Lorentzian dynamics and groups of circle diffeomorphisms

Daniel Monclair

THÈSE

en vue de l'obtention du grade de

Docteur de l'Université de Lyon
délivré par l'École Normale Supérieure de Lyon

Discipline : Mathématiques

Unité de Mathématiques Pures et Appliquées

École doctorale InfoMaths

présentée et soutenue publiquement le 30 juin 2014

par M. Daniel MONCLAIR

Dynamique lorentzienne et groupes de difféomorphismes du cercle

---

Directeur de thèse :	M. Abdelghani ZEGHIB
Après avis de :	M. Thierry BARBOT M. Bertrand DEROIN M. Miguel SÁNCHEZ

Devant la commission d'examen formée de :

M. François BÉGUIN,	*membre*
M. Bertrand DEROIN,	*rapporteur*
M. Etienne GHYS,	*membre*
M. Marc HERZLICH,	*membre*
M. Jean-Marc SCHLENKER,	*membre*
M. Abdelghani ZEGHIB,	*directeur*

Contents

Introduction

Cette thèse comporte deux parties indépendantes qui partagent néanmoins deux fils directeurs : la géométrie lorentzienne et les systèmes dynamiques.

La géométrie lorentzienne est le cadre mathématique de la théorie de la relativité générale. Elle permet d'exprimer l'idée d'Einstein selon laquelle la gravité est la conséquence de la géométrie de l'espace-temps.

La principale différence entre les deux parties de cette thèse réside dans la définition que l'on attribue à "système dynamique".

Dans la première partie, un système dynamique est une action de groupe. Dans le cadre lorentzien, il s'agit d'actions isométriques sur un espace-temps.

Dans la seconde partie, plutôt que d'étudier des systèmes dynamiques sur une variété lorentzienne, nous verrons un espace-temps comme une forme de dynamique que l'on appelle dynamique multi-valuée : au lieu d'envoyer un point sur un autre point, on envoie un point sur un ensemble (son futur causal).

1. Groupes d'isométries des surfaces lorentziennes et dynamique du cercle

Rappelons qu'une **variété lorentzienne** (M, g) est la donnée d'un champ lisse de formes bilinéaires symétriques non dégénérées de signature $(-, +, \cdots, +)$ sur une variété différentielle M. Lorsque la régularité n'est pas précisée, on supposera toujours que g est de classe C^2. On dit qu'un vecteur tangent $v \in T_xM \setminus \{0\}$ est de **type temps** (resp. de **type lumière**, **causal**) si $g_x(v, v) < 0$ (resp. $g_x(v, v) = 0$, $g_x(v, v) \leq 0$). Un **espace-temps** est une variété lorentzienne munie d'une orientation temporelle, i.e. un choix continu d'une composante connexe du cône $g < 0$ (ce qui est équivalent à la donnée d'un champ de vecteurs continu partout de type temps). Un vecteur causal est dit **orienté vers le futur** s'il est dans la composante connexe du cône isotrope déterminée par l'orientation temporelle. Une courbe est dite de type temps (resp. de type lumière, causale, orientée vers le futur) si son vecteur dérivé est de type temps (resp. de type lumière, causal, orienté vers le futur) en tout point.

Le point de départ de la première partie de cette thèse est l'étude des variétés lorentziennes dont le groupe d'isométries a une dynamique riche. La dynamique du groupe d'isométries est une des nombreuses différences entre géométries lorentzienne et riemannienne. Plus précisément, les isométries lorentziennes peuvent agir non proprement. On dit que l'action d'un groupe G sur un espace X est **propre** si l'ensemble $G_K = \{g \in G | g(K) \cap K \neq \emptyset\}$ est compact pour tout compact $K \subset X$. En particulier, si un groupe agit proprement, alors les stabilisateurs sont compacts. Sur une variété riemannienne, le groupe d'isométries agit toujours proprement (l'action est équicontinue car elle préserve la distance associée). Il s'agit même d'une équivalence : si M est une variété différentielle, un groupe de difféomorphismes agit proprement sur M si et seulement si son action préserve une métrique riemannienne.

Dans l'espace de Minkowski $\mathbb{R}^{1,n-1}$ (i.e. $\mathbb{R}^n$ muni de la forme quadratique de signature $(-, +, \cdots, +)$), les stabilisateurs sont isomorphes à $\mathrm{O}(1, n-1)$ qui n'est pas compact, d'où la non propreté de l'action.

Cependant, les géométries présentant beaucoup de symétries sont rares (par exemple, une métrique riemannienne générique n'a aucune isométrie locale, voir **[Sun85]**), et la conjecture vague de Gromov (que l'on trouve dans [**D'AG91**]) suggère que l'on peut les classifier. Plusieurs résultats de classification d'actions non propres sur des variétés lorentziennes existent, en particulier pour les variétés compactes ([**AS97a**], [**AS97b**], [**Zeg99a**] et [**Zeg99b**]). Toutefois, l'étude des variétés lorentziennes ne peut pas se réduire au cas compact, puisqu'un espace-temps compact n'est jamais causal (il existe toujours des courbes fermées de type temps), et perd donc en intérêt d'un point de vue physique. Il existe aussi une classification des groupes de Lie simples agissant sur des variétés lorentziennes non compactes (voir [**Kow96**], et [**DMZ08**] pour l'extension au cas semi-simple).

La motivation du travail présenté dans cette thèse est d'obtenir une classification en partant d'hypothèses sur l'espace-temps plutôt que sur le groupe. Les espace-temps considérés en physique sont globalement hyperboliques, ce qui est plus ou moins équivalent au fait que l'espace des géodésiques isotropes a une structure de variété lisse. Nous posons donc la question suivante : peut-on classifier les espace-temps globalement hyperboliques sur lesquels le groupe d'isométries agit non proprement ?

Une **hypersurface de Cauchy** dans un espace-temps (M, g) est une hypersurface topologique $S \subset M$ qui intersecte toute courbe causale inextensible en un unique point. On dit que (M, g) est **globalement hyperbolique** s'il possède une hypersurface de Cauchy. Si c'est le cas, alors il existe des hypersurfaces de Cauchy lisses (voir [**BS03**]), et elles sont toutes difféomorphes entre elles. Un espace-temps est dit **spatialement compact** (ou Cauchy-compact) s'il est globalement hyperbolique et ses hypersurfaces de Cauchy sont compactes. Pour simplifier le problème, nous ne considèrerons que des espace-temps spatialement compacts.

Lorsque l'on travaille sur un problème de classification, il est d'usage de commencer par les petites dimensions. En géométrie lorentzienne, la plus petite dimension possible est 2 (une d'espace, une de temps). Tous les travaux présentés ici concernent les surfaces lorentziennes spatialement compactes.

Le premier exemple est l'espace de Sitter, i.e. l'hyperboloïde à une nappe $x^2 + y^2 - z^2 = 1$ dans $\mathbb{R}^3$, muni de la restriction de la forme quadratique $dx^2 + dy^2 - dz^2$. Le groupe d'isométries est $\mathrm{SO}^\circ(1, 2) \approx \mathrm{PSL}(2, \mathbb{R})$ (on ne considère que les isométries qui préservent l'orientation et l'orientation temporelle). L'espace dS_2 étant difféomorphe au cylindre $\mathbb{R} \times \mathbb{S}^1$, il admet un unique revêtement dS_2^k d'odre k pour tout $k \in \mathbb{N}$. Le groupe d'isométries de dS_2^k est le revêtement $\mathrm{PSL}_k(2, \mathbb{R})$ de $\mathrm{PSL}(2, \mathbb{R})$. Un des principaux résultats de cette thèse est le fait que l'étude des surfaces spatialement compactes se ramène toujours à ces groupes :

Théorème 1.1. *Soit* (M, g) *une surface lorentzienne spatialement compacte. Si le groupe d'isométries agit non proprement sur* M*, alors* $\mathrm{Isom}(M, g)$ *est isomorphe à un sous-groupe d'un revêtement fini de* $\mathrm{PSL}(2, \mathbb{R})$.

Ici $\mathrm{Isom}(M, g)$ est le groupe des isométries qui préservent l'orientation de M et l'orientation temporelle. Cet isomorphisme se trouve à travers l'étude d'une dynamique unidimensionnelle.

1.1. Actions sur le cercle et modèles conformes. L'espace des géodésiques de type lumière d'un espace-temps globalement hyperbolique (M, g) s'identifie au fibré unitaire tangent $\mathrm{T}^1 S$ d'une hypersurface de Cauchy $S \subset M$ (car les géodésiques isotropes sont des courbes causales inextensibles).

Pour une surface spatialement compacte (M, g), cela signifie que l'espace des géodésiques de type lumière est $\mathrm{T}^1\mathbb{S}^1$, i.e. la réunion disjointe de deux cercles. Le groupe conforme agit sur cet espace, ce qui induit deux représentations $\rho_1^M, \rho_2^M : \mathrm{Conf}(M, g) \to \mathrm{Diff}(\mathbb{S}^1)$, où $\mathrm{Conf}(M, g)$ est le groupe des difféomorphismes conformes qui préservent l'orientation de M et l'orientation temporelle.

Bien qu'elles soient définies sur le groupe conforme, nous utiliserons ces représentations pour décrire le groupe d'isométries. L'outil principal sera la propriété de convergence. Il s'agit d'une description très simple en termes de dynamique des sous-groupes de $\mathrm{Homeo}(\mathbb{S}^1)$ qui sont conjugués à un sous-groupe de $\mathrm{PSL}(2,\mathbb{R})$ (que nous définirons page 9).

La géométrie pseudo-riemannienne en dimension deux se distingue par le fait que toutes les surfaces sont conformément plates. Pour les surfaces lorentziennes, il s'agit d'une simple observation, et c'est un théorème de Gauss pour les surfaces riemanniennes. Nous allons voir que l'on peut trouver un modèle conforme global pour les surfaces spatialement compactes.

Nous appellerons **tore lorentzien plat** le produit $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ muni de la métrique $dxdy$ (qui est conforme à l'espace d'Einstein de dimension deux $(\mathbb{S}^1 \times \mathbb{S}^1, dx^2 - dt^2)$), et **cylindre lorentzien plat** le quotient du plan de Minkowski $(\mathbb{R}^2, dxdy)$ par une translation de type espace (à équivalence conforme près, le quotient ne dépend pas de la translation).

Théorème 1.2. *Soit (M,g) une surface spatialement compacte. Il existe une immersion conforme $p : (M,g) \to (\mathbb{T}^2, dxdy)$ équivariante sous les actions de $\mathrm{Conf}(M,g)$:*

$$\forall \varphi \in \mathrm{Conf}(M,g)\ \forall x \in M \quad p(\varphi(x)) = (\rho_1^M(\varphi)(p_1(x)), \rho_2^M(\varphi)(p_2(x)))$$

Le principal défaut de ce modèle conforme est que cette immersion n'est pas toujours injective. L'application p est injective si et seulement si les deux géodésiques isotropes issues d'un même point ne peuvent jamais se recroiser (autrement dit, il n'existe pas de points conjugués le long de géodésiques de type lumière). Nous utiliserons un second modèle qui donne toujours un plongement.

Théorème 1.3. *Toute surface spatialement compacte se plonge conformément dans le cylindre lorentzien plat.*

Nous utiliserons principalement ce plongement pour montrer que l'étude du groupe d'isométries peut se ramener au cas où l'immersion dans le tore lorentzien plat est injective.

Ces résultats, dont les preuves sont élémentaires, se trouvent dans le chapitre 1 (Proposition 1.3.1 et Théorème 1.4.1).

1.2. Classification des groupes d'isométries. Nous reprenons ici les conventions de [**Ghy87b**] et disons que deux représentations $\rho_1, \rho_2 : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ sont **semi conjuguées** s'il existe une application $h : \mathbb{S}^1 \to \mathbb{S}^1$ non constante et croissante de degré un (i.e. qui se relève en une application croissante $\widetilde{h} : \mathbb{R} \to \mathbb{R}$ telle que $\widetilde{h}(x+1) = \widetilde{h}(x)+1$ pour tout $x \in \mathbb{R}$) telle que $h \circ \rho_1(\gamma) = \rho_2(\gamma) \circ h$ pour tout $\gamma \in \Gamma$. Il est important de remarquer que h n'est pas nécessairement continue.

Le résultat général que l'on obtient est le suivant :

Théorème 1.4. *Soit (M,g) une surface spatialement compacte, et supposons que $\mathrm{Isom}(M,g)$ agit non proprement sur M. Alors ρ_1^M et ρ_2^M sont semi conjuguées, et leurs restrictions à $\mathrm{Isom}(M,g)$ sont fidèles. On peut trouver $k \in \mathbb{N}$ et une représentation fidèle $\rho : \mathrm{Isom}(M,g) \to \mathrm{PSL}_k(2,\mathbb{R})$ qui est semi conjuguée aux restrictions de ρ_1^M et ρ_2^M à $\mathrm{Isom}(M,g)$.*

Bien que général, ce résultat demeure insatisfaisant. Plusieurs informations sur la dynamique ne sont pas transmises par semi conjugaison, telles que la densité des orbites ou le nombre d'orbites périodiques. Sous une hypothèse supplémentaire, on obtient une **conjugaison topologique**, i.e. une conjugaison dans le groupe $\mathrm{Homeo}(\mathbb{S}^1)$.

On dit que le bord conforme de (M, g) est **acausal** si le bord de l'image dans le cylindre lorentzien plat ne contient aucun segment de géodésique isotrope.

Théorème 1.5. *Soit* (M, g) *une surface spatialement compacte, et supposons que* $\mathrm{Isom}(M, g)$ *agit non proprement sur* M. *Supposons de plus que le bord conforme est acausal. Alors* ρ_1^M *et* ρ_2^M *sont fidèles, topologiquement conjuguées entre elles, et leurs restrictions à* $\mathrm{Isom}(M, g)$ *sont topologiquement conjuguées à une représentation dans un revêtement fini de* $\mathrm{PSL}(2, \mathbb{R})$.

La non propreté de l'action du groupe d'isométries n'intervient pas dans l'étude de la dynamique unidimensionnelle, mais plutôt pour s'assurer que l'on peut utiliser le bon modèle conforme. Pour une surface spatialement compacte qui se plonge conformément dans le tore lorentzien plat, on peut retirer cette hypothèse.

Théorème 1.6. *Soit* (M, g) *une surface spatialement compacte qui se plonge conformément dans le tore lorentzien plat. Les représentations* ρ_1^M *et* ρ_2^M *sont semi conjuguées, et leurs restrictions à* $\mathrm{Isom}(M, g)$ *sont fidèles. On peut trouver* $k \in \mathbb{N}$ *et une représentation fidèle* $\rho : \mathrm{Isom}(M, g) \to \mathrm{PSL}_k(2, \mathbb{R})$ *qui est semi conjuguée aux restrictions de* ρ_1^M *et* ρ_2^M *à* $\mathrm{Isom}(M, g)$.

Théorème 1.7. *Soit* (M, g) *une surface spatialement compacte qui se plonge conformément dans le tore lorentzien plat. Si le bord conforme est acausal, alors* ρ_1^M *et* ρ_2^M *sont fidèles, topologiquement conjuguées entre elles , et leurs restrictions à* $\mathrm{Isom}(M, g)$ *sont topologiquement conjuguées à une représentation dans un revêtement fini de* $\mathrm{PSL}(2, \mathbb{R})$.

Le chapitre 3 est dédié aux preuves de ces quatre énoncés (Théorème 3.1.2, Théorème 3.1.3, Théorème 3.1.4 et Théorème 3.1.5).

1.3. Groupes de convergence. On dit qu'une suite $(f_n)_{n\in\mathbb{N}} \in \mathrm{Homeo}(\mathbb{S}^1)^{\mathbb{N}}$ a la **propriété de convergence** si l'on peut trouver deux points $a, b \in \mathbb{S}^1$ tels que $f_n(x) \to b$ pour tout $x \neq a$. Un sous-groupe $G \subset \mathrm{Homeo}(\mathbb{S}^1)$ est un **groupe de convergence** si toute suite dans G a une sous-suite équicontinue ou possédant la propriété de convergence. Il est aisé de voir que $\mathrm{PSL}(2, \mathbb{R})$ est un groupe de convergence, et par conséquent tous ses sous-groupes aussi, ainsi que leurs conjugués dans $\mathrm{Homeo}(\mathbb{S}^1)$. Un résultat célèbre ([**Gab92**], [**CJ94**]) assure que c'est une équivalence : un groupe de convergence $G \subset \mathrm{Homeo}(\mathbb{S}^1)$ est conjugué dans $\mathrm{Homeo}(\mathbb{S}^1)$ à un sous-groupe de $\mathrm{PSL}(2, \mathbb{R})$.

Un des avantages de cette méthode est le fait qu'elle permet de traiter à la fois des groupes discrets et des groupes connexes. La plupart des résultats de classification de groupes d'isométries lorentziens ne concerne que des groupes de Lie connexes (on trouve tout de même des groupes discrets dans [**Zeg99b**] et [**PZ13**]). C'est à l'aide de la propriété de convergence que nous prouverons les théorèmes 1.4, 1.5, 1.6 et 1.7 dans le chapitre 3.

Remarquons que la propriété de convergence peut se définir pour une action sur n'importe quel espace topologique. Cette propriété est au centre de la preuve du Théorème de Ferrand-Obata en géométrie riemannienne conforme (qui résout une conjecture de Lichnerowicz) : toute variété riemannienne sur laquelle le groupe conforme agit non proprement[1] est conformément équivalent à la sphère ronde ou à l'espace euclidien. Dans la preuve de Ferrand (voir [**Fer96**]), ainsi que dans d'autres preuves plus récentes ([**FT02**], [**Fra07**]), une des étapes consiste toujours à montrer que le groupe conforme est un groupe de convergence. Cette étape n'est pas pour autant suffisante, puisqu'en dimension $n \geq 2$, un groupe de convergence n'est pas toujours conjugué à un sous-groupe du groupe conforme de la sphère $\mathrm{O}(1, n + 1)$ (voir [**MarSko89**]). Frances présente des

[1]ou, de façon équivalente, telle que le groupe conforme ne préserve aucune métrique dans la classe conforme

exemples dans [**Fra05b**] qui montrent que l'analogue lorentzien du Théorème de Ferrand-Obata est faux. On retrouve dans son étude (voir aussi [**Fra05a**]) le fait qu'en géométrie lorentzienne, le groupe conforme ne possède pas toujours la propriété de convergence.

1.4. Optimalité. Plusieurs questions découlent de ces résultats. Une conjugaison dans $\mathrm{Homeo}(\mathbb{S}^1)$ reste insatisfaisante, puisque les groupes d'isométries agissent par difféomorphismes sur le cercle.

Un sous-groupe de $\mathrm{Diff}(\mathbb{S}^1)$ qui est conjugué dans $\mathrm{Homeo}(\mathbb{S}^1)$ à un sous-groupe de $\mathrm{PSL}(2,\mathbb{R})$ n'est pas nécessairement conjugué dans $\mathrm{Diff}(\mathbb{S}^1)$ à un tel groupe. Considérons par exemple une dynamique nord/sud, i.e. un difféomorphisme $f \in \mathrm{Diff}(\mathbb{S}^1)$ ayant deux points fixes $N, S \in \mathbb{S}^1$ tels que $f'(N) < 1$ et $f'(S) > 1$. C'est le cas d'un élément hyperbolique de $\mathrm{PSL}(2,\mathbb{R})$, et il est aisé de voir que tous les difféomorphismes nord/sud sont conjugués entre eux dans $\mathrm{Homeo}(\mathbb{S}^1)$. Cependant, si f est conjugué dans $\mathrm{Diff}(\mathbb{S}^1)$ à un élément de $\mathrm{PSL}(2,\mathbb{R})$, alors $f'(N)f'(S) = 1$ (puisque pour un élément hyperbolique de $\mathrm{PSL}(2,\mathbb{R})$, les dérivées aux points fixes sont les carrés des valeurs propres), ce qui n'est pas le cas de toutes les dynamiques nord/sud sur le cercle.

Il existe cepdendant deux résultats garantissant l'existence d'une conjugaison différentiable. Le premier, dû à Herman [**Her79**], assure qu'un difféomorphisme du cercle topologiquement conjugué à une rotation dont l'angle satisfait une condition diophantienne est conjugué dans $\mathrm{Diff}(\mathbb{S}^1)$ à cette rotation. Le second est un théorème dû à Ghys [**Ghy93**], et porte sur les représentations de groupes de surfaces : étant donnée $\rho : \Gamma_g \to \mathrm{Diff}(\mathbb{S}^1)$, où Γ_g est le groupe fondamental de la surface compacte orientable de genre g, si le nombre d'Euler de ρ est maximal, alors ρ est conjuguée dans $\mathrm{Diff}(\mathbb{S}^1)$ à une représentation dans $\mathrm{PSL}(2,\mathbb{R})$.

Dans le chapitre 4, nous étudierons le problème de la conjugaison différentiable dans un cas très particulier, à savoir celui des surfaces conformément équivalentes à l'espace de Sitter dS_2. Nous verrons que cet espace est aussi conformément équivalent à $(\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta, dxdy)$ où $\Delta = \{(x,x) | x \in \mathbb{S}^1\}$ est la diagonale, ce qui implique que $\rho_1^M = \rho_2^M$, et que le bord conforme est acausal.

Dans ce cas, nous verrons aussi que $k = 1$, i.e. $\rho_1^M(\mathrm{Isom}(M,g))$ est topologiquement conjugué à un sous-groupe de $\mathrm{PSL}(2,\mathbb{R})$. Ce résultat est à rapprocher d'un théorème de Navas (Proposition 1.1 dans [**Nav06**]) qui assure la propriété de convergence pour le groupe des homéomorphismes du cercle dont l'action sur $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$ préserve un certain type de mesure (qui n'englobe pas toutes celles associées à une métrique lorentzienne).

Théorème 1.8. *Soit (M,g) une surface lorentzienne conformément équivalente à dS_2. Supposons que $G \subset \mathrm{Isom}(M,g)$ satisfait (au moins) une des conditions suivantes :*
- *$\rho_1^M(G)$ a une orbite dense sur $\mathbb{S}^1$.*
- *g est analytique et $\rho_1^M(G)$ n'a pas d'orbite finie sur $\mathbb{S}^1$.*
- *g est analytique, G est le groupe engendré par $\varphi \in \mathrm{Isom}(M,g)$ et $\rho_1^M(\varphi)$ a exactement deux points fixes sur $\mathbb{S}^1$.*
- *G est le groupe engendré par $\varphi \in \mathrm{Isom}(M,g)$ et $\rho_1^M(\varphi)$ n'a aucun point fixe sur $\mathbb{S}^1$.*

Alors $\rho_1^M(G)$ est conjugué dans $\mathrm{Diff}(\mathbb{S}^1)$ à un sous-groupe de $\mathrm{PSL}(2,\mathbb{R})$.

Il s'agit du Théorème 4.1.1 de la page 63, dont la preuve se sépare en plusieurs énoncés (Proposition 4.1.5, Théorème 4.1.7, Théorème 4.1.8 et Théorème 4.1.9). Ces résultats de rigidité sont tous bien plus simples que les théorèmes de Herman et de Ghys. Nous exhiberons aussi des exemples pour lesquels une telle conjugaison différentiable n'existe pas.

Théorème 1.9. *Il existe des surfaces spatialement compactes (M,g) conformément équivalentes à dS_2 pour lesquelles $\mathrm{Isom}(M,g)$ contient un groupe libre $\mathbb{F}_n$, $n \geq 2$, et telles* *qu'il n'existe pas de conjugaison dans* $\mathrm{Diff}(\mathbb{S}^1)$ *entre* $\rho_1^M(\mathrm{Isom}(M,g))$ *et un sous-groupe de* $\mathrm{PSL}(2,\mathbb{R})$*.*

Ce résultat est la conséquence à la fois du Théorème 4.1.10 et du Théorème 4.1.14. On peut facilement trouver des exemples explicites pour lesquels le groupe d'isométries est engendré par un élément, ou un groupe à un paramètre, mais la construction d'exemples avec un groupe plus compliqué demande plus de travail. La stratégie de la preuve s'appuie grandement sur une construction due à Étienne Ghys, utilisée à la fois pour construire des flots d'Anosov en dimension 3 dont les feuilletages stable faible et instable faible sont lisses dans [**Ghy92**], et pour démontrer le théorème de rigidité pour les actions de groupes de surfaces sur le cercle cité ci-dessus dans [**Ghy93**]. L'idée de cette construction est de relier actions sur le cercle et flots hyperboliques en dimension 3, le prototype étant le flot géodésique sur $\mathrm{T}^1\mathbb{H}^2/\Gamma$ que l'on associe à l'action du groupe $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ sur le cercle.

Finalement, on peut se demander si une conjugaison topologique existe toujours, même sans hypothèse sur le bord conforme. Encore une fois, la réponse est négative, y compris pour des groupes libres à plusieurs générateurs.

Théorème 1.10. *Il existe des surfaces spatialement compactes* (M,g) *pour lesquelles* $\mathrm{Isom}(M,g)$ *contient un groupe libre* $\mathbb{F}_n$*,* $n \geq 2$*, et telles qu'il n'existe pas de conjugaison topologique entre* $\rho_1^M(\mathrm{Isom}(M,g))$ *et un sous-groupe d'un* $\mathrm{PSL}_k(2,\mathbb{R})$*.*

La stratégie de la preuve de ce résultat (Théorème 5.2.1, page 100) est la même que celle du Théorème 1.9.

On peut alors se demander s'il est possible de classifier les groupes d'isométries de surfaces spatialement compactes à conjugaison topologique près. Nous étudierons cette question pour l'action d'un élément du groupe d'isométrie, dans un cas particulier (Proposition 5.1.4, page 97).

Théorème 1.11. *Soit* (M,g) *une surface lorentzienne spatialement compacte qui se plonge conformément dans le tore lorentzien plat, telle que le bord dans le tore soit homéomorphe à un cercle. Toute isométrie* $\varphi \in \mathrm{Isom}(M,g)$ *satisfait une des propositions suivantes :*

- $\rho_1^M(\varphi)$ *est topologiquement conjugué à un élément de* $\mathrm{PSL}(2,\mathbb{R})$*.*
- $\rho_1^M(\varphi)$ *est topologiquement conjugué à un élément parabolique de* $\mathrm{PSL}_2(2,\mathbb{R})$*.*
- $\rho_1^M(\varphi)$ *a trois points fixes* $a<b<c<a$*, les points* a *et* b *étant hyperboliques et* c *parabolique.*
- $\rho_1^M(\varphi)$ *a quatre points fixes* $a<b<c<d<a$*, les points* a *et* c *étant hyperboliques et* b,d *paraboliques.*

Ces propositions caractérisent $\rho_1^M(\varphi)$ à conjugaison topologique près, et nous verrons que toutes ces situations se produisent.

1.5. Reformulation en termes de dynamique du cercle. L'étude des isométries de surfaces lorentziennes spatialement compactes mène à des questions qui peuvent se reformuler en termes de dynamique du cercle (i.e. en oubliant le caractère lorentzien).

C'est le point de vue que nous adopterons dans le chapitre 4, où toutes les surfaces lorentziennes sont conformément équivalentes à $(\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta, dxdy)$. Une métrique $\omega(x,y)dxdy$ dans la classe conforme de $dxdy$ peut aussi s'interpréter comme une forme volume $\omega(x,y)dx \wedge dy$. Le groupe d'isométries se défini alors comme le groupe des difféomorphismes du cercle $f \in \mathrm{Diff}(\mathbb{S}^1)$ tels que :

$$\forall x \neq y \ \frac{\omega(x,y)}{\omega(f(x),f(y))} = f'(x)f'(y)$$

Jusqu'ici, les questions se posaient en partant d'une métrique lorentzienne et en essayant de comprendre les sous-groupes de $\mathrm{Diff}(\mathbb{S}^1)$ qui lui sont associés, mais il est aussi possible de partir d'un sous-groupe de $\mathrm{Diff}(\mathbb{S}^1)$ et de se demander sous quelle(s) condition(s) son action diagonale sur $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$ préserve une forme volume.

2. Dynamique multi-valuée et causalité

La seconde partie de cette thèse s'appuie sur la simple observation qu'un espace-temps peut être considéré comme un système dynamique multi-valué, pour lequel l'image d'un point est son futur causal. Un **système dynamique multi-valué** est une application $\varphi : X \to \mathfrak{P}(X)$, autrement dit à un point $x \in X$ on associe un ensemble $\varphi(x) \subset X$. Si l'on considère un tel système comme l'analogue d'un système dynamique discret, on peut alors trouver un équivalent infinitésimal : à un point $x \in X$ (où X est une variété différentielle) on associe un sous-ensemble $C(x)$ de l'espace tangent T_xX. On appelle un tel système une **inclusion différentielle**, que l'on peut considérer comme une équation différentielle avec une incertitude sur le champ de vecteurs. Une **courbe intégrale** de C est une courbe localement lipschitzienne $\gamma : I \to X$ (où $I \subset \mathbb{R}$ est un intervalle) telle que $\dot\gamma(x) \in C(\gamma(x))$ pour presque tout $x \in I$. Son comportement lors de passages à la limite fait que l'on préfère la régularité lipschitzienne à la classe C^1.

2.1. La dynamique d'un espace-temps. Étant donné un espace-temps (M, g), on note $C(x)$ l'ensemble des vecteurs causaux orientés vers le futur tangents en $x \in M$. Les courbes intégrales sont précisément les courbes causales futures (au sens topologique). Le système C est équivalent à la donnée de la classe conforme $[g]$ (car deux formes quadratiques réelles non dégénérées et non définies ont le même cône isotrope si et seulement si elles sont proportionnelles).

Il est possible d'associer des systèmes multi-valués (discrets) à une inclusion différentielle, tout comme on associe un flot à un champ de vecteurs : pour $0 < t < T$, on note $J^+_{t,T}(x)$ l'ensemble des extrémités $\gamma(1)$ de courbes causales futures $\gamma : [0\,,1] \to M$ telles que $\gamma(0) = x$ et $t < \ell_h(\gamma) < T$.

Cette définition dépend du choix d'une métrique riemannienne auxiliaire h. Si h est complète, alors $\overline{J^+_{t,T}(x)}$ est compact. Si l'on choisit bien cette métrique riemannienne, ce système est continu.

Théorème 2.1. *Soit (M, g) un espace-temps. Il existe une métrique riemannienne complète h sur M telle que l'application $x \mapsto \overline{J^+_{t,T}(x)}$ est continue pour la topologie de Hausdorff sur les compacts de M pour tous $0 < t < T$.*

Cette analogie entre géométrie lorentzienne et systèmes dynamiques multi-valués est motivée par la ressemblance entre les notions de fonction temps en théorie de la causalité et de fonction de Lyapunov en systèmes dynamiques. Une **fonction temps** sur un espace-temps (M, g) est une fonction continue $\tau : M \to \mathbb{R}$ qui est strictement croissante le long de toute courbe causale future. D'après le Théorème de Hawking, l'existence d'une telle fonction est équivalente à la causalité stable, i.e. la non existence de courbe causale fermée pour des métriques proches. Une fonction de Lyapunov pour un système dynamique classique est une fonction continue qui décroit le long de toutes les orbites, et qui décroit strictement le long de certaines orbites (physiquement, on l'interprète comme une énergie). Cependant, un système dynamique pour lequel il existe une fonction continue strictement décroissante le long de toute orbite est considéré comme ayant une dynamique pauvre, alors qu'un espace-temps est intéressant d'un point de vue physique lorsqu'il satisfait certaines conditions de causalité. Le but de la seconde partie de cette thèse est d'utiliser les techniques de construction de fonctions de Lyapunov (principalement dues à Conley, [**Con88**]) pour obtenir des fonctions temps en géométrie lorentzienne. Comme ces techniques ne permettent pas d'obtenir directement une fonction croissante le long de toute courbe causale future, nous introduisons une notion plus générale de fonction temps.

Soit (M, g) un espace-temps, et soit $E \subset M$ un sous-ensemble. On dit qu'une application continue $\tau : M \to \mathbb{R}$ est une **fonction temps pour** E si elle satisfait les deux conditions suivantes :

(1) $\forall x \in M \ \forall y \in J^+(x) \quad \tau(y) \geq \tau(x)$
(2) $\forall x \in E \ \forall y \in J^+(x) \setminus \{x\} \quad \tau(y) > \tau(x)$

Charles Conley a montré dans [**Con88**] que l'existence de fonctions de Lyapunov est liée à la récurrence par chaine. Ses travaux sont valables pour un flot sur un espace métrique compact, et Hurley a étendu ces constructions au cas d'un espace métrique séparable (voir [**Hur92**],[**Hur95**],[**Hur98**], et [**CCP02**] pour quelques corrections). Les variétés lorentziennes compactes manquant d'intérêt d'un point de vue physique (elles ne sont jamais causales), nous ne pourrons pas nous contenter du cas compact.

Un point x d'un espace-temps (M, g) est dit **récurrent par chaine** s'il existe une suite finie $(\gamma_1, \dots, \gamma_N)$ de courbes causales futures de longueurs plus grandes que T telles que $d(\gamma_{i+1}(0), \gamma_i(1)) < \varepsilon(\gamma_i(1))$, $\gamma_1(0) = x$ et $d(x, \gamma_N(1)) < \varepsilon(x)$ pour tout $T > 0$ et toute fonction continue $\varepsilon : M \to]0\,, +\infty[$. On note $R(g)$ l'ensemble des points récurrents par chaine. Le résultat suivant est l'équivalent lorentzien du Théorème de Conley.

Théorème 2.2. *Soit (M, g) un espace-temps. Il existe une fonction temps pour $M \setminus R(g)$. De plus, il existe une fonction temps si et seulement si $R(g) = \emptyset$.*

La démonstration s'appuiera sur la stratégie de preuve de Conley, au coeur de laquelle on trouve la notion d'ensemble attracteur.

En utilisant le Théorème de Hawking, on voit que $R(g) = \emptyset$ si et seulement si (M, g) est stablement causal. Nous verrons que les techniques utilisée pour construire une fonction temps lorsque $R(g) = \emptyset$ (attracteurs et fonctions temps pour des sous-ensembles) s'appliquent aussi dans un espace-temps stablement causal, et donnent donc une nouvelle démonstration du sens direct du Théorème de Hawking (un espace-temps stablement causal admet une fonction temps).

La question de la différentiabilité des fonctions temps est un problème classique et difficile. Dans [**HE73**], on trouve la construction de Hawking d'une fonction temps continue pour tout espace-temps stablement causal. Pour l'existence d'une fonction temporelle, i.e. une fonction lisse dont le gradient est de type temps, il a fallu attendre les travaux de Bernal et Sanchez ([**BS03**],[**BS05**]) qui traitent les différentes variantes de cette question. Fathi et Siconolfi proposent dans [**FS12**] un preuve différente de l'existence d'une fonction temporelle, en s'appuyant sur des méthodes issues de la théorie KAM faible. On y trouve d'ailleurs l'idée de considérer un espace-temps comme une inclusion différentielle. Ce rapprochement avec la théorie KAM faible a aussi été utilisée par Pageault dans [**Pag09**] afin de donner une construction de fonctions de Lyapunov pour des flots.

Il parait naturel de se demander si les fonctions temps généralisées obtenues dans cette thèse peuvent être lissées, mais nous n'aborderons pas cette question.

2.2. Conjugaison entre systèmes multi-valués. Le dernier chapitre de cette thèse propose d'étudier la conjugaison entre systèmes dynamiques multi-valués, dans un cadre non lorentzien. De façon générale, les notions étudiées en dynamique (classique) n'ont pas toujours un équivalent canonique en dynamique multi-valuée. C'est le cas par exemple des itérées d'un système, ou encore de la notion de stabilité structurelle. Dans ce chapitre, nous proposons d'étudier deux notions de conjugaison.

La première notion, que l'on appelle **conjugaison forte**, est l'existence d'un homéomorphisme $h : X_1 \to X_2$ tel que $F_2(h(x)) = h(F_2(x))$ pour tout $x \in X_1$, où $F_i : X_i \to \mathfrak{P}(X_i)$ est un système dynamique multi-valué pour $i = 1, 2$. Cette notion parait la plus naturelle, mais elle se révèlera trop restrictive.

Étant donné un système multi-valué $F : X \to \mathfrak{P}(X)$, on peut lui associer un système dynamique classique $\sigma_F \in \mathrm{Homeo}(\Sigma_F)$, appelé le **shift associé** à F, où $\Sigma_F = \{(x_n)_{n\in\mathbb{N}} | x_{n+1} \in F(x_n)\}$ et $\sigma_F((x_n)_{n\in\mathbb{N}}) = (x_{n+1})_{n\in\mathbb{N}}$. On dit que deux systèmes multi-valués sont **shift-conjugués** si leurs shifts associés sont conjugués par un homéomorphisme. La conjugaison forte implique la shift-conjugaison.

Nous verrons que dans un cadre d'applications intervalles sur la droite, la shift-conjugaison est strictement plus souple que la conjugaison forte, et qu'elle permet de garder une certaine stabilité structurelle. Nous verrons aussi que pour des applications similaires sur le cercle, la shift-conjugaison présente une certaine rigidité (et en particulier n'est pas une notion triviale).

Liste des travaux de l'auteur

Les résultats de la Partie 1 font l'objet de trois prépublications. L'article (1) contient l'étude de la différentiabilité de la conjugante présente dans le chapitre 4. Les résultats du chapitre 3 sur la classification des groupes d'isométries à semi conjugaison près se retrouvent dans (2). Les exemples non topologiquement fuchsiens du chapitre 5 sont présentés dans (3).

(1) D. Monclair, *Differentiable conjugacy for groups of area preserving circle diffeomorphisms*, arXiv:1402.0424 (2014)
(2) D. Monclair, *Isometries of Lorentz surfaces and convergence groups*, arXiv:1402.7179 (2014)
(3) D. Monclair, *Convergence groups and semi conjugacy*, arXiv:1404.2829 (2014)

Part 1

Isometry groups of Lorentz surfaces and circle dynamics

CHAPTER 1

Conformal models for spatially compact surfaces

One of our main tools will be the use of global conformal models for spatially compact surfaces. Lorentz surfaces are locally conformally flat, but in the globally hyperbolic case, one can find two natural immersions in flat spaces which are globally defined.

The first one consists of a conformal immersion in the flat Lorentzian torus $p : (M, [g]) \to (\mathbb{S}^1 \times \mathbb{S}^1, [dxdy])$. It is defined for all spatially compact surfaces. In this model, a conformal diffeomorphism $\varphi \in \mathrm{Isom}(M, g)$ acts on $p(M)$ by the diffeomorphism $(x, y) \mapsto (\rho_1^M(\varphi)(x), \rho_2^M(\varphi)(y))$ where $\rho_1^M, \rho_2^M : \mathrm{Conf}(M, g) \to \mathrm{Diff}(\mathbb{S}^1)$ are representations that we will use to characterise the isometry group (even though they are defined on the whole conformal group). In the case where p is an embedding, the boundary of $p(M)$ consists in the graphs of non decreasing maps of degree one which provide a semi-conjugacy between ρ_1^M and ρ_2^M.

The second conformal model is a conformal embedding in the flat Lorentzian cylinder (quotient of the Minkowski space by a spacelike translation). We will use it in order to see that our study can be reduced to the case where the first conformal model is an embedding.

Note that in this chapter, we only deal with the conformal structure of Lorentz surfaces, the study of the isometry group will start in chapter 3.

1. Lorentz surfaces

Many questions about Lorentz surfaces have been studied, both in the compact case (e.g. completeness [**CR94**], conjugate points [**BM13**], closed geodesics [**Suh13**]) and in the non compact case (see [**Wei96**]).

1.1. Lorentzian background. We will only use some basic notions of Lorentzian geometry (for more details, see [**BEE96**], [**HE73**], [**O'N83**], or the introduction chapter of [**O'N95**]). A Lorentz manifold is a manifold M equipped with a symmetric $(2, 0)$-tensor g of signature $(-, +, \dots, +)$, called a **Lorentz metric**. If the regularity is not explicitly stated, we will assume Lorentz metrics to be C^2. A tangent vector $u \in T_xM$ is said to be **timelike** (resp. **spacelike**, **lightlike**, **causal**) if $g_x(u, u) < 0$ (resp. $g_x(u, u) > 0$, $g_x(u, u) = 0$, $g_x(u, u) \leq 0$). A curve or a submanifold of M is said to be timelike, spacelike, lightlike or causal if all its tangent vectors have the corresponding type.

A **time orientation** of (M, g) is a timelike vector field. We will call a time oriented Lorentz manifold a **spacetime**. It allows us to define **future** (resp. **past**) **directed** causal vectors as vectors in the same connected component of the cone $g < 0$ (resp. in the other connected component). A future (resp. past) directed curve is a curve whose tangent vectors are future (resp. past) directed. Given a point $x \in M$, we define its future $J^+(x)$ (resp. its past $J^-(x)$) to be the set of endpoints of future (resp. past) curves starting at x.

A conformal transformation is said to be **time orientation preserving** if it sends future directed vectors on future directed vectors. We will denote by $\mathrm{Conf}(M, g)$ (resp. $\mathrm{Isom}(M, g)$) the set of orientation and time orientation preserving conformal diffeomorphisms (resp. isometries). The isometry group is always a finite dimensional Lie Group ([**Ada01**]).

A spacetime (M, g) is said to be **globally hyperbolic** if there is a topological hypersurface $S \subset M$ (called a Cauchy hypersurface) such that every inextensible causal curve intersects S exactly once. Smooth Cauchy hypersurfaces always exist ([**BS03**]), moreover they are diffeomorphic to each other and M is diffeomorphic to $\mathbb{R} \times S$. We say that it is **spatially compact** if it is globally hyperbolic and it has a compact Cauchy hypersurface. If (M, g) is globally hyperbolic and $x, y \in M$, then $J^+(x) \cap J^-(y)$ is compact.

Just as in the Riemannian case, a Lorentz metric defines a Levi-Civita connection, and we can define geodesics. They can be timelike, spacelike or lightlike. Unparametrised lightlike geodesics are preserved by conformal transformations. There is also a notion of curvature (and constant sectional curvature implies local isometry with a model space).

Because of the connection, isometries are defined by their 1-jet: if an isometry f has a fixed point x such that Df_x is the identity map, then f is the identity. This property is important as it is the reason for which a Lorentz metric can be considered a rigid geometric structure.

1.2. Dimension two. Let us start by mentioning a few generalities about Lorentz surfaces.

Let (M, g) be a Lorentz surface, and let $p \in M$. The light cone at p is the union of two straight lines. Locally, on a small neighbourhood U of p, one can define two smooth vector fields X_1, X_2 such that the union $\mathbb{R}X_1(q) \cup \mathbb{R}X_2(q)$ is equal to the light-cone $g_q^{-1}(\{0\})$ for all $q \in U$. Let φ_1^t and φ_2^t be the local flows associated to X_1 and X_2.

If U is small enough, then for every $q \in U$ there is a unique pair (t_1, t_1') such that $\varphi_2^{t_1}(q) = \varphi_1^{t_1'}(p)$ (i.e. the orbit of q for X_2 meets the orbit of p for X_1 at exactly one point). Similarly, there is a unique pair (t_2, t_2') such that $\varphi_1^{t_2}(q) = \varphi_2^{t_2'}(p)$. Let $F : U \to \mathbb{R}^2$ be the map $q \mapsto (t_1', t_2')$. It is a diffeomorphism onto its image, and it sends isotropic lines in M (i.e. orbits of X_1 and X_2) to horizontal and vertical lines in $\mathbb{R}^2$, i.e. isotropic lines for the Lorentz metric $dxdy$. This shows that the map $F : (U, [g]) \to (\mathbb{R}^2, [dxdy])$ is conformal.

We just proved that Lorentz surfaces are locally conformally flat. This implies that the local structure of conformal diffeomorphisms between surfaces is the same as for conformal diffeomorphisms of $(\mathbb{R}^2, [dxdy])$. Let $U, V \subset \mathbb{R}^2$ be open sets and let $\varphi : (U, [dxdy]) \to (V, [dxdy])$ be a conformal diffeomorphism. Write $\varphi(x, y) = (\varphi_1(x, y), \varphi_2(x, y))$. Since φ is conformal, it sends isotropic vectors to isotropic vectors. Applying this to $(1, 0)$ and $(0, 1)$, we get the following conditions:

$$\frac{\partial \varphi_1}{\partial x}\frac{\partial \varphi_2}{\partial x} = 0 \text{ and } \frac{\partial \varphi_1}{\partial y}\frac{\partial \varphi_2}{\partial y} = 0$$

To simplify the problem, assume that U is a rectangle $I \times J$ where $I, J \subset \mathbb{R}$ are open intervals. Taking the first equation, we see that either $\frac{\partial \varphi_1}{\partial x} = 0$ everywhere on U either $\frac{\partial \varphi_2}{\partial x} = 0$. In the latter case, the fact that φ is a diffeomorphism implies that $\frac{\partial \varphi_2}{\partial y} \neq 0$, hence $\frac{\partial \varphi_1}{\partial y} = 0$. Since U is convex, this implies that there are maps $f : I \to \mathbb{R}$ $g : J \to \mathbb{R}$ that are diffeomorphisms onto their images such that:

$$\forall (x, y) \in U \quad \varphi(x, y) = (f(x), g(y))$$

In the case where $\frac{\partial \varphi_1}{\partial x} = 0$, we find maps $f : I \to \mathbb{R}$ $g : J \to \mathbb{R}$ such that:

$$\forall (x, y) \in U \quad \varphi(x, y) = (g(y), f(x))$$

If φ is orientation preserving and time orientation preserving, then only the first case is possible, and f, g are both orientation preserving (i.e. increasing).

Finally, since every Lorentz surfaces is locally isometric to $(U, \omega(x, y)dxdy)$ where U is an open subset of $\mathbb{R}^2$, we will use this fact to obtain simple formulae for geodesics and curvature. Indeed, the curvature of the metric $\omega(x,y)dxdy$ has a simple expression:

$$K = \frac{2}{\omega}\frac{\partial^2 \mathrm{Log}\omega}{\partial x \partial y}$$

The geodesics equations can also be simplified (see [**CR94**]):

$$\begin{aligned} x'' + \frac{1}{\omega}\frac{\partial \omega}{\partial x}x'^2 &= 0 \\ y'' + \frac{1}{\omega}\frac{\partial \omega}{\partial y}y'^2 &= 0 \end{aligned}$$

For lightlike geodesics, say $x = x_0$ for some $x_0 \in \mathbb{R}$, we find that the geodesic $(x_0, y(t))$ has a first integral: the quantity $\omega(x_0, y(t))y'(t)$ is constant. This will allow us to define parametrisations of lightlike geodesics for Lorentz surfaces even when the metric is only continuous.

2. The lightlike foliations and actions on the circle

Let (M, g) be an oriented and time oriented Lorentz surface. At any point $x \in M$, the isotropic cone is the union of two straight lines. We can define isotropic vectors $X_1(x), X_2(x)$ in a smooth way by requiring that X_1, X_2 are future directed and that the basis $(X_1(x), X_2(x))$ of T_xM is direct.

This allows us to define two foliations $\mathcal{F}_1, \mathcal{F}_2$ of M as the orbits of the vector fields X_1, X_2.

We now assume that (M, g) is globally hyperbolic. In this case, its is diffeomorphic to $\mathbb{R} \times S$ where S is a connected one-dimensional manifold, therefore diffeomorphic to $\mathbb{R}$ or $\mathbb{S}^1$. In particular, the manifold M is orientable, and the foliations $\mathcal{F}_1, \mathcal{F}_2$ can be globally defined.

Let $F_i = M/\mathcal{F}_i$ be the quotient and $p_i : M \to F_i$ the associated projection. The restriction of p_i to a Cauchy hypersurface is a diffeomorphism (because an isotropic geodesic is an inextendible curve). Therefore, if (M, g) is spatially compact, then F_i is diffeomorphic to $\mathbb{S}^1$.

A conformal diffeomorphism sends (unparametrised) lightlike geodesics to lightlike geodesics, i.e. it preserves $\mathcal{F}_1 \cup \mathcal{F}_2$. If it preserves the orientation and the time orientation, then it preserves $\mathcal{F}_1$ and $\mathcal{F}_2$, and it acts on the quotients, which gives two representations $\rho_i^M : \mathrm{Conf}(M, g) \to \mathrm{Diff}(F_i) \approx \mathrm{Diff}(\mathbb{S}^1)$, $i = 1, 2$.

Note that ρ_1^M, ρ_2^M are well defined up to conjugacy in $\mathrm{Diff}(\mathbb{S}^1)$.

3. Immersion in the flat torus

The map $x \mapsto (p_1(x), p_2(x))$ sends M into $F_1 \times F_2$. By choosing identifications between F_1, F_2 and $\mathbb{S}^1$, it gives us a map from M to $\mathbb{T}^2$. We will denote by p the map obtained after reversing the orientation on F_1 (i.e. $p(x) = (-p_1(x), p_2(x))$).

Proposition 1.3.1. *The map $p : (M, [g]) \to (\mathbb{T}^2, [dxdy])$ is a conformal immersion. If it is injective, then it is a conformal diffeomorphism onto its image.*

PROOF. The kernel of d_xp_i is the tangent space to $\mathcal{F}_i$ at x, therefore $\ker d_xp$ is the intersection of two transversal lines and is equal to $\{0\}$, and p is an immersion. The image of isotropic vectors in (M, g) being isotropic vectors for $(\mathbb{T}^2, dxdy)$, we see that the metric g is sent to a (not necessarily positive) multiple of $dxdy$. Since we changed the orientation of F_1 in the definition of p, the future is given by moving negatively along the x-axis and positively along the y-axis, i.e. the metric is conformal to $+dxdy$.

By equality of dimensions, the immersion p is an open map, and it only needs to be injective in order to be a diffeomorphism onto its image. ☐

We will call $(\mathbb{T}^2, dxdy)$ the **flat Lorentzian torus** (note that it is conformal to the Einstein Universe $\mathrm{Ein}^{1,1}$), even though it is not the only flat Lorentzian metric on the torus.

The injectivity of p is equivalent to the following property: two lightlike geodesics have at most one intersection point (i.e. there are no null conjugate points).

Since all globally hyperbolic open sets of $\mathbb{T}^2$ satisfy this property, we see that p is injective if and only if (M, g) embeds conformally in $(\mathbb{T}^2, dxdy)$. This makes p canonical in some sense: given a spatially compact surface that embeds conformally in $\mathbb{T}^2$, there are many conformal embeddings, but we can choose one that satisfies certain properties.

4. Embedding in the flat cylinder

4.1. Definition. The problem that we encounter with the previous conformal model is that it is not always an embedding, but only an immersion. However, we will now see that there is another conformal model that always give an embedding.

The **flat Lorentzian cylinder** is the quotient of the Minkowski plane by a spacelike translation (note that these quotients are not all isometric to each other, but they are conformally equivalent).

Theorem 1.4.1. *All spatially compact surfaces embed conformally in the flat Lorentzian cylinder.*

Proof. Let (M, g) be a spatially compact surface, and consider its universal cover $(\widetilde{M}, \widetilde{g})$. It also has two foliations by lightlike geodesics $\widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2$, and the quotients $\widetilde{F}_1, \widetilde{F}_2$ are diffeomorphic to the real line $\mathbb{R}$. This gives us a conformal immersion $\widetilde{p} : (\widetilde{M}, [\widetilde{g}]) \to (\mathbb{R}^2, [dxdy])$. This time, however, it is always an embedding: two distinct lightlike geodesics on a simply connected Lorentz surface intersect at most once (see p.51 in **[Wei96]**).

Let $F \in \mathrm{Isom}(\widetilde{M}, \widetilde{g})$ be a generator of $\pi_1(M)$. It is a conformal diffeomorphism for $[dxdy]$, therefore it can be written $F(x, y) = (f_1(x), f_2(y))$ for some $f_1, f_2 \in \mathrm{Diff}(\mathbb{R})$. Since the quotient of $\widetilde{M}$ by F is causal, we see that $(x, y) \in p(\widetilde{M})$ and $F(x, y)$ are not causally related (i.e. they are not in the future or in the past of each other). This shows that either $f_1(x) > x$ and $f_2(y) > y$, either $f_1(x) < x$ and $f_2(y) < y$ (for one (x, y), hence for all (x, y) because $\widetilde{M}$ is connected). So up to replacing F by F^{-1}, we can assume that $f_1(x) > x$ and $f_2(y) > y$ for all $x, y \in \mathbb{R}$. This implies that they are both differentially conjugate to the translation $x \mapsto x + 1$, and we can assume that $F(x, y) = (x + 1, y + 1)$. This shows that (M, g) embeds in the quotient of $\mathbb{R}^2$ by the map F, which is the flat Lorentzian cylinder. $\square$

Even though this conformal model gives an embedding for M, we will mostly use the map $\widetilde{p}$ defined on the universal cover $\widetilde{M}$.

4.2. Conformal classification. We have a simple characterisation of the image in $\mathbb{R}^2$. We call an open set $U \subset \mathbb{R}^2$ **canonically embedded** if there is a spatially compact surface (M, g) such that $U = \widetilde{p}(\widetilde{M})$.

Proposition 1.4.2. *Let $U \subset \mathbb{R}^2$ be a canonically embedded open set. There are non decreasing maps $\widetilde{h}_{\leftarrow}, \widetilde{h}_{\downarrow} : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ and $\widetilde{h}_{\rightarrow}, \widetilde{h}_{\uparrow} : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ that commute with the translation $x \mapsto x + 1$ such that :*

$$\begin{aligned} U &= \{(x, y) \in \mathbb{R}^2 | \widetilde{h}_{\downarrow}(x) < y < \widetilde{h}_{\uparrow}(x)\} \\ &= \{(x, y) \in \mathbb{R}^2 | \widetilde{h}_{\leftarrow}(y) < x < \widetilde{h}_{\rightarrow}(y)\} \end{aligned}$$

PROOF. Every $x \in \mathbb{R}$ defines a vertical line in U: there are $\widetilde{h}_\downarrow(x) \in [-\infty\,, +\infty[$ and $\widetilde{h}_\uparrow(x) \in \,]-\infty\,, +\infty]$ such that:

$$\{x\} \times \mathbb{R} \cap U = \{x\} \times \,]\widetilde{h}_\downarrow(x)\,, \widetilde{h}_\uparrow(x)[$$

Since U is invariant under the map $(x, y) \mapsto (x + 1, y + 1)$, we see that $\widetilde{h}_\uparrow(x + 1) = \widetilde{h}_\uparrow(x) + 1$ and $\widetilde{h}_\downarrow(x + 1) = \widetilde{h}_\downarrow(x) + 1$ for all $x \in \mathbb{R}$.

Assume that there is $x_0 \in \mathbb{R}$ such that $\widetilde{h}_\downarrow(x_0) \neq -\infty$. Let $x > x_0$, and assume by contradiction that $(x, \widetilde{h}_\downarrow(x_0)) \in U$. If $y > \widetilde{h}_\downarrow(x_0)$, the diamond $J^+(x, \widetilde{h}_\downarrow(x_0)) \cap J^-(x_0, y)$ contains $(x_0, \widetilde{h}_\downarrow(x_0))$ and is not included in U (see Figure 1.1), which implies that U is not globally hyperbolic and is absurd. Hence $(x, \widetilde{h}_\downarrow(x_0)) \notin U$ and $\widetilde{h}_\downarrow(x) \geq \widetilde{h}_\downarrow(x_0) > -\infty$. This shows that $\widetilde{h}_\downarrow$ is non decreasing. Reversing the time orientation shows that $\widetilde{h}_\uparrow$ is also non decreasing.

By exchanging the roles of x and y, we define $\widetilde{h}_\leftarrow$ and $\widetilde{h}_\rightarrow$ in the same way. □

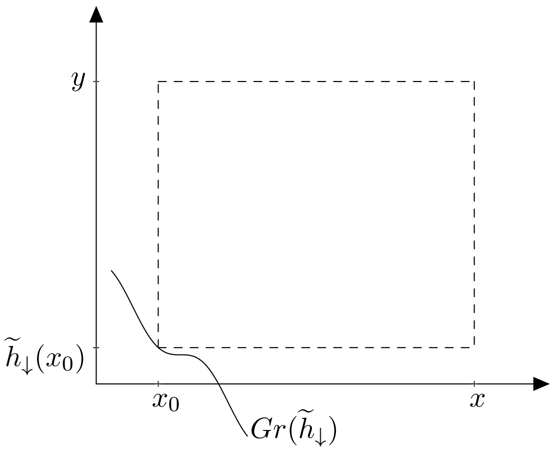

FIGURE 1.1. $\widetilde{h}_\downarrow$ is non decreasing

This implies that a spatially compact surface is conformally diffeomorphic to an open set of the flat cylinder delimited by non timelike curves. Note that if there is $x_0 \in \mathbb{R}$ such that $\widetilde{h}_\downarrow(x_0) = -\infty$ (resp. $\widetilde{h}_\uparrow(x_0) = +\infty$), then $\widetilde{h}_\downarrow(x) = -\infty$ (resp. $\widetilde{h}_\uparrow(x) = +\infty$) for all $x \in \mathbb{R}$.

The map $\widetilde{h}_\uparrow$ (resp. $\widetilde{h}_\downarrow$) is always left continuous (resp. right continuous). All such maps can be obtained: given $\widetilde{h}_-, \widetilde{h}_+ : \mathbb{R} \to \mathbb{R}$ non decreasing that commute with $x \mapsto x+1$, such that $\widetilde{h}_- < \widetilde{h}_+$ and $\widetilde{h}_+$ (resp. $\widetilde{h}_-$) is left continuous (resp. right continuous), we obtain a spatially compact surface M whose universal cover is the set of points $(x, y) \in \mathbb{R}^2$ such that $\widetilde{h}_-(x) < y < \widetilde{h}_+(x)$. It is unique up to a conformal diffeomorphism.

The boundary $\partial U \subset \mathbb{R}^2$ is the union of the graphs of $\widetilde{h}_\downarrow$ and $\widetilde{h}_\uparrow$ and of the vertical lines joining discontinuities (we set the graphs of the constants $\pm\infty$ to be the empty set). We can define the conformal boundary ∂M of a spatially compact surface (M, g) to be the projection of the boundary of U in the flat cylinder. It is an achronal set (two points cannot be joined by a timelike curve). However, if $\widetilde{h}_\downarrow$ or $\widetilde{h}_\uparrow$ is constant on an interval, then the boundary can contain causal curves. Such causal curves in the boundary can only arise when $\widetilde{h}_\downarrow$ and $\widetilde{h}_\uparrow$ fail to be homeomorphisms.

Definition 1.4.3. *Let (M, g) be a spatially compact surface. We say that the conformal boundary is acausal if the maps $\widetilde{h}_\downarrow, \widetilde{h}_\uparrow$ are either homeomorphisms of $\mathbb{R}$ either infinite.*

Note that this boundary falls in the general concept of conformal boundary for spacetimes (see [**FHS11**]). It is a general fact that the conformal boundary of a globally hyperbolic spacetime is achronal, but not necessarily acausal.

4.3. From the cylinder to the torus. Given a spatially compact surface (M, g), we have defined a conformal embedding $\widetilde{p} : \widetilde{M} \to \mathbb{R}^2$ and a conformal immersion $p : M \to \mathbb{T}^2$. If we denote by $\pi : \widetilde{M} \to M$ the (Lorentzian) universal cover and by $\pi_0 : \mathbb{R}^2 \to \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ the universal cover of the flat torus, then it follows from the definitions that $\pi_0 \circ \widetilde{p} = p \circ \pi$.

The projections of (x, y) and $(x, y+1)$ in the flat cylinder are different points on the same lightlike geodesics. This shows that M embeds conformally in the torus if and only if $\widetilde{h}_\uparrow(x) - \widetilde{h}_\downarrow(x) \leq 1$ for all $x \in \mathbb{R}$. In this case, if we denote by $h_\uparrow, h_\downarrow$ the associated maps on the circle, then we find that $p(M) = \{(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1 | h_\downarrow(x) < y < h_\uparrow(x) \leq h_\downarrow(x)\}$. We also see that if the conformal boundary of (M, g) is acausal, then $h_\uparrow$ and $h_\downarrow$ are circle homeomorphisms.

Proposition 1.4.4. *Let (M, g) be a spatially compact surface with an acausal conformal boundary. If (M, g) embeds conformally in the torus, then the representations ρ_1^M and ρ_2^M are topologically conjugate.*

Proof. Given $\varphi \in \mathrm{Conf}(M, g)$ and $(x, y) \in p(M)$, we can write:

$$p \circ \varphi \circ p^{-1}(x, y) = (\rho_1^M(\varphi)(x), \rho_2^M(\varphi)(y))$$

Since for all $x \in \mathbb{S}^1$ there is $y \in \mathbb{S}^1$ such that $(x, y) \in p(M)$, the fact that $p \circ \varphi \circ p^{-1}$ preserves $\partial p(M)$ implies that $\rho_2^M \circ h_\downarrow = h_\downarrow \circ \rho_1^M$ and $\rho_2^M \circ h_\uparrow = h_\uparrow \circ \rho_1^M$. $\square$

Since the maps $\widetilde{h}_\uparrow, \widetilde{h}_\downarrow, \widetilde{h}_\rightarrow, \widetilde{h}_\leftarrow$ commute with $x \mapsto x + 1$, they define maps on $\mathbb{S}^1$ as soon as they are finite, which is the case when (M, g) embeds in the torus. Let $h_\uparrow, h_\downarrow, h_\rightarrow, h_\leftarrow$ be the induced maps on the circle. Using the invariance of $\widetilde{p}(\widetilde{M}) \subset \mathbb{R}^2$ by lifts of conformal diffeomorphisms, we obtain the following relations:

Proposition 1.4.5. *Let (M, g) be a spatially compact surface that embeds conformally in $\mathbb{T}^2$. If $\varphi \in \mathrm{Conf}(M, g)$, then the maps $h_\uparrow, h_\downarrow, h_\rightarrow, h_\leftarrow$ satisfy:*

$$\begin{aligned}
\rho_2^M(\varphi) \circ h_\downarrow &= h_\downarrow \circ \rho_1^M(\varphi) \\
\rho_2^M(\varphi) \circ h_\uparrow &= h_\uparrow \circ \rho_1^M(\varphi) \\
\rho_1^M(\varphi) \circ h_\leftarrow &= h_\leftarrow \circ \rho_2^M(\varphi) \\
\rho_1^M(\varphi) \circ h_\rightarrow &= h_\rightarrow \circ \rho_2^M(\varphi)
\end{aligned}$$

Note that when the conformal boundary is acausal, we automatically have $h_\rightarrow = h_\downarrow^{-1}$ and $h_\leftarrow = h_\uparrow^{-1}$.

4.4. Link with (G, X)-structures. What is at the heart of the two conformal models that we have defined is the fact that surfaces are conformally flat. In general, it is only a local condition, i.e. we have local conformal diffeomorphisms with the Minkowski space, but we have shown that we can find a global embedding for globally hyperbolic surfaces. This translates in terms of completeness of a (G, X)-structure.

If G is a group acting by diffeomorphisms on a simply connected manifold X, then a (G, X)-structure on a manifold M is an atlas of local diffeomorphisms with X, the transitions maps being elements of G. If the action is analytic (i.e. elements of G are uniquely defined by their action on a small open set), then we can define a holonomy morphism $h : \pi_1(M) \to G$ and an equivariant map $D : \widetilde{M} \to X$ called the developing map. A (G, X)-structure is said to be complete if D is injective (which means that M is a quotient of an open set of X).

What is interesting here is that we have shown completeness for a structure where even the existence of the developing map is not given by the general theory (the action of the conformal group is not analytic).

4.5. Example of the de Sitter space. The natural definition of the two dimensional de Sitter space dS_2 is the quadric $q = 1$ in $\mathbb{R}^3$ where $q(x, y, z) = x^2 + y^2 - z^2$, endowed with the restriction of q to tangent spaces of dS_2. In this model, the lightlike foliations $\mathcal{F}_1, \mathcal{F}_2$ are the two foliations of the one sheeted hyperboloïd dS_2 by straight lines. If two straight lines have two or more common points, then they are equal. This implies that dS_2 embeds conformally in the flat torus $\mathbb{T}^2$.

Proposition 1.4.6. *The image* $p(\mathrm{dS}_2)$ *is equal to* $\mathbb{S}^1\times\mathbb{S}^1\setminus\Delta$ *where* $\Delta = \{(x, x)|x \in \mathbb{S}^1\}$ *is the diagonal. The action of* $\mathrm{Isom}(\mathrm{dS}_2) = \mathrm{SO}^\circ(2, 1)$ *on* $p(\mathrm{dS}_2)$ *is given by the isomorphism* $\mathrm{SO}^\circ(2, 1) \approx \mathrm{PSL}(2, \mathbb{R})$ *and the diagonal action of* $\mathrm{PSL}(2, \mathbb{R})$.

Proof. One way of obtaining this result would be to compute the coordinates of isotropic geodesics and find an explicit isomorphism.

For a proof considering the homogeneous space $\mathrm{dS}_2 = \mathrm{PSL}(2, \mathbb{R})/\mathrm{SO}(1, 1)$, see [**DG00**].

There are several ways to see that the action of $\mathrm{SO}^\circ(2, 1)$ on the quotients F_1, F_2 are conjugate to the projective action of $\mathrm{PSL}(2, \mathbb{R})$ on $\mathbb{RP}^1$. First, we can use the uniqueness of the transitive action of $\mathrm{PSL}(2, \mathbb{R})$ on $\mathbb{S}^1$ by diffeomorphisms up to conjugacy in $\mathrm{Diff}(\mathbb{S}^1)$. The uniqueness up to conjugacy in $\mathrm{Homeo}(\mathbb{S}^1)$ is shown in [**Ghy01**]. If $h \in \mathrm{Homeo}(\mathbb{S}^1)$ conjugates a transitive action of $\mathrm{PSL}(2, \mathbb{R})$ by diffeomorphisms to the projective action, then the set of points where h is differentiable is invariant, therefore is equal to $\mathbb{S}^1$. Similarly, the set of points of continuity of h' is invariant and non empty, so h is a diffeomorphism.

Another way of understanding the action on F_1, F_2 consists in considering the stereographic projection of $\mathbb{H}^2 = \{q = -1\}$ on the Poincaré disk. At every point of the boundary passes exactly one line of $\mathcal{F}_1$ and one of $\mathcal{F}_2$. This shows that the action of $\mathrm{SO}^\circ(2, 1)$ on F_1, F_2 is conjugate to the action of $\mathrm{Isom}(\mathbb{H}^2)$ on the circle at infinity $\partial_\infty\mathbb{H}^2$, which is the projective action.

This shows that the action of $\mathrm{Isom}(\mathrm{dS}_2)$ on $p(M)$ is given by the diagonal action of $\mathrm{PSL}(2, \mathbb{R})$. Since this action has two orbits (Δ and $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$) and $p(M)$ is invariant, we see that $p(M) = \mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$ because it is not compact. □

Note that $\partial_\infty\mathbb{H}^2 \times \partial_\infty\mathbb{H}^2 \setminus \Delta$ is also the space of oriented geodesics of $\mathbb{H}^2$ (a geodesic is determined by its two limit points at infinity). The Lorentzian structure can be recovered from hyperbolic geometry. First, the lightlike lines $\{x\} \times \partial_\infty\mathbb{H}^2 \setminus \{x\}$ and $\partial_\infty\mathbb{H}^2 \setminus \{y\} \times \{y\}$ correspond to the horocyclic foliations, which gives a Lorentzian conformal structure. To fix a metric in this conformal class, we have to choose a volume form. It can be seen as the projection of $d\lambda$ from $\mathrm{T}^1\mathbb{H}^2$ to the quotient by the geodesic flow (which is the space of oriented geodesics), where λ is the Liouville form on $\mathrm{T}^1\mathbb{H}^2$. We will discuss the geodesic flows of hyperbolic surfaces page 28.

In projective coordinates, the metric is $\frac{4}{(x-y)^2}dxdy$ (since there is a unique invariant metric up to multiplication by a scalar, one only has to check that it is invariant under the diagonal action of homographies, and that the curvature is 1).

CHAPTER 2

Background in circle dynamics

We denote by $\mathrm{Homeo}(\mathbb{S}^1)$ (resp. $\mathrm{Diff}(\mathbb{S}^1)$) the set of orientation preserving homeomorphisms (resp. diffeomorphisms) of the circle $\mathbb{S}^1$.

1. Closed invariant sets

An important object in the study of groups of circle homeomorphisms is a minimal closed invariant set. Given a group $G \subset \mathrm{Homeo}(\mathbb{S}^1)$, exactly one of the following conditions is satisfied (see **[Ghy01]** for a proof):

(1) G has a finite orbit
(2) All orbits of G are dense
(3) There is a compact G-invariant subset $K \subset \mathbb{S}^1$ which is infinite and different from $\mathbb{S}^1$, such that the orbits of points of K are dense in K.

In the first case, all finite orbits have the same cardinality. In the third case, the set K, called an **exceptional minimal set**, is unique and is homeomorphic to a Cantor set. We can call a group $G \subset \mathrm{Homeo}(\mathbb{S}^1)$ **non elementary** if it does not have any finite orbit, and use L_G to denote $\mathbb{S}^1$ in the second case and the G-invariant compact set K in the third case. We call L_G the **limit set**.

2. Dynamics of a single homeomorphism

2.1. Rotations. The most simple homeomorphisms of the circle are rotations. With the identification $\mathbb{S}^1 \approx \mathbb{R}/\mathbb{Z}$, the **rotation** of angle $\alpha \in \mathbb{R}/\mathbb{Z}$ is the map $R_\alpha : x \mapsto x + \alpha$. Their dynamic behaviour depends on the angle α.

If $\alpha = \frac{p}{q}$ is rational (we always assume that $q > 1$ and $0 \leq p < q$), then all points of $\mathbb{S}^1$ are periodic for R_α, of period q. If α is irrational, then all orbits are dense in $\mathbb{S}^1$.

Two rotations R_α, R_β are conjugate in $\mathrm{Homeo}(\mathbb{S}^1)$ if and only if $\alpha = \beta$. This can be understood by looking at orbits: if $\alpha = \frac{p}{q} \in \mathbb{Q}$ is such that R_α is conjugate to R_β, then $\beta = \frac{p'}{q}$ because all points are q-periodic. By looking at the cyclic order of the orbits, i.e. the permutation $\sigma \in \mathfrak{S}_{q-1}$ such that $x < R_\alpha^{\sigma(1)} < \cdots < R_\alpha^{\sigma(q-1)} < x$ for all $x \in \mathbb{S}^1$, we see that $p = p'$, i.e. $\alpha = \beta$. If $\alpha \notin \mathbb{Q}$, we can also see that R_β is conjugate to R_α if and only if $\alpha = \beta$.

The angle of a rotation is an invariant under topological conjugacy. We will see that this invariant can be extended to $\mathrm{Homeo}(\mathbb{S}^1)$.

2.2. Rotation number.

Proposition 2.2.1. *Let $F \in \mathrm{Homeo}(\mathbb{R})$ be such that $F(x+1) = F(x)+1$ for all $x \in \mathbb{R}$. The limit $\widetilde{\rho}(F) = \lim_{n\to+\infty} \frac{F^n(x)-x}{n}$ exists for all $x \in \mathbb{R}$ and is independent of x.*

*If $f \in \mathrm{Homeo}(\mathbb{S}^1)$ and $F \in \mathrm{Homeo}(\mathbb{R})$ is a lift to the universal cover, then the class $\rho(f)$ of $\widetilde{\rho}(F)$ in $\mathbb{R}/\mathbb{Z}$, called the **rotation number** of f, does not depend on the lift F.*

The rotation number has several interesting properties.

Proposition 2.2.2. *The map $\rho : \mathrm{Homeo}(\mathbb{S}^1) \to \mathbb{S}^1$ is continuous.*
If $f, g \in \mathrm{Homeo}(\mathbb{S}^1)$, then $\rho(g^{-1}fg) = \rho(f)$.

$\rho(R_\alpha) = \alpha$.
$\rho(f) \in \mathbb{Q}$ *if and only if* f *has a periodic point. More precisely, if* $q \in \mathbb{N}$, *then* $\rho(f) = \frac{p}{q}$ *for some* $0 \leq p < q$ *if and only if there is* $x \in \mathbb{S}^1$ *such that* $f^q(x) = x$.

From the last statement, we see that $f \in \mathrm{Homeo}(\mathbb{S}^1)$ has a fixed point if and only $\rho(f) = 0$. Construction a homeomorphism of the circle with a fixed point is equivalent to constructing a homeomorphism of the real line (because the circle minus a point is homeomorphic to the real line). This shows that there are many homeomorphism of the circle with a fixed point, i.e. such that $\rho(f) = 0$, that are not the identity, i.e. that are not conjugate to R_0. Similarly, for all $\frac{p}{q} \in \mathbb{Q}$, it is rather easy to construct $f \in \mathrm{Homeo}(\mathbb{S}^1)$ such that $\rho(f) = \frac{p}{q}$ and that is not conjugate to $R_{\frac{p}{q}}$. The key is to find f with some periodic points and some points that are not periodic. Notice that all periodic points have the same period. Homeomorphisms conjugate to a rational rotation are characterised by periodicity.

Proposition 2.2.3. *A homeomorphism* $f \in \mathrm{Homeo}(\mathbb{S}^1)$ *is conjugate to a rational rotation if and only if there is* $k \in \mathbb{N}$ *such that* $f^k = Id$.

A homeomorphism with irrational number is not necessarily conjugate to a rotation either. We will discuss this more in 5.2.

3. Action of $\mathrm{PSL}(2,\mathbb{R})$ and its subgroups

3.1. The projective action on the circle. The group $\mathrm{PSL}(2,\mathbb{R})$ acts on the projective space $\mathbb{RP}^1$, which is diffeomorphic to the circle $\mathbb{S}^1$. A simple formula can be given for this action: if we identify $\mathbb{RP}^1$ with $\mathbb{R} \cup \{\infty\}$, then the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts by the homography $x \mapsto \frac{ax+b}{cx+d}$.

With this formula, we see that the action is the extension to the circle at infinity $\partial_\infty \mathbb{H}^2$ of the isometric action of $\mathrm{PSL}(2,\mathbb{R})$ on the hyperbolic plane $\mathbb{H}^2$ (in the upper half plane model). This gives a link between hyperbolic geometry and the action of $\mathrm{PSL}(2,\mathbb{R})$ on the circle.

This action is uniquely transitive on triples of points, and the stabiliser of a point $x \in \mathbb{S}^1$ is isomorphic to the affine group $\mathrm{Aff}(\mathbb{R})$.

3.2. Elements of $\mathrm{PSL}(2,\mathbb{R})$. In particular, if $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ has three fixed points on $\mathbb{S}^1$, then $\gamma = Id$. We say that $\gamma \in \mathrm{PSL}(2,\mathbb{R}) \setminus \{Id\}$ is **hyperbolic** if it has two fixed points on $\mathbb{S}^1$, **parabolic** if it has one fixed point on $\mathbb{S}^1$, and **elliptic** if it has no fixed point on $\mathbb{S}^1$.

This classification can also be determined by considering elements of $\mathrm{PSL}(2,\mathbb{R})$ as matrices. Indeed, $\gamma \in \mathrm{PSL}(2,\mathbb{R}) \setminus \{Id\}$ is hyperbolic if and only $|\mathrm{Tr}(\gamma)| > 2$, parabolic if and only if $|\mathrm{Tr}(\gamma)| = 2$, and elliptic if and only if $|\mathrm{Tr}(\gamma)| < 2$.

A third way of understanding this classification is by looking at the action on the hyperbolic plane. An element $\gamma \in \mathrm{PSL}(2,\mathbb{R}) \setminus \{Id\}$ is elliptic if and only if it has a fixed point in $\mathbb{H}^2$, and it is hyperbolic if and only if it preserves a geodesic in $\mathbb{H}^2$ without fixing its points (the geodesic whose limits on the circle at infinity are the fixed points).

The dynamics of an element of $\mathrm{PSL}(2,\mathbb{R})$ are quite simple. If $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ is hyperbolic, then it has two two fixed points $N, S \in \mathbb{S}^1$. The derivatives $\gamma'(N)$ and $\gamma'(S)$ are simply the squares of the eigenvalues of the matrix associated to γ. This implies that they are different from 1, and that $\gamma'(N)\gamma'(S) = 1$. Hyperbolic elements of $\mathrm{PSL}(2,\mathbb{R})$ are the main example of north/south dynamics (see Figure 2.1): if $\gamma'(N) < 1$, and $x \in \mathbb{S}^1 \setminus \{N, S\}$, then $\gamma^n(x) \to N$ as $n \to +\infty$ and $\gamma^n(x) \to S$ as $n \to -\infty$.

North/south homeomorphisms of $\mathbb{S}^1$ are all conjugate to each other in $\mathrm{Homeo}(\mathbb{S}^1)$.

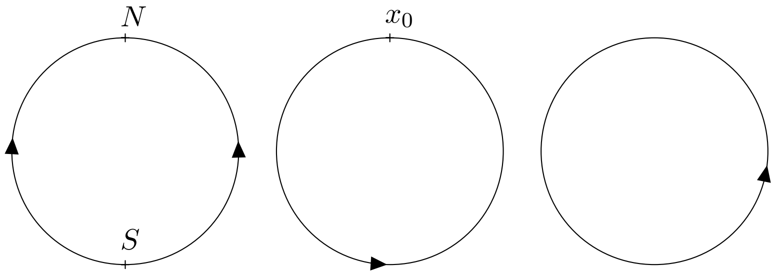

FIGURE 2.1. Dynamics of elements of $\mathrm{PSL}(2,\mathbb{R})$

However, hyperbolic elements of $\mathrm{PSL}(2,\mathbb{R})$ are not all conjugate in $\mathrm{PSL}(2,\mathbb{R})$ (conjugates have the same eigenvalues).

If γ is parabolic, let $x_0 \in \mathbb{S}^1$ be its unique fixed point. For any $x \in \mathbb{S}^1$, both positive and negative iterates accumulate on x_0. All parabolic elements are conjugate to each other in $\mathrm{PSL}(2,\mathbb{R})$.

If $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ is elliptic, then it is conjugate in $\mathrm{PSL}(2,\mathbb{R})$ to a unique element R_α of $\mathrm{SO}(2,\mathbb{R})$, i.e. a rotation.

3.3. The limit set. The limit set L_Γ of a subgroup $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ has an interpretation in terms of limit points of orbits in $\mathbb{H}^2$ (hence the name limit set). Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a non elementary group (in the sense defined above, i.e. with no finite orbit on $\mathbb{S}^1$) that does not fix a point of $\mathbb{H}^2$ (i.e. that is not conjugate to a subgroup of $\mathrm{SO}(2,\mathbb{R})$, according to §7.39 in [**Bea83**]). In this case, L_Γ is the set of accumulation points on $\partial_\infty \mathbb{H}^2$ of the orbit $\Gamma.x$ for any $x \in \mathbb{H}^2$.

Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be non elementary and non minimal (i.e. L_Γ is an exceptional minimal set). In this case, the complement of L_Γ in $\mathbb{S}^1$ is a countable union of open intervals: $\mathbb{S}^1 \setminus L_\Gamma = \bigcup_{n\in\mathbb{N}} I_n$. To each of these intervals we can associate a geodesic γ_n in $\mathbb{H}^2$ (whose endpoints on $\partial_\infty \mathbb{H}^2$ are the endpoints of I_n). The region of $\mathbb{H}^2$ bounded by these geodesics is called the **convex hull** of Γ, denoted by C_Γ. We say that Γ is **convex cocompact** if the quotient C_Γ/Γ is compact. A particular case of Ahlfors' Finiteness Theorem (see [**Ahl64**] or [**Ber65**]) states that any finitely generated discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ with only hyperbolic elements is convex cocompact.

3.4. Geodesic flow. The action of a subgroup $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is linked to the dynamics of the geodesic flow on $\mathrm{T}^1\mathbb{H}^2/\Gamma$. First, let us give a few definitions for hyperbolic flows.

3.4.1. *Hyperbolic flows.* Let φ^t be a complete flow generated by a vector field X on a manifold M. We say that a compact invariant set $K \subset M$ is **hyperbolic** if there are positive constants C, λ and a decomposition of tangent spaces $T_xM = E_x^s \oplus E_x^u \oplus \mathbb{R}.X$ for each $x \in K$ such that:

$$\forall x \in K\ \forall v \in E_x^s\ \forall t \geq 0\ \ \|D\varphi_x^t(v)\| \leq Ce^{-\lambda t}\|v\|$$

$$\forall x \in K\ \forall v \in E_x^u\ \forall t \leq 0\ \ \|D\varphi_x^t(v)\| \leq Ce^{\lambda t}\|v\|$$

The norm $\|.\|$ denotes the norm given by any Riemannian metric on M (since K is compact, the definition does not depend on the choice of a Riemannian metric). If the whole manifold M is a hyperbolic set, then we say that φ^t is an **Anosov** flow.

Let φ^t be a smooth flow on a manifold M. If $K \subset M$ is a compact hyperbolic set and $x \in K$, then we define the stable and unstable manifolds through x:

$$W^s(x) = \{z \in M | d(\varphi^t(x), \varphi^t(z)) \underset{t \to +\infty}{\longrightarrow} 0\}$$

$$W^u(x) = \{z \in M | d(\varphi^t(x), \varphi^t(z)) \underset{t \to -\infty}{\longrightarrow} 0\}$$

The Stable Manifold Theorem states that they are submanifolds of M tangent to E^s and E^u at x (see [**HP69**]).

The most important fact for us is that the limit $d(\varphi^t(x), \varphi^t(z)) \to 0$ is a uniformly decreasing exponential: for all compact set A and all $\varepsilon > 0$, there is a constant $C' > 0$ such that:

$$\forall x \in K \ \forall z \in W^s(x) \cap A \ \forall t \geq 0 \ \ d(\varphi^t(x), \varphi^t(z)) \leq C' e^{-(\lambda - \varepsilon)t}$$

$$\forall x \in K \ \forall z \in W^u(x) \cap A \ \forall t \leq 0 \ \ d(\varphi^t(x), \varphi^t(z)) \leq C' e^{(\lambda - \varepsilon)t}$$

We will denote by $W^s(K)$ (resp. $W^u(K)$) the union $W^s(K) = \bigcup_{x \in K} W^s(x)$ (resp. $W^u(K) = \bigcup_{x \in K} W^u(x)$).

3.4.2. *Geodesic flow of a hyperbolic surface.* The **geodesic flow** φ^t of the hyperbolic plane is a flow on its unit bundle $\mathrm{T}^1\mathbb{H}^2$ defined as follow: if $v \in \mathrm{T}^1\mathbb{H}^2$, consider the unique geodesic c_v of $\mathbb{H}^2$ such that $\dot{c}_v(0) = v$, and set $\varphi^t(v) = \dot{c}_v(t)$.

This flow can be introduced in a different way that will be very practical when we consider perturbations. Let Σ_3 be the set of ordered triples of $\mathbb{S}^1$:

$$\Sigma_3 = \{(x_-, x_0, x_+) \in (\mathbb{S}^1)^3 | x_- < x_0 < x_+ < x_-\}$$

We can identify $\mathrm{T}^1\mathbb{H}^2$ and Σ_3 in the following way: given a unit vector $v \in \mathrm{T}^1\mathbb{H}^2$, we consider x_- and x_+ the limits at $-\infty$ and $+\infty$ of the geodesic c_v given by v, and x_0 is the limit at $+\infty$ of the geodesic $c_\perp$ passing through the base point of v in an orthogonal direction, oriented to the right of v (see Figure 2.2). The map $v \mapsto (x_-, x_0, x_+)$ is a diffeomorphism from $\mathrm{T}^1\mathbb{H}^2$ to Σ_3.

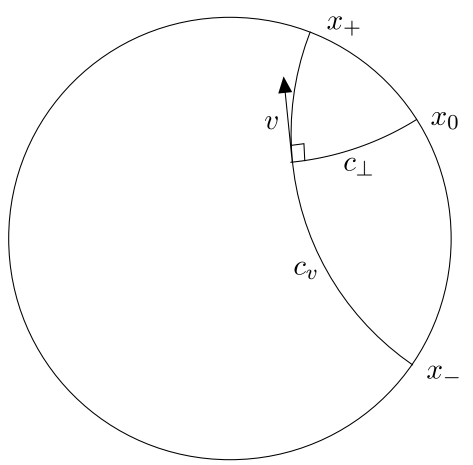

FIGURE 2.2. Identification between $\mathrm{T}^1\mathbb{H}^2$ and Σ_3

On Σ_3, the geodesic vector field is a rescaling of the constant vector field $(0, 1, 0)$, and the action α of $\mathrm{PSL}(2, \mathbb{R})$ is the diagonal action.

Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a discrete subgroup. The quotient $M = \mathrm{T}^1\mathbb{H}^2/\Gamma$ is a three-manifold (even when $\mathrm{T}^1\mathbb{H}^2$ is not a proper surface). The **geodesic flow** φ^t on $\mathrm{T}^1\mathbb{H}^2/\Gamma$ is the projection of the geodesic flow of $\mathrm{T}^1\mathbb{H}^2$ on $\mathrm{T}^1\mathbb{H}^2/\Gamma$.

The **non wandering set** Ω_φ of a flow is the set of points x such that there are sequences $x_n \to x$ and $t_n \to \infty$ satisfying $\varphi^{t_n}(x_n) \to x$. For the geodesic flow, it can be described as follows: its lift to $\mathrm{T}^1\mathbb{H}^2$ is the set of vectors tangent to a geodesic that lies entirely in C_Γ. Identifying $M = \mathrm{T}^1\mathbb{H}^2/\Gamma$ with $\Sigma_3/\alpha(\Gamma)$, the image of a point (x_-, x_0, x_+) in M is in Ω_φ if and only if $(x_-, x_+) \in L_\Gamma \times L_\Gamma$, and it is in $\mathrm{Per}(\varphi)$ if and only if (x_-, x_+) is the pair of fixed points of an element $\gamma \in \Gamma$.

3.4.3. *Convex cocompact groups.* The important property for the geodesic flow is that when the non wandering set Ω_φ of the geodesic flow on $M = \mathrm{T}^1\mathbb{H}^2/\Gamma$ is compact (i.e. when Γ is convex cocompact), it is a hyperbolic set. Since the closure of periodic orbits $Per(\varphi)$ is equal to Ω_φ, it is an **Axiom A** flow (Axiom A flows are a generalisation of Anosov flows that can be defined even on non compact manifolds). If $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is cocompact, then $M = \Omega_\varphi$ is compact, and φ^t is an Anosov flow.

3.5. Finite covers of $\mathrm{PSL}(2,\mathbb{R})$**.** From a topological point of view, $\mathrm{PSL}(2,\mathbb{R})$ is diffeomorphic to $\mathbb{R}^2 \times \mathbb{S}^1$, therefore it admits a unique covering of order k for all $k \in \mathbb{N}$. This covering $\mathrm{PSL}_k(2,\mathbb{R})$ acts on the circle: the group $\mathrm{PSL}_k(2,\mathbb{R})$ is the group of lifts of elements of $\mathrm{PSL}(2,\mathbb{R})$ to the k-covering of the circle (which is diffeomorphic to the circle).

The centre of $\mathrm{PSL}_k(2,\mathbb{R})$ is isomorphic to $\mathbb{Z}/k\mathbb{Z}$, and $\mathrm{PSL}(2,\mathbb{R})$ is the quotient of $\mathrm{PSL}_k(2,\mathbb{R})$ by its centre.

An element $\gamma \in \mathrm{PSL}_k(2,\mathbb{R})$ is elliptic (resp. parabolic, hyperbolic) if its image in $\mathrm{PSL}(2,\mathbb{R})$ is elliptic (resp. parabolic, hyperbolic).

An elliptic element of $\mathrm{PSL}_k(2,\mathbb{R})$ acts on the circle via a rotation.

A parabolic element of $\mathrm{PSL}(2,\mathbb{R})$ has k lifts in $\mathrm{PSL}_k(2,\mathbb{R})$, characterised by their rotation numbers $\frac{i}{k}$ for $0 \leq i < k$. They all have the same periodic points (fixed when $i = 0$) which form an orbit for the centre of $\mathrm{PSL}_k(2,\mathbb{R})$ (hence of cardinal k) . Similarly, a hyperbolic element of $\mathrm{PSL}(2,\mathbb{R})$ has k lifts, all of which has $2k$ periodic points.

4. Convergence groups

The convergence property is a simple dynamic description of subgroups of $\mathrm{Homeo}(\mathbb{S}^1)$ that are topologically conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. We will identify $\mathrm{PSL}(2,\mathbb{R})$ with its image in $\mathrm{Diff}(\mathbb{S}^1)$, and call a subgroup $G \subset \mathrm{Homeo}(\mathbb{S}^1)$ **Fuchsian** if $G \subset \mathrm{PSL}(2,\mathbb{R})$ (note that we do not ask for G to be discrete, although it will be the case in most of the interesting examples). We say that G is **topologically Fuchsian** if there is $h \in \mathrm{Homeo}(\mathbb{S}^1)$ such that $h^{-1}Gh \subset \mathrm{PSL}(2,\mathbb{R})$.

A sequence $(f_n)_{n\in\mathbb{N}} \in \mathrm{Homeo}(\mathbb{S}^1)^\mathbb{N}$ has the **convergence property** if there are $a, b \in \mathbb{S}^1$ such that, up to a subsequence, $f_n(x) \to b$ for all $x \neq a$. A group $G \subset \mathrm{Homeo}(\mathbb{S}^1)$ is a **convergence group** if every sequence in G either satisfies the convergence property or has an equicontinuous subsequence.

The classical definition of a convergence group also involves the sequence of the inverses f_n^{-1} and some uniform convergence, but it is not necessary in the case of $\mathbb{S}^1$. There is a second definition of convergence groups. Let $\Theta_3(\mathbb{S}^1)$ be the set of distinct triples of $\mathbb{S}^1$:

$$\Theta_3(\mathbb{S}^1) = \{(a,b,c) \in (\mathbb{S}^1)^3 | a \neq b \neq c \neq a\}$$

The convergence property can be checked via the diagonal action on $\Theta_3(\mathbb{S}^1)$. A group G acts **properly** on X if the set $G_K = \{g \in G | g.K \cap K \neq \emptyset\}$ is compact for all compact subset $K \subset X$.

Proposition 2.4.1. *A group* $G \subset \mathrm{Homeo}(\mathbb{S}^1)$ *is a convergence group if and only if the action of* G *on the space of distinct triples* $\Theta_3(\mathbb{S}^1)$ *is proper.*

Note that the definition of the properness of an action depends on a topology on the group. Here, the two candidates are the topology of $\mathrm{Homeo}(\mathbb{S}^1)$ and the compact open topology of $\mathrm{Homeo}(\Theta_3(\mathbb{S}^1))$, which happen to be identical.

For the proof of this equivalence, see [**Bow96**]. The idea is that if $f_n \to \infty$, then $f_n(a, b, c) \to \infty$ in $\Theta_3(\mathbb{S}^1)$ for $(a, b, c) \in \Theta_3(\mathbb{S}^1)$, which implies that there are at most two different limits.

From both definitions, it is clear that a subgroup of a convergence group still is a convergence group. There are several ways of seeing that $\mathrm{PSL}(2,\mathbb{R})$ is a convergence group. By using the second definition, we simply notice that the action of $\mathrm{PSL}(2,\mathbb{R})$ on the space of distinct triples $\Theta_3(\mathbb{S}^1)$ preserves the following Riemannian metric:

$$\frac{(y-z)^2}{(x-y)^2(x-z)^2}dx^2 + \frac{(z-x)^2}{(y-z)^2(y-x)^2}dy^2 + \frac{(x-y)^2}{(z-x)^2(z-y)^2}dz^2$$

Since the isometry group of a Riemannian metric preserves a distance, it always acts properly.

Something rather interesting is that most proofs that show that a certain subgroup of $\mathrm{Homeo}(\mathbb{S}^1)$ has the convergence property (in order to show that it is topologically Fuchsian) apply to $\mathrm{PSL}(2,\mathbb{R})$.

From the first definition, it is clear that the convergence property is invariant under topological conjugacy, and that a subgroup of a convergence group is a convergence group, so all topologically Fuchsian groups are convergence groups. The interesting fact is that the converse is true.

Theorem 2.4.2. *A convergence group $G \subset \mathrm{Homeo}(\mathbb{S}^1)$ is topologically Fuchsian.*

This theorem was proved by Gabai and Casson-Jungreis ([**Gab92**],[**CJ94**]), concluding the work of many others.

5. Semi conjugacy

Let us recall a few results of [**Ghy87b**] on semi conjugacy. A map $h : \mathbb{S}^1 \to \mathbb{S}^1$ is **non decreasing of degree one** if it is non constant and it admits a non decreasing lift $\widetilde{h} : \mathbb{R} \to \mathbb{R}$ such that $\widetilde{h}(x + 1) = \widetilde{h}(x) + 1$. We say that $\rho_1 : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ is **semi conjugate** to $\rho_2 : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ if there is a non decreasing map of degree one $h : \mathbb{S}^1 \to \mathbb{S}^1$ such that $h \circ \rho_1(\gamma) = \rho_2(\gamma) \circ h$ for all $\gamma \in \Gamma$.

If $h : \mathbb{S}^1 \to \mathbb{S}^1$ is non decreasing of degree one, then we denote by h_l (resp. h_r) the map that is left (resp. right) continuous and equal to h except at points where h is not left (resp. right) continuous. Recall that the set of discontinuity points of h is at most countable, so $h(x) = h_l(x) = h_r(x)$ except on a countable set.

If $a, b \in \mathbb{S}^1$, the we denote by $]a\,, b[$ the set $\{x \in \mathbb{S}^1 | a < x < b \leq a\}$ (and define $[a\,, b]$, $[a\,, b[$, $]a\,, b]$ similarly). We define $G(h) \subset \mathbb{S}^1 \times \mathbb{S}^1$ as the union of the segments $\{x\} \times [h_l(x)\,, h_r(x)]$ for all $x \in \mathbb{S}^1$. If $\rho_1, \rho_2 : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ are semi conjugate via the map h, then h_l and h_r also provide a semi conjugacy between ρ_1 and ρ_2. This shows that the important object in a semi conjugacy is not the map h but its "graph" $G(h)$.

The advantage of this definition of semi conjugacy (contrary to the standard definition where h is asked to be continuous) is that it is an equivalence relation.

For semi conjugacy between two homeomorphisms (i.e. when $\Gamma = \mathbb{Z}$), the rotation number is a complete invariant: two homeomorphisms are semi conjugate if and only if they have the same rotation number. Consequently, all homeomorphisms are semi conjugate to a rotation.

For elementary groups, we have a simple characterisation of semi conjugacy:

Proposition 2.5.1. *Let $\rho : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ have a finite orbit $E \subset \mathbb{S}^1$ with at least two elements. A representation $\tau : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ is semi conjugate to ρ if and only if it has a finite orbit $F \subset \mathbb{S}^1$ such that there is a cyclic order preserving bijection from E to F which is equivariant under the actions of Γ.*

An implication of this is that if ρ and τ are semi conjugate, then ρ is non elementary if and only if τ is non elementary.

5.1. Collapsing non wandering intervals. When considering a representation $\rho : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ with an exceptional minimal set K, it is standard to consider the collapsed action $\mathrm{Col}(\rho)$ defined by collapsing the connected components of $\mathbb{S}^1 \setminus K$ to points. More precisely, we can consider a continuous map $h : \mathbb{S}^1 \to \mathbb{S}^1$ that is non decreasing of degree one, such that the intervals where h is constant are exactly the connected components of $\mathbb{S}^1 \setminus K$. It induces a unique representation $\mathrm{Col}(\rho) : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ such that $\mathrm{Col}(\rho) \circ h = h \circ \rho$. The interest in considering this collapsed action is that the orbits of $\mathrm{Col}(\rho)$ are dense in $\mathbb{S}^1$. It is well defined up to a conjugacy in $\mathrm{Homeo}(\mathbb{S}^1)$.

If ρ has values in $\mathrm{Diff}(\mathbb{S}^1)$, it is not clear whether $\mathrm{Col}(\rho)$ can also be asked to be differentiable. However, if ρ has values in $\mathrm{PSL}(2, \mathbb{R})$, then the collapsed action $\mathrm{Col}(\varphi)$ still has the convergence property. This implies that $\mathrm{Col}(\rho)$ is topologically conjugate to a subgroup of $\mathrm{PSL}(2, \mathbb{R})$. In other words, the class of topologically Fuchsian representations is stable under collapsing.

5.2. Opening orbits. It is also possible to go backwards. Given a representation $\rho : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$, and a point $x_0 \in \mathbb{S}^1$, we can open the orbit of x to intervals on which the action of $\mathrm{Stab}(x_0)$ can be chosen. Indeed, choose an increasing map of degree one $h : \mathbb{S}^1 \to \mathbb{S}^1$ whose points of discontinuity are exactly the points of the orbit of x. For every $x \in \Gamma.x_0$, let I_x be the interval between the left and right limits of h at x. Fix an action $\alpha : \mathrm{Stab}(x_0) \to \mathrm{Homeo}(I_{x_0})$. For every $x \in \Gamma.x_0$, we choose δ_x such that $x = \rho(\delta_x)(x_0)$ and a homeomorphism $h_x : I_{x_0} \to I_x$. We define $\hat{\rho} : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ such that:

(1) $\hat{\rho}(\gamma)(h(x)) = h(\rho(\gamma)(x))$ if $x \notin \Gamma.x_0$
(2) $\hat{\rho}(\gamma)(y) = \alpha(\gamma)(y)$ if $y \in I_{x_0}$ and $\gamma \in \mathrm{Stab}(x_0)$
(3) $\hat{\rho}(\gamma)(y) = h_{\rho(\gamma)(x)}(\alpha(\delta^{-1}_{\rho(\gamma)(x)} \gamma \delta_x)(h_x^{-1}(y)))$ for $y \in I_x$ and $x \in \Gamma.x_0$

This defines an action $\hat{\rho} : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ such that $\hat{\rho} \circ h = h \circ \rho$.

Note that if ρ has dense orbits, then $\mathrm{Col}(\hat{\rho}) = \rho$ (because the collapsed action is unique, but there are different possibilities for opening orbits).

Once again, if ρ has values in $\mathrm{Diff}(\mathbb{S}^1)$, then it is not easy to construct $\hat{\rho}$ in a differentiable way. A well known construction of Denjoy ([**Den32**]) shows that if we start with an irrational rotation, then we can open an orbit in a C^1 way. However, Denjoy also showed that it is not possible to obtain a C^2 diffeomorphism. A theorem of Matsumoto (see [**Mat87**]) states that if Γ_g is the fundamental group of the closed oriented surface Σ_g of genus $g \geq 2$, and $\rho : \Gamma_g \to \mathrm{PSL}(2, \mathbb{R})$ is the the representation given by a hyperbolic metric on Σ_g, then it is not possible to open an orbit with a C^2 action (because the Euler number stays the same).

CHAPTER 3

Towards a topological classification

1. Results

From the point of view of group isomorphisms, we get the following classification for isometry groups of spatially compact surfaces:

Theorem 3.1.1. *Let (M, g) be a spatially compact surface such that $\mathrm{Isom}(M, g)$ acts non properly on M. The isometry group $\mathrm{Isom}(M, g)$ is isomorphic to a subgroup of a finite cover of $\mathrm{PSL}(2, \mathbb{R})$.*

However, a group isomorphism does not give much information about the geometry.

The goal of this section is to classify the images $\rho_1^M(\mathrm{Isom}(M, g))$ of isometry groups $\mathrm{Isom}(M, g)$ of spatially compact surfaces up to topological conjugacy. We will only get partial results in this direction. However, we will obtain a classification up to semi conjugacy.

Theorem 3.1.2. *Let (M, g) be a spatially compact surface such that $\mathrm{Isom}(M, g)$ acts non properly on M. Then ρ_1^M and ρ_2^M are semi conjugate to each other, and the restrictions to $\mathrm{Isom}(M, g)$ are faithful. There are $k \in \mathbb{N}$ and a faithful representation $\rho : \mathrm{Isom}(M, g) \to \mathrm{PSL}_k(2, \mathbb{R})$ that is semi conjugate to the restrictions of ρ_1^M and ρ_2^M to $\mathrm{Isom}(M, g)$.*

Theorem 3.1.1 is a straightforward consequence of this result (because ρ is faithful).

Theorem 3.1.2 raises an important question: can we replace semi conjugacy with conjugacy in $\mathrm{Homeo}(\mathbb{S}^1)$? Under the assumption that the conformal boundary is acausal, we obtain such a conjugacy.

Theorem 3.1.3. *Let (M, g) be a spatially compact surface such that $\mathrm{Isom}(M, g)$ acts non properly on M. Assume that the conformal boundary of (M, g) is acausal. Then ρ_1^M and ρ_2^M are faithful, topologically conjugate to each other, and the restrictions to $\mathrm{Isom}(M, g)$ are topologically conjugate to a representation in a finite cover of $\mathrm{PSL}(2, \mathbb{R})$.*

However, we will see in chapter 5 that this result does not hold if we do not ask for the conformal boundary to be acausal.

We will constantly make use of the conformal models constructed in the previous chapter. The main use of the embedding in the flat cylinder will be to show that we can restrict our study to spatially compact surfaces that embed conformally in the flat torus. In this case, we no longer need non properness of the action to classify isometry groups up to semi conjugacy.

Theorem 3.1.4. *Let (M, g) be a spatially compact surface that embeds conformally in the flat torus. Then ρ_1^M and ρ_2^M are semi conjugate to each other, and the restrictions to $\mathrm{Isom}(M, g)$ are faithful. There are $k \in \mathbb{N}$ and a faithful representation $\rho : \mathrm{Isom}(M, g) \to \mathrm{PSL}_k(2, \mathbb{R})$ that is semi conjugate to the restrictions of ρ_1^M and ρ_2^M to $\mathrm{Isom}(M, g)$.*

Theorem 3.1.5. *Let (M, g) be a spatially compact surface that embeds conformally in the flat torus. Assume that the conformal boundary of (M, g) is acausal. Then ρ_1^M and ρ_2^M are faithful, topologically conjugate to each other, and the restrictions to $\mathrm{Isom}(M, g)$ are topologically conjugate to a representation in a finite cover of $\mathrm{PSL}(2, \mathbb{R})$.*

All these results deal with continuous conjugacy, which seems rather unsatisfying since ρ_1^M and ρ_2^M take values in $\mathrm{Diff}(\mathbb{S}^1)$. Chapter 4 will be devoted to the study of ρ_1^M and ρ_2^M up to differentiable conjugacy, and chapter 5 deals with the necessity of the acausal character of the conformal boundary in order to obtain a topological conjugacy.

2. First examples

2.1. The constant curvature model spaces. A particularity of two dimensional Lorentzian geometry is that, from the isometry group point of view, there are only two constant curvature geometries. Indeed, if (M, g) is a Lorentz surface, then $-g$ is also a Lorentz metric. If K_g is the curvature of g, then the curvature of $-g$ is $K_{-g} = -K_g$. This shows that the positive and negative constant curvature spaces have the same isometry group.

The flat simply connected model is the Minkowski space $\mathbb{R}^{1,1} = (\mathbb{R}^2, dxdy)$. The spatially compact model is the flat cylinder: it is the quotient of $\mathbb{R}^{1,1}$ by a spacelike translation. The second conformal model described above shows that any spatially compact surface is conformal to an open set of the flat cylinder bounded by two non timelike curves.

As we mentioned earlier, the positive and negative constant curvature spaces share the same isometry group. However, the causality of a metric g does not imply the causality of $-g$. In the classic model spaces, the positive model is the one sheeted hyperboloïd in $\mathbb{R}^{1,2}$ which is spatially compact. But the negative curvature model is not causal, and for this reason we will only consider the positive curvature model.

2.2. Open sets of dS_2 and subgroups of $\mathrm{PSL}(2, \mathbb{R})$. We wish to understand which spatially compact surfaces (M, g) can satisfy $\rho_1^M = \rho_2^M$ and $\rho_1^M(\mathrm{Isom}(M, g)) \subset \mathrm{PSL}(2, \mathbb{R})$. We will first focus on open sets of dS_2 and conformal changes of the metric on these open sets.

Given $h \in \mathrm{Homeo}(\mathbb{S}^1)$, the connected components of $\mathbb{S}^1 \times \mathbb{S}^1 \setminus (\Delta \cup Gr(h))$ are globally hyperbolic (because h is orientation preserving), so they have two possible topologies: either a plane or a cylinder. If h has no fixed point, then $\mathbb{S}^1 \times \mathbb{S}^1 \setminus (\Delta \cup Gr(h))$ has two connected components, both spatially compact. If h has at least one fixed point, then there is one spatially compact component, which we denote it by Ω_h (see Figure 3.1). The action of a group $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ preserves Ω_h if and only h commutes with every element of Γ. This shows that looking for open sets preserved by a subgroup of $\mathrm{PSL}(2, \mathbb{R})$ can be achieved by constructing homeomorphisms that commute with every element of the group. Note that Γ acts non properly on Ω_h if and only if it acts non properly on dS_2, which is equivalent to Γ not being relatively compact (this will be shown in Proposition 3.5.3).

If $g_\sigma = e^\sigma g_{\mathrm{dS}_2}$, then the action of $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is isometric for g_σ if and only if σ is invariant under Γ.

Note that all subgroups cannot have a non trivial commuting homeomorphism or invariant function. If $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ (or any lattice), then it has dense orbits on dS_2, which shows that any continuous invariant function is constant, and the only commuting homeomorphism is the identity. More generally, any commuting homeomorphism is the identity on the limit set.

Proposition 3.2.1. *Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a subgroup such that $L_\Gamma \neq \mathbb{S}^1$. Then there is $h \in \mathrm{Homeo}(\mathbb{S}^1) \setminus \{Id\}$ that commutes with every element of Γ.*

Proof. We write $\mathbb{S}^1 \setminus L_\Gamma = \bigcup_{i \in \mathbb{N}} I_i$ as the union of its connected components. This induces an action of Γ on $\mathbb{N}$, and let $R \subset \mathbb{N}$ be a fundamental domain. For $i \in R$, we set $h_{/I_i}$ to be a homeomorphisms fixing the endpoints of I_i that commutes with the elements

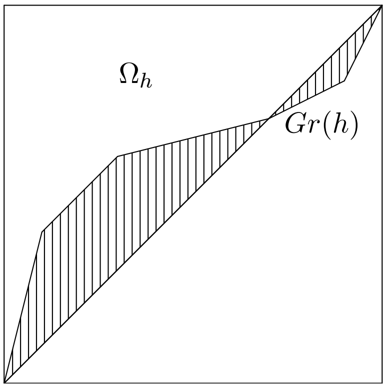

FIGURE 3.1. Ω_h

of Γ stabilizing I_i and that is not the identity. For $\gamma \in \Gamma$, we set $h_{/\gamma(I_i)} = \gamma \circ h_{/I_i} \circ \gamma^{-1}$. It is well defined because if $\gamma_1(I_i) \cap \gamma_2(I_i) \neq \emptyset$, then $\gamma_2^{-1}\gamma_1$ stabilizes I_i, hence commutes with $h_{/I_i}$, and $\gamma_1 \circ h_{/I_i} \circ \gamma_1^{-1} = \gamma_2 \circ h_{/I_i} \circ \gamma_2^{-1}$.

It is continuous because it is the identity on L_Γ, and it commutes with all elements of Γ. $\square$

If $h \neq Id$, then Ω_h is not conformal to dS_2, so we cannot expect all the Lorentz surfaces under study to be conformal to dS_2, even if the group is non elementary.

It is easy to see that the conformal boundary is not necessarily acausal: if $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is non elementary and $L_\Gamma \neq \mathbb{S}^1$, then write $\mathbb{S}^1 \setminus L_\Gamma = \bigcup_{n\in\mathbb{N}}]a_n\,, b_n[$, and let h be the identity on L_Γ and $h(x) = b_n$ for $x \in\,]a_n\,, b_n[$. It is a non decreasing map of degree one that commutes with Γ, and it bounds an open set of dS_2 invariant by Γ.

Proposition 3.2.2. *Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a subgroup such that $L_\Gamma \neq \mathbb{S}^1$. Then there is a non constant smooth Γ-invariant function $\sigma : \mathrm{dS}_2 \to \mathbb{R}$.*

PROOF. Start by writing $\mathbb{S}^1 \setminus L_\Gamma = \bigcup_{i\in\mathbb{N}} I_i$ as the union of its connected components. We start by setting $\sigma = 0$ on $(L_{\rho(\Gamma)} \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L_{\rho(\Gamma)}) \setminus \Delta$ and on $I_i \times I_i \setminus \Delta$ for $i \in \mathbb{N}$. For $x \in I_i \times I_j$ with $i \neq j$, consider R_1, R_2, R_3, R_4 the four rectangles that have x as one corner and a corner of $I_i \times I_j$ as the opposite corner (see Figure 3.2). Let $\sigma(x) = \omega(R_1)\omega(R_2)\omega(R_3)\omega(R_4)$ where ω is the volume form associated to the de Sitter metric. By using the explicit formula $\omega([a\,, b] \times [c\,, d]) = 4\mathrm{Log}([a, b, c, d])$ where $[a, b, c, d] = \frac{a-c}{a-d}\frac{b-d}{b-c}$ is the cross-ratio, we see that σ is continuous. The function σ is smooth in the interior of rectangles $I_i \times I_j$, i.e. where it is non zero. If $F : \mathbb{R} \to \mathbb{R}$ is smooth and constant on a neighbourhood of 0 sufficiently small so that $F \circ \sigma$ is not constant, then $F \circ \sigma$ is Γ-invariant, non constant and smooth.

There are many other ways of constructing invariant functions. We could set $\sigma(x)$ on $I_i \times I_i$ to be $F(\omega(R))$ where R is the rectangle amongst R_1, R_2, R_3 and R_4 defined above that is included in $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$ (see Figure 3.2).

Finally, we could also choose σ arbitrarily on the squares $I_i \times I_i$ where i is in a fundamental domain for the action of Γ on the connected components of $\mathbb{S}^1 \setminus L_\Gamma$, and let σ be constant on rectangles $I_i \times I_j$ with $i \neq j$. $\square$

This result implies that the curvature of a Lorentz surface with a non proper action of the isometry group is not necessarily constant.

By choosing the function σ arbitrarily on the rectangles $I_i \times I_j$ where (i, j) lies in a fundamental domain for the action of Γ on $\mathbb{N}^2$ and extending by invariance (and always composing with a cut-off function in $\mathbb{R}$ to get smoothness), we could show that there are Lorentz metrics such that all the local isometries are restrictions of elements of Γ (by

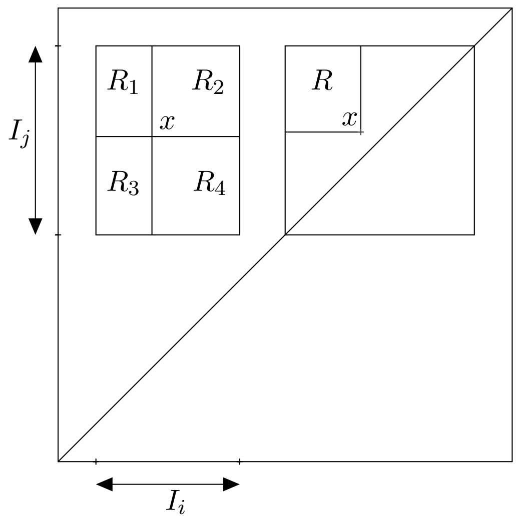

FIGURE 3.2. Construction of invariant functions

showing that such a metric is generic in the set of Γ-invariant metrics).

Note that the same method would work on Ω_h and could produce an invariant metric g_σ that is not extendible to $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$, so maximal spatially compact surfaces can have a non trivial conformal boundary, eventually not acausal.

2.3. Finite covers. If (M, g) is a spatially compact surface, then so are its finite covers. By applying this to the de Sitter space, we find spacetimes for which the representations ρ_1^M, ρ_2^M have values in $\mathrm{PSL}_k(2,\mathbb{R})$ (the order k covering of $\mathrm{PSL}(2,\mathbb{R})$, its elements are the lifts of elements of $\mathrm{PSL}(2,\mathbb{R})$ to the order k covering of $\mathbb{S}^1$, which is still a circle).

By taking finite covers of the examples constructed above, we obtain groups that are covers of subgroups of $\mathrm{PSL}(2,\mathbb{R})$. However, by using similar constructions starting with the k-covering of dS_2, we obtain subgroups of $\mathrm{PSL}_k(2,\mathbb{R})$ that are not covers of subgroups of $\mathrm{PSL}(2,\mathbb{R})$.

2.4. Extensions of dS_2. All the examples that we have defined so far embed conformally in the flat torus. However, it is not always the case.

Proposition 3.2.3. *There are spatially compact surfaces (M, g) with a non proper action of $\mathrm{Isom}(M, g)$ that do not embed conformally in $\mathbb{T}^2$.*

PROOF. Start with $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ a parabolic element. Let $x_0 \in \mathbb{S}^1$ be its fixed point. We consider the open sets $U = \{(x,y) \in \mathbb{S}^1 \times \mathbb{S}^1 | x_0 < y < x < x_0\}$ and $V = \{(x,y) \in \mathbb{S}^1 \times \mathbb{S}^1 | x_0 < \gamma(x) < y < x_0\}$ (see Figure 3.3). Up to replacing γ by γ^{-1}, we can assume that $\Delta \setminus \{(x_0, x_0)\} \subset V$. Let g_U be the restriction to U of the de Sitter metric. Let g_V be a metric on V preserved by γ that is equal to the de Sitter metric in a neighbourhood of the axes of γ (such a metric exists because γ acts properly on the complement of the axes). Let M be the manifold obtained by gluing U and V along $\{x_0\} \times \mathbb{S}^1 \cup \mathbb{S}^1 \times \{x_0\}$. Since the metrics g_U and g_V are equal on a neighbourhood of the glued parts, they endow M with a Lorentz metric g that is preserved by γ. The lightlike geodesics leaving from a point of $U \cap V$ meet again, and γ acts non properly on M. The graph of any rotation R_α is a Cauchy hypersurface in M, therefore it is spatially compact. $\square$

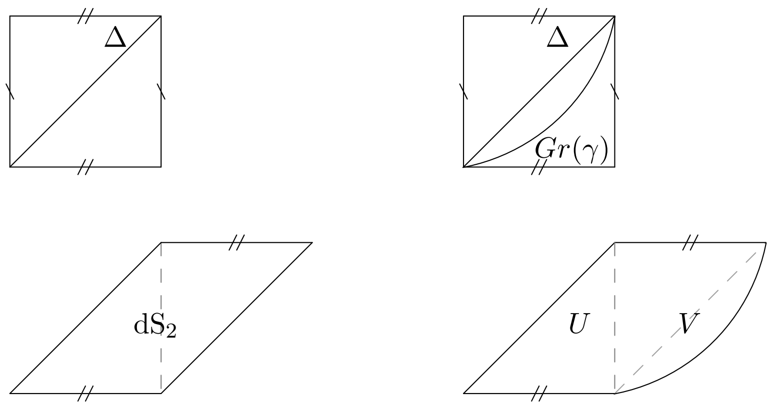

FIGURE 3.3. Extension of dS_2

We constructed this example with a parabolic element, but the same method would apply with a hyperbolic element, or more generally with any subgroup $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ such that $L_\Gamma \neq \mathbb{S}^1$.

3. Study of a specific case

Before dealing with the general case, we will prove Theorem 3.1.4 and Theorem 3.1.5 when $h_\downarrow = h_\uparrow$. The specificity of this case is that we will not need to consider finite covers of $\mathrm{PSL}(2,\mathbb{R})$, so we can directly apply the convergence property. In this section, we will not require any smoothness on the Lorentz metric, continuity will be enough.

3.1. Metrics in the conformal class of dS_2. Let us start by proving an even more specific case of Theorem 3.1.5: assume that the spatially compact surface (M,g) embeds conformally in the flat torus and that $h_\downarrow = h_\uparrow = Id$, in other words (M,g) is conformally equivalent to dS_2. In this case, $\rho_1^M = \rho_2^M$, which implies that they are faithful.

Let us reformulate the problem in terms of the action of the isometry group. Indeed, the diagonal action of a group $G \subset \mathrm{Diff}(\mathbb{S}^1)$ preserves a Lorentz metric $g(x,y)dxdy$ on $\mathcal{C} = \mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$ if and only if it preserves the measure associated to it. A result of Navas (Proposition 1.1 in [**Nav06**]) states that for a certain type of measure, the action is topologically Fuchsian (i.e. topologically conjugate to the projective action of a subgroup of $\mathrm{PSL}(2,\mathbb{R})$).

Theorem 3.3.1 (Navas)**.** *Let μ be a measure on $\mathcal{C}$ that is finite on compact sets, such that horizontal and vertical lines are negligible and such that $\mu([a\,,b[\times]b\,,c]) = \infty$ for $a < b < c < a$ in $\mathbb{S}^1$. The group G_μ of circle homeomorphisms that preserve μ is topologically Fuchsian.*

Navas used this result in [**Nav02**] to show that infinite Kazhdan groups cannot act on the circle by C^2 diffeomorphisms. When dealing with Lorentz metrics on $\mathcal{C}$, we only obtain measures that are absolutely continuous with respect to the Lebesgue measure with a positive density.

If ω is a continuous volume form on $\mathcal{C}$, then we will denote by G_ω the group of circle homeomorphisms f such that the map $(x,y) \mapsto (f(x), f(y))$ of $\mathcal{C}$ preserves the measure defined by ω.

Theorem 3.1.5 in the case where $h_\downarrow = h_\uparrow = Id$ can be formulated in terms of G_ω.

Theorem 3.3.2. *Let ω be a continuous volume form on $\mathcal{C}$. The group G_ω is topologically Fuchsian.*

Lemma 3.3.3. *If ω is a continuous volume form, then $G_\omega \subset \mathrm{Diff}(\mathbb{S}^1)$*

PROOF. Since the map (f, f) preserves a measure in the class of the Lebesgue measure on $\mathcal{C}$, it is absolutely continuous, and so is f on $\mathbb{S}^1$. The derivative of f satisfies the relation $\omega(f(x), f(y))f'(x)f'(y) = \omega(x, y)$ for almost every x, y, therefore f' is continuous and f is C^1. A bootstrap argument shows that if ω is C^k with $k \geq 0$, then $G_\omega \subset \mathrm{Diff}^{k+1}(\mathbb{S}^1)$. $\square$

The fact that G_ω is a group of diffeomorphisms gives us a more practical definition:

$$G_\omega = \{f \in \mathrm{Diff}(\mathbb{S}^1) | \forall x \neq y \ \omega(f(x), f(y))f'(x)f'(y) = \omega(x, y)\}$$

By choosing the appropriate definition, it is very easy to show that G_ω has the convergence property.

PROOF OF THEOREM 3.3.2. Let h be the Riemannian metric on $\Theta_3(\mathbb{S}^1)$ defined by:

$$h_{(x,y,z)} = \frac{\omega(x, y)\omega(x, z)}{\omega(y, z)} dx^2 + \frac{\omega(y, z)\omega(y, x)}{\omega(z, x)} dy^2 + \frac{\omega(z, x)\omega(z, y)}{\omega(x, y)} dz^2$$

It is a Riemannian metric on $\Theta_3(\mathbb{S}^1)$ that is preserved by the action of G_ω. This implies that this action is proper (it is a straightforward consequence of Ascoli's Theorem). Recall that this is one of the definitions of a convergence group (section 4 page 30), therefore G_ω is topologically Fuchsian.

$\square$

3.2. Complement of one curve in $\mathbb{T}^2$. We now study the case where $h_\downarrow = h_\uparrow$ is any non decreasing map of degree one. Once again, we will deal with continuous metrics. For this purpose, we will need to reprove some classical results.

If $h : \mathbb{S}^1 \to \mathbb{S}^1$ is non decreasing of degree one, then we denote by $G(h) \subset \mathbb{S}^1 \times \mathbb{S}^1$ the union of its graph and of vertical segments joining discontinuities. We set $M_h = \mathbb{S}^1 \times \mathbb{S}^1 \setminus G(h)$. Once again, a Lorentz metric in the conformal class of $dxdy$ on M_h can be seen as a volume form ω on M_h. We will denote by G_ω the isometry group of the Lorentz metric associated to ω.

Recall that h_l (resp. h_r) denotes the left continuous (resp. right continuous) non decreasing map of degree one that is equal to h except at points where it is not left (resp. right) continuous.

We are going to prove the following cases of Theorem 3.1.4 and Theorem 3.1.5:

Proposition 3.3.4. *Let ω be a continuous volume form on $M_h = \mathbb{S}^1 \times \mathbb{S}^1 \setminus G(h)$. The representations $\rho_1, \rho_2 : G_\omega \to \mathrm{Diff}(\mathbb{S}^1)$ are faithful.*

Theorem 3.3.5. *Let ω be a continuous volume form on $M_h = \mathbb{S}^1 \times \mathbb{S}^1 \setminus G(h)$. There is a representation $\rho : G_\omega \to \mathrm{PSL}(2, \mathbb{R})$ that is semi conjugate to ρ_1.*

Theorem 3.3.6. *Let ω be a continuous volume form on $M_h = \mathbb{S}^1 \times \mathbb{S}^1 \setminus G(h)$. If $h \in \mathrm{Homeo}(\mathbb{S}^1)$, then ρ_1 is topologically Fuchsian.*

3.2.1. *Faithfulness of the actions.* The standard proof of the fact that pseudo Riemannian isometries are determined by their 1-jet at any point uses the local existence and uniqueness of geodesics, which is not true for continuous metrics. We will see that this result remains true in our case, by using the fact that we can still define isotropic geodesics.

Lemma 3.3.7. *Let ω be a continuous volume form on $M_h = \mathbb{S}^1 \times \mathbb{S}^1 \setminus G(h)$. If $\varphi \in G_\omega \setminus \{Id\}$ fixes a point $(x_0, y_0) \in M_h$, then $\rho_1(\varphi)'(x_0)\rho_2(\varphi)'(y_0) = 1$ and $\rho_1(\varphi)'(x_0) \neq 1$.*

PROOF. Write $f = \rho_1(\varphi)$ and $g = \rho_2(\varphi)$.

The identity $\omega(f(x), g(y))f'(x)g'(y) = \omega(x, y)$ considered at (x_0, y_0) shows that $f'(x_0)g'(y_0) = 1$. Assume that $f'(x_0) = 1$ (hence $g'(y_0) = 1$). Since $\varphi \neq Id$, let us assume that $f \neq Id$ (the case where $g \neq Id$ is similar).

Let $x(t)$ be a maximal solution of the Cauchy problem:

$$\begin{cases} x'(t) = & \frac{1}{\omega(x(t), y_0)} \\ x(0) = & x_0 \end{cases}$$

Not only does x exist (Cauchy-Peano Theorem), but it is also unique (so are solutions to all equations $y' = F(y)$ in $\mathbb{R}$ where $F > 0$). It should be seen as a parametrisation of the geodesic $(\mathbb{S}^1 \times \{y_0\}) \cap M_h$ for the pseudo Riemannian metric associated to ω. Since $x' > 0$, it is a diffeomorphism from an open interval $I \subset \mathbb{R}$ onto its image $\mathbb{S}^1 \setminus K$ where K is the set of points $x \in \mathbb{S}^1$ such that $(x, y_0) \in G(h)$. Let $\alpha = x^{-1} \circ f \circ x$. A simple calculation shows that $\alpha'(t) = 1$ for all $t \in I$. Since $\alpha(0) = 0$, we see that $\alpha = Id$ and $f(x) = x$ for all x such that $(x, y_0) \in M_h$.

Let $K = [a_1\,, a_2]$ and let $[b_1\,, b_2] = [h_l(x_0)\,, h_r(x_0)]$. If $a_1 = a_2$, then f is the identity on a dense subset of $\mathbb{S}^1$, so $f = Id$. If $a_1 \neq a_2$, then $(a_1, b_1) \in M_h$ is a fixed point of φ such that $f'(a_1) = 1$, so f is the identity on all points x such that $(x, b_1) \in M_h$, which includes $[a_1\,, a_2]$, so $f = Id$.

$\square$

The first consequence of this rigidity is the faithfulness of the actions ρ_1 and ρ_2.

PROOF OF PROPOSITION 3.3.4. Assume that $\rho_1(\varphi) = Id$. Then $g = \rho_2(\varphi)$ satisfies $g \circ h = h$, which implies that g has fixed points. If $g(y) = y$, then choose $x \in \mathbb{S}^1$ such that $(x, y) \notin G(h)$. Then (x, y) is a fixed point of φ such that $f'(x) = 1$. Lemma 3.3.7 implies that $\varphi = Id$. $\square$

3.3. Elementary groups. We will now give a proof of Theorem 3.3.5 for elementary groups. We start with stabilizers of points.

Recall that the affine group $\mathrm{Aff}(\mathbb{R})$ can be realised as a subgroup of $\mathrm{PSL}(2, \mathbb{R})$ as the stabilizer of a point in $\mathbb{S}^1$.

Lemma 3.3.8. *Let h be an non decreasing map of degree one of $\mathbb{S}^1$, and let ω be a continuous volume form on $M_h = \mathbb{S}^1 \times \mathbb{S}^1 \setminus G(h)$. Let $x_0 \in \mathbb{S}^1$, and set $G = \{\varphi \in G_\omega | \rho_1(\varphi)(x_0) = x_0\}$. There is a representation $\rho : G \to \mathrm{Aff}(\mathbb{R}) \subset \mathrm{PSL}(2, \mathbb{R})$ that is semi conjugate to ρ_1.*

PROOF. Let us assume that G is non trivial.

Let $b = h(x_0)$. If $\varphi \in G$, then $\rho_2(\varphi)(b) = b$, i.e. φ preserves the horizontal line $\mathbb{S}^1 \times \{b\}$. Fix $a \in \mathbb{S}^1$ such that $(a, b) \in M_h$, and let x be a maximal solution of the Cauchy problem:

$$\begin{cases} x'(t) = & \frac{1}{\omega(x(t), b)} \\ x(0) = & a \end{cases}$$

Just as in Lemma 3.3.7, it is a parametrisation of the horizontal geodesic passing through (a, b), and it is a diffeomorphism from an open interval $I \subset \mathbb{R}$ onto $\mathbb{S}^1 \setminus \overline{h^{-1}(\{b\})}$. If $\varphi \in G$, then a simple calculation shows that $(x^{-1} \circ \rho_1(\varphi) \circ x)'(t) = \frac{1}{\rho_2(\varphi)'(b)}$ for all $t \in \mathbb{R}$ such that $x(t)$ is defined. This shows that x conjugates the action of G on $\mathbb{S}^1 \setminus \overline{h^{-1}(\{b\})}$ with a subgroup of $\mathrm{Aff}(\mathbb{R})$.

Since G is non trivial, the interval I has non trivial affine diffeomorphisms, so I is either $\mathbb{R}$, either affinely equivalent to $]-\infty\,, 0[$ or $]0\,, +\infty[$, in which case the action of the affine group is differentially conjugate to the action of $\mathbb{R}$ on itself by translations. Therefore, up to changing I and x (while preserving the affine structure), we can assume that $I = \mathbb{R}$.

Let $\psi : \mathbb{S}^1 \to \mathbb{S}^1 = \mathbb{R} \cup \{\infty\}$ be defined by $\psi = x^{-1}$ on $\mathbb{S}^1 \setminus \overline{h^{-1}(\{b\})}$ and $\psi \equiv \infty$ on $\overline{h^{-1}(\{b\})}$. It provides a semi conjugacy between $\rho_1(G)$ and a representation $\rho : G \to \mathrm{Aff}(\mathbb{R})$.

□

Proposition 3.3.9. *Let h be an non decreasing map of degree one of $\mathbb{S}^1$, and let ω be a continuous volume form on $M_h = \mathbb{S}^1 \times \mathbb{S}^1 \setminus G(h)$. Assume that $G \subset G_\omega$ is elementary. Then there is a representation $\rho : G \to \mathrm{PSL}(2, \mathbb{R})$ that is semi conjugate to ρ_1.*

Proof. Let $L_1 \subset \mathbb{S}^1$ be a finite orbit for $\rho_1(G)$. If $\sharp L_1 = 1$, then Lemma 3.3.8 applies. If $\sharp L_1 = k \geq 2$, then let $L_1 = \{x_{\bar{1}}, \dots, x_{\bar{k}}\}$ (where the indices are in $\mathbb{Z}/k\mathbb{Z}$, and $x_{\bar{1}} < \cdots < x_{\bar{k}} < x_{\bar{1}}$). Since elements of $\rho_1(G)$ preserve the cyclic order, there is a morphism $\sigma : G \to \mathbb{Z}/k\mathbb{Z}$ such that $\rho_1(\varphi)(x_i) = x_{i+\sigma(\varphi)}$ for all $i \in \mathbb{Z}/k\mathbb{Z}$ and $\varphi \in G$. Since G acts transitively on L_1, we necessarily have $\sigma(G) = \mathbb{Z}/k\mathbb{Z}$. Then $\varphi \mapsto R_{\frac{\sigma(\varphi)}{k}}$ is a representation of G in $\mathrm{SO}(2, \mathbb{R}) \subset \mathrm{PSL}(2, \mathbb{R})$ that is semi conjugate to ρ_1 by Proposition 2.3 of **[Ghy87b]**. □

Note that the reason why we had to start with stabilizers of points is that the semi conjugacy defined in the proof of Proposition 3.3.9 is a constant map in the case where $\sharp L_1 = 1$, hence does not satisfy our definition of a non decreasing map of degree one.

The representations in $\mathrm{PSL}(2, \mathbb{R})$ that we have chosen may not be faithful. We will be more precise while dealing with the general case in order to obtain faithful representations.

3.4. The general case. We can now give a proof of Theorem 3.3.5.

Proof of Theorem 3.3.5. If h has a finite number of values, then G_ω is elementary, and we can apply Proposition 3.3.9. We can now assume that h is not finite valued.

Let U_1 be the union of the open intervals where h is constant, and let U_2 be the reunion of open intervals between discontinuities of h. The complement of U_1 (resp. of U_2) is a closed ρ_1-invariant (resp. ρ_2-invariant) set.

Let $p_1, p_2 : \mathbb{S}^1 \to \mathbb{S}^1$ be continuous non decreasing maps of degree one such that the intervals where p_i is constant are exactly the connected components of U_i. They induce representations $\hat{\rho}_1, \hat{\rho}_2 : G_\omega \to \mathrm{Homeo}(\mathbb{S}^1)$ such that $\hat{\rho}_i \circ p_i = p_i \circ \rho_i$, and we now have a homeomorphism $\hat{h}$ such that $\hat{h} \circ \hat{\rho}_1 = \hat{\rho}_2 \circ \hat{h}$. We are going to show that $\hat{\rho}_1$ and $\hat{\rho}_2$ are topologically Fuchsian, i.e. that they satisfy the convergence property.

Let $(\varphi_n)_{n \in \mathbb{N}} \in G_\omega^{\mathbb{N}}$ be a sequence such that $\hat{\rho}_1(\varphi_n)$ has no equicontinuous subsequence.
To simplify the notations, we will set $f_n = \rho_1(\varphi_n)$, $g_n = \rho_2(\varphi_n)$, $\hat{f}_n = \hat{\rho}_1(\varphi_n)$ and $\hat{g}_n = \hat{\rho}_2(\varphi_n)$.

We are first going to show that the sequences $\hat{f}_n(x)$ have at most two distinct limit points. Indeed, assume that there are three distinct points $\hat{\alpha} < \hat{\beta} < \hat{\gamma} < \hat{\alpha}$ in $\mathbb{S}^1$ and $\hat{a}, \hat{b}, \hat{c} \in \mathbb{S}^1$ such that $\hat{f}_n(\hat{a}) \to \hat{\alpha}$, $\hat{f}_n(\hat{b}) \to \hat{\beta}$ and $\hat{f}_n(\hat{c}) \to \hat{\gamma}$. We consider a subsequence such that $f_n(a) \to \alpha$, $f_n(b) \to \beta$ and $f_n(c) \to \gamma$ for some lifts $a, b, c, \alpha, \beta, \gamma$ with respect to p_1.

We also set $\hat{\alpha}' = \hat{h}(\hat{\alpha}), \hat{a}' = \hat{h}(\hat{a}), \dots$ and choose lifts $a', b', c', \alpha', \beta', \gamma'$ with respect to p_2.

First, let us assume that $g_n'(a') \to 0$. Let K be a compact interval of $\mathbb{S}^1 \setminus \{\alpha\}$ that contains β and γ in its interior. There is $n_0 \in \mathbb{N}$ such that $f_n(x) \in K$ for all $x \in [b\,, c]$ and all $n \geq n_0$. This implies that the sequence $(f_n(x), g_n(a'))$ stays in a compact set of M_h, and:

$$f_n'(x) = \frac{1}{g_n'(a')} \frac{\omega(x, a')}{\omega(f_n(x), g_n(a'))} \to +\infty$$

Fatou's Lemma implies that $\int_b^c f_n'(x)dx \to \infty$, which is impossible because $\int_{\mathbb{S}^1} f_n'(x)dx = 1$. This shows that the sequence $g_n'(a')$ cannot converge to 0, nor can any subsequence, and there is a constant $C > 0$ such that $g_n'(a') \geq C$ for all $n \in \mathbb{N}$. Since a, b and c have similar roles, we can also assume that $g_n'(b') \geq C$ and $g_n'(c') \geq C$.

We now see that $f_n'(x) = \frac{1}{g_n'(a')}\frac{\omega(x,a')}{\omega(f_n(x),g_n(a'))}$ is uniformly bounded on $[b\,,c]$, therefore f_n is equicontinuous on this interval and up to a subsequence we can assume that f_n converges uniformly on $[b\,,c]$ (Ascoli's Theorem). Since b and c have a similar role to a, there is a subsequence that converges uniformly on $[c\,,a]$ and on $[a\,,b]$, therefore on all of $\mathbb{S}^1$, which is impossible because we assumed that $\hat{f}_n$ has no equicontinuous subsequence.

We now know that there are at most two possible limits for f_n, say $\hat{\alpha}$ and $\hat{\beta}$ (we keep similar notations for $\hat{a}$, $\hat{a}'$, a, a', b ...). Let A (resp. B) be the set of points $x \in \mathbb{S}^1$ such that $\hat{f}_n(x) \to \hat{\alpha}$ (resp. $\hat{f}_n(x) \to \hat{\beta}$).

If $x, y \in A$, then one of the two intervals $[\hat{f}_n(x)\,,\hat{f}_n(y)]$ and $[\hat{f}_n(y)\,,\hat{f}_n(x)]$ shrinks to $\{\hat{\alpha}\}$. This implies that one of the intervals $[x\,,y]$ and $[y\,,x]$ is included in A, hence A is connected, and it is an interval of $\mathbb{S}^1$. The same goes for B.

Assume that neither A nor B is reduced to a point or void. Let $\hat{y} \in \hat{h}(\mathring{B})$, and $y \in p_2^{-1}(\hat{y})$. First, assume that $g_n'(y) \to 0$. Then $f_n'(x) \to \infty$ for all $x \in p_1^{-1}(A)$, and $\int_{p_1^{-1}(A)} f_n'(x)dx \to \infty$, which is absurd. This shows that there is $C > 0$ such that $g_n'(y) \geq C$ for all $n \in \mathbb{N}$. Then f_n' is uniformly bounded on $p_1^{-1}(A)$, and the sequence $\hat{f}_n$ is equicontinuous on A. Similarly, the sequence $\hat{g}_n$ is equicontinuous on $\hat{h}(B)$, and $\hat{f}_n$ is equicontinuous on B. Since we can choose a subsequence such that $A \cup B = \mathbb{S}^1$ (by making f_n convergent on a dense countable subset of $\mathbb{S}^1$), this implies that $\hat{f}_n$ is equicontinuous, which is absurd. Therefore A or B contains at most one point, and the sequence $\hat{f}_n$ satisfies the convergence property. □

As a corollary of the proof, we get the convergence property for ρ_1 and ρ_2 when $h \in \mathrm{Homeo}(\mathbb{S}^1)$, i.e. Theorem 3.3.6.

PROOF OF THEOREM 3.3.6. If $h \in \mathrm{Homeo}(\mathbb{S}^1)$, then $U_1 = \emptyset$ and $U_2 = \emptyset$ in the proof of Theorem 3.3.5. This implies that $\hat{\rho}_1 = \rho_1$, so $\rho_1(G_\omega) = \hat{\rho}_1(G_\omega)$ is topologically Fuchsian. □

4. Rotation number of $h_{\to\uparrow}$

In order to introduce a generalised notion of convergence groups, we will need a homeomorphism that commutes with ρ_1^M. If the conformal boundary is acausal, then such a homeomorphism is given by $h_{\to\uparrow} = h_{\to} \circ h_{\uparrow}$. Its class under differentiable conjugacy is a conformal invariant.

The principal invariant under topological conjugacy for circle homeomorphisms is the rotation number. We will see that if the isometry group acts non properly, then there is a restriction on the values that it can take for $h_{\to\uparrow}$.

For the k-cover of dS_2, we find $h_{\to} = Id$ and $h_{\uparrow} = R_{\frac{1}{k}}$, so $h_{\to\uparrow} = R_{\frac{1}{k}}$. We will see that this is part of a more general property.

Proposition 3.4.1. *Let (M, g) be a spatially compact surface with an acausal conformal boundary that embeds conformally in $\mathbb{T}^2$. Assume that* $\mathrm{Isom}(M, g)$ *acts non properly on M. Then there is $k \in \mathbb{N}$ such that the rotation number of $h_{\to\uparrow}$ is $\frac{1}{k}$.*

PROOF. Let α be the rotation number of $h_{\to\uparrow}$, and assume that α is not equal to $\frac{1}{k}$ for some $k \in \mathbb{N}$. Since the cyclic order of the elements of an orbit for $h_{\to\uparrow}$ is the same as for the rotation R_α, there are $x \in \mathbb{S}^1$ and $n \in \mathbb{Z}$ such that $x < (h_{\to\uparrow})^n(x) < h_{\to\uparrow}(x) < x$. By setting $y = h_{\to}(x)$, we find that $h_{\downarrow}(y) < (h_{\to\uparrow})^n(h_{\downarrow}(y)) < h_{\uparrow}(y) < h_{\downarrow}(y)$, i.e.

$(y, (h_{\rightarrow\uparrow})^n(h_\downarrow(y))) \in p(M)$.

If the graphs of $(h_{\rightarrow\uparrow})^n \circ h_\downarrow$ and $h_\downarrow$ (resp. $h_\uparrow$) were to intersect, then $(h_{\rightarrow\uparrow})^n$ (resp. $(h_{\rightarrow\uparrow})^{n-1}$) would have a fixed point, i.e. its rotation number $n\alpha$ (resp. $(n-1)\alpha$) is equal to 0. If α is irrational, this is impossible. If α is rational, then n can be chosen such that $n\alpha = \frac{1}{k}$ for some $k \geq 2$, hence $n\alpha \neq 0$ and $(n-1)\alpha \neq 0$.

This implies that $K = Gr((h_{\rightarrow\uparrow})^n \circ h_\downarrow) \subset p(M)$ is a $\mathrm{Conf}(M,g)$-invariant compact set in M. Since it is a spacelike circle, we can define a distance on K as the infimum of lengths of spacelike curves joining two points. It is preserved by the isometry group, which shows that the action of $\mathrm{Isom}(M,g)$ on K is equicontinuous. The projection of K onto the first and second coordinates of $\mathbb{S}^1 \times \mathbb{S}^1$ shows that the action of $\mathrm{Isom}(M,g)$ on $K \approx \mathbb{S}^1$ is topologically conjugate to ρ_1^M and ρ_2^M, hence $\rho_1^M(\mathrm{Isom}(M,g))$ and $\rho_2^M(\mathrm{Isom}(M,g))$ are compact. This implies that $\mathrm{Isom}(M,g)$ acts properly on $p(M)$, hence on M. $\square$

5. Properness and compactness

Lemma 3.5.1. *Let (M,g) be a spatially compact surface that embeds conformally in $\mathbb{T}^2$. Assume that the conformal boundary is acausal. Then the maps $\rho_1^M, \rho_2^M : \mathrm{Isom}(M,g) \to \mathrm{Homeo}(\mathbb{S}^1)$ are proper.*

Proof. We identify M with $p(M) \subset \mathbb{T}^2$. Let $\varphi_n \to \infty$ in $\mathrm{Isom}(M,g)$. By contradiction, let us assume that the sequence $\rho_1^M(\varphi_n)$ is equicontinuous. Then, up to a subsequence, it converges to $f_1 \in \mathrm{Homeo}(\mathbb{S}^1)$. Since ρ_1^M and ρ_2^M are topologically conjugate, we see that there is $f_2 \in \mathrm{Homeo}(\mathbb{S}^1)$ such that $\rho_2^M(\varphi_n) \to f_2$. Let $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by $\varphi(x,y) = (f_1(x), f_2(y))$. Then $\varphi_n \to \varphi$ uniformly on $\mathbb{T}^2$, and $\varphi(M) = M$. Since $\mathrm{Isom}(M,g)$ is a closed subgroup of $\mathrm{Homeo}(M)$ (see [**Ada01**]), we see that $\varphi \in \mathrm{Isom}(M,g)$ and $\varphi_n \to \varphi$ in $\mathrm{Homeo}(M)$ hence in $\mathrm{Isom}(M,g)$, which is absurd because $\varphi_n \to \infty$. Hence $\rho_1^M(\varphi_n) \to \infty$. This shows that $\rho_1^M : \mathrm{Isom}(M,g) \to \mathrm{Homeo}(\mathbb{S}^1)$ is a proper map, and the same goes for ρ_2^M. $\square$

As a consequence of this, we see that the groups $\rho_i^M(\mathrm{Isom}(M,g))$ are closed subgroups of $\mathrm{Homeo}(\mathbb{S}^1)$. With the same proof, we would obtain the same for $\mathrm{Conf}(M,g)$ by replacing $\mathrm{Homeo}(\mathbb{S}^1)$ with $\mathrm{Diff}(\mathbb{S}^1)$ (because $\mathrm{Conf}(M,g)$ is closed in $\mathrm{Diff}(M)$, but not necessarily in $\mathrm{Homeo}(M)$).

Let us quote a technical result that we will use.

Lemma 3.5.2. *Let $\alpha \in \mathbb{S}^1$ and let $f \in \mathrm{Homeo}(\mathbb{S}^1)$ have α as its rotation number. There is $x \in \mathbb{S}^1$ such that $f(x) = R_\alpha(x)$.*

For the proof, see Lemme 4.1.3 in [**Her79**].

Proposition 3.5.3. *Let (M,g) be a spatially compact surface with an acausal conformal boundary that embeds conformally in $\mathbb{T}^2$. Let $G \subset \mathrm{Isom}(M,g)$ be a subgroup. The following statements are equivalent:*

(1) G acts properly on M
(2) $G \subset \mathrm{Homeo}(M)$ is relatively compact
(3) $\rho_1^M(G) \subset \mathrm{Homeo}(\mathbb{S}^1)$ is relatively compact
(4) $\rho_2^M(G) \subset \mathrm{Homeo}(\mathbb{S}^1)$ is relatively compact

Proof. (2) $\Rightarrow$ (1) It follows from the definition of a proper action that relatively compact groups always act properly.
(3) $\iff$ (4) comes from the fact that ρ_1^M and ρ_2^M are topologically conjugate.
(2) $\iff$ (3) is a straightforward consequence of the fact that the map $\rho_1^M : \mathrm{Isom}(M,g) \to \mathrm{Homeo}(\mathbb{S}^1)$ is proper.
(1) $\Rightarrow$ (2) We start by considering:

$$W = \{f \in \mathrm{Homeo}(\mathbb{S}^1) | \forall x \in \mathbb{S}^1 \ x < f(x) < h_\rightarrow \circ h_\uparrow(x) \leq x\}$$

It is a non empty open set of $\mathrm{Homeo}(\mathbb{S}^1)$. For α small enough and positive, the rotation R_α is in W. Let α be such that $R_\alpha \in W$.

We note $K = Gr(h_\downarrow \circ R_\alpha)$. If $x \in \mathbb{S}^1$, then applying $h_\downarrow$ to the inequalities $x < R_\alpha(x) < h_\to \circ h_\uparrow(x) \leq x$ shows that $(x, h_\downarrow \circ R_\alpha(x)) \in p(M)$, hence K is a compact subset of M.

Let $\varphi \in \mathrm{Isom}(M,g)$. Since the rotation number is an invariant under conjugacy, Lemma 3.5.2 implies that there is $x \in \mathbb{S}^1$ such that:

$$\rho_1^M(\varphi) \circ R_\alpha \circ \rho_1^M(\varphi)^{-1}(x) = R_\alpha(x)$$

Since $h_\downarrow \circ \rho_1^M(\varphi) = \rho_2^M(\varphi) \circ h_\downarrow$, we see that:

$$\rho_2^M(\varphi) \circ (h_\downarrow \circ R_\alpha) \circ \rho_1^M(\varphi)^{-1}(x) = h_\downarrow \circ R_\alpha(x)$$

This means that $(x, h_\downarrow \circ R_\alpha(x)) \in Gr(\rho_2^M(\varphi) \circ (h_\downarrow \circ R_\alpha) \circ \rho_1^M(\varphi)^{-1}) = \varphi(K)$, i.e. $(x, h_\downarrow \circ R_\alpha(x)) \in K \cap \varphi(K)$. We have shown that there is a compact set $K \subset M$ such that $K \cap \varphi(K) \neq \emptyset$ for all $\varphi \in \mathrm{Isom}(M,g)$. This shows that if $G \subset \mathrm{Isom}(M,g)$ acts properly on M, then it is relatively compact. ☐

This result does not hold when (M,g) does not embed conformally in the torus: the isometry group of the flat cylinder is $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$ which is not compact, but it acts properly on the cylinder.

We can now prove Theorem 3.1.5 in the case where the isometry group acts properly.

Proposition 3.5.4. *Let (M,g) be a spatially compact surface that embeds conformally in $\mathbb{T}^2$, with an acausal conformal boundary. If $\mathrm{Isom}(M,g)$ acts properly on M, then ρ_1^M is topologically conjugate to a representation in $\mathrm{SO}(2,\mathbb{R}) \subset \mathrm{PSL}(2,\mathbb{R})$.*

Proof. By Lemma 3.5.1, we see that $\rho_1^M(\mathrm{Isom}(M,g))$ is closed in $\mathrm{Homeo}(\mathbb{S}^1)$, and Proposition 3.5.3 implies that it is compact, therefore topologically conjugate to a subgroup of $\mathrm{SO}(2,\mathbb{R})$. ☐

6. The (h,k)-convergence property

The convergence property allows us to show that some subgroups of $\mathrm{Homeo}(\mathbb{S}^1)$ are topologically conjugate to subgroups of $\mathrm{PSL}(2,\mathbb{R})$. In the case of finite covers of $\mathrm{PSL}(2,\mathbb{R})$, we have to generalise the notion of convergence groups to the case where there can be more limit points. The most simple generalisation would be to keep the same definition but to have k possible limit points, as it is the case for subgroups of $\mathrm{PSL}_k(2,\mathbb{R})$. However, this cannot be enough since the limit points are linked to each other by the rotation of angle $\frac{1}{k}$ in this case. This tells us that we need to add some more data for a proper generalisation of convergence groups.

Definition 3.6.1. *Let $k \in \mathbb{N}^*$ and let $h \in \mathrm{Homeo}(\mathbb{S}^1)$ have rotation number $\frac{1}{k}$. A sequence $(f_n)_{n\in\mathbb{N}} \in \mathrm{Homeo}(\mathbb{S}^1)$ has the (h,k)-convergence property if there are $a, b \in \mathbb{S}^1$ such that, up to a subsequence:*

- $h^k(a) = a$ *and* $h^k(b) = b$
- $\forall i \in \{0, \dots, k-1\}\ \forall x \in]h^i(a), h^{i+1}(a)[\ f_n(x) \to h^i(b)$

A group $G \subset \mathrm{Homeo}(\mathbb{S}^1)$ is a (h,k)-convergence group if all elements of G commute with h and all sequence in G either has the (h,k)-convergence property or has an equicontinuous subsequence.

If $k = 1$, then a (h,k)-convergence group is a convergence group, hence topologically Fuchsian. Of course, the main interest we have in this notion is that it is satisfied by the isometry groups of spatially compact surfaces. If h is topologically conjugate to a rotation, then we immediately obtain an analog of Theorem 2.4.2.

Lemma 3.6.2. *Let $k \in \mathbb{N}^*$ and let h be topologically conjugate to the rotation of angle $\frac{1}{k}$. If $G \subset \mathrm{Homeo}(\mathbb{S}^1)$ is a (h,k)-convergence group, then G is topologically conjugate to a subgroup of $\mathrm{PSL}_k(2,\mathbb{R})$.*

Proof. Let π be the projection of the circle $\mathbb{S}^1$ onto its quotient by h. Since h is topologically conjugate to a rational rotation, it is a finite covering. Since the quotient is homeomorphic to the circle, the image $\pi(G) \subset \mathrm{Homeo}(\mathbb{S}^1/h)$ is a subgroup of $\mathrm{Homeo}(\mathbb{S}^1)$.

Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $\pi(G)$ that leaves every compact set. Choose a sequence $(f_n)_{n\in\mathbb{N}}$ of lifts in G. Since $f_n \to \infty$, there are $a, b \in \mathbb{S}^1$ such that $f_n(x) \to h^i(b)$ for all $x \in \,]h^i(a)\,, h^{i+1}(a)[$. Let $x \in \mathbb{S}^1 \setminus \pi(a)$. Let $z \in \pi^{-1}(\{x\})$. Then z is not in the orbit of a for h, hence $f_n(z) \to h^i(b)$ for some $i \in \{0, \dots, k-1\}$, and $g_n(x) = \pi(f_n(z)) \to \pi(h^i(b)) = \pi(b)$.

We have shown that $\pi(G) \subset \mathrm{Homeo}(\mathbb{S}^1)$ is a convergence group, therefore there is $\varphi \in \mathrm{Homeo}(\mathbb{S}^1)$ such that $\varphi^{-1}\pi(G)\varphi \subset \mathrm{PSL}(2,\mathbb{R})$. If $\psi \in \mathrm{Homeo}(\mathbb{S}^1)$ is a lift of φ, then $\psi^{-1}G\psi \subset \pi^{-1}(\mathrm{PSL}(2,\mathbb{R}))$. Since h is topologically conjugate to a rotation of angle $\frac{1}{k}$, $\pi^{-1}(\mathrm{PSL}(2,\mathbb{R}))$ is topologically conjugate to $\mathrm{PSL}_k(2,\mathbb{R})$. □

The definition of the (h,k)-convergence property can actually be simplified by only looking at convergence on one interval.

Lemma 3.6.3. *Let $h \in \mathrm{Homeo}(\mathbb{S}^1)$. Let $G \subset \mathrm{Homeo}(\mathbb{S}^1)$ be a group that commutes with h. Let $(f_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}}$. Assume that there are $a, b \in \mathbb{S}^1$ such that $f_n(x) \to b$ for all $x \in \,]a\,, h(a)[$. Then there is $k \in \mathbb{N}$ such that the rotation number of h is $\frac{1}{k}$, and $(f_n)_{n\in\mathbb{N}}$ has the (h,k)-convergence property.*

Proof. Let $(f_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}}$ be such a sequence. Consider $a, b \in \mathbb{S}^1$ such that $f_n(x) \to b$ for all $x \in \,]a\,, h(a)[$.

If $x \in \,]h^i(a)\,, h^{i+1}(a)[$, then $h^{-i}(x) \in \,]a\,, h(a)[$, which shows that $f_n(x) = h^i \circ f_n \circ h^{-i}(x) \to h^i(b)$.

If the rotation number is not equal to some $\frac{1}{k}$, then there is $i \in \mathbb{N}$ such that $h^i(a) \in \,]a\,, h(a)[$. Let $x \in \,]a\,, h(a)[$ be close enough to a so that $h^i(x) \in \,]a\,, h(a)[$. We find that $f_n(x) \to b$, and that $f_n(h^i(x)) \to b$. Since f_n commutes with h, we also find that $f_n(h^i(x)) \to h^i(b)$, hence $h^i(b) = b$. Now let $y \in \,]h^{-1}(a)\,, a[$ be close enough to a so that $h^i(a) \in \,]a\,, h(a)[$. We now have $f_n(y) \to h^{-1}(b)$, so $f_n(h^i(b)) \to h^{i-1}(b)$ and $h^{i-1}(b) = b$. This implies that $h(b) = b$, so h has a fixed point, therefore its rotation number is $\frac{1}{1}$, which contradicts our assumption.

We now have to show that a and b are periodic points of h.

If a were not a periodic point of h, then $h^k(a)$ would belong to an interval $]h^i(a)\,, h^{i+1}(a)[$ for some $i \in \mathbb{N}$. In this case, let $x \in \,]a\,, h(a)[$ be close enough to a so that $h^k(x) \in \,]h^i(a)\,, h^{i+1}(a)[$. We then have $f_n(x) \to b$ and $f_n(h^k(x)) \to h^i(b)$, so that $h^k(b) = h^i(b)$. This implies that $i = nk$ for some $n \in \mathbb{N}$, and $h^k(b) = b$. Now let $y \in \,]h^{-1}(a)\,, a[$ be such that $h^k(y) \in \,]h^i(a)\,, h^{i+1}(a)[$. We now have $f_n(y) \to h^{-1}(b)$ and $f_n(h^k(y)) \to h^i(b) = b$, i.e. $h^{k-1}(b) = b$ which is absurd because all periodic points have the same period.

We have shown that $h^k(a) = a$. For any $x \in \,]a\,, h(a)[$, we have $h^k(x) \in \,]a\,, h(a)[$, which shows that $f_n(h^k(x)) \to b$. Since $f_n(x) \to b$, we also have $f_n(h^k(x)) \to h^k(b)$, so finally $h^k(b) = b$. □

The (h,k)-convergence property can be seen as a one dimensional hyperbolic behaviour (there are attracting and repelling points). The key in showing that the isometry groups under study satisfy this property will consist in exhibiting a hyperbolic behaviour for sequences of isometries. In order to find hyperbolic points (attracting in one direction and repelling in the other), we will use a result on non equicontinuous sequences of affine diffeomorphisms.

Lemma 3.6.4. *Let M be a connected manifold equipped with an an affine connection. Let $(f_n)_{n\in\mathbb{N}}$ be a non equicontinuous sequence of affine diffeomorphisms (i.e. that preserve the connection). If there is a converging sequence $p_n \to p$ in M such that the sequence $f_n(p_n)$ lies in a compact set, then, up to a subsequence, $Df_n(p_n) \to \infty$ or $Df_n(p_n) \to 0$.*

We will not detail the proof (which can be found in [**Kob72**]), however the idea of it is quite simple: since affine diffeomorphisms are linearisable via the exponential map, the behaviour of the derivative at one point dictates the behaviour of the diffeomorphism on the whole manifold.

In our context, we get a more precise result.

Lemma 3.6.5. *Let (M,g) be a spatially compact surface that embeds conformally in $\mathbb{T}^2$. Let $\varphi_n \in \mathrm{Isom}(M,g)$ be such that $\varphi_n \to \infty$, and let $(x_0,y_0) \in M$ be such that $\varphi_n(x_0,y_0) \to (x_1,y_1) \in M$. Then, up to a subsequence, one of the following is satisfied:*

- $\rho_1^M(\varphi_n)'(x_0) \to \infty$ *and* $\rho_2^M(\varphi_n)'(y_0) \to 0$
- $\rho_1^M(\varphi_n)'(x_0) \to 0$ *and* $\rho_2^M(\varphi_n)'(y_0) \to \infty$

Proof. The derivative $D\varphi_n(x_0,y_0)$ is given by the diagonal matrix with coefficients $\rho_1^M(\varphi_n)'(x_0)$ and $\rho_2^M(\varphi_n)'(y_0)$. Up to a subsequence, Lemma 3.6.4 gives us four cases.

If $D\varphi_n(x_0,y_0) \to 0$, then either $\rho_1^M(\varphi_n)'(x_0) \to 0$, either $\rho_2^M(\varphi_n)'(y_0) \to 0$. Let us write the metric $g = g(x,y)dxdy$. The fact that the maps φ_n are isometries gives us:

$$g(\varphi_n(x_0,y_0))\rho_1^M(\varphi_n)'(x_0)\rho_2^M(\varphi_n)'(y_0) = g(x_0,y_0)$$

Since $g(\varphi_n(x_0,y_0)) \to g(x_1,y_1) \in \mathbb{R}_+^*$, we find that the Jacobian product $\rho_1^M(\varphi_n)'(x_0)\rho_2^M(\varphi_n)'(y_0)$ is bounded in $\mathbb{R}_+^*$, hence the fact that one term converges to 0 implies that the other tends to ∞.

In the case where $D\varphi_n(x_0,y_0) \to \infty$, one has either $\rho_1^M(\varphi_n)'(x_0) \to \infty$ or $\rho_2^M(\varphi_n)'(y_0) \to \infty$, and the fact that the product $\rho_1^M(\varphi_n)'(x_0)\rho_2^M(\varphi_n)'(y_0)$ is bounded in $\mathbb{R}_+^*$ implies that when one term tends to ∞, the other converges to 0. □

This result is exactly the hyperbolic behaviour that we were looking for: we find attraction in one direction and repulsion in the other. A useful fact is that the stable and unstable foliations are simply the lightlike foliations. We are now ready to show that when the conformal boundary is acausal, the isometry groups are convergence groups.

Proposition 3.6.6. *Let (M,g) be a spatially compact surface with an acausal conformal boundary that embeds in $\mathbb{T}^2$. Assume that the homeomorphisms $h_\downarrow, h_\uparrow$ defining the boundary in $\mathbb{T}^2$ are such that the rotation number of $h_{\to\uparrow}$ is $\frac{1}{k}$. Then $\rho_1^M(\mathrm{Isom}(M,g))$ is a $(h_{\to\uparrow},k)$-convergence group.*

Proof. Let $(\varphi_n)_{n\in\mathbb{N}} \in \mathrm{Isom}(M,g)^{\mathbb{N}}$ be a sequence such that $\rho_1^M(\varphi_n) \to \infty$ in $\mathrm{Homeo}(\mathbb{S}^1)$ (i.e. $\varphi_n \to \infty$ because of Proposition 3.5.3). We start with $(x_0,y_0) \in M \subset \mathbb{T}^2$, and consider a subsequence such that $\varphi_n(x_0,y_0) \to (x_1,y_1) \in \overline{M} \subset \mathbb{T}^2$.

First case: Assume that $(x_1,y_1) \in M$.

By Lemma 3.6.5, there are two subcases.

First subcase: $\rho_1^M(\varphi_n)'(x_0) \to 0$ and $\rho_2^M(\varphi_n)'(y_0) \to \infty$

Let $x \in \mathbb{S}^1$ be such that $(x,y_0) \in M$, i.e. $x \in \,]h_\leftarrow(y_0)\,, h_\to(y_0)[$, and consider the geodesic γ such that $\gamma(0) = (x_0,y_0)$ and $\gamma(1) = (x,y_0)$. The geodesic $\gamma_n = \varphi_n \circ \gamma$ has initial data $\gamma_n(0) \to (x_1,y_1) \in M$ and $\gamma_n'(0) \to 0$, hence γ_n converges uniformly to a constant geodesic. This implies that $\gamma_n(1) \to (x_1,y_1)$, i.e. $\rho_1^M(\varphi_n)(x) \to x_1$.

Let $a = h_\leftarrow(y_0)$. We have shown that $\rho_1^M(\varphi_n)(x) \to x_1$ for all $x \in \,]a\,, h_{\to\uparrow}(a)[$. Lemma 3.6.3 implies that the sequence $(\rho_1^M(\varphi_n))_{n\in\mathbb{N}}$ has the $(h_{\to\uparrow},k)$-convergence property.

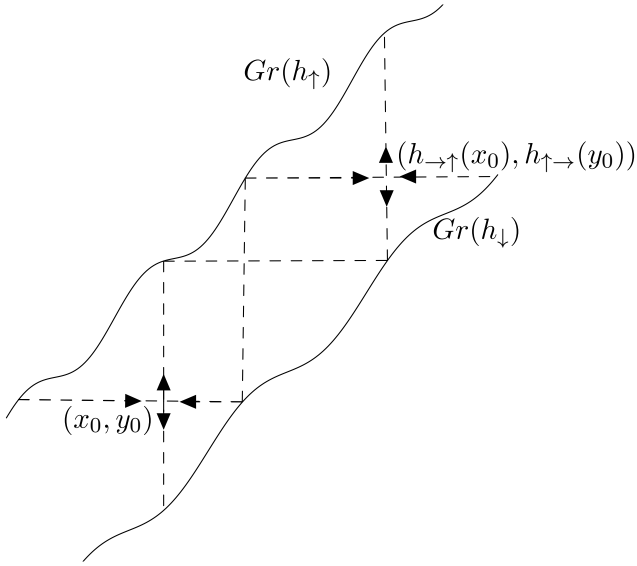

FIGURE 3.4. The dynamics of the isometry group

Second subcase: $\rho_1^M(\varphi_n)'(x_0) \to \infty$ and $\rho_2^M(\varphi_n)'(y_0) \to 0$

In this case, for $x \in \mathbb{S}^1$ such that $(x, y_0) \in M$, the geodesic joining (x_0, y_0) to (x, y_0) is now dilated by the sequence φ_n, which shows that $\rho_1^M(\varphi_n)(x)$ converges to $h_\leftarrow(y_1)$ for $x \in [h_\leftarrow(y_0)\,, x_0[$ and to $h_\rightarrow(y_1)$ for $x \in]x_0\,, h_\rightarrow(y_0)]$.

If $x \in [h_\rightarrow(y_0)\,, h_{\rightarrow\uparrow}(x_0)[$, then $h_{\rightarrow\uparrow}^{-1}(x) \in [h_\leftarrow(y_0)\,, x_0[$ which shows that $\rho_1^M(\varphi_n)(x) \to h_{\rightarrow\uparrow}(h_\leftarrow(y_1)) = h_\rightarrow(y_1)$.

We have shown that $\rho_1^M(\varphi_n)(x) \to h_\rightarrow(y_0)$ for all $x \in]x_0\,, h_{\rightarrow\uparrow}(x_0)[$. Lemma 3.6.3 implies that $(\rho_1^M(\varphi_n))_{n\in\mathbb{N}}$ has the $(h_{\rightarrow\uparrow}, k)$-convergence property.

We now know that if $(x_1, y_1) \in M$, then $(\rho_1^M(\varphi_n))_{n\in\mathbb{N}}$ has the $(h_{\rightarrow\uparrow}, k)$-convergence property.

Second case: Assume that $(x_1, y_1) \notin M$.

If there is $x \in \mathbb{S}^1$ such that $(x, y_0) \in M$ and $\rho_1^M(\varphi_n)(x) \to x'$ with $(x', y_1) \in M$, then the first case shows that $(\rho_1^M(\varphi_n))_{n\in\mathbb{N}}$ has the $(h_{\rightarrow\uparrow}, k)$-convergence property. Therefore we can assume that there is no such x. In this case, the only limit points of $\rho_1^M(\varphi_n)(x)$ are $h_\leftarrow(y_1)$ and $h_\rightarrow(y_1)$. We now have three subcases.

First subcase: $\rho_1^M(\varphi_n)(x) \to h_\leftarrow(y_1)$ for all $x \in]h_\leftarrow(y_0)\,, h_\rightarrow(y_0)[$.

Since $h_\rightarrow(y_0) = h_{\rightarrow\uparrow}(h_\leftarrow(y_0))$, Lemma 3.6.3 implies that $(\rho_1^M(\varphi_n))_{n\in\mathbb{N}}$ has the $(h_{\rightarrow\uparrow}, k)$-convergence property.

Second subcase: $\rho_1^M(\varphi_n)(x) \to h_\rightarrow(y_1)$ for all $x \in]h_\leftarrow(y_0)\,, h_\rightarrow(y_0)[$.

The argument is the same as in the previous case since we only change the limit.

Third subcase: The two limits are possible.

If $\rho_1^M(\varphi_n)(u) \to h_\leftarrow(y_1)$ for some $u \in]h_\leftarrow(y_0)\,, h_\rightarrow(y_0)[$, then $\rho_1^M(\varphi_n)(x) \to h_\leftarrow(y_1)$ for all $x \in]h_\leftarrow(y_0)\,, u]$. Similarly, if $\rho_1^M(\varphi_n)(v) \to h_\rightarrow(y_1)$ for some $v \in]h_\leftarrow(y_0)\,, h_\rightarrow(y_0)[$, then $\rho_1^M(\varphi_n)(x) \to h_\rightarrow(y_1)$ for all $x \in [v\,, h_\rightarrow(y_0)[$. This implies that there is a point $z \in]h_\leftarrow(y_0)\,, h_\rightarrow(y_0)[$ such that $\rho_1^M(\varphi_n)(x) \to h_\leftarrow(y_1)$ for all $x \in]h_\leftarrow(y_0)\,, z[$ and $\rho_1^M(\varphi_n)(x) \to h_\rightarrow(y_1)$ for all $x \in]z\,, h_\rightarrow(y_0)[$.

If $x \in]h_\rightarrow(y_0)\,, h_{\rightarrow\uparrow}(z)[$, then $h_{\rightarrow\uparrow}^{-1}(x) \in]h_\leftarrow(y_0)\,, z[$, which shows that $\rho_1^M(\varphi_n)(x) \to h_{\rightarrow\uparrow}(h_\leftarrow(y_1)) = h_\rightarrow(y_1)$. Once again, Lemma 3.6.3 implies that $(\rho_1^M(\varphi_n))_{n\in\mathbb{N}}$ has the $(h_{\rightarrow\uparrow}, k)$-convergence property.

□

Note that the strategy consisting in separating the cases depending on the possible limit points can also be found in Theorem 2.5 of [**Bar96**].

We do not know if the (h, k)-convergence property implies topological conjugacy with a subgroup of $\mathrm{PSL}_k(2, \mathbb{R})$, but it will still be a crucial tool in our proof for isometry groups of spatially compact surfaces. We already obtain Theorem 3.1.5 in a special case (when $k = 1$).

Corollary 3.6.7. *Let (M, g) be a spatially compact surface with an acausal conformal boundary that embeds conformally in $\mathbb{T}^2$. Assume that the boundary of $p(M) \subset \mathbb{T}^2$ is connected. Then $\rho_1^M(\mathrm{Isom}(M, g))$ is topologically conjugate to a subgroup of* $\mathrm{PSL}(2, \mathbb{R})$.

Proof. If the boundary of $p(M)$ is connected, then $h_{\to\uparrow}$ has a fixed point, i.e. $k = 1$. We showed that $\rho_1^M(\mathrm{Isom}(M, g))$ is a $(h_{\to\uparrow}, 1)$-convergence group, i.e. a convergence group, hence topologically Fuchsian. $\square$

7. Reducing the problem to open sets of $\mathbb{T}^2$

Before we go further, we will see that under the assumption that the isometry group acts non properly, isometries are completely described by the representations ρ_1^M and ρ_2^M, i.e. their restrictions to $\mathrm{Isom}(M, g)$ are faithful. In the study of surfaces that do not embed conformally in $\mathbb{T}^2$, we will make use of the second conformal model (embedding in the flat cylinder, defined in section 4 of chapter 1, page 20).

Lemma 3.7.1. *Let (M, g) be a spatially compact surface that is conformal to the flat cylinder. Then* $\mathrm{Isom}(M, g)$ *acts properly on M.*

Proof. If the action were not proper, we could find a sequence f_n in $\mathrm{Isom}(M, g)$ that leaves every compact set, and converging sequences $a_n \to a$ and $b_n \to b$ in M such that $f_n(a_n) = b_n$. We can lift everything to $\widetilde{M} = \mathbb{R}^2$: we choose lifts $\widetilde{a}_n, \widetilde{a}, \widetilde{b}_n, \widetilde{b}$ such that $\widetilde{a}_n \to \widetilde{a}$ and $\widetilde{b}_n \to \widetilde{b}$, and lifts $\widetilde{f}_n$ of f_n such that $\widetilde{f}_n(\widetilde{a}_n) = \widetilde{b}_n$. We write the lifts $\widetilde{f}_n(x, y) = (\alpha_n(x), \beta_n(y))$ and $\widetilde{a}_n = (x_n, y_n), \widetilde{b} = (u, v)$. Since $\widetilde{f}_n \to \infty$, Lemma 3.6.4 implies that we have $\alpha_n'(x_n) \to \infty$ or $\beta_n'(y_n) \to \infty$.

In the first case, we have $\beta_n'(y_n) \to 0$ (just as in Lemma 3.6.5). Consider the geodesic going from (x_n, y_n) to $(x_n, y_n + 1)$. The image of this geodesic by $\widetilde{f}_n$ converges towards a constant geodesic (the initial vector shrinks). This shows that $\widetilde{f}_n(x_n, y_n + 1) \to (u, v)$, i.e. $\beta_n(y_n + 1) \to v$, which is incompatible with $\beta_n(y_n + 1) = \beta_n(y_n) + 1 \to v + 1$.

The same reasoning applied to the geodesic joining (x_n, y_n) and $(x_n + 1, y_n)$ treats the other case. Therefore $\mathrm{Isom}(M, g)$ acts properly on M. $\square$

Associating this and the following proposition, we obtain the first parts of Theorem 3.1.2 and Theorem 3.1.3.

Proposition 3.7.2. *Let (M, g) be a spatially compact surface that is not conformal to the flat cylinder. Then ρ_1^M and ρ_2^M are semi conjugate to each other, and their restrictions to* $\mathrm{Isom}(M, g)$ *are faithful. Moreover, if the conformal boundary is acausal, then they are topologically conjugate.*

Proof. Since (M, g) is not conformally equivalent to the flat cylinder, at least one of $\widetilde{h}_\downarrow, \widetilde{h}_\uparrow$ is not a constant and provides a semi conjugacy between ρ_1^M and ρ_2^M.

Let $f \in \ker(\rho_1^M)$, and let $\widetilde{f}$ be a lift to $\widetilde{M}$, written $\widetilde{f}(x, y) = (\alpha(x), \beta(y))$. Since $\rho_1^M(f) = Id$, we find that $\alpha(x) - x \in \mathbb{Z}$ for all $x \in \mathbb{R}$. The continuity of f implies that there is $n \in \mathbb{Z}$ such that $\alpha(x) = x + n$ for all $x \in \mathbb{R}$. Consider $A = T^{-n} \circ \widetilde{f}$ (where $T(x, y) = (x + 1, y + 1) \in \mathrm{Isom}(\widetilde{M}, \widetilde{g})$). It is also a lift of f, that can be written $A(x, y) = (x, \gamma(y))$ where γ is semi conjugate to the identity via $\widetilde{h}_\uparrow$ or $\widetilde{h}_\downarrow$, therefore γ has fixed points. If $\gamma(y) = y$, then we choose $x \in \mathbb{R}$ such that $(x, y) \in \widetilde{M}$. Since

$A(x,y)=(x,y)$ and A is an isometry, the Jacobian at (x,y) is equal to 1, i.e. $\gamma'(y)=1$. This implies that A is an isometry with a fixed point $(x,y)\in M$ where the differential is the identity, therefore $A=Id$, and $f=Id$, i.e. ρ_1^M is injective. The same goes for ρ_2^M. □

This result gives a similarity with the case where M embeds in the torus. We will now see that from the isometry group point of view, there is no difference.

Proposition 3.7.3. *Let (M,g) be a spatially compact surface that does not embed conformally in $\mathbb{T}^2$, such that the action of* $\mathrm{Isom}(M,g)$ *on M is non proper. Then there is an open set $U\subset M$ such that:*

- *U is invariant under* $\mathrm{Conf}(M,g)$
- *$(U,g_{/U})$ is spatially compact*
- *The restriction map $r:\mathrm{Isom}(M,g)\to\mathrm{Isom}(U,g_{/U})$ is injective*
- *$\rho_1^U\circ r=\rho_1^M$ and $\rho_2^U\circ r=\rho_2^M$*
- *$(U,g_{/U})$ embeds conformally in $(\mathbb{T}^2,dxdy)$*
- *If the conformal boundary of (M,g) is acausal, then the same goes for $(U,g_{/U})$. In this case, the image of U in $\mathbb{T}^2$ has a connected boundary.*

This implies that all the results in the case where (M,g) embeds in the torus still apply when (M,g) does not embed in the torus, provided that the isometry group acts non properly. Therefore, Theorem 3.1.4 implies Theorem 3.1.2 and Theorem 3.1.5 implies Theorem 3.1.3. The proof will make use of two intermediate results.

Lemma 3.7.4. *Let (M,g) be a spatially compact surface. Consider its universal cover $\widetilde{M}=\{(x,y)\in\mathbb{R}^2|\widetilde{h}_\downarrow(x)<y<\widetilde{h}_\uparrow(x)\}$. Let $(x_0,y_0)\in M$, let $(f_n)_{n\in\mathbb{N}}$ be a sequence in* $\mathrm{Isom}(M,g)$ *such that $f_n\to\infty$ and let $\widetilde{f_n}$ be a sequence of lifts to $\widetilde{M}$. Assume that $\widetilde{f_n}(x_0,y_0)\to(x_1,y_1)\in\mathbb{R}^2$. If $(x_0,y_0+1)\in\widetilde{M}$, or $(x_0,y_0-1)\in\widetilde{M}$, then $(x_1,y_1)\notin\widetilde{M}$.*

Proof. Write $\widetilde{f_n}(x,y)=(\alpha_n(x),\beta_n(y))$. Let us assume that $(x_1,y_1)\in\widetilde{M}$, and that $(x_0,y_0+1)\in\widetilde{M}$ (the case where $(x_0,y_0-1)\in\widetilde{M}$ is identical). In that case, $\alpha_n'(x_0)\to\infty$ or $\beta_n'(y_0)\to\infty$. Assume that $\alpha_n'(x_0)\to\infty$. Then $\beta_n'(y_0)\to 0$. Let γ be the geodesic with starting point $\gamma(0)=(x_0,y_0)$ and end point $\gamma(1)=(x_0,y_0+1)$. The image $\eta_n=\widetilde{f_n}\circ\gamma$ has initial value $\eta_n(0)\to(x_1,y_1)$ and $\eta'(0)=\beta_n'(y)\gamma'(0)\to 0$, hence η converges uniformly to a constant geodesic, and $\widetilde{f_n}\circ\gamma(1)\to(x_1,y_1)$, which is absurd because $\beta_n(y_0+1)=\beta_n(y_0)+1\to y_1+1\neq y_1$. In the case where $\beta_n'(y_0)\to\infty$, then $\alpha_n'(x_0)\to 0$ and we use the same reasoning with the geodesic joining (x_0,y_0) to $(x_0-1,y_0)\in\widetilde{M}$. This shows that $(x_1,y_1)\notin\widetilde{M}$.

□

We have already seen in Lemma 3.7.1 that the conformal geometry can give obstructions to the non properness of the action of the isometry group. The consequences of the next result can be explained in terms of intersection of lightlike geodesics: the fact that $\widetilde{h}_\uparrow(x)-\widetilde{h}_\downarrow(x)\leq 2$ for all $x\in\mathbb{R}$ means that there are no lightlike geodesics with three or more intersection points, and the fact that there is some $x_0\in\mathbb{R}$ such that $\widetilde{h}_\uparrow(x_0)-\widetilde{h}_\downarrow(x_0)\leq 1$ means there are some lightlike geodesics with exactly one intersection point.

Lemma 3.7.5. *Let (M,g) be a spatially compact surface. Consider its universal cover $\widetilde{M}=\{(x,y)\in\mathbb{R}^2|\widetilde{h}_\downarrow(x)<y<\widetilde{h}_\uparrow(x)\}$. If* $\mathrm{Isom}(M,g)$ *acts non properly on M, then:*

- *$\widetilde{h}_\downarrow\neq-\infty$ and $\widetilde{h}_\uparrow\neq+\infty$*
- *$\forall x\in\mathbb{R}$ $\widetilde{h}_\uparrow(x)-\widetilde{h}_\downarrow(x)\leq 2$*

If the conformal boundary of (M,g) is acausal, then there is $x_0 \in \mathbb{R}$ such that $\widetilde{h}_\uparrow(x_0) - \widetilde{h}_\downarrow(x_0) \leq 1$

PROOF. Note that if $\widetilde{h}_\uparrow = \infty$ (resp. $\widetilde{h}_\downarrow = -\infty$), then $(x, y+1) \in \widetilde{M}$ (resp. $(x, y-1) \in \widetilde{M}$) for all $(x,y) \in \widetilde{M}$. In this case, Lemma 3.7.4 shows that the action of $\mathrm{Isom}(M,g)$ on M is proper.

Let us assume that there is $x_0 \in \mathbb{R}$ such that $\widetilde{h}_\uparrow(x_0) - \widetilde{h}_\downarrow(x_0) > 2$ Let $y_0 \in \mathbb{R}$ be such that $\widetilde{h}_\downarrow(x_0) < y_0 - 1 < y_0 + 1 < \widetilde{h}_\uparrow(x_0)$. Let $f_n \to \infty$ in $\mathrm{Isom}(M,g)$ and let $\widetilde{f_n}$ be a sequence of lifts to $\widetilde{M}$ such that the sequence $\widetilde{f_n}(x_0,y_0)$ lies in a compact set of $\mathbb{R}^2$. Up to a subsequence, we consider that $\widetilde{f_n}(x_0,y_0) \to (x_1,y_1)$. By Lemma 3.7.4, we know that $(x_1,y_1) \notin \widetilde{M}$, hence $y_1 = \widetilde{h}_\uparrow(x_1)$ or $y_1 = \widetilde{h}_\downarrow(x_1)$. In the case where $y_1 = \widetilde{h}_\uparrow(x_1)$, we have $\beta_n(y_0) \leq \beta_n(y_0+1) \leq \widetilde{h}_\uparrow(\alpha_n(x_0))$, which shows that $\beta_n(y_0+1) \to y_1$. This is impossible because $\beta_n(y_0+1) = \beta_n(y_0) + 1 \to y_1 + 1$. Similarly, if $y_1 = \widetilde{h}_\downarrow(x_1)$, we find $\beta_n(y_1 - 1) \to y_1$, which is absurd.

For the third statement, notice that if $\widetilde{h}_\uparrow(x) - \widetilde{h}_\downarrow(x) > 1$ for all $x \in \mathbb{R}$, then $L = \{(x, \widetilde{h}_\downarrow(x)+1) | x \in \mathbb{R}\} \subset \widetilde{M}$ is homeomorphic to the real line and preserved by the conformal group. Since it is spacelike (when the conformal boundary is acausal), the isometry group preserves a distance on L that is bi-lipschitz to the euclidian distance, and it acts properly on M. □

PROOF OF PROPOSITION 3.7.3. By Lemma 3.7.5, we know that $\widetilde{h}_\uparrow$ and $\widetilde{h}_\downarrow$ are not constants. Let $\widetilde{h}(x)$ denote $\widetilde{h}_\uparrow(x)$ if $\widetilde{h}_\uparrow(x) \leq \widetilde{h}_\downarrow(x)+1$ and $\widetilde{h}_\downarrow(x)+1$ if $\widetilde{h}_\uparrow(x) < \widetilde{h}_\downarrow(x)+1$. It is a non decreasing map such that $\widetilde{h}(x+1) = \widetilde{h}(x)+1$, and it commutes with lifts of conformal diffeomorphisms of M. Consider $V = \{(x,y) \in \mathbb{R}^2 | \widetilde{h}_\downarrow(x) < y < \widetilde{h}(x)\} \subset \widetilde{M}$ and let U be its image in M. It is an open set invariant under $\mathrm{Conf}(M)$. Since it contains some Cauchy surfaces of M and it is causally convex (i.e. an inextensible causal curve in U is the intersection of U and an inextensible causal curve in M), it is spatially compact. Since $\widetilde{h}(x) - \widetilde{h}_\downarrow(x) \leq 1$, we see that lightlike geodesics in U cannot have several intersections, therefore U embeds conformally in $\mathbb{T}^2$.

The lightlike geodesics of U are the lightlike geodesic of M, it follows immediately that $\rho_1^U \circ r = \rho_1^M$ and $\rho_2^U \circ r = \rho_2^M$. In particular, we find that $\ker(r) \subset \ker(\rho_1^M) = \{Id\}$, so r is injective.

If M does not embed in the torus and the conformal boundary is acausal, then there is $x_0 \in \mathbb{R}$ such that $\widetilde{h}_\uparrow(x_0) - \widetilde{h}_\downarrow(x_0) > 1$, hence $\widetilde{h}(x_0) - \widetilde{h}_\downarrow(x_0) = 1$. This shows that the image in the torus has a connected boundary.

□

Corollary 3.7.6. *Let (M,g) be a spatially compact surface with an acausal conformal boundary that does not embed conformally in $\mathbb{T}^2$, such that the action of $\mathrm{Isom}(M,g)$ on M is non proper. Then $\rho_1^M(\mathrm{Isom}(M,g))$ is a convergence group, hence topologically Fuchsian.*

PROOF. Let $U \subset M$ be the open set given by Proposition 3.7.3. By Proposition 3.6.7, we see that $\rho_1^M(\mathrm{Isom}(M,g)) = \rho_1^U \circ r(\mathrm{Isom}(M,g))$ is topologically Fuchsian. □

8. Proof of Theorem 3.1.2

8.1. Elementary groups. We wish to prove Theorem 3.1.2 for elementary groups. Just as in the proof of Theorem 3.3.5 (the case where $h_\downarrow = h_\uparrow$), we start with the stabilizer of a point.

Lemma 3.8.1. *Let (M,g) be a spatially compact surface that embeds conformally in $\mathbb{T}^2$. Assume that $G \subset \mathrm{Isom}(M,g)$ is such that $\rho_1^M(G)$ fixes a point $x_0 \in \mathbb{S}^1$. There is a faithful representation $\rho : G \to \mathrm{PSL}(2,\mathbb{R})$ that is semi conjugate to the restriction of ρ_1^M to G.*

Proof. Since $\rho_1^M(G)$ fixes x_0, the representation $\rho_2^M(G)$ fixes $y_0 = h_\uparrow(x_0)$. This implies that G fixes the horizontal geodesic $(\mathbb{S}^1 \times \{y_0\}) \cap M$.

The parametrisation of this geodesic gives a representation $\rho : G \to \mathrm{Aff}(\mathbb{R}) \subset \mathrm{PSL}(2,\mathbb{R})$ and a diffeomorphism $\alpha :]h_\leftarrow(y_0)\,, h_\rightarrow(y_0)[\to \mathbb{R}$ such that $\alpha \circ \rho_1^M = \rho \circ \alpha$ (just as in the proof of Lemma 3.3.8, we can assume that we get the real line $\mathbb{R}$ and not just any open interval because G acts non trivially on this geodesic).

Let us show that ρ is faithful. If $\varphi \in \ker(\rho)$, then φ fixes all points on the horizontal geodesic $(\mathbb{S}^1 \times \{y_0\}) \cap M$. If $(x,y_0) \in M$, we then get $\varphi(x,y_0) = (x,y_0)$ and $\rho_1^M(\varphi)'(x) = 1$, therefore $\rho_2^M(\varphi)'(y_0) = 1$ (because the Jacobian is equal to 1) and φ is an isometry having a fixed point where its derivative is the identity, therefore $\varphi = Id$.

Let $\psi : \mathbb{S}^1 \to \mathbb{S}^1 = \mathbb{R} \cup \{\infty\}$ be defined by $\psi = \alpha$ on $]h_\leftarrow(y_0)\,, h_\rightarrow(y_0)[$ and $\psi \equiv \infty$ on $[h_\rightarrow(y_0)\,, h_\leftarrow(y_0)]$. It provides a semi conjugacy between $\rho_1^M(G)$ and the action of ρ on the circle. □

Proposition 3.8.2. *Let (M,g) be a spatially compact surface that embeds conformally in $\mathbb{T}^2$. Assume that $G \subset \mathrm{Isom}(M,g)$ is elementary. There are $k \in \mathbb{N}$ and a faithful representation $\rho : G \to \mathrm{PSL}_k(2,\mathbb{R})$ that is semi conjugate to the restriction of ρ_1^M to G.*

Proof. Up to considering the closure $\overline{G}$, we can assume that G is closed. Let $L_1 \subset \mathbb{S}^1$ be a finite orbit of $\rho_1^M(G)$.

Lemma 3.8.1 treats the case where $\sharp L_1 = 1$, therefore we can assume that $\sharp L_1 \geq 2$. Let $n = \sharp L_1$, and consider $L_1 = \{x_{\overline{1}}, x_{\overline{2}}, \dots, x_{\overline{n}}\}$ where the indices are taken in $\mathbb{Z}/n\mathbb{Z}$.

Since ρ_1^M preserves the cyclic ordering, there is a morphism $\sigma : G \to \mathbb{Z}/n\mathbb{Z}$ such that $\rho_1^M(\varphi)(x_i) = x_{i+\sigma(\varphi)}$ for all $\varphi \in G$ and $i \in \mathbb{Z}/n\mathbb{Z}$. Since G acts transitively on L_1, we necessarily have $\sigma(G) = \mathbb{Z}/n\mathbb{Z}$.

Let $\varphi_1 \in G$ be such that $\sigma(\varphi_1) = \overline{1}$, and let $H = \ker \varphi = \mathrm{Stab}(x_1)$.

If $H = \{Id\}$, then $G = \langle \varphi_1 \rangle$ and it is semi conjugate any element of $\mathrm{PSL}_n(2,\mathbb{R})$ having the same rotation number as φ_1. Such an element can be chosen to be of finite order (if $\varphi_1^n = Id$) or not, so that the corresponding subgroup of $\mathrm{PSL}_n(2,\mathbb{R})$ is isomorphic to G.

We now assume that H is non trivial. The proof of Lemma 3.8.1 shows that the group H is isomorphic to a closed subgroup of $\mathrm{Aff}(\mathbb{R})$ hence isomorphic to either $\mathbb{Z}$, $\mathbb{R}$ or $\mathrm{Aff}(\mathbb{R})$. If it is isomorphic to $\mathrm{Aff}(\mathbb{R})$, then its orbits are dense in M, so (M,g) has constant curvature and $\mathrm{Isom}(M,g)$ is differentially conjugate to a subgroup of $\mathrm{PSL}_k(2,\mathbb{R})$. Indeed, the developing map $D : \widetilde{M} \to N$ (where N is either $\mathbb{R}^{1,1}$ or $\widetilde{\mathrm{dS}_2}$) is the map $\widetilde{p}$ defined page 20. This implies that D is injective, so M is the quotient of an open set of $\mathbb{R}^{1,1}$ or $\widetilde{\mathrm{dS}_2}$, and the representations of the isometries in $\mathrm{Diff}(\mathbb{S}^1)$ are either in some $\mathrm{PSL}_k(2,\mathbb{R})$ or in $\mathrm{SO}(2,\mathbb{R}) \subset \mathrm{PSL}(2,\mathbb{R})$.

We now assume that H is either isomorphic to $\mathbb{Z}$ or to $\mathbb{R}$. The group K generated by φ_1 acts on H by conjugacy, which shows that G is a semi direct product $H \rtimes K$.

If the action of K on H is trivial, i.e. if $G \approx H \times K$, then it is isomorphic and semi conjugate to a subgroup of $\mathrm{PSL}_n(2,\mathbb{R})$, taking either the group generated by an element of the center of $\mathrm{PSL}(2,\mathbb{R})$ and the corresponding subgroup of $\mathrm{Aff}(\mathbb{R})$ (seen as the stabilizer of a point in $\mathrm{PSL}_n(2,\mathbb{R})$), when $K \approx \mathbb{Z}/n\mathbb{Z}$, either the group generated by a parabolic element of $\mathrm{PSL}_n(2,\mathbb{R})$ with same rotation number $\frac{1}{n}$ and the corresponding subgroup of $\mathrm{Aff}(\mathbb{R})$, when $K \approx \mathbb{Z}$.

We now assume that K acts non trivially on H. Since $\varphi_1^n \in H$ and H is abelian, this implies that the action of K on H is done by a finite order automorphism of K. There is only one such non trivial element (the map $x \mapsto -x$ in $\mathbb{Z}$ or $\mathbb{R}$), and it is of order two. This implies that there is $k \in \mathbb{N}$ such that $n = 2k$. One can realise such a group in $\mathrm{PSL}_k(2,\mathbb{R})$ by considering the group generated by a hyperbolic element and an elliptic element that exchanges its fixed points.

□

8.2. Non elementary groups.

8.2.1. *The collapsed actions.* Recall (see 5.1 page 31) that if $\rho : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ is a non elementary representation with an invariant Cantor set $L_{\rho(\Gamma)}$, then one can construct a minimal representation $\hat{\rho} : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ by considering a continuous non decreasing map of degree one $\pi : \mathbb{S}^1 \to \mathbb{S}^1$ obtained by collapsing the connected components of $\mathbb{S}^1 \setminus L_{\rho(\Gamma)}$ to points. We then define $\hat{\rho}$ so that it satisfies $\hat{\rho} \circ \pi = \pi \circ \rho$.

Let (M,g) be a spatially compact surface that embeds conformally in $\mathbb{T}^2$, and assume that $\mathrm{Isom}(M,g)$ is non elementary. We denote by $\pi_1, \pi_2 : \mathbb{S}^1 \to \mathbb{S}^1$ and $\hat{\rho}_1^M, \hat{\rho}_2^M$ the maps and representations obtained for the representations ρ_1^M, ρ_2^M. Note that $\hat{\rho}_1^M$ and $\hat{\rho}_2^M$ are representations of $\mathrm{Isom}(M,g)$, i.e. they do not necessarily extend to $\mathrm{Conf}(M,g)$, since the conformal group does not necessarily preserve the minimal sets $L_{\rho_1^M(\mathrm{Isom}(M,g))}$ and $L_{\rho_2^M(\mathrm{Isom}(M,g))}$.

In general, the fact that ρ is faithful does not imply that $\hat{\rho}$ is. However, it is the case for representations associated to spatially compact surfaces.

Proposition 3.8.3. *Let (M,g) be a spatially compact surface that embeds conformally in $\mathbb{T}^2$. If $\mathrm{Isom}(M,g)$ is non elementary, then the collapsed representations $\hat{\rho}_1^M$ and $\hat{\rho}_2^M$ are faithful.*

Proof. Let $\varphi \in \mathrm{Isom}(M,g)$ be such that $\hat{\rho}_1^M(\varphi) = Id$. If $x \in L_{\rho_1^M(\mathrm{Isom}(M,g))}$, then there are two possibilities. Either x bounds an interval I of $\mathbb{S}^1 \setminus L_{\rho_1^M(\mathrm{Isom}(M,g))}$, in which case the fact that $\hat{\rho}_1^M(\varphi)(\hat{x}) = \hat{x}$ implies that $\rho_1^M(\varphi)$ is equal to x or to the other endpoint of I; either x can be approached in both directions by elements of $L_{\rho_1^M(\mathrm{Isom}(M,g))}$, in which case $\hat{y} = \hat{x}$, therefore $\rho_1^M(x) = x$.

If $x \in L_{\rho_1^M(\mathrm{Isom}(M,g))}$ is of the first kind, then it can be approached by a sequence x_n in $L_{\rho_1^M(\mathrm{Isom}(M,g))}$ such that $\rho_1^M(\varphi)(x_n) \to x$, therefore $\rho_1^M(\varphi)(x) = x$ for all $x \in L_{\rho_1^M(\mathrm{Isom}(M,g))}$. This implies that $\rho_1^M(\varphi)'(x) = 1$ for all $x \in L_{\rho_1^M(\mathrm{Isom}(M,g))}$.

The fixed points of $\rho_2^M(\varphi)$ contain the closure of $h_{\to}(L_{\rho_1^M(\mathrm{Isom}(M,g))})$ which is ρ_2^M-invariant, which implies that $\rho_2^M(\varphi)(y) = y$ and $\rho_2^M(\varphi)'(y) = 1$ for all $y \in L_{\rho_2^M(\mathrm{Isom}(M,g))}$. Taking $(x,y) \in M \cap (L_{\rho_1^M(\mathrm{Isom}(M,g))} \times L_{\rho_2^M(\mathrm{Isom}(M,g))})$, we have a fixed point of the isometry φ where the derivative is the identity, hence $\varphi = Id$. This shows that $\hat{\rho}_1^M$ is faithful, and so is $\hat{\rho}_2^M$. □

We will use the fact that the map $h_{\to\uparrow}$ gives a homeomorphism that commutes with the collapsed representation $\hat{\rho}_1^M$.

Proposition 3.8.4. *Let (M,g) be a spatially compact surface that embeds conformally in $\mathbb{T}^2$. Assume that $\mathrm{Isom}(M,g)$ is non elementary. There is $\hat{h}_{\to\uparrow} \in \mathrm{Homeo}(\mathbb{S}^1)$ such that $\hat{h}_{\to\uparrow} \circ \pi_1 = \pi_1 \circ h_{\to\uparrow}$. It commutes with $\hat{\rho}_1^M$.*

Proof. Let $\hat{x} = \pi_1(x) \in \mathbb{S}^1$. We wish to show that $\pi_1 \circ h_{\to\uparrow}(x)$ only depends on $\hat{x}$. It is enough to show that if $I =]a\,,b[$ is a connected component of $\mathbb{S}^1 \setminus L_{\rho_1^M(\mathrm{Isom}(M,g))}$, then $h_{\to\uparrow}(\overline{I})$ is included in the closure of a connected component of $\mathbb{S}^1 \setminus L_{\rho_1^M(\mathrm{Isom}(M,g))}$. If it were not the case, there would be $y \in L_{\rho_1^M(\mathrm{Isom}(M,g))}$ such that $h_{\to\uparrow}(a) < y < h_{\to\uparrow}(b) \leq h_{\to\uparrow}(a)$.

Since $\overline{h_{\to\uparrow}(L_{\rho_1^M(\mathrm{Isom}(M,g))})}$ is closed an invariant under ρ_1^M, it contains $L_{\rho_1^M(\mathrm{Isom}(M,g))}$, so there is $z \in h_{\to\uparrow}(L_{\rho_1^M(\mathrm{Isom}(M,g))})$ such that $h_{\to\uparrow}(a) < z < h_{\to\uparrow}(b) \leq h_{\to\uparrow}(a)$. If $z = h_{\to\uparrow}(u)$ with $u \in L_{\rho_1^M(\mathrm{Isom}(M,g))}$, then we find $u \in I$, which is absurd, and shows that $\hat{h}_{\to\uparrow}$ is well defined.

Notice that $\hat{\rho}_1^M \circ \hat{h}_{\to\uparrow} \circ \pi_1 = \pi_1 \circ \rho_1^M \circ h_{\to\uparrow} = \pi_1 \circ h_{\to\uparrow} \circ \rho_1^M = \hat{h}_{\to\uparrow} \circ \hat{\rho}_1^M \circ \pi_1$. Since π_1 is onto, this shows that $\hat{h}_{\to\uparrow}$ commutes with $\hat{\rho}_1^M$.

Since $\hat{h}_{\to\uparrow}$ is non decreasing of degree one, the union of the open intervals where it is constant is invariant under $\hat{\rho}_1^M$, and therefore is empty, so $\hat{h}_{\to\uparrow}$ is injective. Similarly, it is onto, and continuous, so $\hat{h}_{\to\uparrow} \in \mathrm{Homeo}(\mathbb{S}^1)$. □

We can also define $\hat{h}_{\to} = \pi_1 \circ h_{\to}$ and $\hat{h}_{\leftarrow} = \pi_1 \circ h_{\leftarrow}$. We will use the fact that they are linked by $\hat{h}_{\to\uparrow}$.

Proposition 3.8.5. $\hat{h}_{\to\uparrow} \circ \hat{h}_{\leftarrow} = \hat{h}_{\to}$

Lemma 3.8.6. *Let $I \subset \mathbb{S}^1$ be an interval such that $h_{\leftarrow}$ only takes a finite number of values on I. Then $\mathring{I} \cap L_{\rho_1^M(\mathrm{Isom}(M,g))} = \emptyset$.*

Proof. Assume that $\mathring{I} \cap L_{\rho_2^M(\mathrm{Isom}(M,g))}$ is non empty. Let $x \in L_{\rho_1^M(\mathrm{Isom}(M,g))}$. Since the action of $\mathrm{Isom}(M,g)$ on $L_{\rho_1^M(\mathrm{Isom}(M,g))}$ is minimal, the orbit of x meets $\mathring{I} \cap L_{\rho_1^M(\mathrm{Isom}(M,g))}$, so x has a neighbourhood on which $h_{\leftarrow}$ only takes a finite number of values. Since $L_{\rho_1^M(\mathrm{Isom}(M,g))}$ is compact, this implies that $h_{\leftarrow}(L_{\rho_1^M(\mathrm{Isom}(M,g))})$ is a finite set invariant under ρ_2^M, which is absurd. □

Proof of Proposition 3.8.5. First, we see that $\hat{h}_{\to\uparrow} \circ \hat{h}_{\leftarrow} = \pi_1 \circ h_{\to\uparrow} \circ h_{\leftarrow} = \hat{h}_{\to} \circ h_{\uparrow\leftarrow}$. We wish to show that $\hat{h}_{\to} \circ h_{\uparrow\leftarrow} = \hat{h}_{\to}$.

Let $x \in \mathbb{S}^1$ be such that $h_{\uparrow\leftarrow}(x) \neq x$. This means that x lies on an interval where $h_{\leftarrow}$ is constant. If $\hat{h}_{\to} \circ h_{\uparrow\leftarrow}(x) \neq \hat{h}_{\to}(x)$, then the interval $]h_{\to} \circ h_{\uparrow\leftarrow}(x)\, , h_{\to}(x)[$ intersects $L_{\rho_1^M(\mathrm{Isom}(M,g))}$, but $h_{\leftarrow}$ is constant on this interval. According to the previous lemma, this is a contradiction. Therefore $\hat{h}_{\to\uparrow} \circ \hat{h}_{\leftarrow} = \hat{h}_{\to}$. □

8.2.2. *Convergence property for the collapsed actions.*

Lemma 3.8.7. *Let (M,g) be a spatially compact surface that embeds conformally in $\mathbb{T}^2$. Assume that $\mathrm{Isom}(M,g)$ is non elementary. Then either $\hat{\rho}_1^M(\mathrm{Isom}(M,g))$ is topologically conjugate to a subgroup of $\mathrm{SO}(2,\mathbb{R})$, either the rotation number of $\hat{h}_{\to\uparrow}$ is equal to $\frac{1}{k}$ for somme $k \in \mathbb{N}$, and $\hat{\rho}_1^M(\mathrm{Isom}(M,g))$ is a $(\hat{h}_{\to\uparrow}, k)$-convergence group.*

Proof. If $\hat{\rho}_1^M(\mathrm{Isom}(M,g))$ is compact, then it is topologically conjugate to a subgroup of $\mathrm{SO}(2,\mathbb{R})$, in particular it has the convergence property. We can therefore assume that $\hat{\rho}_1^M(\mathrm{Isom}(M,g))$ is non compact, i.e. that sequences φ_n such that $\hat{\rho}_1^M(\varphi_n) \to \infty$ exist.

Let $(x_0, y_0) \in M$. Consider a sequence $(\varphi_n)_{n\in\mathbb{N}} \in \mathrm{Isom}(M,g)^{\mathbb{N}}$ such that $\hat{\rho}_1^M(\varphi_n)$ has no equicontinuous subsequence. Since $\hat{\rho}_1^M$ is continuous, this implies that $\varphi_n \to \infty$.

Up to a subsequence, we can assume that there are $x_1, y_1 \in \mathbb{S}^1$ such that $\rho_1^M(\varphi_n)(x_0) \to x_1$ and $\rho_2^M(\varphi_n)(y_0) \to y_1$.

First case: Assume that $(x_1, y_1) \in M$.

By Lemma 3.6.5, there are two subcases.

First subcase: $\rho_1^M(\varphi_n)'(x_0) \to 0$ and $\rho_2^M(\varphi_n)'(y_0) \to \infty$

Just as in the proof of Proposition 3.6.6, the horizontal geodesic passing through (x_0, y_0) is shrunk to the point (x_1, y_1), i.e. $\rho_1^M(\varphi_n)(x) \to x_1$ for all $x \in]h_{\leftarrow}(y_0)\, , h_{\to}(y_0)[$. Let $z = h_{\leftarrow}(y_0)$.

On the collapsed circle, we see that $\hat{\rho}_1^M(\varphi_n)(\hat{x}) \to \hat{x}_1$ for all $\hat{x} \in \,]\hat{z}\,, \hat{h}_{\to\uparrow}(\hat{z})[$, and Lemma 3.6.3 implies that the rotation number of $\hat{h}_{\to\uparrow}$ is some $\frac{1}{k}$ and that $(\hat{\rho}_1^M(\varphi_n))_{n\in\mathbb{N}}$ has the $(\hat{h}_{\to\uparrow}, k)$-convergence property.
Second subcase: $\rho_1^M(\varphi_n)'(x_0) \to \infty$ and $\rho_2^M(\varphi_n)'(y_0) \to 0$

The horizontal geodesic passing through (x_0, y_0) is now dilated. If $\hat{x} \in \,]\hat{x}_0\,, \hat{h}_{\to}(y_0)[$, then $\hat{\rho}_1^M(\varphi_n)(x) \to \hat{h}_{\to}(y_1)$. If $\hat{x} \in \,]\hat{h}_{\to}(y_0)\,, \hat{h}_{\to\uparrow}(\hat{x}_0)[$, then $\hat{h}_{\to\uparrow}^{-1}(\hat{x}) \in \,]\hat{h}_{\leftarrow}(y_0)\,, \hat{x}_0[$, so $\hat{\rho}_1^M(\varphi_n)(\hat{h}_{\to\uparrow}^{-1}(\hat{x})) \to \hat{h}_{\leftarrow}(y_1)$, and $\hat{\rho}_1^M(\varphi_n)(\hat{x}) \to \hat{h}_{\to\uparrow}(\hat{h}_{\leftarrow}(y_1)) = \hat{h}_{\to}(y_1)$.

We have shown that $\hat{\rho}_1^M(\varphi_n)(\hat{x}) \to \hat{h}_{\to}(y_1)$ for all $\hat{x} \in \,]\hat{x}_0\,, \hat{h}_{\to\uparrow}(\hat{x}_0)[\, \setminus \{\hat{h}_{\to}(y_0)\}$. By monotonicity, we have convergence on the whole interval, so Lemma 3.6.3 can once again be applied.

Second case: Assume that $(x_1, y_1) \notin M$.
Just as in Proposition 3.6.6, we can assume that there is no $x \in \mathbb{S}^1$ such that $(x, y_0) \in M$ and such that the sequence $\rho_1^M(\varphi_n)(x)$ has a limit point $z \in \mathbb{S}^1$ satisfying $(z, y_1) \in M$. This implies that for all $\hat{x} \in \,]\hat{h}_{\leftarrow}(y_0)\,, \hat{h}_{\to}(y_0)[$, the only limit points of the sequence $\hat{\rho}_1^M(\varphi_n)(\hat{x})$ are $\hat{h}_{\leftarrow}(y_1)$ and $\hat{h}_{\to}(y_1)$. Up to a subsequence, we have three possibilities.
First subcase: $\hat{\rho}_1^M(\varphi_n)(\hat{x}) \to \hat{h}_{\leftarrow}(y_1)$ for all $\hat{x} \in \,]\hat{h}_{\leftarrow}(y_0)\,, \hat{h}_{\to}(y_0)[$.

Since $\hat{h}_{\to}(y_0) = \hat{h}_{\to\uparrow}(\hat{h}_{\leftarrow}(y_0))$, Lemma 3.6.3 implies that the rotation number of $\hat{h}_{\to\uparrow}$ is some $\frac{1}{k}$ and that $(\hat{\rho}_1^M(\varphi_n))_{n\in\mathbb{N}}$ has the $(\hat{h}_{\to\uparrow}, k)$-convergence property..
Second subcase: $\hat{\rho}_1^M(\varphi_n)(\hat{x}) \to \hat{h}_{\to}(y_1)$ for all $\hat{x} \in \,]\hat{h}_{\leftarrow}(y_0)\,, \hat{h}_{\to}(y_0)[$.

The reasoning is exactly the same as in the previous case.
Third subcase: The two limits are possible.

As in Proposition 3.6.6, there is $z \in \,]h_{\leftarrow}(y_0)\,, h_{\to}(y_0)[$ such that $\hat{\rho}_1^M(\varphi_n)(\hat{x}) \to \hat{h}_{\leftarrow}(y_1)$ for all $\hat{x} \in \,]\hat{h}_{\leftarrow}(y_0)\,, \hat{z}[$ and $\hat{\rho}_1^M(\varphi_n)(\hat{x}) \to \hat{h}_{\to}(y_1)$ for all $\hat{x} \in \,]\hat{z}\,, \hat{h}_{\to}(y_0)[$. This implies that $\hat{\rho}_1^M(\varphi_n)(\hat{x}) \to \hat{h}_{\to}(y_1)$ for all $\hat{x} \in \,]\hat{z}\,, \hat{h}_{\to\uparrow}(\hat{z})[$, and we once again conclude with Lemma 3.6.3. □

We now have all the ingredients for the proof of Theorem 3.1.4, which implies Theorem 3.1.2 because of Proposition 3.7.3.

Proof of Theorem 3.1.4. Let (M, g) be a spatially compact surface that embeds conformally in $\mathbb{T}^2$.

Proposition 3.7.2 implies that ρ_1^M and ρ_2^M are semi conjugate, and that their restrictions to $\mathrm{Isom}(M, g)$ are faithful.

If $\mathrm{Isom}(M, g)$ is elementary, then Proposition 3.8.2 states that there is a faithful representation $\rho : \mathrm{Isom}(M, g) \to \mathrm{PSL}_k(2, \mathbb{R})$ for some $k \in \mathbb{N}$ that is semi conjugate to ρ_1^M.

If $\mathrm{Isom}(M, g)$ is non elementary, then Lemma 3.8.7 assures that either $\hat{\rho}_1^M$ is topologically conjugate to a representation in $\mathrm{SO}(2, \mathbb{R}) \subset \mathrm{PSL}(2, \mathbb{R})$, either the rotation number of $\hat{h}_{\to\uparrow}$ is equal to some $\frac{1}{k}$ and that $\hat{\rho}_1^M(\mathrm{Isom}(M, g))$ is a $(\hat{h}_{\to\uparrow}, k)$-convergence group. Since the periodic points of $\hat{h}_{\uparrow}$ form a non empty closed set invariant under $\hat{\rho}_1^M(\mathrm{Isom}(M, g))$, it is equal to $\mathbb{S}^1$ and $\hat{h}_{\to\uparrow}^k = Id$, therefore $\hat{h}_{\to\uparrow}^k$ is topologically conjugate to the rotation of angle $\frac{1}{k}$, and Lemma 3.6.2 states that $\hat{\rho}_1^M(\mathrm{Isom}(M, g))$ is topologically conjugate to a subgroup of $\mathrm{PSL}_k(2, \mathbb{R})$. Since the collapsed action $\hat{\rho}_1^M$ is faithful (Proposition 3.8.3) and semi conjugate to ρ_1^M, we have finished the proof of Theorem 3.1.2. □

9. Conjugacy for elementary groups

The goal of this section is to prove Theorem 3.1.3 in the case of elementary groups. If (M, g) is a spatially compact surface and $G \subset \mathrm{Isom}(M, g)$ is a subgroup, we will identify G and $\rho_1^M(G)$.

9.1. Classification of elements and finite invariant sets. Rather than looking at finite orbits, it will be more practical to consider certain finite invariant sets on which the group may not act transitively.

Lemma 3.9.1. *Let $k \in \mathbb{N}$, and let $h \in \mathrm{Homeo}(\mathbb{S}^1)$ have rotation number $\frac{1}{k}$. Let $G \subset \mathrm{Homeo}(\mathbb{S}^1)$ be an elementary (h,k)-convergence group. Then G has a finite invariant set L_G satisfying one of the following properties:*

(1) L_G has more than $2k$ points
(2) $L_G = \{x_0, y_0, h(x_0), h(y_0), \dots, h^{k-1}(x_0), h^{k-1}(y_0)\}$ where x_0 and y_0 are periodic points for h such that $x_0 < y_0 < h(x_0)$
(3) $L_G = \{x_0, h(x_0), \dots, h^{k-1}(x_0)\}$ where $h^k(x_0) = x_0$.

Proof. Let $E \subset \mathbb{S}^1$ be a finite invariant set. If $\#E > 2k$, then we are in the first case. If $h(E) \neq E$, then G also preserves $h(E)$. If E has elements that are not periodic for h, then E preserves $E \cup h(E) \cup \dots \cup h^n(E)$ for all n. For n large enough, it has more than $2k$ elements. Therefore we can assume that all elements of E are periodic for h, and by adding the iterates under h we can assume that $\#E$ is a multiple of k. If it is $3k$ or more, then we are in the first case. If $\#E = 2k$, then the second condition is satisfied. Finally, if $\#E = k$, then the third condition is satisfied. $\square$

Applying this to the group generated by one element, we obtain a classification of elements similar to the case of $\mathrm{PSL}(2,\mathbb{R})$. If G is a (h,k)-convergence group, and $f \in G \setminus \{Id\}$, then we say that f is

- Hyperbolic if f has exactly $2k$ periodic points.
- Parabolic if f has exactly k periodic points.
- Elliptic if it is not hyperbolic or parabolic.

Note that if γ is elliptic, then the group generated by γ is not always elementary (think of irrational rotations).

9.2. The elliptic case: $\#L_G > 2k$.

Lemma 3.9.2. *Let $k \in \mathbb{N}$ let $h \in \mathrm{Homeo}(\mathbb{S}^1)$ have rotation number $\frac{1}{k}$. Let $G \subset \mathrm{Homeo}(\mathbb{S}^1)$ be a closed elementary (h,k)-convergence group. If G has a finite invariant set $L_G \subset \mathbb{S}^1$ with more than $2k$ elements, then G is compact.*

Proof. If $\#L_G > 2k$, then we can find three points $x_1, x_2, x_3 \in L_G$ such that $x_1 < x_2 < x_3 < h(x_1)$. Let us assume that there is a sequence (f_n) in G such that $f_n \to \infty$. Since the images of x_1, x_2, x_3 under the f_n belong to the finite set L_G, up to a subsequence there are $y_1, y_2, y_3 \in L_G$ such that $f_n(x_i) = y_i$ for $i = 1, 2, 3$. This shows that the sequence (f_n) does not satisfy the (h,k)-convergence property, which is impossible because G is a (h,k)-convergence group. Therefore there is no sequence (f_n) in G such that $f_n \to \infty$, i.e. G is relatively compact. Since it is closed, it is compact. $\square$

9.3. The hyperbolic case: $\#L_G = 2k$.

Lemma 3.9.3. *Let (M,g) be a spatially compact surface with an acausal conformal boundary that embeds in $\mathbb{T}^2$ and let $G \subset \mathrm{Isom}(M,g)$ be an elementary closed subgroup. Let $h_\uparrow, h_\downarrow$ be the homeomorphisms that define the boundary in $\mathbb{T}^2$, and assume that the rotation number of $h_{\to\uparrow}$ is $\frac{1}{k}$. If G has a finite invariant set $L_G \subset \mathbb{S}^1$ such that $L_G = \bigcup_{i=0}^{k-1} \{h^i_{\to\uparrow}(x_0), h^i_{\to\uparrow}(y_0)\}$ where $h^k_{\to\uparrow}(x_0) = x_0$ and $h^k_{\to\uparrow}(y_0) = y_0$, then G is topologically conjugate to a subgroup of $\mathrm{PSL}_k(2,\mathbb{R})$.*

Proof. We note $H = \{f \in G | f(x_0) = x_0 \text{ and } f(y_0) = y_0\}$.

First step: Show that $H \approx \mathbb{R}$ or $H \approx \mathbb{Z}$.
Let $\Phi : H \to \mathbb{R}$ be defined by $\Phi(f) = \mathrm{Log}(f'(x_0))$. If $\Phi(f) = 0$, then f fixes the point

$(x_0, h_\downarrow(y_0)) \in M$ and its derivative at this point is the identity, hence $f = Id$. We showed that Φ is an injective homomorphism, and H is isomorphic to a closed subgroup of $\mathbb{R}$.

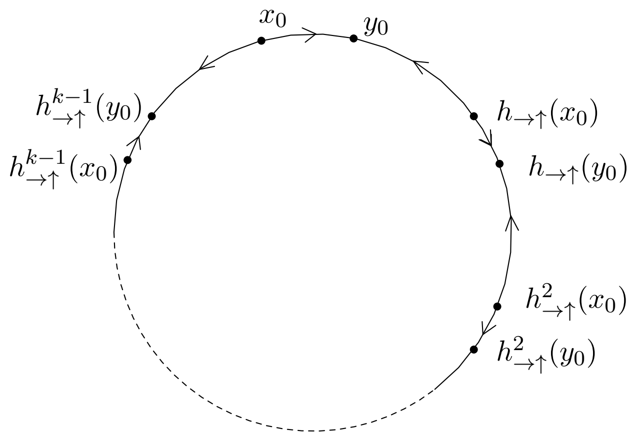

FIGURE 3.5. Dynamics of an element of H

Second step: Find a conjugacy for H
First case: When $H \approx \mathbb{R}$. We set $H = \{f_t | t \in \mathbb{R}\}$. We start by choosing $\widetilde{x}_0, \widetilde{y}_0 \in \mathbb{S}^1$ such that $\widetilde{x}_0 < \widetilde{y}_0 < \widetilde{x}_0 + \frac{1}{k}$ and γ_t the (unique) one parameter subgroup of $\mathrm{PSL}_k(2, \mathbb{R})$ such that $\gamma_t(\widetilde{x}_0) = \widetilde{x}_0$, $\gamma_t(\widetilde{y}_0) = \widetilde{y}_0$ and $\gamma_1'(\widetilde{x}_0) = f_1'(x_0)$ (i.e. γ_t has the same dynamics as f_t, see Figure 3.5). We also choose $z_i \in]h^i_{\to\uparrow}(x_0)\,, h^i_{\to\uparrow}(y_0)[$, $z_i' \in]h^i_{\to\uparrow}(y_0)\,, h^{i+1}_{\to\uparrow}(x_0)[$, $\widetilde{z}_i \in]\widetilde{x}_0 + \frac{i}{k}\,, \widetilde{y}_0 + \frac{i}{k}[$, and $\widetilde{z}_i' \in]\widetilde{y}_0 + \frac{i}{k}\,, \widetilde{x}_0 + \frac{i+1}{k}[$.

If $x \in]h^i_{\to\uparrow}(x_0)\,, h^i_{\to\uparrow}(y_0)[$, then we consider $t_x \in \mathbb{R}$ such that $x = f_{t_x}(z_i)$ and we set $\varphi(x) = \gamma_{t_x}(\widetilde{z}_i)$. If $x \in]h^i_{\to\uparrow}(y_0)\,, h^{i+1}_{\to\uparrow}(x_0)[$, then we consider $t_x \in \mathbb{R}$ such that $x = f_{t_x}(z_i')$ and we set $\varphi(x) = \gamma_{t_x}(\widetilde{z}_i')$. Since $t_{f_s(x)} = t + s$, we see that $\varphi \circ f_t = \gamma_t \circ \varphi$, and φ is a homeomorphism.
Second case: When $H \approx \mathbb{Z}$. We set $H = \{f_1^n | n \in \mathbb{Z}\}$. We start by choosing $\widetilde{x}_0, \widetilde{y}_0 \in \mathbb{S}^1$ such that $\widetilde{x}_0 < \widetilde{y}_0 < \widetilde{x}_0 + \frac{1}{k}$ and $\gamma_1 \in \mathrm{PSL}_k(2, \mathbb{R})$ a hyperbolic element such that $\gamma_1(\widetilde{x}_0) = \widetilde{x}_0$, $\gamma_1(\widetilde{y}_0) = \widetilde{y}_0$ and $\gamma_1'(\widetilde{x}_0) = f_1'(x_0)$. We also choose $z_i \in]h^i_{\to\uparrow}(x_0)\,, h^i_{\to\uparrow}(y_0)[$, $z_i' \in]h^i_{\to\uparrow}(y_0)\,, h^{i+1}_{\to\uparrow}(x_0)[$, $\widetilde{z}_i \in]\widetilde{x}_0 + \frac{i}{k}\,, \widetilde{y}_0 + \frac{i}{k}[$, and $\widetilde{z}_i' \in]\widetilde{y}_0 + \frac{i}{k}\,, \widetilde{x}_0 + \frac{i+1}{k}[$. Finally, we can choose arbitrarily the restrictions $\varphi : [z_i\,, f_1(z_i)] \to [\widetilde{z}_i\,, \gamma_1(\widetilde{z}_i)]$ and $\varphi : [z_i'\,, f_1(z_i')] \to [\widetilde{z}_i'\,, \gamma_1(\widetilde{z}_i')]$.

If $x \in]h^i_{\to\uparrow}(x_0)\,, h^i_{\to\uparrow}(y_0)[$, then we consider the unique $n_x \in \mathbb{Z}$ such that $f_1^{n_x}(x) \in [z_i\,, f_1(z_i)[$, and we set $\varphi(x) = \gamma_1^{-n_x} \circ \varphi \circ f_1^{n_x}(x)$. Similarly, if $x \in]h^i_{\to\uparrow}(y_0)\,, h^{i+1}_{\to\uparrow}(x_0)[$, then we consider the unique $n_x \in \mathbb{Z}$ such that $f_1^{n_x}(x) \in [z_i'\,, f_1(z_i')[$, and we set $\varphi(x) = \gamma_1^{-n_x} \circ \varphi \circ f_1^{n_x}(x)$. This construction gives a homeomorphism φ such that $\varphi \circ f_1 = \gamma_1 \circ \varphi$.

Third step: Show that if $f \in G$, then $f(x_0) = h^i_{\to\uparrow}(x_0)$ for some $i \in \mathbb{N}$.
Let $f_1 \in H$ have x_0 as an attracting fixed point. We first wish to show that f_1 and f commute. For this, we see that $f^{-1} \circ f_1 \circ f(x_0) = f(x_0)$, i.e. $f^{-1} \circ f_1 \circ f \in H$. If $H =< f_1 >\approx \mathbb{Z}$, then there is $n \in \mathbb{N}$ such that $f^{-1} \circ f_1 \circ f = f_1^n$. But $f^k \in H$, therefore f^k commutes with f_1, which shows that $f_1 = f_1^n$, hence $n = 1$. If $H \approx \mathbb{R}$, then there is t such that $f^{-1} \circ f_1 \circ f = f_t$ and $f^k \in H$ shows $f_1 = f_t$, hence $t = 1$.
Now that we know that f and f_1 commute, we choose $x \in \mathbb{S}^1$ sufficiently close to x_0 so that $f_1^n(x) \to x_0$ and $f(x) \neq h^j_{\to\uparrow}(y_0)$ for all j. Then $f_1^n(x) \to h^i_{\to\uparrow}(x_0)$ for some i, and

$f(x_0) = \lim f(f_1^n(x)) = \lim f_1^n(f(x)) = h^i_{\rightarrow\uparrow}(x_0)$.

Fourth step: Find a conjugacy for H and one element of $G \setminus H$.
Note that we showed in the third step that H is contained in the center of G. Let $f \in G \setminus H$, and let $\gamma \in \mathrm{PSL}_k(2,\mathbb{R})$ have the same rotation number as f and have $\widetilde{x}_0, \widetilde{y}_0$ as periodic points. If x_0 is attracting (resp. repelling) for f^k, then we choose γ such that $\widetilde{x}_0$ is attracting (resp. repelling) for γ^k (i.e. f and γ have the same dynamics).
First case: $H \approx \mathbb{R}$. We wish to choose φ such that $\gamma \circ \varphi(z_j) = \varphi \circ f(z_j)$ and $\gamma \circ \varphi(z'_j) = \varphi \circ f(z'_j)$. The left side of the equation is $\gamma \circ \varphi(z_j) = \gamma(\widetilde{z}_j)$. On the right side, we have $\varphi \circ f(z_j)$. Let $t_j \in \mathbb{R}$ be such that $f(z_j) = f_{t_j}(z_{i+j})$. We see that $\varphi \circ f(z_j) = \gamma_{t_j}(\widetilde{z}_{i+j})$. Hence the first equation holds if and only if $\widetilde{z}_{i+j} = \gamma_{-t_j} \circ \gamma(\widetilde{z}_j)$. This shows that we can fix the family (z_j) arbitrarily, then choose $\widetilde{z}_j$ for one j in each class in $\mathbb{Z}/i\mathbb{Z}$, and set $\widetilde{z}_{i+j} = \gamma_{-t_j} \circ \gamma(\widetilde{z}_j)$. The same goes for the z'_j and $\widetilde{z}'_j$.

For $x \in]h^j_{\rightarrow\uparrow}(x_0)\,, h^j_{\rightarrow\uparrow}(y_0)[$, let $t \in \mathbb{R}$ be such that $x = f_t(z_j)$.

$$\begin{aligned}
\varphi \circ f(x) = \varphi \circ f \circ f_t(z_j) &= \varphi \circ f_t \circ f(z_j) && f_t \in Z(G)\\
&= \gamma_t \circ \varphi \circ f(z_j) && \varphi \text{ conjugates } f_t \text{ and } \gamma_t\\
&= \gamma_t \circ \gamma \circ \varphi(z_j) && \text{Choice of } \widetilde{z}_{i+j}\\
&= \gamma \circ \gamma_t \circ \varphi(z_j) && \gamma_t \text{ commutes with } \gamma\\
&= \gamma \circ \varphi(x) && \text{Definition of } \varphi
\end{aligned}$$

The same calculations hold for $x \in]h^j_{\rightarrow\uparrow}(y_0)\,, h^{j+1}_{\rightarrow\uparrow}(x_0)[$. This shows that $\varphi \circ f = \gamma \circ \varphi$.
Second case: $H \approx \mathbb{Z}$. We wish to choose φ such that $\gamma \circ \varphi(u) = \varphi \circ f(u)$ for $u \in [z_j\,, f_1(z_j)[\cup [z'_j\,, f_1(z'_j)[$. For $u \in [z_j\,, f_1(z_j)[$, we consider $n_u \in \mathbb{Z}$ such that $f(u) = f_1^{n_u}(x_u)$ where $x_u \in [z_{i+j}\,, f_1(z_{i+j})[$. We get $\varphi \circ f(u) = \gamma \circ \varphi(u)$ if and only if $\varphi(x_u) = \gamma_1^{-n_u} \circ \gamma \circ \varphi(u)$. Hence we only need to choose φ on $[z_j\,, f_1(z_j)[$ for one j in each class modulo i, and set φ on $[z_{i+j}\,, f_1(z_{i+j})[$ by the formula $\varphi(x_u) = \gamma_1^{-n_u} \circ \gamma \circ \varphi(u)$. We do the same on $[z'_j\,, f_1(z'_j)[$.

Finally, for $x \in]h^j_{\rightarrow\uparrow}(x_0)\,, h^j_{\rightarrow\uparrow}(y_0)[$, we consider $n \in \mathbb{Z}$ such that $x = f_1^n(u)$ where $u \in [z_j\,, f_1(z_j)[$.

$$\begin{aligned}
\varphi \circ f(x) = \varphi \circ f \circ f_1^n(u) &= \varphi \circ f_1^n \circ f(u)\\
&= \gamma_1^n \circ \varphi \circ f(u)\\
&= \gamma_1^n \circ \gamma \circ \varphi(u)\\
&= \gamma \circ \gamma_1^n \circ \varphi(u)\\
&= \gamma \circ \varphi(x)
\end{aligned}$$

The same calculations holds for $x \in]h^j_{\rightarrow\uparrow}(y_0)\,, h^{j+1}_{\rightarrow\uparrow}(x_0)[$. This shows that $\varphi \circ f = \gamma \circ \varphi$.

Fifth step: Show that it provides a conjugacy for G.
Let $F \subset \mathbb{Z}/k\mathbb{Z}$ be the set of classes of numbers $i \in \mathbb{Z}$ such that there is $f \in G$ satisfying $f(x_0) = h^i_{\rightarrow\uparrow}(x_0)$. It is a subgroup of $\mathbb{Z}/k\mathbb{Z}$, hence there is $n \in \{0, \ldots, k-1\}$ such that $F = n\mathbb{Z}/k\mathbb{Z}$. Let $f \in G$ be such that $f(x_0) = h^n_{\rightarrow\uparrow}(x_0)$, and let $\varphi \in \mathrm{Homeo}(\mathbb{S}^1)$ be a conjugacy for H and for f.
If u is another element of $G \setminus H$, then there is $i \in \mathbb{Z}$ such that $u \circ f^{-i} \in H$, hence $\varphi \circ u \circ \varphi^{-1} = (\varphi \circ (u \circ f^{-i}) \circ \varphi^{-1}) \circ (\varphi \circ f^i \circ \varphi^{-1}) \in \mathrm{PSL}_k(2,\mathbb{R})$. □

9.4. The parabolic case: $\#L_G = k$.

Lemma 3.9.4. *Let (M,g) be a spatially compact surface with an acausal conformal boundary that embeds in $\mathbb{T}^2$ and let $G \subset \mathrm{Isom}(M,g)$ be an elementary closed subgroup. Let $h_\uparrow, h_\downarrow$ be the homeomorphisms that define the boundary in $\mathbb{T}^2$, and assume that the* *rotation number of* $h_{\to\uparrow} = h_\to \circ h_\uparrow$ *is* $\frac{1}{k}$. *If* G *has a finite invariant set* $L_G \subset \mathbb{S}^1$ *such that* $L_G = \{x_0, h_{\to\uparrow}(x_0), \ldots, h_{\to\uparrow}^{k-1}(x_0)\}$ *where* $h_{\to\uparrow}^k(x_0) = x_0$, *then* G *is topologically conjugate to a subgroup of* $\mathrm{PSL}_k(2,\mathbb{R})$.

PROOF. Let $H = \{f \in G | f(x_0) = x_0\}$.

First step: Assume that H contains a hyperbolic element f.
Up to replacing f by f^{-1}, we can assume that x_0 is an attracting point for f. If all elements of H fix the other fixed point y_0 of f, then we have an invariant set with $2k$ elements and Lemma 4.1.6 shows that G is topologically conjugate to a subgroup of $\mathrm{PSL}_k(2,\mathbb{R})$. Hence we can assume that there is $u \in H$ that does not fix y_0.
First case: Assume that u is parabolic. Up to replacing u by u^{-1}, assume that $u(y_0) \in]y_0\,, x_0[$. We consider the sequence $u_n = f^{-n} \circ u \circ f^n$. We have $u_n(x_0) = x_0$ and $u_n(y_0) = f^{-n}(u(y_0)) \to y_0$. If $x \in]x_0\,, y_0[$, then $x_0 < f^n(x) < u(f^n(x)) < y_0 < x_0$ gives $x_0 < f^{-n}(f^n(x)) = x < u_n(x) < y_0$ (see Figure 3.6), hence the sequence $u_n(x)$ does not have x_0 as a limit point. If $y \in]y_0\,, h_{\to\uparrow}(x_0)[$, then we find $y_0 < y < u_n(y) < h_{\to\uparrow}(x_0) < y_0$ and the sequence $u_n(y)$ does not have y_0 as a limit point. This shows that the sequence u_n does not have the $(h_{\to\uparrow}, k)$-convergence property, hence it is equicontinuous. This implies that G is not discrete. Since it is a Lie group, it has dimension at least one, and there is a one parameter subgroup of parabolic elements. The orbit of any point of M under this one parameter subgroup intersects the axes of f, hence the curvature of M is constant.

Consider the developing map $D : \widetilde{M} \to N$ where N is either $\mathbb{R}^{1,1}$ or $\widetilde{dS_2}$. Notice

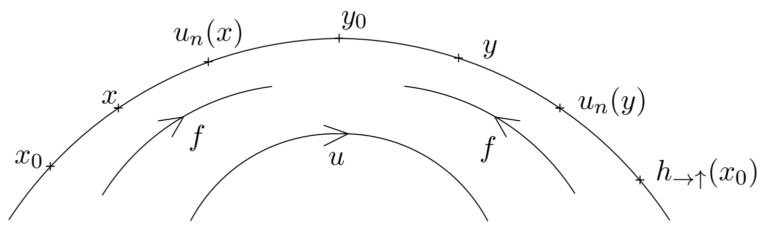

FIGURE 3.6. Dynamics in the parabolic case

that D is the conformal embedding $\widetilde{p}$ defined in Theorem 1.4.1, so it is injective and M is a quotient of an open set of $\mathbb{R}^{1,1}$ or $\widetilde{dS_2}$. Since it has an isometry group acting non properly, it is an open set of $\widetilde{dS_2}$ and $\rho_1^M(\mathrm{Isom}(M,g))$ is differentially conjugate to a quotient of $\widetilde{\mathrm{PSL}}(2,R)$. The fact that it has isometries with k fixed points on the circle implies that $\rho_1^M(\mathrm{Isom}(M,g))$ is differentially conjugate to a subgroup of $\mathrm{PSL}_k(2,\mathbb{R})$.
Second case: Assume that u is hyperbolic. The commutator $[u,f] \in H$ satisfies $[u,f](x_0) = x_0$ and $[u,f]'(x_0) = 1$, therefore it is either parabolic, either the identity. Since u and f have different fixed points, we have $[u,f](y_0) \neq y_0$, and H possesses a parabolic element.

Second step: H has only parabolic elements.
Since H preserves the affine structure on $\{x_0\} \times]h_\downarrow(x_0)\,, h_\uparrow(x_0)[$ defined by the parametrisation of geodesics, we see that H is isomorphic to a subgroup of the affine group $\mathrm{Aff}(\mathbb{R})$. Parabolic elements are sent to translations in $\mathrm{Aff}(\mathbb{R})$, therefore H is isomorphic to a subgroup of $\mathbb{R}$, and it is either $\mathbb{Z}$ either $\mathbb{R}$. By using exactly the same methods as in the hyperbolic case (second to fifth steps), we see first that H, then G are topologically conjugate to subgroups of $\mathrm{PSL}_k(2,\mathbb{R})$. $\square$

As a byproduct of the proof, we obtain the following result that will simplify our search for counter examples when we study the differential conjugacy problem.

Corollary 3.9.5. *Let (M,g) be a spatially compact surface with an acausal conformal boundary that embeds in $\mathbb{T}^2$. Let $G \subset \mathrm{Isom}(M,g)$ be an elementary subgroup (i.e. $\rho_1^M(G)$ is elementary) that acts non properly on M. If $\rho_1^M(G)$ is not differentially conjugate to a subgroup of some $\mathrm{PSL}_k(2,\mathbb{R})$, then there is a finite index subgroup of G that is isomorphic to $\mathbb{Z}$ or $\mathbb{R}$.*

10. Conjugacy for non elementary groups

We now wish to show Theorem 3.1.3 in the non elementary case. We will deal only with surfaces that embed conformally in $\mathbb{T}^2$, and conclude with Proposition 3.7.3. The main tool will be the (h,k)-convergence property, but we face the problem that h is not necessarily a rotation. If it were the case, then the result would be immediate by Lemma 3.6.2.

Our goal is to see that there is a homeomorphism $\widetilde{h}$ that is topologically conjugate to a rotation, such that $\rho_1^M(\mathrm{Isom}(M,g))$ is a $(\widetilde{h},k)$-convergence group. Our strategy is to keep $h_{\to\uparrow}$ on the limit set, and to change it outside the limit set in order to make all points periodic. First, we will see that $h_{\to\uparrow}$ is of order k on the limit set.

Lemma 3.10.1. *Let $k \in \mathbb{N}^*$ and let $h \in \mathrm{Homeo}(\mathbb{S}^1)$ have rotation number $\frac{1}{k}$. Let $G \subset \mathrm{Homeo}(\mathbb{S}^1)$ be a non elementary (h,k)-convergence group. Let F be the set of points $b \in \mathbb{S}^1$ such that there is a sequence $f_n \to \infty$ in G and $a \in \mathbb{S}^1$ satisfying:*

- $h^k(a) = a$ *and* $h^k(b) = b$
- $\forall i \in \{0,\dots,k-1\}\ \forall x \in]h^i(a)\,,h^{i+1}(a)[\ f_n(x) \to h^i(b)$

Then $L_G \subset \overline{F}$. In particular, all points of L_G are periodic for h.

Proof. Since the minimal invariant closed set is unique in the non elementary case, we only need to show that F is G-invariant. If f_n is a sequence in G defining $b \in F$, then the sequence $g \circ f_n$ shows that $g(b) \in F$. □

Lemma 3.10.2. *Let (M,g) be a spatially compact surface with an acausal conformal boundary that embeds in $\mathbb{T}^2$ and assume that $G = \rho_1^M(\mathrm{Isom}(M,g))$ is a non elementary subgroup. Let $h_\uparrow, h_\downarrow$ be the homeomorphisms that define the boundary in $\mathbb{T}^2$, and assume that the rotation number of $h_{\to\uparrow}$ is $\frac{1}{k}$. There is $h \in \mathrm{Homeo}(\mathbb{S}^1)$ such that:*

- $h^k = Id$
- $h(x) = h_{\to\uparrow}(x)$ *for all* $x \in L_G$
- $h \circ f = f \circ h$ *for all* $f \in G$

Consequently, G is a (h,k)-convergence group.

Proof. If $L_G = \mathbb{S}^1$, then $h_{\to\uparrow}^k = Id$ and we can take $h = h_{\to\uparrow}$. We now assume that L_G is a Cantor set.

If $I \subset \mathbb{S}^1$ is a connected component of $\mathbb{S}^1 \setminus L_G$, then so is $h_{\to\uparrow}(I)$, and $h_{\to\uparrow}^k(I) = I$ (because $h_{\to\uparrow}^k$ is the identity on L_G). We use the decomposition into connected components $\mathbb{S}^1 \setminus L_G = \bigcup_{n\in\mathbb{N}}(I_n \sqcup h_{\to\uparrow}(I_n) \sqcup \cdots \sqcup h_{\to\uparrow}^{k-1}(I_n))$. Let $E \subset \mathbb{N}$ be a fundamental domain for the action of G.

Let $n \in E$. Let G_n be the set of elements of G that preserve the union $I_n \sqcup \cdots \sqcup h_{\to\uparrow}^{k-1}(I_n)$. It is a closed elementary group with an invariant set containing $2k$ elements. Lemma 4.1.6 implies that it is topologically conjugate to subgroup Γ_n of $\mathrm{PSL}_k(2,\mathbb{R})$. We write $G_n = \varphi^{-1}\Gamma_n\varphi$. Let R be the element of the center of $\mathrm{PSL}_k(2,\mathbb{R})$ of rotation number $\frac{1}{k}$. Set h on $h_{\to\uparrow}^i(I_n)$ to be $\varphi^{-1}R\varphi$. Since $R^k = Id$, we find $h^k = Id$. Since R commutes with Γ_n, we see that h commutes with G_n.

On $f(h^i_{\to\uparrow}(I_n))$ for $f \in G$, we set $h_{/f(h^i_{\to\uparrow}(I_n))} = f \circ h_{/h^i_{\to\uparrow}(I_n)} \circ f^{-1}$. Since h commutes with the elements that preserve $I_n \sqcup \cdots \sqcup h^{k-1}_{\to\uparrow}(I_n)$, we see that h commutes with G. By construction, it is equal to $h_{\to\uparrow}$ on L_G, and it is a homeomorphism. $\square$

We can now achieve the proof Theorem 3.1.5, which implies Theorem 3.1.3 because of Proposition 3.7.3.

Proof of Theorem 3.1.5. Let (M, g) be a spatially compact surface with an acausal conformal boundary that embeds conformally in the flat torus.

Proposition 3.7.2 implies that ρ_1^M and ρ_2^M are faithful and topologically conjugate to each other.

If $\mathrm{Isom}(M, g)$ acts properly on M, then Proposition 3.5.4 concludes. If $\mathrm{Isom}(M, g)$ acts non properly on M, then according to Proposition 3.4.1, the rotation number of $h_{\to\uparrow}$ is equal to $\frac{1}{k}$ for some $k \in \mathbb{N}$.

If $\rho_1^M(\mathrm{Isom}(M, g))$ is elementary, then Lemma 3.9.1 assures that we can apply Lemma 4.1.5, 4.1.6 or 3.9.4 to show that $\rho_1^M(\mathrm{Isom}(M, g))$ is topologically conjugate to a subgroup of a finite cover of $\mathrm{PSL}(2, \mathbb{R})$.

If $\rho_1^M(\mathrm{Isom}(M, g))$ is non elementary, then Lemma 3.10.2 shows that there is $h \in \mathrm{Homeo}(\mathbb{S}^1)$ such that $h^k = Id$ and $\rho_1^M(\mathrm{Isom}(M, g))$ is a (h, k)-convergence group. Since $h^k = Id$, it is topologically conjugate to a rotation, and Lemma 3.6.2 shows that $\rho_1^M(\mathrm{Isom}(M, g))$ is topologically conjugate to a subgroup of $\mathrm{PSL}_k(2, \mathbb{R})$. $\square$

CHAPTER 4

Differentiability of the conjugacy

1. Introduction

In order to study ρ_1^M and ρ_2^M up to differential conjugacy, we will restrict ourselves to the case where (M, g) is conformal to dS_2 (just as in section 3.1 of chapter 3). Studying a Lorentz surface conformal to dS_2 is equivalent to studying a volume form ω on $\mathcal{C} = \mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$.

We will study various notions of conjugacy to a Fuchsian group.

1.1. Fuchsian groups and generalisations. Recall that a group action on the circle $\rho : \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ is said to be **Fuchsian** if $\rho(\Gamma) \subset \mathrm{PSL}(2, \mathbb{R})$, and **topologically Fuchsian** if there is $h \in \mathrm{Homeo}(\mathbb{S}^1)$ such that $h^{-1}\rho(\Gamma)h \subset \mathrm{PSL}(2, \mathbb{R})$.

1.1.1. *Differential conjugacy.* When considering actions by diffeomorphisms, the natural notion of conjugacy is the conjugacy in the group $\mathrm{Diff}(\mathbb{S}^1)$. We will say that $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ is **differentially Fuchsian** if there is $h \in \mathrm{Diff}(\mathbb{S}^1)$ such that $h^{-1}\rho(\Gamma)h \subset \mathrm{PSL}(2, \mathbb{R})$ (in the absence of precision, $\mathrm{Diff}(\mathbb{S}^1)$ denotes the group of C^∞ diffeomorphisms).

There is no general condition under which a topologically Fuchsian representation $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ is automatically differentially Fuchsian. However, there are two known results assuring the existence of a differential conjugacy under specific hypothesis: a theorem of Herman on diffeomorphisms conjugate to irrational rotations, and a theorem of Ghys on representations of surface groups.

1.1.2. *Area-preserving groups.* We will say that an action $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ is **area-preserving** if the diagonal action on $\mathcal{C} = \mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$ preserves a smooth volume form.

Theorem 3.3.2 states that an area-preserving representation is topologically Fuchsian. If $h \in \mathrm{Diff}(\mathbb{S}^1)$ and $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ preserves the volume form ω on $\mathcal{C}$, then $h^{-1}\rho h$ preserves the volume form $h^\star\omega$. If h is only continuous, then $h^\star\omega$ is only a measure, it is not always absolutely continuous with respect to the Lebesgue measure.

Since the action of $\mathrm{PSL}(2, \mathbb{R})$ preserves a volume form, all differentially Fuchsian representations are area-preserving.

We will show that under some specific hypotheses, it is an equivalence.

Theorem 4.1.1. *Assume that $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ satisfies (at least) one of the following conditions:*

- *There is a dense orbit on $\mathbb{S}^1$.*
- *$\rho(\Gamma) \subset \mathrm{Diff}^\omega(\mathbb{S}^1)$ and Γ has no finite orbit on $\mathbb{S}^1$.*
- *$\Gamma = \mathbb{Z}$, $\rho(1) \in \mathrm{Diff}^\omega(\mathbb{S}^1)$ and $\rho(1)$ has exactly two fixed points.*
- *$\Gamma = \mathbb{Z}$ and $\rho(1)$ has no fixed point on $\mathbb{S}^1$.*

Then ρ is area-preserving if and only if it is differentially Fuchsian.

Here, if $\rho(\Gamma) \subset \mathrm{Diff}^\omega(\mathbb{S}^1)$, then we say that ρ is area-preserving if it preserves an analytic volume form on $\mathcal{C}$. The proof is obtained by combining Proposition 4.1.5, Theorem 4.1.7, Theorem 4.1.8 and Theorem 4.1.9. We will also see that this equivalence is not always true.

1.1.3. *L-Differential conjugacy.* It will be very simple to find some area preserving representations that are not differentially Fuchsian, but the first examples will be elementary. To find non elementary examples, we will have to consider representations with an invariant Cantor set.

However, the examples that we will give share a property with minimal actions (i.e. all orbits on $\mathbb{S}^1$ are dense): the conjugacy is always differentiable along the limit set.

Definition 4.1.2. *We will say that two representations $\rho_1, \rho_2 : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ with no finite orbits are L-differentially conjugate if there is $h \in \mathrm{Homeo}(\mathbb{S}^1)$ such that $h^{-1}\rho_2 h = \rho_1$ and such that there is $\varphi \in \mathrm{Diff}(\mathbb{S}^1)$ with the same restriction $\varphi_{/L_{\rho_1(\Gamma)}} = h_{/L_{\rho_1(\Gamma)}}$.*
We say that $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ is L-differentially Fuchsian if it is L-differentially conjugate to a Fuchsian action.

Knowing that L-differentially Fuchsian actions are not necessarily differentially Fuchsian, the following statement shows that area-preserving actions are not necessarily differentially Fuchsian.

Theorem 4.1.3. *If $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ is L-differentially conjugate to a convex cocompact representation in $\mathrm{PSL}(2, \mathbb{R})$, then ρ is area-preserving.*

1.1.4. *Spectral conditions.* Finally, a weaker generalisation of Fuchsian actions consists in looking only at the derivatives at fixed points. A hyperbolic element $\gamma \in \mathrm{PSL}(2, \mathbb{R})$ has exactly two fixed points $N, S \in \mathbb{S}^1$. The derivatives satisfy $\gamma'(N)\gamma'(S) = 1$ and $\gamma'(N) \neq 1$.

Definition 4.1.4. *We say that $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ is spectrally Möbius-like if non trivial elements have at most two fixed points, and if elements γ with two fixed points N, S satisfy $\rho(\gamma)'(N)\rho(\gamma)'(S) = 1$ and $\rho(\gamma)'(N) \neq 1$.*

This is a condition that concerns individual elements of the group rather than the group structure (hence the terminology, in reference to Möbius-like actions, i.e. such that every element is topologically conjugate to an element of $\mathrm{PSL}(2, \mathbb{R})$). Differentially Fuchsian and L-differentially Fuchsian actions are spectrally Möbius-like. It is also quite straightforward to see that area-preserving actions are spectrally Möbius-like (see Proposition 4.1.6).

One can also define the spectrum $S(\rho) : \Gamma \to \mathbb{R}^2$ as the data of the derivatives at fixed points for all elements of Γ.

1.2. The case of a single diffeomorphism. The problem of knowing when a diffeomorphism that is topologically conjugate to a rotation is differentially conjugate to this rotation has been deeply studied. A well known theorem of Herman ([**Her79**]) states that a differentiable conjugacy always exists provided the diffeomorphism has its rotation number in a certain set of full Lebesgue measure (more precisely, if it satisfies a Diophantine condition, see [**Yoc84**] for an exact description), but there are smooth examples where a differentiable conjugacy does not exist. In the area-preserving case, we do not have different behaviours:

Proposition 4.1.5. *Let $f \in \mathrm{Diff}(\mathbb{S}^1)$ be a fixed point free diffeomorphism. If f is area-preserving, then it is differentially conjugate to a rotation.*

This result does not extend to diffeomorphisms with fixed points: there are some area-preserving circle diffeomorphisms that are not differentially conjugate to an element of $\mathrm{PSL}(2, \mathbb{R})$. The following result treats the case corresponding to hyperbolic elements of $\mathrm{PSL}(2, \mathbb{R})$.

Proposition 4.1.6. *Let $f \in \mathrm{Diff}(\mathbb{S}^1)$ have exactly two fixed points N and S. It is area-preserving if and only if it is spectrally Möbius-like*

For parabolic diffeomorphisms (i.e. having one fixed point), the situation is more complicated. We will see there are some area-preserving examples that are not differentially conjugate to elements of $\mathrm{PSL}(2,\mathbb{R})$, but that some diffeomorphisms with one fixed point do not preserve any volume form on the cylinder $\mathcal{C}$.

1.3. The analytic case. The counter examples produced by Proposition 4.1.6 never give an analytic volume form. Indeed, it appears that the analytic case is rigid.

We say that $\rho : \Gamma \to \mathrm{Diff}^{\omega}(\mathbb{S}^1)$ is **analytically Fuchsian** if there is a real analytic diffeomorphism $h \in \mathrm{Diff}^{\omega}(\mathbb{S}^1)$ such that $h^{-1}\rho(\Gamma)h \subset \mathrm{PSL}(2,\mathbb{R})$.

Theorem 4.1.7. *Let $f \in \mathrm{Diff}^{\omega}(\mathbb{S}^1)$ have exactly two fixed points. If f preserves an analytic volume form on $\mathcal{C}$, then f is analytically conjugate to a hyperbolic element of* $\mathrm{PSL}(2,\mathbb{R})$.

For parabolic diffeomorphisms, there are some straightforward analytic counter examples. However, for non elementary representations, i.e. without any finite orbit on $\mathbb{S}^1$, there is also a rigidity phenomenon:

Theorem 4.1.8. *If $\rho : \Gamma \to \mathrm{Diff}^{\omega}(\mathbb{S}^1)$ is a non elementary representation preserving an analytic volume form on $\mathcal{C}$, then ρ is analytically Fuchsian.*

The treatment of the non elementary case will be very different from the case of a single diffeomorphism, mainly since the preserved volume form is unique for an analytic non elementary group.

1.4. The topologically transitive case. A theorem of Ghys, proved in [**Ghy93**], states that any representation of a surface group (i.e. the fundamental group of a compact surface without boundary) into $\mathrm{Diff}(\mathbb{S}^1)$ with maximal Euler number is differentially Fuchsian. One particularity of these representations is that they are topologically transitive. Given the condition of preserving a volume on $\mathcal{C}$, we also obtain a rigidity result.

Theorem 4.1.9. *Let $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ be a topologically transitive representation that preserves a C^2 volume form on $\mathcal{C}$. Then ρ is differentially Fuchsian.*

Remark. This result actually contains Proposition 4.1.5, since diffeomorphisms that are topologically conjugate to a rational rotation are automatically differentially conjugate to this rotation, and irrational rotations are topologically transitive.

The C^2 regularity hypothesis is not only practical for the proof (it ensures the existence of curvature), but it is important as there are some counter examples if we do not ask for enough regularity on the volume form.

1.5. The exceptional minimal set case. The case of a single diffeomorphism suggests that the preservation of a volume form on $\mathcal{C}$ can be understood by looking at the fixed points. In the setting of Theorem 4.1.9, fixed points (when they exist) are dense in $\mathbb{S}^1$. We will now study groups for which the closure of fixed points is a Cantor set.

1.5.1. *Differential structure on the Cantor set.* The definition of L-differential conjugacy suggests that we define a notion of diffeomorphisms between Cantor sets.

If $C \subset \mathbb{S}^1$ is a closed set, then a function $f : C \to \mathbb{S}^1$ is C^k in the Whitney sense if f admits a Taylor development of order k at every point of C, the coefficients being continuous functions. This is equivalent to asking that f is the restriction to C of a C^k function on $\mathbb{S}^1$.

We say that $f : C_1 \to C_2$ (where C_1 and C_2 are two Cantor sets in $\mathbb{S}^1$) is a C^k diffeomorphism if f is a cyclic order preserving homeomorphism such that f and f^{-1} are C^k in the Whitney sense. This is equivalent to asking that f is the restriction to C_1 of a circle diffeomorphism.

With this definition, we see that two non elementary representations $\rho_1, \rho_2 : \Gamma \to$

$\mathrm{Diff}(\mathbb{S}^1)$ are L-differentially conjugate if there is a homeomorphism $h \in \mathrm{Homeo}(\mathbb{S}^1)$ such that $h\rho_1 h^{-1} = \rho_2$ and such that the restriction $h_{/L_{\rho_1(\Gamma)}} : L_{\rho_1(\Gamma)} \to L_{\rho_2(\Gamma)}$ is a diffeomorphism.

If $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ is L-differentially Fuchsian, then let $h \in \mathrm{Homeo}(\mathbb{S}^1)$ be such that $\rho_0 = h\rho h^{-1}$ is Fuchsian and such that $h_{/L_{\rho(\Gamma)}} : L_{\rho(\Gamma)} \to h(L_{\rho(\Gamma)})$ is a diffeomorphism. Let $\varphi \in \mathrm{Diff}(\mathbb{S}^1)$ be such that $\varphi_{/L_{\rho(\Gamma)}} = h_{/L_{\rho(\Gamma)}}$. We set $h_1 = \varphi \circ h^{-1}$ and $\rho_1 = h_1\rho_0 h_1^{-1} = \varphi\rho\varphi^{-1}$. Since ρ_1 and ρ are differentially conjugate, we see that ρ is area-preserving if and only if ρ_1 is area-preserving. That way, we reduced the problem to a representation ρ_1 such that $\rho_1 = h_1\rho_0 h_1^{-1}$ where ρ_0 is Fuchsian and h_1 is the identity on $L_{\rho_0(\Gamma)}$. We get a reformulation of Theorem 4.1.3 which we will use for its proof.

Theorem 4.1.10. *Let $\rho : \Gamma \to \mathrm{PSL}(2, \mathbb{R})$ be a convex cocompact representation and let $h \in \mathrm{Homeo}(\mathbb{S}^1)$ be such that $h_{/L_{\rho(\Gamma)}} = Id$ and $\rho_1 = h\rho h^{-1}$ has values in $\mathrm{Diff}(\mathbb{S}^1)$. Then ρ_1 preserves a C^2 volume form on $\mathcal{C}$.*

We will also show that some specific deformations of Schottky groups provide non differentially Fuchsian representations that satisfy the hypothesis of this theorem. The proof of Theorem 4.1.10 will take a substantial part of this chapter (sections 5 and 6). Because of the lower regularity examples in the topologically transitive case mentioned above, it will be necessary to pay particular attention to the regularity of the obtained volume form.

A natural development would be to ask whether the converse is true.

Question 4.1.11. *If $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ is non elementary and area-preserving, is it L-differentially Fuchsian?*

1.5.2. *Infinitesimal rigidity.* Even though we do not have an answer to this exact question, we will see that there is some rigidity on the limit set by observing order three derivatives. The Schwarzian derivative, defined by $S(f) = \left(\frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2\right)dx^2$, is a quadratic differential that vanishes only for $f \in \mathrm{PSL}(2, \mathbb{R})$. We obtain the following:

Theorem 4.1.12. *If $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ is a non elementary representation that preserves a smooth volume form on $\mathcal{C}$, then there is $h \in \mathrm{Diff}(\mathbb{S}^1)$ such that $S(h \circ \rho(\gamma) \circ h^{-1})(x) = 0$ for all $\gamma \in \Gamma$ and $x \in L_{h\rho(\Gamma)h^{-1}}$.*

1.5.3. *Spectrally Möbius-like groups.* In the case of a single hyperbolic diffeomorphism, preserving a volume form on $\mathcal{C}$ is equivalent to a condition on the derivatives at the fixed points. We can ask ourselves if it is also the case for more complicated groups.

So far, it seems that spectrally Möbius-like is the weakest of all the properties defined above. However, for a group generated by a hyperbolic diffeomorphism, it is equivalent to being area-preserving. A natural question is to ask whether it is true for all group actions.

Question 4.1.13. *If $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ is topologically Fuchsian and spectrally Möbius-like, is it area-preserving?*

Note that even though they seem to be indicating different directions, there is no obvious contradiction between this statement and Question 4.1.11 (i.e. we can ask whether spectrally Möbius-like actions are L-differentially Fuchsian).

We will see that there is a positive answer to Question 4.1.13 for actions close to Fuchsian actions. For convenience, we will only treat the case of free groups.

Theorem 4.1.14. *Let $\rho_0 : \mathbb{F}_n \to \mathrm{PSL}(2, \mathbb{R})$ be a convex cocompact representation. If $\rho_1 : \mathbb{F}_n \to \mathrm{Diff}(\mathbb{S}^1)$ is sufficiently C^1-close to ρ_0 and spectrally Möbius-like, then ρ_1 is area-preserving.*

Note that the hypothesis that ρ_0 is Fuchsian could be weakened by asking for ρ_0 to be L-differentially Fuchsian.

For representations of surfaces groups, a theorem of Ghys in **[Ghy92]** (which preceded the result mentioned above) states that given $\rho_0 : \Gamma_g \to \mathrm{PSL}(2, \mathbb{R})$ defined by a hyperbolic metric on the surface of genus g, any C^1-close representation $\rho_1 : \Gamma_g \to \mathrm{Diff}(\mathbb{S}^1)$ is differentially Fuchsian (notice that this does not mean that ρ_1 is differentially conjugate to ρ_0, but to another Fuchsian representation). In our context, we could ask if a representation $\rho_1 : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ that is spectrally Möbius-like and C^1-close to a convex cocompact representation $\rho_0 \to \mathrm{PSL}(2, \mathbb{R})$ is L-differentially Fuchsian. As in the case of surface groups, this does not mean that the existing topological conjugacy is a diffeomorphism between the limit sets. For this to be true, elements should have the same derivatives at their fixed points.

Similarly, given $\rho_0, \rho_1 : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ such that ρ_0 is Fuchsian and that are topologically conjugate, if we assume that ρ_0 and ρ_1 have the same spectrum, are ρ_1 and ρ_0 L-differentially conjugate? In the context of hyperbolic dynamics, this is linked to understanding differentiable conjugacy by looking at the periodic data, i.e. the eigenvalues of the derivatives at periodic points (for Anosov diffeomorphisms of surfaces, the periodic data defines the system up to smooth conjugacy, see **[LMM88]** and **[dlL92]**).

1.6. Structure of this chapter. We will start by studying the case of a single diffeomorphism. In section 3, we will introduce an important for the study of the non elementary case: projective structures. The rigidity results concerning the non-elementary case, i.e. Theorem 4.1.9, Theorem 4.1.8 and Theorem 4.1.12, will be proved in section 4. Finally, we will prove Theorem 4.1.10 in sections 5 and 6, and Theorem 4.1.14 in section 7.

2. The elementary case

In this section, we study the problem of differentiable conjugacy for a single diffeomorphism preserving a volume form on $\mathcal{C}$. Because such an element is topologically conjugate to an element of $\mathrm{PSL}(2, \mathbb{R})$, we know that if it fixes at least three points, then it is the identity. We will study separately diffeomorphisms with a different number of fixed points. This corresponds to the classification of elements in $\mathrm{PSL}(2, \mathbb{R})$: elliptic (no fixed point), parabolic (one fixed point) or hyperbolic (two fixed points).

2.1. The elliptic case. We first look at the elliptic case, i.e. fixed point free diffeomorphisms. Recall that elliptic elements of $\mathrm{PSL}(2, \mathbb{R})$ are conjugate (in $\mathrm{PSL}(2, \mathbb{R})$, hence in $\mathrm{Diff}(\mathbb{S}^1)$) to rotations. The problem of knowing when a diffeomorphism topologically conjugate to a rotation is differentially conjugate to it has been studied deeply. There are examples for which a smooth conjugacy does not exist (including some irrational rotation numbers), however Herman proved that a smooth conjugacy exists when the rotation number lies in a set of full Lebesgue measure (**[Her79]** discusses the general problem of differential conjugacy with a rotation). Luckily for us, the volume preserving case is much more simple.

Propositon 4.1.5. *Let φ be a fixed point free diffeomorphism of $\mathbb{S}^1$. If it preserves a C^k volume form on $\mathcal{C}$, then it is C^{k+1} conjugate to a rotation.*

Proof. Let ω be a volume form on $\mathcal{C}$ preserved by φ. We can define a Riemannian metric on $\mathbb{S}^1$ by $\|h\|_x^2 = \omega(x, \varphi(x))\varphi'(x)h^2$. It is preserved by φ, therefore φ is differentially conjugate to a rotation (because all C^k Riemannian metrics on the circle are C^{k+1} homothetic to the euclidian metric whose isometries are rotations). $\square$

Note that the Riemannian metric that we used can be seen as the restriction of the Lorentzian metric $\omega(x,y)dxdy$ on $\mathcal{C}$ to the graph of φ.

2.2. The parabolic case. We now deal with a diffeomorphism φ that has exactly one fixed point $x_0 \in \mathbb{S}^1$. Unlike the elliptic case, we will see that there is no rigidity. We can start by observing that the proof of the elliptic case does not apply here: the graph of φ is not included in $\mathcal{C}$, therefore the Riemannian metric that we used is only defined on $\mathbb{S}^1 \setminus \{x_0\}$ and it only gives a conjugacy on $\mathbb{S}^1 \setminus \{x_0\}$ with a translation of the real line, which only extends to a continuous conjugacy on $\mathbb{S}^1$ with a parabolic element of $\mathrm{PSL}(2,\mathbb{R})$, but this conjugacy is (in general) not smooth.

There are immediate counter examples to differential conjugacy: we can consider the family of diffeomorphisms $\varphi(x) = x(1+x^n)^{-\frac{1}{n}}$ (for n odd) of $\mathbb{RP}^1 = \mathbb{R}\cup\{\infty\}$. A preserved volume form is given by $|x^n - y^n|^{-1-\frac{1}{n}}\, dx \wedge dy$. For $n \neq 1$, these diffeomorphisms are not differentially conjugate to an element of $\mathrm{PSL}(2,\mathbb{R})$.

However, all diffeomorphisms with one fixed point do not preserve a volume form on $\mathcal{C}$.

Proposition 4.2.1. *We see $\mathbb{S}^1$ as $\mathbb{R} \cup \{\infty\}$. Let $f \in \mathrm{Diff}(\mathbb{S}^1)$ be such that:*

(1) $\mathrm{Fix}(f) = \{0\}$

(2) $\forall x \in \,]0\,,1]\ f(x) = (\mathrm{Log}(1+e^{x^{-2}}))^{-\frac{1}{2}}$

(3) $\forall x \in [-1\,,0[\ f(x) = -(\mathrm{Log}(1+e^{x^{-4}}))^{-\frac{1}{4}}$

Then f does not preserve any continuous volume form on $\mathcal{C}$.

Proof. Start by considering sequences $x_n \in \,]0\,,1]$ and $y_n \in [-1\,,0[$ such that $x_n \to x \neq 0$ and $v_n = f^n(y_n) \to v \in [-1\,,0[$ (this implies that $f^n(x_n) \to 0$ and $y_n \to 0$).

If f preserves a volume form ω on $\mathcal{C}$, then we find:

$$(*) \qquad (f^n)'(x_n)(f^n)'(y_n) = \frac{\omega(x_n,y_n)}{\omega(f^n(x_n),f^n(y_n))} \to \frac{\omega(x,0)}{\omega(0,v)} \in \,]0\,,+\infty[$$

By rewriting $(f^n)'(y_n) = 1/(f^{-n})'(v_n)$, we see that computing the product $(f^n)'(x_n)(f^n)'(y_n)$ only uses f on $[-1\,,1]$.

For $x \in \,]0\,,1]$, we find $f^n(x) = (\mathrm{Log}(n+e^{x^{-2}}))^{-\frac{1}{2}}$ for all $n>0$, which gives:

$$(f^n)'(x) = \frac{1}{x^3}\frac{1}{1+ne^{-x^{-2}}}(\mathrm{Log}(n+e^{x^{-2}}))^{-\frac{3}{2}}$$

Similarly, for $y \in [-1\,,0[$, we find $f^{-n}(y) = -(\mathrm{Log}(n+e^{y^{-4}}))^{-\frac{1}{4}}$ and

$$(f^{-n})'(y) = \frac{-1}{y^5}\frac{1}{1+ne^{-y^{-4}}}(\mathrm{Log}(n+e^{y^{-4}}))^{-\frac{5}{4}}$$

This shows that:

$$(f^n)'(x_n)(f^n)'(y_n) = \frac{(f^n)'(x_n)}{(f^{-n})'(v_n)} \sim \frac{-v^5}{x^3}e^{x^{-2}-v^{-4}}(\mathrm{Log}(n))^{-\frac{1}{4}} \to 0$$

This is in contradiction with $(*)$. $\square$

We will not try to give a necessary and sufficient condition for a diffeomorphism with one fixed point to preserve a volume form on $\mathcal{C}$. Note that the example in Proposition 4.2.1 is C^∞-tangent to the identity at its fixed point. The same calculations could give a smooth preserved volume form for a diffeomorphism that is not infinitely tangent to the identity, as well as for some examples that are infinitely tangent to the identity. It seems that the key for preserving a volume form on $\mathcal{C}$ is having the same behaviour on each side of the fixed point.

2.3. The hyperbolic case. In the hyperbolic case (i.e. a diffeomorphism with two fixed points), we can start by seeing that all north/south diffeomorphisms cannot preserve a smooth volume.

Lemma 4.2.2. *Let $f \in \mathrm{Diff}(\mathbb{S}^1)$ have exactly two fixed points N and S. If f preserves a continuous volume form on $\mathcal{C}$, then $f'(N) \neq 1$ and $f'(N)f'(S) = 1$.*

Proof. This is Lemma 3.3.7 (page 39) when $h = Id$. $\square$

This property is satisfied by a hyperbolic element of $\mathrm{PSL}(2,\mathbb{R})$ (the derivatives at the fixed points are the squares of the eigenvalues of the matrix), and therefore by any diffeomorphism that is differentially conjugate to a hyperbolic element of $\mathrm{PSL}(2,\mathbb{R})$, but there are examples of diffeomorphisms satisfying this property that have no differential conjugate in $\mathrm{PSL}(2,\mathbb{R})$.

Indeed, start with $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ a hyperbolic element. Let N and S be its fixed points. Let $\varphi \in \mathrm{Homeo}(\mathbb{S}^1)$ be such that:

- φ fixes N and S
- φ is a diffeomorphism on $\mathbb{S}^1 \setminus \{S\}$
- φ is the identity in a neighbourhood of N
- φ commutes with γ in a neighbourhood of S

Set $f = \varphi^{-1}\gamma\varphi \in \mathrm{Diff}(\mathbb{S}^1)$. If f were differentially conjugate to an element of $\mathrm{PSL}(2,\mathbb{R})$, then this element could be chosen to be γ. If $h^{-1}fh = \gamma$, then $\varphi \circ h$ is a diffeomorphism of $\mathbb{S}^1 \setminus \{S\}$ that commutes with γ. This implies that there is some $t \in \mathbb{R}$ such that $\varphi \circ h = \gamma_t$ on $\mathbb{S}^1 \setminus \{S\}$ where γ_s is the one parameter subgroup of $\mathrm{PSL}(2,\mathbb{R})$ generated by γ. Indeed, in projective charts, we can see $\varphi \circ h$ as a diffeomorphism that commutes with a non trivial homothecy $x \mapsto \lambda x$. The derivative is a continuous function on $\mathbb{R}$ invariant under $x \mapsto \lambda x$, hence constant, and $\varphi \circ h$ fixes 0, hence is equal to some $x \mapsto \mu x$ in projective charts.

By continuity, the equality $\varphi \circ h = \gamma_t$ holds on all $\mathbb{S}^1$, and φ is differentiable. Hence, if we choose φ non differentiable, then f is not differentially conjugate to an element of $\mathrm{PSL}(2,\mathbb{R})$.

The obstruction for a diffeomorphism with two fixed points to be differentially conjugate to an element of $\mathrm{PSL}(2,\mathbb{R})$ is encoded in an element of $\mathrm{Diff}(\mathbb{S}^1)/\mathrm{PSL}(2,\mathbb{R})$ called the Mather invariant (see **[Yoc95]** for more details).

Knowing this, the following result shows that preserving a volume form on $\mathcal{C}$ is not enough in order to be differentially conjugate to a homography.

Proposition 4.1.6. *Let $f \in \mathrm{Diff}^{k+1}(\mathbb{S}^1)$ ($k \geq 0$) have exactly two fixed points N and S. It preserves a C^k volume form on $\mathcal{C}$ if and only if $f'(N)f'(S) = 1$ and $f'(N) \neq 1$.*

Proof. Let $\lambda = f'(N)$ and let $h_N : \mathbb{S}^1 \setminus \{S\} \to \mathbb{R}$ and $h_S : \mathbb{S}^1 \setminus \{N\} \to \mathbb{R}$ be the linearizations of f at N and S (i.e. $h_N \circ f \circ h_N^{-1}(x) = \lambda x$ and $h_S \circ f \circ h_S^{-1}(x) = \lambda^{-1}x$). Let U_1 (resp. U_2) be a neighbourhood of (N,S) (resp. (S,N)) in $\mathcal{C}$ delimited by graphs of maps that commute with f (hence invariant by f). The linearizations give us invariant volume forms (take $dx \wedge dy$ in coordinates) on U_1 and U_2. Since the action of f on the complement of $U_1 \cup U_2$ is proper (it is differentially conjugate to a translation on the plane), we can find a smooth invariant volume form on $\mathcal{C}$ that coincides on U_1 and U_2 with the ones chosen above. $\square$

2.4. Analytic conjugacy. In the fixed point free case, the conjugacy obtained is analytic when the diffeomorphism and the volume form are analytic. The previous construction in the hyperbolic case can never give a real analytic metric (given that the diffeomorphism is real analytic). Indeed, in this construction, the curvature is constant in a neighbourhood of the axes, therefore any analytic prolongation to the whole cylinder would have constant curvature and the isometry group (that contains the diffeomorphism f) would be analytically Fuchsian.

Lorentzian metrics are examples of rigid geometric structures. We will use the fact that for an analytic rigid geometric structure, local vector fields generating isometries can be extended.

Theorem 4.1.7. *Let f be an analytic diffeomorphism of $\mathbb{S}^1$ with exactly two fixed points. If it preserves an analytic volume form on $\mathcal{C}$, then it is analytically conjugate to an element of* $\mathrm{PSL}(2,\mathbb{R})$.

PROOF. Let ω be an analytic volume form preserved by f. By Lemma 4.2.2, if N and S are the fixed points of f, then $\lambda = f'(N) \neq 1$ and $f'(S) = \lambda^{-1}$. By considering the linearizations of f around its fixed points, we see that the diagonal action of f is analytically conjugate in a neighbourhood of (N, S) to the map $(x, y) \mapsto (\lambda x, \lambda^{-1} y)$ in a neighbourhood of $(0, 0)$. Since it preserves the volume form $dx \wedge dy$ in those coordinates, we can write $\omega = e^{\sigma} dx \wedge dy$ in coordinates where σ is an analytic function that satisfies $\sigma(\lambda x, \lambda^{-1} y) = \sigma(x, y)$. By writing σ in its power series around $(0, 0)$ and considering the invariance equation, we see that all the terms in $x^n y^p$ with $n \neq p$ must have zero as their coefficient, therefore we can write $\sigma = g(xy)$ where g is an analytic function, and the form ω is preserved (around the fixed point (N, S)) by the one parameter group associated to f.

We will now apply the main result of [**Amo79**]: a local Killing field (i.e. a vector field that generates a flow of isometries) on a simply connected real analytic Lorentz manifold admits a unique extension to the whole manifold (the paper treats the more general case of finite type G-structures, which includes Lorentz metrics).

In order to apply this result, consider a map from $[N\,, S]$ to $[S\,, N]$ that commutes with the (topological) one parameter group associated to f, and let U be the complement of the graph of this map. It is simply connected open set of $\mathcal{C}$ that is invariant under the one parameter group associated to f and that contains (N, S) and (S, N). There is a vector field $\mathfrak{X}$ on U that preserves ω and such that the time one map is (f, f). Since the vector field $\mathfrak{X}$ has the form $\mathfrak{X}(x, y) = (\mathfrak{x}(x), \mathfrak{x}(y))$ where $\mathfrak{x}$ is defined on all $\mathbb{S}^1$, it is complete, and the map f is the time 1 of the flow of the analytic vector field $\mathfrak{x}$, hence f is analytically conjugate to an element of $\mathrm{PSL}(2,\mathbb{R})$ (the Mather invariant of the time one map of a flow is trivial, see [**Yoc95**]).

□

However, there are non Fuchsian examples in the parabolic case. Indeed, for $n \in \mathbb{N}$ odd and greater than 1, consider the examples $f(x) = x(1 + x^n)^{-1/n}$ discussed in the differentiable case. It is analytic on $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$ (because $\frac{1}{f}$ is analytic in a neighbourhood of -1). It preserves the volume form $|x^n - y^n|^{-1-1/n} dx \wedge dy$ which extends analytically to $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$.

The example of a parabolic diffeomorphism that does not preserve a volume form given in Proposition 4.2.1 is not analytic. We suspect that in the parabolic case, all analytic diffeomorphisms preserve an analytic volume form on $\mathcal{C}$.

3. Projective structures and curvature

Lemma 4.3.1. *Let $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ be a representation that preserves a C^2 volume form on $\mathcal{C}$. Assume that there is a least one hyperbolic element. Then the curvature K is constant on $(L_{\rho(\Gamma)} \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L_{\rho(\Gamma)}) \setminus \Delta$.*

PROOF. Let ω be such a volume form. If $\gamma \in \Gamma$ and $\rho(\gamma)$ has two fixed points N, S in $\mathbb{S}^1$, then we can consider the fixed point $p = (N, S) \in \mathcal{C}$. The orbits of points of the axes $\{N\} \times \mathbb{S}^1 \setminus \{N\}$ and $\mathbb{S}^1 \setminus \{S\} \times \{S\}$ accumulate on p, therefore the curvature at these points have the same value $K(p)$. Given two hyperbolic elements of Γ, the axes meet, therefore the curvature has the same value on the axes of all hyperbolic elements of Γ. Since a fixed point of a hyperbolic element has a dense orbit in $L_{\rho(\Gamma)}$, we find that K is constant on $(L_{\rho(\Gamma)} \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L_{\rho(\Gamma)}) \setminus \Delta$ $\square$

Note that the exact same proof works for any continuous function on $\mathcal{C}$ invariant under the action of Γ. The specificity of the curvature is that when it is constant, the metric is locally isometric to a model space. We will now see how this can give a global conjugacy for the isometry group. It is in general more difficult to have global results on constant curvature Lorentz manifolds than on Riemannian manifolds, because the associated (G, X)-structure is not always complete (the developing map may not be a covering map, whereas it is always the case for Riemannian isometries).

Horizontal and vertical lines in $\mathcal{C} = \mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$ are geodesics (because they are the only isotropic curves), which gives us some specific parametrisations. We will translate them in terms of projective structures on one dimensional manifolds.

A **projective structure** on a one-dimensional manifold I is an atlas (U_i, f_i) with $f_i : U_i \to \mathbb{RP}^1$ such that the transition maps $f_i \circ f_j^{-1}$ are projective diffeomorphisms (i.e. restrictions of elements of $\mathrm{PSL}(2, \mathbb{R})$). If f is a diffeomorphism between two projective one-dimensional manifolds I and J, then one can define a quadratic differential $s(f)$ on I, called the **Schwarzian derivative** of f, by $s(f) = \left(\frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2\right) dx^2$ in projective charts. Then f is a projective diffeomorphism (i.e. f has the form $x \mapsto \frac{ax+b}{cx+d}$ in projective charts) if and only if $s(f) = 0$ (see **[Ghy93]** for more details).

Note that some links between the Schwarzian derivative and Lorentzian geometry have been studied, mostly concerning the geodesic curvature (see **[DO00]**).

Geodesics inherit a projective structure, the charts being given by the different parametrisations of the geodesic (the coordinate changes are affine, therefore projective). Recall that the geodesic equations are the following:

$$\begin{aligned} x'' + \frac{1}{\omega}\frac{\partial \omega}{\partial x} x'^2 &= 0 \\ y'' + \frac{1}{\omega}\frac{\partial \omega}{\partial y} y'^2 &= 0 \end{aligned}$$

A representation $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ is differentially Fuchsian if and only if it preserves a projective structure on $\mathbb{S}^1$ that is equivalent to the standard structure on $\mathbb{RP}^1$ (because a conjugacy between ρ and a Fuchsian representation is the same as a projective diffeomorphism with $\mathbb{RP}^1$). Therefore in order to show that a representation is differentially Fuchsian, we can proceed in two steps: first we find an invariant projective structure, then we show that it is equivalent to the standard projective structure on $\mathbb{RP}^1$. This is what we will use in the proof of the following result.

Lemma 4.3.2. *Let $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ be a representation that preserves a C^2 volume form on $\mathcal{C}$. Assume that its curvature is constant. Then ρ is differentially Fuchsian.*

Proof. Given $y \in \mathbb{S}^1$, we consider a diffeomorphism $f_y : \mathbb{S}^1 \setminus \{y\} \to \mathbb{R}$ given by a parametrisation of the horizontal circle $\mathbb{S}^1 \setminus \{y\} \times \{y\}$ as a geodesic for the Lorentz metric associated to ω. This gives us an atlas of $\mathbb{S}^1$, and we will first show that it is a projective structure, i.e. that the transition maps $f_{y'} \circ f_y^{-1}$ are projective. For any sequence $y_1, \ldots, y_n$, we can decompose $f_{y'} \circ f_y^{-1}$:

$$f_{y'} \circ f_y^{-1} = (f_{y'} \circ f_{y_n}^{-1}) \circ (f_{y_n} \circ f_{y_{n-1}}^{-1}) \circ \cdots \circ (f_{y_1} \circ f_y^{-1})$$

Since the composition of projective maps is projective, it is enough to show that $f_{y'} \circ f_y^{-1}$ is projective when y and y' are sufficiently close.

Given $(x, y) \in \mathcal{C}$, we can find a local isometry with the model space of constant curvature, which can also be seen (locally) as a volume form on $\mathcal{C}$ ($dx \wedge dy$ for zero curvature, $4c\frac{dx\wedge dy}{(x-y)^2}$ for curvature $c \neq 0$). An isometry sends parametrized geodesics onto parametrized geodesics, hence $f_{y'} \circ f_y^{-1}$ is equal to the analogue in the model space, and it is projective because it is the case in the model space.

Given an element $\gamma \in \Gamma$, we know that $f_y \circ \rho(\gamma)$ is also the inverse of the parametrisation of a geodesic, therefore $f_{y'} \circ \rho(\gamma) \circ f_y^{-1}$ is projective, and the projective structure that we defined is preserved by ρ.

To conclude, we separate two cases. If there is an element of Γ with a fixed point in $\mathbb{S}^1$, then Lemma 5.1 of [**Ghy93**] concludes that the projective structure is equivalent to the standard structure on $\mathbb{RP}^1$, and ρ is differentially Fuchsian.

If all elements are elliptic, then applying Theorem 3.3.2 shows that ρ is topologically conjugate to a representation in $\mathrm{PSL}(2, \mathbb{R})$ with only elliptic elements, and it is therefore conjugate to a subgroup of $\mathrm{SO}(2, \mathbb{R})$ (see §7.39 in [**Bea83**]). In particular, it is abelian, and the same argument as in Proposition 4.1.5 (ρ preserves the Riemannian metric $\omega(x, \rho(\gamma_0)x)\rho(\gamma_0)'(x)dx^2$ on $\mathbb{S}^1$ where γ_0 is any element in $\Gamma \setminus \{e\}$) shows that ρ is differentially conjugate to a representation in $\mathrm{SO}(2, \mathbb{R}) \subset \mathrm{PSL}(2, \mathbb{R})$. ☐

One can ask if the projective structures given by the lightlike geodesics are the same, so that they could provide a projective structure on $\mathbb{S}^1$ invariant under the isometry group. We already know that it is not possible: if it were the case, then the group would be differentially Fuchsian, and we found some counter examples. However, we mentioned that it is the case when the curvature is constant. We will see that it is the only case where it happens.

Given a parametrisation $g_y : \mathbb{R} \to \mathbb{S}^1 \setminus \{y\}$ of the horizontal geodesic $(\mathbb{S}^1 \setminus \{y\}) \times \{y\}$, we wish to know when the transition maps $g_{y'}^{-1} \circ g_y$ are projective, i.e. when their Schwarzian derivative vanishes.

The cocycle relation of the Schwarzian derivative gives:

$$\begin{aligned} s(g_{y'}^{-1} \circ g_y) &= {g_y}^\star s(g_{y'}^{-1}) + s(g_y) \\ &= -{g_y}^\star (g_{y'}^{-1})^\star s(g_{y'}) + s(g_y) \\ &= {g_y}^\star \left((g_y^{-1})^\star s(g_y) - (g_{y'}^{-1})^\star s(g_{y'}) \right) \end{aligned}$$

This shows that $s(g_{y'}^{-1} \circ g_y) = 0$ for all y, y' if and only if $(g_y^{-1})^\star s(g_y)$ does not depend on y. The advantage of this formulation is that we can compute $(g_y^{-1})^\star s(g_y)$ without having an explicit expression of g_y.

Recall that g_y is a solution of the geodesic equation:

$$g_y'' + \frac{1}{\omega}\frac{\partial \omega}{\partial x} g_y'^2 = 0$$

The defintion of $s(g_y)$ is:

$$s(g_y) = \left(\frac{g_y'''}{g_y'} - \frac{3}{2}\left(\frac{g_y''}{g_y'}\right)^2 \right) dx^2$$

The geodesic equation simplifies the term on the right:

$$\frac{g_y''}{g_y'} = -\frac{1}{\omega}\frac{\partial \omega}{\partial x} g_y'$$

Taking the derivative of the geodesic expression, we can also simplify the other term:

$$\frac{g_y'''}{g_y'} = \left(3\left(\frac{1}{\omega}\frac{\partial \omega}{\partial x}\right)^2 - \frac{1}{\omega}\frac{\partial^2 \omega}{\partial x^2} \right) (g_y')^2$$

Combining the two, we get:

$$s(g_y) = \left(\frac{3}{2}\left(\frac{1}{\omega}\frac{\partial\omega}{\partial x}\right)^2 - \frac{1}{\omega}\frac{\partial^2\omega}{\partial x^2}\right)(g_y')^2 dx^2$$

Finally, we find:

$$(g_y^{-1})^\star s(g_y) = \left(\frac{3}{2}\left(\frac{1}{\omega}\frac{\partial\omega}{\partial x}\right)^2 - \frac{1}{\omega}\frac{\partial^2\omega}{\partial x^2}\right) dx^2$$

Let us write $(g_y^{-1})^\star s(g_y) = F(x,y)dx^2$. Note that it does not on the choice of the parametrisation g_y of the geodesic. We mentioned earlier that the parametrisations of different geodesics give the same projective structure on $\mathbb{S}^1$ if and only if $(g_y^{-1})^\star s(g_y)$ does not depend on y, i.e. if and only if $\frac{\partial F}{\partial y} = 0$. This derivative happens to be exactly the derivative of the curvature.

Proposition 4.3.3. *The Schwarzian derivative of g_y and the curvature are linked by the formula:*

$$\frac{\partial F}{\partial y} = -\frac{\omega}{2}\frac{\partial K}{\partial x}$$

PROOF. Recall the formula for the curvature:

$$K = \frac{2}{\omega}\frac{\partial^2 \mathrm{Log}\omega}{\partial x \partial y} = \frac{2}{\omega^2}\frac{\partial^2\omega}{\partial x\partial y} - \frac{2}{\omega^3}\frac{\partial\omega}{\partial x}\frac{\partial\omega}{\partial y}$$

This immediately gives the right hand side of the equation:

$$\frac{\partial K}{\partial x} = -\frac{6}{\omega^3}\frac{\partial\omega}{\partial x}\frac{\partial^2\omega}{\partial x\partial y} + \frac{2}{\omega^2}\frac{\partial^3\omega}{\partial x^2\partial y} + \frac{6}{\omega^4}\left(\frac{\partial\omega}{\partial x}\right)^2\frac{\partial\omega}{\partial y} - \frac{2}{\omega^3}\frac{\partial^2\omega}{\partial x^2}\frac{\partial\omega}{\partial y}$$

The derivative of $F(x,y) = \frac{3}{2}\left(\frac{1}{\omega}\frac{\partial\omega}{\partial x}\right)^2 - \frac{1}{\omega}\frac{\partial^2\omega}{\partial x^2}$ can be broken into two parts.

$$\begin{aligned}
\frac{\partial}{\partial y}\left(\frac{1}{\omega}\frac{\partial\omega}{\partial x}\right) &= -\frac{1}{\omega^2}\frac{\partial\omega}{\partial y}\frac{\partial\omega}{\partial x} + \frac{1}{\omega}\frac{\partial^2\omega}{\partial x\partial y}\\
\frac{\partial}{\partial y}\left(\frac{1}{\omega}\frac{\partial^2\omega}{\partial x^2}\right) &= -\frac{1}{\omega^2}\frac{\partial\omega}{\partial y}\frac{\partial^2\omega}{\partial x^2} + \frac{1}{\omega}\frac{\partial^3\omega}{\partial x^2\partial y}
\end{aligned}$$

This allows us to compute the derivative of F.

$$\begin{aligned}
\frac{\partial F}{\partial y} &= 3\left(\frac{1}{\omega}\frac{\partial\omega}{\partial x}\right)\left(-\frac{1}{\omega^2}\frac{\partial\omega}{\partial y}\frac{\partial\omega}{\partial x} + \frac{1}{\omega}\frac{\partial^2\omega}{\partial x\partial y}\right) + \frac{1}{\omega^2}\frac{\partial\omega}{\partial y}\frac{\partial^2\omega}{\partial x^2} - \frac{1}{\omega}\frac{\partial^3\omega}{\partial x^2\partial y}\\
&= \frac{3}{\omega^2}\frac{\partial\omega}{\partial x}\frac{\partial^2\omega}{\partial x\partial y} - \frac{1}{\omega}\frac{\partial^3\omega}{\partial x^2\partial y} - \frac{3}{\omega^3}\left(\frac{\partial\omega}{\partial x}\right)^2\frac{\partial\omega}{\partial y} + \frac{1}{\omega^2}\frac{\partial^2\omega}{\partial x^2}\frac{\partial\omega}{\partial y}\\
&= -\frac{\omega}{2}\frac{\partial K}{\partial x}
\end{aligned}$$

□

This shows that horizontal geodesics define the same projective structure on $\mathbb{S}^1$ if and only if the curvature is constant along horizontal geodesics. In this case, the curvature is a continuous function on $\mathbb{S}^1$ invariant under the action of the isometry group. If it possesses a non elliptic element, then such a function is necessarily constant.

4. Rigidity results for non elementary groups

4.1. Topologically transitive actions. In the topologically transitive case, i.e. when the limit set is the whole circle, the situation is rigid (provided sufficient regularity). We will use the results stated above to show that the curvature is constant.

Theorem 4.1.9. *Let $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ be a topologically transitive representation that preserves a C^2 volume form on $\mathcal{C}$. Then ρ is differentially Fuchsian.*

Proof. If there is a hyperbolic element, then Lemma 4.3.1 states that the curvature is constant on $\mathcal{C}$ and Lemma 4.3.2 allows us to conclude.

We now treat the case where there are no hyperbolic elements, i.e. all elements are elliptic or parabolic. First assume that there is a parabolic element γ. Let $x_0 \in \mathbb{S}^1$ be its fixed point. If there is another parabolic element δ with a different fixed point, then either $\gamma\delta$ or $\gamma^{-1}\delta$ is hyperbolic, hence we can assume that all parabolic elements fix x_0. Since the group is not elementary, there is a non trivial elliptic element α. The conjugate $\alpha\gamma\alpha^{-1}$ is a parabolic element whose fixed point is $\rho(\alpha)(x_0) \neq x_0$, and as we just showed this implies the existence of a hyperbolic element in Γ. We have shown that the existence of a parabolic element in a non elementary group preserving a volume form on $\mathcal{C}$ implies the existence of a hyperbolic element.

We are left with the case where all elements are elliptic, where we simply notice that we did not use the fact that the curvature is constant in this case in the proof of Lemma 4.3.2. ☐

The regularity of the preserved volume form is essential in this result. If (S, h) is a smooth compact Riemannian surface of negative curvature, then the fundamental group $\pi_1(S)$ acts isometrically on the universal cover $\widetilde{S}$, hence it acts on its boundary at infinity $\partial_\infty \widetilde{S} \approx \mathbb{S}^1$. To find an invariant volume form, consider the space of oriented geodesics of $\widetilde{S}$. It can be seen as $\mathrm{T}^1\widetilde{\mathrm{S}}/\mathbb{R}$ where the action of $\mathbb{R}$ is the geodesic flow, and $\pi_1(S)$ preserves the form $\omega = d\lambda$ where λ is the projection of the Liouville 1-form on $\mathrm{T}^1\widetilde{\mathrm{S}}$. An oriented geodesic is given by a starting point and an end point on $\partial_\infty \widetilde{S}$, which gives an identification between $\mathrm{T}^1\widetilde{\mathrm{S}}/\mathbb{R}$ and $\mathcal{C} = \partial_\infty \widetilde{S} \times \partial_\infty \widetilde{S} \setminus \Delta$. This identification is only a C^1-diffeomorphism (its regularity is exactly the regularity of the weak stable and weak unstable foliations of the geodesic flow), so the volume form obtained on $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$ is only continuous. A result of Ghys in **[Ghy87a]** states that if the identification $\mathrm{T}^1\widetilde{\mathrm{S}}/\mathbb{R} \approx \mathcal{C}$ is C^2, then (S, h) has constant curvature.

It is not even clear whether the regularity required in Theorem 4.1.9 can be lowered to $C^{1,1}$ (i.e. C^1 with a Lipschitz derivative). In this case, the curvature is defined almost everywhere, and is locally L^∞. To ensure that such a function is constant almost everywhere, the right notion is no longer topological transitivity but ergodicity. A group action by diffeomorphisms on a manifold is **ergodic** if all invariant measurable sets are either negligible or of full measure (for the class of the Lebesgue measure). If an action on the circle $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ is ergodic, then the diagonal action on $\mathcal{C}$ also is. The question of knowing whether topologically transitive actions on the circle are ergodic is very important in the theory of circle diffeomorphisms. It has been proven to be true for analytic actions of finitely generated free groups (in **[Her79]** for $\mathbb{Z}$, and **[DKN09]**,**[DKN13]** for $\mathbb{F}_n$, $n \geq 2$), and it is expected to be true for C^2 actions of finitely presented groups. This could be applied in our situation (if the metric is $C^{1,1}$, then isometries are $C^{2,1}$), but we would still have to prove that if the curvature is constant almost everywhere, then we have an isometry with the model space.

4.2. Analytic rigidity. As in the elementary case, analyticity also provides more rigidity in the non elementary case.

Theorem 4.1.8. *Let $\rho : \Gamma \to \mathrm{Diff}^\omega(\mathbb{S}^1)$ be a non elementary representation that preserves an analytic volume form on $\mathcal{C}$. Then ρ is analytically Fuchsian.*

Proof. Applying Lemma 4.3.1 we see that the curvature is constant on the set $(L_{\rho(\Gamma)} \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L_{\rho(\Gamma)}) \setminus \Delta$. The analyticity of the curvature implies that it is constant on $\mathcal{C}$ (consider the function along horizontal and vertical lines and the fact that $L_{\rho(\Gamma)}$ is without isolated points), and Lemma 4.3.2 implies that ρ is analytically Fuchsian. $\square$

4.3. Infinitesimal rigidity on the limit set. We saw that the curvature is constant on $(L_{\rho(\Gamma)} \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L_{\rho(\Gamma)}) \setminus \Delta$, but we cannot have anything better than this. Indeed, Proposition 3.2.2 implies that we cannot expect the curvature to be constant everywhere.

The question of differentiable conjugacy appears to be difficult and a way of dealing with a more simple problem is to linearise the conjugacy equation, i.e. considering the derivatives of the equations $\rho_1(\gamma) = h^{-1} \circ \rho_0(\gamma) \circ h$ where $\rho_1 : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ is the data and $h \in \mathrm{Diff}(\mathbb{S}^1)$ and $\rho_0 : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ are the unknowns. First and second order derivatives remain quite complicated, but the third order is more simple because elements of $\mathrm{PSL}(2,\mathbb{R})$ can be defined as the solutions of a third order differential equation. But since we know that it is not always possible to have a differentiable conjugacy on the whole circle (the proof will be exposed in sections 5 and 6), we can only look at subsets of the circle. In the counter example that we will construct, the conjugacy is differentiable along the limit set. This is interesting because the limit set is the subset of the circle that contains the non trivial dynamical behaviour.

We have already seen that a volume form on $\mathcal{C}$ endows the horizontal and vertical lines with projective structures. We showed that in the constant curvature case, they give the same projective structure on $\mathbb{S}^1$. Before we give a statement of a result, we will reformulate this.

We will denote by E^1 (resp. E^2) the sub-bundle of $T\mathcal{C}$ consisting of horizontal (resp. vertical) lines. If $p \in X$ and $u \in E^2(p)$, then α_u is the geodesic with initial condition u, and $\mathcal{C}_t^u$ is the horizontal circle passing through $\alpha_u(t)$. We will consider the holonomy map $H_t^u : \mathcal{C}_0^u \to \mathcal{C}_t^u$ (which is defined everywhere on the circle except at two points, see Figure 4.1). The Schwarzian derivative $K_u(t) = S(H_t^u)$ relatively to the projective structure on $\mathcal{C}_t^u$ given by the Lorentzian metric is a field of quadratic forms on E^1, and we will mostly consider $k_u(t) = K_u(t)(p) \in S^2(E^1(p))$. Note that if ρ were Fuchsian, then $k_u(t)$ would vanish everywhere (this is what we have shown in the constant curvature case). If it were L-differentially Fuchsian, then it would vanish when the base point of u is in $L_\Gamma \times L_\Gamma$, therefore the following result can be interpreted as a rigidity result.

Theorem 4.4.1. *If $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ preserves a smooth volume form on $\mathcal{C}$, and if $\rho(\Gamma)$ is non elementary, then $k_u(t) = 0$ for all $p \in L_{\rho(\Gamma)} \times L_{\rho(\Gamma)} \setminus \Delta$ and all $u \in E^2(p), t \in \mathbb{R}$.*

Proof. If $\gamma \in \Gamma$, then $H_t^{\gamma.u} = \gamma \circ H_t^u \circ \gamma^{-1}$. Since the group Γ acts isometrically with respect to the Lorentz metric, it preserves the projective structures, and the cocycle relation on the Schwarzian derivative gives us $K_{\gamma.u}(t) = \gamma_* K_u(t)$.

Let us now remark that since the space $S^2(E^1(p))$ is one-dimensional, we can write $k_u(t)(v) = F(u,t) < u, v >^2$ for all $v \in E^1(p)$ (where $< \cdot, \cdot >$ is the Lorentz metric associated to the preserved volume form). The relation $K_{\gamma.u}(t) = \gamma_* K_u(t)$ gives us $F(\gamma.u, t) = F(u,t)$.

If $a > 0$, then we have $\alpha_{au}(t) = \alpha_u(at)$, which gives us $K_{au}(t) = K_u(at)$.

We will now study the case where p is a fixed point of γ. We write $p = (x, y)$ and $\gamma'(x) = \lambda^{-1}$, $\gamma'(y) = \lambda$, with $\lambda \neq 1$. Since $\gamma.u = \lambda u$, we have $k_u(\lambda t) = k_{\lambda u}(t) =$

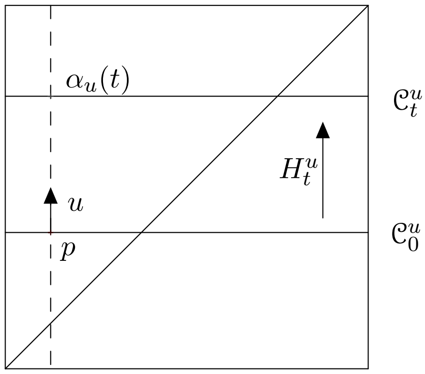

FIGURE 4.1. The holonomy map H_t^u

$k_{\gamma.u}(t) = \gamma_* k_u(t) = \lambda^2 k_u(t)$, which implies that $F(u, \lambda t) = \lambda^2 F(u,t)$, therefore (because of the differentiability of the map $t \mapsto F(u,t)$) there is a real number $c(u)$ such that $F(u,t) = c(u)t^2$.

We now wish to extend this to $L_{\rho(\Gamma)} \times L_{\rho(\Gamma)} \setminus \Delta$. If we fix $t \in \mathbb{R}$ and $k > 2$, the function $\frac{\partial^k}{\partial t^k} F(u,t)$ is invariant under the action of Γ, and it is equal to 0 on all vectors tangent to fixed points of Γ, therefore by continuity it is equal to 0 on $L_{\rho(\Gamma)} \times L_{\rho(\Gamma)} \setminus \Delta$, i.e. $F(u,t) = a(u) + b(u)t + c(u)t^2$. Since the coefficients are continuous, we have $a(u) = b(u) = 0$, i.e. $F(u,t) = c(u)t^2$.

We will finally compute $k_u(t+s)$ in two ways in order to conclude. We choose $p \in L_{\rho(\Gamma)} \times L_{\rho(\Gamma)} \setminus \Delta$ and $t > 0$ such that $\alpha_u(t) \in L_{\rho(\Gamma)} \times L_{\rho(\Gamma)} \setminus \Delta$. For $s \in \mathbb{R}$, we have $H_{t+s}^u = H_s^{\alpha'_u(t)} \circ H_t^u$, hence $k_u(t+s) = k_u(t) + (H_t^u)^* K_{\alpha'_u(t)}(s)$, which we can write:

$$c(u)(t+s)^2 < u, v >^2 = c(u)t^2 < u, v >^2 + c(\alpha'_u(t))s^2 < dH_t^u(v), \alpha'_u(t) >^2$$

By computing the derivative with respect to s at $s = 0$ on both sides, we obtain $c(u) = 0$, i.e. $k_u(t) = 0$. □

We can now prove Theorem 4.1.12, that can be slightly reformulated:

Theorem 4.1.12. *If $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ is a non elementary representation that preserves a smooth volume form on $\mathcal{C}$, then there is a projective structure on $\mathbb{S}^1$, equivalent to the standard structure on $\mathbb{RP}^1$, such that $S(\rho(\gamma))(x) = 0$ for all $\gamma \in \Gamma$ and $x \in L_{\rho(\Gamma)}$.*

PROOF. Let I be a connected component of $\mathbb{S}^1 \setminus L_{\rho(\Gamma)}$ and let $x_0, x_-, x_+ \in I$ be such that $x_- < x_0 < x_+ < x_-$ and the interval consisting of points x such that $x_- < x < x_+ < x_-$ is included in I. We can choose a parametrization $\varphi : \mathbb{S}^1 \setminus \{x_0\}$ of the horizontal geodesic $\mathbb{S}^1 \setminus \{x_0\} \times \{x_0\}$ such that the image $\varphi(\mathbb{S}^1 \setminus]x_-, x_+[)$ is equal to $[-1, 1]$.

Let $\psi : \mathbb{S}^1 \to \mathbb{RP}^1$ be a diffeomorphism such that the restriction of ψ to $\mathbb{S}^1 \setminus]x_-, x_+[$ is equal to the restriction of φ. It equips $\mathbb{S}^1$ with a projective structure equivalent to the standard structure on $\mathbb{RP}^1$.

Let $x \in L_{\rho(\Gamma)}$ and let $\gamma \in \Gamma$. Since $L_{\rho(\Gamma)} \subset \mathbb{S}^1 \setminus]x_-, x_+[$, the projective structure is defined by φ. Hence it is sent by γ to a parametrization of another horizontal geodesic, and the Schwarzian derivative of γ at x is the Schwarzian derivative of the holonomy at x, and it is equal to 0. □

5. Actions on the circle and flows in dimension 3

The two following sections are dedicated to Theorem 4.1.10, which we recall:

Theorem 4.1.10. *Let $\rho_0 : \Gamma \to \mathrm{PSL}(2, \mathbb{R})$ be a convex cocompact representation and let $h \in \mathrm{Homeo}(\mathbb{S}^1)$ be such that $h_{/L_{\rho_0(\Gamma)}} = Id$ and $\rho_1 = h\rho_0 h^{-1}$ has values in $\mathrm{Diff}(\mathbb{S}^1)$. Then ρ_1 preserves a C^2 volume form on $\mathcal{C}$.*

The main ingredient in this proof is to construct a flow on a 3-manifold (a deformation of the geodesic flow on $\mathrm{T}^1\mathbb{H}^2/\rho_0(\Gamma)$) that has a transverse structure given by ρ_1. This construction follows an idea of Ghys used in two different settings. The first one, found in [**Ghy93**], was to show a rigidity theorem for actions of surface groups on the circle, and the second was the construction of (the only) exotic Anosov flows with smooth weak stabe and weak unstable foliations on 3-manifolds in [**Ghy92**], called quasi-Fuchsian flows. However, Ghys used a local construction (given a certain atlas on $\mathrm{T}^1\mathbb{H}^2/\rho_0(\Gamma)$), whereas we will take a global approach.

We will see in 6.4 that there are some non differentially Fuchsian examples satisfying the hypothesis of Theorem 4.1.10.

5.1. A cohomological reformulation. Searching for an invariant volume form is equivalent to solving a cohomological equation. Let ω_0 be a volume form on $\mathcal{C}$. Any other volume form on $\mathcal{C}$ is a multiple of ω_0, hence if $\gamma \in \Gamma$, then we can write $\rho(\gamma)^*\omega_0 = e^{-\alpha_\gamma}\omega_0$. The chain rule shows that α_γ satisfies the cocycle relation $\alpha_{\gamma'\gamma} = \alpha_{\gamma'} \circ \rho(\gamma) + \alpha_\gamma$.

Let $\omega = e^\sigma \omega_0$ be a volume form on $\mathcal{C}$. We can compute the pull back $\rho(\gamma)^*\omega = e^{\sigma\circ\rho(\gamma)}\rho(\gamma)^*\omega_0 = e^{\sigma\circ\rho(\gamma)-\sigma-\alpha_\gamma}\omega$, hence ω is preserved by Γ if and only if $\sigma\circ\rho(\gamma) - \sigma = \alpha_\gamma$ for all $\gamma \in \Gamma$. In other words, we wish to show that the cocycle α_γ is a coboundary.

The issue with this formulation of the problem is that we do not know much about the cohomology of Γ. We will now see how we can translate the problem to a cohomology equation for a hyperbolic flow, which is a much more simple situation. In this setting, a cocycle is a smooth function $\alpha : M \to \mathbb{R}$ (where M is the manifold on which we study a flow φ^t), and we look for a smooth function $\sigma : M \to \mathbb{R}$ such that $\sigma(\varphi^t(x)) - \sigma(x) = \int_0^t \alpha(\varphi^s(x))ds$ for all $(x, t) \in M \times \mathbb{R}$.

There is a first necessary condition for the existence of a solution: if $x \in \mathrm{Per}(\varphi)$, i.e. if there is $T > 0$ such that $\varphi^T(x) = x$, then $\int_0^T \alpha(\varphi^s(x))ds = 0$. Livšic's Theorem states that this condition is sufficient in order to find a solution on a compact hyperbolic set (as defined page 28).

Theorem 4.5.1. *Let φ^t be a smooth flow on a manifold M, and let K be a compact hyperbolic set, such that the action on K has a dense orbit. If $\alpha : K \to \mathbb{R}$ is a Hölder continuous function such that $\int_0^T \alpha(\varphi^s(x))ds = 0$ for all $x \in K$ such that $\varphi^T(x) = x$, then there is a unique Hölder continuous function $\sigma : K \to \mathbb{R}$ such that $\sigma(\varphi^t(x)) - \sigma(x) = \int_0^t \alpha(\varphi^s(x))ds$ for all $(x, s) \in K \times \mathbb{R}$.*

As stated, the proof can be found in [**KH95**] (Livšic's work in [**Liv71**] deals with Anosov flows on compact manifolds). We will discuss the different versions of Livšic's Theorem (especially concerning regularity conditions) in section 7.

However, Livšic's Theorem will not be of any use in the proof of Theorem 4.1.10, because we will already have a solution on the hyperbolic set (but we will use it in section 7 for Theorem 4.1.14). Instead, we will show that given a solution on a compact hyperbolic set K, we can extend it to $W^s(K) \cup W^u(K)$. When translating the problem back to the action on $\mathcal{C} = \mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta$, this will give a volume form invariant at points of $L_\Gamma \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L_\Gamma$, and there will still be some work involved in order to extend the solution to $\mathcal{C}$ (which is the content of section 6).

5.2. The flow associated to ρ_1. From now on, we consider a convex cocompact representation $\rho_0 : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ (as introduced page 27) and another representation $\rho_1 : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ such that there is $h \in \mathrm{Homeo}(\mathbb{S}^1)$ satisfying $h_{/L_{\rho_0(\Gamma)}} = Id$ and $\rho_1 = h\rho_0 h^{-1}$. Let us start by remarking that $L_{\rho_0(\Gamma)}$ is a compact invariant set for ρ_1. Because of the uniqueness of the minimal invariant compact set, we see that $L_{\rho_1(\Gamma)} \subset L_{\rho_0(\Gamma)}$. Since the actions ρ_0 and ρ_1 restricted to $L_{\rho_0(\Gamma)}$ are equal and have dense orbits, we have $L_{\rho_1(\Gamma)} = L_{\rho_0(\Gamma)}$. We will call this set L_Γ.

We are now going to construct a flow ψ^t on a 3-manifold N that will have the same relation to ρ_1 as the geodesic flow φ^t on $M = \mathrm{T}^1\mathbb{H}^2/\rho_0(\Gamma)$ has with ρ_0. We consider $\Sigma = \{(x_-, x_0, x_+) \in (\mathbb{S}^1)^3 | x_- < h^{-1}(x_0) < x_+ < x_-\}$, and the action α_1 of Γ on Σ given by:

$$\alpha_1(\gamma)(x_-, x_0, x_+) = (\rho_1(\gamma)(x_-), \rho_0(\gamma)(x_0), \rho_1(\gamma)(x_+))$$

The quotient N is a smooth manifold homeomorphic to M: consider the map $\widetilde{H}$: $\Sigma_3 \to \Sigma$ defined by $\widetilde{H}(x_-, x_0, x_+) = (h(x_-), x_0, h(x_+))$. It is a homeomorphism satisfying $\widetilde{H} \circ \alpha_0 = \alpha_1 \circ \widetilde{H}$ that is differentiable in restriction to $L_\Gamma \times \mathbb{S}^1 \times L_\Gamma$. It induces a homeomorphism $H : M \to N$.

The projection on N of the constant vector field $(0, 1, 0)$ on Σ can be reparametrised into a smooth flow ψ^t. The homeomorphism H sends φ^t to a reparametrisation of ψ^t and is a diffeomorphism from Ω_φ to Ω_ψ (recall that the non wandering set Ω_Φ of the flow Φ^t is the set of points x such that there are sequences $x_n \to x$ and $t_n \to \infty$ satisfying $\Phi^{t_n}(x_n) \to x$). From this we deduce that Ω_ψ is a compact hyperbolic set for ψ^t. If the image $x \in N$ of $(x_-, x_0, x_+) \in \Sigma$ is in Ω_ψ, then the stable (resp. unstable) manifold of x is the set of images of points (y_-, y_0, y_+) such that $y_+ = x_+$ (resp. $y_- = x_-$).

The classical result for solving cohomological equation for hyperbolic flows is Livšic's Theorem. However, it only provides solutions on the hyperbolic set, and we already have an invariant volume on Ω_ψ (because the flow ψ^t and the geodesic flow φ^t are differentially conjugate on their non wandering sets). The hyperbolicity gives us an extension to $W^s(\Omega_\psi) \cup W^u(\Omega_\psi)$, which consists of projections of points $(x_-, x_0, x_+) \in \Sigma$ such that $x_- \in L_\Gamma$ or $x_+ \in L_\Gamma$.

Lemma 4.5.2. *There is a smooth volume form ω_1 on N that is invariant under ψ^t at points of $W^s(\Omega_\psi) \cup W^u(\Omega_\psi)$.*

Proof. The differentiable conjugacy on the non wandering set implies that there is a smooth volume form ω_0 on N that is preserved by the flow at points of the non wandering set. Hence, if $\psi^{t*}\omega_0 = e^{-A(t,x)}\omega_0$ and $\alpha(x) = \frac{\partial A}{\partial t}(0, x)$, then $\alpha = 0$ on Ω_ψ. We will now construct a smooth function σ on N such that $\sigma(\psi^t(x)) - \sigma(x) = \int_0^t \alpha(\psi^s(x))ds$ for all $x \in W^s(\Omega_\psi) \cup W^u(\Omega_\psi)$, so that $\omega_1 = e^\sigma \omega_0$ meets our requirements.

If $x \in W^s(z)$ with $z \in \Omega_\psi$, and if we have found such a function σ, then $\sigma(\psi^t(x)) \approx \sigma(\psi^t(z)) = 0$ for t large enough, hence $\sigma(x) = -\int_0^\infty \alpha(\psi^t(x))dt$. We will use this formula as a definition of σ. If it is well defined, then it satisfies the cohomology equation.

Let $C > 0$ be such that $d(\psi^t(x), \psi^t(z)) \leq Ce^{-t}$ (locally C can be chosen independently from x and z). Let k be a Lipschitz constant for α in a neighbourhood U of Ω_ψ. For t such that $\psi^t(x) \in U$ (which is locally uniform in x), we have:

$$|\alpha(\psi^t(x))| \leq \underbrace{|\alpha(\psi^t(z))|}_{=0} + k\, \underbrace{d(\psi^t(x), \psi^t(z))}_{\leq Ce^{-t}}$$

This gives us uniform convergence, hence σ is well defined and continuous. By applying the same reasoning with negative times, we define σ on $W^u(\Omega_\psi)$.

We now wish to see that it is differentiable (i.e. it is the restriction to $W^s(\Omega_\psi) \cup W^u(\Omega_\psi)$ of a differentiable function). Since the problem of differentiation is local, we can assume that the underlying manifold is $\mathbb{R}^3$ (so that tangent vectors at z and at x can be identified). Let k' be a Lipschitz constant for $d^2\alpha$ in U. For t large enough, we have:

$$\begin{aligned} d\alpha_{\psi^t(x)}(d\psi^t_x(v)) &= \underbrace{d\alpha_{\psi^t(z)}(d\psi^t_x(v))}_{=0} \\ &+ \int_0^1 \underbrace{d^2\alpha_{\psi^t(z)+s(\psi^t(x)-\psi^t(z))}}_{\leq k'Ce^{-t}}(\underbrace{\psi^t(x)-\psi^t(z)}_{\leq Ce^{-t}},\underbrace{d\psi^t_x(v)}_{\leq C'e^t})ds \end{aligned}$$

hence

$$|d\alpha_{\psi^t(x)}(d\psi^t_x(v))| \leq C''e^{-t}$$

and σ is C^1. By iterating this reasoning (to estimate $d^k\alpha$ we have to use a Taylor development at order $2k$, so that we have k terms dominated by e^t and $k+1$ terms dominated by e^{-t}), we show that σ is C^∞. □

6. Non Fuchsian examples

6.1. Going back from N to $\mathcal{C}$. Now that we have found an invariant volume form on a larger set for the flow ψ^t, we need to translate it in terms of the action on $\mathcal{C}$.

Lemma 4.6.1. *If there is a C^r volume form v on N preserved by ψ^t at points of $W^s(\Omega_\psi)\cup W^u(\Omega_\psi)$, then there is a C^r volume form ω_2 on $\mathcal{C}$ preserved by $\rho_1(\Gamma)$ at points of $L_\Gamma \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L_\Gamma$.*

Proof. We have defined a smooth volume form $\omega_1 = e^\sigma\omega_0$ that is invariant at points of $W^s(\Omega_\psi)\cup W^u(\Omega_\psi)$. Let $\widetilde{\omega}_1$ be its lift to Σ_3 and write:

$$\widetilde{\omega}_1 = \widetilde{\omega}_1(x_-,x_0,x_+)dx_-\wedge dx_0\wedge dx_+$$

If x_- or x_+ is in L_Γ, then the image in N is in $W^s(\Omega_\psi)\cup W^u(\Omega_\psi)$, and the invariance under the flow ψ^t gives us $\widetilde{\omega}_1(x_-,x_0,x_+) = \widetilde{\omega}_1(x_-,x_0',x_+)$ for all x_0' such that $(x_-,x_0',x_+)\in\Sigma_3$.

Choose a smooth map $i_0 : \mathcal{C} \to \mathbb{S}^1$ such that $(x_-, i_0(x_-,x_+), x_+) \in \Sigma_3$ for all $(x_-,x_+)\in\mathcal{C}$ (such as a convex combination of x_- and x_+), and let $\omega_2(x_-,x_+) = \widetilde{\omega}_1(x_-,i_0(x_-,x_+),x_+)$ for $(x_-,x_+)\in\mathcal{C}$. If x_- or x_+ is in L_Γ and $\gamma\in\Gamma$, then the invariance under ψ^t gives us:

$$\begin{aligned} & \omega_2(\rho_1(\gamma)(x_-),\rho_1(\gamma)(x_+))\rho_1(\gamma)'(x_-)\rho_1(\gamma)'(x_+) \\ = \;& \widetilde{\omega}_1(\rho_1(\gamma)(x_-), i_0(\rho_1(\gamma)(x_-),\rho_1(\gamma)(x_+)),\rho_1(\gamma)(x_+))\rho_1(\gamma)'(x_-)\rho_1(\gamma)'(x_+) \\ = \;& \widetilde{\omega}_1(\rho_1(\gamma)(x_-),\rho_1(\gamma)(i_0(x_-,x_+)),\rho_1(\gamma)(x_+))\rho_1(\gamma)'(x_-)\rho_1(\gamma)'(x_+) \\ = \;& \widetilde{\omega}_1(x_-,i_0(x_-,x_+),x_+) \\ = \;& \omega_2(x_-,x_+) \end{aligned}$$

We have defined a smooth volume form ω_2 on $\mathcal{C}$ that is $\rho_1(\Gamma)$-invariant at points of $(L_\Gamma\times\mathbb{S}^1\cup\mathbb{S}^1\times L_\Gamma)\setminus\Delta$. □

6.2. Extension to vertical strips. The first step in extending ω_2 to all of $\mathcal{C}$ is to extend it to vertical strips delimited by elements of L_Γ, so that we only need to deal with invariance under one element of the group.

Lemma 4.6.2. *Let I be a connected component of $\mathbb{S}^1\setminus L_\Gamma$, and let $\gamma\in\Gamma$ be a generator of its stabilizer. There is a smooth volume form ω on $\overline{I}\times\mathbb{S}^1\setminus\Delta$ that is invariant by γ and that is equal to ω_2 on $L_\Gamma\times\mathbb{S}^1\cup\mathbb{S}^1\times L_\Gamma$.*

Proof. By Proposition 4.1.6, there is a smooth volume form ω_γ on $\mathcal{C}$ that is invariant under $\rho_1(\gamma)$.

Let $a \in L_\Gamma \setminus \overline{I}$. The interval $[a\,,\rho_1(\gamma)(a)[$ is a fondamental domain for the action of γ on $\mathbb{S}^1 \setminus \overline{I}$, i.e. for every $y \in \mathbb{S}^1 \setminus \overline{I}$ there is a unique $n_y \in \mathbb{Z}$ such that $\rho_1(\gamma^{n_y})(y) \in [a\,,\rho_1(\gamma)(a)[$. We set $\omega = \omega_2$ on $\overline{I} \times [a\,,\rho_1(\gamma)(a)[$ and extend ω to $\overline{I} \times (\mathbb{S}^1 \setminus \overline{I})$ by using the equivariance formula:

$$\frac{\omega(x,y)}{\omega_2(\rho_1(\gamma^{n_y})(x),\rho_1(\gamma^{n_y})(y))} = \rho_1(\gamma^{n_y})'(x)\rho_1(\gamma^{n_y})'(y)$$

We have to show that ω is smooth. First, remark that it is continuous on $\overline{I} \times [a\,,\rho_1(\gamma)(a)[$: if $(x_n,y_n) \to (a,y)$ with $\rho_1(\gamma)(x_n) \in [a\,,\rho_1(\gamma)(a)[$, using $a \in L_\Gamma$, we see that the volume ω_2 is preserved at (a,y) and we get:

$$\begin{aligned} \omega(x_n,y_n) &= \omega_2(\rho_1(\gamma)(x_n),\rho_1(\gamma)(y_n))\rho_1(\gamma)'(x_n)\rho_1(\gamma)'(y_n) \\ &\to \omega_2(\rho_1(\gamma)(a),\rho_1(\gamma)(y))\rho_1(\gamma)'(a)\rho_1(\gamma)'(y) \\ &= \omega_2(a,y) = \omega(a,y) \end{aligned}$$

This shows that ω is continuous on $\overline{I} \times (\mathbb{S}^1 \setminus \overline{I})$. For the derivatives, we have:

$$\begin{aligned} \frac{\partial \omega}{\partial x}(x_n,y_n) &= \frac{\partial \omega_2}{\partial x}(\rho_1(\gamma)(x_n),\rho_1(\gamma)(y_n))\rho_1(\gamma)'(x_n)^2\rho_1(\gamma)'(y_n) \\ &\quad +\omega_2(\rho_1(\gamma)(x_n),\rho_1(\gamma)(y_n))\rho_1(\gamma)''(x_n)\rho_1(\gamma)'(y_n) \\ &\to \frac{\partial \omega_2}{\partial x}(\rho_1(\gamma)(a),\rho_1(\gamma)(y))\rho_1(\gamma)'(a)^2\rho_1(\gamma)'(y) \\ &\quad +\omega_2(\rho_1(\gamma)(a),\rho_1(\gamma)(y))\rho_1(\gamma)''(a)\rho_1(\gamma)'(y) \\ &= \frac{\partial \omega_2}{\partial x}(a,y) = \frac{\partial \omega}{\partial x}(a,y) \end{aligned}$$

The last line comes from the fact that the derivatives of ω_2 satisfy the associated equivariance relations on $L_\Gamma \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L_\Gamma$. This is true because all points of L_Γ are accumulation points (it is a Cantor set). The same can be applied to all the derivatives, which shows that ω is smooth on $\overline{I} \times (\mathbb{S}^1 \setminus \overline{I})$.

If $(x_k,y_k) \to (x,y) \in \mathcal{C}$ with $y \in \partial I$, then set $n_k = n_{y_k}$, as well as $u_k = \rho_1(\gamma^{n_k})(x_k)$ and $v_k = \rho_1(\gamma^{n_k})(y_k)$. By definition, we have:

$$\omega(x_k,y_k) = \omega_2(u_k,v_k)\rho_1(\gamma^{n_k})'(x_k)\rho_1(\gamma^{n_k})'(y_k)$$

Since ω_γ is invariant under $\rho_1(\gamma)$, we have:

$$\rho_1(\gamma^{n_k})'(x_k)\rho_1(\gamma^{n_k})'(y_k) = \frac{\omega_\gamma(x_k,y_k)}{\omega_\gamma(u_k,v_k)}$$

These two equalities give us:

$$\omega(x_k,y_k) = \frac{\omega_2(u_k,v_k)}{\omega_\gamma(u_k,v_k)}\omega_\gamma(x_k,y_k)$$

The continuity of ω_γ gives us $\omega_\gamma(x_k,y_k) \to \omega_\gamma(x,y)$.

Since $y_k \to y \in \partial I$, we have $n_k \to \infty$ and $u_k \to u$ where u is the other extremal point of I. By using the uniform continuity of ω_2 and ω_γ on $\overline{I} \times [a\,,\rho_1(\gamma)(a)]$, we obtain:

$$\omega(x_k,y_k) \sim \frac{\omega_2(u,v_k)}{\omega_\gamma(u,v_k)}\omega_\gamma(x,y)$$

We now only have to deal with the restrictions of ω_2 and ω_γ to the axes $\{u\}\times\mathbb{S}^1\cup\mathbb{S}^1\times\{y\}$ (see Figure 4.2), where continuous volume forms invariant under $\rho_1(\gamma)$ are unique up

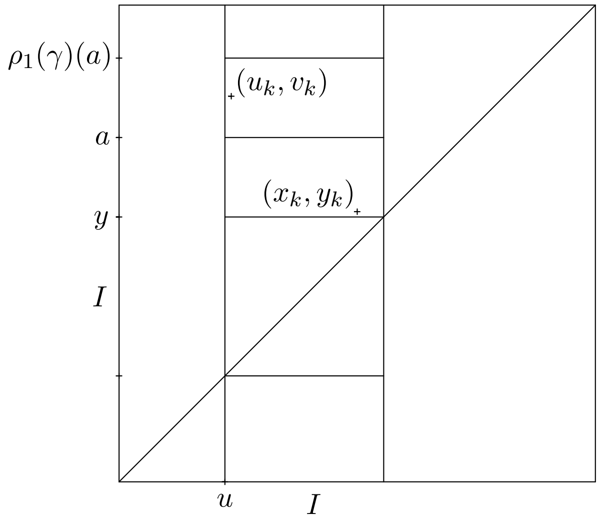

FIGURE 4.2. Defining ω on vertical strips

to multiplication by a constant: there is $\lambda > 0$ such that $\omega_2(s,t) = \lambda\omega_\gamma(s,t)$ whenever $s = u$ or $t = y$. We can finally conclude:

$$\omega(x_k, y_k) \to \lambda\omega_\gamma(x,y) = \omega_2(x,y) = \omega(x,y)$$

We have shown that ω is continuous on $(\overline{I} \times \mathbb{S}^1 \setminus I) \setminus \Delta$. For the derivatives., we will use the notation $f_x = \frac{\partial \mathrm{Log}\omega}{\partial x}$ and define f_y, f_{xy} and so on in the same way. We also define f_x^γ, f_y^γ, f_{xy}^γ, etc… the derivatives of $\mathrm{Log}\omega_\gamma$. The equivariance relation for f_x is:

$$f_x(x,y) = f_x(\rho_1(\gamma)(x), \rho_1(\gamma)(y))\rho_1(\gamma)'(x) + \frac{\rho_1(\gamma)''(x)}{\rho_1(\gamma)'(x)}$$

We keep the same notations u_k, v_k as above, and find:

$$f_x(x_k, y_k) - f_x^\gamma(x_k, y_k) = \rho_1(\gamma^{n_k})'(x_k)(f_x(u_k, v_k) - f_x^\gamma(u_k, v_k))$$

The Mean Value Theorem gives us $u_k', u_k'' \in [u\,, u_k]$ such that:

$$f_x(u_k, v_k) - f_x(u, v_k) = (u_k - u)f_{xx}(u_k', v_k)$$

And:

$$f_x^\gamma(u_k, v_k) - f_x^\gamma(u, v_k) = (u_k - u)f_{xx}^\gamma(u_k'', v_k)$$

The forms ω and ω_γ are proportional on the axis $\{u\} \times \mathbb{S}^1 \setminus \{u\}$. This implies that $f_x(u, v_k) = f_x^\gamma(u, v_k)$ (the multiplicative constant disappears because we consider the derivative of the logarithm). Finally, we obtain:

$$f_x(x_k, y_k) - f_x^\gamma(x_k, y_k) = \underbrace{\rho_1(\gamma^{n_k})'(x_k)(u - u_k)}_{\text{bounded}}\underbrace{(f_{xx}(u_k', v_k) - f_{xx}^\gamma(u_k'', v_k))}_{\to 0}$$

Since f_x^γ is continuous, we see that f_x also is. The same technique (applying several times the Mean Value Theorem to get rid of the term $\rho_1(\gamma^{n_k})'(x_k)$ or $\rho_1(\gamma^{n_k})'(y_k)$ which explodes) shows that ω is smooth on $(\overline{I} \times \mathbb{S}^1 \setminus I) \setminus \Delta$.

Finally, we can extend ω to $\overline{I} \times \mathbb{S}^1 \setminus \Delta$ in a similar manner: we fix ω on a fondamental domain $[b\,, \rho_1(\gamma)(b)[\, \times \overline{I} \setminus \Delta$ for some $b \in I$, making sure that the derivatives on the boundary allow the extension on $\overline{I} \times \overline{I} \setminus \Delta$ to be smooth. $\square$

6.3. From vertical strips to $\mathcal{C}$. We can now extend ω to $\mathcal{C}$. Getting an invariant volume form is not complicated, however its regularity requires some work.

6.3.1. *Continuity.* Our proof of the regularity of ω on vertical strips relied on the existence of a smooth invariant form by any element of Γ. To deal with the invariance under the whole group, we will need a different method.

Proposition 4.6.3. *There is a continuous invariant form ω on $\mathcal{C}$ that is invariant under $\rho_1(\Gamma)$ and that is equal to ω_2 on $L_\Gamma \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L_\Gamma$.*

Proof. The action of Γ on the set of connected components of $\mathbb{S}^1 \setminus L_\Gamma$ has a finite number of orbits (each orbit correspond to a half cylinder in the surface $\mathbb{H}^2/\rho_0(\Gamma)$). Let $I_1, \dots, I_n$ be a choice of an interval of each orbit. Note that the stabilizer of I_i is always non empty (a generator of the stabilizer corresponds to a closed geodesic bounding a half cylinder in the surface $\mathbb{H}^2/\rho_0(\Gamma)$). By Lemma 5.2.7, there is a smooth volume form ω on $\overline{I}_i \times \mathbb{S}^1 \setminus \Delta$ that is equal to ω_2 in restriction to $L_\Gamma \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L_\Gamma$ and that is invariant under the stabilizer of I_i. If $\gamma \in \Gamma$, then we define ω on $\rho_1(\gamma)(\overline{I}_i) \times \mathbb{S}^1 \setminus \Delta$ to be $\rho_1(\gamma)_*\omega$. This defines a volume form ω on $\mathcal{C}$ that is $\rho_1(\Gamma)$-invariant, smooth on all vertical strips $I \times \mathbb{S}^1 \setminus \Delta$ where I is a connected component of $\mathbb{S}^1 \setminus L_\Gamma$ and equal to ω_2 on $L_\Gamma \times \mathbb{S}^1 \cup \mathbb{S}^1 \cup L_\Gamma$.

To show that ω is continuous, assume that $(x_k, y_k) \to (x, y)$ with $x \in L_\Gamma$ (if $x \notin L_\Gamma$, then there is a connected component I of $\mathbb{S}^1 \setminus L_\Gamma$ such that $x_k \in I$ for k large enough, which gives us $\omega(x_k, y_k) \to \omega(x, y)$, and the same for the derivatives of ω). If $x_k \in L_\Gamma$ for all k, then $\omega(x_k, y_k) = \omega_2(x_k, y_k)$ and we already have the continuity, hence we can assume that $x_k \notin L_\Gamma$ for all k. Up to considering a finite number of subsequences, we can assume that there is $\gamma_k \in \Gamma$ such that $u_k = \rho_1(\gamma_k)(x_k) \in I_1$. By composing γ_k with an element of the stabilizer of I_1, we can take u_k in a compact interval $K \subset I$ (take a fundamental domain $K = [a\,, \rho_1(\delta)(a)]$ where δ is a generator of $\mathrm{Stab}(I_1)$).
Let $v_k = \rho_1(\gamma_k)(y_k)$. The definition of ω is:

$$\omega(x_k, y_k) = \omega(u_k, v_k)\rho_1(\gamma_k)'(x_k)\rho_1(\gamma_k)'(y_k)$$

We have already seen that ω is continuous on $\overline{I}_1 \times \mathbb{S}^1 \setminus \Delta$ and $u_k \in I_1$. The problem in finding the limit of $\omega(x_k, y_k)$ is the control of the Jacobian product $\rho_1(\gamma_k)'(x_k)\rho_1(\gamma_k)'(y_k)$. However, we know that ω is continuous on $L_\Gamma \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L_\Gamma$. We will use this fact to get rid of the derivatives: if x'_k and y'_k are sequences in L_Γ such that $x'_k \neq y'_k$, $x'_k \neq y_k$ and $x_k \neq y'_k$, then we set $u'_k = \rho_1(\gamma_k)(x'_k)$ and $v'_k = \rho_1(\gamma_k)(y'_k)$. The equivariance equation for ω gives us:

$$\frac{\omega(x_k, y_k)}{\omega(x_k, y'_k)} \frac{\omega(x'_k, y'_k)}{\omega(x'_k, y_k)} = \frac{\omega(u_k, v_k)}{\omega(u_k, v'_k)} \frac{\omega(u'_k, v'_k)}{\omega(u'_k, v_k)} \tag{1}$$

We are now looking for suitable points x'_k and y'_k. Let $I_1 =]a\,, b[$, and assume that v_k does not admit a as a limit point (up to considering two subsequences and replacing a by b in the following discussion, we can always assume that it is the case), i.e. that v_k lies in a compact interval $J \subset \mathbb{S}^1 \setminus \{a\}$. Let $u'_k = a$ and $x'_k = \rho_1(\gamma_k^{-1})(a) \to x$. If $y_k \in L_\Gamma$, then we choose $y'_k = y_k$. If $y_k \notin L_\Gamma$, then we set y'_k to be an extremal point of the connected component of $\mathbb{S}^1 \setminus L_\Gamma$ containing y_k, in a way such that $v'_k = \rho_1(\gamma_k)(y'_k) \in J$.

We now have $x'_k \to x$ and $x_k \to x$, which gives:

$$\frac{\omega(x_k, y_k)}{\omega(x_k, y'_k)} \frac{\omega(x'_k, y'_k)}{\omega(x'_k, y_k)} \sim \frac{\omega(x_k, y_k)}{\omega(x, y'_k)} \frac{\omega(x, y'_k)}{\omega(x, y)} = \frac{\omega(x_k, y_k)}{\omega(x, y)}$$

We wish to show that this quantity converges to 1 as $k \to \infty$. The compact set $E = \{b\} \times J \cup K \times \mathbb{S}^1 \setminus I_1$ of $\mathcal{C}$ contains the sequences (u_k, v_k), (u_k, v'_k), (u'_k, v_k) and (u'_k, v'_k). Consequently, the ratio (1) lies in a compact set of $]0\,, +\infty[$, and it is enough to see that its only possible limit is 1. If there is a subsequence such that the ratio (1) converges to $\lambda \in]0\,, +\infty[$, then up to another subsequence, we can assume that the sequence γ_k has the convergence property: there are $N, S \in \mathbb{S}^1$ such that $\rho_1(\gamma_k)(z) \to N$

for all $z \neq S$. Since $\rho_1(\gamma_k^{-1})(z) \to x$ for all $z \in I_1$, we see that S in necessarily equal to x, hence the sequences v_k and v'_k converge to $N \in \mathbb{S}^1$. We get:

$$\frac{\omega(u_k, v_k)}{\omega(u_k, v'_k)} \frac{\omega(u'_k, v'_k)}{\omega(u'_k, v_k)} \to \frac{\omega(u, N)}{\omega(u, N)} \frac{\omega(a, N)}{\omega(a, N)} = 1$$

This shows that $\lambda = 1$, therefore $\omega(x_k, y_k) \to \omega(x, y)$ and ω is continuous. □

6.3.2. *Differentiability.* For higher regularity of ω, we will keep the same notations as in the proof of Proposition 5.2.8 to show that we also have $\frac{\partial^{n+m}\omega}{\partial x^n \partial y^m}(x_k, y_k) \to \frac{\partial^{n+m}\omega_2}{\partial x^n \partial y^m}(x, y)$. By considering the restrictions of ω to horizontal and vertical circles, this will show that the partial derivatives of ω are well defined, and that they are continuous, which implies the smoothness of ω. To simplify the calculations, we will use the notation $f_x = \frac{\partial \mathrm{Log}\omega}{\partial x}$ and define f_y, f_{xy} and so on in the same way. We will make use repeatedly of an intermediate result.

Lemma 4.6.4. *Let $g, h : \mathcal{C} \to \mathbb{R}$ be functions such that:*

- *The restrictions of g to vertical strips $\overline{I} \times \mathbb{S}^1 \setminus \Delta \to \mathbb{R}$ where I is a connected component of $\mathbb{S}^1 \setminus L_\Gamma$ are C^1.*
- *The restriction of g, h and the derivatives of g to $L_\Gamma \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L_\Gamma$ are continuous.*

If h is a function such that $h(x_k, y_k) = g(u_k, v_k)\rho_1(\gamma_k)'(x_k) + h_k(x_k)$ for some function $h_k : \mathbb{S}^1 \to \mathbb{R}$ and for any choice of the sequences u_k, v_k defined above, then h is continuous.

Proof. The Mean Value Theorem gives us $w_k \in [v_k\,, v'_k]$ such that:

$$h(x_k, y_k) - h(x_k, y'_k) = \rho_1(\gamma_k)'(x_k)(v_k - v'_k)\frac{\partial g}{\partial y}(u_k, w_k)$$

A change of variables $s = \rho_1(\gamma_k)(t)$ allows us to compute $v_k - v'_k$, by setting $y_k^t = (1-t)y'_k + ty_k$:

$$v_k - v'_k = \int_{v'_k}^{v_k} ds = \int_{y'_k}^{y_k} \rho_1(\gamma_k)'(t)dt = (y_k - y'_k)\int_0^1 \rho_1(\gamma_k)'(y_k^t)dt$$

Let $v_k^t = \rho_1(\gamma_k)(y_k^t)$.

$$\begin{aligned} h(x_k, y_k) - h(x_k, y'_k) &= \rho_1(\gamma_k)'(x_k)(y_k - y'_k)\left(\int_0^1 \rho_1(\gamma_k)'(y_k^t)dt\right)\frac{\partial g}{\partial y}(u_k, w_k) \\ &= (y_k - y'_k)\frac{\partial g}{\partial y}(u_k, w_k)\int_0^1 \frac{\omega(x_k, y_k^t)}{\omega(u_k, v_k^t)}dt \end{aligned}$$

This shows that the sequence $h(x_k, y_k)$ is bounded, so all that we have to show is that it only has one limit point. Up to a subsequence, we can assume that $y'_k \to y' \in L_\Gamma$ and that $u_k \to u$.

$$h(x_k, y_k) - h(x_k, y'_k) \to (y - y')\frac{\partial g}{\partial y}(u, N)\int_0^1 \frac{\omega(x, y^t)}{\omega(u, N)}dt$$

We now only have to show that the limit does not depend on y' and u. To see this, we first notice that since the expression is independent on the choice of u_k and v_k (which are defined up to composition with an element of $\mathrm{Stab}(I_1)$), and since $(y - y')\int_0^1 \omega(x, y^t)dt \neq 0$, the function $\frac{1}{\omega}\frac{\partial g}{\partial y}$ is invariant under $\rho_1(\Gamma)$. Since it is continuous on $L_\Gamma \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L_\Gamma$, it is constant on this set, and $N \in L_\Gamma$. This shows that the limit only depends on x, y and y', hence is the same for constant sequences, and it is $h(x, y) - h(x, y')$. Since $h(x_k, y'_k) \to h(x, y')$ (because $y'_k \in L_\Gamma$), h is continuous. □

We achieve the proof of Theorem 4.1.10 by showing that ω is differentiable.

Proposition 4.6.5. *ω is C^2.*

PROOF. If $\gamma \in \Gamma$ and $(x, y) \in \mathcal{C}$, then the derivative of the equivariance relation $\omega(\rho_1(\gamma)(x), \rho_1(\gamma)(y))\rho_1(\gamma)'(x)\rho_1(\gamma)'(y) = \omega(x, y)$ with respect to x is:

$$\begin{aligned}\frac{\partial\omega}{\partial x}(x,y) &= \frac{\partial\omega}{\partial x}(\rho_1(\gamma)(x), \rho_1(\gamma)(y))\rho_1(\gamma)'(x)^2\rho_1(\gamma)'(y)\\ &\quad +\omega(\rho_1(\gamma)(x), \rho_1(\gamma)(y))\rho_1(\gamma)''(x)\rho_1(\gamma)'(y)\end{aligned}$$

Applied to the sequence (x_k, y_k), we get:

$$f_x(x_k, y_k) = f_x(u_k, v_k)\rho_1(\gamma_k)'(x_k) + \frac{\rho_1(\gamma_k)''(x_k)}{\rho_1(\gamma_k)'(x_k)} \tag{2}$$

Lemma 4.6.4 show that $f_x(x_k, y_k)$ converges to $f_x(x, y)$. For f_y, we have:

$$f_y(x_k, y_k) = f_y(u_k, v_k)\rho_1(\gamma_k)'(y_k) + \frac{\rho_1(\gamma_k)''(y_k)}{\rho_1(\gamma_k)'(y_k)} \tag{3}$$

Just as in Lemma 4.6.4, we see that $f_y(x_k, y_k) - f_y(x'_k, y_k) \to 0$ (because $x_k - x'_k \to 0$), and we now know that ω is C^1. Derivating once more with respect to y, we get:

$$\begin{aligned}f_{yy}(x_k, y_k) - f_{yy}(x'_k, y_k) &= \rho_1(\gamma_k)'(y_k)^2(f_{yy}(u_k, v_k) - f_{yy}(u_k, v'_k))\\ &\quad +3(f_y(x_k, y_k) - f_y(x'_k, y_k))\frac{\rho_1(\gamma_k)''(y_k)}{\rho_1(\gamma_k)'(y_k)}\end{aligned}$$

Since $\rho_1(\gamma_k)'(y_k) \to 0$ (if were not the case, then $\rho_1(\gamma_k)$ would be equicontinuous, which is impossible because $\rho_1(\Gamma)$ is discrete in $\mathrm{Homeo}(\mathbb{S}^1)$), we see that the first term tends to 0. The equivariance formula (3) for f_y shows that the ratio $\frac{\rho_1(\gamma_k)''(y_k)}{\rho_1(\gamma_k)'(y_k)}$ has a limit as $k \to \infty$, hence is bounded. This shows that $f_{yy}(x_k, y_k) - f_{yy}(x'_k, y_k) \to 0$, i.e. that f_{yy} is continuous.

For the crossed derivative f_{xy}, we use the derivative with respect to y of (2):

$$\begin{aligned}f_{xy}(x_k, y_k) &= f_{xy}(u_k, v_k)\rho_1(\gamma_k)'(x_k)\rho_1(\gamma_k)'(y_k)\\ &= f_{xy}(u_k, v_k)\frac{\omega(x_k, y_k)}{\omega(u_k, v_k)}\end{aligned}$$

Since ω is continuous, we have:

$$f_{xy}(x_k, y_k) \to f_{xy}(u, N)\frac{\omega(x, y)}{\omega(u, N)}$$

This limit gives the impression that it depends on u, however the curvature function $\frac{1}{\omega}f_{xy}$ is $\rho_1(\Gamma)$-invariant, and continuous on $L_\Gamma \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L_\Gamma$, hence constant on this set (the proof of Lemma 4.3.1 can be applied) and the limit does not depend on u (because $N \in L_\Gamma$). This shows that f_{xy} is continuous. To get the convergence for f_{xx}, we first notice that it is sufficient to show that f_{xxy} converges:

$$\begin{aligned}f_{xx}(x_k, y_k) &= f_{xx}(x_k, y'_k) + \int_{y'_k}^{y_k} f_{xxy}(x_k, t)dt\\ &\to f_{xx}(x, y') + \int_{y'}^{y} f_{xxy}(x, t)dt = f_{xx}(x, y)\end{aligned}$$

The reason why we consider f_{xxy} rather than f_{xx} is to get a control on the term $\rho_1(\gamma_k)'(x_k)^2$ by multiplying it with $\rho_1(\gamma_k)'(y_k)$. The equivariance formula is:

$$\begin{aligned}f_{xxy}(x_k, y_k) &= f_{xxy}(u_k, v_k)\rho_1(\gamma_k)'(x_k)^2\rho_1(\gamma_k)'(y_k)\\ &\quad +f_{xy}(u_k, v_k)\rho_1(\gamma_k)''(x_k)\rho_1(\gamma_k)'(y_k)\end{aligned}$$

If we consider $g = \frac{1}{\omega} f_{xxy}$ and $h = \frac{1}{\omega} f_{xy}$, we can simplify:

$$g(x_k, y_k) = g(u_k, v_k)\rho_1(\gamma_k)'(x_k) + h(u_k, v_k)\frac{\rho_1(\gamma_k)''(x_k)}{\rho_1(\gamma_k)'(x_k)}$$

The equavariance relation (2) for f_x allows us to get rid of the term $\frac{\rho_1(\gamma_k)''(x_k)}{\rho_1(\gamma_k)'(x_k)}$:

$$g(x_k, y_k) = \rho_1(\gamma_k)'(x_k)\left(g(u_k, v_k) - f_x(u_k, v_k)h(u_k, v_k)\right) + f_x(x_k, y_k)h(u_k, v_k)$$

We now set $k = g - f_x h$ so that we have (by using the fact that h is $\rho_1(\Gamma)$-invariant):

$$g(x_k, y_k) = k(u_k, v_k)\rho_1(\gamma_k)'(x_k) + f_x(x_k, y_k)h(x_k, y_k)$$

Lemma 4.6.4 gives the convergence of the first term, and we have already shown that f_x and $h = \frac{1}{\omega} f_{xy}$ are continuous. This shows that ω is C^2. □

To get a smooth ω, first show that we can get $\frac{\partial^{n+m}}{\partial x^n \partial y^m}\mathrm{Log}\omega$ when $m > n$, then integrate with respect to y to get all derivatives.

6.4. Constructing an example. In order to make Theorem 4.1.10 relevant, we will see that such examples of groups exist. Start with $\rho_0 : \mathbb{F}_2 = \langle a, b\rangle \to \mathrm{PSL}(2,\mathbb{R})$ a convex cocompact representation generated by two hyperbolic elements $\rho_0(a) = \gamma_1, \rho_0(b) = \gamma_2$ (e.g. a Schottky group). Consider two circle diffeomorphisms φ_1, φ_2 that are the identity on the limit set $L_{\rho_0(\mathbb{F}_2)}$, and set $\widetilde{\gamma}_i = \varphi_i^{-1}\gamma_i\varphi_i$. We define the representation $\rho_1 : \mathbb{F}_2 \to \mathrm{Diff}(\mathbb{S}^1)$ by $\rho_1(a) = \widetilde{\gamma}_1$ and $\rho_1(b) = \widetilde{\gamma}_2$.

Lemma 4.6.6. *ρ_1 is differentially Fuchsian if and only if $\varphi_1 = \varphi_2$.*

PROOF. If $\varphi_1 = \varphi_2$, then φ_1 is a differentiable conjugacy between ρ_0 and ρ_1, so ρ_1 is differentially Fuchsian.

Assume that ρ_1 is differentially Fuchsian. Let $\varphi \in \mathrm{Diff}(\mathbb{S}^1)$ be such that $\varphi^{-1}\rho_1(\mathbb{F}_2)\varphi \subset \mathrm{PSL}(2,\mathbb{R})$. Up to composing φ with an element of $\mathrm{PSL}(2,\mathbb{R})$, we can assume that $\varphi^{-1}\rho_1(a)\varphi = \rho_0(a)$. This implies that $\varphi_1^{-1} \circ \varphi$ commutes with γ_1, hence that there is $t \in \mathbb{R}$ such that $\varphi_1^{-1} \circ \varphi = \gamma_1^t$ (where γ_1^t denotes the one parameter subgroup of $\mathrm{PSL}(2,\mathbb{R})$ generated by γ_1, see 1.1 for a proof). Similarly, there is $s \in \mathbb{R}$ such that $\varphi_2^{-1} \circ \varphi = \gamma_2^s$ (an element of the one parameter group generated by γ_2).

The equality $\varphi_2 \circ \gamma_2^s = \varphi_1 \circ \gamma_1^t$ applied to the fixed points of γ_1 and γ_2 shows that $s = t = 0$, hence $\varphi_1 = \varphi_2$. □

Proposition 4.6.7. *There is $h \in \mathrm{Homeo}(\mathbb{S}^1)$ such that $h_{/L_{\rho_0(\mathbb{F}_2)}} = Id$ and $\rho_1 = h\rho_0 h^{-1}$.*

PROOF. Let $\mathbb{S}^1 \setminus L_{\rho_0(\mathbb{F}_2)} = \bigcup_{i\in\mathbb{N}} I_i$ its decomposition into connected components, and let $A \subset \mathbb{N}$ be a fundamental domain for the action of $\mathbb{F}_2$ on the set of connected components of $\mathbb{S}^1 \setminus L_{\rho_0(\mathbb{F}_2)}$. Given $i \in A$, set $h_{/I_i}$ any homeomorphism that fixes the endpoints of I_i such that $h_{/I_i} \circ \rho_0(\delta) = \rho_1(\delta) \circ h_{/I_i}$ for δ in the stabilizer of I_i. For $\gamma \in \mathbb{F}_2$, set $h = \rho_1(\gamma) \circ h_{/I_i} \circ \rho_0(\gamma^{-1})$ on $\rho_0(\gamma)(I_i) = \rho_1(\gamma)(I_i)$. This defines an element $h \in \mathrm{Homeo}(\mathbb{S}^1)$ that fixes all points of $L_{\rho_0(\mathbb{F}_2)}$ such that $h^{-1}\rho_1 h = \rho_0$. □

Note that we proved here that $\rho_1(\mathbb{F}_2)$ remains a free group, which is a general fact for a C^1 perturbation of a Schottky group (see **[Sul85]**).

7. Spectrally Möbius-like deformations

In the proof of Theorem 4.1.10, we used the fact that the conjugacy is the identity on the limit set for two purposes: in order to find an invariant volume form on $L_\Gamma \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L_\Gamma \setminus \Delta$, and in order to show that Ω_ψ is a hyperbolic set. In the case of spectrally Möbius-like actions, we only have an invariant volume form on pairs of fixed points of elements of Γ.

In the context of the flow ψ, this means that we need to find an invariant volume form on Ω_ψ, starting with a data on periodic orbits. This is exactly the context of Livšic's Theorem. However, we still need hyperbolicity for the flow ψ, which is why we only prove Theorem 4.1.14 for small perturbations of Fuchsian groups.

Given a representation $\rho_0 : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ of a finitely generated group Γ, we say that $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ is C^1-close to ρ_0 if the images under ρ of a system of generators of Γ are close to the images under ρ_0 in the C^1 topology.

Theorem 4.1.14. *Let $\rho_0 : \mathbb{F}_n \to \mathrm{PSL}(2,\mathbb{R})$ be a convex cocompact representation. If $\rho_1 : \mathbb{F}_n \to \mathrm{Diff}(\mathbb{S}^1)$ is sufficiently C^1-close to ρ_0, and if ρ_1 is spectrally Möbius-like, then ρ_1 is area-preserving.*

PROOF. The central argument is the fact that the flow ψ associated to ρ_1 is C^1-close to the geodesic flow φ. Since hyperbolicity is stable under C^1 perturbations, it will imply that Ω_ψ is a hyperbolic set for ψ.

In the definitions of these flows, they seem to be defined on different manifolds. We will start by giving a slightly different construction so that they live on the same manifold.

Consider a path $\rho_u : \mathbb{F}_n \to \mathrm{Diff}(\mathbb{S}^1)$ for $u \in [0,1]$ defined as convex combinations of ρ_0 and ρ_1 (we chose free groups so that such a path can be easily defined). Recall the definition of Σ_3:

$$\Sigma_3 = \{(x_-, x_0, x_+) \in (\mathbb{S}^1)^3 | x_- < x_0 < x_+ < x_-\}$$

We can define an action of Γ on $\Sigma_3 \times [0\,,1]$ by:

$$\gamma.(x_-, x_0, x_+, u) = (\rho_u(\gamma)(x_-), \rho_u(\gamma)(x_0), \rho_u(\gamma)(x_+), u)$$

This action preserves the slices $\Sigma_3 \times \{u\}$, which gives a map on the quotient $\pi : \Sigma_3 \times [0\,,1]/\Gamma \to [0\,,1]$ which is a submersion. Each fiber $\pi^{-1}(\{u\})$ is diffeomorphic to the manifold N_u on which the flow ψ_u^t associated with the representation ρ_u is defined.

If $U \subset \Sigma_3$ is a relatively compact neighbourhood of $(L_{\rho_0(\Gamma)} \times \mathbb{S}^1 \times L_{\rho_0(\Gamma)}) \cap \Sigma_3$, then the restriction of π to $U \times [0\,,1]$ is a proper submersion onto $[0\,,1]$, hence a trivial fibration, i.e. there is a diffeomorphism $\Phi : U \times [0\,,1]/\Gamma \to N \times [0\,,1]$ such that projection on the second factor is equal to π. This shows that the flows ψ_u (restricted to a neighbourhood of the non wandering set) can be considered as flows on the manifold N, that vary continuously with u in the C^1 topology. Therefore, if ρ_1 is sufficiently close to ρ_0, then Ω_{ψ_1} is a hyperbolic set for ψ_1.

We will now use the notation ψ for the flow associated to ρ_1, and α_1 for the diagonal action of Γ on Σ_3 (note that it is not exactly the same flow as defined in the proof of Theorem 4.1.10, where we kept the action ρ_0 on the middle factor of Σ_3 so that the conjugacy with the geodesic flow would be differentiable along all the non wandering set).

Given a volume ω_0 on N, we set $\psi^{t*}\omega_0 = e^{-A(t,x)}\omega_0$. To find a volume $\omega_1 = e^\sigma \omega_0$ that is invariant under ψ at points of Ω_ψ, we have to solve the equation $\sigma(\psi^t(x)) - \sigma(x) = A(t,x)$ for all $x \in \Omega_\psi$. A necessary condition on the cocycle A is that $A(T,x) = 0$ whenever $\psi^T(x) = x$. Livšic's Theorem states that this condition is sufficient.

Let us show that $A(T,x) = 0$ for periodic orbits $\psi^T(x) = x$. Since $A(T,x) = -\mathrm{Log}\det(D\psi_x^T)$, we have to show that the Jacobian $\det(D\psi_x^T)$ is equal to 1.

To compute this Jacobian, we consider the lift $\widetilde{\psi}^t$ to Σ_3, and $p : \Sigma_3 \to \Sigma_3/\Gamma$ the covering map. Since the flow $\widetilde{\psi}^t$ is a reparametrisation of the vector field $(0,1,0)$, it can be written:

$$\widetilde{\psi}^t(x_-, x_0, x_+) = (x_-, f(t, x_-, x_0, x_+), x_+)$$

If $\psi^T(x) = x$, then a lift $\widetilde{x} = (x_-, x_0, x_+) \in p^{-1}(\{x\})$ satisfies $\widetilde{\psi}^T(\widetilde{x}) = \alpha_1(\gamma)(\widetilde{x})$ for some $\gamma \in \Gamma$. For all $y \in \mathbb{S}^1$ such that $(x_-, y, x_+) \in \Sigma_3$, we get $\widetilde{\psi}^T(x_-, y, x_+) =$

$(x_-, \rho_1(\gamma)(y), x_+)$, which shows that the matrix of $D\widetilde{\psi}^T_{\widetilde{x}}$ has the form:

$$\begin{pmatrix} 1 & * & 0 \\ 0 & \rho_1(\gamma)'(x_0) & 0 \\ 0 & * & 1 \end{pmatrix}$$

Consequently, its determinant is $\rho_1(\gamma)'(x_0)$. The derivative $D\psi^T_x$ is similar to $(D\alpha_1(\gamma)_{\widetilde{x}})^{-1}D\widetilde{\psi}^T_{\widetilde{x}}$. The matrix of $D\alpha_1(\gamma)_{\widetilde{x}}$ is the diagonal matrix:

$$\begin{pmatrix} \rho_1(\gamma)'(x_-) & 0 & 0 \\ 0 & \rho_1(\gamma)'(x_0) & 0 \\ 0 & 0 & \rho_1(\gamma)'(x_+) \end{pmatrix}$$

Since the action ρ_1 is spectrally Möbius-like and x_- and x_+ are fixed points of $\rho_1(\gamma)$, we have $\rho_1(\gamma)'(x_-)\rho_1(\gamma)'(x_+) = 1$, hence $\det(D\alpha_1(\gamma)_{\widetilde{x}}) = \rho_1(\gamma)'(x_0)$, and $\det(D\psi^T_x) = 1$.

In order to apply Livšic's Theorem, one has to be precise on the exact setting, as well as on the required regularity. The first result, proved by Livšic in [**Liv71**], concerns transitive Anosov flows, and deals with Hölder solutions. Smooth solutions for transitive Anosov flows are given in [**LMM86**]. Concerning compact topologically transitive hyperbolic sets, the existence of a Hölder-continuous and even C^1 solutions can be found in [**KH95**] (Theorem 19.2.4 and Theorem 19.2.5). The main difficulty appears while studying crossed derivatives for C^2 regularity. For smoothness outside of the Anosov setting (i.e. when the hyperbolic set is not the whole manifold), the only result concerns diffeomorphisms of surfaces in [**NT07**]. However, flows on three-manifolds are analogous to diffeomorphisms on surfaces.

Lemma 3.3 of [**NT07**] states that there is a continuous solution σ that is differentiable in restriction to stable and unstable leaves ([**NT07**] deals with diffeomorphisms of surfaces, but the same proof, up to replacing discrete sums by integrals, works for flows on three-manifolds). Going back to the cylinder $\mathcal{C}$, we get a function that is (uniformly) differentiable in restriction to leaves $\{x\} \times \mathbb{S}^1$ and $\mathbb{S}^1 \times \{y\}$ for $x, y \in L_{\rho_1(\Gamma)}$. Theorem 1.5 of [**NT07**] implies that this solution is smooth on $\mathbb{S}^1 \times L_{\rho_1(\Gamma)} \cup L_{\rho_1(\Gamma)} \times \mathbb{S}^1$ in the Whitney sense (i.e. that it is the restriction of a smooth function on $\mathcal{C}$).

From there, Lemma 5.2.7, Proposition 5.2.8 and Proposition 4.6.5 show that ρ_1 is area-preserving. $\square$

8. Transverse structure of Anosov flows

The use of a hyperbolic flow in the proof of Theorem 4.1.10 introduced the link between flows on three manifolds, actions on the circle and Lorentz surfaces. We studied the problem starting with a group action on the circle and used the flow in dimension 3 in order to find a Lorentz surface. Starting from an Anosov flow, we always find a Lorentzian conformal structure, and a Lorentz metric if the flow preserves a volume form. Let (M, φ^t) be an Anosov flow on a compact 3-manifold. It is known that the quotient space Q^φ of lifts of orbits to the universal cover is diffeomorphic to $\mathbb{R}^2$ (see [**Bar95**]). The stable and unstable foliations determine two transversal one-dimensional foliations of Q^φ, hence a conformal Lorentz structure (whose isotropic lines are the leaves of these foliations).

The fundamental group $\pi_1(M)$ acts naturally on Q^φ by preserving this structure. In fact, Anosov flows up to topological equivalence tend to be classified by this conformal action of the fundamental group ([**Bar95**]).

If φ^t preserves a volume form, then we get a Lorentz metric on Q^φ which is in general $C^{1+\text{Zygmund}}$ (and thus $C^{1+\alpha}$, for any $0 < \alpha < 1$). Higher regularity (C^2, or even $C^{1+\text{zygmund}}$) implies not only rigidity for the Lorentz space Q^φ (the curvature is constant), but also rigidity for the flow itself (smooth conjugacy with a reparametrisation of an algebraic flow, [**HK90**] and [**Ghy87a**]). The fundamental group acts isometrically for this Lorentz metric.

Examples of such flows are abundant, and exist for instance on many hyperbolic manifolds (see [**FH13**]). Theses examples belong to a special subcategory of Anosov flows: they are $\mathbb{R}$-covered ([**Bar01**]). It means that the quotient space of Q^φ by one of the foliations defined by the stable or unstable foliation (and automatically by both), is Hausdorff, and hence homeomorphic to $\mathbb{R}$. This is equivalent to saying that the Lorentz space Q^φ is globally hyperbolic.

In these examples, the Lorentz metric is globally hyperbolic but not spatially compact, and the action of $\pi_1(M)$ gives diffeomorphisms of the real line. However, on some examples, circle diffeomorphisms arise, mostly for geodesic flows on negatively curved surfaces, and for the action of the fundamental group of a Seifert piece in a graph manifold (see [**Bar96**]).

CHAPTER 5

Non convergence examples

So far, we have classified isometry groups of spatially compact surfaces up to semi conjugacy, and we discussed the question of differentiable conjugacy in a specific case where the semi conjugacy is always a topological conjugacy. We will now investigate the problem of knowing when the semi conjugacy is or is not a topological conjugacy. In particular, we will exhibit examples for which there is no topological conjugacy with a subgroup of a finite cover of $\mathrm{PSL}(2,\mathbb{R})$.

We will restrict this study to the most simple case where the conformal boundary is not necessarily acausal, which is the case where $h_\downarrow = h_\uparrow$. Once again, we will deal with continuous metrics. We already had a look at this case in section 3 of chapter 3, where we obtained a semi conjugacy with a representation in $\mathrm{PSL}(2,\mathbb{R})$ even when the metric is only continuous.

We will keep the notations of section 3 of chapter 3: if $h : \mathbb{S}^1 \to \mathbb{S}^1$ is non decreasing of degree one, then we denote by $G(h) \subset \mathbb{S}^1 \times \mathbb{S}^1$ the union of its graph and of vertical segments joining discontinuities. We set $M_h = \mathbb{S}^1 \times \mathbb{S}^1 \setminus G(h)$. A Lorentz metric in the conformal class of $dxdy$ on M_h can be seen as a volume form ω on M_h. We will denote by G_ω the isometry group of the Lorentz metric associated to ω.

Recall that h_l (resp. h_r) denotes the left continuous (resp. right continuous) non decreasing map of degree one that is equal to h except at points where it is not left (resp. right) continuous.

1. Elements of G_ω

Since we know that all elements of $\mathrm{PSL}(2,\mathbb{R})$ can appear, we are going to focus on elements of $\rho_1(G_\omega)$ that are not conjugate in $\mathrm{Homeo}(\mathbb{S}^1)$ to elements in $\mathrm{PSL}(2,\mathbb{R})$. We will construct three types of examples: first introducing a parabolic fixed point in a hyperbolic element of $\mathrm{PSL}(2,\mathbb{R})$, then opening the fixed point of a parabolic element of $\mathrm{PSL}(2,\mathbb{R})$, and finally considering the lift of a parabolic element of $\mathrm{PSL}(2,\mathbb{R})$ to the two sheeted cover $\mathrm{PSL}_2(2,\mathbb{R})$.

1.1. Hyperbolic elements.

Let $f, g \in \mathrm{Diff}(\mathbb{S}^1)$ be such that:

- f has three fixed points $a_1 < b_1 < c_1 < a_1$
- g has two fixed points a_2, b_2
- $f'(a_1)g'(b_2) = 1$ and $f'(b_1)g'(a_2) = 1$
- $f'(a_1) < 1$ and $f'(b_1) > 1$

Let $h : \mathbb{S}^1 \to \mathbb{S}^1$ be defined by:

- $h(x) = b_2$ for $x \in [b_1\,, c_1[$
- $h(x) = a_2$ for $x \in [c_1\,, a_1[$
- $h : [a_1\,, b_1] \to [a_2\,, b_2]$ is an orientation preserving homeomorphism (or a non decreasing map such that $h(a_1) = a_2$ and $h(b_1) = b_2$) such that $g \circ h = h \circ f$

Proposition 5.1.1. *The map $(x, y) \mapsto \varphi(x, y) = (f(x), g(y))$ preserves a continuous volume form on M_h.*

Proof. We start by dividing M_h into several subsets, as shown in Figure 5.2. Let $\alpha_1, \alpha_2 : [a_1\,, b_1] \to [b_2\,, a_2]$, $\beta : [b_1\,, c_1] \to [a_2\,, b_2]$ and $\gamma : [c_1\,, a_1] \to [a_2\,, b_2]$ be decreasing

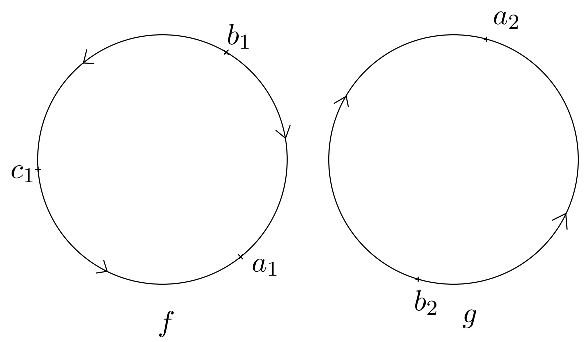

FIGURE 5.1. Dynamics of f and g

homeomorphisms whose graphs are invariant under φ (i.e. that conjugate f and g). We choose α_1 and α_2 so that $b_2 < \alpha_1(x) < \alpha_2(x) < a_2 < b_2$ for all $x \in]a_1\,,b_1[$.

Let U, V, W, X be the open subsets of M_h delimited by $G(h)$ and the graphs of

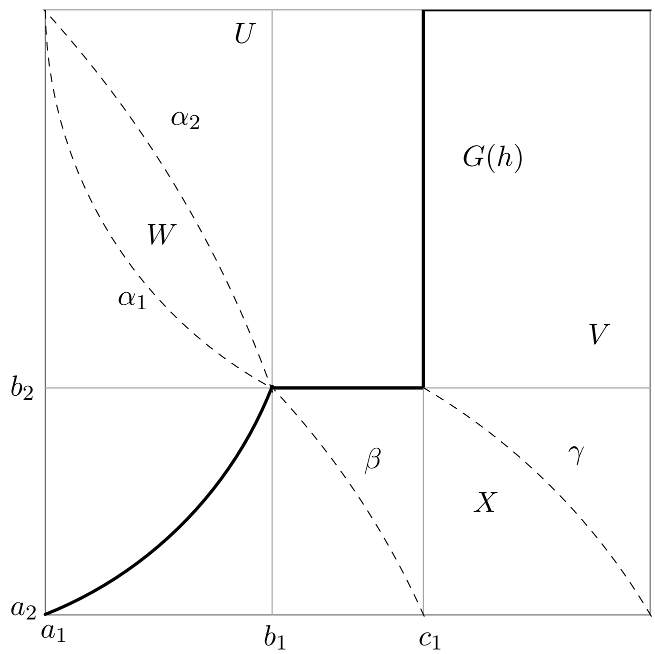

FIGURE 5.2. Constructing ω on M_h

α_1, α_2, β and γ:

$$\begin{aligned}
U &= \{(x,y) \in]a_1\,,b_1] \times \mathbb{S}^1 | y \in]\alpha_2(x)\,,h(x)[\} \\
&\qquad \cup \{(x,y) \in [b_1\,,c_1[\times \mathbb{S}^1 | y \in]b_2\,,\beta(x)[\} \\
V &= \{(x,y) \in]c_1\,,a_1] \times \mathbb{S}^1 | y \in]\gamma(x)\,,a_2[\} \\
&\qquad \cup \{(x,y) \in [a_1\,,b_1[\times \mathbb{S}^1 | y \in]h(x)\,,\alpha_1(x)[\} \\
W &= \{(x,y) \in]a_1\,,b_1[\times \mathbb{S}^1 | y \in]\alpha_1(x)\,,\alpha_2(x)[\} \\
X &= \{(x,y) \in]b_1\,,c_1] \times \mathbb{S}^1 | y \in]\beta(x)\,,b_2[\} \\
&\qquad \cup \{(x,y) \in [c_1\,,a_1[\times \mathbb{S}^1 | y \in]a_2\,,\gamma(x)[\}
\end{aligned}$$

Consider the linearisation maps $\tau^f_{a_1} :]c_1\,,b_1[\, \to \mathbb{R}$, $\tau^f_{b_1} :]a_1\,,c_1[\, \to \mathbb{R}$, $\tau^g_{a_2} : \mathbb{S}^1 \setminus \{b_2\} \to \mathbb{R}$ and $\tau^g_{b_2} : \mathbb{S}^1 \setminus \{a_2\} \to \mathbb{R}$. They are smooth maps such that:

$$\begin{aligned}
\tau^f_{a_1} \circ f \circ (\tau^f_{a_1})^{-1}(x) &= f'(a_1)x \\
\tau^f_{b_1} \circ f \circ (\tau^f_{b_1})^{-1}(x) &= f'(b_1)x \\
\tau^g_{a_2} \circ g \circ (\tau^g_{a_2})^{-1}(x) &= g'(a_2)x \\
\tau^g_{b_2} \circ g \circ (\tau^g_{b_2})^{-1}(x) &= g'(b_2)x
\end{aligned}$$

The map $(x, y) \mapsto (\tau^f_{b_1}(x), \tau^g_{a_2}(y))$ sends U to an open set of $\mathbb{R}^2$ and conjugates φ with $(x, y) \mapsto (f'(b_1)x, g'(a_2)y)$. Since $f'(b_1)g'(a_2) = 1$, this map preserves $dx \wedge dy$ on $\mathbb{R}^2$, and φ preserves the pull-back ω_U on U.

The map $(x, y) \mapsto (\tau^f_{a_1}(x), \tau^g_{b_2}(y))$ sends V to an open set of $\mathbb{R}^2$ and conjugates φ with $(x, y) \mapsto (f'(a_1)x, g'(b_2)y)$. Since $f'(a_1)g'(b_2) = 1$, this map preserves $dx \wedge dy$ on $\mathbb{R}^2$, and φ preserves the pull-back ω_V on V.

To extend ω to W, we notice that the action of φ on $]a_1\,,b_1[\,\times\,]b_2\,,a_2[$ is conjugate to a translation in $\mathbb{R}^2$, so the quotient is diffeomorphic to a cylinder. The images of U and V on the cylinder are open sets $\hat{U}$, $\hat{V}$ each bounded by a curve on which volume forms $\hat{\omega}_U$, $\hat{\omega}_V$ are defined. Simply consider a continuous volume form $\hat{\omega}$ on the cylinder that is equal to $\hat{\omega}_U$ on $\hat{U}$, and equal to $\hat{\omega}_V$ on $\hat{V}$ (this is possible because $\hat{\omega}_U$ and $\hat{\omega}_V$ can be defined on open sets larger than U and V). This lifts to a volume form ω on $]a_1\,,b_1[\,\times\,]b_2\,,a_2[$ that is invariant under φ and that is equal to ω_U in a neighbourhood of the axes $\{a_1\}\times]b_2\,,a_2[\cup]a_1\,,b_1[\times\{b_2\}$ and to ω_V on a neighbourhood of $\{b_1\}\times]b_2\,,a_2[\cup\,]a_1\,,b_1[\,\times\{a_2\}$.

We now only have to extend ω to X. If $x \in\,]a_2\,,b_2[$, then $]b_1\,,a_1[\,\times\,[x\,,g(x)[$ is a fundamental domain for the action of φ on $]b_1\,,a_1[\,\times\,]a_2\,,b_2[$, which shows that the quotient is diffeomorphic to the cylinder, and we can extend ω to X in the same way that we did for W. ☐

The same could also be done for f with four fixed points $a_1 < b_1 < c_1 < d_1 < a_1$ such that b_1 and d_1 are parabolic, a_1 is attractive and c_1 repulsive. We then choose g with two hyperbolic fixed points a_2, b_2 such that $f'(a_1)g'(b_2) = 1$ and $f'(c_1)g'(a_1) = 1$. We set $h \equiv a_2$ on $[d_1\,,b_1[$ and $h \equiv b_2$ on $[b_1\,,d_1[$. The same kind of division of M_h into four invariant open sets U, V, W, X gives a (f, g)-invariant volume form on M_h (see Figure 5.3).

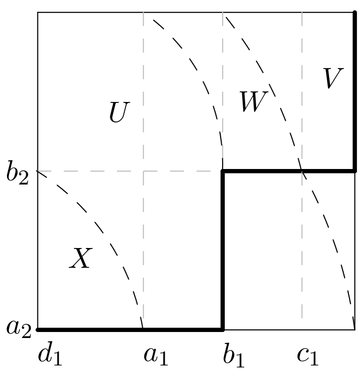

FIGURE 5.3. Example with four fixed points

1.2. Parabolic elements. Let $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ be parabolic, and let $x_0 \in \mathbb{S}^1$ be its fixed point. We will denote by ω_0 the volume form on $\mathbb{S}^1\times\mathbb{S}^1\setminus\Delta$ preserved by $\mathrm{PSL}(2,\mathbb{R})$.

Let $I \subset \mathbb{S}^1$ be a compact interval and define a continuous function $h : \mathbb{S}^1 \to \mathbb{S}^1$ such that $h(I) = \{x_0\}$ et $h : \mathbb{S}^1\setminus I \to \mathbb{S}^1\setminus\{x_0\}$ is an affine diffeomorphism. It is non decreasing of degree one.

We can define $f \in \mathrm{Diff}^{1,1}(\mathbb{S}^1)$ such that the restriction to I is an orientation preserving diffeomorphism with $f' = 1$ at the endpoints and $h^{-1}\circ\gamma\circ h$ on the complement of I. We have $h\circ f = \gamma\circ h$, so the map (f,γ) acts on M_h.

Proposition 5.1.2. *The map $(x, y) \mapsto (f(x), \gamma(y))$ preserves a continuous volume form on M_h.*

Proof. First, define ω on $M_h\setminus(\mathring{I}\times\mathbb{S}^1)$ to be $\omega(x, y) = \omega_0(h(x), y)$. It is a continuous volume form (even Lipschitz), and let us show that $(x, y) \mapsto (f(x), \gamma(y))$ preserves ω. If $x \notin \mathring{I}$, then we get:

$$\begin{aligned}
\omega(f(x),\gamma(y))f'(x)\gamma'(y) &= \omega_0(h\circ f(x),\gamma(y))f'(x)\gamma'(y)\\
&= \omega_0(\gamma(h(x)),\gamma(y))\gamma'(h(x))\gamma'(y)\frac{f'(x)}{\gamma'(h(x))}\\
&= \omega_0(h(x),y)\frac{f'(x)}{\gamma'(h(x))}\\
&= \omega(x,y)\frac{h'(x)}{h'(f(x))}\\
&= \omega(x,y)
\end{aligned}$$

To extend ω to M_h, we simply notice that the quotient of $I\times\mathbb{S}^1\setminus G(h)$ by (f,γ) is diffeomorphic to a cylinder, on which we have defined a volume form on the boundary. We can extend it to a volume form on the cylinder, then lift it to a continuous volume form on $I\times\mathbb{S}^1\setminus G(h)$ that is (f,γ)-invariant and equal to ω on the boundary. This way, we have defined a continuous volume form on M_h that is (f,γ)-invariant. $\square$

Remark. In all of the other examples, the invariant metrics can actually be constructed in a smooth way. However, in the case of the opening of the fixed point of a parabolic, it is not clear whether it is possible to find a smooth metric.

1.3. Lifts of parabolics to $\mathrm{PSL}_2(2,\mathbb{R})$. The two-sheeted covering of the circle is still a circle, which induces a two-sheeted covering group $\mathrm{PSL}_2(2,\mathbb{R})$ of $\mathrm{PSL}(2,\mathbb{R})$. Lifts of parabolic elements either have two fixed points, or two (not fixed) periodic points. Let $R \in \mathrm{PSL}_2(2,\mathbb{R})$ be a generator of the centre (R is of order two).

If $\gamma \in \mathrm{PSL}_2(2,\mathbb{R})$ is a lift of a parabolic element without fixed points, then it has two periodic points $x_0, R(x_0)$. Let $U = \{(x, y)|y \in \,]x\,, R(x)[\}$. The quotient by the map $(x, y) \mapsto (R(x), R(y))$ of U endowed with the action of $\mathrm{PSL}(2,\mathbb{R})$ is equivariant to the diagonal action on $\mathbb{S}^1\times\mathbb{S}^1\setminus\Delta$, therefore has a volume form preserved by $\mathrm{PSL}(2,\mathbb{R})$, which can be lifted to U as a volume form ω_0 invariant under $\mathrm{PSL}_2(2,\mathbb{R})$.

Let $h : \mathbb{S}^1 \to \mathbb{S}^1$ be defined by $h(x) = x_0$ if $x \in [x_0\,, R(x_0)[$ and $h(x) = R(x_0)$ if $x \in [R(x_0)\,, x_0[$. It is non decreasing of degree one, and it commutes with γ.

Proposition 5.1.3. *The map $(x, y) \mapsto (\gamma(x), \gamma(y))$ preserves a continuous volume form on M_h.*

Proof. In order to extend ω_0 to M_h, we start by considering the open set $V = \{(x, y)|x \in \,]x_0\,, R(x_0)[, y \in \,]x\,, x_0[\}$ and its image $R(V)$ under the map $(x, y) \mapsto (R(x), R(y))$ (see Figure 5.4).

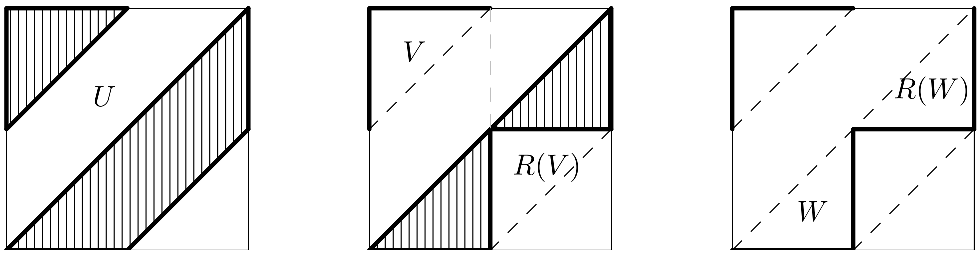

FIGURE 5.4. Invariant volume form for a parabolic element of $\mathrm{PSL}_2(2,\mathbb{R})$

Since the action of γ on $V \cup R(V)$ is proper (the quotient is a cylinder), we can extend ω_0 from $U \cap (V \cup R(V))$ to a volume form ω on $U \cup V \cup R(V)$.

Similarly, we set $W = \{(x,y) | x \in \,]x_0\,, R(x_0)[, y \neq x_0\}$ and extend ω to $W \cup R(W)$, so that ω is now defined on M_h. $\square$

1.4. Classification of elements of $\rho_1(G_\omega)$ up to topological conjugacy. We will now see that we have described all of the examples.

Proposition 5.1.4. *Let $h : \mathbb{S}^1 \to \mathbb{S}^1$ be non decreasing of degree one, and let ω be a continuous volume form on M_h. For any $\varphi \in G_\omega$, $\rho_1(\varphi)$ is topologically conjugate to one of the examples described above, i.e. it satisfies one of the following propositions:*

- *$\rho_1(\varphi)$ is topologically conjugate to an element of* $\mathrm{PSL}(2,\mathbb{R})$.
- *$\rho_1(\varphi)$ is topologically conjugate to a parabolic element of* $\mathrm{PSL}_2(2,\mathbb{R})$.
- *$\rho_1(\varphi)$ has three fixed points $a < b < c < a$ such that a, b are hyperbolic and c is parabolic.*
- *$\rho_1(\varphi)$ has four fixed points $a < b < c < d < a$ such that a, c are hyperbolic and b, d are parabolic.*

Let $h : \mathbb{S}^1 \to \mathbb{S}^1$ be non decreasing of degree one, let ω be a continuous volume form on M_h, and let $\varphi = (f, g) \in G_\omega \setminus \{Id\}$. We will classify them according to their numbers of fixed points.

As we saw in Lemma 3.3.7, fixed points of φ in M_h are hyperbolic. This implies that there cannot be too many of them.

Lemma 5.1.5. *If $a_1 \in \mathbb{S}^1$ is fixed by f, then there is at most one fixed point $b_2 \in \mathbb{S}^1$ of g such that $(a_1, b_2) \in M_h$.*

PROOF. Assume that there are two fixed points a_2, b_2 of g such that $(a_1, a_2) \in M_h$ and $(a_1, b_2) \in M_h$. According to Lemma 3.3.7, a_1 is hyperbolic, so up to replacing φ by φ^{-1} we can assume that a_1 is attractive for f. This implies that a_2 and b_2 are both repulsive for g, so g has non repulsive fixed points $c_2 \in \,]a_2\,, b_2[$ and $d_2 \in \,]b_2\,, a_2[$. Since they are not repulsive, $(a_1, c_2) \notin M_h$ and $(a_1, d_2) \notin M_h$. This is absurd because the set of $y \in \mathbb{S}^1$ such that $(a_1, y) \in M_h$ is connected. $\square$

PROOF OF PROPOSITION 5.1.4. We will distinguish different cases depending on the number of fixed points of f and g.

One fixed point: First, we assume that g has exactly one fixed point $y_0 \in \mathbb{S}^1$. The interval $h^{-1}(\{y_0\})$ is stabilised by f, so its endpoints are fixed by f. If f had a fixed point x outside of $h^{-1}(\{y_0\})$, then $h(x)$ is fixed by g, which is impossible, therefore f has no fixed point out of $h^{-1}(\{y_0\})$, and it is topologically conjugate to an example described in 1.2.

Two fixed points: Now, assume that both f and g have at least two fixed points. Let $a_1 \neq b_1$ be fixed points of f and $a_2 \neq b_2$ be fixed points of g.

First, let us assume that φ has no fixed point in M_h. In this case, non of the four points (a_1, a_2), (a_1, b_2), (b_1, a_2), (b_1, b_2) are in M_h. This implies that h is constant on both intervals $]a_1\,, b_1[$ and $]b_1\,, a_1[$, therefore f (as well as g) has exactly two fixed points, so it either has north/south dynamics (in which case it is topologically conjugate to a hyperbolic element of $\mathrm{PSL}(2,\mathbb{R})$), or it is topologically conjugate to a parabolic element of $\mathrm{PSL}_2(2,\mathbb{R})$.

We can now assume that φ has a fixed point in M_h. Up to renaming the fixed points, we can assume that $(a_1, b_2) \in M_h$. Lemma 3.3.7 implies that they are hyperbolic fixed points for f and g. If they each have only two fixed points, then they have north/south dynamics, and are topologically conjugate to a hyperbolic element in $\mathrm{PSL}(2,\mathbb{R})$. Up to replacing φ by φ^{-1}, we can assume that a_1 is attractive for f and b_2 is repulsive for g.

Assume that f has a third fixed point $c_1 \in\,]b_1\,, a_1[$. Because of Lemma 5.1.5, the points (b_1, b_2) and (c_1, b_2) are not in M_h, which implies that $h \equiv b_2$ on $]b_1\,, c_1[$. Up to replacing b_1 and c_1 by the edges of the interval $h^{-1}(\{b_2\})$ (which are fixed by f), we can assume that $\overline{h^{-1}(\{b_2\})} = [b_1\,, c_1]$. Lemma 5.1.5 implies that a_1 is the only fixed point of f in $]c_1\,, a_1[$, and that f has at most one fixed point in $]b_1\,, c_1[$, and that if it exists, then it is hyperbolic. This implies that f has either three or four fixed points and is topologically conjugate to one of the examples described in 1.1.

Notice that in this case (where f has at least three fixed points), g has only two fixed points (an extra fixed point would generate a point in M_h which cannot exist because of Lemma 5.1.5).

No fixed point: We have treated all cases where f and g have fixed points (note that since f and g are semi conjugate, if one has a fixed point then so does the other). They share the same rotation number $\alpha \in \mathbb{R}/\mathbb{Z}$. If they have no fixed points, then $\alpha \neq 0$.

If $\alpha = \frac{p}{q}$ is rational, then the number of fixed points of f^q and g^q are at least q and are multiples of q, and $(f^q, g^q) \in G_\omega$. This shows that if $f^q \neq Id$, then $\alpha = \frac{1}{2}$ (because either f^q or g^q has exactly two fixed points) and f^2 either has north/south dynamics, has four fixed points, or is topologically conjugate to a parabolic element of $\mathrm{PSL}_2(2,\mathbb{R})$. In the latter case, f is itself topologically conjugate to a parabolic element of $\mathrm{PSL}_2(2,\mathbb{R})$.

If f^2 has north/south dynamics, let a_1, b_1 be its fixed points. Then f conjugates f^2 on a neighbourhood of a_1 with f^2 on a neighbourhood of a_2, which is absurd because one is attractive and the other is repulsive.

If f^2 has some hyperbolic points (which is the case if it has four fixed points), then let a_1 be a hyperbolic periodic points of f and $b_1 = f(a_1)$. They satisfy $(f^2)'(a_1) = f'(b_1)f'(a_1) = (f^2)'(b_1)$. This is impossible since one is attractive for f^2 and the other is repulsive. Therefore, if f has no fixed points and $\alpha = \frac{p}{q} \in \mathbb{Q}$, then either $f^q = Id$, and f is topologically conjugate to a rotation, either f is topologically conjugate to a parabolic element of $\mathrm{PSL}_2(2,\mathbb{R})$.

If $\alpha \notin \mathbb{Q}$, then there are two possibilities: either f is topologically conjugate to a rotation, either f has an invariant Cantor set and is semi conjugate to a rotation (a Denjoy example, which cannot be the case if f is C^2, see **[Nav11]** for a thorough treatment of Denjoy diffeomorphisms). We are going to show that f and g are both topologically conjugate to a rotation.

Assume that f is a Denjoy example. Let $K \subset \mathbb{S}^1$ be the invariant Cantor set, and let I be a connected component of $\mathbb{S}^1 \setminus K$. Using the fact that $f^n(I) \cap f^p(I) = \emptyset$ if $n \neq p$, a simple calculation (see exercise 3.5.24 in [**Nav11**]) shows that:

$$\int_I \left(\sum_{n\in\mathbb{Z}} (f^n)'(x) \right) dx = \sum_{n\in\mathbb{Z}} |f^n(I)| \leq 1$$

In particular, the set of points $x \in \mathbb{S}^1$ such that $(f^n)'(x) \to 0$ as $n \to \infty$ is a set of full Lebesgue measure in $\mathbb{S}^1 \setminus K$, hence non empty. The points $y \in \mathbb{S}^1$ such that the sequence $((g^n)'(y))_{n\in\mathbb{Z}}$ is bounded form a non empty set ([**Her79**], chapter X) invariant under g.

Note that $G(g\circ h) \subset M_h$. Indeed, if h is continuous at x, then $g(h(x)) \neq h(x)$, and if h is discontinuous at x, then the interval $]h_l(x)\,, h_r(x)[$ is wandering for g, therefore cannot intersect its image $g(]h_l(x)\,, h_r(x)[)$. Since $\varphi^2 \in G_\omega$, we also have $G(g^2 \circ h) \subset M_h$. Up to replacing φ by φ^{-1}, we can assume that $K = \{(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1 | y \in [g(h_l(x))\,, g^2(h_r(x))]\}$ is a compact subset of M_h, invariant under φ, with non empty interior. Let $x \in \mathbb{S}^1$ be such that $(f^n)'(x) \to 0$ as $n \to \infty$, and let $y \in \mathbb{S}^1$ be such that the sequence $((g^n)'(y))_{n\in\mathbb{Z}}$ is bounded and $(x, y) \in K$. Let $M = \max_K \omega$ and $m = \min_K \omega$. We then have:

$$\underbrace{(f^n)'(x)(g^n)'(y)}_{\to 0} = \frac{\omega(x, y)}{\omega(f^n(x), g^n(y))} \in \left[\frac{m}{M}, \frac{M}{m}\right]$$

This is a contradiction, therefore f cannot be a Denjoy example, so it is topologically conjugate to a rotation, which is an element of $\mathrm{PSL}(2, \mathbb{R})$. □

2. Non elementary examples

We are now going the prove the following :

Theorem 5.2.1. *There are $h : \mathbb{S}^1 \to \mathbb{S}^1$ non decreasing of degree one and ω a continuous volume form on M_h such that G_ω contains a free group $\mathbb{F}_2$, and such that there is no topological conjugacy between $\rho_1(G_\omega)$ and any subgroup of any finite cover of* $\mathrm{PSL}(2, \mathbb{R})$.

The main idea in constructing a non elementary example consists in starting with an appropriately chosen representation $\rho_1 : \mathbb{F}_3 \to \mathrm{PSL}(2, \mathbb{R})$, then considering $\rho_2 : \mathbb{F}_3 \to \mathrm{Diff}(\mathbb{S}^1)$ where we only modified one of the generators by introducing a parabolic fixed point (just as in the elementary case). Proposition 5.1.1 shows that the images of the generators by (ρ_1, ρ_2) each preserve a volume form, and the difficulty consists in showing that we can find one that is preserved by all three. The proof is almost identical to the proof of Theorem 4.1.10.

2.1. Choice of ρ_1 and construction of ρ_2. Let T be the twice punctured torus. Its fundamental group is the free group on three generators $\mathbb{F}_3 = \langle a, b, c\rangle$. Given a complete hyperbolic structure on T such that neighbourhoods of the omitted points have infinite volume, we obtain a convex cocompact representation $\rho_1 : \mathbb{F}_3 \to \mathrm{PSL}(2, \mathbb{R})$.

Let δ_1 and δ_2 be the simple loops going around the omitted points (see Figure 5.5). If I is a connected component of $\mathbb{S}^1 \setminus L_{\rho_1(\mathbb{F}_3)}$ and γ stabilises I, then γ is conjugate to a power of δ_1 or δ_2. It is explained in [**But00**] how the generators can be chosen in a way the $\delta_1 = abc$ and $\delta_2 = cba$. This shows that a, b, δ_1 freely generate $\pi_1(T)$.

Let $N, S \in \mathbb{S}^1$ be the fixed points of $\rho_1(\delta_1)$, such that $]N\,, S[$ is a connected component of $\mathbb{S}^1 \setminus L_{\rho_1(\mathbb{F}_3)}$. Let $f \in \mathrm{Diff}(\mathbb{S}^1)$ be equal to $\rho_1(a)$ on $[S\,, N]$, and have exactly one fixed point P_a in $]N\,, S[$. We define $\rho_2 : \mathbb{F}_3 \to \mathrm{Diff}(\mathbb{S}^1)$ by $\rho_2(a) = \rho_1(a)$, $\rho_2(b) = \rho_1(b)$ and $\rho_2(\delta_1) = f$.

Since f coincides with $\rho_1(\delta_1)$ on the limit set, we see that $\rho_2(\gamma)$ and $\rho_1(\gamma)$ coincide on $L_{\rho_1(\mathbb{F}_3)}$ for all $\gamma \in \mathbb{F}_3$. More precisely, they are equal everywhere except on intervals that are bounded by images of N and S. Therefore, for any $\gamma \in \mathbb{F}_3$ that is not a power of a conjugate of δ_1, $\rho_2(\gamma)$ has two fixed points, which are hyperbolic.

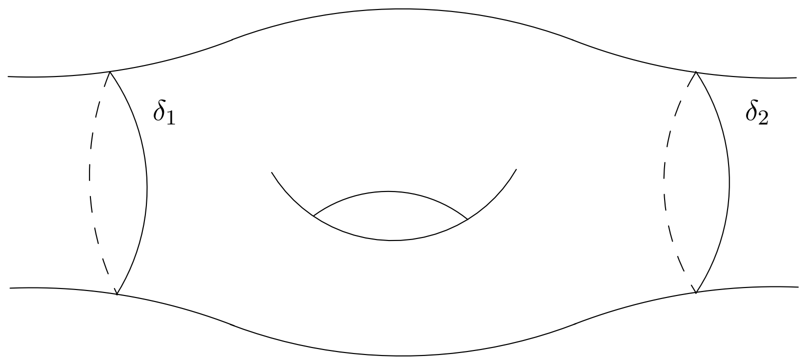

FIGURE 5.5. The twice punctured torus

Lemma 5.2.2. $L_{\rho_2(\mathbb{F}_3)} = L_{\rho_1(\mathbb{F}_3)}$.

PROOF. The compact set $L_{\rho_1(\mathbb{F}_3)}$ is invariant under ρ_2, which shows $L_{\rho_2(\mathbb{F}_3)} \subset L_{\rho_1(\mathbb{F}_3)}$ because of the uniqueness of the minimal compact invariant set. Since the actions on $L_{\rho_1(\mathbb{F}_3)}$ are equal, the orbits are dense and $L_{\rho_2(\mathbb{F}_3)} = L_{\rho_1(\mathbb{F}_3)}$. □

We will denote this set by $L = L_{\rho_1(\mathbb{F}_3)} = L_{\rho_2(\mathbb{F}_3)}$.

Lemma 5.2.3. *The representations ρ_1 and ρ_2 are semi conjugate by a map that is the identity on L.*

PROOF. We start by setting h to be the identity on L. Let $I_1 =]N, S[$ be the connected component of $\mathbb{S}^1 \setminus L$ stabilised by δ_1.

Set $h : I_1 \to I_1$ to be a semi conjugacy between $\rho_1(\delta_1)$ and $\rho_2(\delta_1)$, and extend h on images $\rho_1(\gamma)(I_1)$ for $\gamma \in \mathbb{F}_3$ by $h_{/\rho_1(\gamma)(I_1)} = \rho_2(\gamma) \circ h_{/I_1} \circ \rho_1(\gamma^{-1})$.

Finally, set h to be the identity on the other connected components of $\mathbb{S}^1 \setminus L$. It provides a semi conjugacy between ρ_1 and ρ_2 that is the identity on L. □

Given $h : \mathbb{S}^1 \to \mathbb{S}^1$ non increasing of degree one such that h is the identity on L and such that (ρ_1, ρ_2, h) is a semi conjugate triple, we wish to show that it is area preserving.

Since semi conjugacy is an equivalence relation, we can also consider $h^* : \mathbb{S}^1 \to \mathbb{S}^1$ non decreasing of degree one such that $\rho_1 \circ h^* = h^* \circ \rho_2$. Since we can construct it in the same way as for h, we can assume that h^* is the identity on L.

Remark. Such a construction is not possible if we start with a Fuchsian representation of the fundamental group of a compact surface, because the limit set is the whole circle. This implies that we have to work with a surface of finite type, i.e. a closed surface of genus $g \geq 0$ with r omitted points, so that $2g + r > 2$ (to ensure the existence of a hyperbolic structure, and that the representation in $\mathrm{PSL}(2, \mathbb{R})$ is non elementary). In the construction of ρ_2, we used two technical properties of ρ_1. First, it is important to work with a free group, so that a choice of images of the generators in $\mathrm{Diff}(\mathbb{S}^1)$ always corresponds to a representation of the fundamental group considered. This is always the case with a non compact surface of finite type. The second important condition is that we can choose a system of generators so that one of the generators represents a simple loop around one of the omitted points. This is not possible with a simple punctured torus (where the fundamental group is $\mathbb{F}_2 = \langle a, b\rangle$ and the simple loop around the omitted point is the commutator $[a, b] = a^{-1}b^{-1}ab$). This is why we used the twice punctured torus, but the sphere with three omitted points (i.e. a pair of pants) could also have been used. More generally, this construction works for a sphere with three or more removed points, or a closed surface of genus $g \geq 2$ with two or more removed points.

2.2. The flow associated to (ρ_1, ρ_2)**.** We consider the following three manifold:

$$\Sigma_h = \{(a, b, c) \in (\mathbb{S}^1)^3 | a < b < z < a \ \forall z \in h^{-1}(\{c\})\}$$

The group $\mathbb{F}_3$ acts on Σ_h by $\gamma.(a, b, c) = (\rho_1(\gamma)(a), \rho_1(\gamma)(b), \rho_2(\gamma)(c))$.

Proposition 5.2.4. *The action of* $\mathbb{F}_3$ *on* Σ_h *is properly discontinuous.*

PROOF. Assume that $\gamma_k \to \infty$ and that there is a sequence $(a_k, b_k, c_k) \to (a, b, c) \in \Sigma_h$ such that $\gamma_k.(a_k, b_k, c_k) \to (u, v, w) \in \Sigma_h$. Up to a subsequence and up to replacing γ_k with γ_k^{-1}, we can assume that $\rho_1(\gamma_k)(x) \to v$ for all $x \neq u$.

Assume that a does not bound an interval where $\rho_1 \neq \rho_2$. In that case, one can find a compact interval $K \subset \mathbb{S}^1$ bounded by points of L such that $a \notin K$ and $c \in \mathring{K}$. The sequence of intervals $\rho_1(\gamma_k)(K)$ collapses to the point $\{v\}$. Since $\rho_2(\gamma_k)(K) = \rho_1(\gamma_k)(K)$ and $c_k \in K$, we see that $\rho_2(\gamma_k)(c_k) \to v$, hence $w = v = h(v)$, which is absurd because $(u, v, w) \in \Sigma_h$.

We now know that $u = N$ where $I =]N, S[$ is an interval bounded by fixed points for a conjugate δ of δ_1, such that the third fixed point P is in I. If c_k were not in I for k large enough, then we could still find a compact interval K as above, which is impossible, hence $c_k \in I$ for k large enough. This implies that γ_k stabilizes I, which gives us $v = S$ and $w = P$. This is also impossible because $(N, S, P) \notin \Sigma_h$ by construction of h. $\square$

We can now consider the quotient manifold $N_h = \Sigma_h/\mathbb{F}_3$.

2.3. Invariant volume on the hyperbolic set. The projection on N_h of the constant vector field $(0, 1, 0)$ on Σ_h can be reparametrised into a smooth flow ψ^t. Consider the map $\widetilde{H}^* : \Sigma_h \to \Sigma_3$ defined by $\widetilde{H}^*(x, y, z) = (x, y, h^*(z))$. It induces a map $H^* : N_h \to M = \mathrm{T}^1\mathbb{H}^2/\rho_1(\mathbb{F}_3)$. Its restriction to Ω_ψ is a diffeomorphism onto Ω_φ that sends ψ^t to a reparametrisation of φ^t. From this we deduce that Ω_ψ is a compact hyperbolic set for ψ^t. If the image $x \in N_h$ of $(x_-, x_0, x_+) \in \Sigma_h$ is in Ω_ψ, then the stable (resp. unstable) manifold of x is the set of images of points (y_-, y_0, y_+) such that $y_+ = x_+$ (resp. $y_- = x_-$).

Once again, we will use this flow in order to extend the volume form to the stable and unstable manifolds of the non wandering set.

Lemma 5.2.5. *There is a continuous volume form* ω_1 *on* N_h *that is invariant under* ψ^t *at points of* $W^s(\Omega_\psi) \cup W^u(\Omega_\psi)$.

PROOF. The differentiable conjugacy on the non wandering set implies that there is a smooth volume form ω_0 on N_h that is preserved by the flow at points of the non wandering set. Hence, if $\psi^{t*}\omega_0 = e^{-A(t,x)}\omega_0$ and $\alpha(x) = \frac{\partial A}{\partial t}(0, x)$, then $\alpha = 0$ on Ω_ψ. We will now construct a smooth function σ on N_h such that $\sigma(\psi^t(x)) - \sigma(x) = \int_0^t \alpha(\psi^s(x))ds$ for all $x \in W^s(\Omega_\psi) \cup W^u(\Omega_\psi)$, so that $\omega_1 = e^\sigma\omega_0$ meets our requirements.

If $x \in W^s(z)$ with $z \in \Omega_\psi$, and if we have found such a function σ, then $\sigma(\psi^t(x)) \approx \sigma(\psi^t(z)) = 0$ for t large enough, hence $\sigma(x) = -\int_0^\infty \alpha(\psi^t(x))dt$. We will use this formula as a definition of σ. If it is well defined, then it satisfies the cohomology equation.

Let $C > 0$ be such that $d(\psi^t(x), \psi^t(z)) \leq Ce^{-t}$ (locally C can be chosen independently from x and z). Let k be a Lipschitz constant for α in a neighbourhood U of Ω_ψ. For t such that $\psi^t(x) \in U$ (which is locally uniform in x), we have:

$$|\alpha(\psi^t(x))| \leq \underbrace{|\alpha(\psi^t(z))|}_{=0} + k\,\underbrace{d(\psi^t(x), \psi^t(z))}_{\leq Ce^{-t}}$$

This gives us uniform convergence, hence σ is well defined and continuous. By applying the same reasoning with negative times, we define σ on $W^u(\Omega_\psi)$. $\square$

2.4. Going back from the flow to M_h. Now that we have found an invariant volume form on a larger set for the flow ψ^t, we need to translate it in terms of the action on M_h.

Lemma 5.2.6. *If there is a continuous volume form v on N_h preserved by ψ^t at points of $W^s(\Omega_\psi) \cup W^u(\Omega_\psi)$, then there is a continuous volume form ω on M_h preserved by (ρ_1, ρ_2) at points of $L \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L$.*

PROOF. Let $\omega_1 = e^\sigma \omega_0$ be a continuous volume form on N_h that is invariant at points of $W^s(\Omega_\psi) \cup W^u(\Omega_\psi)$. Let $\widetilde{\omega}_1$ be its lift to Σ_h and write:

$$\widetilde{\omega}_1 = \widetilde{\omega}_1(x_-, x_0, x_+) dx_- \wedge dx_0 \wedge dx_+$$

If x_- or x_+ is in L, then the image in N_h is in $W^s(\Omega_\psi) \cup W^u(\Omega_\psi)$, and the invariance under the flow ψ^t gives us $\widetilde{\omega}_1(x_-, x_0, x_+) = \widetilde{\omega}_1(x_-, x_0', x_+)$ for all x_0' such that $(x_-, x_0', x_+) \in \Sigma_h$.

Choose a continuous map $i_0 : M_h \to \mathbb{S}^1$ such that $(x_-, i_0(x_-, x_+), x_+) \in \Sigma_h$ for all $(x_-, x_+) \in M_h$, and let $\omega_2(x_-, x_+) = \widetilde{\omega}_1(x_-, i_0(x_-, x_+), x_+)$ for $(x_-, x_+) \in M_h$. If x_- or x_+ is in L and $\gamma \in \mathbb{F}_3$, then the invariance under ψ^t gives us:

$$\begin{aligned}
& \omega_2(\rho_1(\gamma)(x_-), \rho_2(\gamma)(x_+))\rho_1(\gamma)'(x_-)\rho_2(\gamma)'(x_+) \\
= \; & \widetilde{\omega}_1(\rho_1(\gamma)(x_-), i_0(\rho_1(\gamma)(x_-), \rho_2(\gamma)(x_+)), \rho_2(\gamma)(x_+))\rho_1(\gamma)'(x_-)\rho_2(\gamma)'(x_+) \\
= \; & \widetilde{\omega}_1(\rho_1(\gamma)(x_-), \rho_1(\gamma)(i_0(x_-, x_+)), \rho_2(\gamma)(x_+))\rho_1(\gamma)'(x_-)\rho_2(\gamma)'(x_+) \\
= \; & \widetilde{\omega}_1(x_-, i_0(x_-, x_+), x_+) \\
= \; & \omega_2(x_-, x_+)
\end{aligned}$$

We have defined a continuous volume form ω_2 on M_h that is (ρ_1, ρ_2)-invariant at points of $(L \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L) \cap M_h$. $\square$

2.5. Horizontal strips. The first step in extending ω to all of M_h is to extend it to horizontal strips delimited by elements of L, so that we only need to deal with invariance under one element of the group.

Lemma 5.2.7. *Let I be a connected component of $\mathbb{S}^1 \setminus L$, and let $\gamma \in \mathbb{F}_3$ be a generator of its stabilizer. There is a continuous volume form ω on $\mathbb{S}^1 \times \overline{I} \setminus G(h)$ that is invariant by $(\rho_1(\gamma), \rho_2(\gamma))$ and that is equal to ω_2 on $L \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L$.*

PROOF. If γ is conjugate to δ_1, then Proposition 5.1.1, states that there is a continuous volume form ω_γ on M_h that is invariant under $(\rho_1(\gamma), \rho_2(\gamma))$. If γ is conjugate to δ_2, then Proposition 4.1.6 gives the same result.

Let $a \in L \setminus \overline{I}$. The interval $[a\,, \rho_1(\gamma)(a)[$ is a fondamental domain for the action of $\rho_1(\gamma)$ on $\mathbb{S}^1 \setminus \overline{I}$, i.e. for every $y \in \mathbb{S}^1 \setminus \overline{I}$ there is a unique $n_y \in \mathbb{Z}$ such that $\rho_1(\gamma^{n_y})(y) \in [a\,, \rho_1(\gamma)(a)[$. We set $\omega = \omega_2$ on $[a\,, \rho_1(\gamma)(a)[\times \overline{I}$ and extend ω to $(\mathbb{S}^1 \setminus \overline{I}) \times \overline{I}$ by using the equivariance formula:

$$\frac{\omega(x, y)}{\omega_2(\rho_1(\gamma^{n_y})(x), \rho_2(\gamma^{n_y})(y))} = \rho_1(\gamma^{n_y})'(x)\rho_2(\gamma^{n_y})'(y)$$

We have to show that ω is continuous. First, remark that it is continuous at every point of $[a\,, \rho_1(\gamma)(a)[\times \overline{I}$: if $(x_n, y_n) \to (a, y)$ with $\rho_1(\gamma)(x_n) \in [a\,, \rho_1(\gamma)(a)[$, then using the fact that $a \in L$ and because the volume ω_2 is preserved at (a, y), we get:

$$\begin{aligned}
\omega(x_n, y_n) &= \omega_2(\rho_1(\gamma)(x_n), \rho_1(\gamma)(y_n))\rho_1(\gamma)'(x_n)\rho_1(\gamma)'(y_n) \\
&\to \omega_2(\rho_1(\gamma)(a), \rho_1(\gamma)(y))\rho_1(\gamma)'(a)\rho_1(\gamma)'(y) \\
&= \omega_2(a, y) = \omega(a, y)
\end{aligned}$$

If $(x_k, y_k) \in I \times \overline{I} \to (x, y) \in M_h$ with $x \in \partial I$, then set $n_k = n_{x_k}$, as well as $u_k = \rho_1(\gamma^{n_k})(x_k)$ and $v_k = \rho_1(\gamma^{n_k})(y_k)$. By definition, we have:

$$\omega(x_k, y_k) = \omega_2(u_k, v_k)\rho_1(\gamma^{n_k})'(x_k)\rho_1(\gamma^{n_k})'(y_k)$$

Since ω_γ is invariant under $\rho_1(\gamma)$, we have:

$$\rho_1(\gamma^{n_k})'(x_k)\rho_1(\gamma^{n_k})'(y_k) = \frac{\omega_\gamma(x_k, y_k)}{\omega_\gamma(u_k, v_k)}$$

These two equalities give us:

$$\omega(x_k, y_k) = \frac{\omega_2(u_k, v_k)}{\omega_\gamma(u_k, v_k)}\omega_\gamma(x_k, y_k)$$

The continuity of ω_γ gives us $\omega_\gamma(x_k, y_k) \to \omega_\gamma(x, y)$.

Since $x_k \to x \in \partial I$, we have $n_k \to \infty$ and $v_k \to v$ where v is the other extremal point of I. By using the uniform continuity of ω_2 and ω_γ on $[a\,, \rho_1(\gamma)(a)] \times \overline{I}$, we obtain:

$$\omega(x_k, y_k) \sim \frac{\omega_2(u_k, v)}{\omega_\gamma(u_k, v)}\omega_\gamma(x, y)$$

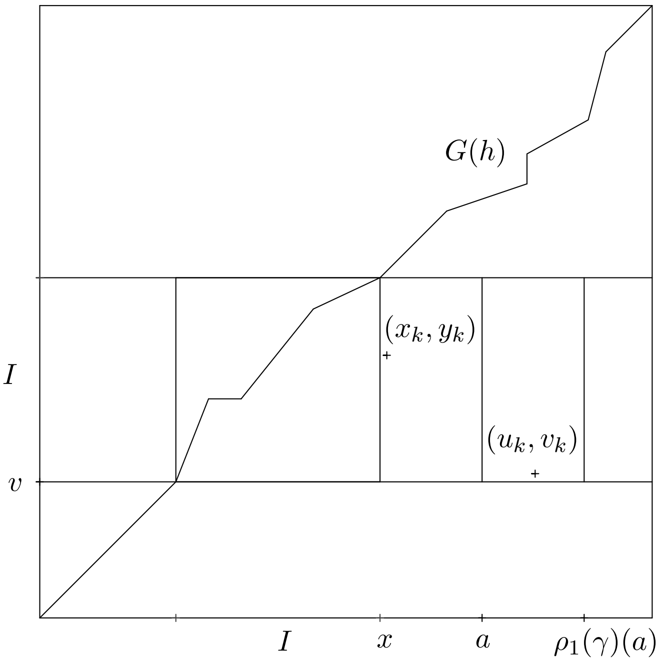

FIGURE 5.6. Defining ω on horizontal strips

We now only have to deal with the restrictions of ω_2 and ω_γ to the axes $\{x\}\times\mathbb{S}^1\cup\mathbb{S}^1\times\{v\}$ (see Figure 5.6), where continuous volume forms invariant under $(\rho_1(\gamma), \rho_2(\gamma))$ are unique up to multiplication by a constant: there is $\lambda > 0$ such that $\omega_2(s, t) = \lambda\omega_\gamma(s, t)$ whenever $s = x$ or $t = v$. We can finally conlude:

$$\omega(x_k, y_k) \to \lambda\omega_\gamma(x, y) = \omega_2(x, y) = \omega(x, y)$$

Finally, we can extend ω to $\mathbb{S}^1 \times \overline{I} \setminus G(h)$ by setting $\omega = \lambda\omega_\gamma$ on $\overline{I} \times \overline{I} \setminus G(h)$. $\square$

2.6. Extending to M_h. We can now extend ω to M_h. Getting an invariant volume form is not complicated, however its regularity requires some work.

Our proof of the regularity of ω on horizontal strips relied on the existence of a continuous invariant form by any element of $\mathbb{F}_3$. To deal with the invariance under the whole group, we will need a different method.

Proposition 5.2.8. *There is a continuous invariant form ω on M_h that is invariant under (ρ_1, ρ_2) and that is equal to ω_2 on $L \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L$.*

PROOF. The action of $\mathbb{F}_3$ on the set of connected components of $\mathbb{S}^1 \setminus L$ has two orbits. Let I_1, I_2 be the components preserved by δ_1 and δ_2. By Lemma 5.2.7, there is a continuous volume form ω on $\mathbb{S}^1 \times \overline{I}_i \setminus G(h)$ that is equal to ω_2 in restriction to $L \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L$ and that is invariant under the stabilizer of I_i. If $\gamma \in \mathbb{F}_3$, then we define ω on $\mathbb{S}^1 \times \rho_2(\gamma)(\overline{I}_i) \setminus G(h)$ to be $(\rho_1(\gamma), \rho_2(\gamma))_*\omega$. This defines a volume form ω on M_h that is (ρ_1, ρ_2)-invariant, continuous on all horizontal strips $\mathbb{S}^1 \times \overline{I} \setminus G(h)$ where I is a connected component of $\mathbb{S}^1 \setminus L$ and equal to ω_2 on $L \times \mathbb{S}^1 \cup \mathbb{S}^1 \cup L$.

To show that ω is continuous, assume that $(x_k, y_k) \to (x, y)$ with $y \in L$ (if $y \notin L$, then there is a connected component I of $\mathbb{S}^1 \setminus L$ such that $y_k \in I$ for k large enough, which gives us $\omega(x_k, y_k) \to \omega(x, y)$). If $y_k \in L$ for all k, then $\omega(x_k, y_k) = \omega_2(x_k, y_k)$ and we already have the continuity, hence we can assume that $y_k \notin L$ for all k. Up to considering two subsequences, we can assume that there is $\gamma_k \in \mathbb{F}_3$ such that $v_k = \rho_2(\gamma_k)(y_k) \in I_1$. By composing γ_k with an element of the stabilizer of I_1, we can take v_k in a compact interval $K \subset I_1$.

Let $u_k = \rho_1(\gamma_k)(x_k)$. The definition of ω is:

$$\omega(x_k, y_k) = \omega(u_k, v_k)\rho_1(\gamma_k)'(x_k)\rho_2(\gamma_k)'(y_k)$$

We have already seen that ω is continuous on $\mathbb{S}^1 \times \overline{I}_1 \setminus G(h)$ and $v_k \in I_1$. The problem in finding the limit of $\omega(x_k, y_k)$ is the control of the Jacobian product $\rho_1(\gamma_k)'(x_k)\rho_2(\gamma_k)'(y_k)$. However, we know that ω is continuous on $L \times \mathbb{S}^1 \cup \mathbb{S}^1 \times L$. We will use this fact to get rid of the derivatives: if x'_k and y'_k are sequences in L such that $(x'_k, y'_k) \notin G(h)$, $(x'_k, y_k) \notin G(h)$ and $(x_k, y'_k) \notin G(h)$, then we set $u'_k = \rho_1(\gamma_k)(x'_k)$ and $v'_k = \rho_2(\gamma_k)(y'_k)$. The equivariance equation for ω gives us:

$$\frac{\omega(x_k, y_k)}{\omega(x_k, y'_k)} \frac{\omega(x'_k, y'_k)}{\omega(x'_k, y_k)} = \frac{\omega(u_k, v_k)}{\omega(u_k, v'_k)} \frac{\omega(u'_k, v'_k)}{\omega(u'_k, v_k)} \tag{4}$$

We are now looking for suitable points x'_k and y'_k. Let $I_1 =]a\,, b[$, and assume that u_k does not admit a as a limit point (up to considering two subsequences and replacing a by b in the following discussion, we can always assume that it is the case), i.e. that a_k lies in a compact interval $J \subset \mathbb{S}^1 \setminus \{a\}$. Let $v'_k = a$ and $y'_k = \rho_2(\gamma_k^{-1})(a) \to y$. If $x_k \in L$, then we choose $x'_k = x_k$. If $x_k \notin L$, then we set x'_k to be an extremal point of the connected component of $\mathbb{S}^1 \setminus L$ containing x_k, in a way such that $u'_k = \rho_1(\gamma_k)(x'_k) \in J$.

We now have $y'_k \to y$ and $y_k \to y$, so we get:

$$\frac{\omega(x_k, y_k)}{\omega(x_k, y'_k)} \frac{\omega(x'_k, y'_k)}{\omega(x'_k, y_k)} \sim \frac{\omega(x_k, y_k)}{\omega(x, y)} \frac{\omega(x'_k, y)}{\omega(x'_k, y)} = \frac{\omega(x_k, y_k)}{\omega(x, y)}$$

We wish to show that this quantity converges to 1 as $k \to \infty$. The compact set $E = J \times \{b\} \cup \mathbb{S}^1 \setminus I_1 \times K$ of M_h contains the sequences (u_k, v_k), (u_k, v'_k), (u'_k, v_k) and (u'_k, v'_k). Consequently, the ratio (4) lies in a compact set of $]0\,, +\infty[$, and it is enough to see that its only possible limit is 1. If there is a subsequence such that the ratio (4) converges to $\lambda \in]0\,, +\infty[$, then up to another subsequence, we can assume that the sequence γ_k has the convergence property: there are $N, S \in \mathbb{S}^1$ such that $\rho_1(\gamma_k)(z) \to N$ for all $z \neq S$. Since $\rho_1(\gamma_k^{-1})(z) \to x$ for all $z \in I_1$, we see that S in necessarily equal to

x, hence the sequences v_k and v'_k converge to $N \in \mathbb{S}^1$. We get:

$$\frac{\omega(u_k, v_k)}{\omega(u_k, v'_k)} \frac{\omega(u'_k, v'_k)}{\omega(u'_k, v_k)} \to \frac{\omega(u, N)}{\omega(u, N)} \frac{\omega(a, N)}{\omega(a, N)} = 1$$

This shows that $\lambda = 1$, therefore $\omega(x_k, y_k) \to \omega(x, y)$ and ω is continuous. $\square$

We only showed that ω is continuous, but just as in the proof of 4.1.10, one can show that it is possible to obtain smooth volume form.

Part 2

Multi valued dynamics and causality

CHAPTER 6

Multi valued dynamical systems

1. Definitions

A **discrete dynamical system** is a map $\varphi : X \to X$, usually a bijection, and preserving some structure (e.g. a measure, a topology, a geometric structure, etc…). This type of dynamical system is single valued: to one point, we associate exactly one other point. A **multi valued dynamical system** is a map $\varphi : X \to \mathfrak{P}(X)$, i.e. to one point $x \in X$ we associate a set $\varphi(x) \subset X$. A typical case where this type of consideration is natural is for random perturbations of a discrete dynamical system: if f is a homeomorphism of the metric space X and $\varepsilon > 0$, then we can consider the multi valued system $F_\varepsilon(x) = B(f(x), \varepsilon)$.

A **continuous dynamical system** is a flow $\varphi : X \times \mathbb{R} \to X$, i.e. an action of the group $(\mathbb{R}, +)$. If X is a differentiable manifold and φ is smooth, then there is a unique vector field V such that φ is the solution of the differential equation $\frac{d}{dt}\varphi^t(x) = V(\varphi^t(x))$ with initial condition $\varphi^0(x) = x$. The equivalent notion for multi valued system is the notion of **differential inclusion**: to a point $x \in X$ we associate a subset $C(x)$ of the tangent space T_xX. A differential inclusion can be seen as a differential equation with uncertainty on the vector field. We will define **integral curves** of C as locally Lipschitz curves $\gamma : I \to X$ (where $I \subset \mathbb{R}$ is an interval) such that $\dot\gamma(x) \in C(\gamma(x))$ for almost every $x \in I$. We choose Lipschitz regularity over C^1 curves for its nice behaviour when considering converging sequences of curves.

2. Lorentzian conformal classes

If (M, g) is a time oriented Lorentz manifold, then let $C(x)$ be the set of future oriented causal vectors tangent at $x \in M$. Integral curves are exactly future directed causal curves (in the topological sense). The data of C is equivalent to the data of the conformal structure $[g]$ (two non degenerate quadratic forms of non definite signature share the same isotropic cone if and only if they differ from a multiplicative constant).

Starting from a point $x \in M$, the set of endpoints of integral curves starting at x is the causal future $J^+(x)$.

If we compare this situation to the classical setting of a vector field on a manifold, we realise that we are missing something: special parametrisations of integral curves. Since we consider all causal curves, as opposed to only timelike curves, we cannot use Lorentzian arc length. For this purpose, we will always consider an auxiliary Riemannian metric h on M, and denote by $J_t^+(x)$ the subset of M consisting of endpoints $\gamma(1)$ of future directed causal curves $\gamma : [0\,,1] \to M$ such that $\gamma(0) = x$ and $\ell_h(\gamma) > t$.

When there is an ambiguity about the Riemannian metric, we will call this set $J_{t,h}^+(E)$. Let us remark that this definition depends strongly on the auxiliary Riemmanian metric, and so will the definitions to come. With a little more work and some slightly different definitions, some of these notions would not depend on such a choice (if we replace constants T by positive continuous functions then we avoid the scale problem), but it is not necessary for the applications that we propose. As we will see later, it will be important to choose a Riemannian metric with some special properties.

If E is a subset of M, then we let $J_t^+(E) = \bigcup_{x\in E} J_t^+(x)$. If $0 < t < T$, we will also consider the set $J^+_{t,T}(x)$ of endpoints $\gamma(1)$ of future directed causal curves $\gamma : [0\,,1] \to M$ such that $\gamma(0) = x$ and $t < \ell_h(\gamma) < T$.

3. Time function vs Lyapunov function

A time function in Lorentzian geometry is a function that increases along a future directed causal curve. Hawking's Theorem states that the existence of such a function is equivalent to stable causality. A Lyapunov function in classical dynamical systems is a function that is non increasing along orbits, and decreasing along certain orbits (physically it is seen as an energy function). One of the main differences is that in dynamical systems, a flow that carries a function decreasing along all orbits is said to have poor dynamics, and isn't very interesting. On the opposite, a Lorentzian manifold is physically relevant if it satisfies a certain number of causality conditions. Since the goal of our work is to use the techniques of construction of Lyapunov functions in order to construct time functions, our work will take place in a general setting with no particular causality conditions. Therefore we cannot construct time functions right away, and we need to define some notion of partial time function.

Definition 6.3.1. *Let (M, g) be a spacetime (i.e. a time oriented Lorentzian manifold), and E a subset of M. We call E-time function a continuous map $\tau : M \to \mathbb{R}$ such that :*

(1) $\forall x \in M\ \forall y \in J^+(x) \quad \tau(y) \geq \tau(x)$
(2) $\forall x \in E\ \forall y \in J^+(x) \setminus \{x\} \quad \tau(y) > \tau(x)$

In **[Con88]**, Charles Conley proved that the existence of Lyapunov functions is related to attractors and chain recurrence. His work was done in the case of a flow on a compact metric space, and it was extended to separable metric spaces by Hurley (see **[Hur92]**,**[Hur95]**,**[Hur98]**, and **[CCP02]** for some corrections). Since compact Lorentzian manifolds are not physically relevant (they cannot be causal), we will not restrict ourselves to the compact case. It implies changing a few definitions, in some places constants must be replaced by continuous functions $\varepsilon : M \to]0\,,+\infty[$.

It is noteworthy that Hawking's method for constructing time functions could not provide E-time functions, as some causality is important in order to prove the continuity of the function in **[HE73]**.

4. Causality and time functions

If (M, g) is a spacetime, recall that the **chronological future** $I^+(p)$ of a point $p \in M$ is the set of endpoints of future directed timelike curves starting at p. Its **chronological past** $I^-(p)$ is the set of endpoints of past directed timelike curves starting at p.

The **causal future** $J^+(p)$ (resp. **causal past** $J^-(p)$) is the set of endpoints of future directed (resp. past directed) causal curves starting at p.

Let us state a few basic properties for the chronological and causal futures (see **[Pen72]** for a proof).

Proposition 6.4.1. *If $y \in J^+(x)$, then $J^+(y) \subset J^+(x)$ and $I^+(y) \subset I^+(x)$.*
The chronological future $I^+(p)$ is open.

The causal future is not necessarily closed, but we always have the double inclusion $I^+(p) \subset J^+(p) \subset \overline{I^+(p)}$.

A spacetime (M, g) is said to be **chronological** if there is no closed timelike curve, i.e. if $p \notin I^+(p)$ for all $p \in M$. We say that (M, g) is **causal** if there is no closed causal curve. The first implication of the chronological character of a spacetime concerns its topological nature.

Proposition 6.4.2. *If (M, g) is chronological, then M is not compact.*

Proof. Consider a time orientation T on M, i.e. a timelike vector field, and let φ^t be its flow. If M is compact, then the flow φ^t has recurrent points. Let $p \in M$ be recurrent. Since $I^-(p)$ is open, it contains $\varphi^{t_0}(p)$ for some $t_0 > 0$. This shows that $p \in I^+(\varphi^{t_0}(p)) \subset I^+(p)$, therefore (M, g) is not chronological. □

We say that (M, g) is **strongly causal** if every $p \in M$ has arbitrarily small neighbourhoods U such that the intersection of U with a causal curve is always connected. Such a neighbourhood U is called **causally convex**. Since small neighbourhoods do not contain closed causal curves (one can find charts such that the first coordinate is increasing along future directed causal curves), a strongly causal spacetime is causal.

Let us define the partial order $<$ on the set of Lorentz metrics on M by $g < g'$ if every non zero causal vector for g is timelike for g' (i.e. if the light cone of g' is larger that the light cone of g). We say that (M, g) is **stably causal** if there is $g < g'$ such that (M, g) is causal.

Proposition 6.4.3. *A stably causal spacetime is strongly causal.*

Proof. Let (M, g) be a stably causal spacetime, and let $p \in M$. Let $g' > g$ be a causal metric. Let U be a chart around p with coordinates $(t, x_1, \dots, x_{n-1})$ such that $g_p = -dt^2 + dx_1^2 + \dots + dx_{n-1}^2$ (such coordinates can be obtained with the exponential map). For $\alpha > 0$, we denote by g_α the metric $-\alpha dt^2 + dx_1^2 + \cdots dx_{n-1}^2$ on U. If $\alpha' > \alpha > 1$ and α' is sufficiently close to 1, then $g < g_\alpha < g_{\alpha'} < g'$ on U.

For $q \in U$ and h a Lorentz metric on U, we denote by $I^\pm_{U,h}(q)$ the chronological past and future of q in the spacetime (U, h).

Assume that $p = (0, \dots, 0)$ in coordinates, and let $p_+ = (\delta, 0, \dots, 0)$ for $\delta > 0$. Let $\varepsilon > 0$ be small enough so that every future (resp. past) directed causal curve starting at p and escaping U meets the level $\{t = \varepsilon\}$ (resp. $\{t = -\varepsilon\}$).

If $\delta > 0$ is small enough so that $(\delta + \varepsilon)^2 < \frac{\alpha'}{\alpha}\varepsilon^2$, then we obtain:

$$I^-_{U,g_\alpha}(p_+) \cap \{t = -\varepsilon\} \subset I^-_{U,g_{\alpha'}}(p) \cap \{t = -\varepsilon\}$$

Since $g < g_\alpha < g_{\alpha'} < g'$, this implies that:

$$I^-_{U,g}(p_+) \cap \{t = -\varepsilon\} \subset I^-_{U,g'}(p) \cap \{t = -\varepsilon\}$$

If γ is a future directed causal curve (for g), and $\gamma(1) = p_+$, then there is $t < 1$ such that $\gamma(t) \in I^-_{g'}(p)$ (see Figure 6.1).

Similarly, if δ is small enough, then the point $p_- = (-\delta, 0, \dots, 0)$ satisfies:

$$I^+_{U,g}(p_-) \cap \{t = \varepsilon\} \subset I^+_{U,g}(p) \cap \{t = \varepsilon\}$$

If γ is a future directed causal curve (for g), and $\gamma(0) = p_-$, then there is $t > 0$ such that $\gamma(t) \in I^+_{g'}(p)$.

Now let $W = I^-_{U,g}(p_+) \cap I^+_{U,g}(p_-)$. If γ were a closed causal curve whose intersection with W is disconnected, then we can assume that $\gamma(0) = p_-$ and $\gamma(1) = p_+$. There are $t_1 > 0$ such that $\gamma(t_1) \in I^+_{g'}(p)$ and $t_1 < t_2 < 1$ such that $\gamma(t_2) \in I^-_{g'}(p)$. This implies that $p \in I^+_{g'}(p)$, which is absurd because (M, g') is causal. This shows that W is causally convex, hence the strong causality of (M, g).

□

A **time function** is a continuous function $\tau : M \to \mathbb{R}$ such that $\tau \circ \gamma$ is increasing for any future directed causal curve $\gamma : I \subset \mathbb{R} \to M$. The existence of a time function implies that (M, g) is causal. It is also easy to see that it implies strong causality (the sets $\tau^{-1}(]a, b[)$ are causally convex). A famous theorem of Hawking states that the right condition is stable causality.

Theorem 6.4.4. *A spacetime admits a time function if and only if it is stably causal.*

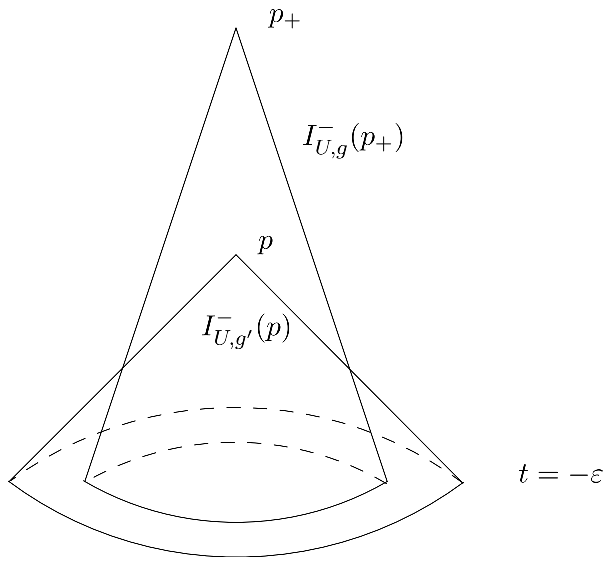

FIGURE 6.1. Finding a causally convex neighbourhood

Note that both implications are non trivial. A **temporal function** is a smooth function $\tau : M \to \mathbb{R}$ whose gradient $\nabla\tau$ is timelike (the gradient $\nabla\tau$ is the unique vector field such that $d\tau_p(v) = g_p(\nabla\tau(p), v)$ for all $(p, v) \in TM$). A temporal function is a time function, but a smooth time function needs not be a temporal function. However, existence of one or the other is equivalent.

Theorem 6.4.5. *A spacetime admits a time function if and only if it admits a temporal function.*

Theorem 6.4.5 was finally proved by Bernal and Sanchez in a series of papers ([**BS03**], [**BS05**]) resolving several classical yet unsolved problems around temporal functions. Hawking proved that stable causality implies the existence of a time function, but he did not prove the converse (in [**HE73**], one can also find the statement of Theorem 6.4.5, however the proof quotes a paper which contains mistakes). In [**FS12**], Fathi and Siconolfi give a direct proof of the fact that a stably causal spacetime admits a temporal function (the proof is completely independent from any previous work, and is based on some tools of weak KAM theory). So far, it seems that the only way to prove that a spacetime admitting a time function is stably causal uses Theorem 6.4.5 (indeed, a temporal function is still a temporal function for nearby metrics, hence the stable causality). We will discuss this problem in Chapter 8.

5. Overview of Part 2

While working with flows on metric spaces, continuity plays a major role. For this purpose, we will show that the map $x \mapsto \overline{J^+_{t,T}(x)}$ is continuous for the Hausdorff topology on compact subsets of M, provided the auxiliary Riemannian metric is appropriately chosen. This will be the content of Chapter 7. Chapter 8 follows the work of Conley and deals with the central notion of attractors, and their relationship with E-time functions. As an application, we get an analog of Conley's Theorem in Lorentzian geometry, and a new proof of one of the implications in Hawking's Theorem (a stably causal spacetime admits a time function). Finally, Chapter 9 is an introduction to the notion of conjugacy between multi valued dynamical systems, and does not deal with Lorentzian geometry at all.

CHAPTER 7

Continuity of the future

1. Choosing the right Riemannian metric

We will start by choosing a Riemannian metric that has a nice properties when studying lengths of limits of causal curves.

Definition 7.1.1. *Let (M, g) be a time oriented Lorentz manifold. We say that a Riemannian metric h on M is adapted to g if :*

- *For any point $p \in M$, there is a coordinate chart around p and a constant $\lambda > 0$ such that, in coordinates, g at p is $\lambda(-dx_1^2 + dx_2^2 + \cdots + dx_n^2)$ and h at p is $dx_1^2 + dx_2^2 + \cdots + dx_n^2$*
- *h is complete*

The standard example is the Euclidian metric on $\mathbb{R}^n$ that is adapted to the Minkowsky metric. It is quite easy to see that such a metric always exist.

Proposition 7.1.2. *Let (M, g) be a time oriented Lorentz manifold. Then there exists a Riemannian metric h adapted to g.*

Proof. Let T be an everywhere timelike and future directed vector field. For $p \in M$, we will denote by $E(p)$ the orthogonal space for g of $T(p)$ (hence $T_pM = Vect(T(p)) \oplus E(p)$). Let $h(g)$ be the Riemannian metric defined by:

$$h(g)_p(xT(p) + \underbrace{u}_{\in E(p)}, yT(p) + \underbrace{v}_{\in E(p)}) = -xy\underbrace{g_p(T(p), T(p))}_{<0} + \underbrace{g_p(u, v)}_{\geq 0}$$

The idea of this construction (called a Wick rotation) is that with the Minkowski metric $-dx_1^2+dx_2^2+\cdots+dx_n^2$ and the vector field $\frac{\partial}{\partial x_1}$, we find the Eucildian metric $dx_1^2+\cdots+dx_n^2$. Let us remark that if we multiply g by a positive function, then we also multiply $h(g)$ by the same function. Since every conformal class of Riemannian metrics contains a complete metric, we can find $\widetilde{g}$ in the conformal class of g such that $h(\widetilde{g})$ is complete. Since the notion of an adapted metric depends only on the conformal class, we can now work exclusively with $\widetilde{g}$. Consider $p \in M$, and $(e_1, e_2, \ldots, e_n)$ an orthonormal frame of T_pM for $\widetilde{g}$ such that $e_1 = \frac{T(p)}{\sqrt{-\widetilde{g}_p(T(p),T(p))}}$. It is also an orthonormal frame for $h(\widetilde{g})$. Applying the exponential map for $\widetilde{g}$ shows that $h(\widetilde{g})$ is adapted to $\widetilde{g}$ (with constant $\lambda = 1$), and therefore to g.

□

We will now only consider adapted Riemannian metrics, unless specified. We will be particularly interested in properties of such metrics regarding the length of limits of sequences of causal curves. Since adaptiveness is a local property, our results will all start with a local version, and then will be extended globally.

The length function is always upper semi continuous, i.e. $\ell_h(\gamma) \leq \liminf \ell_h(\gamma_k)$ for a sequence (γ_k) converging to γ in the compact open topology, but it is not lower semi continuous. We will show that we have a weak version of lower semi continuity for sequences of causal curves.

Since we are going to consider limits of sequences of causal curves, we are going to need to extend the notion of causal curve to some continuous curves. We say that a curve $\gamma : I \subset \mathbb{R} \to M$ is **future directed** if it is locally Lipschitz and its derivative is almost everywhere in the future cone of g. An important fact is that this definition does not change the future or past: all points that are reachable by a future directed curve (in the topological sense) are reachable by a smooth future curve.

Proposition 7.1.3. *Let $(\gamma_k)_{k\in\mathbb{N}} \in C(I,M)^{\mathbb{N}}$ be a sequence of future directed curves defined on an interval $I \subset \mathbb{R}$. If (γ_k) converges in the compact open topology to a curve γ, then γ is future directed.*

Proposition 7.1.4. *Let $(\gamma_k)_{k\in\mathbb{N}} \in C(\mathbb{R},M)^{\mathbb{N}}$ be a sequence of future directed curves. Up to changes of parameters, there is a subsequence that converges towards a future directed curve γ in the compact open topology.*

The proofs are in section 7 of [**Bar05**].

Proposition 7.1.5. *Let h be a Riemannian metric adapted to g, and let $(\gamma_k)_{k\in\mathbb{N}}$ be a sequence of future directed causal curves converging to γ in the compact open topology, then $\ell_h(\gamma) \geq \frac{1}{\sqrt{2}} \limsup \ell_h(\gamma_k)$*

Before we prove this result, let us prove a local version.

Lemma 7.1.6. *Let $p \in M$ and $C \in \left]0\,,\frac{1}{\sqrt{2}}\right[$. There is a neighbourhood U_p of p such that for all sequence (γ_k) of future curves in U converging to γ, we have $\ell_h(\gamma) \geq C \limsup \ell_h(\gamma_k)$.*

PROOF. Let us consider U a coordinate neighbourhood of p given by the definition of an adapted metric, and choose $a \in \left]0\,,1\right[$ and $b > 1$. We will denote by $\widetilde{h}$ the Euclidian metric on U. If we reduce U sufficiently, then we have $a\widetilde{h}_q(u,u) \leq h_q(u,u) \leq b\widetilde{h}_q(u,u)$ for all $q \in U$ and $u \in T_qM$ because of the continuity of both metrics and the equality at p.

Let us also choose $\alpha > 1$ and denote by $\widetilde{g}$ the constant Lorentzian metric on U given by $-\alpha dx_1^2 + dx_2^2 + \cdots + dx_n^2$. If we reduce U sufficiently, then $g < \widetilde{g}$ on U (i.e. a non zero causal vector for g is timelike for $\widetilde{g}$), since g has the same lightcone at p as $-dx_1^2 + dx_2^2 + \cdots + dx_n^2$.

Let us now consider a sequence (γ_k) of future curves in U converging to γ. Since these curves are causal for g and therefore for $\widetilde{g}$, they can be parametrized by the first coordinate: $\gamma_k(t) = \gamma_k(0) + (t, x_2^k(t), \ldots, x_n^k(t))$ and $\gamma(t) = \gamma(0) + (t, x_2(t), \ldots, x_n(t))$. Since these curves are causal for $\widetilde{g}$, we have $\dot{x}_2^k(t)^2 + \cdots + \dot{x}_n^k(t)^2 \leq \alpha$. Let us denote by x_k (resp. x) the first coordinate of the endpoint of γ_k (resp. γ). We have:

$$\begin{aligned} \ell_{\widetilde{h}}(\gamma_k) &= \int_0^{x_k} \underbrace{\sqrt{1 + \dot{x}_2^k(t)^2 + \cdots + \dot{x}_n^k(t)^2}}_{\leq\sqrt{1+\alpha}}\, dt \\ &\leq x_k\sqrt{1+\alpha} \end{aligned}$$

We also have $\ell_{\widetilde{h}}(\gamma) = \int_0^x \underbrace{\sqrt{1 + \dot{x}_2(t)^2 + \cdots + \dot{x}_n(t)^2}}_{\geq 1}\, dt \geq x$, therefore:

$$\begin{aligned} \ell_{\widetilde{h}}(\gamma) &\geq x \\ &\geq \lim x_k \\ &\geq \frac{1}{\sqrt{1+\alpha}} \limsup \ell_{\widetilde{h}}(\gamma_k) \end{aligned}$$

By choosing a, b and α such that $\sqrt{\frac{a}{b(1+\alpha)}} \geq C$, we obtain :

$$\begin{aligned} \ell_h(\gamma) &\geq \sqrt{a}\, \ell_{\widetilde{h}}(\gamma) \\ &\geq \sqrt{\frac{a}{1+\alpha}} \limsup \ell_{\widetilde{h}}(\gamma_k) \\ &\geq \sqrt{\frac{a}{b(1+\alpha)}} \limsup \ell_h(\gamma_k) \\ &\geq C \limsup \ell_h(\gamma_k) \end{aligned}$$

□

Proof of Proposition 7.1.5. Let us consider a sequence (γ_k) of future causal curves converging to γ. For all $p \in M$, let us choose a neighbourhood given by Lemma 7.1.6. Let $[a\,,b]$ be a compact interval in the domain of these curves. Since $\gamma([a\,,b])$ is compact, we can consider a finite cover $\gamma([a\,,b]) = \bigcup_{1 \leq i \leq m} U_{\gamma(t_i)}$ with $t_1 < \cdots < t_m$. Since γ is continuous, we can find numbers $a = s_1 < s_2 < \cdots < s_{m+1}$ such that $\gamma([s_i\,, s_{i+1}]) \subset U_{\gamma(t_i)}$. Let γ^i (resp. γ_k^i) be the restriction of γ (resp. γ_k) to $[s_i\,, s_{i+1}]$.

Let $C \in \,]0\,, \frac{1}{\sqrt{2}}[$. Given $i \in \{1, \ldots, m\}$, then for k sufficiently large, the curve γ_k^i lies in $U_{\gamma(t_i)}$ and therefore $\ell_h(\gamma^i) \geq C \limsup \ell_h(\gamma_k^i)$. After a sum over i, we obtain $\ell_h(\gamma) \geq C \limsup \ell_h(\gamma_k)$ and the proposition is proved by considering the limit when C tends to $\frac{1}{\sqrt{2}}$.

□

Even though this result will be useful in the next section, it will not be enough for our purpose, and we need to show that we can choose another curve joining the same points that is longer.

Proposition 7.1.7. *Let $(\gamma_k : [0\,,1] \to M)$ be a sequence of future directed causal curves with constant speed, of length $\ell_k > 0$ converging to $\ell > 0$, and such that γ_k converges uniformally to a curve γ. Then for all $\varepsilon > 0$, there exists a future directed causal curve η_ε such that :*

- $\eta_\varepsilon(0) = \gamma(0)$
- $\eta_\varepsilon(1) = \gamma(1)$
- $\ell(1-\varepsilon) \leq \ell_h(\eta_\varepsilon) \leq \ell(1+\varepsilon)$

Once again, let us start by formulating and proving a local version of this result.

Lemma 7.1.8. *Let $p \in M$ and let $\varepsilon > 0$. There is a closed neighbourhood V of p such that for any sequence $(\gamma_k : [0\,,1] \to V)$ of future directed causal curves with constant speed, of length $\ell_k > 0$ converging to $\ell > 0$, and such that γ_k converges uniformally to a curve γ, there exists a future directed causal curve η such that :*

- $\eta(0) = \gamma(0)$
- $\eta(1) = \gamma(1)$
- $\ell(1-\varepsilon) \leq \ell_h(\eta) \leq \ell(1+\varepsilon)$

Proof. Let us consider a coordinate neighbourhood U of p given by Definition 7.1.1. For $\alpha \in [0\,,1]$, let us denote by g_α the constant Lorentzian metric $-(1+\alpha)dx_1^2 + dx_2^2 + \cdots + dx_n^2$, by $g_{-\alpha}$ the metric $-(1-\alpha)dx_1^2 + dx_2^2 + \cdots + dx_n^2$, by ξ the euclidian metric on U, by ξ_α (resp. $\xi_{-\alpha}$ the Riemannian metric $(1+\alpha)\xi$ (resp. $(1-\alpha)\xi$) on U. We will choose α small enough so that it will satisfy the following inequalities:

(1) $(1+\alpha)^2 \leq 1+\varepsilon$
(2) $\frac{1-\alpha}{1+\alpha}\frac{\sqrt{2-\alpha}}{\sqrt{2+\alpha}} \geq 1-\varepsilon$
(3) $\frac{1-\alpha}{1+\alpha}\frac{\sqrt{2-\alpha}-\sqrt{1-\alpha}+\sqrt{1-\alpha}}{\sqrt{2-\alpha}} \geq 1-\varepsilon$

Let V be a closed ball centered at p for the infinite norm in coordinates, small enough so that $g_{-\alpha} < g < g_\alpha$ and $\xi_{-\alpha} \leq h \leq \xi_\alpha$ on V.

We will denote by f the first coordinate function on V.

First step: Estimation of $\lambda = f(\gamma(1)) - f(\gamma(0))$

Let $c : [0\,,1] \to V$ be a future directed causal curve. It is also causal for g_α. If $c(t) = (c_1(t), \dots, c_n(t))$ in coordinates, then $-(1+\alpha)\dot{c}_1^2 + \dot{c}_2^2 + \cdots + \dot{c}_n^2 \leq 0$ and $\dot{c}_1 \geq 0$. We have:

$$\begin{aligned} \ell_h(c) &\leq (1+\alpha)\ell_\xi(c) \\ &\leq (1+\alpha)\int_0^1 \sqrt{\dot{c}_1^2 + \cdots + \dot{c}_n^2}dt \\ &\leq (1+\alpha)\int_0^1 \sqrt{(2+\alpha)\dot{c}_1^2}dt \\ &\leq (1+\alpha)\sqrt{2+\alpha}(f(c(1)) - f(c(0))) \end{aligned}$$

We also have:

$$\begin{aligned} \ell_h(c) &\geq (1-\alpha)\ell_\xi(c) \\ &\geq (1-\alpha)\int_0^1 \sqrt{\dot{c}_1^2 + \cdots + \dot{c}_n^2}dt \\ &\geq (1-\alpha)\int_0^1 \sqrt{\dot{c}_1^2}dt \\ &\geq (1-\alpha)(f(c(1)) - f(c(0))) \end{aligned}$$

By combining these inequalities and applying to γ_k, we obtain $\frac{1}{(1+\alpha)\sqrt{2+\alpha}}\ell_k \leq f(\gamma_k(1)) - f(\gamma_k(0)) \leq \frac{1}{1-\alpha}\ell_k$.The continuity of f gives us:

$$\frac{\ell}{(1+\alpha)\sqrt{2+\alpha}} \leq \lambda \leq \frac{\ell}{1-\alpha}$$

Second step: First case: if $\gamma(1) \in J^+_{V,g_{-\alpha}}(\gamma(0))$.

In this case, we construct η as a piecewise causal geodesic for $g_{-\alpha}$. We will consider the intersection S of a horizontal hyperplane (in coordinates) located between $\gamma(0)$ and $\gamma(1)$ that meets the intersection of the light cones for $J^+_{V,g_{-\alpha}}$ of $\gamma(0)$ and $\gamma(1)$, and of the half cone $J^+_{g_{-\alpha}}(\gamma(0))$. For a point $p \in S$, we consider the curve η_p obtained as the concatenation of the straight lines joining $\gamma(0)$ to p and p to $\gamma(1)$ (see Figure 7.1). The curve η_p is causal for $g_{-\alpha}$, and therefore causal (actually timelike) for g. The maximum Euclidian length of η_p is obtained when p lies on the border of the cone (i.e. η_p is a null curve for $g_{-\alpha}$). In this case, we have $\ell_\xi(\eta_p) = \sqrt{2-\alpha}\lambda \geq \frac{1}{1+\alpha}\frac{\sqrt{2-\alpha}}{\sqrt{2+\alpha}}\ell$ (this inequality is given by the first step), hence $\ell_h(\eta_p) \geq \frac{1-\alpha}{1+\alpha}\frac{\sqrt{2-\alpha}}{\sqrt{2+\alpha}}\ell$. The minimum euclidian length is obtained when η_p is a straight line, in which case we have $\ell_\xi(\eta_p) = d_\xi(\gamma(0), \gamma(1))$, hence $\ell_h(\eta_p) \leq (1+\alpha)d_\xi(\gamma(0), \gamma(1))$. By using the integral expression of $\ell_h(\eta_p)$, one can see that it is a continuous function of p. Therefore, the values of this map contains the interval $J = [(1+\alpha)d_\xi(\gamma(0), \gamma(1))\,, \frac{1-\alpha}{1+\alpha}\frac{\sqrt{2-\alpha}}{\sqrt{2+\alpha}}\ell]$. In order to conclude, we wish to see that $J \cap [\ell(1-\varepsilon)\,, \ell(1+\varepsilon)] \neq \emptyset$ (after that, we choose p such that η_p has length between $\ell(1-\varepsilon)$ and $\ell(1+\varepsilon)$ and we set $\eta = \eta_p$). By using $\gamma(t) = \lim \gamma_k(t)$, we obtain

$d_\xi(\gamma(0),\gamma(1)) \leq \frac{1}{1-\alpha}\ell \leq (1+\alpha)\ell$, therefore $(1+\alpha)d_\xi(\gamma(0),\gamma(1)) \leq (1+\alpha)^2\ell \leq (1+\varepsilon)\ell$ (this is the first required inequality). We also chose α such that $\frac{1-\alpha}{1+\alpha}\frac{\sqrt{2-\alpha}}{\sqrt{2+\alpha}}\ell \geq (1-\varepsilon)\ell$, which concludes to prove that $J \cap [\ell(1-\varepsilon), \ell(1+\varepsilon)] \neq \emptyset$.

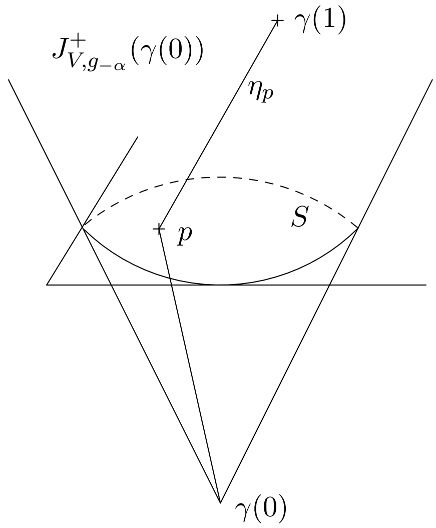

FIGURE 7.1. Construction of the curve η

Third step: Second case: if $\gamma(1) \notin J^+_{V,g_{-\alpha}}(\gamma(0))$.

In this case, we will simply show that the curve γ already satisfies the desired properties. We have $\ell_h(\gamma) \leq \ell$ by upper semi continuity of the Riemannian length, which gives us the desired upper bound on $\ell_h(\gamma)$. We also have $\ell_h(\gamma) \geq (1-\alpha)\ell_\xi(\gamma) \geq (1-\alpha)d_\xi(\gamma(0),\gamma(1))$ where d_ξ is the euclidian distance in coordinates. The rest of the proof is Euclidian geometry in coordinates. Let us consider the vertical plane P containing $\gamma(0)$ and $\gamma(1)$ and the horizontal hyperplane S containing $\gamma(1)$. Let x (resp. y) be intersection of P and $\partial J^+_{V,g_{-\alpha}}(\gamma(0))$ (resp. $\partial J^+_{V,g_\alpha}(\gamma(0))$) that is closest to $\gamma(1)$. Let z be the intersection of S and the vertical line passing through $\gamma(0)$. Since $\gamma(1) \notin J^+_{V,g_{-\alpha}}(\gamma(0))$ and $\gamma(1) \in J^+_{V,g_\alpha}(\gamma(0))$, we have $d_\xi(x,\gamma(1)) \leq d_\xi(x,y)$.

Since x, y and z are on the same line, we have $d_\xi(x,y) = d_\xi(z,y) - d_\xi(z,x)$. By using Pythagora's Theorem, we obtain $d_\xi(z,x) = \sqrt{1-\alpha}\lambda$ and $d_\xi(z,y) = \sqrt{1-\alpha}\lambda$. We now have:

$$\begin{aligned} d(\gamma(0),\gamma(1)) &\geq d(\gamma(0),x) - d(x,\gamma(1)) \\ &\geq \sqrt{2-\alpha}\lambda - d(x,y) \\ &\geq (\sqrt{2-\alpha} - \sqrt{1-\alpha} + \sqrt{1-\alpha})\lambda \end{aligned}$$

The first step and the third required inequality on α give us:

$$\begin{aligned}\ell_h(\gamma) &\geq (1-\alpha)(\sqrt{2-\alpha}-\sqrt{1-\alpha}+\sqrt{1-\alpha})\lambda \\ &\geq \frac{1-\alpha}{1+\alpha}\frac{\sqrt{2-\alpha}-\sqrt{1-\alpha}+\sqrt{1-\alpha}}{\sqrt{2-\alpha}}\ell \\ &\geq \ell(1-\varepsilon)\end{aligned}$$

This gives the desired lower bound on $\ell_h(\gamma)$, which concludes the proof.

□

Proof of Proposition 7.1.7. Let (γ_k) be a sequence of future directed curves converging to γ, and such that $\ell_h(\gamma_k) \to \ell$. Let $\varepsilon > 0$. For $t \in [0\,,1]$, let us consider a neighbourhood U_t given by Lemma 7.1.8. The open covering $\gamma([0\,,1]) \subset \bigcup_{t\in[0\,,1]} U_t$ admits a finite subcover $\gamma([0\,,1]) \subset \bigcup_{1\leq i\leq m} U_{t_i}$ with $t_1 < t_2 < \cdots < t_m$. Let $0 = s_0 < s_2 < \dots s_m = 1$ such that $\gamma([s_i\,, s_{i+1}]) \subset U_{t_i}$ for all i. Let us denote by γ^i (resp. γ_k^i) the restriction of γ (resp. γ_k) to $[s_i\,, s_{i+1}]$. For all i, there is a curve η^i joining $\gamma(s_i)$ and $\gamma(s_{i+1})$ such that $(s_{i+1}-s_i)\ell(1-\varepsilon) \leq \ell_h(\eta^i) \leq (s_{i+1}-s_i)\ell(1+\varepsilon)$. The concatenation η of the curves η^i joins $\gamma(0)$ and $\gamma(1)$ and satisfies $\ell(1-\varepsilon) \leq \ell_h(\eta) \leq \ell(1+\varepsilon)$. □

We will also use the following result that shows that, locally, causal curves cannot be arbitrarily long.

Proposition 7.1.9. *Let h be an adapted Riemannian metric to g and let $x \in M$. For all $\varepsilon > 0$, there is a neighbourhood U of x such that for all causal curve γ included in U, we have $\ell_h(\gamma) \leq \varepsilon$.*

Proof. Once again, the idea is that it is simple in the Minkowski case, with an explicit neighbourhood. Let U be a coordinate neighbourhood of x given by the definition of an adapted metric. By reducing U, we can assume that $g < g_1$ and $h \leq \xi_1$ on U (we use the same notations as in the previous proposition), and that $-\frac{\varepsilon}{4\sqrt{3}} \leq x_1 \leq \frac{\varepsilon}{4\sqrt{3}}$ for all $(x_1, \dots, x_n) \in U$. Let γ be a future directed causal curve in U, and let us write $\gamma(t) = (x_1(t), \dots, x_n(t))$. Since γ is timelike for g_1, we can consider that $x_1(t) = t$.

$$\begin{aligned}\ell_h(\gamma) &\leq 2\ell_\xi(\gamma) \\ &\leq 2\int_{t_0}^{t_1}\sqrt{1+\dot{x}_2^2(t)+\cdots+\dot{x}_n^2(t)}dt \\ &\leq (1-\alpha)\int_{t_0}^{t_1}\sqrt{1+2}dt \\ &\leq 2\sqrt{3}(t_1-t_0) \\ &\leq \varepsilon\end{aligned}$$

□

With a hypothesis on causality, this tells us that causal curves that stay in a compact set have bounded length.

Corollary 7.1.10. *If h is an adapted metric to g and if (M, g) is strongly causal, then for any compact set $K \subset M$, there is a constant $\ell > 0$ such that for all causal curve γ included in K, we have $\ell_h(\gamma) \leq \ell$.*

Proof. For all $x \in K$ we consider a neighbourhood U_x of x given by Proposition 7.1.9 with $\varepsilon = 1$. Since (M, g) is strongly causal, by reducing U_x we can assume that U_x is causally convex. From the open covering $K \subset \bigcup_{x\in K} U_x$ we can extract a finite cover $K \subset \bigcup_{i=1}^n U_{x_i}$ where $x_1, \dots, x_n \in K$. Let $\ell = n$. If γ is a causal curve included in K, we can divide γ in a finite number k of curves γ_i such each of these curves is included in one U_{x_j}. Since they are causally convex, whe have $k \leq n$, and their length is at most 1 by definition of U_{x_j}, therefore $\ell_h(\gamma) = \sum_{i=1}^k \ell_h(\gamma_i) \leq k \leq \ell$.

□

2. Hausdorff topology and continuity of $\overline{J^+_{t,T}}$

Let us recall that if K_1 and K_2 are two non empty compact subsets of a metric space X, then the Hausdorff distance between K_1 and K_2 is given by $d_H(K_1, K_2) = \inf\{\varepsilon > 0 | K_1 \subset V_\varepsilon(K_2) \text{ and } K_2 \subset V_\varepsilon(K_1)\}$, where $V_\varepsilon(K)$ denotes the ε-neighbourhood of K. It defines a metric on the space $\mathrm{Comp}(X)$ of non empty compact subsets of X. The topology inherent to this metric, called the Hausdorff topology, does not depend on the choice of a metric on X, as long as it defines the same topology.

The following result will be important later on.

Proposition 7.2.1. *Let X be a proper metric space (i.e. closed balls are compact). Let us consider a continuous map $f : Y \to \mathbb{R}$. Then the map $g : \mathrm{Comp}(X) \to \mathbb{R}$ defined by $g(K) = \max\{f(x) | x \in K\}$ is continuous.*

Proof. We will separate the proofs of upper and lower semi continuity. Let $K_0 \in \mathrm{Comp}(X)$ and $\varepsilon > 0$. Let $x_0 \in K_0$ such that $f(x_0) = g(K_0)$. Let $\delta > 0$ such that $d(x, x_0) < \delta$ implies $f(x) \geq f(x_0) - \varepsilon$. Let us denote by φ the map $K \mapsto d(x_0, K)$. It is easy to check that φ is 1-Lipschitz for the Hausdorff metric and therefore continuous. Since $\varphi(K_0) = 0$, let W be a neighbourhood of K_0 in $\mathrm{Comp}(X)$ such that $\varphi(K) < \delta$ for $K \in W$. For $K \in W$, we have $d(x_0, K) < \delta$ and therefore $B(x_0, \delta) \cap K \neq \emptyset$. Let $x \in B(x_0, \delta) \cap K$, whe have $f(x) \geq f(x_0) - \varepsilon$ and $g(K) \geq f(x)$, hence $g(K) \geq g(K_0) - \varepsilon$. This concludes the proof of lower semi continuity.

Let us now prove by contradiction that g is upper semi continuous. If it is not, then we can find $K \in \mathrm{Comp}(X)$, $\varepsilon > 0$ and a sequence K_n in $\mathrm{Comp}(X)$ such that $\lim K_n = K$ and $g(K_n) \geq g(K) + \varepsilon$ for all n. Let $C = \{x \in X / d(x, K) \leq 1\}$. Since X is a proper metric space, C is compact because it is closed and $\mathrm{diam}(C) \leq \mathrm{diam}(K) + 2 < \infty$. For n large enough, we have $K_n \subset V_1(K) \subset C$. Let $x_n \in K_n$ such that $f(x_n) = g(K_n)$. Since $x_n \in C$, up to the choice of a subsequence, we can assume that x_n tends to $x \in C$. For any $\eta > 0$, we have $x_n \in K_n \subset V_\eta(K)$ for n sufficiently large, and therefore $d(x_n, K) \leq \eta$. This shows that $\lim d(x_n, K) = 0$, therefore $x \in K$. We now have $g(K) \geq f(x) = \lim f(x_n) = \lim g(K_n) \geq g(K) + \varepsilon$ which is impossible. Therefore g is lower semi continuous. □

Since the composition of continuous functions is continuous, the following is now obvious.

Corollary 7.2.2. *Let X be a topological space and Y a proper metric space. Let us consider continuous maps $F : X \to \mathrm{Comp}(Y)$ and $f : Y \to \mathbb{R}$. Then the map $g : X \to \mathbb{R}$ given by $g(x) = \max\{f(y)/y \in F(x)\}$ is continuous.*

We now wish to prove that, in some sense, the map that associates to a point its future is continuous. If we try to deal with the whole future, then we face a major problem: it is not generally continuous, and its continuity is actually related to causality conditions (see **[MinSán08]**). This is why we consider the map $x \mapsto \overline{J^+_{t,T}(x)}$. To prove its continuity, we will need some results that rely on the fact that we consider Riemannian metrics adapted to g.

Lemma 7.2.3. *Consider $0 < t < T$, a sequence $(x_k)_{k\in\mathbb{N}} \in M^{\mathbb{N}}$ converging to $x \in M$, and a sequence $(y_k)_{k\in\mathbb{N}} \in M^{\mathbb{N}}$ converging to $y \in M$, such that $y_k \in J^+_{t,T}(x_k)$ for all $k \in \mathbb{N}$. Then $y \in \overline{J^+_{t,T}(x)}$.*

PROOF. For $k \in \mathbb{N}$, consider a future curve $\gamma_k : [0,1] \to M$ parametrized by arc length, of length ℓ_k between t et T, such that $\gamma_k(0) = x_k$ and $\gamma_k(1) = y_k$. Up to the choice of a sub sequence, we can assume that (γ_k) converges uniformally to a curve γ, and that ℓ_k converges to $\ell \in [t\,, T]$. We have $\gamma(0) = x$ et $\gamma(1) = y$.

Let $\varepsilon > 0$. By Proposition 7.1.7, there is a future curve η_ε of length $\widetilde{\ell} \in [\ell - \varepsilon\,, \ell + \varepsilon]$ joining x and y. We can either shorten or extend η_ε to a future curve of length ℓ with endpoint z_ε satisfying $z_\varepsilon \in J^+_{t,T}(x)$ and $d(y, z_\varepsilon) \leq \varepsilon$, therefore $y \in V_\varepsilon(J^+_{t,T}(x))$ for all $\varepsilon > 0$ and $y \in \overline{J^+_{t,T}(x)}$.

□

Corollary 7.2.4. *For all $x \in M$, $t > 0$, $T > t$ and $\varepsilon > 0$, there is a neighbourhood V of x such that for all $y \in V$, $J^+_{t,T}(y)$ lies in the ε-neighbourhood of $J^+_{t,T}(x)$.*

PROOF. Let us assume that this statement is false, so that there exists $x \in M$, $t > 0$, $T > t$, $\varepsilon > 0$, a sequence (x_k) converging to x and a sequence (y_k) such that $y_k \in J^+_{t,T}(x_k)$ and $d(y_k, J^+_{t,T}(x)) \geq \varepsilon$. Since $y_k \in \overline{B(x_k, T)} \subset \overline{B(x, T+1)}$ for k sufficiently large, we can assume up to extraction that (y_k) converges to $y \in M$ (let us recall that since the metric h is complete, by the Hopf Rinow Theorem, closed balls are compact). The previous result states that $y \in \overline{J^+_{t,T}(x)}$, but $d(y, J^+_{t,T}(x)) \geq \varepsilon$, which is absurd.

□

Lemma 7.2.5. *For all $x \in M$, $t > 0$, $T > t$ et $\varepsilon > 0$, there is a neighbourhood V of x such that for all $y \in V$, $J^+_{t,T}(x) \subset V_\varepsilon(J^+_{t,T}(y))$.*

PROOF. Let us once again prove this result by contradiction (it allows us to consider one point of $J^+_{t,T}(x)$ instead of the whole set). Let us assume that there is a sequence (x_k) converging to x, $\varepsilon > 0$ and a sequence (y_k) such that $y_k \in J^+_{t,T}(x)$ and $y_k \notin V_\varepsilon(J^+_{t,T}(x_k))$ for all k. Let (γ_k) be a sequence of future curves with length between t and T such that $\gamma_k(0) = x$ and $\gamma_k(1) = y_k$. Up to extraction, we can assume that (γ_k) converges to a future curve γ and that $(\ell_h(\gamma_k))$ converges to $\ell \in [t, T]$. Let y be the limit of y_k. By proposition 7.1.7 there is a future curve parametrized by arc length η joining x and y with length between $t - \frac{\varepsilon}{4}$ and $T + \frac{\varepsilon}{4}$. Let ν_t be a time dependent locally Lipschitz everywhere causal vector field with constant norm such that $\nu_t(\eta(t)) = \dot{\eta}(t)$. Let φ_t denote the isotopy of ν_t. The map φ_1 is continuous, therefore $\varphi_1(x_k)$ converges to $\varphi_1(x) = y$. Therefore $y \in V_{\frac{\varepsilon}{4}}(J^+_{t-\frac{\varepsilon}{4}, T+\frac{\varepsilon}{4}}(x_k)) \subset V_{\frac{\varepsilon}{2}}(J^+_{t,T}(x_k))$ for k large enough, and $y_k \in V_\varepsilon(J^+_{t,T}(x_k))$, which is absurd.

□

By combining Corollary 7.2.4 and Lemma 7.2.5, we obtain the following result.

Theorem 7.2.6. *Let h be an adapted Riemannian metric and let $0 < t < T$. The map $x \mapsto \overline{J^+_{t,T}(x)}$ is continuous with respect to the Hausdorff topology.*

CHAPTER 8

Attractors, chain recurrence and Hawking's Theorem

1. Attractors in spacetimes

1.1. Pre attractors, attractors and basin of attraction. At the heart of Conley's proof of the existence of Lyapunov functions lies the notion of attractors. Sets that attract orbits will be natural candidates to be the place where a Lyapunov functions reaches its minimum, and the Lyapunov function can be thought of as a distance to an attractor.

Definition 8.1.1. *An open set $U \subset M$ is said to be a pre attractor if there is $t_0 > 0$ such that $\overline{J^+_{t_0}(U)} \subset U$.*
The set $A = \bigcap_{t \geq t_0} \overline{J^+_t(U)}$ is called the attractor.
The set $B(A,U) = \bigcup_{t \geq 0} \{p \in M | \overline{J^+_t(p)} \subset U\}$ is called the basin of U-attraction.
If $\mathfrak{U}$ is the set of pre attractors sharing the same attractor A, then the basin of attraction is $B(A) = \bigcup_{U \in \mathfrak{U}} B(A,U)$

Note that the attractor A may be empty. The central result of this chapter is the following theorem, which is the equivalent in Lorentzian geometry of Conley's Theorem for flows.

Theorem 8.1.2. *If A is an attractor in a spacetime (M,g) (for an adapted Riemannian metric), then there is a $B(A) \setminus A$-time function.*

The proof consists mainly in observing that attractors attract long curves that start in the basin of attraction. We start by working with $B(A,U) \setminus A$ for a given pre attractor U, then extend to $B(A)$. When considering long curves, the choice of the Riemannian metric becomes important, which is why we will only consider adapted Riemannian metrics.

Let us start with a lemma that justifies the name attractor.

Lemma 8.1.3. *Let U be a pre attractor and A its attractor. Let $(\gamma_i)_{i \in \mathbb{N}}$ be a sequence of future directed causal curves with length tending to $+\infty$ and such that $\gamma_i(0) \in U$. Then for any neighbourhood V of A and for any compact set K, if we have $z_i = \gamma_i(1) \in K$ for all i, then $z_i \in V$ for all i sufficiently large.*

Proof. We will prove this result by contradiction. Let us assume that we can extract a sub-sequence (still written z_i) such that $z_i \notin V$. After a second extraction, we can assume that z_i converges to a $z \in K$.

Consider $i \in \mathbb{N}$. If j is large enough, then $\ell(\gamma_j) \geq \ell(\gamma_i)$, hence $z_j \in J^+_{\ell(\gamma_i)}(U)$. Therefore $z \in \overline{J^+_{\ell(\gamma_i)}(U)}$ for all i, and $z \in A$, which gives us $z_i \in V$ for any large i, which is a contradiction with our hypothesis. □

This result means that long futur directed curves that start in the pre attractor and that do not go to infinity will be attracted by A. Another important fact is that the basin of attraction is open.

Corollary 8.1.4. *Let h be a Riemannian metric adapted to g, and let U be a pre attractor and A its attractor. Then the basin of U-attraction $B(A,U)$ is open.*

PROOF. Let $B_s = \{x \in M | J^+_s(x) \subset U\}$, so that $B(A,U) = \bigcup_{s>0} B_s$. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $M \setminus B_s$. Assume that $x_k \to x \in M$. For all $k \in \mathbb{N}$, there is a future directed curve $\gamma_k : [0\,,1] \to M$ such that $\gamma_k(0) = x_k$ and $\gamma_k(1) \notin U$, of length $\ell_h(\gamma_k) > s$. Since $J^+_T(U) \subset U$, we can also assume that $\ell_h(\gamma_k) \leq s + T$, therefore, up to changes of parameters, we can assume that γ_k converges towards a future directed curve γ. According to Proposition 7.1.7, we can find a future curve η such that $\eta(0) = x$ and $\eta(1) = \gamma(1) \notin U$ of length greater than s, which shows that $x \notin B_s$, i.e. B_s is open, and so is $B(A,U) = \bigcup_{s>0} B_s$. □

1.2. Construction of a $B(A) \setminus A$-time function. We start with a refined version of Urysohn's Lemma adapted to attractors. Conley's construction of a Lyapunov function for an attractor in [**Con88**] associated to a flow φ^t consists in considering the function $\sup_{t\geq 0} f(\varphi^t(x))$ where $f(x) = \frac{d(x,A)}{d(x,A)+d(x,B(A)^c)}$. In the non compact case, Hurley noticed in [**Hur98**] that one has to choose a different function f.

Lemma 8.1.5. *Let U be a pre attractor and A its attractor. Then there exists a continuous fonction $f : M \to [0,1]$ such that:*

(1) $f^{-1}(0) = A$
(2) $f^{-1}(1) = M \setminus B(A,U)$
(3) For all $x \in B(A,U)$, there is a neighbourhood N of x such that:

$$\forall \varepsilon > 0 \ \exists t_0 > 0 \ \forall y \in J^+_{t_0}(N) \quad f(y) < \varepsilon$$

PROOF. We will first look for a function Ψ such that the following function will almost satisfy our conditions:

$$f(x) = \frac{d(x,A)}{d(x,A) + \Psi(x) d(x, M \setminus B(A,U))}$$

First step: Construction of Ψ

We start by writing $M = \bigcup_{n\in\mathbb{N}} K_n$ where the K_n are compact and where for all n, K_n lies in the interior of K_{n+1}.

Let $T > 0$ be such that $\overline{J^+_T(U)} \subset U$. The goal is to obtain a continuous fonction $\Psi : M \to [1,+\infty[$ such that $\Psi(x)d(x, M \setminus B(A,U)) \geq n$ on $\overline{J^+_T(U)} \cap M \setminus K_n$

For $i \in \mathbb{N}$, we denote by U_i a relatively compact open set of M, such that $\overline{U_i} \subset B(A,U)$ and $(K_i \setminus K_{i-1}) \cap \overline{J^+_T(U)} \subset U_i$, and $U_i \cap K_{i-2} = \emptyset$ (e.g. an ε neighbourhood of $(K_i \setminus K_{i-1}) \cap \overline{J^+_T(U)}$ with ε small enough).

We now consider $U_\infty = M \setminus \overline{J^+_T(U)}$ so that we have an open cover $M = \bigcup_{i\in\mathbb{N}\cup\{\infty\}} U_i$, and let θ_i be a partition of unity associated to this open cover.

Finally, we consider $\alpha_i = \min(\frac{i}{\inf_{U_i} d(.,M\setminus B(A,U))}, 1)$ and $\alpha_\infty = 1$, and let $\Psi = \sum \theta_i \alpha_i$. Since $\alpha_i \geq 1$ for all i, we have $\psi \geq 1$, and ψ is continuous (even smooth) because the sum is locally finite.

Let $x \in \overline{J^+_T(U)} \cap M \setminus K_n$. If $x \in U_i$, then $x \in K_i$, therefore $i > n$. We have:

$$\begin{aligned}\psi(x)d(x,M\setminus B(A,U)) &= d(x,M\setminus B(A,U))\sum_{x\in U_i}\theta_i(x)\alpha_i\\ &\geq d(x,M\setminus B(A,U))\sum_{x\in U_i}\theta_i(x)\frac{i}{\inf_{U_i} d(.,M\setminus B(A,U))}\\ &\geq \sum_{x\in U_i} i\theta_i(x)\underbrace{\frac{d(x,M\setminus B(A,U))}{\inf_{U_i} d(.,M\setminus B(A,U))}}_{\geq 1}\\ &\geq \sum_{x\in U_i} i\theta_i(x) > n\sum_{x\in U_i}\theta_i(x) = n\end{aligned}$$

Second step: Construction of f

Define $\mu : M \to \mathbb{R}$ by $\mu(x) = \min(1, d(x, A))$ (set $\mu(x) = 1$ if $A = \emptyset$) and consider the function $f : M \to \mathbb{R}$ such that:

$$f(x) = \frac{\mu(x)}{\mu(x) + \Psi(x)d(x, M\setminus B(A,U))}$$

If $B(A,U) = M$, then we set $f(x) = \frac{\mu(x)}{\mu(x)+\Psi(x)}$. The function f is continuous, has values in $[0,1]$ and satisfies the two first requirements.

Third step: Checking the last requirement

Let $x \in B(A,U)$. We know that $I^-(x)\cap B(A,U) \neq \emptyset$ since $B(A,U)$ is open and x lies in the closure of $I^-(x)$. Let $y \in I^-(x)\cap B(A,U)$, and let N be a compact neighbourhood of x included in $I^-(x)\cap B(A,U)$.

Since $y \in B(A,U)$, there is $t_1 > 0$ such that $J^+_{t_1}(y) \subset U$, therefore $J^+_{t_1}(N) \subset U$ and $\overline{J^+_{t_2}(N)} \subset \overline{J^+_T(U)}$ where $t_2 = t_1 + T$.

Let $\varepsilon > 0$. Because of Lemma 8.1.3, if we consider $n > \frac{1}{\varepsilon}$ and V the ε-neighbourhood of A, then there is $t_0 > t_2$ such that $J^+_{t_0}(N) \subset V \cup (M \setminus K_n)$ (if it was not the case, we could construct a sequence γ_i of curves with length growing to infinity and with endpoints in K_n but not in V, which would be a contradiction).

Therefore if $y \in J^+_{t_0}(N)$, then either $y \in V$, in which case $M(y) < \varepsilon$ results in $f(y) < \varepsilon$, either $y \notin K_n$ in which case $f(y) \leq \frac{1}{0+n} < \varepsilon$. □

We will now use this function to construct a $B(A,U)\setminus A$-time function.

Lemma 8.1.6. *Let us consider an attractor A and f the function given by Lemma 8.1.5. For $t \geq 0$ we consider:*

$$g_t(x) = \sup_{J^+_t(x)} f$$

The function g_t is continuous.

Proof. The idea is to see that locally, we can find $t' > t$ such that $g_t(x) = \max_{\overline{J^+_{t,t'}(x)}} f$, and use the continuity of the map $x \mapsto \overline{J^+_{t,t'}(x)}$ to conclude.

Let us start by considering the case where $x \in B(A,U)$. If $g_t(x) > 0$, then let U be a small compact neighbourhood of x, and set:

$$C = \min_{y\in U}\max_{z\in\overline{J^+_{t,t+1}(y)}} f(z)$$

Let us show that if U is small enough, then $C > 0$. If not, then we can find a sequence $x_k \to x$ such that $\max_{\overline{J^+_{t,t+1}(x_k)}} f = 0$, therefore by continuity of $z \to \overline{J^+_{t,t+1}(z)}$, we have $\max_{\overline{J^+_{t,t+1}(x)}} f = 0$ and $\overline{J^+_{t,t+1}(x)} \subset A$, therefore $\overline{J^+_t}(x) \subset A$ and $g_t(x) = 0$, which is absurd. We choose U such that $C > 0$. According to Lemma 8.1.5, there is a neighbourhood N of x and $t_0 > 0$ such that $f(y) \leq \frac{C}{2}$ for all $y \in J^+_{t_0}(N)$. Therefore, for $y \in N \cap U$, we have $g_t(y) = \max_{\overline{J^+_{t,t_0}(y)}} f$ which is a continuous function, and g_t is continuous at x.

If $g_t(x) = 0$, let $\varepsilon > 0$. By Lemma 8.1.5, there is a neighbourhood N of x and $t_0 > t$ such that $f(y) \leq \varepsilon$ for $y \in J^+_{t_0}(N)$. Let W be a neighbourhood of x such that $\max_{\overline{J^+_{t,t_0}(y)}} f < \varepsilon$ for $y \in W$ (recall that this map is continuous and has value 0 at x). For $y \in W \cap N$, we have $g_t(y) \leq \varepsilon$, therefore g_t is continuous at x.

Let us now consider the case where $x \notin B(A, U)$. First, let us show that $g_t(x) = 1$. If $g_t(x) < 1$, then $\overline{J^+_t}(x) \subset f^{-1}([0\,,1[) = B(A, U)$. Consider the set E of endpoints of future causal curves starting at x of length t. Then E is relatively compact and $\overline{E} \subset \overline{J^+_t}(x) \subset B(A, U)$. Since $B(A, U) = \bigcup_{s \geq 0} B_s$ where $B_s = \{x \in M | J^+_s(x) \subset U\}$ is open (see the proof of Corollary 8.1.4), we have a finite cover $E \subset \bigcup_{1 \leq i \leq k} B_{t_i}$. Set $t' = t + \max t_i$, we find that $J^+_{t'}(x) \subset U$ and $x \in B(A, U)$ which is absurd. Therefore $g_t(x) = 1$.

Finally, consider $T > t$ such that $g_t(x) = \sup_{J^+_{t,T}(x)} f$. Let $\varepsilon > 0$ and let U be a neighbourhood of x such that $\max_{\overline{J^+_{t,T}(y)}} f > 1 - \varepsilon$ for $y \in U$. We have $g_t(y) \geq \max_{\overline{J^+_{t,T}(y)}} f \geq 1 - \varepsilon$ for $y \in U$, therefore g_t is continuous at x.

□

Let us see how we can obtain a $B(A) \setminus A)$-time function.

Proposition 8.1.7. *Let A be an attractor. Let $x \in B(A, U) \setminus A$ and $y \in J^+(x) \setminus \{x\}$, there is an interval I of real numbers with non empty interior such that $g_s(y) < g_s(x)$ for all $s \in I$.*

Proof. Let $t > 0$ such that $y \in J^+_t(x)$. Let $\varepsilon = \frac{f(x)}{2}$ and consider:

$$a = \inf\{u > 0 / \ g_u(x) \leq \varepsilon\}$$

The continuity of f implies that $a > 0$. Let $I =]\max(a - t, 0)\,, a[$. If $s \in I$, then $g_s(x) > \varepsilon$ and $s + t > a$, therefore $g_s(y) \leq g_{s+t}(x) \leq g_a(x) \leq \varepsilon < g_s(x)$, and $g_s(y) < g_s(x)$. □

Corollary 8.1.8. *Let A be an attractor. Consider $(u_q)_{q \in \mathbb{Q}^*_+}$ a sequence of positive real numbers such that $\sum_{q \in \mathbb{Q}^*_+} u_q = 1$. Then $\tau_A = \sum_{q \in \mathbb{Q}^*_+} u_q(1 - g_q)$ is a $B(A, U) \setminus A$-time function.*

Proof. The function τ_A is continuous because the g_q are continuous and the sum converges normally. Let $x \in M$, and let us consider $y \in J^+(x)$. For $t \geq 0$, we have $J^+_t(y) \subset J^+_t(x)$, which gives us $g_t(y) \leq g_t(x)$, and $\tau_A(y) \geq \tau_A(x)$. If $x \in B(A, U) \setminus A$ and $y \neq x$, then Proposition 8.1.7 gives us an interval with non empty interior I such that $g_t(y) < g_t(x)$ for $t \in I$. Let $q_0 \in I \cap \mathbb{Q}$.

$$\begin{aligned}
\tau_A(y) &= u_{q_0}(1 - g_{q_0}(y)) + \sum_{q \neq q_0} u_q(1 - g_q(y)) \\
&\leq u_{q_0}(1 - g_{q_0}(y)) + \sum_{q \neq q_0} u_q(1 - g_q(x)) \\
&< u_{q_0}(1 - g_{q_0}(x)) + \sum_{q \neq q_0} u_q(1 - g_q(x)) \\
&< \tau_A(x)
\end{aligned}$$

□

We can easily extend this to the basin $B(A)$.

Corollary 8.1.9. *Let (M, g) be a spacetime and let A be an attractor. There is a $B(A) \setminus A$-time function.*

Proof. Since $B(A)$ is a subset of M, it is a separable metric space and therefore it satisfies the Lindelöf property: of any open cover we can extract a countable cover. This allows us to choose a sequence of pre attractors $(U_n)_{n\in\mathbb{N}}$ with attractor A such that $B(A) = \bigcup_{n\in\mathbb{N}} B(A, U_n)$. For $n \in \mathbb{N}$, let τ_n be a $B(A, U_n) \setminus A$-time function. The function $\tau = \sum_{n\in\mathbb{N}} 2^{-n}\tau_n$ is a $B(A) \setminus A$-time function. □

The same technique provides a time function for the union of all basins of attraction.

Corollary 8.1.10. *Let (M, g) be a spacetime and let $\mathcal{A}$ be the set of all attractors. There is a $\bigcup_{A\in\mathcal{A}} B(A, U) \setminus A$-time function.*

Proof. Let $U = \bigcup_{A\in\mathcal{A}} B(A)\setminus A$. By the Lindelöf property, we can choose a sequence of attractors $(A_n)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ such that $U = \bigcup_{n\in\mathbb{N}} B(A_n) \setminus A_n$. For $n \in \mathbb{N}$, let τ_n be a $B(A_n) \setminus A_n$-time function. The function $\tau = \sum_{n\in\mathbb{N}} 2^{-n}\tau_n$ is a U-time function.

□

2. Chain recurrence, and a Lorentzian Conley Theorem

Closed future curves are an obvious obstruction to the existence of a time function. Classically, the existence of a time function is linked to stable causality, which has an inconvenient: its definition involves other metrics, whereas the existence of a time function does not. We will see that we can find an obstruction to the existence of a time function that does not involve nearby metrics: chain recurrence. The idea of chains consists in joining points by sequences of long curves and small jumps from the end of a curve to the beginning of the next.

We follow [**Hur92**] and define $\mathcal{P}(M)$ as the set of continuous functions from M to $]0, +\infty[$.

Definition 8.2.1. *Let $\varepsilon \in \mathcal{P}(M)$, $T > 0$ and $p, q \in M$.*
An (ε, T)-chain from p to q is a finite sequence of future directed causal curves $(\gamma_i : [0, 1] \to M)_{i=1,\dots,k}$ of length at least T such that:

(1) $d(p, \gamma_1(0)) \leq \varepsilon(p)$
(2) $d(\gamma_i(1), \gamma_{i+1}(0)) \leq \varepsilon(\gamma_i(1))$ for all $i < k$
(3) $d(\gamma_k(1), q) \leq \varepsilon(q)$

A point $p \in M$ is said to be chain recurrent if for any $\varepsilon \in \mathcal{P}(M)$ and $T > 0$ there is an (ε, T)-chain from p to p. We will denote by $R(g)$ the set of chain recurrent points.

Conley noticed that chain recurrence is linked to attractors. The same goes for spacetimes.

Proposition 8.2.2. *Let $x \notin R(g)$. There exists an attractor A such that $x \in B(A) \setminus A$.*

Proof. Since $x \notin R(g)$, let us consider $\varepsilon \in \mathcal{P}(M)$ and $T > 0$ such that there is no (ε, T)-chain from x to x. Let U be the set of points $y \in M$ such that there is an (ε, T)-chain from x to y. It follows from the definition of (ε, T)-chains that U is open. We will start by proving that U is a pre attractor.

Let $y \in \overline{J_T^+(U)}$. Let us consider a function $\delta \in \mathcal{P}(M)$ such that $\delta \leq \frac{\varepsilon}{2}$ and that $d(y, z) < \delta(z)$ implies $\varepsilon(z) > \frac{\varepsilon(y)}{2}$ (the existence of such a function, which can be seen as continuous continuity modulus of ε, is proved in [**CCP02**]). Let $z \in B(y, \delta(y)) \cap J_T^+(U)$. We can write $z = \gamma(1)$ where γ is a future curve with length at least T and such that

$\gamma(0) \in U$. Since there is an (ε, T)-chain $\gamma_1, \ldots, \gamma_k$ from x to $\gamma(0)$ (by definition of U), the choice of the function δ was made in such a way that $\gamma_1, \ldots, \gamma_k, \gamma$ is an (ε, T)-chain from x to y, therefore $y \in U$.

If γ is a future curve of length at least T such that $\gamma(0) = x$, then $\gamma(1) \in U$ (because γ itself is an (ε, T)-chain). Therefore $J_T^+(x) \subset U$, and $x \in B(A, U)$.

Since there is no (ε, T)-chain from x to x, we know that $x \notin U$, but $A \subset U$, hence $x \notin A$. We have shown that $x \in B(A, U) \setminus A \subset B(A) \setminus A$. □

By combining this result and Corollary 8.1.10, we obtain the following:

Theorem 8.2.3. *Let (M, g) be a spacetime. There exists a $M \setminus R(g)$-time function. Particularly, if $R(g) = \emptyset$, then there exists a time function.*

As mentioned earlier, chain recurrence is an obstruction to the existence of a time function, therefore the last statement of this theorem is an equivalence.

Theorem 8.2.4. *Let (M, g) be a spacetime that admits a time function. Then for any $T > 0$ there is a function $\varepsilon \in \mathcal{P}(M)$ such that there is no (ε, T)-chain with same end points, and therefore $R(g) = \emptyset$.*

Proof. Let f be a time function. If $K \subset M$ is compact and $T > 0$ let $\alpha_{K,T} = \inf\{|f(y) - f(x)|/y \in K \text{ and } y \in J_T^+(x)\}$.

Since h is adapted to g, we have $\overline{J_T^+(x)} \subset J_{T/\sqrt{2}}^+(x)$, which shows that $\alpha_{K,T} > 0$.

Let us fix $x_0 \in M$ and write $M = \bigcup_{n \in \mathbb{N}} K_n$ where $K_n = \overline{B}(x_0, nT)$. For $x \in M$ and $T > 0$, we will denote by $n(x)$ the smallest integer n such that $x \in \mathring{K}_n$.

We will construct a function $\varepsilon \in \mathcal{P}(M)$ such that for all $x \in M$ and for all $y \in B(x, \varepsilon(x))$, we have $f(y) \leq f(x) + \frac{1}{2}\alpha_{K_{n(x)},T}$.

Let us consider $x \in M$ and U_x a relatively compact open neighbourhood of x that lies in $\mathring{K}_{n(x)}$. For $y \in U_x$, we have $n(y) \leq n(x)$ hence $\alpha_{K_{n(y)}} \geq \alpha_{K_{n(x)}}$. The compacity of $\overline{U_x}$ and the continuity of f assure the existence of $\delta_x > 0$ such that for all $y \in U_x$ and $z \in B(y, \delta_x)$, we have $f(z) \leq f(y) + \frac{\alpha_{K_{n(x)}}}{2} \leq f(y) + \frac{\alpha_{K_{n(y)}}}{2}$. From the open covering $M = \bigcup_{x \in M} U_x$, we extract a locally finite covering $M = \bigcup_{i \in I} U_{x_i}$. Let $\varepsilon_i = \inf\{\delta_j / \ U_{x_i} \cap U_{x_j} \neq \emptyset\}$ and let $(\theta_i)_{i \in I}$ be a partition of unity subordinate to $M = \bigcup_{i \in I} U_{x_i}$. The function $\varepsilon = \sum_{i \in I} \varepsilon_i \theta_i$ meets our requirements.

Let $x \in M$ and consider an (ε, T)-chain from x to x. It can be seen as a sequence of points $(x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{k-1})$ in M such that $x_1 = x = x_k$, $y_i \in J_T^+(x_i)$ and $d(y_i, x_{i+1}) < \varepsilon(y_i)$.

We have $f(x_{i+1}) \leq f(y_i) + \frac{\alpha_{K_{n(y_i)}}}{2}$, but $|f(y_i) - f(x_i)| \leq \alpha_{K_{n(y_i)}}$, therefore $f(x_{i+1}) \leq f(x_i) - \frac{\alpha_{K_{n(y_i)}}}{2} < f(x_i)$, which implies $f(x_k) < f(x_1)$, i.e. $f(x) < f(x)$, which is absurd.

We have shown that for any $x \in M$, there is no (ε, T)-chain from x to x.

□

3. Hawking's Theorem

Hawking's Theorem states that spacetime admits a time function if and only if it is stably causal. Though the important result is the existence of a time function for a stably causal spacetime, the necessity of stable causality is non trivial, and was not proved by Hawking. So far, it seems that the only available proof of this necessity is to first use Bernal and Sanchez's Theorem from **[BS05]** that shows the existence of a temporal function, and it is easy to see that a temporal function is a time function for all close metrics. The problem in the topological case is that a time function is not necessarily a time function for close metrics (see for example a linear function with past directed lightlike gradient in Minkowski space).

In the previous section, we showed that the absence of chain recurrence is equivalent to the existence of a (continuous) time function. It would be interesting to a find a proof of that a time function implies stable causality without using differentiable functions, and a possibility would be to show that the absence of chain recurrence implies stable causality.

We are now going to prove the direct sense in Hawking's Theorem: stable causality implies the existence of a time function. By using Corollary 8.1.10 all we have to see is that the set $\bigcup_{A\in\mathcal{A}} B(A)\setminus A$ is the whole manifold M.

Lemma 8.3.1. *Let $g' > g$ and $x \in M$. The chronological future $U = I^+_{g'}(x)$ of x for g' is a pre attractor.*

Proof. We will show that $\overline{J^+_1(U)} \subset U$. If $y \in \overline{J^+_1(U)}$, we can find a sequence γ_k of past directed causal curves (for g), with unit speed (for h) such that $\gamma_k(0) \to y$ and $\gamma_k(t_k) \in U$ for some $t_k \geq 1$. Since a causal curve for g is also causal for g', we have $\gamma_k(1) \in U$. Up to a sub sequence, we can assume that $\gamma_{k/[0,1]}$ converges uniformly to a past directed causal curve γ. Let $z = \gamma(1)$. We have $z \in J^-_g(y)$, therefore $z \in I^-_{g'}(y)$. Since $I^-_{g'}(y)$ is open and $\gamma_k(1) \to z$, we have $\gamma_k(1) \in I^-_{g'}(y)$ for k large enough, and $\gamma_k(1) \in U$ implies $y \in U$. □

Proposition 8.3.2. *Let (M, g) be a stably causal spacetime. Then for all $x \in M$, there is an attractor A such that $x \in B(A) \setminus A$.*

Proof. Let $x \in M$ and let $g' > g$ be a strongly causal metric (this is possible because g is stably causal, see Proposition 6.4.3). Let W be a neighbourhood of x in U such that the intersection of any g'-causal curve with W is connected, small enough to satisfy Proposition 7.1.9 (there is an upper bound on the length of g-causal curves in W), and let us consider $z \in I^-_g(x) \cap W$. Then $U = I^+_{g'}(z)$ is a pre attractor, and $x \in U \subset B(A, U)$, where A is the attractor associated to U.

Let us show that $x \notin A$. If it were the case, then for all $t > 0$ there would be a g-causal future curve γ_t of length at least t such that $\gamma_t(0) \in U$ and $d(\gamma_t(1), x) \leq 1/t$. Since $\gamma_t(0) \in U = I^+_{g'}(z)$, we can consider a g'-causal future curve η_t such that $\eta_t(0) = z$ and $\eta_t(1) = \gamma_t(1)$ (the concatenation of a g'-timelike curve from z to $\gamma_t(0)$ and of γ_t). If t is large enough, then $\eta_t(1) \in W$, and therefore $\eta_t(s) \in W$ for all $s \in [0, 1]$, hence $\gamma_t(s) \in W$ for all $s \in [0, 1]$, but this is impossible because $\ell_h(\gamma_t) \to +\infty$. Therefore $x \notin A$. □

Note that we used the fact that a stably causal spacetime is strongly causal in order to construct a time function, even though the classical proof uses a time function. This is why we gave a direct proof of Proposition 6.4.3. However, we could have used a weaker notion than strong causality (future distinguishing) that is used in Hawking's proof.

To complete the proof of the direct sense in Hawking's Theorem, notice that according to Proposition 8.3.2, the union $\bigcup_{A\in\mathcal{A}} B(A)\setminus A$, where $\mathcal{A}$ is the set of attractors, is equal to the whole manifold M. Corollary 8.1.10 implies that there is time function.

CHAPTER 9

Conjugacy for multi valued systems

1. Introduction

We will now study multi valued dynamical systems, i.e. to a point x in a space X, we associate a set $F(x) \subset X$. Several notions of regularity make sense, such as continuity for the Hausdorff topology if X is a metric space and $F(x)$ is always compact. Another classical regularity is to ask for the graph $G(F) = \{(x, y) \in X^2 | y \in F(x)\}$ to be closed (which is equivalent to a semi continuity for the Hausdorff topology).

The results in chapter 8 revolved around a generalisation of the notion of Lyapunov functions to multi valued dynamical systems. It is not clear which notion of classical dynamics makes sense in the multi valued context. Take iterations for example, or more generally the composition of two systems $F, G : X \to \mathfrak{P}(X)$. One could define $G \circ F(x)$ as $\bigcup_{y \in F(x)} G(y)$, or as the intersection. In order to choose the right definition, we have to choose which properties we want to keep from classical dynamics. A natural one is that they agree in the classical setting, i.e. when $F(x) = \{f(x)\}$ where $f \in \mathrm{Homeo}(X)$. In this chapter, we are going to discuss two notions of conjugacy.

In order to study these systems, we introduce the space of orbits $\Sigma_F = \{(x_n)_{n\in\mathbb{Z}} \in X^{\mathbb{Z}} | \forall n \in \mathbb{Z}\ x_{n+1} \in F(x_n)\}$. This space is naturally endowed with a homeomorphism $\sigma_F((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$, which we will call the **shift associated to** F.

In the single valued case, i.e. if $F(x) = \{f(x)\}$ where $f \in \mathrm{Homeo}(X)$, the shift σ_F is topologically conjugate to f.

We will study notions of conjugacy for two multi valued systems F_1, F_2 defined on spaces X_1, X_2. The first natural notion, which we will call **strong conjugacy**, is the existence of a homeomorphism $h : X_1 \to X_2$ such that $h(F_1(x)) = F_2(h(x))$ for all $x \in X_1$. This notion is very restrictive: if $X = \mathbb{R}$ and $F(x) = [f(x), g(x)]$ where $f, g \in \mathrm{Homeo}(\mathbb{R})$ do not have fixed points, then strong conjugacy between F and the equivalent system where f and g are translations is equivalent to the fact that f and g commute, which is usually not the case. We have lost the notion of structural stability for such homeomorphisms while looking at strong conjugacy.

We will say that $F_1 : X_1 \to \mathfrak{P}(X_1)$ and $F_2 : X_2 \to \mathfrak{P}(X_2)$ are **shift conjugate** if the associated shifts $\sigma_{F_1} \in \mathrm{Homeo}(\Sigma_{F_1})$ and $\sigma_{F_2} \in \mathrm{Homeo}(\Sigma_{F_2})$ are conjugate by a homeomorphism. Clearly, strong conjugacy implies shift conjugacy. We will show that in the case mentioned above, for homeomorphisms of the real line, shift conjugacy is a strictly weaker notion than strong conjugacy, then we will see that it is not trivial, i.e. that it allows us to distinguish some systems.

2. A flexible case: segments of the real line

This section is devoted to the proof of the following result:

Proposition 9.2.1. *All systems $F(x) = [f(x), g(x)]$ where $f, g \in \mathrm{Homeo}(\mathbb{R})$ satisfy $x < f(x) < g(x)$ for all $x \in \mathbb{R}$ and such that there is $x_0 \in \mathbb{R}$ satisfying $f \circ f(x_0) > g(x_0)$ are shift conjugate to each other.*

Remark. The fact that there is $x_0 \in \mathbb{R}$ such that $f \circ f(x_0) > g(x_0)$ means that the interval $F(x_0)$ is not too large. For example, if we start with $T > 0$ and we open a small interval around the translation $x+T$, i.e. if we consider a multi valued system $F(x) = [f(x)\,,g(x)]$ where $x + \frac{3T}{4} < f(x) < g(x) < x + \frac{5T}{4}$, this hypothesis is met. We then find a result that is similar to the structural stability of translations (however the shift σ_F is not conjugate to a translation of the real line, because the underlying spaces are not homeomorphic). It is therefore natural that the proofs are similar (we will look for a common fundamental domain).

PROOF. First, notice that Σ_F is homeomorphic to $\mathbb{R} \times [0\,,1]^{\mathbb{Z}}$, where the shift σ_F can be written:

$$\sigma_F(x,(t_n)_{n\in\mathbb{Z}}) = (t_0 f(x) + (1-t_0)g(x),(t_{n+1})_{n\in\mathbb{Z}})$$

Consider the two domains:

$$D_1 = [x_0\,,f(x_0)[\, \times [0\,,1]^{\mathbb{Z}}$$

$$D_2 = \{(x,(t_n)_{n\in\mathbb{Z}}) \in [f(x_0)\,,g(x_0)[\, \times [0\,,1]^{\mathbb{Z}} | \mathbb{Z}.(x,(t_n)_{n\in\mathbb{Z}}) \cap D_1 = \emptyset\}$$

First step : $D = D_1 \cup D_2$ is a fundamental domain (i.e. it meets every orbit of σ_F exactly once)

Every orbit meets $[x_0\,,g(x_0)[\, \times [0\,,1]^{\mathbb{Z}}$ (because $[x_0\,,g(x_0)]$ is a fundamental domain for the action of g on $\mathbb{R}$). If the orbit does not meet D_1, then it meets D_2 by defintion. Hence D intersects every orbit.

If two elements of the same orbit are in D, then it follows from the definition of D_2 that both points are in the same subset D_1 or D_2.

If they are in D_1, then we can find $x, y \in \mathbb{R}$ such that $x_0 \leq x < y < f(x_0)$ and $y \geq f(x)$, which contradicts the fact that f is increasing.

If both points are in D_2, then we can find $x, y \in \mathbb{R}$ such that $f(x_0) \leq x < y < g(x_0)$ and $f(x) \leq y \leq g(x)$, hence $y \geq f(x) \geq f(f(x_0)) > g(x_0)$, which is absurd. Therefore D can only meet an orbit once.

Define the domain D_3 by:

$$D_3 = \{(x,(t_n)_{n\in\mathbb{Z}}) \in [f(x_0)\,,g(x_0)[\, \times [0\,,1]^{\mathbb{Z}} / x < t_{-1}f(x_0) + (1-t_{-1})g(x_0)\}$$

Second step : $D_2 = D_3$

Let $(x,(t_n)) \in D_2$. The map σ^{-1} is defined by:

$$\sigma^{-1}(x,(t_n)) = ((t_{-1}f + (1-t_{-1})g)^{-1}(x),(t_{n-1}))$$

Since $\sigma^{-1}(x,(t_n)) \notin D$, we get that:

$$(t_{-1}f + (1-t_{-1})g)^{-1}(x) < x_0$$

This shows that $(x,(t_n)) \in D_3$.

Let $(x,(t_n)) \in D_3$. The fact that $(t_{-1}f + (1-t_{-1})g)^{-1}(x) < x_0$ shows that $\sigma^{-1}(x,(t_n)) \notin D_1$, and $\sigma^{-k}(x,(t_n)) \notin D_1$ for $k \geq 1$. Since $x \geq f(x_0)$ implies that $\sigma^k(x,(t_n)) \notin D_1$ pour $k \geq 0$, we see that $(x,(t_n)) \in D_2$.

Third step : Homeomorphism between the fundamental domains

Consider two systems $F_i(x) = [f_i(x)\,,g_i(x)]$ for $i \in \{1,2\}$, both satisfying the hypothesis of the initial statement. Let $x_i \in \mathbb{R}$ be such that $f_i(f_i(x_i)) > g_i(x_i)$, and let $D^i = D_1^i \cup D_2^i$ be the fundamental domain considered above. Let $\varphi : [x_1\,,g_1(x_1)] \to [x_2\,,g_2(x_2)]$ be an increasing homeomorphism, piecewise affine, such that $\varphi(f_1(x_1)) = f_2(x_2)$ (as we will notice later, it is important for φ to be affine on $[f_1(x_1)\,,g_1(x_1)]$). Let Φ be the map from $[x_1\,,g_1(x_1)]\times[0\,,1]^{\mathbb{Z}}$ to $[x_2,g_2(x_2)]\times[0\,,1]^{\mathbb{Z}}$ defined by $\Phi(x,(t_n))=(\varphi(x),(t_n))$. It is a homeomorphism. Let us show that $\Phi(D^1)\subset D^2$.

Let $(x,(t_n))\in D^1$. If $(x,(t_n))\in D^1_1$, then immediatly $\Phi(x,(t_n))\in D^2_1$. Assume that $(x,(t_n))\in D^1_2$. Since $D^1_2=D^1_3$, we see that:

$$x<t_{-1}f_1(x_1)+(1-t_{-1})g_1(x_1)$$

Since φ is affine on $[f_1(x_1)\,,g_1(x_1)]$, we get:

$$\begin{aligned}\varphi(t_{-1}f_1(x_1)+(1-t_{-1})g_1(x_1)) &= t_{-1}\varphi(f_1(x_1))+(1-t_{-1})\varphi(g_1(x_1))\\ &= t_{-1}f_2(x_2)+(1-t_{-1})g_2(x_2)\end{aligned}$$

Since φ is invreasing, we obtain:

$$\varphi(x)<\varphi(t_{-1}f_1(x_1)+(1-t_{-1})g_1(x_1))=t_{-1}f_2(x_2)+(1-t_{-1})g_2(x_2)$$

Hence $\Phi(x,(t_n))\in D^2_2$.

Since the inverse of Φ can be constructed in the same way, it follows that $\Phi(D^1)=D^2$, and $\Phi(\overline{D^1})=\overline{D^2}$.

Fourth step : Conjugacy between the shifts

For all $x\in\mathbb{R}\times[0\,,1]^{\mathbb{Z}}$, there is a unique $n_x\in\mathbb{Z}$ such that $\sigma_1^{n_x}(x)\in D_1$, and we set $h(x)=\sigma_2^{-n_x}(\Phi(\sigma_1^{n_x}(x)))$. The map h is a homeomorphism that conjugates σ_1 and σ_2. □

3. A rigid case: multi valued rotations on the circle

Rotations of the circle are only conjugate to each other (by an orientation preserving homeomorphism) when they are equal. This is easy to see once the right invariant has been found: the rotation number. We will see that there is an analog for multi valued systems, that is invariant under shift conjugacy.

For $a\neq b$ in $\mathbb{S}^1=\mathbb{R}/\mathbb{Z}$, we set $[a\,,b]=\{x\in\mathbb{S}^1|a\leq x\leq b<a\}$.

Proposition 9.3.1. *Consider* $0<\alpha_i<\beta_i<1$ *for* $i\in\{1,2\}$*, and* $F_i(x)=[x+\overline{\alpha}_i\,,x+\overline{\beta}_i]$ *for* $x\in\mathbb{S}^1$*. Then* F_1 *and* F_2 *are shift conjugate if and only if* $F_1=F_2$*.*

Proof. Consider $\Sigma=\mathbb{S}^1\times[0,1]^{\mathbb{Z}}$ where the shift is defined by:

$$\sigma_{F_i}(x,(t_n))=(x+t_0\overline{\alpha}_i+(1-t_0)\overline{\beta}_i,(t_{n+1}))$$

Note that the universal cover of Σ is $\widetilde{\Sigma}=\mathbb{R}\times[0\,,1]^{\mathbb{Z}}$.

First step : Definition and characterisation of the rotation set

The easiest is to define the rotation number of a periodic orbit.

Let $\underline{x}=(x,(t_n))$ be a periodic point for σ_{F_i}. Let q be the period of $\underline{x}$. Since $\sigma^q_{F_i}(\underline{x})=(x+\sum_{j=0}^{q-1}t_j\overline{\alpha}_i+(1-t_j)\overline{\beta}_i,(t_{n+k}))=\underline{x}$, there is $p\in\mathbb{Z}$ such that $\sum_{j=0}^{q-1}t_j\alpha_i+(1-t_j)\beta_i=q$. Let $\rho(\underline{x})=\overline{p/q}\in\mathbb{R}/\mathbb{Z}$. It lies in $[\overline{\alpha}_i\,,\overline{\beta}_i]$. Let $\rho(F_i)$ be the set of $\rho(\underline{x})$ for all periodic points of σ_{F_i}, which we call the rotation set of F_i. We see that $\rho(F_i)\subset[\overline{\alpha}_i\,,\overline{\beta}_i]\cap\mathbb{Q}$. By considering $(x,(\gamma))$ for $\gamma\in[\overline{\alpha}_i\,,\overline{\beta}_i]\cap\mathbb{Q}$, we see that $\rho(F_i)=[\overline{\alpha}_i\,,\overline{\beta}_i]\cap\mathbb{Q}$.

Second step : Invariance of the rotation set under shift conjugacy

Let $h:\Sigma\to\Sigma$ be a conjugacy between σ_{F_1} and σ_{F_2}. Let $\underline{x}=(x,(t_n))$ be a periodic point for σ_{F_1}. Then $h(\underline{x})$ is a periodic point for σ_{F_2}, with the same period q.

Let $\widetilde{h} : \widetilde{\Sigma} \to \widetilde{\Sigma}$ be a lift to the universal cover. Note that $\rho(\underline{x})$ can be defined in the following way: if $(\widetilde{x}, (t_n))$ is a lift to $\widetilde{\Sigma}$, then $\rho(\underline{x}) = \lim_{n\to+\infty} \frac{x_n - x_0}{n}$ where $x_n = \widetilde{x} + \sum_{j=0}^{n-1} t_j\alpha_1 + (1-t_j)\beta_1$.

Let $\underline{y} = h(\underline{x})$ and let $(\widetilde{y}, (s_n))$ be a lift to the universal cover. By noting π_0 the map $(z, (u_n)) \mapsto z$, we now have:

$$\begin{aligned}
\rho(\underline{y}) &= \lim_{n\to+\infty} \frac{y_n - y_0}{n} \\
&= \lim_{n\to+\infty} \frac{\pi_0(h(\sigma_1^n(\underline{x}))) - \pi_0(h(\underline{x})}{n} \\
&= \lim_{n\to+\infty} \frac{\pi_0(h(\sigma_1^n(\underline{x}))) - \pi_0(\sigma_1^n(\underline{x}))}{n} + \frac{\pi_0(\sigma_1^n(\underline{x})) - \pi_0(\underline{x})}{n} + \frac{\pi_0(\underline{x}) - \pi_0(h(\underline{x}))}{n} \\
&= 0 + \rho(\underline{x}) + 0
\end{aligned}$$

The fact that the two peripheral terms tend to 0 is due to the fact that $\pi_0(h(\underline{x})) - \pi_0(\underline{x})$ is continuous and periodic in space, hence bounded (because $\mathbb{S}^1 \times [0,1]^{\mathbb{Z}}$ is compact).

Third step : Conclusion

If F_1 and F_2 are shift conjugate, then $\rho(h(\underline{x})) = \rho(\underline{x})$ for all σ_{F_1}-periodic point $\underline{x}$, hence $\rho(F_1) = \rho(F_2)$, therefore $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, i.e. $F_1 = F_2$. $\square$

Bibliography

[Ada01] S. Adams: *Dynamics on Lorentz manifolds*, World Scientific, Singapore, 2001

[Ahl64] L.V. Ahlfors: *Finitely generated Kleinian groups*, Amer. J. Math., **86** (1964), p. 413-429

[Amo79] A.M. Amores: *Vector fields of a finite type G-structure*, J. Differential Geometry, **14** (1979), p. 1-6

[AS97a] S. Adams, G. Stuck: *The isometry group of a compact Lorentz manifold*, Inventiones Mathematicae, **129** (1997), p. 239-261

[AS97b] S.Adams, G. Stuck: *The isometry group of a compact Lorentz manifold II*, Inventiones Mathematicae, **129** (1997), p. 263-287

[Bar95] T. Barbot: *Caractérisation des flots d'Anosov en dimension* 3 *par leurs feuilletages faibles*, Ergodic Theory Dynam. Systems, **15** (1995), no. 2, p.247-270

[Bar96] T. Barbot: *Flots d'Anosov sur les variétés graphées au sens de Waldhausen*, Ann. Inst. Fourier, **46** (1996), p. 1451-1517

[Bar01] T. Barbot: *Plane affine Geometry of Anosov flows*, Annales Scientifiques de l'Ecole Normale Supérieure, **34** (2001), no. 6, p. 871-889

[Bar05] T. Barbot, *Géométrie lorentzienne, causalité et convexité*, http://www.umpa.ens-lyon.fr/∼barbot/COURS/ADS2/Notescours.pdf, lecture notes, 2005-2006

[Bea83] A.F. Beardon: *The geometry of discrete groups*, Springer Verlag, New York, 1983

[Ber65] L. Bers: *Automorphic forms and Poincaré series for infinitely generated Fuchsian groups*, Amer. J. Math., **87** (1965), p. 196-214

[Bow96] B. H. Bowditch: *Convergence groups and configuration spaces*, Geometric group theory down under (Canberra, 1996), de Gruyter, Berlin, 1999, p. 23–54

[But00] J. Button: *Matrix representations and the Teichmüller space of the twice punctured torus*, Conformal Geometry and Dynamics, **4** (2000), p. 97-107

[BEE96] J. K. Beem, P. E. Ehrlich, K. L. Easley: *Global Lorentzian Geometry*, Pure and Applied Mathematics, Marcel Dekker, 2^{nd} edition, 1996

[BM13] C. Bavard, P. Mounoud: *Sur les surfaces lorentziennes compactes sans points conjugués*, Geometry and Topology, **17** (2013), p. 469-492

[BS03] A. N. Bernal, M. Sánchez: *On Smooth Cauchy Hypersurfaces and Geroch's Splitting Theorem*, Comm. Math. Phys., **243** (2003), no.3, p. 437-456

[BS05] A. N. Bernal, M. Sánchez: *Smoothness of Time Functions and the Metric Splitting of Globally Hyperbolic Spacetimes*, Comm. Math. Phys., **257** (2005), no.1, p. 43-50

[Con88] C. Conley: *The Gradient Structure of a Flow : I*, in *Charles Conley Memorial Volume*, Special Issue of Ergodic Theory and Dynamical Systems, 1988

[CCP02] S. K. Choi, C-K. Chu, J. S. Park: *Chain Recurrent Set for Flows on Non-Compact Spaces*, J. Dynam. Differential Equations, **14** (2002), no.3, p. 597-611

[CJ94] A. Casson, D. Jungreis: *Convergence groups and Seifert fibered* 3*-manifolds*, Invent. Math., **118** (1994), no. 3, p.441-456

[CR94] Y. Carrière, L. Rozoy: *Complétude des métriques lorentziennes de* $\mathbb{T}^2$ *et difféomorphismes du cercle*, Bol. Soc. Bras. Mat., **25** (1994), no. 2, p. 223-235

[Den32] A. Denjoy: *Sur les courbes définies par les équations différentielles à la surface du tore*, J. Math. Pures Appl., (9) **11** (1932), p. 333-375

[DG00] C. Duval, L. Guieu: *The Virasoro group and Lorentzian surfaces: the hyperboloid of one sheet*, Journal of geometry and physics, **33** (2000), p. 103-127

[D'AG91] G. D'Ambra, M. Gromov: *Lectures on transformation groups: geometry and dynamics*, Surveys in Differential Geometry, **1** (1991), p. 19-111

[DKN09] B. Deroin, V. Kleptsyn, A. Navas: *On the question of ergodicity for minimal group actions on the circle*, Moscow Mathematical Journal, **9** (2009), no. 2, p. 263-303

[DKN13] B. Deroin, V. Kleptsyn, A. Navas: *Towards the solution of some fundamental questions concerning group actions on the circle and codimension-one foliations*, preprint, 2013, arXiv:1312.4133

[DMZ08] M. Deffaf, K. Melnick, A. Zeghib: *Actions of noncompact semisimple groups on Lorentz manifolds*, Geom. Funct. Anal., **18** (2008), no. 2, p. 463-488

[DO00] C. Duval, V. Ovsienko: *Lorentzian worldlines and the Schwarzian derivative*, Functional Analysis and its applications, **34** (2000), no. 2, p. 135-137

[Fer96] J. Ferrand: *The action of conformal transformations on a Riemannian manifold*, Math. Ann., **304** (1996), p. 277-291

[Fra05a] C. Frances: *Lorentzian Kleinian groups*, Comment. Math. Helv., **80** (2005), no. 4, p. 883-910.

[Fra05b] C. Frances: *Des contre-exemples au théorème de Ferrand-Obata en géométrie Lorentzienne conforme*, Math. Ann., **332** (2005), no. 1, p. 103-119

[Fra07] C. Frances: *Sur le groupe d'automorphismes des géométries paraboliques de rang un*, Ann. Scient. Ec. Norm. Sup., **40** (2007)

[FH13] P. Foulon, B. Hasselblatt: *Contact Anosov flows on hyperbolic* 3*-manifolds*, Geometry & Topology, **17** (2013), p. 1225-1252

[FHS11] J.L. Flores, J. Herrera, M. Sánchez: *On the final definition of the causal boundary and its relation with the conformal boundary*, Adv. Theor. Math. Phys., **15** (2011), p. 991-1058

[FS12] A. Fathi, A. Siconolfi: *On smooth time functions*, Mathematical Proceedings of the Cambridge Philosophical Society, **152** (2012), no. 2, p. 303-339

[FT02] C. Frances, C. Tarquini: *Autour du théorème de Ferrand-Obata*, Annals of Global Analysis and Geometry, **21** (2002), p. 51–62

[Gab92] D. Gabai: *Convergence groups are Fuchsian groups*, Ann. of Math. **136** (1992), p. 447-510

[Ghy87a] E. Ghys: *Flots d'Anosov dont les feuilletages stables sont différentiables*, Ann. Scient. Ec. Norm. Sup., (4) **20** (1987), no. 2, p. 251-270

[Ghy87b] E. Ghys: *Groupes d'homéomorphismes du cercle et cohomologie bornée*, Contemporary Mathematics, **58** (1987), Part III, p. 81-106

[Ghy92] E. Ghys: *Déformations de flots d'Anosov et de groupes fuchsiens*, Ann. Inst. Fourier, **42** (1992), p. 209-247

[Ghy93] E. Ghys: *Rigidité différentiable des groupes fuchsiens*, Publ. Math. de l'I.H.E.S., **78** (1993), p. 163-185

[Ghy01] E. Ghys: *Groups acting on the circle*, Enseign. Math., (2) **47** (2001), no. 3-4, p. 329-407

[Her79] M.R. Herman: *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Publ. Math. de l'I.H.E.S., **49** (1979), p. 5-234

[Hur92] M. Hurley: *Non Compact Chain Recurrence and Attraction*, Proc. Amer. Math. Soc., **115** (1992), no.4, P. 1139-1148

[Hur95] M. Hurley: *Chain Recurrent Semiflows and Gradients*, J. Dynam. Differential Equations, **7** (1995), no.3, p. 437-456

[Hur98] M. Hurley: *Lyapunov Functions and Attractors in Arbitrary Metric Spaces*, Proc. Amer. Math. Soc., **126** (1998), no.1, p. 245-256

[HE73] S. W. Hawking, G. F. R. Ellis: *The large scale structure of space-time*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 1973

[HK90] S. Hurder, A. Katok: *Differentiability, rigidity, and Godbillon-Vey classes for Anosov flows*, Publ. Math. de l'I.H.E.S., **72** (1990), p. 5-61

[HP69] M. Hirsh, C. Pugh : *Stable manifolds for hyperbolic sets*, Bull. Amer. Math. Soc., **75**, no. 1 (1969), p. 149-152

[Kob72] S. Kobayashi: *Transformation Groups in Differential Geometry*, Springer-Verlag, 1972

[Kow96] N. Kowalsky, *Actions of non-compact simple groups on Lorentz manifolds and other geometric manifolds*, Annals of Mathematics, (2) **144** (1996), p. 611-640

[KH95] A. Katok, B. Hasselblatt: *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia of Mathematics and its Applications, **54**, Cambridge University Press, 1995

[dlL92] R. de la Llave, *Smooth conjugacy and S-R-B measures for uniformly and non-uniformly hyperbolic systems*, Comm. Math. Phys., **150** (1992), no. 2, p. 289-320

[Liv71] A. N. Livšic: *Homology properties of U systems*, Math. Notes, **10** (1971), p. 758-763

[LMM86] R. de la Llave, J. M. Marco, R. Moriyón: *Canonical perturbation theory of Anosov systems and regularity results for the Livšic cohomology equation*, Ann. of Math., (2) **123** (1986), no. 3, p. 537–611

[LMM88] R. de la Llave, J. M. Marco, R. Moriyón: *Invariants for smooth conjugacy of hyperbolic dynamical systems*, I-IV., Comm. Math. Phys., **109**, **112**, **116** (1987, 1988)

[Mat87] S. Matsumoto: *Some remarks on foliated S^1 bundles*, Invent. Math., **90** (1987), p. 343-358

[MarSko89] G. Martin, R. Skora: *Group actions of the* 2*-sphere*, Am. Jour. Math., **111** (1989), no. 3, p. 387-402

[MinSán08] E. Minguzzi, M. Sánchez: *The causal hierarchy of spacetimes*, Recent developments in pseudo-Riemannian geometry, ESI Lect. Math. Phys., Eur. Math. Soc., Zürich, 2008 p. 299-358

[Nav02] A. Navas: *Actions de groupes de Kazhdan sur le cercle*, Ann. Scient. Ec. Norm. Sup., **35** (2002), p. 749-758

[Nav06] A. Navas: *Reduction of cocycles and groups of diffeomorphisms of the circle*, Bull. Belg. Math. Soc., **13** (2006), p. 193-205

[Nav11] A. Navas: *Groups of circle diffeomorphisms*, Chicago Lectures in Mathematics (2011)

[NT07] M. Nicol, A. Török: *Whitney regularity for solutions to the coboundary equation on Cantor sets*, Math. Phys. Electron. J., **13** (2007), Paper 6

[O'N83] B. O'Neill: *Semi-Riemannian Geometry With Applications To Relativity*, Pure and Applied Mathematics, Elsevier Science, 1983

[O'N95] B. O'Neill: *The Geometry of Kerr Black Holes*, A K Peters Ltd, 1995

[Pag09] P. Pageault: *Conley barriers and their applications: Chain-recurrence and Lyapunov functions*, Topology and its Applications, **156** (2009), no. 15, p. 2426-2442

[Pen72] R. Penrose: *Techniques of Differential Topology in Relativity*, Regional Conference Series in Applied Mathematics, Society for Industrial and Applied Mathematics, 1972

[PZ13] P. Piccione, A. Zeghib: *Actions of discrete groups on stationary Lorentz manifolds*, Ergodic Theory Dynam. Systems, available on CJO2013. doi:10.1017/etds.2013.17.

[Suh13] S. Suhr: *Closed geodesics in Lorentzian surfaces*, Trans. Amer. Math. Soc., **365** (2013), p. 1469-1486

[Sul85] D. Sullivan: *Quasiconformal homeomorphisms and dynamics II: Structural stability implies hyperbolicity for Kleinian groups*, Acta Math., **155** (1985), p. 243-260

[Sun85] T. Sunada: *Riemannian coverings and isospectral manifolds*, The Annals of Mathematics, **121** (1985), no. 1, p. 169-186

[Wei96] T. Weinstein: *An Introduction to Lorentz Surfaces*, De Gruyter expositions in mathematics, **22** (1996)

[Yoc84] J.C. Yoccoz: *Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne*, Ann. Scient. Ec. Norm. Sup., **17** (1984), p. 333-359

[Yoc95] J.C. Yoccoz: *Petits diviseurs en dimension* 1, Astérisque, **231** (1995)

[Zeg99a] A. Zeghib: *Isometry groups and geodesic foliations of Lorentz manifolds. Part I: Foundations of Lorentz dynamics*, Geom. Funct. Anal., **9** (1999), p. 775-822

[Zeg99b] A. Zeghib: *Isometry groups and geodesic foliations of Lorentz manifolds. Part II: Geometry of analytic Lorentz manifolds with large isometry groups*, Geom. Funct. Anal., **9** (1999), p. 823-854