-volume limit $N \to \infty$, where the degeneracy becomes infinite. Indeed, in this limit, the basis becomes the set of all points on the unit sphere. One can make good sense of the resulting space of infinite-volume quantum states by using, instead of the mutually orthogonal Hilbert spaces $\mathcal{H}_{\vec{v}}$, the geometry induced by averages of local operators[14]. In this geometry, the distance between states in directions $\vec{v}$ and $\vec{v}'$ is smoothly related to the distances between the vectors $\vec{v}$ and $\vec{v}'$ on the unit sphere, hence the unit sphere is a good representation of the space. This geometry is the correct one for discussing the entanglement entropy, because, as is developed in [11] (based on earlier works [8]), the entanglement entropy can be evaluated from the average of a local (unitary) permutation operator.

Linear combinations can in general involve infinitely many directions $\vec{v}$, with appropriate integration measures on the unit sphere. Such linear combinations may occur in quite natural ways: for instance, when a ground state of the infinite-volume one-dimensional Heisenberg ferromagnet is reached by a certain adiabatic lowering of the anisotropy of the XXZ model, one obtains an integration over a great circle on the unit sphere [12]. Furthermore, imperfections in the way a certain direction $\vec{v}_0$ is physically chosen may lead to linear combinations involving many directions $\vec{v}$ around $\vec{v}_0$. Also, since for $N \to \infty$ the Hilbert spaces $\mathcal{H}_{\vec{v}}$ are orthogonal to each other, from the viewpoint of averages of local operators, linear combinations are equivalent to statistical ensembles, with probabilities given by the modulus squared of the coefficients. The entanglement entropy is then just the usual von Neumann entropy of the ensemble on the subsystem $\mathcal{A}$.

Although each state $|\Psi_{\vec{v}}\rangle$ has zero entanglement entropy (since it is factorisable), linear combinations do not, and linear combinations involving infinitely many directions $\vec{v}$ should have growing entanglement entropy as $\ell \to \infty$. What is the $\ell \to \infty$ behaviour of the entanglement entropy of a ground state of the Heisenberg ferromagnet with such infinite linear combinations?

With the picture of the unit sphere in mind for the basis set, we can get some intuition as to the behaviour of the entanglement entropy. Let $\mathcal{A}$ be composed of $m$ degrees of freedom (and $N = \infty$). Clearly, any minimal-energy state has a symmetry under exchange of any two sites, hence the entanglement entropy depends on $m$ but not on the particular sites chosen. We may take the $m$ sites to form a continuum of dimension $D$, so that we may write $m = \ell^D$. First note that a large-$m$ divergence $\frac{1}{2}\log m$ of the entanglement entropy was found in [11] for the state formed by an integration over a great circle. Second, recall that when there is spontaneous symmetry breaking, there are massless excitations in the spectrum, the Goldstone bosons. In general, our intuition for the form of the divergence comes from the idea that linear combinations composed of all points along an arc on the unit sphere should be interpreted as representing the presence of a zero-energy Goldstone boson corresponding to the continuous motion along this arc. Moreover, every linearly independent local direction on the unit sphere corresponds to a linearly independent Goldstone boson, each of which can be seen as a universal degree of freedom. Hence, the result for the great-circle state just mentioned, and the observations about critical ground states reviewed above, suggest a divergence of the form $\frac{d}{2}\log m$, where $d$ is the number of Goldstone degrees of freedom present in the linear combinations. This number is simply the dimension of the support of the linear combination on the unit sphere. As is the case for the central charge as a measure of a number of degrees of freedom, the number of Goldstone degrees of freedom $d$ is not restricted to the integers: the support of the linear combination may have a fractal (Hausdorff) dimension $0 < d < 2$.

The main point of the paper is to argue that indeed, the large-$m$ (equivalently large-$\ell$) behaviour is exactly, for all $n$,

$$S_n = \frac{d}{2}\log m + O(1) = \frac{dD}{2}\log \ell + O(1) \qquad (3)$$

where $d$ is the fractal dimension of the support of the linear combination. The case of the great-circle state corresponds to $d = 1$. A ground state obtained under the influence of both a small directional magnetic field and a small temperature may correspond to a two-dimensional support for a Maxwell distribution centered around a fixed direction of the spins, hence to $d = 2$. Also, various ratios of strengths of magnetic field and temperature may give rise to sets of directions (again centered around a fixed one) with various densities or fractal dimensions. Hence, it is conceivable that other values of $d$ between 0 and 2 may be physically obtained in this way.

Beyond this simple example, our derivation below makes it clear that this formula hold much more generally. Indeed, it holds whenever it is possible to choose a basis for the lowest-energy subspace where the entanglement entropy of each basis element is zero, and where each basis element lies in a variety (the basis set) whose geometry is in accordance with that induced by averages of local operators. The fractal dimension $d$ in (3) is that of the support of the linear combination, as a subvariety of the basis set. For instance, in systems with a degenerate ground state that breaks a global Lie-group symmetry, the basis set is the manifold describing the representation space of the Lie group (in the Heisenberg ferromagnet, this was the unit sphere). Also, for a quantum field in a potential with degenerate minima, the basis

set is the set of minima, with a geometry induced by that of the target space. The set of minima may be smooth, like for the Mexican-hat potential (a circle), or may be a fractal itself, like for quenched random potentials. The latter case is perhaps the most interesting potential application of formula (3), as it may provide useful additional information in the study of glasses.

In general, it may be impossible to find a basis set for the lowest-energy state where all basis elements have zero entanglement. For instance, this would be the case in a spin-1/2 system if the lowest-energy subspace were a spin-$k$ representation with $k < N/2$. If the entanglement stays finite at infinite volume for every basis element, we expect formula (3) to still hold. Otherwise, we may expect additional terms in (3) representing the entanglement of the basis elements themselves. These extra terms may be dominant in higher dimensions, and may be determined in accordance with the known results for critical systems reviewed above. That is, we expect to have, in general, two contributions to the asymptotic form of the entanglement entropy of highly degenerate ground states: that of the large degeneracy, and that coming from the local interaction. Formula (3) describes the large-degeneracy contribution.

In the rest of this paper, we provide the main lines of the derivation of (3). A more precise proof and statement will be given in a separate work [12].

## SKETCH OF THE DERIVATION FOR THE HEISENBERG FERROMAGNET

Let us consider the Heisenberg ferromagnet (2), a model with a highly degenerated ground state.

Given a point $\vec{v}$ on the unit sphere $S^2$, let us denote by $|\psi_{\vec{v}}\rangle^{(N)} = \otimes_{i=1}^{N} |\psi_{\vec{v}}\rangle_i$ the normalised vector where all spins point in the direction $\vec{v}$, and by $\psi_{\vec{v}} \in \mathcal{F}$ the quantum state corresponding to its infinite-length ($N \to \infty$) limit. As developed in [11], a quantum state is a linear functional on the space of finitely-supported operators, which evaluates the average; e.g. $\psi_{\vec{v}}(\mathcal{O}) = \lim_{N\to\infty} {}^{(N)}\langle\psi_{\vec{v}}|\mathcal{O}|\psi_{\vec{v}}\rangle^{(N)}$. We can write $\psi_{\vec{v}}$ as a product of single-site quantum states, all acting in the same way:

$$\psi_{\vec{v}} = \bigotimes_{i\in\mathbb{Z}} \psi_{\vec{v};i}. \tag{4}$$

For chains of infinite length, vectors pointing in different directions have zero overlap:

$$\lim_{N\to\infty} {}^{(N)}\langle\psi_{\vec{v}}|\psi_{\vec{v'}}\rangle^{(N)} = 0$$

for $\vec{v} \neq \vec{v'}$. This holds as well with insertions of finitely-supported operators, so the infinite-length limit of linear combinations $\sum_{\vec{v}} a_{\vec{v}} |\psi_{\vec{v}}\rangle^{(N)}$ gives the quantum state

$$\psi_{\{a_{\vec{v}}\}} := \sum_{\vec{v}} |a_{\vec{v}}|^2 \psi_{\vec{v}}. \tag{5}$$

Note that we require $\sum_{\vec{v}} |a_{\vec{v}}|^2 = 1$, so that this has the correct normalisation.

In order to evaluate the Rényi entanglement entropy associated to the ground state $\psi_{\{a_{\vec{v}}\}}$ we recall the approach developed in [11]. There, the Rényi entropy of a region $A$ in a quantum state $\psi$ was expressed as an average on the $n^{\text{th}}$ tensor power of $\psi$:

$$S_n = \frac{1}{1-n} \log\left(\psi^{\otimes n}(\mathcal{T}_A)\right). \tag{6}$$

The operator averaged is $\mathcal{T}_A = \prod_{i\in A} \mathcal{T}_i$, where $\mathcal{T}_i$ are *local cyclic replica permutation operators* which act on site $i$ of the quantum spin chain by cyclicly permuting the spins of the $n$ replicas of the model at that particular site. One of the results of [11] was the closed formula

$$\mathcal{T}_i = \mathrm{Tr}_{\text{aux}} \prod_{\alpha=1}^{n} \sum_{\epsilon_1,\epsilon_2} E_{\text{aux}}^{\epsilon_1\epsilon_2} E_{\alpha,i}^{\epsilon_2\epsilon_1}, \tag{7}$$

where $E_V^{\epsilon_2\epsilon_1}$ represent elementary $2\times 2$ matrices with a single non-vanishing entry at row $\epsilon_2$, column $\epsilon_1$, acting on space $V = \alpha, i$ (site $i$ tensor copy $\alpha$) or $V = \text{aux}$ (auxiliary space).

It is now a simple matter to evaluate the Rényi entropy associated to the quantum state $\psi_{\{a_{\vec{v}}\}}$. We write

$$\psi_{\{a_{\vec{v}}\}}^{\otimes n} = \sum_{\{\vec{v}_\alpha:\alpha=1,\ldots,n\}} \left(\prod_{\alpha=1}^{n} |a_{\vec{v}_\alpha}|^2\right) \bigotimes_{\alpha=1}^{n} \psi_{\vec{v}_\alpha}$$

and use (4). From the trace expression (7) for the permutation operators we find

$$\bigotimes_{\alpha} \psi_{\vec{v}_\alpha}(\mathcal{T}_A) = \prod_{i\in A} \mathrm{Tr}_{\text{aux}} \prod_{\alpha} \sum_{\epsilon_1,\epsilon_2} E_{\text{aux}}^{\epsilon_1\epsilon_2} \psi_{\vec{v}_\alpha;i}(E_i^{\epsilon_2\epsilon_1}). \tag{8}$$

Clearly, $\psi_{\vec{v}_\alpha;i}(E_i^{\epsilon_2\epsilon_1})$ is independent of $i$. Writing $|\psi_{\vec{v}}\rangle = s_{\vec{v},1}|\uparrow\rangle + s_{\vec{v},2}|\downarrow\rangle$, we find $\psi_{\vec{v}_\alpha,i}\left(E_{\alpha,i}^{\epsilon_2\epsilon_1}\right) = s^*_{\vec{v}_\alpha,\epsilon_2} s_{\vec{v}_\alpha,\epsilon_1}$, and tracing over the auxiliary space we obtain

$$\mathrm{Tr}_{\text{aux}} \prod_{\alpha} \sum_{\epsilon_1,\epsilon_2} E_{\text{aux}}^{\epsilon_1\epsilon_2} \psi_{\vec{v}_\alpha;i}(E_i^{\epsilon_2\epsilon_1}) = \prod_{\alpha} \langle\psi_{\vec{v}_\alpha}|\psi_{\vec{v}_{\alpha+1}}\rangle.$$

Hence, we find

$$S_n = \frac{1}{1-n} \log\left(\sum_{\{\vec{v}_\alpha\}} \left[\prod_{\alpha} |a_{\vec{v}_\alpha}|^2\right] \left[\prod_{\alpha} \langle\psi_{\vec{v}_\alpha}|\psi_{\vec{v}_{\alpha+1}}\rangle\right]^m\right) \tag{9}$$

where $m$ is the number of sites in $A$ (and with $\vec{v}_{n+1} := \vec{v}_1$). It saturates at large $m$ to

$$\lim_{m\to\infty} S_n = \frac{1}{1-n} \log\left(\sum_{\vec{v}} |a_{\vec{v}}|^{2n}\right). \tag{10}$$

The expression above shows that for any ground state given by a finite linear combination of basic zero entropy

states, the entanglement entropy reaches a finite maximum as $m \to \infty$.

More interesting behaviours are obtained from "infinite linear combinations" of basic states, generalising (5). For instance, given a smooth, self-avoiding path $\vec{\gamma} : [0,1] \to S^2$ on the unit sphere and a smooth function $f : S^2 \to \mathbb{R}^+$ with $\int_0^1 dt\, f(\vec{\gamma}(t)) = 1$, the following integral of quantum states can be defined and is itself a quantum state (a ground state):

$$\psi^{(1)} := \int_0^1 dt\, f(\vec{\gamma}(t))\, \psi_{\vec{\gamma}(t)}. \tag{11}$$

Further, we can parametrize points on a surface on $S^2$ by using a pair of coordinates $\lambda \in [0,1]$ and $\phi \in [0,1]$. Let $\vec{\mu} : [0,1] \times [0,1] \to S^2$ be a (two-dimensional) smooth curve such that $\int_0^1 \int_0^1 d\lambda d\phi f(\vec{\mu}(\lambda,\phi)) = 1$. Then, the following is a ground state:

$$\psi^{(2)} := \int_0^1 \int_0^1 d\lambda d\phi\, f(\vec{\mu}(\lambda,\phi))\, \psi_{\vec{\mu}(\lambda,\phi)}. \tag{12}$$

Generalising, we may consider a situation where the set of non-zero coefficients in the infinite linear combination is a fractal set $\mathcal{W} \subset S^2$ with fractal dimension [13] $d$. With $d\mathcal{H}(\vec{v})$ the corresponding Hausdorff integration measure, we may write

$$\psi^{(d)} := \int_{\mathcal{W}} d\mathcal{H}(\vec{v})\, f(\vec{v})\, \psi_{\vec{v}} \;\text{ with }\; \int_{\mathcal{W}} d\mathcal{H}(\vec{v})\, f(\vec{v}) = 1. \tag{13}$$

The Hausdorff measure is expected to occur naturally in taking the large-volume limit of the chain, if the set of vectors $\vec{v}$ such that $a_{\vec{v}} \neq 0$ becomes a fractal set.

Computing the Rényi entropy of $\psi^{(1)}, \psi^{(2)}$ or in general $\psi^{(d)}$ yields a simple generalisation of (9) where the sums over the vectors $\vec{v}_\alpha$ are replaced by integrations and the coefficients $|a_{\vec{v}_\alpha}|^2$ by the functions $f(\vec{v}_\alpha)$. The logarithmic factor in (9) becomes (in the general case)

$$\log\left(\int_{\mathcal{W}} \prod_\alpha d\mathcal{H}(\vec{v}_\alpha)\, f(\vec{v}_\alpha) \left(\prod_\alpha \langle\psi_{\vec{v}_\alpha}|\psi_{\vec{v}_{\alpha+1}}\rangle\right)^m\right). \tag{14}$$

The large $m$ asymptotics of these expressions will however be radically different from (10) in that there will be no saturation. Instead, we will recover the behaviour highlighted in (3), with $d = 1, 2$ or the fractal dimension, respectively. This can be shown using a saddle-point analysis, as was done in [11] in the particular case considered there, but here, in general, generalised to integrals over fractal domains.

The explicit calculations use the following parametrisation. Let $\vec{v} =: (x, y, z)$, and $\theta$ be the angle around the $z$ axis starting at $x = 0$ on the $y = 0$ plane; that is, $x + iy = \sqrt{1-z^2}\, e^{i\theta}$ if $\vec{v}$ is a unit vector. Then, we may take $|\psi_{\vec{v}}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1+z} \\ \sqrt{1-z}\, e^{i\theta} \end{pmatrix}$. One can see explicitly from this that $|\langle\psi_{\vec{v}}|\psi_{\vec{w}}\rangle| \leq 1$, with equality if and only if $\vec{v} = \vec{w}$. The saddle-point analysis from (14) is done by expanding the overlaps $\langle\psi_{\vec{v}_\alpha}|\psi_{\vec{v}_{\alpha+1}}\rangle$ around $\vec{v}_\alpha = \vec{v}_{\alpha+1}$, and re-writing the product $\prod_\alpha$ of these overlaps as an exponential where in the exponent only the leading term is kept (this leading term is vanishing because the maximum of the overlaps is 1). We get

$$\prod_{\alpha=1}^n \langle\psi_{\vec{v}_\alpha}|\psi_{\vec{v}_{\alpha+1}}\rangle = \exp\left[-\frac{1}{8}\sum_{\alpha=1}^n |\vec{v}_{\alpha+1} - \vec{v}_\alpha|^2 + \dots\right], \tag{15}$$

where the ellipsis stand for terms that are order-2 and antisymmetric, and higher order terms. The order-2 antisymmetric terms vanish when the integrations in (14) are performed.

In the case of $\psi^{(1)}$ for instance, we may use the assumptions relating to $f$ and $\gamma$: both are smooth, and the curve $\gamma$ is self-avoiding. Hence, with $\vec{v}_\alpha = \vec{\gamma}(t_\alpha)$, the maximum occurs when $t_\alpha = t_{\alpha+1}$ for all $\alpha = 1, \dots, n$. In this case we find

$$\prod_{\alpha=1}^n \langle\psi_{\vec{\gamma}(t_\alpha)}|\psi_{\vec{\gamma}(t_{\alpha+1})}\rangle = \exp\left[-\frac{|\dot{\vec{\gamma}}|^2}{8}\sum_{\alpha=1}^n (t_{\alpha+1} - t_\alpha)^2 + \dots\right], \tag{16}$$

where $|\dot{\vec{\gamma}}|^2$ is evaluated at $t = t_1$. The saddle-point analysis is then performed as follows. We need to raise the quantity above to the power $m$ and substitute into the integral (14). We can replace $f(t_\alpha)$ by $f(t_1)$ for all $\alpha$, since $f$ is smooth. Changing variables to $t_1 = t_1$ and $\hat{t}_i = \sqrt{m}(t_i - t_1)$ with $i = 2, \dots, n$ guarantees that larger positive powers of $\hat{t}_\alpha$ give lower-order contributions at large $m$. We obtain

$$S_n = \frac{1}{1-n}\log\left(\frac{1}{m^{\frac{n-1}{2}}}\int_0^1 dt_1 f(\vec{\gamma}(t_1))^n \int_{-\infty}^{\infty} d^{n-1}\hat{t}\, e^{-\frac{|\dot{\vec{\gamma}}|^2}{8}\left[\sum_{\alpha=2}^{n-1}(\hat{t}_{\alpha+1}-\hat{t}_\alpha)^2 + \hat{t}_2^2 + \hat{t}_n^2\right] + O(\hat{t}^3/\sqrt{m})}\right), \tag{17}$$

where the integrals over $\hat{t}_2, \dots, \hat{t}_n$ have been extended to $(-\infty, \infty)$ (the resulting correction terms are exponentially small). These integrals are of standard gaussian type and can be carried out explicitly. The final result is

$$S_n \sim \frac{1}{2}\log\left(\frac{m}{8\pi}\right) + \frac{1}{1-n}\log\left(\sqrt{n}\int_0^1 dt\, \frac{f(\vec{\gamma}(t))^n}{|\dot{\vec{\gamma}}(t)|^{n-1}}\right), \tag{18}$$

where higher order corrections would be $O(m^{-1/2})$.

A very similar computation can be carried out for the state $\psi^{(2)}$ leading to the following aysmptotics:

$$S_n \sim \log\left(\frac{m}{8\pi}\right) + \frac{1}{1-n}\log\left(n\int_0^1 d\lambda d\phi\, \frac{g(\vec{\mu}(\lambda,\phi))^n}{w(\lambda,\phi)^{n-1}}\right), \tag{19}$$

with $w(\lambda,\phi) := \sqrt{|\vec{\mu}|_\lambda^2 |\vec{\mu}|_\phi^2 - (\vec{\mu}_\lambda \cdot \vec{\mu}_\phi)^2}$. Both (18) and (19) are special cases of (3) with $d = 1, 2$. The functions

$|\dot{\vec{\gamma}}(t)|$ and $w(\lambda,\phi)$ are nothing but the element of volume in one and two dimensions, guaranteeing invariance under reparametrizations of the curves $\vec{\gamma}(t)$ and $\vec{\mu}(\lambda,\phi)$, respectively.

The calculation for fractal sets follows similar lines. A crucial feature of the Hausdorff measure is its scaling property; for the measure $\mathcal{H}'$ on the plane, this can be expressed in general as

$$s^d\,\mathcal{H}'(\mathcal{W}) = \mathcal{H}'(s\mathcal{W}+\vec{u}). \tag{20}$$

That is, for a set $\mathcal{W}$, scaling by a factor $s$ and translating by $\vec{u}$ gives a scaling of the Hausdorff measure by a factor $s^d$. For the measure $\mathcal{H}$ on $S^2$, the equivalent is an asymptotic behaviour, leading to the measure $\mathcal{H}'$ on the tangent plane; using

$$\lim_{m\to\infty} m^{d/2}\,d\mathcal{H}(\hat{\vec{v}}_i/\sqrt{m}+\vec{v}_1) = d\mathcal{H}'(\hat{\vec{v}}_i) \tag{21}$$

we find

$$S_n \;\sim\; \frac{1}{1-n}\log\left(\frac{1}{m^{\frac{d(n-1)}{2}}}\int_{\mathcal{W}} d\mathcal{H}(\vec{v}_1)f(\vec{v}_1)^n \right. \tag{22}$$
$$\left. \int_{\mathcal{W}'} d^{n-1}\mathcal{H}'(\hat{\vec{v}})\, e^{-\frac{1}{8}\left[\sum_{\alpha=2}^{n-1}|\hat{\vec{v}}_{\alpha+1}-\hat{\vec{v}}_\alpha|^2+|\hat{\vec{v}}_2|^2+|\hat{\vec{v}}_n|^2\right]+\ldots}\right)$$

where $\mathcal{W}' = \sqrt{m}\mathcal{W}-\vec{v}_1$ (for $m$ large). Although we cannot evaluate the integral in general, and its large-$m$ limit may also not exist, we know that it is bounded. This boundedness is enough to establish the leading asymptotics (3). It would be desirable to investigate more precisely the nature of the constant corrections to this leading behaviour; we hope to return to this in a future work.


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