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A Strategic Implementation of the Sequential Equal Surplus Division Rule for Digraph Cooperative Games[1]

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Abstract

We provide a strategic implementation of the sequential equal surplus division rule (Béal et al., 2014). Precisely, we design a non-cooperative mechanism of which the unique subgame perfect equilibrium payoffs correspond to the sequential equal surplus division outcome of a superadditive rooted tree TU-game. This mechanism borrowed from the bidding mechanism designed by Pérez-Castrillo and Wettstein (2001), but takes into account the direction of the edges connecting any two players in the rood tree, which reflects some dominance relation between them.

Keywords: Bidding approach – Implementation – Rooted tree TU-games – Sequential equal surplus division

1 Introduction

The Nash program, intended to bridge the gap between cooperative and non-cooperative game theory, has in recent literature been influenced much by the work of Pérez-Castrillo and Wettstein (2001), which implements the Shapley value (Shapley, 1953) through a mechanism consisting of a bidding stage followed by a proposal stage. Follow-up this seminal article, Ju and Wettstein (2009) provide a framework for implementing and comparing several solution concepts for transferable utility cooperative game (shortly, TU-games) by using a class of bidding mechanisms. On each of these bidding mechanisms, modeled through a non-cooperative extensive form game, the outcome of each sub-game perfect equilibrium (SPE) outcome coincides with the allocation of a value among a set of players that a solution concept for TU-games recommends.

In TU-games, the primitive information to allocate a value is what productive value is generated by each possible group of players or coalition. However, an important characteristic of social and economic situations is that players are part of a relational structure which possibly affects the cooperation possibilities. That is, alternative graph structures connecting the same set of players might lead to very different costs and benefits for coalitions. Thus, in many situations it is important to account for graph structures and not just coalition functions. Myerson (1977) makes a seminal contribution in adapting the cooperative game theory structure to accommodate information about the graph connecting players. The way in which he does this is by augmenting a cooperative game by an undirected graph over the player set. The Myerson value of an undirected graph TU-game is the Shapley value of the Myerson restricted game, being the TU-game that results from taking into account the fact that players can cooperate only if they are connected in

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the graph. Adapting the mechanism of Pérez-Castrillo and Wettstein (2001), Slikker (2007) and van den Brink et al. (2013) construct bidding mechanisms to implement several allocation rules for cooperative environments with an undirected graph structure.

Nevertheless, many of the aforementioned relational structures are better represented by a directed graph rather than by an undirected graph. A typical example is the river sharing problem in which the importance of directional flow is pointed out by Ambec and Sprumont (2002). Specifically, a group of players (firms, farmers, cities, countries) are located along a river with one source and one sink. The benefit of the players depend on the amount of water they consume, but they have unequal access to water. On the one hand, as players can only consume water entering upstream to their location, upstream players have a dominance advantage in water consumption. On the other hand, because of rainfall and inflow from tributaries, the river flow picks up volume along its course. Water inflow at the territory of downstream players cannot be consumed by upstream players. These directional constraints can be well represented by a directed-line graph. Ambec and Sprumont (2002) study the fair distribution of welfare resulting from the optimal allocation of water among the riparian players by modeling the river sharing problem as a cooperative TU-game augmented by a directed-line graph.

Béal et al. (2014) consider the scenario where the TU-game is augmented by a rooted tree. As a directed line is a special case of a rooted tree, our class of rooted tree TU-games are well suited to deal with the river sharing problem where the river has one source and multiple bifurcations. A new allocation rule, called the sequential equal surplus division rule, is introduced for rooted tree TU-games. It is constructed sequentially by following the direction of the edges of the rooted tree. The root player of this tree has possibly several successors, each of them initiating a separate branch of the tree, viewed as a coalition. Together with each branch, one for each successor, the root player achieves some surplus (positive or negative depending on the properties of the TU-game), measured by the difference between the worth achieved by the entire component and the sum of the worths achieved separately by each branch. In order to distribute this surplus, the root as well as each branch are considered as single bargaining entities. The participation of each entity is necessary to attain this surplus. Therefore, it seems natural to give to each entity an equal share of the surplus in addition to the worth it can secure in the absence of cooperation. This is equivalent to reward each entity by the well-known equal surplus division for TU-games. For each branch, the obtained total payoff is what remains to be shared among its members. The sequential equal surplus division rule then consists of applying recursively the above step to the root of all (sub-)branches of the tree. The distribution of the surplus of cooperation between two or more coalitions is also at the heart of the construction of many of the solutions proposed in the literature. Béal et al. (2014) provide two axiomatic characterizations of the sequential equal surplus division rule, and then apply it to solve the river sharing problem.

The aim of the present article is to implement the sequential equal surplus division rule through a bidding mechanism for the subclass of rooted-tree superadditive TU-games. Since the seminal article by Pérez-Castrillo and Wettstein (2001), several bidding mechanisms have been constructed for implementing solutions for TU-games. Most, if not all, of these bidding mechanisms share a set of desirable properties which prove very useful to demonstrate the uniqueness of equilibrium payoffs. In a first step, we identify a sufficient condition under which a bidding mechanism, where each winner of the bidding stage induces a sub-game, possesses these properties. In a second step, we design these sub-games in order to construct a bidding mechanism implementing the sequential surplus division rule. The main differences between our bidding mechanism and the existing ones for solutions of (undirected graph) TU-games are twofold.

First, the direction of the edges are taken into account in the construction of the bidding mechanism. The latter starts at the root of the tree. The root player bargains over the surplus of cooperation between her and the coalitions formed by her respective sets of subordinates. At this end of this bargaining game, the root player obtains his final payoff in the bidding mechanism. Then, the bidding mechanism continues its route on each branch of the tree: each successor of the root player is involved in another bargaining game to share a surplus achieved when this successor cooperates with her respective coalitions of subordinates in the rooted tree.

Second, each player, except the root, is involved in exactly two interrelated and local bargaining games. Consider any player different from the root. This player first bargains with her unique predecessor and the other successors of her predecessor, and then starts another bargaining game with all her successors. In the first bargaining game, this player represents her subordinates so that the payoff she obtains is a temporary payoff. In the second bargaining game, this player bargains with her own successors (if any) over a surplus that takes into account the payoff she has obtained in the first bargaining game. At the end of this second bargaining game, this player receives her final payoff and leaves the bidding mechanism.

Each bargaining game contains a bidding stage and a bargaining stage. The bidding stage is identical to the bidding stage designed by Pérez-Castrillo and Wettstein (2001), and the bargaining stage is a “take it or leave it” procedure.

The rest of the article is organized as followed. Section 2 provides preliminaries on TU-games and directed graph TU-games, and then introduces the sequential equal surplus division rule. Contrary to Béal et al (2014), this rule is defined recursively. This formulation will help to see the connection between this rule and our bidding mechanism that we also defined recursively. Section 3 singles out an abstract class of bidding mechanisms possesses a set of desirable properties. As mentioned above, these properties are common to all specific bidding mechanisms inspired from Perez-Castrillo and Wettstein (2001). These findings are then used in section 4 to design our bidding mechanism that implements the sequential equal surplus division rule in environments where the underlying TU-game is superadditive. Section 5 is devoted to the proof of this claim.

2 The cooperative situation and the allocation rule

2.1 Cooperative TU-games

Notations For a finite set A, the notation $|A|$ stands for the number of elements of A. Weak set inclusion is denoted by $\subseteq$, whereas proper set inclusion is denoted by $\subset$. For the ease of notation, we often write the singleton $\{i\}$ as i.

A *cooperative game* with transferable utility (henceforth called a TU-game) is a pair (N, v) consisting of a finite player set $N \subseteq \mathbb{N}$ of size n and a coalition function $v : 2^N \longrightarrow \mathbb{R}$ satisfying $v(\emptyset) = 0$. An element S of 2^N is a coalition, and $v(S)$ is the maximal worth that the members of S can obtain by cooperating. Denote by $\mathcal{C}$ the set of all TU-games. A TU-game (N, v) is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for any pair of disjoint coalitions S and T. For any two TU-games (N, v) and (N, w) defined on the same player set N, and for any $\alpha \in \mathbb{R}$, the TU-game $(N, \alpha v + w) \in \mathcal{C}$ is defined as follows: for each $S \subseteq N$, $(\alpha v + w)(S) = \alpha v(S) + w(S)$. For any non-empty coalition $T \subseteq N$, the *Dirac TU-game* $(N, \delta_T) \in \mathcal{C}$ is defined as: $\delta_T(T) = 1$, and $\delta_T(S) = 0$ for each other $S \subseteq N$.

In the TU-game (N, v), each player $i \in N$ may receive a payoff $z_i \in \mathbb{R}$. A *payoff vector* $z = (z_i)_{i \in N} \in \mathbb{R}^n$ lists a payoff z_i for each $i \in N$. For any nonempty coalition $S \in 2^N$ the notation z_S stands for $\sum_{i \in S} z_i$. An *allocation rule* Φ on the class of all TU-games $\mathcal{C}$ is a map that assigns to each TU-game $(N, v) \in \mathcal{C}$ a payoff vector $\Phi(N, v) \in \mathbb{R}^n$.

2.2 Digraph TU-games

In several social situations there is an underlying ordering of the players, which describes some social, technical, or communicational structure. In this article, we assume that the social structure is represented by a digraph on the player set representing some dominance relation between these players.

Precisely, a digraph TU-game is a triple (N, v, D) where $(N, v) \in \mathcal{C}$ and (N, D) is a directed graph. A *directed graph or digraph* is a pair (N, D), where N is a finite set of nodes (representing the players) and $D \subseteq N \times N$ is a binary relation on N. An ordered pair of elements $(i, j) \in D$ represents a *directed edge* from i to j. We assume the digraph to be irreflexive, i.e., $(i, i) \notin D$ for all $i \in N$. Let $E_i \subseteq D$ be the set of directed edges to which $i \in N$ is incident, i.e., directed edges of the form $(i, j) \in D$ or $(j, i) \in D$. For any subset $C \subseteq N$, the *subdigraph* induced on (N, D) by C is the pair (C, D_C) where $D_C = \{(i, j) \in D : i, j \in C\}$. For $i \in N$, the nodes in $S_D(i) = \{j \in N : (i, j) \in D\}$ are called the *successors* of i, and the nodes in $P_D(i) = \{j \in N : (j, i) \in D\}$ are called the *predecessors* of i in (N, D). A *directed path* from i to j in N is a sequence of distinct nodes $(i_1, \ldots, i_p)$, $p \geq 2$, such that $i_1 = i$, $i_{q+1} \in S_D(i_q)$ for $q = 1, \ldots, p-1$, and $i_p = j$. The number $p - 1$ is the *length* of the path. Given two nodes $i, j \in N$, j is a *subordinate* of i in (N, D) if there is a directed path from i to j. The set of i's subordinates is denoted by $\hat{S}_D(i)$, and we will use the notation $\hat{S}_D[i]$ to represent the union of $\hat{S}_D(i)$ and $\{i\}$. Note that for any $i \in N$, $S_D(i) \subseteq \hat{S}_D(i)$. We refer to the players in $\hat{P}_D(i) = \{j \in N : i \in \hat{S}_D(j)\}$ as the *superiors* of i in (N, D). We have $P_D(i) \subseteq \hat{P}_D(i)$.

A digraph (N, D) is a *rooted tree* if each node in N except one node called the root and denoted by r, has exactly one predecessor. If (N, D) is a tree rooted at $r \in N$, $p_D(i)$ refers to the unique predecessor of $i \in N \setminus r$. The *depth* of a node $i \in N$ in a rooted tree (N, D) is the length of the unique directed path from r to i, with the convention that the depth of r is set to 0. The *depth of a rooted tree* (N, D) is the depth of its deepest nodes. In a rooted tree (N, D), for each $i \in N$, note that the subdigraph $(\hat{S}_D[i], D_{\hat{S}_D[i]})$ induced on (N, D) by the subset of players $\hat{S}_D[i]$ is itself a rooted tree where the root is i, and $(\hat{S}_D[r], D_{\hat{S}_D[r]})$ coincides with (N, D). Denote by $\mathcal{D}$ the set of rooted trees.

In this article, we restrict our attention to the digraph TU-games (N, v, D) such that $(N, v) \in \mathcal{C}$ and $(N, D) \in \mathcal{D}$. Denote this set of digraph TU-games by $\mathcal{CD}$. An *allocation rule* Φ on $\mathcal{CD}$ is a map that assigns to each digraph TU-game $(N, v, D) \in \mathcal{CD}$ a payoff vector $\Phi(N, v, D) \in \mathbb{R}^n$.

2.3 The sequential equal surplus division rule

Béal et al. (2014) introduce the sequential equal surplus division rule, denoted by Φ^e, on $\mathcal{CD}$. Before defining formally Φ^e, we need to introduce the following quantity for each player $i \in N$ involved in (N, v, D):

$$\mathbb{S}_i(N, v, D) = v(\hat{S}_D[i]) - v(\{i\}) - \sum_{j \in S_D(i)} v(\hat{S}_D[j]),$$

which represents the *surplus* created by i and each coalition of subordinates $\hat{S}_D[j]$, $j \in S_D(i)$, when they decide to cooperate in the coalition $\hat{S}_D[i]$. Of course, $\mathbb{S}_i(N, v, D) \geq 0$ when the underlying TU-game (N, v) is superadditive.

For each $(N, v, D) \in \mathcal{CD}$, Φ^e is defined recursively from the root r as follows:

1. For the root $r \in N$,
$$\Phi^e_r(N, v, D) = v(\{r\}) + \frac{\mathbb{S}_r(N, v, D)}{|S_D[r]|}.$$

2. For each other $i \in N \setminus \{r\}$,

$$\Phi_i^e(N, v, D) = \Phi_i^e(\hat{S}_D[j], v_j, D_{\hat{S}_D[j]}),$$

where j is the unique successor of the root r of (N, D) such that $i \in \hat{S}_D[j]$. The TU-game $(\hat{S}_D[j], v_j)$ is as follows:

$$\forall S \subseteq \hat{S}_D[j], \quad v_j(S) = v(S) + \frac{\mathbb{S}_r(N, v, D)}{|S_D[r]|}\delta_{\hat{S}_D[j]}(S).$$

In other words, $v_j(S)$ is identical to $v(S)$ for each $S \subset \hat{S}_D[j]$ and $v_j(\hat{S}_D[j])$ is the sum of $v(\hat{S}_D[j])$ and an equal share of the surplus $\mathbb{S}_r(N, v, D)$.

Example 1 *Let $N = \{1, \ldots, 4\}$ be the player set, and let (N, D) be the rooted tree represented in Figure 1. Let (N, v) be a TU-game such that $v(S) = |S|^2$ for each $S \subseteq N$. For the root 1, we*

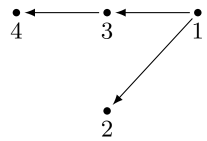

Figure 1: The rooted tree (N, D).

have:

$$\mathbb{S}_1(N, v, D) = v(\hat{S}_D[1]) - v(\{1\}) - \sum_{j \in S_D(1)} v(\hat{S}_D[j]) = 16 - 1 - 1 - 4 = 10.$$

By point 1 of the definition of Φ^e applied to (N, v, D), we get:

$$\begin{aligned} \Phi_1^e(N, v, D) &= 1 + \frac{10}{3} \\ &= \frac{13}{3}. \end{aligned}$$

Next, consider player 2. By point 2 of the definition of Φ^e applied to (N, v, D), we have to use the digraph TU-game $(\hat{S}_D[2], v_2, D_{\hat{S}_D[2]}) = (\{2\}, v_2, \emptyset)$, where

$$v_2 = v + \frac{\mathbb{S}_1(N, v, D)}{3}\delta_{\{2\}}.$$

By point 2 of the definition of Φ^e applied to (N, v, D),

$$\Phi_2^e(N, v, D) = \Phi_2^e(\{2\}, v_2, \emptyset).$$

By point 1 of the definition of Φ^e applied to $(\{2\}, v_2, \emptyset)$,

$$\begin{aligned} \Phi_2^e(\{2\}, v_2, \emptyset) &= v_2(\{2\}) + \frac{\mathbb{S}_2(\{2\}, v_2, \emptyset)}{1} \\ &= v_2(\{2\}) + 0 \\ &= 1 + \frac{10}{3} \\ &= \frac{13}{3}. \end{aligned}$$

Next, consider player 3. By point 2 of the definition of Φ^e *applied to* (N, v, D), *we have to use the digraph TU-game* $(\hat{S}_D[3], v_3, D_{\hat{S}_D[2]}) = (\{3,4\}, v_3, \{(3,4)\})$, *where*

$$v_3 = v + \frac{\mathbb{S}_1(N, v, D)}{3}\delta_{\{3,4\}}.$$

By point 2 of the definition of Φ^e *applied to* (N, v, D),

$$\Phi^e_3(N, v, D) = \Phi^e_3(\{3,4\}, v_3, \{(3,4)\}), \text{ and also } \Phi^e_4(N, v, D) = \Phi^e_4(\{3,4\}, v_3, \{(3,4)\}).$$

By point 1 of the definition of Φ^e *applied to* $(\{3,4\}, v_3, \{(3,4)\}$,

$$\Phi^e_3(\{3,4\}, v_3, \{(3,4)\}) = v_3(\{3\}) + \frac{\mathbb{S}_3(\{3,4\}, v_3, \{(3,4)\})}{2},$$

where

$$\begin{aligned} \mathbb{S}_3(\{3,4\}, v_3, \{(3,4)\}) &= v_3(\{3,4\}) - v_3(3) - v_3(4) \\ &= v(\{3,4\}) + \frac{10}{3} - v(\{3\}) - v(\{4\}) \\ &= \frac{16}{3}. \end{aligned}$$

Therefore, we obtain:

$$\begin{aligned} \Phi^e_3(N, v, D) &= \Phi^e_3(\{3,4\}, v_3, \{(3,4)\}) \\ &= 1 + \frac{16}{3 \times 2} \\ &= \frac{11}{3}. \end{aligned}$$

At last, consider player 4. We have to compute $\Phi^e_4(\{3,4\}, v_3, \{(3,4)\})$. *By point 2 of the definition of* Φ^e *applied to* $(\{3,4\}, v_3, \{(3,4)\})$, *we have* $\Phi^e_4(\{3,4\}, v_3, \{(3,4)\}) = \Phi^e_4(\{4\}, (v_3)_4, \emptyset)$, *where*

$$\begin{aligned} (v_3)_4 &= v_3 + \frac{\mathbb{S}_3(\{3,4\}, v_3, \{(3,4)\})}{2}\delta_4 \\ &= v + \frac{\mathbb{S}_3(\{3,4\}, v_3, \{(3,4)\})}{2}\delta_4. \end{aligned}$$

From this, we see that:

$$(v_3)_4(\{4\}) = 1 + \frac{8}{3}.$$

By point 1 of definition of Φ^e *applied to* $(\{4\}, (v_3)_4, \emptyset)$, *we obtain:*

$$\begin{aligned} \Phi^e(\{4\}, (v_3)_4, \emptyset) &= 1 + \frac{8}{3} + \frac{\mathbb{S}_4(\{4\}, (v_3)_4, \emptyset)}{1} \\ &= 1 + \frac{8}{3} + 0 \\ &= \frac{11}{3}. \end{aligned}$$

It follows that the payoff allocation vector induced by Φ^e *on* (N, v, D) *is given by:*

$$\Phi^e(N, v, D) = \left(\frac{13}{3}, \frac{13}{3}, \frac{11}{3}, \frac{11}{3}\right).$$

□

To close this section, we will use the fact that Φ^e satisfies the axiom of *(strict) aggregate mononoticity* (Meggido, 1974) on $\mathcal{CD}$ saying that the allocation of each player strictly increases when the worth of the grand coalition N increases while the worth of the other coalitions remains fixed. This axiom is well-known in problems of fair division. Formally, for each constant $c > 0$, it holds that:

$$\forall i \in N, \quad \Phi^e_i(N, v + c\delta_N, D) > \Phi^e_i(N, v, D). \tag{1}$$

To verify that Φ^e indeed satisfies this axiom on $\mathcal{CD}$, it suffices to note that $\mathbb{S}_r(N, v, D) < \mathbb{S}_r(N, v + c\delta_N, D)$ and to apply the recursive definition for Φ^e.

3 A general class of bidding mechanisms

Since the seminal article by Pérez-Castrillo and Wettstein (2001), several mechanism including a bidding stage followed by a proposal stage and an acceptance stage have been constructed to determine how the surplus generated by cooperation is to be shared in environments with transferable utility (see, e.g., Ju, Wettstein, 2009, Ju, 2012, Brink van den et al., 2013). It turns out that all these bidding mechanisms exhibit the same interesting properties that we will use later in this article. Here, we collect these properties for more abstract or disembodied bidding mechanisms.

Mechanism (A) Assume that the player set is $S \subseteq N$, $s \geq 2$.

Stage 1: Each player $i \in S$ makes $s-1$ bids $b^i_j \in \mathbb{R}$ to each other player $j \in S \setminus \{i\}$. For each $i \in S$, define the *net bid* of player i by

$$B^i = \sum_{j \in S\setminus\{i\}} b^i_j - \sum_{j \in S\setminus\{i\}} b^j_i.$$

Define by Ω_S the subset of players with the highest net bid. Pick any player i_s at random from Ω_S.

Stage 2: Player i_s induces some (sequential) non-cooperative game G_{i_s} on the player set S; the payoffs received in G_{i_s} are denoted by $(g^{i_s}_j)_{j\in S}$.

The rewards $(r^{i_s}_j)_{j\in S}$ resulting from **Stage 1** and **Stage 2** are given by:

$$r^{i_s}_j = \begin{cases} g^{i_s}_j + b^{i_s}_j & \text{if } j \in S \setminus \{i_s\}, \\ g^{i_s}_{i_s} - \sum_{l \in S\setminus\{i_s\}} b^{i_s}_l & \text{if } j = i_s. \end{cases}$$

The final payoff m_j received by each player $j \in S$ in Mechanism (A) is computed by taking the average of the rewards $r^{i_s}_j$ over Ω_S, i.e.

$$\forall j \in S, \quad m_j = \frac{\sum_{i_s \in \Omega_S} r^{i_s}_j}{|\Omega_S|}.$$

■

Proposition 1 *Consider Mechanism (A) and assume that on each SPE of G_{i_s}, $i_s \in \Omega_S$, of Stage 2, the payoff vector is the same and equal to $(\hat{g}^{i_s}_j)_{j\in S}$. Then, at each SPE of Mechanism (A), i.e. including Stage 1 and Stage 2, it holds that:*

1. *For each $j \in S$, player j's net bid is such that $\hat{B}^j = 0$, which implies that $\Omega_S = S$.*

2. *The rewards* $(\hat{r}_j^{i_s})_{j\in S}$ *induced by* $(\hat{g}_j^{i_s})_{j\in S}$, $i_s \in \Omega_S$, *and the bids do not depend on the identity of the proposer:*

$$\forall i_s, i'_s \in \Omega_S, \forall j \in S, \quad \hat{r}_j^{i_s} = \hat{r}_j^{i'_s}.$$

3. *For each* $j \in S$, *the induced expected payoff* $\hat{m}_j$ *is equal to:*

$$\hat{m}_j = \frac{\sum_{k\in S} \hat{g}_j^k}{|S|}.$$

4. *For each pair of distinct players* $i, j \in S$, *the bid* $\hat{b}_j^i$ *of* i *to* j *is equal to:*

$$\hat{b}_j^i = \frac{\sum_{k\in S} \hat{g}_j^k}{|S|} - \hat{g}_j^i.$$

The proof of Proposition 1 is relegated to the Appendix. The next proposition indicates the existence of such a SPE for Mechanism (A) is also ensured under the hypothesis of the statement of Proposition 1.

Proposition 2 *Under the hypothesis of Proposition 1, there exists a SPE for Mechanism (A) such that the bid of each player* i *to each player* $j \in S \setminus \{i\}$ *is given by* $\hat{b}_j^i$ *as in point 4 of Proposition 1.*

Proof. Assume that at **Stage 2** each player plays his or her equilibrium strategy in each G_{i_s}, $i_s \in \Omega_S$. Then, it suffices to verify that at **Stage 1** of the Mechanism (A), no player has a strict interest to deviate unilaterally from $\{\hat{b}_j^i\}_{j\in S\setminus\{i\}}$. Pick any $i \in S$ and any set of bids $\{b_j^i\}_{j\in S\setminus\{i\}}$ distinct from $\{\hat{b}_j^i\}_{j\in S\setminus\{i\}}$. Two cases arise.
Case (a) Assume that:

$$\sum_{j\in S\setminus\{i\}} b_j^i > \sum_{j\in S\setminus\{i\}} \hat{b}_j^i.$$

By point 1 of Proposition 1, player i becomes the unique proposer. His or her final payoff in Mechanism (A) becomes:

$$\hat{g}_i^i - \sum_{j\in S\setminus\{i\}} b_j^i < \hat{g}_i^i - \sum_{j\in S\setminus\{i\}} \hat{b}_j^i = \hat{r}_i^i = \hat{m}_i,$$

where the last equality follows from point 2 of Proposition 1 and the definition of the expected payoff $\hat{m}_i$.

Case (b) Assume that:

$$\sum_{j\in S\setminus\{i\}} b_j^i \leq \sum_{j\in S\setminus\{i\}} \hat{b}_j^i.$$

Because the set $\{b_j^i\}_{j\in S\setminus\{i\}}$ is distinct from the set $\{\hat{b}_j^i\}_{j\in S\setminus\{i\}}$, there is at least one $j \in S \setminus \{i\}$ such that $b_j^i < \hat{b}_j^i$, so that $B(j) > B(i)$. This implies that player i is not a member of Ω_S anymore. But this fact does not affect i's reward in G_{i_s}, $i_s \in \Omega_S$. Indeed, pick any possible proposer i_s (necessarily different from i), and denote by $r_i^{i_s}$ player i's reward in G_{i_s} when he or she plays $\{b_j^i\}_{j\in S\setminus\{i\}}$ at **Stage 1**:

$$r_i^{i_s} = \hat{g}_i^{i_s} - \hat{b}_i^{i_s} = \hat{g}_i^{i_s} - \left(\frac{\sum_{k\in S} \hat{g}_j^k}{|S|} - \hat{g}_i^{i_s}\right) = \hat{m}_i,$$

where the second equality follows from point 4 of Proposition 1 and the last equality follows from point 3 of Proposition 1.

By combining cases (a) and (b), the proof is complete. ■

4 A bidding mechanism for implementing Φ^e

In order to construct a mechanism which implements Φ^e, we need to specify **Stage 2** in Mechanism (A), from now on denoted by (Ae). Given a digraph TU-game $(N, v, D) \in \mathcal{CD}$ and the root $r \in N$, we proceed as follows.

Mechanism (Ae) The player set is $S_D[r]$.

Stage 1: this stage corresponds to stage 1 in Mechanism A applied to player set $S_D[r]$.

Stage 2: Player i_s induces a non-cooperative game G_{i_s} on the player set $S_D[r]$. The game G_{i_s} consists in two stages.

Stage 2.1: Player i_s makes an offer $x_j^{i_s} \in \mathbb{R}$ to each other player $j \in S_D[r] \setminus \{i_s\}$.

Stage 2.2: The players in $S_D[r] \setminus \{i_s\}$ observe the offer $x_j^{i_s}$ and, sequentially, either accept or reject the offer. If each of these players accepts the offer, then the game ends and the payoffs are given by:

$$\forall j \in S_D[r] \setminus \{i_s\}, \quad g_j^{i_s} = x_j^{i_s}, \quad \text{and} \quad g_{i_s}^{i_s} = \mathbb{S}_r(N, v, D) - \sum_{j \in S_D[r] \setminus \{i_s\}} x_j^{i_s}.$$

If at least one player in $S_D[r] \setminus \{i_s\}$ rejects the offer, then:

$$\forall j \in S_D[r], \quad g_j^{i_s} = 0.$$

The rewards $(r_j^{i_s})_{j \in S_D[r]}$ resulting from **Stage 1** and **Stage 2**, and the final expected payoffs $(m_j)_{j \in S_D[r]}$ are defined as in Mechanism (A). ■

Three remarks are in order.

Remark 1 At **Stage 2** of Mechanism (Ae), the non-cooperative game G_{i_s} is a "take it or leave it" game where the collective acceptation of an offer made by i_s requires unanimity. The game G_{i_s} admits several SPE. Nevertheless, on each of these equilibria the proposer i_s makes the offer $x_j^{i_s} = 0$ to each $j \in S_D[r] \setminus \{i_s\}$, and each player $j \in S_D[r] \setminus \{i_s\}$ accepts this offer. Out of the equilibrium path, several action profiles are possible. It follows that the equilibrium payoffs of G_{i_s} are such that: $\hat{g}_{i_s}^{i_s} = \mathbb{S}_r(N, v, D)$ and, for each $j \in S_D[r] \setminus \{i_s\}$, $\hat{g}_j^{i_s} = 0$.

Remark 2 By Remark 1, each G_{i_s}, $i_s \in \Omega_{S_D[r]}$, satisfies the hypothesis of the statement of Proposition 1. Therefore, by applying point 3 of Proposition 1 to Mechanism (Ae), we obtain that at each SPE of Mechanism (Ae), the expected payoff of each involved player is:

$$\frac{\mathbb{S}_r(N, v, D)}{|S_D[r]|}.$$

Furthermore, such a SPE exists by Proposition 2.

Remark 3 Mechanism (Ae) applied on (N, v, D) is defined only for the player set $S_D[r]$. The reason lies on the fact that Mechanism (B) that we will construct below for implementing Φ^e has a recursive structure: Mechanism (B) first calls Mechanism (Ae) and then calls Mechanism (B) in a self-similar way but applied to digraph TU-games constructed from the outcome of Mechanism (Ae) and (N, v, D).

Mechanism (B) Let $(N, v, D) \in \mathcal{CD}$.

1. If $N = \{i\}$, then player i's payoff in Mechanism (B) is $v(\{i\})$.

2. Otherwise, pick the root r of the rooted tree (N, D).

   (a) Members of $S_D[r]$ are involved in Mechanism (Ae) applied to (N, v, D). Players in $S_D[r]$ other than r receive the expected payoffs $(m_j)_{j\in S_D(r)}$ obtained in Mechanism (Ae) applied to (N, v, D), while player r receives his or her final payoff $m_r + v(\{r\})$ in Mechanism (B).

   (b) Each $i \in N \setminus \{r\}$ is involved in Mechanism (B) applied to the digraph TU-game $(\hat{S}_D[j], \tilde{v}_j, D_{S_D[j]})$ where j is the unique successor of the root r of (N, D) such that $i \in \hat{S}_D[j]$, and $\tilde{v}_j$ is such that:

$$\forall S \subseteq \hat{S}_D[j], \quad \tilde{v}_j(S) = v(S) + m_j \delta_{\hat{S}_D[j]}(S).$$

   (c) The final payoff obtained by i in Mechanism (B) applied to (N, v, D) coincides with the final payoff he or she obtains in Mechanism (B) applied to $(\hat{S}_D[j], \tilde{v}_j, D_{S_D[j]})$.

Example 2 *Consider the digraph TU-game (N, v, D), where the rooted tree is represented in Figure 1. Mechanism (B) applied to (N, v, D) runs as follows. Because N is not a singleton, go to point 2 in Mechanism (B).*

Point 2(a) indicates that players in $S_D[1] = \{1, 2, 3\}$ are involved in Mechanism (Ae) applied to (N, v, D). In (Ae), players 2 and 3 receive payoffs m_2 and m_3, respectively. The root 1 receives his final payoff in Mechanism (B) applied to (N, v, D), which is equal to his stand-alone worth $v(\{1\})$ plus his expected payoff m_1 obtained in Mechanism (Ae).

Next, go to point 2(b). At this step, Mechanism (B) indicates that player 2 is involved in Mechanism (B) applied to the digraph TU-game $(\{2\}, \tilde{v}_2, \emptyset)$, where $\tilde{v}_2(\{2\}) = v(\{2\}) + m_2$. By point 1 of Mechanism (B), the final payoff of player 2 is $v(\{2\}) + m_2$. Because player 2 obtains his final payoff in Mechanism (B) applied to $(\{2\}, \tilde{v}_2, \emptyset)$, go to point 3 of Mechanism (B). It indicates that $\tilde{v}(\{2\})$ is also the final payoff of player 2 in Mechanism (B) applied to (N, v, D).

By point (2b), players 3 and 4 are both involved in Mechanism (B) applied to $(S_{D[3]}, \tilde{v}_3, D_{S_D[3]})$, where

$$\forall S \in S_{D[3]}, \quad \tilde{v}_3(S) = v(S) + m_3 \delta_{S_D[3]}.$$

Because $S_{D[3]}$ is not a singleton, go to point 2(a) of Mechanism (B). By point 2(a) of Mechanism B applied to $(S_{D[3]}, \tilde{v}_3, D_{S_D[3]})$, the root 3 of $(S_{D[3]}, D_{S_D[3]})$ and his unique successor 4 play Mechanism (Ae) applied to $(S_{D[3]}, \tilde{v}_3, D_{S_D[3]})$. Player 4 receives his expected payoff m_4 obtained in Mechanism (Ae) applied to $(S_{D[3]}, \tilde{v}_3, D_{S_D[3]})$. Player 3 receives his final payoff $m'_3 + \tilde{v}_3(\{3\})$ in Mechanism (B) applied to $(S_{D[3]}, \tilde{v}_3 D_{S_D[3]})$, where m'_3 represents his expected payoff obtained in Mechanism (Ae) applied to $(S_{D[3]}, \tilde{v}_3, D_{S_D[3]})$. Note that $\tilde{v}_3(\{3\}) = v(\{3\})$ so that his final payoff in Mechanism (B) applied to $(S_{D[3]}, \tilde{v}_3, D_{S_D[3]})$ is $m'_3 + \tilde{v}_3(\{3\})$. Because player 3 obtains his final payoff in Mechanism (B) applied to $(S_{D[3]}, \tilde{v}_3, D_{S_D[3]})$, go to point 3 of Mechanism (B). It indicates that $m'_3 + \tilde{v}_3(\{3\})$ is also his final payoff in Mechanism (B) applied to (N, v, D).

Next, go to point 2(b). At this step, Mechanism (B) applied to $(S_{D[3]}, \tilde{v}_3, D_{S_D[3]})$ indicates that player 4 is involved in Mechanism (B) applied to the digraph TU-game $(\{4\}, \widetilde{(\tilde{v}_3)}_4, \emptyset)$, where

$$\widetilde{(\tilde{v}_3)}_4(\{4\}) = \tilde{v}_3(\{4\}) + m_4 = v(\{4\}) + m_4.$$

By point 1 of Mechanism (B), the final payoff of player will be equal to $v(\{4\}) + m_4$. Because player 4 obtains his final payoff in Mechanism (B) applied to $(\{4\}, \widetilde{(\tilde{v}_3)}_4, \emptyset)$, go to point 3 of Mechanism (B). It indicates that $v(\{4\}) + m_4$ is also the final payoff of player 4 in Mechanism (B) applied to (N, v, D). Mechanism (B) applied to (N, v, D) ends. □

5 Main result

We will show that for any $(N, v, D) \in \mathcal{CD}$ such that (N, v) is superadditive, the outcome of the subgame perfect equilibria (SPE) of Mechanism (B) applied to (N, v, D) coincide with the payoff vector $\Phi^e(N, v, D)$ as described in section 2.3.

Proposition 3 *Mechanism (B) implements the sequential equal surplus division payoff vector $\Phi^e(N, v, D)$ of a superadditive digraph TU-game $(N, v, D) \in \mathcal{CD}$ in SPE.*

Proof. We proceed in two steps.
Step (a): Uniqueness of the equilibrium payoffs. We first show that if Mechanism (B) admits a SPE, then on each SPE the final payoffs of the players coincide with Φ^e. Pick any superadditive digraph TU-game $(N, v, D) \in \mathcal{CD}$, and assume that a SPE in played in Mechanism (B). We proceed by induction on the depth p of the tree (N, D).

Initial step If $p = 0$, the digraph TU-game has only one player, say i, so that the associated digraph is empty. By Mechanism (B), this player gets $v(\{i\})$, which coincides with $\Phi^e_i(\{i\}, v, \emptyset)$ by definition of Φ^e.
Induction hypothesis Fix $p \geq 0$, and assume that the claim is true for any depth $p' \leq p$.
Before proceeding to the induction step, we need the following lemma which uses the induction hypothesis.

Lemma 1 *Under the induction hypothesis, assume that a SPE is played in Mechanism (B) applied to $(N, v, D) \in \mathcal{CD}$. Then, the corresponding equilibrium strategies induce a SPE in Mechanism (Ae).*

Proof. Pick any $(N, v, D) \in \mathcal{CD}$. Assume that a SPE is played in Mechanism (B), and denote by u_i the corresponding final payoff of player $i \in N$. Consider Mechanism (Ae) applied on (N, v, D) and where the player set is $S_D[r]$. Assume by contradiction that there exists a player $i \in S_D[r]$ who has an incentive to deviate unilaterally from his or her (induced) equilibrium strategy. Denote by m_j his expected final payoff in Mechanism (Ae) when he plays his or her equilibrium strategy in Mechanism (B), and denote by $\hat{m}_j$ his or her expected final payoff in Mechanism (Ae) when he or she has chosen to deviate from his or her equilibrium strategy in Mechanism (Ae). There are two cases: either $\hat{m}_j > m_j$ or $\hat{m}_j < m_j$. We deal with the first case, the second one is similar and so left to the reader.

Assume that $\hat{m}_j > m_j$. We distinguish two cases.

Case (a) Assume that $j = r$. Since $\hat{m}_r > m_r$, by definition of Mechanism (B), we obtain:

$$u_r = v(\{r\}) + m_r < v(\{r\}) + \hat{m}_r,$$

which contradicts that players play a SPE in Mechanism (B).

Case (b) Assume that $j \in S_D(r)$. Then, by point (c) of Mechanism (B), j's final payoff u_j in Mechanism (B) applied to (N, v, D) coincides with the final payoff he or she obtains in Mechanism (B) applied to $(\hat{S}_D[j], \tilde{v}_j, D_{S_D[j]})$. Note that the depth of $(\hat{S}_D[j], D_{S_D[j]})$ is inferior of equal to p. By the induction hypothesis, we obtain:

$$u_j = \Phi^e_j(S_D[j], v + m_j \delta_{\hat{S}_D[j]}, D_{S_D[j]}), \quad \text{and} \quad \hat{u}_j = \Phi^e_j(S_D[j], v + \hat{m}_j \delta_{\hat{S}_D[j]}, D_{S_D[j]})$$

where $\hat{u}_j$ denotes j's final payoff in Mechanism (B) whe he or she deviates from his equilibrium strategy in Mechanism (Ae). By the fact that $m_j < \hat{m}_j$ and strict aggregate monotonocity of Φ^e, we obtain $u_j < \hat{u}_j$, which, once again, contradicts that players play a SPE in Mechanism (B). ■

We now have the material for the induction step.

INDUCTION STEP Take an instance $(N,v,D) \in \mathcal{CD}$ such that the depth of the tree (N,D) is $p+1$, and assume that the players play a SPE of Mechanism (B). Then, by Lemma 1, the strategies induce a SPE in Mechanism (Ae). In particular, the root r obtains $v(\{r\}) + \hat{m}_r$. By Remarks 1 and 2,

$$\hat{m}_r = \frac{\mathbb{S}_r(N,v,D)}{|S_D[r]|}. \tag{2}$$

So, at the SPE, r's final payoff coincides with $\Phi^e_r(N,v,D)$. Regarding each other player $i \in N\setminus\{r\}$, his of her final payoff in Mechanism (B) is by definition the final payoff he or she obtains in Mechanism (B) applied to $(\hat{S}_D[j], \tilde{v}_j, D_{S_D[j]})$, where j is the unique successor of the root r of (N,D) such that $i \in \hat{S}_D[j]$. By definition of j, the depth of $(\hat{S}_D[j], D_{S_D[j]})$ is inferior or equal to p. By the induction hypothesis, i's final payoff in Mechanism (B) is equal to $\Phi^e_i(\hat{S}_D[j], \tilde{v}_j, D_{S_D[j]})$, where

$$\forall S \subseteq \hat{S}_D[j], \quad \tilde{v}_j(S) = v(S) + \hat{m}_j \delta_{\hat{S}_D[j]}(S),$$

where $\hat{m}_j$ is the expected payoff obtained in Mechanism (Ae) applied to (N,v,D). Once again, by Remarks and 1 and 2, we know that:

$$\forall j \in S_D(r), \quad \hat{m}_j = \frac{\mathbb{S}_r(N,v,D)}{|S_D[r]|}.$$

Conclude that $\tilde{v}_j = v_j$ (see the definition of Φ^e_i in section 2.3), and so:

$$\Phi^e_i(\hat{S}_D[j], \tilde{v}_j, D_{S_D[j]}) = \Phi^e_i(\hat{S}_D[j], v_j, D_{S_D[j]}).$$

At last, by definition of $\Phi^e_i(N,v,D)$, we obtain that:

$$\Phi^e_i(\hat{S}_D[j], v_j, D_{S_D[j]}) = \Phi^e_i(N,v,D).$$

Therefore, i's final payoff in Mechanism (B) coincides with $\Phi^e_i(N,v,D)$ at a SPE. This completes the proof of **Step** (a).

Step (b) Existence of a SPE. We show that Mechanism (B) admits a SPE. Once again, we proceed by induction of the depth of (N,D) of a digraph TU-game $(N,v,D) \in \mathcal{CD}$.

INITIAL STEP If $p=0$, then the results holds trivially.

INDUCTION HYPOTHESIS Assume that Mechanism (B) admits a SPE for each $(N,v,D) \in \mathcal{CD}$ such that the depth of (N,D) is at most equal to p.

INDUCTION STEP Assume that the depth of the tree of the chosen digraph TU-game is equal to $p+1$. Consider any SPE profile in Mechanism (Ae). By Remarks 1 and 2 such a profile exists and the induced digraph TU-games $(S_D[j], v + \tilde{m}_j \delta_{\hat{S}_D[j]}, D_{S_D[j]})$, $j \in S_D[r]$, are such that $v + \tilde{m}_j \delta_{\hat{S}_D[j]} = v_j$ for each $j \in S_D[r]$. The depth of each tree $(S_D[j], D_{S_D[j]})$ is at most p. By the induction hypothesis, there is a SPE for Mechanism (B) applied to $(S_D[j], v_j, D_{S_D[j]})$. By definition of a SPE, no player has an incentive to deviate in Mechanism (B) applied to $(S_D[j], v_j, D_{S_D[j]})$. Next, assume that a player $j \in S_D[r]$ contemplates the possibility to deviate in Mechanism (Ae) given the choice of the equilibrium strategies in Mechanism (B) applied to each $(S_D[j], v_j, D_{S_D[j]})$, $j \in S_D[r]$. Clearly, he or she cannot increase his or her expected payoff in Mechanism (Ae), which is equal to

$$\frac{\mathbb{S}_r(N,v,D)}{|S_D[r]|},$$

and so, has no possibility to increase his final payoff in Mechanism (B) applied to (N,v,D). This completes the proof of **Step** (b).

By combining **Step** (a) and **Step** (b), we obtain that Mechanism (B) implements in SPE the Sequential Surplus Division Rule for each superadditive rooted tree TU-game. ■

6 Appendix: Proof of Proposition 1

Consider the Mechanism (A) and assume that on each SPE of G_{i_s}, $i_s \in \Omega_S$ of Stage 2, the payoff vector is the same and equal to $(\hat{g}_j^{i_s})_{j\in S}$.

Point 1. By definition of the net bid:

$$\sum_{i\in S} \hat{B}^i = 0.$$

Therefore, it is sufficient to prove that for each pair of distinct players $i, j \in S$, $\hat{B}^i = \hat{B}^j$ on each SPE of the Mechanism (A). Consider any SPE of the Mechanism (A). We proceed by contradiction by assuming that the assertion is false. This implies that $|\Omega_S| < |S|$. Pick any player $i_s \in \Omega_S$ and any player $j_{s-\omega_s}$ in the set

$$\arg \max_{l\in S\setminus\Omega_S} \{\hat{B}^l\}.$$

Focus the attention on the reward $\hat{r}_{i_s}^{i_s}$. Two exclusive cases arise.

Case (a). Assume first that this reward is strictly inferior to the expected payoff:

$$\hat{r}_{i_s}^{i_s} < \hat{m}_{i_s} = \sum_{l\in\Omega_S} \frac{\hat{r}_{i_s}^l}{|\Omega_S|}. \tag{3}$$

Then, player i_s can strictly improve his/her expected payoff by unilaterally changing his/her set of bids. Indeed, choose any positive $\varepsilon < \hat{B}^{i_s} - \hat{B}^{j_{s-\omega_s}}$, which is possible by assumption, and consider the following new set of bids for i_s:

(i) $\tilde{b}_{j_{s-\omega_s}}^{i_s} = \hat{b}_{j_{s-\omega_s}}^{i_s} - \varepsilon$;

(ii) $\tilde{b}_j^{i_s} = \hat{b}_j^{i_s}$ for each other $j \in S \setminus \{j_{s-\omega_s}\}$.

We obtain:

(iii) $\tilde{B}^{i_s} = \hat{B}^{i_s} - \varepsilon$;

(iv) $\tilde{B}^{j_{s-\omega_s}} = \hat{B}^{j_{s-\omega_s}} + \varepsilon$;

(v) $\tilde{B}^j = \hat{B}^j$ for each other $j \in S \setminus \{i_s, j_{s-\omega_s}\}$.

(vi) Player i_s is not the proposer anymore, and the set of potential proposers becomes $\Omega_S \setminus \{i_s\}$.

It follows that i_s's expected payoff, denoted by $\tilde{m}_{i_s}$, is as follows:

$$\tilde{m}_{i_s} = \sum_{l\in\Omega_S\setminus\{i_s\}} \frac{\hat{r}_{i_s}^l}{|\Omega_S| - 1}. \tag{4}$$

From inequality (3), we deduce that:

$$(|\Omega_S| - 1)\hat{r}_{i_s}^{i_s} < \sum_{l\in\Omega_S\setminus\{i_s\}} \hat{r}_{i_s}^l \iff \hat{r}_{i_s}^{i_s} < \sum_{l\in\Omega_S\setminus\{i_s\}} \frac{\hat{r}_{i_s}^l}{|\Omega_S| - 1} = \tilde{m}_{i_s}.$$

Therefore, by (4), we obtain:

$$\tilde{m}_{i_s} = \left(\frac{|\Omega_S| - 1}{\Omega_S}\right)\tilde{m}_{i_s} + \frac{1}{|\Omega_S|}\tilde{m}_{i_s} > \sum_{l\in\Omega_S\setminus\{i_s\}} \frac{\hat{r}_{i_s}^l}{|\Omega_S|} + \frac{\hat{r}_{i_s}^{i_s}}{|\Omega_S|} = \hat{m}_{i_s},$$

which contradicts that $\hat{m}_{i_s}$ is i_s's equilibrium expected payoff.

Case (b). From case (a), one can assume that:

$$\hat{r}_{i_s}^{i_s} \geq \hat{m}_{i_s} = \sum_{l\in\Omega_S} \frac{\hat{r}_{i_s}^l}{|\Omega_S|}. \tag{5}$$

As in case (a), one can show that player i_s can strictly improve his/her expected payoff by unilaterally changing his/her set of bids. Indeed, consider the following set of bids for i_s: for any positive ε such that $2|\Omega_S|\varepsilon < \hat{B}^{i_s} - \hat{B}^{j_{s-\omega_s}}$,

(i) $\tilde{b}^{i_s}_{j_{s-\omega_s}} = \hat{b}^{i_s}_{j_{s-\omega_s}} - (2|\Omega_S| - 1)\varepsilon$;

(ii) $\tilde{b}^{i_s}_j = \hat{b}^{i_s}_j + 2\varepsilon$ for each $j \in \Omega_S \setminus \{i_s\}$;

(iii) $\tilde{b}^{i_s}_j = \hat{b}^{i_s}_j$ for each $j \in S \setminus (\Omega_S \cup \{j_{s-\omega_s}\})$.

We obtain:

(iv) $\tilde{B}^{i_s} = \hat{B}^{i_s} - \varepsilon$;

(v) $\tilde{B}^{j_{s-\omega_s}} = \hat{B}^{j_{s-\omega_s}} + (2|\Omega_S| - 1)\varepsilon$;

(vi) $\tilde{B}^j = \hat{B}^j - 2\varepsilon$ for each $j \in \Omega_S \setminus \{i_s\}$;

(vii) $\tilde{B}^j = \hat{B}^j$ for each $j \in S \setminus (\Omega_S \cup \{j_{s-\omega_s}\})$.

(viii) Player i_s becomes the unique proposer.

Equipped with this new set of bids, i_s's final payoff becomes:

$$\hat{g}^{i_s}_{i_s} - \sum_{l \in S\setminus\{i_s\}} \tilde{b}^{i_s}_l = \hat{g}^{i_s}_{i_s} - \left(\sum_{l \in S\setminus\{i_s\}} \hat{b}^{i_s}_l - \varepsilon \right) = \hat{r}^{i_s}_{i_s} + \varepsilon > \hat{r}^{i_s}_{i_s} \geq \hat{m}_{i_s},$$

which contradicts that $\hat{m}_{i_s}$ is i_s's equilibrium payoff.

From case (a) and case (b), we conclude that $\Omega_S < S$ is impossible at the equilibrium. Therefore, $\Omega_S = S$ and so $\hat{B}^i = 0$ for each $i \in S$.

Point 2 Pick any $j \in S$. Assume that there exist i_s and i'_s in Ω_S such that $\hat{r}^{i_s}_j > \hat{r}^{i'_s}_j$. Without loss of generality, assume that i_s is such that $\hat{r}^{i_s}_j > \hat{r}^{i'_s}_j$ for each other $i'_s \in \Omega_S \setminus \{i_s\}$. Two cases arise.

Case (a) Assume that $i_s \neq j$. Then, j can strictly improve his/her expected payoff by unilaterally changing his/her set of bids. Indeed, consider the following set of bids for j: for any positive ε,

(i) $\tilde{b}^j_{i_s} = \hat{b}^j_{i_s} - \varepsilon$;

(ii) $\tilde{b}^j_i = \hat{b}^j_i$ for each $i \in S \setminus \{i_s\}$.

We have:

(iii) $\tilde{B}^j = \hat{B}^j - \varepsilon$;

(iv) $\tilde{B}^{i_s} = \hat{B}^{i_s} + \varepsilon$;

(v) $\tilde{B}^i = \hat{B}^i$ for each other $i \in S \setminus \{i_s, j\}$;

(vi) Player i_s becomes the unique proposer.

Under this scenario and for sufficiently small ε, player's j final payoff becomes $\hat{r}^{i_s}_j - \varepsilon > \hat{m}_j$, which contradicts that $\hat{m}_j$ is the equilibrium payoff.

Case (b). The previous case forces $i_s = j$. Then, j can strictly improve his/her expected payoff by unilaterally changing his/her set of bids. Indeed, consider the following set of bids for j: for any positive ε,

(i) $\tilde{b}^j_i = \hat{b}^j_i + \varepsilon$ for each $i \in S \setminus \{j\}$,

from which we obtain:

(iii) $\tilde{B}^j = \hat{B}^j + (|S| - 1)\varepsilon$;

(iv) $\tilde{B}^i = \hat{B}^i - \varepsilon$;

(v) Player j becomes the unique proposer.

Under this scenario, player j's final payoff is $\hat{r}^j_j - (|S| - 1)\varepsilon$. For a sufficiently small ε, this final payoff is strictly greater than $\hat{m}_j$.

By combining case (a) and case (b), we deduce that the initial assumption is impossible at the equilibrium. Therefore, the desired result follows: for $i_s, i'_s \in \Omega_S$, it holds that $r^{i_s}_j = r^{i'_s}_j$.

Point 3 Pick any $j \in S$. From **Point 1** and **Point 2**, we have:

$$\hat{m}_j|S| = \sum_{k\in S} \hat{r}_j^k = \sum_{k\in S} \hat{g}_j^k + \hat{B}^i = \sum_{k\in S} \hat{g}_j^k \Longrightarrow \hat{m}_j = \frac{\sum_{k\in S} \hat{g}_j^k}{|S|}.$$

Point 4 From **Point 1**, $\Omega_S = S$. Thus, for $i, j \in S$, we obtain by using **Point 2** and **Point 3**,

$$\hat{g}_i^j + \hat{b}_i^j = \hat{r}_i^j = \hat{m}_i = \frac{\sum_{k\in S} \hat{g}_i^k}{|S|} \Longrightarrow \hat{b}_i^j = \frac{\sum_{k\in S} \hat{g}_i^k}{|S|} - \hat{g}_i^j,$$

as asserted.

References

[1] Ambec, S., Sprumont, Y., 2002. Sharing a river, *Journal of Economic Theory* 107:453–462.

[2] Béal, S., Ghintran, A., Rémila, R., Solal, P., 2014. The sequential equal surplus division for rooted forest games and an application to sharing a river with bifurcations. *Theory and Decision*, DOI 10.1007/s11238-014-9463-y.

[3] Brink, R. van den, Funaki, Y., and Ju, Y., 2013. Reconciling marginalism with egalitarianism: consistency, monotonicity, and implementation of egalitarian shapley values, *Social Choice and Welfare* 40:693–714.

[4] Brink, R. van den, van der Laan, G., Moes, N, 2013. A strategic implementation of the average tree solution for cycle-free graph games, *Journal of Economic Theory*, 148:2737–2748.

[5] Herings, J.J, van der Laan, G., Talman, D, 2008. The average tree solution for cycle-free graph games. *Games and Economic Behavior* 62:77–92.

[6] Ju, Y., Wettstein, D., 2009. Implementing cooperative solution concepts: a generalized bidding approach. *Economic Theory* 39:307–330.

[7] Megiddo, N., 1974. On the nonmonotocity of the bargaining set, the kernel, and the nucleolus of a game. *SIAM Journal of Applied Mathematics* 27:355–358.

[8] Myerson,R., 1977. Graphs and cooperation in games. *Mathematics of Operations Research* 2:225–229.

[9] Pérez-Castrillo, D., Wettstein, D., 2001. Bidding for the surplus: a non-cooperative approach to the Shapley value. *Journal of Economic Theory* 100:274–294.

[10] Shapley, L.S., 1953. A value for n-person games. In: Kuhn, H.W., Tucker, A.W. (Eds.), Contributions to the Theory of Games II. Princeton University Press, Princeton, NJ, pp. 307–317.

[11] Slikker, M., 2007. Bidding for surplus in network allocation problems. *Journal of Economic Theory* 137:493–511.