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# Two-step values for games with two-level communication structure*

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**Abstract**

TU games with two-level communication structure, in which a two-level communication structure relates fundamentally to the given coalition structure and consists of a communication graph on the collection of the a priori unions in the coalition structure, as well as a collection of communication graphs within each union, are considered. For such games we introduce two families of two-step values inspired by the two-step procedures staying behind the Owen value (Owen, 1977) and the two-step Shapley value (Kamijo, 2009) for games with coalition structures. Our approach is based on the unified treatment of several component efficient values for games with communication structure and it generates two-stage solution concepts that apply component efficient values for games with communication structure on both distribution levels. Comparable axiomatic characterizations are provided.

**Keywords:** TU game with two-level communication structure, Owen value, two-step Shapley value, component efficiency, deletion link property

**JEL Classification Number:** C71

**Mathematics Subject Classification 2000:** 91A12, 91A43

## 1 Introduction

In classical cooperative game theory it is assumed that any coalition of players may form. However, in many practical situations a collection of feasible coalitions is restricted by some social, economical, hierarchical, communicational, or technical structure. The study of transferable utility (TU) games with limited cooperation introduced by means of coalition structures, or in other terms a priori unions, was initiated in the 1970's first by Aumann and Drèze [1] and then Owen [13]. In these papers a coalition structure is given by a partition of the set of players. Another model of a game with limited cooperation presented by means of undirected communication graphs was introduced in Myerson [12]. Various studies in both directions were done during the last four decades, but mostly either within one model or another. Vázquez-Brage, García-Jurado and Carreras [16] is the first study that combines both models by considering a TU game endowed with, independent of each other, both a coalition structure and a communication graph on the set of players. For this class of games they propose a solution by applying the Owen value for games with coalition structure to the Myerson restricted game of the game with communication graph.

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Another model of a TU game endowed with both a coalition structure and a communication graph, the so-called game with two-level communication structure, is considered in Khmelnitskaya [9]. In contrast to Vázquez-Brage et al. [16], in this model a two-level communication structure relates fundamentally to the given coalition structure and consists of a communication graph on the collection of the a priori unions in the coalition structure, as well as a collection of communication graphs within each union. It is assumed that communication is only possible either among the entire a priori unions or among single players within any a priori union. No communication and therefore no cooperation is allowed between proper subcoalitions, in particular single players, of distinct elements of the coalition structure.[1] This approach allows to model different network situations, in particular, telecommunication problems, distribution of goods among different cities (countries) along highway networks connecting the cities and local road networks within the cities, or sharing an international river with multiple users but without international firms, i.e., when no cooperation is possible among single users located at different levels along the river, and so on. Communication structures under scrutiny are given by combinations of graphs of different types both undirected–arbitrary graphs and cycle-free graphs, and directed–line-graphs with linearly ordered players, rooted forests and sink forests. The proposed solution concepts reflect a two-stage distribution procedure when, first, a priori unions collect their shares through the upper level bargaining based only on the cumulative interests of all members of every involved entire a priori union, and second, the players collect their individual payoffs through the lower level bargaining over the distribution of the unions' shares within the unions. Following Myerson [12] it is assumed that cooperation is possible only between connected participants and different combinations of known component efficient values, such as the Myerson value, the position value, the average tree solution, etc., are applied on both communication levels. However, as discussed in [9], the two-stage distribution procedure based on the application of component efficient values on both levels suffers from severe restrictions in cases when some a priori unions are internally not connected, because each union always has to distribute its total share among the members. Another solution concept for TU games with two-level communication structure introduced by means of undirected graphs, the so-called Owen-type value for games with two-level communication structure, is considered in van den Brink, Khmelnitskaya, and van der Laan [4] under a weaker assumption concerning the communication on the level of a priori unions, when in the upper level bargaining between a priori unions for one union it is allowed to be presented by its proper subcoalition. This solution can be seen as an adaptation of the two-step procedure determining the Owen value [13] for games with coalition structure which takes into account the limited cooperation represented by two-level communication structure replacing twice the Shapley value by the Myerson value.

In this paper we assume that two-level communication structures are given by combinations of graphs of different types, both undirected and directed, and we introduce two families of two-step values for games with two-level communication structure adapting the two-step procedures staying behind two values for games with coalition structure, the Owen value introduced in Owen (1977) and the two-step Shapley value introduced in Kamijo (2009). Our approach is based on the unified treatment of several component efficient values for games with communication structure and it generates two-stage solution concepts that apply component efficient values for games with communication structure on both distribution levels. In fact the newly introduced family of the Owen-type values is the generalization of the Owen-type value for games with two-level communication structures of van den Brink et al. [4], when on both communication levels not only the Myerson value, but different component efficient values for games with communication structure can be applied. The incorporation of different solutions for games with communication structure aims not only to enrich the solution concepts for games with two-level communication structure, but it also opens a broad diversity of applications impossible otherwise,

[1] A similar model, but at other and quite special assumptions concerning the ability of players to cooperate under given communication constraints is also studied in Kongo [10].

because there exists no universal solution concept for games with communication structure that is applicable to the full variety of possible undirected and directed graph structures. Moreover, it allows to choose, depending on types of graph structures under scrutiny, the most preferable, in particular, the most computationally efficient combination of values among others suitable. We provide axiomatic characterizations of the two-step values of the considered two families that have some axioms in common, which in turn allows to compare the peculiarities of both families.

The rest of the paper is organized as follows. Basic definitions and notation are introduced in Section 2. Section 3 provides the uniform approach to several known component efficient values for games with communication structure, which allows also to consider within a unified framework different deletion link properties with respect to the values for games with two-level communication structure. In Sections 4 we introduce the family of the Owen-type values axiomatically and present an explicit formula representation, while in Section 5 we do the same for the family of the Kamijo-type values. Section 6 concludes.

# 2 Preliminaries

## 2.1 TU games and values

A *cooperative game with transferable utility*, or *TU game*, is a pair $\langle N, v\rangle$, where $N \subset \mathbb{N}$ is a finite set of $n$ players and $v\colon 2^N \to \mathbb{R}$ is a *characteristic function* with $v(\emptyset) = 0$, assigning to every *coalition* $S \subseteq N$ of $s$ players its *worth* $v(S)$. The set of TU games with fixed player set $N$ is denoted by $\mathcal{G}_N$. For simplicity of notation and if no ambiguity appears we write $v$ when we refer to a TU game $\langle N, v\rangle$. A *subgame* of a TU game $v \in \mathcal{G}_N$ with a player set $T \subseteq N$ is the TU game $v|_T \in \mathcal{G}_T$ defined by $v|_T(S) = v(S)$ for every $S \subseteq T$. A *payoff vector* is a vector $x \in \mathbb{R}^N$ with $x_i$ the payoff to player $i \in N$. A single-valued solution, called a *value*, is a mapping $\xi\colon \mathcal{G}_N \to \mathbb{R}^N$ that assigns for every finite set $N \subset \mathbb{N}$ and every TU game $v \in \mathcal{G}_N$ a payoff vector $\xi(v) \in \mathbb{R}^N$. A value $\xi$ is *efficient* if $\sum_{i\in N} \xi_i(v) = v(N)$ for every $v \in \mathcal{G}_N$ and $N \subset \mathbb{N}$. The best-known efficient value is the Shapley value [14] given by

$$Sh_i(v) = \sum_{S\subseteq N\setminus\{i\}} \frac{s!(n-s-1)!}{n!}(v(S\cup\{i\}) - v(S)), \qquad \text{for all } i \in N.$$

In the sequel we denote the cardinality of a given set $A$ by $|A|$, along with lower case letters like $n = |N|$, $m = |M|$, $n_k = |N_k|$, $s = |S|$, $c = |C|$, $c' = |C'|$, and so on, and we use the standard notation $x(S) = \sum_{i\in S} x_i$ for any $x \in \mathbb{R}^N$ and $S \subseteq N$.

## 2.2 Games with coalition structure

A *coalition structure*, or in other terms a *system of a priori unions*, on $N \subset \mathbb{N}$ is given by a partition $\mathcal{P} = \{N_1, ..., N_m\}$ of $N$, i.e., $N_1 \cup ... \cup N_m = N$ and $N_k \cap N_l = \emptyset$ for $k \neq l$. Let $\mathfrak{P}_N$ denote the set of all coalition structures on $N$, and let $\mathcal{G}^P_N = \mathcal{G}_N \times \mathfrak{P}_N$. A pair $\langle v, \mathcal{P}\rangle \in \mathcal{G}^P_N$ constitutes a *game with coalition structure*, or simply *P-game*, on $N$. Remark that $\langle v, \{N\}\rangle$ represents the same situation as $v$ itself. A *P-value* is a mapping $\xi\colon \mathcal{G}^P_N \to \mathbb{R}^N$ that assigns for every $N \subset \mathbb{N}$ and every $P$-game $\langle v, \mathcal{P}\rangle \in \mathcal{G}^P_N$ a payoff vector $\xi(v, \mathcal{P}) \in \mathbb{R}^N$. A $P$-value $\xi$ is *efficient* (E) if $\sum_{i\in N} \xi_i(v, \mathcal{P}) = v(N)$ for every $v \in \mathcal{G}^P_N$ and $N \subset \mathbb{N}$. In what follows denote by $M = \{1, \dots, m\}$ the index set of all a priori unions in $\mathcal{P}$; for every $P$-game $\langle v, \mathcal{P}\rangle \in \mathcal{G}^P_N$ and every $k \in M$ let $v_k$ denote the subgame $v|_{N_k}$; for every $i \in N$ let $k(i)$ be defined by the relation $i \in N_{k(i)}$; and for every $x \in \mathbb{R}^N$ let $x^{\mathcal{P}} = \big(x(N_k)\big)_{k\in M} \in \mathbb{R}^M$ stand for the vector of total payoffs to a priori unions.

One of the best-known values for games with coalition structure is the Owen value [13] that can be seen as a two-step procedure in which the Shapley value applies twice. Namely, the Owen value assigns to player $i \in N$ his Shapley value in the game $\bar{v}_{k(i)}$, i.e.,

$$Ow_i(v, \mathcal{P}) = Sh_i(\bar{v}_{k(i)}), \qquad \text{for all } \ i \in N,$$

while for every a priori union $N_k$, $k \in M$, the game $\bar{v}_k \in \mathcal{G}_{N_k}$ on the player set $N_k$ is given by

$$\bar{v}_k(S) = Sh_k(\hat{v}_S), \qquad S \subseteq N_k,$$

where for every $S \subseteq N_k$ the game $\hat{v}_S \in \mathcal{G}_M$ on the player set $M$ of a priori unions is defined by

$$\hat{v}_S(Q) = \begin{cases} v(\cup_{h\in Q} N_h), & k \notin Q, \\ v(\cup_{h\in Q\setminus\{k\}} N_h \cup S), & k \in Q, \end{cases} \qquad \text{for all } \ Q \subseteq M.$$

It is well-known that the Owen value is efficient and satisfies the quotient game property which means that for every a priori union the total payoff to the players within that union is determined by applying the Shapley value to the so-called *quotient game* being the game $v_{\mathcal{P}} \in \mathcal{G}_M$ in which the unions act as individual players,

$$v_{\mathcal{P}}(Q) = v(\cup_{k\in Q} N_k), \qquad \text{for all } \quad Q \subseteq M.$$

Notice that for every $k \in M$ the game $\hat{v}_{N_k}$ is equal to the quotient game $v_{\mathcal{P}}$.

Another value for games with coalition structure that also can be seen as a two-step procedure in which the Shapley value applies twice is the so-called two-step Shapley value $\psi$ introduced in Kamijo [7]. The two-step Shapley value first allocates to player $i \in N$ his Shapley value in the subgame on the a priori union $N_{k(i)}$ he belongs to and then distributes what remains of the Shapley value of its union in the quotient game equally among the union's members, i.e., for any $P$-game $\langle v, \mathcal{P}\rangle \in \mathcal{G}_N^P$,

$$Ka_i(v, \mathcal{P}) = Sh_i(v_{k(i)}) + \frac{Sh_{k(i)}(v_{\mathcal{P}}) - v(N_{k(i)})}{n_{k(i)}}. \tag{1}$$

The Kamijo's two-step Shapley value is efficient and meets the quotient game property.

### 2.3 Games with communication structure

A *communication structure* on $N$ is specified by a graph $\Gamma$, undirected or directed, on $N$. A *graph* on $N$ consists of $N$ as the set of nodes and for an *undirected graph* a collection of unordered pairs $\Gamma \subseteq \{\{i, j\} \,|\, i, j \in N, \ i \neq j\}$ as the set of links between two nodes in $N$, and for a *directed graph*, or a *digraph*, a collection of ordered pairs $\Gamma \subseteq \{(i, j) \,|\, i, j \in N, \ i \neq j\}$ as the set of directed links from one node to another node in $N$. When it is necessary to specify the set of nodes $N$ in a graph $\Gamma$, we write $\Gamma_N$ instead of $\Gamma$. Let $\mathfrak{G}_N$ denote the set of all communication structures, undirected or directed, on $N$, and let $\mathcal{G}_N^\Gamma = \mathcal{G}_N \times \mathfrak{G}_N$. A pair $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ constitutes a *game with graph (communication) structure*, or simply a *graph game*, or a *Γ-game*, on $N$. A *Γ-value* is a mapping $\xi \colon \mathcal{G}_N^\Gamma \to \mathbb{R}^N$ that assigns for every $N \subset \mathbb{N}$ and every *Γ*-game $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ a payoff vector $\xi(v, \Gamma) \in \mathbb{R}^N$.

In a graph $\Gamma$ a sequence of different nodes $(i_1, \ldots, i_r)$, $r \geq 2$, is a *path* in $\Gamma$ from node $i_1$ to node $i_r$ if for $h{=}1, \ldots, r{-}1$ it holds that $\{i_h, i_{h+1}\} {\in} \Gamma$ when $\Gamma$ is undirected and $\{(i_h, i_{h+1}), (i_{h+1}, i_h)\} \cap \Gamma \neq \emptyset$ when $\Gamma$ is directed. In a digraph $\Gamma$ a path $(i_1, \ldots, i_r)$ is a *directed path* from node $i_1$ to node $i_r$ if $(i_h, i_{h+1}) \in \Gamma$ for all $h{=}1, \ldots, r{-}1$. In a digraph $\Gamma$, $j \neq i$ is a *successor* of $i$ and $i$ is a *predecessor* of $j$ if there exists a directed path from $i$ to $j$. Given a digraph $\Gamma$ on $N$ and $i \in N$, the sets of predecessors and successors of $i$ in $\Gamma$ we denote correspondingly by $P^\Gamma(i)$ and $S^\Gamma(i)$; moreover, $\bar{P}^\Gamma(i) = P^\Gamma(i) \cup \{i\}$ and $\bar{S}^\Gamma(i) = S^\Gamma(i) \cup \{i\}$.

Given a graph $\Gamma$ on $N$, two nodes $i$ and $j$ in $N$ are *connected* if either there exists a path from node $i$ to node $j$, or $i$ and $j$ coincide. Graph $\Gamma$ on $N$ is *connected* if any two nodes in $N$ are connected. For a graph $\Gamma$ on $N$ and a coalition $S \subseteq N$, the *subgraph of* $\Gamma$ *on* $S$ is the graph $\Gamma|_S = \{\{i,j\} \in \Gamma \mid i,j \in S\}$ on $S$ when $\Gamma$ is undirected and the digraph $\Gamma|_S = \{(i,j) \in \Gamma \mid i,j \in S\}$ on $S$ when $\Gamma$ is directed. Given a graph $\Gamma$ on $N$, a coalition $S \subseteq N$ is *connected* if the subgraph $\Gamma|_S$ is connected. For a graph $\Gamma$ on $N$ and coalition $S \subseteq N$, $C^\Gamma(S)$ is the set of all connected subcoalitions of $S$, $S/\Gamma$ is the set of maximal connected subcoalitions of $S$, called the *components of* $S$, and $(S/\Gamma)_i$ is the component of $S$ containing player $i \in S$. Notice that $S/\Gamma$ is a partition of $S$. For any $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$, a payoff vector $x \in \mathbb{R}^N$ is *component efficient* if $x(C) = v(C)$, for every $C \in N/\Gamma$.

Following Myerson [12], we assume that for *Γ*-games cooperation is possible only among connected players and concentrate on component efficient *Γ*-values. A *Γ*-value $\xi$ is *component efficient* (CE) if for any $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$, for all $C \in N/\Gamma$, $\sum_{i \in C} \xi_i(v, \Gamma) = v(C)$. Below for a *Γ*-game $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$ we also consider the introduced in Myerson [12] *restricted game* $v^\Gamma \in \mathcal{G}_N$ defined as

$$v^\Gamma(S) = \sum_{C \in S/\Gamma} v(C), \qquad \text{for all } S \subseteq N. \tag{2}$$

Hereinafter along with communication structures given by arbitrary undirected graphs we consider also those given by cycle-free undirected graphs and by directed graphs – line-graphs with linearly ordered players, rooted and sink forests. In an undirected graph $\Gamma$ a path $(i_1, \ldots, i_r)$, $r \geq 3$, is a *cycle* in $\Gamma$ if $\{i_r, i_1\} \in \Gamma$. An undirected graph is *cycle-free* if it contains no cycles. A directed graph $\Gamma$ is a *rooted tree* if there is one node in $N$, called a *root*, having no predecessors in $\Gamma$ and there is a unique directed path in $\Gamma$ from this node to any other node in $N$. A directed graph $\Gamma$ is a *sink tree* if the directed graph composed by the same set of links as $\Gamma$ but with the opposite orientation is a rooted tree; in this case the root of a tree changes its meaning to the absorbing sink. A directed graph is a *rooted/sink forest* if it is composed by a number of disjoint rooted/sink trees. A *line-graph* is a directed graph that contains links only between subsequent nodes. Without loss of generality we may assume that in a line-graph nodes are ordered according to the natural order from 1 to $n$, i.e., line-graph $\Gamma \subseteq \{(i, i+1) \mid i = 1, \ldots, n-1\}$.

For ease of notation given graph $\Gamma$ and link $\{i,j\} \in \Gamma$ if $\Gamma$ is undirected, or $(i,j) \in \Gamma$ if $\Gamma$ is directed, the subgraph $\Gamma \backslash \{\{i,j\}\}$, correspondingly $\Gamma \backslash \{(i,j)\}$, is denoted by $\Gamma|_{-ij}$.

## 2.4 Games with two-level communication structure

We now consider situations in which the players are partitioned into a coalition structure $\mathcal{P}$ and are linked to each other by communication graphs. First, there is a communication graph $\Gamma_M$ on the set of a priori unions determined by the partition $\mathcal{P}$. Second, for each a priori union $N_k$, $k \in M$, there is a communication graph $\Gamma_{N_k}$ between the players in $N_k$. In what follows for simplicity of notation and when it causes no ambiguity we denote graphs $\Gamma_{N_k}$ within a priori unions $N_k$, $k \in M$, by $\Gamma_k$. Given a player set $N \subset \mathbb{N}$ and a coalition structure $\mathcal{P} \in \mathfrak{P}_N$, a *two-level graph (communication) structure* on $N$ is a tuple $\Gamma_{\mathcal{P}} = \langle \Gamma_M, \{\Gamma_k\}_{k \in M} \rangle$. For every $N \subset \mathbb{N}$ and $\mathcal{P} \in \mathfrak{P}_N$ by $\mathfrak{G}_N^{\mathcal{P}}$ we denote the set of all two-level graph structures on $N$ with fixed $\mathcal{P}$. Let $\mathfrak{G}_N^P = \bigcup_{\mathcal{P} \in \mathfrak{P}_N} \mathfrak{G}_N^{\mathcal{P}}$ be the set of all two-level graph structures on $N$, and let $\mathcal{G}_N^{P\Gamma} = \mathcal{G}_N \times \mathfrak{G}_N^P$. A pair $\langle v, \Gamma_{\mathcal{P}} \rangle \in \mathcal{G}_N^{P\Gamma}$ constitutes a *game with two-level graph (communication) structure*, or simply a *two-level graph game* or a *PΓ-game*, on $N$. A *PΓ-value* is a mapping $\xi \colon \mathcal{G}_N^{P\Gamma} \to \mathbb{R}^N$ that assigns for every $N \subset \mathbb{N}$ and every *PΓ*-game $\langle v, \Gamma_{\mathcal{P}} \rangle \in \mathcal{G}_N^{P\Gamma}$ a payoff vector $\xi(v, \Gamma_{\mathcal{P}}) \in \mathbb{R}^N$.

Given a $P\Gamma$-game $\langle v, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{P\Gamma}$, one can consider the *quotient $\Gamma$-game* $\langle v_{\mathcal{P}}, \Gamma_M\rangle \in \mathcal{G}_M^{\Gamma}$ and the *$\Gamma$-games within a priori unions* $\langle v_k, \Gamma_k\rangle \in \mathcal{G}_{N_k}^{\Gamma}$ with $v_k = v|_{N_k}$, $k \in M$, that model the bargaining between a priori unions for their total shares and the bargaining within each a priori union for the distribution of its total worth among the members taking also into account limited cooperation at both communication levels introduced by the communication graphs $\Gamma_M$ and $\Gamma_k$, $k \in M$. Moreover, given a $\Gamma$-value $\phi$, for any $\langle v, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{P\Gamma}$ with a graph structure $\Gamma_M$ on the level of a priori unions suitable for application of $\phi$ to the corresponding quotient $\Gamma$-game $\langle v_{\mathcal{P}}, \Gamma_M\rangle$, along with a subgame $v_k$ within a priori union $N_k$, $k \in M$, one can also consider a *$\phi_k$-game* $v_k^{\phi}$ defined as

$$v_k^{\phi}(S) = \begin{cases} \phi_k(v_{\mathcal{P}}, \Gamma_M), & S = N_k, \\ v(S), & S \neq N_k, \end{cases} \qquad \text{for all } S \subseteq N_k,$$

where $\phi_k(v_{\mathcal{P}}, \Gamma_M)$ is the payoff to $N_k$ given by $\phi$ in $\langle v_{\mathcal{P}}, \Gamma_M\rangle$. Then a $\Gamma$-game $\langle v_k^{\phi}, \Gamma_k\rangle \in \mathcal{G}_{N_k}^{\Gamma}$ models the bargaining within union $N_k$ for the distribution of its total share among the members taking into account restrictions on cooperation in $N_k$ given by $\Gamma_k$, when the share is obtained by the application of $\Gamma$-value $\phi$ at the upper level bargaining between a priori unions.

## 3 Deletion link properties for two-level graph games

As it is discussed in Khmenitskaya [9], a number of known component efficient $\Gamma$-values for games with undirected or directed communication structure such as the Myerson value $\mu$ (cf. Myerson [12]) and the position value $\pi$ (cf. Meessen [11], Borm, Owen, and Tijs [3], and Slikker [15]) for arbitrary undirected graph games, the average tree solution AT (cf. Herings, van der Laan, and Talman [6]) and the compensation solution CS (cf. Béal, Rémila, and Solal [2]) for undirected cycle-free graph games, or for directed graph games the upper equivalent solution UE, the lower equivalent solution LE and the equal loss solution EL for line-graph games (cf. van den Brink, van der Laan, and Vasil'ev [5]), the tree value $t$ for rooted forest and the sink value $s$ for sink forest digraph games (cf. Khmelnitskaya [8]), can be approached within the unified framework. Indeed, each one of these $\Gamma$-values is defined for $\Gamma$-games with *suitable* graph structure and is characterized by two axioms, CE and one or another *deletion link* (DL) property (axiom), reflecting the relevant reaction of a $\Gamma$-value on the deletion of a link in the communication graph. The corresponding DL properties are fairness (F), balanced link contributions (BLC), component fairness (CF), relative fairness (RF), upper equivalence (UE), lower equivalence (LE), equal loss property (EL), successor equivalence (SE), and predecessor equivalence (PE), and the characterization results are as follows:

$$\begin{aligned}
\text{CE} + \text{F} \quad &\text{for all undirected } \Gamma\text{-games} &\iff& \mu(v, \Gamma),\\
\text{CE} + \text{BLC} \quad &\text{for all undirected } \Gamma\text{-games} &\iff& \pi(v, \Gamma),\\
\text{CE} + \text{CF} \quad &\text{for undirected cycle-free } \Gamma\text{-games} &\iff& AT(v, \Gamma),\\
\text{CE} + \text{RF} \quad &\text{for undirected cycle-free } \Gamma\text{-games} &\iff& CS(v, \Gamma),\\
\text{CE} + \text{UE} \quad &\text{for line-graph } \Gamma\text{-games} &\iff& UE(v, \Gamma),\\
\text{CE} + \text{LE} \quad &\text{for line-graph } \Gamma\text{-games} &\iff& LE(v, \Gamma),\\
\text{CE} + \text{EL} \quad &\text{for line-graph } \Gamma\text{-games} &\iff& EL(v, \Gamma),\\
\text{CE} + \text{SE} \quad &\text{for rooted forest } \Gamma\text{-games} &\iff& t(v, \Gamma),\\
\text{CE} + \text{PE} \quad &\text{for sink forest } \Gamma\text{-games} &\iff& s(v, \Gamma).
\end{aligned}$$

This observation allows to identify each of the listed above $\Gamma$-values with the corresponding DL axiom. Given a DL axiom, let $\mathcal{G}_N^{DL} \subseteq \mathcal{G}_N^{\Gamma}$ denote a set of all $\langle v, \Gamma\rangle \in \mathcal{G}_N^{\Gamma}$ with $\Gamma$ suitable for DL application.

Then

$$\text{CE} \ + \ \text{DL} \quad \text{on } \mathcal{G}_N^{DL} \iff DL(v,\Gamma).$$

Whence it simply follows that $F(v,\Gamma) = \mu(v,\Gamma)$ and $BLC(v,\Gamma) = \pi(v,\Gamma)$ for all undirected $\Gamma$-games, $CF(v,\Gamma) = AT(v,\Gamma)$ and $RF(v,\Gamma) = CS(v,\Gamma)$ for all undirected cycle-free $\Gamma$-games, $UE(v,\Gamma)$, $LE(v,\Gamma)$, and $EL(v,\Gamma)$ are UE, LE, and EL solutions correspondingly for all line-graph $\Gamma$-games, $SE(v,\Gamma) = t(v,\Gamma)$ for all rooted forest $\Gamma$-games, and $PE(v,\Gamma) = s(v,\Gamma)$ for all sink forest $\Gamma$-games. Remark that all just discussed values are additive.

Next notice that every discussed DL axiom can be equivalently defined by an equality

$$\Psi^{DL}(\xi(v,\Gamma),\Gamma') = 0, \tag{3}$$

where $\Psi^{DL}$ is an operator which for a value $\xi$ defined on $\mathcal{G}_N^{DL}$ and applied to a game $\langle v,\Gamma\rangle \in \mathcal{G}_N^{DL}$ assigns a real number representing one or another evaluation of players' payoff reaction on the deletion of links in $\Gamma$ from a chosen set of links $\Gamma'$. For the mentioned above DL axioms $\Gamma' = \Gamma$ and the corresponding operators are:

**Fairness (F):** For any player set $N \subset \mathbb{N}$, for every $\Gamma$-game $\langle v,\Gamma\rangle \in \mathcal{G}_N^{\Gamma}$, for every link $\{i,j\} \in \Gamma$, it holds that

$$\Psi^F(\xi(v,\Gamma),\Gamma') = \sum_{i,j\in N \mid \{i,j\}\in\Gamma'} \Big| \Big(\xi_i(v,\Gamma) - \xi_i(v,\Gamma|_{-ij})\Big) - \Big(\xi_j(v,\Gamma) - \xi_j(v,\Gamma|_{-ij})\Big) \Big|, \tag{4}$$

**Balanced link contributions (BLC):** For any player set $N \subset \mathbb{N}$, for every $\Gamma$-game $\langle v,\Gamma\rangle \in \mathcal{G}_N^{\Gamma}$ and $i,j \in N$, it holds that

$$\Psi^{BLC}(\xi(v,\Gamma),\Gamma') = \sum_{i,j\in N} \Big| \sum_{h\in N \mid \{i,h\}\in\Gamma'} \Big(\xi_j(v,\Gamma) - \xi_j(v,\Gamma|_{-ih})\Big) - \sum_{h\in N \mid \{j,h\}\in\Gamma'} \Big(\xi_i(v,\Gamma) - \xi_i(v,\Gamma|_{-jh})\Big) \Big|, \tag{5}$$

**Component fairness (CF):** For any player set $N \subset \mathbb{N}$, for every cycle-free $\Gamma$-game $\langle v,\Gamma\rangle \in \mathcal{G}_N^{\Gamma}$, for every link $\{i,j\} \in \Gamma$, it holds that

$$\Psi^{CF}(\xi(v,\Gamma),\Gamma') = \sum_{i,j\in N \mid \{i,j\}\in\Gamma'} \Big| \frac{1}{|(N/\Gamma|_{-ij})_i|} \sum_{h\in(N/\Gamma|_{-ij})_i} \Big(\xi_h(v,\Gamma) - \xi_h(v,\Gamma|_{-ij})\Big) - \frac{1}{|(N/\Gamma|_{-ij})_j|} \sum_{h\in(N/\Gamma|_{-ij})_j} \Big(\xi_h(v,\Gamma) - \xi_h(v,\Gamma|_{-ij})\Big) \Big|, \tag{6}$$

**Relative fairness (RF):** For any player set $N \subset \mathbb{N}$, for every cycle-free $\Gamma$-game $\langle v,\Gamma\rangle \in \mathcal{G}_N^{\Gamma}$, for every link $\{i,j\} \in \Gamma$, it holds that

$$\Psi^{RF}(\xi(v,\Gamma),\Gamma') = \sum_{i,j\in N \mid \{i,j\}\in\Gamma'} \Big| \Big(\xi_i(v,\Gamma) - \frac{1}{|(N/\Gamma|_{-ij})_i|} \sum_{h\in(N/\Gamma|_{-ij})_i} \xi_h(v,\Gamma|_{-ij})\Big) - \Big(\xi_j(v,\Gamma) - \frac{1}{|(N/\Gamma|_{-ij})_j|} \sum_{h\in(N/\Gamma|_{-ij})_j} \xi_h(v,\Gamma|_{-ij})\Big) \Big|, \tag{7}$$

**Upper equivalence (UE):** For any player set $N \subset \mathbb{N}$, for every line-graph $\Gamma$-game $\langle v, \Gamma\rangle \in \mathcal{G}_N^{\Gamma}$, for any $i = 1, \dots, n-1$, for all $j = 1, \dots, i$, it holds that

$$\Psi^{UE}(\xi(v,\Gamma),\Gamma') = \sum_{i=1}^{n-1}\sum_{j=1}^{i} \left|\xi_j(v,\Gamma) - \xi_j(v,\Gamma|_{-i,i+1})\right|. \tag{8}$$

**Lower equivalence (LE):** For any player set $N \subset \mathbb{N}$, for every line-graph $\Gamma$-game $\langle v, \Gamma\rangle \in \mathcal{G}_N^{\Gamma}$, for any $i = 1, \dots, n-1$, for all $j = i+1, \dots, n$, it holds that

$$\Psi^{LE}(\xi(v,\Gamma),\Gamma') = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \left|\xi_j(v,\Gamma) - \xi_j(v,\Gamma|_{-i,i+1})\right|. \tag{9}$$

**Equal loss property (EL):** For any player set $N \subset \mathbb{N}$, for every line-graph $\Gamma$-game $\langle v, \Gamma\rangle \in \mathcal{G}_N^{\Gamma}$, for all $i = 1, \dots, n-1$, it holds that

$$\Psi^{EL}(\xi(v,\Gamma),\Gamma') = \sum_{i=1}^{n-1}\Big|\sum_{j=1}^{i}\Big(\xi_j(v,\Gamma) - \xi_j(v,\Gamma|_{-i,i+1})\Big) - \sum_{j=i=1}^{n}\Big(\xi_j(v,\Gamma) - \xi_j(v,\Gamma|_{-i,i+1})\Big)\Big|. \tag{10}$$

**Successor equivalence (SE):** For any player set $N \subset \mathbb{N}$, for every rooted forest $\Gamma$-game $\langle v, \Gamma\rangle \in \mathcal{G}_N^{\Gamma}$, for every link $\{i,j\} \in \Gamma$, for all $k \in \bar{S}^{\Gamma}(j)$, it holds that

$$\Psi^{SE}(\xi(v,\Gamma),\Gamma') = \sum_{i,j\in N|\{i,j\}\in\Gamma'}\ \sum_{h\in\bar{S}^{\Gamma}(j)} \left|\xi_h(v,\Gamma) - \xi_h(v,\Gamma|_{-ij})\right|. \tag{11}$$

**Predecessor equivalence (PE):** For any player set $N \subset \mathbb{N}$, for every sink forest $\Gamma$-game $\langle v, \Gamma\rangle \in \mathcal{G}_N^{\Gamma}$, for every link $\{i,j\} \in \Gamma$, for all $k \in \bar{P}^{\Gamma}(j)$, it holds that

$$\Psi^{PE}(\xi(v,\Gamma),\Gamma') = \sum_{i,j\in N|\{i,j\}\in\Gamma'}\ \sum_{h\in\bar{P}^{\Gamma}(j)} \left|\xi_h(v,\Gamma) - \xi_h(v,\Gamma|_{-ij})\right|. \tag{12}$$

The definition of deletion link axioms for $\Gamma$-values in terms of equality (3) allows to consider the corresponding deletion link axioms within a unified framework also for $P\Gamma$-values with respect to both communication levels, at the upper level with respect to the quotient $\Gamma$-game determining the total payoffs to a priori unions and at the lower level with respect to $\Gamma$-games within a priori unions determining the distribution of total payoffs within each a priori union. Let $\xi$ be a $P\Gamma$-value. By definition $\xi$ is a mapping $\xi\colon \mathcal{G}_N^{P\Gamma} \to \mathbb{R}^N$ that assigns a payoff vector to any $P\Gamma$-game on the player set $N \subset \mathbb{N}$. For every $N \subset \mathbb{N}$ a mapping $\xi = \{\xi_i\}_{i\in N}$ generates on the domain of $P\Gamma$-games on $N$ a mapping $\xi^{\mathcal{P}}\colon \mathcal{G}_N^{P\Gamma} \to \mathbb{R}^M$, $\xi^{\mathcal{P}} = \{\xi_k^{\mathcal{P}}\}_{k\in M}$, with $\xi_k^{\mathcal{P}} = \sum_{i\in N_k}\xi_i$, $k \in M$, that assigns to every $P\Gamma$-game on $N$ a vector of total payoffs to all a priori unions, and $m$ mappings $\xi^k\colon \mathcal{G}_N^{P\Gamma} \to \mathbb{R}^{N_k}$, $\xi^k = \{\xi_i\}_{i\in N_k}$, $k \in M$, assigning payoffs to players within a priori unions. For a given $(m+1)$-tuple of deletion link axioms $\langle \mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k\in M}\rangle$ for $\Gamma$-values, let $\mathcal{G}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}} \subseteq \mathcal{G}_N^{P\Gamma}$ be the set of $P\Gamma$-games composed of $P\Gamma$-games $\langle v, \Gamma_{\mathcal{P}}\rangle$ with graph structures $\Gamma_{\mathcal{P}} = \langle \Gamma_M, \{\Gamma_k\}_{k\in M}\rangle$ such that $\langle v_{\mathcal{P}}, \Gamma_M\rangle \in \mathcal{G}_M^{DL^{\mathcal{P}}}$ and $\langle v_k^{DL^{\mathcal{P}}}, \Gamma_k\rangle \in \mathcal{G}_{N_k}^{DL^k}$, $k \in M$. Then, given a $(m+1)$-tuple of deletion link axioms $\langle \mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k\in M}\rangle$ for $\Gamma$-values, we define axioms of Quotient DL property and Union DL property for $P\Gamma$-values defined on $\mathcal{G}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ as follows:

**Quotient DL property (QDL)** For any player set $N \subset \mathbb{N}$ and $P\Gamma$-game $\langle v, \Gamma_{\mathcal{P}} \rangle \in \mathcal{G}_N^{P\Gamma}$, $\Gamma_{\mathcal{P}} = \langle \Gamma_M, \{\Gamma_h\}_{h \in M} \rangle$, it holds

$$\Psi^{DL^{\mathcal{P}}}(\xi^{\mathcal{P}}(v, \Gamma_{\mathcal{P}}), \Gamma_M) = 0, \tag{13}$$

where for $\Psi^{DL^{\mathcal{P}}}$ given by one of the operators (4)–(12), for every link $\{k, l\} \in \Gamma_M$ if $\Gamma_M$ is undirected, or $(k, l) \in \Gamma_M$ if $\Gamma_M$ is directed, $\Gamma_{\mathcal{P}}|_{-kl} = \langle \Gamma_M|_{-kl}, \{\Gamma_h\}_{h \in M} \rangle$.

**Union DL property (UDL)** For any player set $N \subset \mathbb{N}$, $P\Gamma$-game $\langle v, \Gamma_{\mathcal{P}} \rangle \in \mathcal{G}_N^{P\Gamma}$, $\Gamma_{\mathcal{P}} = \langle \Gamma_M, \{\Gamma_h\}_{h \in M} \rangle$, and $k \in M$, it holds

$$\Psi^{DL^k}(\xi^k(v, \Gamma_{\mathcal{P}}), \Gamma_k) = 0, \tag{14}$$

where for $\Psi^{DL^k}$ given by one of the operators (4)–(12), for every link $\{i, j\} \in \Gamma_k$ if $\Gamma_k$ is undirected, or $(i, j) \in \Gamma_k$ if $\Gamma_k$ is directed, $\Gamma_{\mathcal{P}}|_{-ij} = \Gamma_{\mathcal{P}}|_{-ij}^k = \langle \Gamma_M, \{\widehat{\Gamma}_h\}_{h \in M} \rangle$ with $\widehat{\Gamma}_h = \Gamma_h$ for $h \neq k$ and $\widehat{\Gamma}_k = \Gamma_k|_{-ij}$.

For example, if we consider the Myerson fairness (F) as a DL axiom, QDL and UDL$^k$ can be denoted QF and UF$^k$ correspondingly. In fact in this case QF coincides with quotient fairness (QF) and $m$-tuple of axioms (UF$^1$, ..., UF$^m$) coincides with union fairness (UF) employed in van den Brink et al. [4].

QLD and UDL axioms for $P\Gamma$-values provide the uniform approach to various deletion link properties on both bargaining levels. This allows to introduce within the unified framework two families of $P\Gamma$-values based on the adaptation of the two-step distribution procedures of Owen and Kamijo respectively for games with coalition structure to the case when the cooperation between and within a priori unions is restricted by communication graphs and when different combinations of known component efficient solution concepts on both communication levels could be applied. Moreover, this allows to include into consideration not only combinations of undirected communication graphs but also combinations including some types of digraphs.

## 4 Owen-type values for two-level graph games

The Owen-type value for $P\Gamma$-games introduced in van den Brink et al. [4] is based on the application of the Myerson value on both communication levels. It is characterized by two fairness axioms just discussed above at the end of the previous section, quotient fairness and union fairness, and two other axioms of quotient component efficiency and of fair distribution of the surplus within unions. Quotient component efficiency requires that each component on the upper level between a priori unions distributes fully its total worth among the players of the a priori unions forming this component.

**Quotient component efficiency (QCE)** For any player set $N \subset \mathbb{N}$, for every $\langle v, \Gamma_{\mathcal{P}} \rangle \in \mathcal{G}_N^{P\Gamma}$, $\Gamma_{\mathcal{P}} = \langle \Gamma_M, \{\Gamma_k\}_{k \in M} \rangle$, it holds

$$\sum_{k \in K} \sum_{i \in N_k} \xi_i(v, \Gamma_{\mathcal{P}}) = v_{\mathcal{P}}(K), \qquad \text{for all } K \in M/\Gamma_M. \tag{15}$$

Fair distribution of the surplus within unions determines distribution of the shares obtained by a priori unions at the upper level bargaining between a priori unions among their members for internally disconnected a priori unions. When a priori unions are negotiating for their shares in the total payoff, their cooperation possibilities are restricted by the communication graph on the level of a priori unions. Also when some a priori union is represented by its proper subset, the cooperation possibilities of this representative subset with the other (full) a priori unions are restricted by the communication graph on the level of a priori unions. Similarly, when players within a subcoalition within some a priori union are negotiating for their share in the union's payoff, their cooperation possibilities are restricted by the communication graph within the a priori union. In all these cases, following Myerson, we assume that only connected participants are able to cooperate when the connectedness is determined by the corresponding underlying communication graph. But the players within each a priori union always have to distribute the total payoff that has been assigned to the a priori union in the game between a priori unions irrespectively of the existing communication links within the union. So, deviating from Myerson, we consider every a priori union as an institution that allows their members to cooperate beyond the bilateral communication links within the a priori union, but this concerns only the a priori union as a whole. In case every a priori union is internally connected, the payoff allocation is determined by traditional efficiency and deletion link axioms. Consider a particular component within any a priori union that is not internally connected. Suppose we know the total payoff of that component when the a priori union is replaced by just the players in this component (and thus this reduced union is internally connected). Doing this for every connected component of this a priori union we can compare how the (positive or negative) excess is shared among the different components. Fair distribution of the surplus within unions requires this excess to be shared proportional to the size of the component.

**Fair distribution of the surplus within unions (FDSU)** For any player set $N \subset \mathbb{N}$, for every $\langle v, \Gamma_{\mathcal{P}} \rangle \in \mathcal{G}_N^{P\Gamma}$, $\Gamma_{\mathcal{P}} = \langle \Gamma_M, \{\Gamma_k\}_{k \in M} \rangle$, $k \in M$, and any two components $C, C' \in N_k/\Gamma_k$, it holds

$$\frac{1}{c} \sum_{i \in C} \Big( \xi_i(v, \Gamma_{\mathcal{P}}) - \xi_i(v_C^k, \Gamma_{\mathcal{P}_C^k}) \Big) = \frac{1}{c'} \sum_{i \in C'} \Big( \xi_i(v, \Gamma_{\mathcal{P}}) - \xi_i(v_{C'}^k, \Gamma_{\mathcal{P}_{C'}^k}) \Big), \tag{16}$$

where for $k \in M$ and component $C \in N_k/\Gamma_k$, $v_C^k$ denotes the subgame $v|_{(N \setminus N_k) \cup C}$ of $v$ with respect to the coalition $(N \setminus N_k) \cup C$, $\mathcal{P}_C^k$ denotes the partition on $(N \setminus N_k) \cup C$ consisting of union $C$ and all unions $N_h$ in $\mathcal{P}$, $h \neq k$, and $\Gamma_{\mathcal{P}_C^k} = \langle \Gamma_M, \{\widetilde{\Gamma}_h\}_{h \in M} \rangle$ with $\widetilde{\Gamma}_k = \Gamma_k|_C$ and $\widetilde{\Gamma}_h = \Gamma_h$ for all $h \in M \setminus \{k\}$, denotes the two-level communication structure that is obtained from $\langle \Gamma_M, \{\Gamma_h\}_{h \in M} \rangle$ by replacing the communication graph $\Gamma_k$ by its restriction on $C \subseteq N_k$.

It is easy to check that considering one component $C \in N_k/\Gamma_k$, the expression

$$\frac{1}{c} \sum_{i \in C} \Big( \xi_i(v, \Gamma_{\mathcal{P}}) - \xi_i(v_C^k, \Gamma_{\mathcal{P}_C^k}) \Big) = \frac{1}{n_k} \sum_{i \in N_k} \left( \xi_i(v, \Gamma_{\mathcal{P}}) - \xi_i(v_{(N_k/\Gamma_k)_i}^k, \Gamma_{\mathcal{P}_{(N_k/\Gamma_k)_i}^k}) \right) \tag{17}$$

provides an alternative representation of the fair distribution of the surplus within unions axiom. Notice that this axiom only states a requirement for the distribution of the total payoff within a priori union $N_k$ when $N_k$ consists of multiple components with respect to the internal communication graph $\Gamma_k$, otherwise the requirement reduces to an identity.

The next theorem extends the Owen-type value for $P\Gamma$-games studied in van den Brink et al. [4] by allowing the application of different combinations of known component efficient solution concepts for $\Gamma$-games on both communication levels.

**Theorem 1** *For any $(m+1)$-tuple of deletion link axioms $\langle \mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k \in M} \rangle$ such that the set of $\mathrm{DL}^k$, $k \in M$, axioms is restricted to F, CF, and RF, there is the unique $P\Gamma$-value defined on $\mathcal{G}_N^{DL^{\mathcal{P}}, \{DL^k\}_{k \in M}}$ that meets axioms QCE, FDSU, QDL, and UDL, and for every $P\Gamma$-game $\langle v, \Gamma_{\mathcal{P}} \rangle \in \mathcal{G}_N^{DL^{\mathcal{P}}, \{DL^k\}_{k \in M}}$ it is given by*

$$\xi_i(v, \Gamma_{\mathcal{P}}) = DL_i^{k(i)}(\tilde{v}_{k(i)}, \Gamma_{k(i)}) + \frac{DL_{k(i)}^{\mathcal{P}}(v_{\mathcal{P}}, \Gamma_M) - \sum\limits_{C \in N_{k(i)}/\Gamma_{k(i)}} \tilde{v}_{k(i)}(C)}{n_{k(i)}}, \quad i \in N, \tag{18}$$

*where for all $k \in M$, $\tilde{v}_k \in \mathcal{G}_{N_k}$ is defined as*

$$\tilde{v}_k(S) = DL_k^{\mathcal{P}}(\hat{v}_S, \Gamma_M), \qquad \textit{for all } S \subseteq N_k,$$

*and for every $S \subseteq N_k$, $\hat{v}_S \in \mathcal{G}_M$ is given by*

$$\hat{v}_S(Q) = \begin{cases} v(\cup_{h \in Q} N_h), & k \notin Q, \\ v(\cup_{h \in Q \setminus \{k\}} N_h \cup S), & k \in Q, \end{cases} \qquad \textit{for all } Q \subseteq M.$$

Notice that in case when the $(m+1)$-tuple of deletion link axioms $\langle \mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k \in M} \rangle$ is given by $\langle \mathrm{F}^{\mathcal{P}}, \{\mathrm{F}^k\}_{k \in M} \rangle$, i.e., the Myerson fairness F is applied on both upper and lower levels, the statement of Theorem 1 coincides with the statement of both Theorem 1 and Theorem 2 and the $P\Gamma$-value $\xi$ given by (18) coincides the Owen-type $P\Gamma$-value of van den Brink et al. [4]. Because of that from now on we refer to the $P\Gamma$-value (18) as to the *Owen-type $\langle DL^{\mathcal{P}}, \{DL^k\}_{k \in M} \rangle$-value*, which we denote further by $Ow^{\langle DL^{\mathcal{P}}, \{DL^k\}_{k \in M} \rangle}$.

Observe also that the Owen-type $\langle DL^{\mathcal{P}}, \{DL^k\}_{k \in M} \rangle$-value admits the similar two-step construction procedure as the Owen value for games with coalition structure. The difference is that instead of two applications of the Shapley value used in the case of the classical Owen value, in the Owen-type $\langle DL^{\mathcal{P}}, \{DL^k\}_{k \in M} \rangle$-value different known component efficient $\Gamma$-values can be applied on both communication levels.

*Proof.* The proof strategy is similar to that applied in van den Brink et al. [4].

I. [Existence]. We show that under the hypothesis of the theorem the $P\Gamma$-value $\xi = Ow^{\langle DL^{\mathcal{P}}, \{DL^k\}_{k \in M} \rangle}$ defined on $\mathcal{G}_N^{DL^{\mathcal{P}}, \{DL^k\}_{k \in M}}$ by (18) meets the axioms QCE, QDL, FDSU, and UDL. Consider an arbitrary $P\Gamma$-game $\langle v, \Gamma_{\mathcal{P}} \rangle \in \mathcal{G}_N^{DL^{\mathcal{P}}, \{DL^k\}_{k \in M}}$.

**QCE**. By the definition (18) of $\xi$ and component efficiency of each $DL^k$-value for all $k \in M$ it holds that

$$\sum_{i \in N_k} \xi_i(v, \Gamma_{\mathcal{P}}) = DL_k^{\mathcal{P}}(v_{\mathcal{P}}, \Gamma_M). \tag{19}$$

Thus, for any $K \in M/\Gamma_M$ we have

$$\sum_{k \in K} \sum_{i \in N_k} \xi_i(v, \Gamma_{\mathcal{P}}) = \sum_{k \in K} DL_k^{\mathcal{P}}(v_{\mathcal{P}}, \Gamma_M) = v_{\mathcal{P}}(K),$$

where the second equality follows from component efficiency of $DL^{\mathcal{P}}$-value.

**QDL**. From (19) we obtain that $\xi^{\mathcal{P}}(v, \Gamma_{\mathcal{P}}) = DL^{\mathcal{P}}(v_{\mathcal{P}}, \Gamma_M)$. Whence it follows that

$$\Psi^{DL^{\mathcal{P}}}(\xi^{\mathcal{P}}(v, \Gamma_{\mathcal{P}}), \Gamma_M) = \Psi^{DL^{\mathcal{P}}}(DL^{\mathcal{P}}(v_{\mathcal{P}}, \Gamma_M)), \Gamma_M) = 0,$$

where the last equality holds true since the $DL^{\mathcal{P}}$-value for $\Gamma$-games meets $\mathrm{DL}^{\mathcal{P}}$.

**UDL**. We need to show that if the set of $\mathrm{DL}^k$, $k \in M$, axioms is restricted to F, CF, and RF, then for all $k \in M$, $\Psi^{DL^k}(\xi^k(v, \Gamma_{\mathcal{P}}), \Gamma_k) = 0$.

Let for some $k \in M$, $\mathrm{DL}^k = \mathrm{F}$. Then since by definition for any link $\{i,j\} \in \Gamma_k$, $\Gamma_{\mathcal{P}}|_{-ij} = \Gamma_{\mathcal{P}}|_{-ij}^k = \langle \Gamma_M, \{\widehat{\Gamma}_h\}_{h \in M} \rangle$ with $\widehat{\Gamma}_h = \Gamma_h$ for $h \neq k$, and $\widehat{\Gamma}_k = \Gamma_k|_{-ij}$, we obtain that

$$\Psi^{DL^k}(\xi^k(v, \Gamma_{\mathcal{P}}), \Gamma_k) = \Psi^F(\xi^k(v, \Gamma_{\mathcal{P}}), \Gamma_k) \overset{(4)}{=}$$

$$\sum_{i,j \in N_k | \{i,j\} \in \Gamma_k} \Big| \Big( \xi_i^k(v, \Gamma_{\mathcal{P}}) - \xi_i^k(v, \Gamma_{\mathcal{P}}|_{-ij}) \Big) - \Big( \xi_j^k(v, \Gamma_{\mathcal{P}}) - \xi_j^k(v, \Gamma_{\mathcal{P}}|_{-ij}) \Big) \Big| \overset{(18)}{=}$$

$$\sum_{i,j\in N_k|\{i,j\}\in\Gamma_k} \Big|\Big(F_i(\tilde{v}_k,\Gamma_k)+\frac{DL_k^{\mathcal{P}}(v_{\mathcal{P}},\Gamma_M)-\sum\limits_{C\in N_k/\Gamma_k}\tilde{v}_k(C)}{n_k}-$$

$$F_i(\tilde{v}_k,\Gamma_k|_{-ij})-\frac{DL_k^{\mathcal{P}}(v_{\mathcal{P}},\Gamma_M)-\sum\limits_{C\in N_k/\Gamma_k|_{-ij}}\tilde{v}_k(C)}{n_k}\Big)-$$

$$\Big(F_j(\tilde{v}_k,\Gamma_k)+\frac{DL_k^{\mathcal{P}}(v_{\mathcal{P}},\Gamma_M)-\sum\limits_{C\in N_k/\Gamma_k}\tilde{v}_k(C)}{n_k}-$$

$$F_j(\tilde{v}_k,\Gamma_k|_{-ij})-\frac{DL_k^{\mathcal{P}}(v_{\mathcal{P}},\Gamma_M)-\sum\limits_{C\in N_k/\Gamma_k|_{-ij}}\tilde{v}_k(C)}{n_k}\Big)\Big|=$$

$$\sum_{i,j\in N_k|\{i,j\}\in\Gamma_k}\Big|\Big(F_i(\tilde{v}_k,\Gamma_k)-F_i(\tilde{v}_k,\Gamma_k|_{-ij})\Big)-\Big(F_j(\tilde{v}_k,\Gamma_k)-F_j(\tilde{v}_k,\Gamma_k|_{-ij})\Big)\Big|\overset{(4)}{=}$$

$$\Psi^F(F(\tilde{v}_k,\Gamma_k),\Gamma_k)=0,$$

where the last equality holds true since the $F$-value for $\Gamma$-games (the Myerson value) meets fairness F.

Using the similar arguments as above we prove that $\Psi^{DL^k}(\xi^k(v,\Gamma_{\mathcal{P}}),\Gamma_k)=0$ also for $k\in M$ for which DL$^k$ coincides with CF, and RF.

**FDSU**. From (18) we obtain that for every $C\in N_k/\Gamma_k$ it holds that

$$\sum_{i\in C}\xi_i(v,\Gamma_{\mathcal{P}})\overset{(18)}{=}\sum_{i\in C}DL_i^k(\tilde{v}_k,\Gamma_k)+\frac{c}{n_k}\left(DL_k^{\mathcal{P}}(v_{\mathcal{P}},\Gamma_M)-\sum_{H\in N_k/\Gamma_k}\tilde{v}_k(H)\right)=$$

$$=\tilde{v}_k(C)+\frac{c}{n_k}\left(DL_k^{\mathcal{P}}(v_{\mathcal{P}},\Gamma_M)-\sum_{H\in N_k/\Gamma_k}\tilde{v}_k(H)\right),$$

where the second equality is due to component efficiency of $DL^k$-value for $\Gamma$-games. Further,

$$\sum_{i\in C}\xi_i(v_C^k,\Gamma_{\mathcal{P}_C^k})\overset{(18)}{=}\sum_{i\in C}DL_i^k(\widetilde{(v_C^k)}_k,\Gamma_k|_C)+\frac{c}{n_k}\left(DL_k^{\mathcal{P}}((v_C^k)_{\mathcal{P}_C^k},\Gamma_M)-\sum_{H\in C/\Gamma_k|_C}\widetilde{(v_C^k)}_k(H)\right)=$$

$$=\widetilde{(v_C^k)}_k(C)+\frac{c}{n_k}\Big(DL_k^{\mathcal{P}}((v_C^k)_{\mathcal{P}_C^k},\Gamma_M)-\widetilde{(v_C^k)}_k(C)\Big)=\tilde{v}_k(S),$$

where the second equality is due to component efficiency of $DL^k$-value for $\Gamma$-games and $C$ being the only component in $\Gamma_k|_C$, and the third equality follows from equality $\widetilde{(v_C^k)}_k(C)=\tilde{v}_k(C)$ which holds true because $\hat{v}_C=\widehat{(v_C^k)}_C$, and equalities $DL_k^{\mathcal{P}}((v_C^k)_{\mathcal{P}_C^k},\Gamma_M)=DL_k^{\mathcal{P}}(\hat{v}_C,\Gamma_M)\overset{\text{def}}{=}\tilde{v}_k(C)$ first of which holds true because $(v_C^k)_{\mathcal{P}_C^k}=\widehat{(v_C^k)}_C=\hat{v}_C$. Thus,

$$\frac{1}{c}\sum_{i\in C}\Big(\xi_i(v,\Gamma_{\mathcal{P}})-\xi_i(v_C^k,\Gamma_{\mathcal{P}_C^k})\Big)=\frac{1}{n_k}\left(DL_k^{\mathcal{P}}(v_{\mathcal{P}},\Gamma_M)-\sum_{H\in N_k/\Gamma_k}\tilde{v}_k(H)\right),$$

where the right side of the latter equality is independent of $C$.

II. [Uniqueness]. Assume that a $(m+1)$-tuple of deletion link axioms $\langle \mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k\in M}\rangle$ such that the set of axioms $\mathrm{DL}^k$, $k \in M$, is restricted to F, CF, and RF, is given. We show that there exists at most one $P\Gamma$-value on $\mathcal{G}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ that satisfies axioms QCE, FDSU, QDL, and UDL. Let $\phi$ be such $P\Gamma$-value on $\mathcal{G}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$. Given an arbitrary $P\Gamma$-game $\langle v, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ we show that the individual payoffs $\phi_i(v,\Gamma_{\mathcal{P}})$, $i \in N$, are uniquely determined.

**Step 1.** We determine the union payoffs $\phi_k^{\mathcal{P}}(v,\Gamma_{\mathcal{P}})$, $k \in M$, by induction on the number of links in $\Gamma_M$ in a similar way as uniqueness of the Myerson value for $\Gamma$-games is shown in Myerson [12].

Initialization: If $|\Gamma_M| = 0$ then for all $k \in M$ the set of neighboring unions $\{h \in M \mid \{h,k\} \in \Gamma\} = \emptyset$, and therefore by QCE and definition of the quotient game $v_{\mathcal{P}}$, $\phi_k^{\mathcal{P}}(v,\Gamma_{\mathcal{P}}) = v_{\mathcal{P}}(\{k\}) = v(N_k)$.

Induction hypothesis: Assume that the values $\phi_k^{\mathcal{P}}(v,\Gamma'_{\mathcal{P}})$, $k \in M$, have been determined for all two-level graph structures $\Gamma'_{\mathcal{P}} = \langle \Gamma', \{\Gamma_h\}_{h\in M}\rangle$ with $\Gamma'$ such that $|\Gamma'| < |\Gamma_M|$.

Induction step: Let $Q \in M/\Gamma_M$ be a component in graph $\Gamma_M$ on $M$. If $Q \subseteq M$ is a singleton, let $Q = \{k\}$, then from QCE it follows that $\phi_k^{\mathcal{P}}(v,\Gamma_{\mathcal{P}}) = v(N_k)$. If $q \geq 2$, then there exists a spanning tree $\widetilde{\Gamma} \subseteq \Gamma_M|_Q$ on $Q$ with the number of links $|\widetilde{\Gamma}| = q-1$. By QDL it holds that

$$\Psi^{DL^{\mathcal{P}}}(\phi^{\mathcal{P}}(v,\Gamma_{\mathcal{P}}),\Gamma_M) = 0. \tag{20}$$

The above equality in fact provides for any $k,l \in M$ in case when $\mathrm{DL}^{\mathcal{P}}$ is given by BLC and for every other possible choice of $\mathrm{DL}^{\mathcal{P}}$ for each link $\{k,l\} \in \Gamma_M$ if $\Gamma_M$ is undirected, or $(k,l) \in \Gamma_M$ if $\Gamma_M$ is directed, some equality relating values of $\phi_h^{\mathcal{P}}(v,\Gamma_{\mathcal{P}})$, $h \in M$, with values of distinct $\phi_h^{\mathcal{P}}(v,\Gamma_{\mathcal{P}}|_{-kl})$, $h \in M$. Since $|\Gamma_M|_{-kl}| = |\Gamma_M|-1$, from the induction hypothesis it follows that for all links $\{k,l\} \in \Gamma_M$ ($(k,l) \in \Gamma_M$) the payoffs $\phi_h^{\mathcal{P}}(v,\Gamma_{\mathcal{P}}|_{-kl})$, $h \in M$, are already determined. Thus with respect to $q-1$ links $\{k,l\} \in \widetilde{\Gamma}$, (20) yields $q-1$ linearly independent linear equations in the $q$ unknown payoffs $\phi_k^{\mathcal{P}}(v,\Gamma_{\mathcal{P}})$, $k \in Q$. Moreover, by QCE it holds that

$$\sum_{k\in Q} \phi_k^{\mathcal{P}}(v,\Gamma_{\mathcal{P}}) = v_{\mathcal{P}}(Q).$$

All these $q$ equations are linearly independent. Whence it follows that for every $Q \in M/\Gamma$, all payoffs $\phi_k^{\mathcal{P}}(v,\Gamma_{\mathcal{P}})$, $k \in Q$, are uniquely determined. Note that in the proof of the induction step, every possible spanning tree $\widetilde{\Gamma}$ yields the same solution for the values $\phi_k^{\mathcal{P}}(v,\Gamma_{\mathcal{P}})$, $k \in Q$, because otherwise a solution does not exist, which contradicts the already proved "existence" part of the proof of the theorem.

**Step 2.** Similarly as in Step 1, for every $k \in M$, for every subset $C \subseteq N_k$ we determine the union payoffs $\phi_k^{\mathcal{P}}(v_C^k, \Gamma_{\mathcal{P}_C^k})$ in the game $(v_C, \Gamma_{\mathcal{P}_C^k})$, where $v_C^k$ denotes the subgame $v|_{(N\setminus N_k)\cup C}$ of $v$ with respect to the coalition $(N \setminus N_k) \cup C$, and $\Gamma_{\mathcal{P}_C^k}$ denotes the two-level communication structure $\langle \Gamma_M, \{\Gamma_h\}_{h\in M}\rangle$, where $\Gamma_M$ is the communication graph on the partition $(\mathcal{P} \setminus \{N_k\}) \cup \{C\}$ (where the 'position' of $N_k$ is taken over by $C$) and with the communication graph $\Gamma_k$ replaced by its restriction on $C \subseteq N_k$. Note that now, for $k \in M$, the union payoff $\phi_k^{\mathcal{P}}(v_C^k, \Gamma_{\mathcal{P}_C^k})$ is the total payoff to the players in $C \subseteq N_k$ in the game $(v_C^k, \Gamma_{\mathcal{P}_C^k})$.

**Step 3.** We determine now the individual payoffs in every a priori union $N_k$, $k \in M$. For this first we show that for each component $C \in N_k/\Gamma_k$ the total payoff to the players in $C$ is uniquely determined. The payoff $\phi_k^{\mathcal{P}}(v,\Gamma_{\mathcal{P}})$ to the a priori union $N_k$ has been determined already in Step 1, so

$$\sum_{i\in N_k} \phi_i(v,\Gamma_{\mathcal{P}}) = \phi_k^{\mathcal{P}}(v,\Gamma_{\mathcal{P}}). \tag{21}$$

If $N_k$ is the unique component in $N_k/\Gamma_k$, then FDSU does not state any requirement. When $N_k/\Gamma_k$ consists of multiple components, then for every component $C \in N_k/\Gamma_k$, from the alternative form (17) of FDSU it follows that

$$\frac{\sum_{i\in C}\phi_i(v,\Gamma_{\mathcal{P}}) - \phi_k^{\mathcal{P}}(v_C^k,\Gamma_{\mathcal{P}_C^k})}{c} = \frac{\phi_k^{\mathcal{P}}(v,\Gamma_{\mathcal{P}}) - \sum_{K\in N_k/\Gamma_k}\phi_k^{\mathcal{P}}(v_K^k,\Gamma_{\mathcal{P}_K^k})}{n_k}. \tag{22}$$

Notice that every payoff $\phi_k^{\mathcal{P}}$ in this equation has been determined in either Step 1, or Step 2, and therefore, $\sum_{i\in C}\phi_i(v,\Gamma_{\mathcal{P}})$ is uniquely determined.

The rest of the proof we proceed by induction similar as in Step 1. Take some $k\in M$. Let $\Gamma'_{\mathcal{P}}$ denote the two-level graph structure $\langle\Gamma_M,\{\Gamma'_h\}_{h\in M}\rangle$ with $\Gamma'_h=\Gamma_h$ if $h\neq k$ and $\Gamma'_k=\Gamma'$ for some graph $\Gamma'$ on $N_k$.

Initialization: If $|\Gamma_k|=0$ then $\{i\}\in N_k/\Gamma_k$ for all $i\in N_k$. FDSU in the alternative form (17) implies that

$$\phi_i(v,\Gamma_{\mathcal{P}}) - \phi_k^{\mathcal{P}}(v_{\{i\}}^k,\Gamma_{\mathcal{P}_{\{i\}}^k}) = \frac{\phi_i^{\mathcal{P}}(v,\Gamma_{\mathcal{P}}) - \sum_{j\in N_k}\phi_k^{\mathcal{P}}(v_{\{j\}}^k,\Gamma_{\mathcal{P}_{\{j\}}^k})}{n_k}, \quad \text{for all } i\in N_k. \tag{23}$$

From Steps 1 and 2 above we know $\phi_k^{\mathcal{P}}(v,\Gamma_{\mathcal{P}})$ and $\phi_k^{\mathcal{P}}(v_{\{j\}}^k,\Gamma_{\mathcal{P}_{\{j\}}^k})$, for all $j\in N_k$. So, equation (23) determines $\phi_i(v,\Gamma_{\mathcal{P}})$ for all $i\in N_k$.

Induction hypothesis: Assume that the values $\phi_i(v,\Gamma'_{\mathcal{P}})$ have been determined for every $\Gamma'$ with $|\Gamma'|<|\Gamma_k|$.

Induction step: Take a component $C\in N_k/\Gamma_k$. If $c=1$, then $C$ is a singleton and the payoff $\phi_i(v,\Gamma_{\mathcal{P}})$ of the only player $i\in C$ is uniquely determined by (22). If $c\geq 2$, then there exists a spanning tree $\widetilde{\Gamma}\subseteq\Gamma_k|_C$ on $C$ with the number of links $|\widetilde{\Gamma}|=c-1$. By UDL it holds that

$$\Psi^{DL^k}(\phi^k(v,\Gamma_{\mathcal{P}}),\Gamma_k)=0. \tag{24}$$

The above equality in fact provides for any link $\{i,j\}\in\Gamma_k$ some equality relating values of $\phi_h(v,\Gamma_{\mathcal{P}})$, $h\in N_k$, with values of distinct $\phi_h(v,\Gamma_{\mathcal{P}}|_{-ij}^k)$. Since $|\Gamma_k|_{-ij}|=|\Gamma_k|-1$, by the induction hypothesis it follows that for all links $\{i,j\}\in\Gamma_k$ the payoffs $\phi_h(v,\Gamma_{\mathcal{P}}|_{-ij}^k)$, $h\in N_k$, are already determined. Thus with respect to $c-1$ links $\{i,j\}\in\widetilde{\Gamma}$, (24) yields $c-1$ linearly independent linear equations in the $c$ unknown payoffs $\phi_i(v,\Gamma_{\mathcal{P}})$, $i\in C$. These $c-1$ equations together with (22) if $C\neq N_k$, or together with (21) when $C=N_k$, yield $c$ linearly independent equations in the $c$ unknown payoffs $\phi_i(v,\Gamma_{\mathcal{P}})$, $i\in C$. Hence, for every $C\in N_k/\Gamma_k$, all payoffs $\phi_i(v,\Gamma_{\mathcal{P}})$, $i\in C$, are uniquely determined. Note that similar as in Step 1, every possible spanning tree $\widetilde{\Gamma}$ yields the same solution for the values $\phi_i(v,\Gamma_{\mathcal{P}})$, $i\in C$, because otherwise a solution does not exist, which contradicts the already proved "existence" part of the proof of the theorem. ∎

**Remark 1** The *PΓ*-value $\xi=Ow^{\langle DL^{\mathcal{P}},\{DL^k\}_{k\in M}\rangle}$ determined by (18) violates UDL property if in the $(m+1)$-tuple of deletion link axioms $\langle \mathrm{DL}^{\mathcal{P}},\{\mathrm{DL}^k\}_{k\in M}\rangle$ among the axioms $\mathrm{DL}^k$, $k\in M$ there are axioms BLC, UE, LE, EL, SE, and PE. The reason is that the second summand in the numerator of the second term in the right-hand side of (18) is sensitive to the deletion of different links in graph $\Gamma_k$, and therefore, for these cases the equality $\Psi^{DL^k}(\xi^k(v,\Gamma_{\mathcal{P}}),\Gamma_k)=\Psi^{DL^k}(DL^k(\tilde{v}_k,\Gamma_k),\Gamma_k)=0$ in general does not hold.

The logical independence of axioms in case $\langle \mathrm{DL}^{\mathcal{P}},\{\mathrm{DL}^k\}_{k\in M}\rangle=\langle \mathrm{F}^{\mathcal{P}},\{\mathrm{F}^k\}_{k\in M}\rangle$ is shown in van den Brink et al. [4] by a number of examples. We skip the proof of logical independence of axioms for the general case since it can be easily obtained by modification of the same examples.

## 5 Kamijo-type values for two-level graph games

In this section we consider another family of $P\Gamma$-values based on the adaptation of the Kamijo's two-step distribution procedure for $P$-games. We introduce these values axiomatically by means of six axioms. The first three axioms are QCE, QDL, and UDL which are also used in the characterization of the Owen-type $\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$-values discussed in the previous section. The three additional axioms are as follows.

The first axiom is a straightforward adaptation of the standard covariance under strategic equivalence to the case of $P\Gamma$-games. Two games $v, w \in \mathcal{G}_N$ are *strategically equivalent* if there are $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}^n$ such that

$$w(S) = av(S) + b(S), \qquad \text{for all } S \subseteq N.$$

**Covariance under strategic equivalence (COV)** For any player set $N \subset \mathbb{N}$ and $P\Gamma$-game $\langle v, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{P\Gamma}$, $\Gamma_{\mathcal{P}} = \langle \Gamma_M, \{\Gamma_k\}_{k\in M}\rangle$, for any $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}^n$ it holds that

$$\xi(av + b, \Gamma_{\mathcal{P}}) = a\xi(v, \Gamma_{\mathcal{P}}) + b,$$

where the game $(av + b) \in \mathcal{G}_N$ is defined by $(av + b)(S) = av(S) + b(S)$ for all $S \subseteq N$.

The next axiom requires equal payoffs to all members of every a priori union $N_k$, $k \in M$, for which all subcoalitions $S \subseteq N_k$ with nonzero worth $v_k(S) \neq 0$ are disconnected. In this case every connected coalition possesses zero worth, i.e., communication between the members of every connected coalition is useless, and so, the asymmetries among the players created by game $v_k$ and by their locations in graph $\Gamma_k$ on $N_k$ vanish. Therefore, it makes sense to treat all players of $N_k$ symmetrically. Remark that the condition that $v(S) \neq 0$ implies $S$ to be disconnected is equivalent to $v_k^{\Gamma_k} \equiv \mathbf{0}$, i.e., the axiom requires that all players in $N_k$ obtain the same payoffs if the Myerson restricted game $v_k^{\Gamma_k}$ is a null game. The latter observation determines the name of the axiom.

**Union null restricted game property (UNRGP)** For any player set $N \subset \mathbb{N}$ and $P\Gamma$-game $\langle v, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{P\Gamma}$, $\Gamma_{\mathcal{P}} = \langle \Gamma_M, \{\Gamma_k\}_{k\in M}\rangle$, if for some $k \in M$, for all $S \subseteq N_k$, $v_k(S) \neq 0$ implies $S \notin C^{\Gamma_k}(N_k)$, then it holds that for all $i, j \in N_k$, $i \neq j$,

$$\xi_i(v, \Gamma_{\mathcal{P}}) = \xi_j(v, \Gamma_{\mathcal{P}}).$$

The last axiom determines the distribution of the total shares obtained by internally disconnected a priori unions at the upper level bargaining between a priori unions among the components of these unions. Imagine that each a priori union $N_k$, $k \in M$, is a public institution (e.g. university, hospital, or firm) of which every component $C \in N_k/\Gamma_k$ is an independent unit (e.g. the faculties within a university, medical departments within a hospital, or production plants within a firm). First public institutions $N_k$, $k \in M$ compete among themselves for their annual budgets from the government. Once obtained the budget, institution $N_k$ has to decide how much to give to each of its independent units. At this stage the independent units of an institution compete against each other for the best possible shares from the institution's budget. Similarly as in the competition among the public institutions, the total payoff to a unit depends on the total productivity of each of the units, but not on the productivity of the smaller collaborating teams within the units. Our last axiom requires the total payoff to a component of any a priori union to be independent of the so-called internal coalitions, each of which is a proper subcoalition of some component of one of the given a priori unions, or more precisely, a coalition $\emptyset \neq S \subseteq N$ is *internal* if there is $k \in M$ such that $S \subset C$ for some $C \in N_k/\Gamma_k$. From now on, given a player set $N \subset \mathbb{N}$, a partition $\mathcal{P} = \{N_1, ..., N_m\}$ of $N$, and a set of communication graphs $\{\Gamma_k\}_{k\in M}$ on a priori unions $N_k$, $k \in M$, the set of all internal coalitions we denote by $\mathrm{Int}(N, \mathcal{P}, \{\Gamma_k\}_{k\in M})$. It is worth to remark that, of course, the worths of internal coalitions play a crucial role in the redistribution of the total payoff obtained by a component among its members, but the axiom does not concern this.

**Union component payoff independence of internal coalitions (UCPIIC)** For any player set $N \subset \mathbb{N}$ and two $P\Gamma$-games $\langle v, \Gamma_{\mathcal{P}}\rangle, \langle w, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{P\Gamma}$ with the same $\Gamma_{\mathcal{P}} = \langle \Gamma_M, \{\Gamma_k\}_{k\in M}\rangle$ and such that $w(S) = v(S)$ for all $S \subseteq N$, $S \notin \mathrm{Int}(N, \mathcal{P}, \{\Gamma_k\}_{k\in M})$, it holds that for every $k \in M$, for all $C \in N_k/\Gamma_k$,

$$\sum_{i\in C} \xi_i(v, \Gamma_{\mathcal{P}}) = \sum_{i\in C} \xi_i(w, \Gamma_{\mathcal{P}}).$$

The next theorem introduces axiomatically another family of $P\Gamma$-values.

**Theorem 2** *For any $(m+1)$-tuple of deletion link axioms $\langle \mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k\in M}\rangle$ such that the set of $\mathrm{DL}^k$, $k \in M$, axioms is restricted to F, CF, and RF, there is the unique $P\Gamma$-value defined on $\mathcal{G}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ that meets axioms QCE, QDL, UDL, COV, UNRGP, and UCPIIC, and for every $P\Gamma$-game $\langle v, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ it is given by*

$$\xi_i(v, \Gamma_{\mathcal{P}}) = DL_i^{k(i)}(v_{k(i)}, \Gamma_{k(i)}) + \frac{DL_{k(i)}^{\mathcal{P}}(v_{\mathcal{P}}, \Gamma_M) - v_{k(i)}^{\Gamma_{k(i)}}(N_{k(i)})}{n_{k(i)}}, \qquad \textit{for all } i \in N. \tag{25}$$

**Remark 2** It is not difficult to trace a relation between the $P\Gamma$-value $\xi$ given by (25) and the two step Shapley value (1). Indeed, for a $P\Gamma$-game $\langle v, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{F^{\mathcal{P}},\{F^k\}_{k\in M}}$, $\xi(v, \Gamma_{\mathcal{P}}) = Ka(v^{\Gamma_{\mathcal{P}}}, \mathcal{P})$, where $v^{\Gamma_{\mathcal{P}}} \in \mathcal{G}_N$ is determined as

$$v^{\Gamma_{\mathcal{P}}}(S) = \begin{cases} v_{\mathcal{P}}^{\Gamma_M}(Q), & Q \subseteq M\colon S = \cup_{q\in Q} N_q, \\ v_k^{\Gamma_k}(S), & S \subseteq N_k,\ k \in M, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{for all } S \subseteq N,$$

i.e., when $S$ is the union of number of a priori unions its worth is defined according to the Myerson restricted quotient game, when $S$ is a subset of some a priori union its worth is defined according to the Myerson restricted game within the union, otherwise the worth of $S$ is zero.

Because of the mentioned similarity, from now on we refer to the $P\Gamma$-value (25) as to the *Kamijo-type $\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$-value*, denoted further by $Ka^{\langle DL^{\mathcal{P}},\{DL^k\}_{k\in M}\rangle}$.

*Proof.* I. [Existence]. We show that under the hypothesis of the theorem the $P\Gamma$-value $\xi = Ka^{\langle DL^{\mathcal{P}},\{DL^k\}_{k\in M}\rangle}$ defined on $\mathcal{G}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ by (25) meets the axioms QCE, QDL, UDL, COV, UNRGP and UCPIICL. Consider an arbitrary $P\Gamma$-game $\langle v, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$.

**QCE, QDL, UDL.** The proof of these axioms for the Kamijo-type $\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$-value is similar to the proof of same axioms for the Owen-type $\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$-value in Theorem 1, and so we skip it.

**COV.** Pick any $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}^n$. Then, for all $i \in N$,

$$\xi_i(av{+}b, \Gamma_{\mathcal{P}}) \overset{(25),(2)}{=} DL_i^{k(i)}((av{+}b)_{k(i)}, \Gamma_{k(i)}){+}\frac{DL_{k(i)}^{\mathcal{P}}((av+b)_{\mathcal{P}}, \Gamma_M) - \sum\limits_{C\in N_{k(i)}/\Gamma_{k(i)}} (av+b)(C)}{n_{k(i)}} =$$

$$aDL_i^{k(i)}(v_{k(i)}, \Gamma_{k(i)}) + b_i + \frac{aDL_{k(i)}^{\mathcal{P}}(v_{\mathcal{P}}, \Gamma_M){+}b(N_{k(i)}){-}\sum\limits_{C\in N_{k(i)}/\Gamma_{k(i)}} (av(C){+}b(C))}{n_{k(i)}} = a\xi_i(v, \Gamma_{\mathcal{P}}) + b,$$

where the second equality is true because each of the considered $DL^k$-values, $k \in M$, and $DL^{\mathcal{P}}$-values meets COV on its domain, and $(av+b)_{\mathcal{P}}(Q) = av_{\mathcal{P}}(Q) + b(\cup_{k\in Q} N_k)$ for all $Q \subseteq M$; and the third equality is due to the equality $\sum_{C \in N_{k(i)}/\Gamma_{k(i)}} b(C) = b(N_{k(i)}) = b\, n_{k(i)}$, since $N_{k(i)}/\Gamma_{k(i)}$ forms a partition of $N_{k(i)}$.

**UNRGP.** Assume that for the chosen $P\Gamma$-game $\langle v, \Gamma_{\mathcal{P}}\rangle$ there exists $k \in M$ such that for all $S \subseteq N_k$, $v_k(S) \neq 0$ implies $S \notin C^{\Gamma_k}(N_k)$. Every of the considered $DL^k$-values, the Myerson value, the average tree solution and the compensation solution, are determined only by worths of connected coalitions, and therefore, $DL^k_i(v_k, \Gamma_k) = 0$ for all $i \in N_k$. Moreover, as it was already mentioned ealier, the above assumption is equivalent to $v_k^{\Gamma_k} \equiv \mathbf{0}$, which implies that $v(C) = 0$ for every $C \in N_k/\Gamma_k$. Hence, from (25) it follows that for all $i \in N_k$,

$$\xi_i(v, \Gamma_{\mathcal{P}}) = \frac{DL^{\mathcal{P}}_k(v_{\mathcal{P}}, \Gamma_M)}{n_k},$$

where the right side is independent of $i$, from which it follows that $P\Gamma$-value $\xi$ meets UNRGP.

**UCPIIC.** Take any $k \in M$ and $C \in N_k/\Gamma_k$. The component efficiency of each of the considered $DL^k$-values implies $DL^k(v_k, \Gamma_k)(C) = v(C)$. Then from (25) it follows that

$$\sum_{i\in C} \xi(v, \Gamma_{\mathcal{P}}) = v(C) + \frac{c}{n_k}\Big(DL^{\mathcal{P}}_k(v_{\mathcal{P}}, \Gamma_M) - v^{\Gamma_k}(N_k)\Big),$$

where the right side is independent of worths of internal coalitions, i.e. $P\Gamma$-value $\xi$ meets UCPIIC.

II. [Uniqueness]. Assume that a $(m+1)$-tuple of deletion link axioms $\langle \mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k\in M}\rangle$ such that the set of axioms $\mathrm{DL}^k$, $k \in M$, is restricted to F, CF, and RF, is given. We show that there exists at most one $P\Gamma$-value on $\mathcal{G}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ that satisfies axioms QCE, QDL, UDL, COV, UNRGP, and UCPIIC. Let $\phi$ be such $P\Gamma$-value on $\mathcal{G}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$. Take an arbitrary $P\Gamma$-game $\langle v, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$. We show that the individual payoffs $\phi_i(v, \Gamma_{\mathcal{P}})$, $i \in N$, are uniquely determined, for which it is enough to show that the individual payoffs in each a priori union $N_k$, $k \in M$, are uniquely determined. Fix some $k \in M$. The rest of the proof is by induction on the number of links in $\Gamma_k$.

Initialization: Assume that $|\Gamma_k| = 0$. Let $v_0$ be the 0-normalization of the TU game $v$, i.e., $v_0(S) = v(S) - \sum_{i\in S} v(\{i\})$ for all $S \subseteq N$, and let $(v_0)_k = v_0|_{N_k}$. As it is shown in Step 1 of the proof of the "uniqueness" part of Theorem 1, QCE and $\mathrm{DL}^{\mathcal{P}}$ together yield that the union payoffs $\phi^{\mathcal{P}}_k(w, \Gamma_{\mathcal{P}})$, $k \in M$, are uniquely determined for any $P\Gamma$-game $\langle w, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$. In particular, the 'union payoffs' $\phi^{\mathcal{P}}_k(v_0, \Gamma_{\mathcal{P}})$, $k \in M$, are uniquely determined. By definition $(v_0)_k(\{i\}) = 0$ for every $i \in N_k$, and therefore, $(v_0)_k^{\Gamma_k} \equiv \mathbf{0}$ since $|\Gamma_k| = 0$. Then, from UNRGP it follows that

$$\phi_i(v_0, \Gamma_{\mathcal{P}}) = \frac{\phi^{\mathcal{P}}_k(v_0, \Gamma_{\mathcal{P}})}{n_k}, \qquad \text{for all } i \in N_k.$$

Whence by COV we obtain

$$\phi_i(v, \Gamma_{\mathcal{P}}) = v(\{i\}) + \frac{\phi^{\mathcal{P}}_k(v^0, \Gamma_{\mathcal{P}})}{n_k}, \qquad \text{for all } i \in N_k,$$

i.e., for every $i \in N_k$, $\phi_i(v, \Gamma_{\mathcal{P}})$ is uniquely determined.

Induction hypothesis: Let $\Gamma'_{\mathcal{P}}$ denote the two-level graph structure $\langle \Gamma_M, \{\Gamma'_h\}_{h\in M}\rangle$ with $\Gamma'_h = \Gamma_h$ if $h \neq k$ and $\Gamma'_k = \Gamma'$ for some graph $\Gamma'$ on $N_k$. Assume that the values $\phi_i(v, \Gamma'_{\mathcal{P}})$ have been determined for every $\Gamma'$ with $|\Gamma'| < |\Gamma_k|$.

INDUCTION STEP: For every $S \in \text{Int}(N, \mathcal{P}, \{\Gamma_k\}_{k\in M})$ let $C_S \in N_k/\Gamma_k$ be the unique component such that $S \subset C_S$. Consider a game $w \in \mathcal{G}_N$ defined as

$$w(S) = \begin{cases} v(S), & S \notin \text{Int}(N, \mathcal{P}, \{\Gamma_k\}_{k\in M}), \\ \dfrac{s\, v(C_S)}{c_S}, & S \in \text{Int}(N, \mathcal{P}, \{\Gamma_k\}_{k\in M}), \end{cases} \quad \text{for all } S \subseteq N. \tag{26}$$

For the 0-normalization $w_0$ of $w$, $(w_0)_k = (w_k)_0$. The subgame $w_k$ is an additive game and, therefore, $(w_k)_0 \equiv \mathbf{0}$. Then, due to UNRGP, similar as in the Initialization step, it follows that

$$\phi_i(w_0, \Gamma_{\mathcal{P}}) = \frac{\phi_k^{\mathcal{P}}(w_0, \Gamma_{\mathcal{P}})}{n_k}, \qquad \text{for all } i \in N_k.$$

Whence by COV we obtain

$$\phi_i(w, \Gamma_{\mathcal{P}}) = w(\{i\}) + \frac{\phi_k^{\mathcal{P}}(w_0, \Gamma_{\mathcal{P}})}{n_k}, \qquad \text{for all } i \in N_k.$$

Consider a component $C \in N_k/\Gamma_k$. Fron the above equality it follows that

$$\sum_{i\in C} \phi_i(w, \Gamma_{\mathcal{P}}) = \sum_{i\in C} w(\{i\}) + \frac{c}{n_k}\phi_k^{\mathcal{P}}(w_0, \Gamma_{\mathcal{P}}) \overset{(26)}{=} v(C) + \frac{c}{n_k}\phi_k^{\mathcal{P}}(w_0, \Gamma_{\mathcal{P}}).$$

By the definition (26) of $w$, $w(S) = v(S)$ for all $S \subseteq N$, $S \notin \text{Int}(N, \mathcal{P}, \{\Gamma_k\}_{k\in M})$. Whence by UCPIIC it follows that

$$\sum_{i\in C} \phi_i(v, \Gamma_{\mathcal{P}}) = v(C) + \frac{c}{n_k}\text{DL}_k^{\mathcal{P}}(w_{0,\mathcal{P}}, \Gamma_M), \qquad \text{for all } C \in N_k/\Gamma_k. \tag{27}$$

Next, if $c = 1$, then $C$ is a singleton and the payoff $\phi_i(v, \Gamma_{\mathcal{P}})$ of the only player $i \in C$ is uniquely determined by (27). If $c \geq 2$, then there exists a spanning tree $\widetilde{\Gamma} \subseteq \Gamma_k|_C$ on $C$ with the number of links $|\widetilde{\Gamma}| = c - 1$. By UDL it holds that

$$\Psi^{DL^k}(\phi^k(v, \Gamma_{\mathcal{P}}), \Gamma_k) = 0. \tag{28}$$

The above equality in fact provides for any link $\{i, j\} \in \Gamma_k$ some equality relating values of $\phi_h(v, \Gamma_{\mathcal{P}})$, $h \in N_k$, with values of distinct $\phi_h(v, \Gamma_{\mathcal{P}}|_{-ij}^k)$. Since $|\Gamma_k|_{-ij}| = |\Gamma_k| - 1$, by the induction hypothesis it follows that for all links $\{i, j\} \in \Gamma_k$ the payoffs $\phi_h(v, \Gamma_{\mathcal{P}}|_{-ij}^k)$, $h \in N_k$, are already determined. Thus with respect to $c - 1$ links $\{i, j\} \in \widetilde{\Gamma}$, (28) yields $c - 1$ linearly independent linear equations in the $c$ unknown payoffs $\phi_i(v, \Gamma_{\mathcal{P}})$, $i \in C$. These $c - 1$ equations together with (27) form a system of $c$ linearly independent equations in the $c$ unknown payoffs $\phi_i(v, \Gamma_{\mathcal{P}})$, $i \in C$. Hence, for every $C \in N_k/\Gamma_k$, all payoffs $\phi_i(v, \Gamma_{\mathcal{P}})$, $i \in C$, are uniquely determined. Note that every possible spanning tree $\widetilde{\Gamma}$ yields the same solution for the values $\phi_i(v, \Gamma_{\mathcal{P}})$, $i \in C$, because otherwise a solution does not exist, which contradicts the already proved "existence" part of the proof of the theorem. ∎

**Logical independence**

Given a $(m+1)$-tuple of deletion link axioms $\langle \text{DL}^{\mathcal{P}}, \{\text{DL}^k\}_{k\in M}\rangle$, such that the set of $\text{DL}^k$, $k \in M$, axioms is restricted to F, CF, and RF, the logical independence of the axioms in Theorem 2 is demonstrated by the following examples of $P\Gamma$-values:

- The $P\Gamma$-value $\xi^{(1)}$ assigning in every $\langle v, \Gamma_{\mathcal{P}} \rangle \in \mathcal{G}_N^{\mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k\in M}}$ to every player $i \in N$ a payoff

$$\xi_i^{(1)}(v, \Gamma_{\mathcal{P}}) = \frac{DL_{k(i)}^{\mathcal{P}}(v_{\mathcal{P}}, \Gamma_M)}{n_{k(i)}}$$

  satisfies all axioms except COV.

- The $P\Gamma$-value $\xi^{(2)}$ assigning in every $\langle v, \Gamma_{\mathcal{P}} \rangle \in \mathcal{G}_N^{\mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k\in M}}$ to every player $i \in N$ a payoff

$$\xi_i^{(2)}(v, \Gamma_{\mathcal{P}}) = v(\{i\}) + \frac{DL_{k(i)}^{\mathcal{P}}(v_{\mathcal{P}}, \Gamma_M) - \sum_{j\in N_{k(i)}} v(\{j\})}{n_{k(i)}}$$

  satisfies every axiom except UCPIIC.

- The $P\Gamma$-value $\xi^{(3)}$ assigning in every $\langle v, \Gamma_{\mathcal{P}} \rangle \in \mathcal{G}_N^{\mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k\in M}}$ to every player $i \in N$ a payoff

$$\xi_i^{(3)}(v, \Gamma_{\mathcal{P}}) = DL_i^{k(i)}(v_{k(i)}, \Gamma_{k(i)}) + \frac{Sh_{k(i)}(v_{(M/\Gamma_M)_{k(i)}}) - v^{\Gamma_{k(i)}}(N_k)}{n_{k(i)}}$$

  satisfies every axiom except QDL.

- The $P\Gamma$-value $\xi^{(4)}$ assigning in every $\langle v, \Gamma_{\mathcal{P}} \rangle \in \mathcal{G}_N^{\mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k\in M}}$ to every player $i \in N$ a payoff

$$\xi_i^{(4)}(v, \Gamma_{\mathcal{P}}) = EL_i^{k(i)}(v_{k(i)}, \Gamma_{k(i)}) + \frac{DL_{k(i)}^{\mathcal{P}}(v_{\mathcal{P}}, \Gamma_M) - v^{\Gamma_{k(i)}}(N_k)}{n_{k(i)}}$$

  satisfies every axiom except UDL.

- The $P\Gamma$-value $\xi^{(5)}$ assigning in every $\langle v, \Gamma_{\mathcal{P}} \rangle \in \mathcal{G}_N^{\mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k\in M}}$ to every player $i \in N$ a payoff

$$\xi_i^{(5)}(v, \Gamma_{\mathcal{P}}) = DL_i^{k(i)}(v_{k(i)}, \Gamma_{k(i)}) + \frac{v(N) - v^{\Gamma_{k(i)}}(N_k)}{n_{k(i)}}$$

  satisfies every axiom except QCE.

- The $P\Gamma$-value $\xi^{(6)}$ assigning in every $\langle v, \Gamma_{\mathcal{P}} \rangle \in \mathcal{G}_N^{\mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k\in M}}$ to every player $i \in N$ a payoff

$$\xi_i^{(6)}(v, \Gamma_{\mathcal{P}}) = Sh_i(v_{(N_{k(i)}/\Gamma_{k(i)})_i}) + \frac{DL_{k(i)}^{\mathcal{P}}(v_{\mathcal{P}}, \Gamma_M) - v^{\Gamma_{k(i)}}(N_k)}{n_{k(i)}}$$

  satisfies every axiom except UNRGP.

To compare the two families of solution concepts introduced by Theorems 1 and 2 observe that all the $P\Gamma$-values meet QSE, QDL, UDL, and COV. However, while the Owen-type $\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$-values satisfy FDSU, but violate UNRGP and UCPIIC, the Kamijo-type $\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$-values vice versa satisfy UNRGP and UCPIIC, but violate FDSU. The summary of the properties is given in the Table 1, where the axioms need for our axiomatizations are marked by $*$.

| solution | QCE | QDL | UDL | COV | FDSU | UNRGP | UCPIIC |
|---|---|---|---|---|---|---|---|
| Owen-type $\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$-value | $+^*$ | $+^*$ | $+^*$ | $+$ | $+^*$ | - | - |
| Kamijo-type $\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$-value | $+^*$ | $+^*$ | $+^*$ | $+^*$ | - | $+^*$ | $+^*$ |

Table 1.

# 6 Conclusion

The paper introduces a unified approach to different two-step solution concepts for games with two-level communication structure when different component efficient values for games with communication structures can be applied at the both communication levels. The $P\Gamma$-values introduced by Theorems 1 and 2 allow to combine any of the component efficient values for games with communication structure discussed in Section 3 at the upper level of the allocation procedure, but at the lower level the list of applicable component efficient values is restricted to the Myerson value, the average tree solution and the compensation solution. Nevertheless, the associated general definitions given by (18) and (25) are not restricted to only these values, but can be formulated for the other values for games with communication structure as well. Moreover, the "uniqueness" parts of both theorems are already independent of the choice of the values for games with communication structures at the both levels of allocation procedure. So, a natural extension of this work could be to find parallel results to Theorems 1 and 2, or even more general results, which may allow to incorporate also the other component efficient solutions for games with communication structure.

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