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December 2015

Working paper No. 2015 – 18

CRESE
30, avenue de l'Observatoire
25009 Besançon
France
http://crese.univ-fcomte.fr/

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Sylvain Béal[a,*], Eric Rémila[b], Philippe Solal[b]

[a] *CRESE EA3190, Univ. Bourgogne Franche-Comté, F-25000 Besançon, France*
[b] *Université de Saint-Etienne, CNRS UMR 5824 GATE Lyon Saint-Etienne, France*

Abstract

This article introduces a discount parameter and a weight function in Myerson's (1977) classical model of cooperative games with restrictions on cooperation. The discount parameter aims to reflect the time preference of the agents while the weight function aims to reflect the importance of each node of a graph. We provide axiomatic characterizations of two types of solution that are inspired by the hierarchical outcomes (Demange, 2004).

Keywords: Graph games, discount parameter, weight function, discounted tree solutions, invariance with respect to cone amalgamation, generalized standardness, δ-reducing agent.
JEL: C71.

1. Introduction

A cooperative game with transferable utility, or simply a TU-game, consists of a finite set of agents and for every coalition of agents a worth representing the total payoff that the coalition can obtain by cooperating. Myerson (1977) introduced TU-games with restricted cooperation possibilities (graph TU-games henceforth) modeled by an undirected graph. Each agent is located at exactly one node of the graph, and the bilateral communication possibilities between the agents are represented by the (undirected) edges. For the sake of presentation, the set of nodes is regarded as the set of agents. Myerson introduced a (graph-)restricted TU-game based on the idea that only connected coalitions are likely to form. In this article, we enrich Myerson's model by two new elements:

- A *discount parameter*, which aims to reflect the time preference of the agents.
- A *weight function*, which aims to reflect the importance of each node of a graph.

On the one hand, many solutions for TU-games such as the Shapley value (1953), the procedural values (2013) and the pyramidal values (2014) are based on the marginal contributions of the agents

*Corresponding author
Email addresses: sylvain.beal@univ-fcomte.fr (Sylvain Béal), eric.remila@univ-st-etienne.fr (Eric Rémila), philippe.solal@univ-st-etienne.fr (Philippe Solal)
URL: https://sites.google.com/site/bealpage/ (Sylvain Béal)

to coalitions. They assume that the payoffs distributed to the agents are sequentially obtained through a dynamic process of coalition formation, in which the agents successively come into play and join the current coalition until the grand coalition is formed.[1] In a sense, these formation processes take time but the aforementioned solutions do not capture explicitly temporal effects. In order to incorporate time into the allocation process for graph TU-games, we follow the approach of Joosten (1996) who introduced the discounted Shapley values. In each such value, the worth associated with smaller coalitions is more discounted than for bigger coalitions. This is consistent with the fact that agents exhibit time preference for present (and are often impatient) as initiated by Samuelson (1937). More specifically, we consider that the payoffs are distributed to the agents once the grand coalition has formed. For an agent who came into play in a small coalition, i.e. formed at an early stage of the process, this means that he/she has to substantially wait before being paid. The discounting reflects the greater or lesser degree of impatience of the agent in this situation. As extreme cases, the discounted Shapley values contain the Shapley value and the equal division value, and thus belong to the popular class of values which create space for solidarity (see Casajus and Huettner (2015) for a recent study of another class of such values). The discount parameter has also a natural interpretation in terms of non-cooperative implementation as shown by van den Brink and Funaki (2015).

Contrary to Joosten (1996), we introduce the discount rate in combination with a tree structure.[2] More specifically, we assume that the grand coalition forms sequentially as described by the hierarchical outcomes defined by Demange (2004). Each node induces a directed tree of which it is the root, and which corresponds to a specific marginal vector of the Myerson restricted TU-game. The agents come into play sequentially according to the partial order associated with the rooted tree, the root agent arriving at the last step. Therefore, the time elapsed since the arrival of an agent is an increasing function of its distance from the root. Through this distance notion, our discounting fits into other models. It is similar to the decay incorporated to utility functions in models of strategic network formation (see section 3.1 in Jackson and Wolinsky (1996) for instance). It can also help to represent the losses incurred in the transmission of electricity between agents in a network (see Lima et al. (2008) and Lim et al. (2009) for a game-theoretic approach and a graph-theoretic approach to this problem, respectively).

We characterize the set of discounted tree solutions, i.e. the linear combinations of such discounted hierarchical outcomes, by means of linearity, proper cone efficiency and the axiom of δ-reducing agent, where δ is the discount parameter (Proposition 5). Linearity and proper cone efficiency are already used in Béal et al. (2010) where the linear combinations of (undiscounted)

[1] A more general model of dynamic coalition formation is proposed by Faigle and Grabisch (2012). There are two essential differences. Firstly, the authors allow for coalition formation with an infinite number of steps, in which the current coalition might either grow or shrink. Secondly, other processes than the Shapley-like model of coalition formation are considered. Furthermore, Faigle and Grabisch (2012) do not incorporate a discount parameter into their analysis contrary to our approach.

[2] The assumption that the cycle-free graph is a tree is only made for the sake of exposition.

hierarchical outcomes are characterized. Proper cone efficiency only requires efficiency for standard characteristic functions associated with the proper cones of a tree, i.e. the connected coalitions resulting from the deletion of any link of a tree. The axiom of δ-reducing agent is a variation of the axiom of null agent in Béal et al. (2010) that accounts for the discount parameter in a similar way as in van den Brink and Funaki (2015). Furthermore, by definition, for each tree and each discount rate, the set of discounted tree solutions forms a linear space. Proposition 5 also establishes that the set of hierarchical outcomes forms a basis for this space, which means that the dimension of this space is equal to the size of the agent set. Two particular cases similar to those arising with the discounted Shapley values are obtained: the so-called marginalist tree solutions (Theorem 1 in Béal et al., 2010) if $\delta = 1$, and the equal division solution if $\delta = 0$. As a by-product, we improve Theorem 1 in Béal et al. (2010) by showing that the axiom of cone equivalence is unnecessary to obtain this result (Lemma 1). Proposition 5 is similar in spirit to the results in Weber (1988) for classical TU-games. Adding the mild axiom of communication ability to the axioms in Proposition 5 and invoking efficiency instead of proper cone efficiency yield a characterization of the average discounted tree solution (Proposition 6), which generalizes the average tree solution for cycle-free graph TU-games introduced by Herings et al. (2008). Communication ability assigns identical payoffs to all agents if all coalitions except the grand coalition enjoy a zero worth. Other generalizations of the average tree solution are due to Béal et al. (2012) for multichoice graph TU-games, to Khmelnitskaya and Talman (2014) for cycle-free directed graph TU-games and to van den Brink et al. (2015) for graph TU-games with a permission tree.

On the other hand, the location of a node in a graph is often essential. For instance, Herings et al. (2005) and van den Brink and Gilles (2008) propose two measures for calibrating the importance of nodes in a graph. Our weight function aims at reflecting this importance. More specifically, here we do not seek to distribute just the worth of the grand coalition. We rather consider that the agents altogether achieve a total output that depends on both the worth of the grand coalition (the production resulting from their cooperation of all agents) and their locations in the graph. More specifically, these locations stimulate or restrain the worth of the grand coalition in proportion to the cumulated weights. The axiom of node-weighted efficiency captures this idea: the sum of the agents' payoff equals the above total output. Thus, beyond the classical communicational effects that are often taken into account in the literature, we also incorporate an external effect of the graph structure, which can be positive or negative depending on the sum of the locations' weight. It is worth noting that weights on nodes are different from weights of edges that appear frequently in other models to reflect communicational effects (see also González-Arangüena (2015) for a recent contribution). Node-weighted model have been investigated in Lindelauf, Hamers and Husslage (2013) and Husslage et al. (2015) to examine to influence of the members of a terrorist network. Contrary to our approach, these authors do not assume an underlying characteristic function: their TU-game is constructed only from the weights of both the edges and the nodes. Our model is also different from Kalai and Samet (1987) where the agents are endowed with weights in that our weights are associated with nodes of a graph and can vary if the graph is altered. Two extra axioms are invoked in addition to node-weighted efficiency. Firstly, invariance with respect to cone amalgamation implements an amalgamation principle as follows. If the agents in a proper cone are amalgamated an act as a single entity, then the payoff of any other agent is not affected. Our amalgamated TU-game is in line with those in the literature initiated by Lehrer (1988). Secondly, generalized standardness extends in a natural way the well-known axiom of standardness (see Hart and Mas-Colell, 1989) to situations with both a discount parameter and a weight function. It turns out that these three axioms characterize the solution defined as the linear combination of discounted hierarchical outcomes such that the coordinates associated with each hierarchical outcome coincide with the weight of the corresponding root (Proposition 3).

The rest of the article is organized as follows. Section 2 is devoted to the definitions and notations. Section 3 contains the results on the model enriched by both the discount parameter and the weight function. Section 4 contains the results on the model enriched by only the discount parameter.

2. Preliminaries

Throughout this article, the cardinality of a finite set S will be denoted by the lower case s, the collection of all subsets of S will be denoted by 2^S, and weak set inclusion will be denoted by $\subseteq$. The complement $S \setminus T$ of a subset T of S is denoted by T^c. Also for notational convenience, we will write singleton $\{i\}$ as i. Given two linear spaces U and V, and $F : V \longrightarrow U$ a linear mapping, denote by $\mathrm{Ker}(F)$ the kernel (or null space) of F, i.e. the set of vectors $v \in V$ such that $F(v) = \mathbf{0}_U$, where $\mathbf{0}_U$ stands for the additive identity element of U.

2.1. TU-games

A situation in which a finite set of agents can obtain certain payoffs by cooperation can be designed by a *cooperative game with transferable utility* or simply a *TU-game*, being a pair (N, v) where $N \subseteq \mathbb{N}$ is a finite set of agents, and $v : 2^N \longrightarrow \mathbb{R}$ is a coalition function on N such that $v(\emptyset) = 0$. Each subset S of N is called a *coalition* and $v(S)$ is called the worth of S, i.e. the transferable utility that the members of coalition S can obtain by agreeing to cooperate.

For any two coalition functions v and w on N and $\alpha \in \mathbb{R}$, the coalition function $\alpha v + w$ on N is defined as follows: for each $S \subseteq N$, $(v + w)(S) = \alpha v(S) + w(S)$. Equipped with these operations of addition and multiplication, the set of coalition functions v on N forms a linear space over the field $\mathbb{R}$ of dimension $2^n - 1$. For any nonempty coalition $T \subseteq N$, the so-called *T-standard TU-game* $(N, 1_T)$, is defined as: $1_T(T) = 1$, and $1_T(S) = 0$ for each other S. Clearly, the collection of coalition functions $\{1_T : T \subseteq N, T \neq \emptyset\}$ forms a basis for the linear space of coalition functions on N. In fact, we have the following linear decomposition of the coalition function v: $v = \sum_{\{T \in 2^N : T \neq \emptyset\}} v(T) 1_T$.

2.2. Node-weighted trees, TU-games, and discount

A well-known game theoretic model of restrictions in coalition formation is that of games on communication graphs introduced by Myerson (1977). Each agent is located at exactly one node of an undirected graph, and the bilateral communication possibilities between the agents are represented by the (undirected) edges on this set of nodes. For the sake of presentation, the set of nodes is regarded as the set of agents. Therefore, an undirected graph on N is defined by a pair (N, L), where N, the set of agents, is viewed as set of nodes, and the set of links L is viewed as a subset of the set L_N of all unordered pairs of elements of N. For each agent $i \in N$, the set $L_i = \{j \in N : \{i, j\} \in L\}$ denotes the *neighborhood* of i in (N, L). A sequence of distinct agents $(i_1, i_2, \ldots, i_p)$, $p \geq 2$, is a *path* in (N, L) if $\{i_q, i_{q+1}\} \in L$ for $q = 1, \ldots, p-1$. Two agents i and j are *connected* in L if $i = j$ or there exists a path from i to j. A maximal set (with respect to set inclusion) of pairwise connected agents is called a *component* of the graph. A graph (N, L) is *connected* if N is the only component of the graph.

A *tree* is a minimally connected graph (N, L) in the sense that if removing any link from L would disconnect the graph. Equivalently, a tree is a connected graph such that only one path connects any two agents. A *leaf* of (N, L) is an agent in N who is incident to only one link. The distance $d(i, j)$ between two distinct nodes i and j in a tree (N, L) is equal to the number of links on the unique path connecting them. We set $d(i, i) = 0$ for each $i \in N$. Following Béal et al. (2010), the set of *cones* of a tree (N, L) consists of N, $\emptyset$ and, for each $\{i, j\} \in L$, of the two connected components that are obtained after deleting (or desactivating) link $\{i, j\}$. Every cone except N is called a *proper cone*. The unique agent of a nonempty proper cone K who has a link with the complementary cone $K^c = N \backslash K$ is called the *head* $h(K)$ of the cone. We will sometimes use the notation $K_{(i,j)}$ to denote the cone with head i that results from deleting the link $\{i, j\}$ from L; the complementary cone $N \backslash K_{(i,j)}$ with head j will be denoted by $K_{(j,i)}$. From this definition, we deduce that the tree (N, L) contains $2(n-1) + 2 = 2n$ cones. Denote by $\Delta_{(N,L)}$ the set of cones of (N, L), and by $\Delta^0_{(N,L)}$ the subset of nonempty proper cones of (N, L). A proper cone $T \in \Delta^0_{(N,L)}$ is a *successor* of a (nonempty) proper cone $K \in \Delta^0_{(N,L)}$ if either $T \subseteq K$ and $\{h(T), h(K)\} \in L$ or $K = \{i\}$ for some $i \in N$ and $T = \emptyset$. The set of successors of a proper cone K is denoted by $s(K)$, where $s(\emptyset) = \emptyset$.

A *tree TU-game* is a triple (N, v, L) where (N, v) is a TU-game and (N, L) is a tree. We introduce two new parameters to complete the description of the model of restrictions in coalition formation:

1. A *discount rate* $\delta \in [0, 1]$. As discussed in the introduction, the discount rate aim to reflect the time preference of the agents. A *tree TU-game augmented by a discount rate* is a 4-tuple (N, v, L, δ). Let $\mathcal{C}$ be the class of tree TU-games with discount rate.
2. A *weight function* $p : N \longrightarrow \mathbb{R}$. For each node $i \in N$, the real weight $p(i) \in \mathbb{R}$ reflects

the importance of its location on the graph[3]. A *node-weighted tree TU-game augmented by discount rate* is a 5-tuple (N, v, L, p, δ). Let $\mathcal{C}^*$ be the class of node-weighted tree TU-games augmented by a discount rate.

2.3. Discounted hierarchical outcomes and their linear combinations

A *payoff vector* $x \in \mathbb{R}^n$ is an n-dimensional vector giving a payoff $x_i \in \mathbb{R}$ to each agent $i \in N$. A *(single-valued) solution* on $\mathcal{C}$ is a function Φ that assigns to each $(N, v, L, \delta) \in \mathcal{C}$ a payoff vector $\Phi(N, v, L, \delta) \in \mathbb{R}^n$. Such a solution Φ represents a method for measuring the value of playing a particular role in (N, v, L, δ). A (single-valued) solution on $\mathcal{C}^*$ is defined in a similar way.

In order to calibrate the importance of each agent in a (node-weighted) tree TU-game augmented by a discount rate, we define specific contribution vectors. To describe these contribution vectors we first give some definitions concerning rooted trees. By a *rooted tree* t_r, we mean a directed tree that arises from a tree (N, L) by selecting a node $r \in N$, called the *root*, and directing all links away from the root. Each $r \in N$ is the root of exactly one rooted tree t_r on (N, L). Note also that for any rooted tree t_r on (N, L), any $i \in N\backslash\{r\}$, there is exactly one directed link (j, i); j is the unique *predecessor* of i and i is a *successor* of j in t_r. Denote by $s_r(i)$ the possibly empty set of successors of $i \in N$ in t_r. A node i is a *subordinate* of $j \in N \setminus \{i\}$ in t_r if there is a directed path from j to i, i.e. if there is a sequence of distinct agents $(i_1, i_2, \ldots, i_p)$ such that $i_1 = j$, $i_p = i$ and for each $q = 1, 2, \ldots, p-1$, $i_{q+1} \in s_r(i_q)$. The set $S_r(j)$ denotes the union of the set of all subordinates of j in t_r and $\{j\}$. So, we have $s_r(j) \subseteq S_r(j)\backslash\{j\}$. Note that:

$$\Delta_{(N,L)} = \big\{S_i(j) : \{i, j\} \in N \times N\big\} \cup \big\{\emptyset\big\}. \tag{1}$$

In particular, for each $i \in N$, $S_i(i) = N$.

Pick any $(N, v, L, \delta) \in \mathcal{C}$, and define the *discounted hierarchical outcome* $h^r(N, v, L, \delta) \in \mathbb{R}^n$ with respect to the root $r \in N$ as:

$$\begin{aligned}
\forall i \in N, \quad h_i^r(N, v, L, \delta) \quad &= \quad \delta^{d(i,r)} v(S_r(i)) - \delta^{d(j,r)} \sum_{j \in s_r(i)} v(S_r(j)) \\
&= \quad \delta^{d(i,r)} \big(v(S_r(i)) - \delta \sum_{j \in s_r(i)} v(S_r(j))\big),
\end{aligned} \tag{2}$$

where the second equality follows from the fact that $d(i, r) = d(i, j) + d(j, r)$ since $j \in s_r(i)$. In case $\delta = 1$, $h^r(N, v, L, \delta)$ coincides with the *hierarchical outcome* defined in Demange (2004), and it is equal to the worth of the coalition consisting of agent i and all his subordinates in t_r minus the sum of the worths of the coalitions consisting of any successor j of i and all subordinates of this successor in t_r. In case $\delta = 0$, $h^r(N, v, L, \delta)$ reduces to the *top value* introduced by Herings et al. (2015), which assigns the worth $v(N)$ of the grand coalition N fully to the (unique) top agent

[3]The function p depends on the tree (N, L), but, for convenience, we prefer to use p instead of $p_{(N,L)}$.

located at the root r of t_r, while the other agents get a payoff of zero, i.e. $h_r^r(N,v,L,\delta) = v(N)$ and for each other $i \in N \setminus \{r\}$, $h_i^r(N,v,L,\delta) = 0$. From (2) we have:

$$\sum_{j \in S_r(i)} h_i^r(N,v,L,\delta) = \delta^{d(i,r)} v(S_r(i)). \tag{3}$$

A single-valued solution Ψ is called a *discounted tree solution* on $\mathcal{C}$ if for each $(N,v,L,\delta) \in \mathcal{C}$, $\Psi(N,v,L,\delta)$ is a linear combination of the hierarchical outcomes $h^r(N,v,L,\delta)$, $r \in N$ in the following sense: for each tree (N,L) and each discount rate $\delta \in [0,1]$, there exists a function $\alpha_{(N,L),\delta} : N \longrightarrow \mathbb{R}$ such that:

$$\Psi(N,v,L,\delta) = \sum_{r \in N} \alpha_{(N,L),\delta}(r) h^r(N,v,L,\delta). \tag{4}$$

The solution Ψ is called the *average discounted tree solution* if, for each (N,v,L,δ) and each $r \in N$, $\alpha_{(N,L),\delta}(r) = n^{-1}$.

In case $\delta = 1$, Ψ belongs to the set of *(marginalist) tree solutions* as introduced and characterized by Béal et al. (2010), i.e. a tree solution is defined as a linear combination of the hierarchical outcomes as above. Among the set of tree solutions lies the *average tree solution*, first introduced and characterized by Herings et al. (2008). The average tree solution is the average of the hierarchical outcomes, i.e. for each $(N,v,L,1)$ and each $r \in N$, $\alpha_{(N,L),1}(r) = n^{-1}$.

In case $\delta = 0$, Ψ is a weighted division value in the sense that for each $(N,v,L,0) \in \mathcal{C}$, the worth of the grand coalition $v(N)$ is distributed among the agents according to the weights given by $\alpha_{(N,L),0}$. If for each $(N,v,L,0) \in \mathcal{C}$, and each $r \in N$, we have $\alpha_{(N,L),0}(r) = n^{-1}$, the weighted division value Ψ coincides with the *equal division value* on $\mathcal{C}$. Several characterizations of the set of weighted division values and of the equal division value can be found, e.g., in Béal et al. (2015a) and van den Brink (2007).

Regarding the class $\mathcal{C}^*$ of node-weighted tree TU-games augmented by a discount rate, the discounted hierarchical outcomes are defined as in (2). The *discounted tree solution* on $\mathcal{C}^*$ is defined in similar way as in (4) except that, for each $(N,v,L,p,\delta) \in \mathcal{C}^*$, the weights defining the linear combination of the discounted hierarchical outcomes are fixed and given by p, i.e. (4) becomes:

$$\Psi(N,v,L,p,\delta) = \sum_{r \in N} p(r) h^r(N,v,L,\delta). \tag{5}$$

There is a major difference between (5) and (4): the linear combination of the hierarchical outcomes is specified by the node-weighted tree TU-game (N,v,L,p,δ). From an axiomatic point of view, this difference has some consequences. Indeed, for the class $\mathcal{C}$, the functions $\alpha_{(N,L),\alpha}$ on the agent set N constitutes a by-product of the axiomatic system in the sense that it emerges from the combination of the axioms. In contrast, when we restrict our attention to the class $\mathcal{C}^*$, the axiomatic system takes explicitly into account the function p in (N,v,L,p,δ), so that a unique solution is defined on $\mathcal{C}^*$. The same remarks as above apply when $\delta \in \{0,1\}$ and $p(r) = n^{-1}$ for each $r \in N$.

3. Axiomatic characterization of the solution Ψ on $\mathcal{C}^*$

In this section we present an axiomatic characterization of the solution Ψ defined as in (5) on $\mathcal{C}^*$. We will use the following notations. For a node-weighted function $p : N \longrightarrow \mathbb{R}$, and a nonempty coalition $S \subseteq N$, p_S stands for $\sum_{i \in S} p(i)$. In a similar way, for $\Phi(N, v, L, p, \delta)$ and a nonempty coalition $S \subseteq N$, $\Phi_S(N, v, L, p, \delta)$ stands for $\sum_{i \in S} \Phi_i(N, v, L, p, \delta)$. We now introduce the following list of axioms for a solution Φ on $\mathcal{C}^*$.

The first axiom that we invoke is a natural variant of the classical axiom of efficiency which accounts for the weight of the agents' location.

Node-weighted efficiency For each $(N, v, L, p, \delta) \in \mathcal{C}^*$, it holds that:

$$\Phi_N(N, v, L, p, \delta) = p_N v(N).$$

Trough node-weighted efficiency, we explicitly make a difference between the worth $v(N)$ produced by the grand coalition and the total output that the agents have to distribute among them as payoffs. The worth $v(N)$ is considered as the quantity jointly created by the agents by cooperating without any help of the structure (N, L) that connects them. To the contrary, the total output is influenced by this structure in that it exhibits constant returns to scale with respect to the total of the agents' weights. Thus, the structure stimulates (or restrains if p_N is less than one) the worth $v(N)$ in proportion to p_N. The obtained total output highlights the importance of the interactive environment: beyond the agents' joint productivity, measured by v, their position in the structure, measured by p, is also decisive.

The total to be distributed among the agents is often influenced by a structure in various ways. For TU-games on communication graphs, Myerson (1977), among others, impose that the worth of each connected component of a graph is split among its members. In Calvo et al. (1999), the structure is a probabilistic graph constructed from independent probabilities for each pair of agents. Then, it is required that each suitably-defined connected component distributes its expected worth according to the probabilistic graph (see also González-Arangüena (2015) and the references therein). In this approach, the weights are associated with links in order to insist on the quality of the connexions between the agents, whereas we associate weights with nodes so as to emphasize the quality of the locations in a structure. The closer approach to ours is maybe Herings et al. (2007) in which each agent is endowed with a weight in each coalition he/she belongs to in order to represent an underlying social structure. A difference with our article is that this underlying structure is not explicitly modeled.

In case $p_N = 1$, node-weighted efficiency reduces to the classical axiom of efficiency.

The second axiom incorporates an amalgamation principle. This principle describes the payoff variation of some agents when two or more agents are amalgamated to act as if they were a single agent. More specifically, our axiom states that if the members of a cone of a tree are amalgamated into one entity, just because they have colluded, then the payoff of any other agent does not change. Formally, consider any situation $(N, v, L, p, \delta) \in \mathcal{C}^*$ such that $n \geq 3$, any link $\{i, j\} \in L$, and assume that agents in $K_{(i,j)}$ collude and act as a single entity so that they are amalgamated into a new agent denoted by $K^*_{(i,j)}$. From this amalgamation, we define a new node-weighted tree TU-game augmented by a discounted rate $(N^{(i,j)}, v^{(i,j)}, L^{(i,j)}, p^{(i,j)}, \delta^{(i,j)}) \in \mathcal{C}^*$, where:

1. The agent set is $N^{(i,j)} = K_{(j,i)} \cup \{K^*_{(i,j)}\}$;
2. The node-weighted tree $(N^{(i,j)}, L^{(i,j)}, p^{(i,j)})$ is such that $L^{(i,j)} = L(K_{(j,i)}) \cup \{\{j, K^*_{(i,j)}\}\}$, where $L(K_{(j,i)}) \subseteq L$ is the subset of links which connect agents of $K_{(j,i)}$ in (N, L); and the function $p^{(i,j)} : N^{(i,j)} \longrightarrow \mathbb{R}$ assigns to each $l \in K_{(j,i)}$, the weight $p^{(i,j)}(l) = p(l)$, and the weight $p^{(i,j)}(K^*_{(i,j)}) = \sum_{l \in K_{(i,j)}} p(l)\delta^{d(i,l)}$.
3. The TU-game $(N^{(i,j)}, v^{(i,j)})$ is defined as:
$$\forall S \subseteq N^{(i,j)}, \quad v^{(i,j)}(S) = \begin{cases} v(S) & \text{if } S \not\ni K^*_{ij}, \\ v((S \setminus \{K^*_{ij}\}) \cup K_{(i,j)}) & \text{otherwise.} \end{cases}$$
4. The discount rate is $\delta^{(i,j)} = \delta$.

Invariance with respect to cone amalgamation For each $(N, v, L, p, \delta) \in \mathcal{C}^*$, and each proper cone $K_{(i,j)}$, $\{i, j\} \in L$, it holds that:

$$\forall l \in K_{(j,i)}, \quad \Phi_l(N, v, L, p, \delta) = \Phi_l(N^{(i,j)}, v^{(i,j)}, L^{(i,j)}, p^{(i,j)}, \delta^{(i,j)}).$$

Invariance with respect to cone amalgamation is in line with other axioms of amalgamation initiated by Lehrer (1988). There are however two main differences. Firstly, the axiom does not compare the payoff of the amalgamated agent with the sum of the payoffs of its constituents in the original situation. Similarly as the axiom of no advantageous downstream merging in Ansink and Weikard (2012), we rather require that the payoff of each other agent is invariant. Secondly, the weight associated with the position of the amalgamated agent incorporates distance effects from the graph of the original situation. As such, the amalgamated game still captures some aspect of the internal structure within the amalgamated agent, the weight of its original members located far from the head of the cone being more discounted than for members located close to the head of the cone.

Remark 2. Combining invariance with respect to cone amalgamation with node-weighted efficiency allows to compare the total payoff of the amalgamated agents before the operation of amalgamation with the payoff of the entity which results from the operation of the amalgamation. Precisely, the combination of invariance to cone amalgamation and node-weighted efficiency has the following consequence. For each $(N, v, L, p, \delta) \in \mathcal{C}^*$, and each proper cone $K_{(i,j)} \in \Delta^0_{(N,L)}$, $\{i, j\} \in L$ it holds that:

$$\Phi_{K_{(i,j)}}(N, v, L, p, \delta) - \Phi_{K^*_{(i,j)}}(N^{(i,j)}, v^{(i,j)}, L^{(i,j)}, p^{(i,j)}, \delta^{(i,j)}) = (p_N - p_N^{(i,j)})v(N). \tag{6}$$

Indeed, on the one hand, by node-weighted efficiency, we have:

$\Phi_N(N, v, L, p, \delta) = p_N v(N)$ and $\Phi_{N^{(i,j)}}(N^{(i,j)}, v^{(i,j)}, L^{(i,j)}, p^{(i,j)}, \delta^{(i,j)}) = p_N v^{\{i,j\}}(N) = p_N^{(i,j)} v(N)$.

On the other hand, by invariance with respect to cone amalgamation, we have:

$$\Phi_{K_{(j,i)}}(N, v, L, p, \delta) = \Phi_{K_{(j,i)}}(N^{(i,j)}, v^{(i,j)}, L^{(i,j)}, p^{(i,j)}, \delta^{(i,j)})$$

Keeping in mind that $K_{(j,i)} \cup K_{(i,j)} = N$ and $K_{(j,i)} \cup \{K^*_{(i,j)}\} = N^{\{i,j\}}$, it follows that:

$$\begin{aligned} v(N)(p_N - p_N^{\{i,j\}}) &= \Phi_N(N, v, L, p, \delta) - \Phi_{N^{(i,j)}}(N^{(i,j)}, v^{(i,j)}, L^{(i,j)}, p^{(i,j)}, \delta^{(i,j)}) \\ &= \Phi_{K_{(i,j)}}(N, v, L, p, \delta) - \Phi_{K^*_{(i,j)}}(N^{(i,j)}, v^{(i,j)}, L^{(i,j)}, p^{(i,j)}, \delta^{(i,j)}), \end{aligned}$$

as asserted.

If $\delta = 1$, it holds that $p_N^{(i,j)} = p_N$, which means that (6) boils down to the standard amalgamation principle in the literature: the amalgamated agent gets the same payoff as the cumulated payoff of its members in the original game. This aspect clearly highlights that invariance with respect to cone amalgamation (together with efficiency) takes some distance effect into account.

The last axiom that we invoke is a generalization of the axiom of standardness of Hart and Mas-Colell (1989) for TU-games to node-weighted tree TU-games augmented by discount rate. Standardness requires that in a two-agent TU-game (N, v), each agent $i \in N$ obtains the payoff

$$v(i) + \frac{1}{2}\Big(v(\{i,j\}) - v(i) - v(j)\Big).$$

Thus each agent gets his/her stand-alone worth in a first step and then what remains of the worth of the grand coalition is split equally in a second step. Many well-known solutions for TU-games such as the Shapley value, the equal surplus division and the nucleolus satisfy this axiom. On the class of TU-games augmented by a connected undirected graph, the Myerson value (Myerson, 1977) and the average tree solution (Herings et al., 2008) satisfy standardness as well. In van den Brink and Funaki (2009), a weaker version of standardness, called α-standardness, is considered. The agents receive similar payoffs as in standardness except that they only get a fraction $\alpha \in [0, 1]$ of their stand-alone worths in the first step.

In our generalization of standardness, the fraction α is replaced by the discount rate δ. Furthermore, it is natural to account for the weights of both agents when splitting the remainder. Finally, we also integrate the fact that these weights are part of the worth creation.

Generalized standardness For each two-agent $(N, v, L, p, \delta) \in \mathcal{C}^*$, it holds for each $i \in N$:

$$\Phi_i(N, v, L, p, \delta) = p_N\left(\delta v(i) + \frac{p(i)}{p_N}\Big(v(\{i,j\}) - \delta v(i) - \delta v(j)\Big)\right). \tag{7}$$

Remark that (7) is undefined for $p_N = 0$, but just because we have chosen a suitable formulation for the interpretation of the axiom below. By developing the first parenthesis in (7), the resulting expression is well-defined even for $p_N = 0$ since the fraction disappears. Now, let us provide the following natural 3-step interpretation of expression (7). In a first step, each agent gets a fraction $\delta \in [0, 1]$ of his/her stand-alone worth. If we follow Shapley's (1953) interpretation of the formation of the grand coalition as a sequential process and if the discount rate expresses time preferences of the agents, then it makes sense that the worth of singleton coalitions (formed at the earliest stage of the procedure) are discounted while the worth of the two-agent grand coalition (formed at the final stage) is not. In a second step, the remainder of the worth of the grand coalition is distributed in proportion to the agent's weight. This reflects the asymmetries created by their intrinsic strength. In a third step, the obtained payoffs are multiplied by factor p_N in order to reflect the impact on the worth of the grand coalition jointly induced by the two agents weights.

Clearly, in case $p(1) = p(2) = 0.5$, generalized standardness coincides with standardness if $\delta = 1$ and with α-standardness if $\alpha = \delta$.

Proposition 1 *There exists at most one solution on $\mathcal{C}^*$ which satisfies, node-weighted efficiency, invariance with respect to cone amalgamation and generalized standardness.*

Proof. Assume that Φ is any solution on $\mathcal{C}^*$ satisfying node-weighted efficiency, node-weighted component fairness, invariance with respect to cone amalgamation and the inessential game axiom for leaves. Consider any $(N, v, L, p, \delta) \in \mathcal{C}^*$. Three cases arise.

Case 1: if N contains only one agent, then by node-weighted efficiency, $\Phi(N, v, L, p, \delta)$ is uniquely determined.

Case 2: if N contains two agents, then by generalized standardness, $\Phi(N, v, L, p, \delta)$ is uniquely determined

Case 3: if N contains at least three agents, then consider any link $\{i, j\} \in L$, and proceed to the amalgamation of the members of $K_{(i,j)}$ in order to obtain $(N^{(i,j)}, v^{(i,j)}, L^{(i,j)}, p^{(i,j)}, \delta^{(i,j)}) \in \mathcal{C}^*$. Then, in $(N^{(i,j)}, v^{(i,j)}, L^{(i,j)}, p^{(i,j)}, \delta^{(i,j)})$ proceed to the amalgamation of the members of $K_{(j,i)}$ in order to obtain the instance

$$(N^{(i,j)^{(j,i)}}, v^{(i,j)^{(j,i)}}, L^{(i,j)^{(j,i)}}, p^{(i,j)^{(j,i)}}, \delta^{(i,j)^{(j,i)}}) \in \mathcal{C}^*,$$

which is as follows:

1. $N^{(i,j)^{(j,i)}} = \{K^*_{(i,j)}, K^*_{(j,i)}\}$ and $\{K^*_{(i,j)}, K^*_{(j,i)}\}$ in the only link of $L^{(i,j)^{(j,i)}}$;
2. $p^{(i,j)^{(j,i)}}(K^*_{(i,j)}) = p^{(i,j)}(K^*_{(i,j)}) = \sum_{l \in K_{(i,j)}} \delta^{d(i,l)} p(l)$, and

   $p^{(i,j)^{(j,i)}}(K^*_{(j,i)}) = \sum_{l \in K_{(j,i)}} \delta^{d(j,l)} p^{(i,j)}(l) = \sum_{l \in K_{(j,i)}} \delta^{d(j,l)} p(l)$;
3. $v^{(i,j)^{(j,i)}}(K^*_{(i,j)}) = v^{(i,j)}(K^*_{(i,j)}) = v(K_{(i,j)})$ and $v^{(i,j)^{(j,i)}}(K^*_{(j,i)}) = v^{(i,j)}(K_{(j,i)}) = v(K_{(j,i)})$.

By invariance with respect to cone amalgamation, we have:

$$\Phi_{K^*_{(i,j)}}(N^{(i,j)}, v^{(i,j)}, L^{(i,j)}, p^{(i,j)}, \delta^{(i,j)}) = \Phi_{K^*_{(i,j)}}(N^{(i,j)(j,i)}, v^{(i,j)(j,i)}, L^{(i,j)(j,i)}, p^{(i,j)(j,i)}, \delta^{(i,j)(j,i)}). \tag{8}$$

Because the combination of node-weighted efficiency with invariance with respect to cone amalgamation implies (6), we have:

$$\Phi_{K_{(i,j)}}(N, v, L, p, \delta) = \Phi_{K^*_{(i,j)}}(N^{(i,j)}, v^{(i,j)}, L^{(i,j)}, p^{(i,j)}, \delta^{(i,j)}) + (p_N - p^{(i,j)}_{N^{(i,j)}})v(N). \tag{9}$$

Combining (8) and (9), we obtain:

$$\Phi_{K_{(i,j)}}(N, v, L, p, \delta) = \Phi_{K^*_{(i,j)}}(N^{(i,j)(j,i)}, v^{(i,j)(j,i)}, L^{(i,j)(j,i)}, p^{(i,j)(j,i)}, \delta^{(i,j)(j,i)}) + (p_N - p^{(i,j)}_{N^{(i,j)}})v(N),$$

Because $N^{(i,j)(j,i)}$ contains two agents, by **Case 2**, the individual payoff

$$\Phi_{K^*_{(i,j)}}(N^{(i,j)(j,i)}, v^{(i,j)(j,i)}, L^{(i,j)(j,i)}, p^{(i,j)(j,i)}, \delta^{(i,j)(j,i)})$$

is uniquely determined by generalized standardness. Therefore, the total payoff $\Phi_{K_{(i,j)}}(N, v, L, p, \delta)$ is uniquely determined. By node-weighted efficiency, the total payoff $\Phi_{K_{(j,i)}}(N, v, L, p, \delta)$ is uniquely determined as well. Proceeding in this fashion for each link of the tree (N, L), we conclude that, for each proper cone $K \in \Delta^0_{(N,L)}$, the total payoff $\Phi_K(N, v, L, p, \delta)$ is uniquely determined by the combination of node-weighted efficiency, invariance with respect to cone amalgamation and generalized standardness. It remains to notice that the payoff vector $\Phi(N, v, L, p, \delta)$ is uniquely determined by the total payoffs $\Phi_K(N, v, L, p, \delta)$, $K \in \Delta^0_{(N,L)}$. To see this, it suffices to use the successor relation $s(\cdot)$ on $\Delta^0_{(N,L)}$ defined in subsection 2.2. This completes the proof of Proposition 1. ■

Proposition 2 *The solution Ψ defined as in (5) on $\mathcal{C}^*$ satisfies node-weighted efficiency, invariance with respect to cone amalgamation and generalized standardness.*

Proof.
Generalized standardness. The fact that the solution Ψ satisfies generalized standardness follows from a simples rewriting of (5) in two-agent cases.
Node-weighted efficiency. The fact that the solution Ψ satisfies node-weighted efficiency follows from (3) and (5).
Invariance with respect to cone amalgamation. Consider any $(N, v, L, p, \delta) \in \mathcal{C}^*$, any $\{i, j\} \in L$, and proceed to the amalgamation of the members of $K_{(i,j)}$ in order to obtain $(N^{(i,j)}, v^{(i,j)}, L^{(i,j)}, p^{(i,j)}, \delta^{(i,j)}) \in \mathcal{C}^*$. We distinguish two cases.
Case 1: the root $r \in N^{(i,j)} \setminus \{K^*_{(i,j)}\}$. Using definition (2), the fact that $\delta = \delta^{(i,j)}$ and the fact that the distance between $l \in K_{(j,i)}$ and r is, by construction, the same in (N, L) and $(N^{(i,j)}, L^{(i,j)})$, we

have:

$$\forall l \in K_{(j,i)},\ h_l^r(N^{(i,j)}, v^{(i,j)}, L^{(i,j)}, p^{(i,j)}, \delta^{(i,j)}) \;=\; \delta^{d(l,r)}\big(v^{(i,j)}(S_r^{(i,j)}(l)) - \delta \sum_{k \in s_r^{(i,j)}(l)} v^{(i,j)}(S_r^{(i,j)}(k))\big) \tag{10}$$

where $S_r^{(i,j)}(l)$ denotes the union of the set of all subordinates of l and $\{l\}$ in the rooted tree induced by r in $(N^{(i,j)}, L^{(i,j)})$. The notations $s_r^{(i,j)}(l)$ and $S_r^{(i,j)}(k)$ must be interpreted in a similar way. Two subcases arise.

Case 1-1: $S_r^{(i,j)}(l)$ does not contain $K^*_{(i,j)}$. Then, $s_r^{(i,j)}(l) = s_r(l)$, and, for each $k \in s_r^{(i,j)}(l)$, $S_r^{(i,j)}(k) = S_r(k)$ and does not contain $K^*_{(i,j)}$. It follows that:

$$v^{(i,j)}(S_r^{(i,j)}(l)) = v(S_r(l)) \text{ and } \forall k \in s_r^{(i,j)}(l) = s_r(l),\ v^{(i,j)}(S_r^{(i,j)}(k)) = v(S_r(k)).$$

From this remark, we obtain:

$$h_l^r(N^{(i,j)}, v^{(i,j)}, L^{(i,j)}, p^{(i,j)}, \delta^{(i,j)}) = h_l^r(N, v, L, p, \delta).$$

By definition of $p^{(i,j)}$, for each $r \in N^{(i,j)} \setminus \{K^*_{(i,j)}\}$, $p^{(i,j)}(r) = p(r)$. Therefore, we have:

$$p^{(i,j)}(r) h_l^r(N^{(i,j)}, v^{(i,j)}, L^{(i,j)}, p^{(i,j)}, \delta^{(i,j)}) = p(r) h_l^r(N, v, L, p, \delta).$$

Case 1-2: $S_r^{(i,j)}(l)$ contains $K^*_{(i,j)}$. Then, there exists only one $k \in s_r^{(i,j)}(l)$ such that $S_r^{(i,j)}(k)$ contains $K^*_{(i,j)}$.

If $l = j$, then $K^*_{(i,j)} \in s_r^{(i,j)}(j)$ and $K^*_{(i,j)}$ is a leaf of $(N^{(i,j)}, L^{(i,j)})$. We also have $s_r^{(i,j)}(j) = (s_r(j) \setminus \{i\}) \cup \{K^*_{(i,j)}\}$. For $k \in s_r(j) \setminus \{i\}$, we proceed as in **Case 1-1** to conclude that $v^{(i,j)}(S_r^{(i,j)}(k)) = v(S_r(k))$. If $k = K^*_{(i,j)}$, then $S_r^{(i,j)}(k) = \{K^*_{(i,j)}\}$ and so we have:

$$v^{(i,j)}(S_r^{(i,j)}(k)) = v^{(i,j)}(\{K^*_{(i,j)}\}) = v(K_{(i,j)}) = v(S_r(i)).$$

Note also that $S_r^{(i,j)}(j) = (S_r(j) \setminus \{K_{(i,j)}\}) \cup \{K^*_{(i,j)}\}$. Thus, by definition of $v^{(i,j)}$, we obtain:

$$v^{(i,j)}(S_r^{(i,j)}(j)) = v\big((S_r^{(i,j)}(j) \setminus \{K^*_{(i,j)}\}) \cup K_{(i,j)}\big) = v(S_r(j)).$$

From the above arguments, we obtain:

$$h_j^r(N^{(i,j)}, v^{(i,j)}, L^{(i,j)}, p^{(i,j)}, \delta^{(i,j)}) = h_j^r(N, v, L, p, \delta).$$

Because $r \in N^{(i,j)} \setminus \{K^*_{(i,j)}\}$, $p^{(i,j)}(r) = p(r)$. Thus, we conclude that:

$$p^{(i,j)}(r) h_j^r(N^{(i,j)}, v^{(i,j)}, L^{(i,j)}, p^{(i,j)}, \delta^{(i,j)}) = p(r) h_j^r(N, v, L, p, \delta).$$

If $l \neq j$, $s_r^{(i,j)}(l) = s_r(l)$. Consider any $k \in s_r(l)$. If $S_r^{(i,j)}(k)$ does not contain $K^*_{(i,j)}$, we proceed as in **Case 1-1** to obtain that $v^{(i,j)}(S_r^{(i,j)}(k)) = v(S_r(k))$. If $S_r^{(i,j)}(k)$ contains $K^*_{(i,j)}$, we proceed as in **Case 2-2** for $v^{(i,j)}(S_r^{(i,j)}(k))$ and $v^{(i,j)}(S_r^{(i,j)}(l))$. From this, we easily conclude that:

$$p^{(i,j)}(r)h_l^r(N^{(i,j)}, v^{(i,j)}, L^{(i,j)}, p^{(i,j)}, \delta^{(i,j)}) = p(r)h_l^r(N, v, L, p, \delta).$$

From **Case 1-1** and **Case 2-2**, we have:

$$\forall l \in K_{(j,i)}, \forall r \in N^{(i,j)} \setminus \{K^*_{(i,j)}\}, \quad p^{(i,j)}(r)h_l^r(N^{(i,j)}, v^{(i,j)}, L^{(i,j)}, p^{(i,j)}, \delta^{(i,j)}) = p(r)h_l^r(N, v, L, p, \delta),$$

which completes **Case 1**.

Case 2: $r = K^*_{(i,j)}$. In this case, the distance between $l \in K_{(j,i)}$ and i in (N, L) is the same as the distance between l and $K^*_{(i,j)}$ in $(N^{(i,j)}, L^{(i,j)})$, and $K^*_{(i,j)}$ does not belong to $S_r^{(i,j)}(l)$. Thus, for each $l \in K_{(j,i)}$, we have:

$$\begin{aligned} h_l^r(N^{(i,j)}, v^{(i,j)}, L^{(i,j)}, p^{(i,j)}, \delta^{(i,j)}) &= \delta^{d(l,r)} v^{(i,j)}(S_r^{(i,j)}(l)) - \delta^{d(k,r)} \sum_{k \in s_r^{(i,j)}(l)} v^{(i,j)}(S_r^{(i,j)}(k)) \\ &= \delta^{d(l,i)} \big(v(S_i(l)) - \delta \sum_{k \in s_i(l)} v(S_i(k)) \big). \end{aligned}$$

By definition, $p^{(i,j)}(K^*_{(i,j)}) = \sum_{q \in K_{(i,j)}} p(q)\delta^{d(i,q)}$, and, for $l \in K_{(j,i)}$ and $q \in K_{(i,j)}$, agent i lies in the unique path between l and q. Thus, $d(l,q) = d(l,i) + d(i,q)$, $S_q(l) = S_i(l)$, $s_q(l) = s_i(l)$, and, for each $k \in s_i(l)$, $S_q(k) = S_i(k)$. Thus,

$$p^{(i,j)}(r)\, h_l^r(N^{(i,j)}, v^{(i,j)}, L^{(i,j)}, p^{(i,j)}, \delta^{(i,j)})$$

is equal to:

$$\begin{aligned} & \Big(\sum_{q \in K_{(i,j)}} p(q)\delta^{d(i,q)} \Big) \delta^{d(l,i)} \Big(v(S_i(l)) - \delta \sum_{k \in s_i(l)} v(S_i(k)) \Big) \\ = & \sum_{q \in K_{(i,j)}} p(q)\delta^{d(i,q)}\delta^{d(l,i)} \Big(v(S_i(l)) - \delta \sum_{k \in s_i(l)} v(S_i(k)) \Big) \\ = & \sum_{q \in K_{(i,j)}} p(q)\delta^{d(l,q)} \Big(v(S_q(l) - \delta \sum_{k \in s_q(l)} v(S_q(k)) \Big) \\ = & \sum_{q \in K_{(i,j)}} p(q) h_l^q(N, v, L, p, \delta). \end{aligned}$$

Combining **Case 1** with **Case 2**, we obtain the set of desired equalities, i.e. for each $l \in K_{(j,i)}$,

$$\begin{aligned} \Psi_l(N^{(i,j)}, v^{(i,j)}, L^{(i,j)}, p^{(i,j)}, \delta^{(i,j)}) &= \sum_{r \in N^{(i,j)}} p^{(i,j)}(r)h_l^r(N^{(i,j)}, v^{(i,j)}, L^{(i,j)}, p^{(i,j)}, \delta^{(i,j)}) \\ &= \sum_{r \in K_{(j,i)}} p(l)h_l^r(N, v, L, p, \delta) + \sum_{r \in K_{(i,j)}} p(r)h_l^r(N, v, L, p, \delta) \\ &= \Psi_l(N, v, L, p, \delta). \end{aligned}$$

$\blacksquare$

Combining Proposition 1 with Proposition 2 we obtain the following characterization result.

Proposition 3 *The solution Ψ defined as in (5) on $\mathcal{C}^*$ is the unique solution satisfying node-weighted efficiency, invariance with respect to cone amalgamation and generalized standardness.*

The logical independence of the axioms invoked in Proposition 3 can be demonstrated by exhibiting the following solutions:

- The solution Φ on $\mathcal{C}^*$ which assigns to each $(N, v, L, p, \delta) \in \mathcal{C}$ the payoff $\Phi(N, v, L, p, \delta) = \Psi(N, v, L, p, \delta)$ if $n \geq 2$ and $\Phi_i(N, v, L, p, \delta) = 0$ if $N = \{i\}$ satisfies invariance with respect to cone amalgamation and generalized standardness but violates node-weighted efficiency.

- The solution Φ on $\mathcal{C}^*$ which assigns to each $(N, v, L, p, \delta) \in \mathcal{C}$ and each $i \in N$ a payoff

$$\Phi_i(N, v, L, p, \delta) = p_N \delta v(i) + p(i)\left(v(N) - \sum_{j \in N} \delta v(j)\right)$$

  satisfies node-weighted efficiency and generalized standardness but violates invariance with respect to cone amalgamation.

- The solution Φ on $\mathcal{C}^*$ which assigns to each $(N, v, L, p, \delta) \in \mathcal{C}$ and each $i \in N$ a payoff $\Phi_i(N, v, L, p, \delta) = p(i)v(N)$ satisfies node-weighted efficiency and invariance with respect to cone amalgamation but violates generalized standardness.

4. Axiomatic characterization of the solutions Ψ on $\mathcal{C}$

Below is a set of axioms for a solution Φ on $\mathcal{C}$. Each axiom is designed for each fixed pair formed by a tree (N, L) and a discount rate $\delta \in [0, 1]$. For each (N, L) and a discount rate $\delta \in [0, 1]$, these axioms specify either the payoff distributed to a certain subset of agents in some specific underlying TU-games, or how the payoffs are related between two tree TU-games augmented by a discount rate when only the underlying TU-games change. In other words, none of these axioms provides information on the way the solution evolves as a function of the tree (N, L) and the discount rate δ.

Linearity For each $(N, v, L, \delta) \in \mathcal{C}$, each $(N, w, L, \delta) \in \mathcal{C}$ and each $a \in \mathbb{R}$, it holds that:

$$\Phi(N, v + w, L, \delta) = \Phi(N, v, L, \delta) + \Phi(N, w, L, \delta) \quad \text{and} \quad \Phi(N, av, L, \delta) = a\Phi(N, v, L, \delta).$$

Two tree TU-games augmented by a discount rate (N, v, L, δ) and (N, w, L, δ) in $\mathcal{C}$ are *cone equivalent* if for each cone $K \in \Delta_{(N,L)}$, it holds that $v(K) = w(K)$.

Cone equivalence If (N, v, L, δ) and (N, w, L, δ) in $\mathcal{C}$ are cone equivalent[4], then it holds that $\Phi(N, v, L, \delta) = \Phi(N, w, L, \delta)$.

Given $(N, v, L, \delta) \in \mathcal{C}$, agent $i \in N$ is a *null agent* if for each proper cone $K \in \Delta^0_{N,L}$ such that $h(K) = i$, it holds that $v(K) = \sum_{T \in s(K)} v(T)$ and

$$v(N) = \sum_{\{K \in \Delta_{(N,L)} : h(K) \in L_i, i \in N \setminus K\}} v(K).$$

Null agent For each $(N, v, L, \delta) \in \mathcal{C}$ and each null agent $i \in N$ in (N, v, L, δ), it holds that $\Phi_i(N, v, L, \delta) = 0$.

Proper cone efficiency For each $(N, v, L, \delta) \in \mathcal{C}$ and each $K \in \Delta^0_{(N,L)}$, it holds that:

$$\sum_{i \in N} \Phi_i(N, 1_K, v, L, \delta) = 0.$$

In the context of tree TU-games (N, v, L), Béal et al. (2010) show that the combination of the above axioms characterizes the set of tree solutions. We rephrase this result in the context of tree TU-games augmented by a discount rate.

Proposition 4 *(Theorem 1 in Béal et al. 2010)*
The set of tree solutions on $\mathcal{C}$ coincides with the set of solutions on $\mathcal{C}$ satisfying linearity, cone equivalence, proper cone efficiency and the null agent axiom.

We now turn our attention to the axiomatic characterization of the discounted tree solutions on $\mathcal{C}$.

Given $(N, v, L, \delta) \in \mathcal{C}$, agent $i \in N$ is a *δ-reducing agent* if for each proper cone $K \in \Delta^0_{N,L}$ such that $h(K) = i$, it holds that $v(K) = \delta \sum_{T \in s(K)} v(T)$ and

$$v(N) = \delta \sum_{\{K \in \Delta_{(N,L)} : h(K) \in L_i, i \in N \setminus K\}} v(K).$$

δ-reducing agent For each $(N, v, L, \delta) \in \mathcal{C}$ and each δ-reducing agent $i \in N$ in (N, v, L, δ), it holds that $\Phi_i(N, v, L, \delta) = 0$.

Two remarks are in order. First, when $\delta = 1$, the axiom of δ-reducing agent reduces to the null agent axiom, and when $\delta = 0$ the δ-reducing agent *nullifies* each proper cone $K \in \Delta^0_{N,L}$ such that $h(K) = i$, and N, i.e. $v(K) = 0$, and $v(N) = 0$. Second, the idea to introduce a similar type of

[4] Béal et al. (2015b) use the equivalent axiom of Invariance to irrelevant coalitions in order to characterize the average tree solution.

δ-reducing agent can been found in van den Brink and Funaki (2015) for the the class of TU-games. In the class of TU-games (N, v) an agent is δ-reducing if for each $S \subseteq N \setminus \{i\}$, $v(S \cup \{i\}) = \delta v(S)$. The axiom of δ-reducing agent for the class of TU-games indicates that an agent gets a zero payoff whenever he or she is δ-reducing in a TU-game. van den Brink and Funaki (2015) use this axiom to characterize the discounted Shapley values introduced by Joosten (1996).

Substituting the axiom of null agent for the axiom of δ-reducing agent in Proposition 4 yields the set of discounted tree solutions. More specifically, the axiom of cone equivalence can even be dropped in both results as a consequence of the following lemma.

Lemma 1 *If a solution Φ on $\mathcal{C}$ satisfies linearity and the axiom of δ-reducing agent, then Φ also satisfies cone equivalence.*

Proof. Choose a solution Φ on $\mathcal{C}$ that satisfies linearity and the axiom of δ-reducing agent. Now consider two cone equivalent tree TU-games augmented by a discount rate $(N, v, L, \delta), (N, w, L, \delta) \in \mathcal{C}$. In $(N, v - w, L, \delta)$, note that $(v - w)(K) = 0$ for each cone $K \in \Delta_{(N,L)}$. Thus, each agent $i \in N$ is a δ-reducing agent. By the axiom of δ-reducing agent, we obtain $\Phi_i(N, v - w, L, \delta) = 0$ for each $i \in N$. By linearity, the latter equality is equivalent to $\Phi_i(N, v, L, \delta) = \Phi_i(N, w, L, \delta)$ for each $i \in N$. Therefore, Φ satisfies cone equivalence. ■

Setting $\delta = 1$, a consequence of Lemma 1 is that the combination of linearity and null agent implies cone equivalence, which means that cone equivalence can be dropped in Proposition 4 (enunciated as Theorem 1 in Béal et al., 2010). Before stating our characterization of the discounted tree solutions in Proposition 5, we also need the following remark.

Remark 1 Recall that the axioms listed below are valid for each fixed pair formed by a tree (N, L) and a discount rate $\delta \in [0, 1]$. Consequently, for each such a pair, we define:

1. The subclass $\mathcal{C}_{(N,L),\delta}$ of tree TU-games augmented by a discounted rate of the form $(N, \cdot, L, \delta)$. Clearly, we have:
$$\mathcal{C} = \bigcup_{(N,L),\delta} \mathcal{C}_{(N,L),\delta}.$$
2. A solution $\Phi(N, \cdot, L, \delta)$ on $\mathcal{C}_{(N,L),\delta}$, which can be viewed as the partial solution at point $((N, L), \delta)$ of a solution Φ on $\mathcal{C}$.
3. The set $\mathcal{E}_{(N,L),\delta}$ given by the set of solutions $\Phi(N, \cdot, L, v)$ on $\mathcal{C}_{(N,L),\delta}$ satisfying linearity, proper cone efficiency and the axiom of δ-reducing agent.

Proposition 5 *The set of discounted tree solutions Ψ on $\mathcal{C}$ coincides with the set of solutions on $\mathcal{C}$ satisfying linearity, proper cone efficiency and the axiom of δ-reducing agent. More precisely, for each fixed tree (N, L) and each discount rate $\delta \in [0, 1]$, the set of discounted tree solutions*

$\Psi(N,\cdot,L,\delta)$ on $\mathcal{C}_{(N,L),\delta}$ is a linear space of dimension n, and the n discounted hierarchical outcomes $h^r(N,\cdot,L,\delta)$, $r\in N$, form a basis for this linear space. In particular, we have:

$$\Psi(N,\cdot,L,\delta)=\sum_{r\in N}\Psi_r(N,1_N,L,\delta)h^r(N,\cdot,L,\delta). \tag{11}$$

The proof strategy of Proposition 5 is different from the one used by Béal et al. (2010) to prove Proposition 4. It is much more transparent and relies heavily on the following two new lemmata. The first lemma contains three points. The first two points study the consequences of the axiom of δ-reducing agent on the distribution of payoffs in tree TU-games augmented by a discount rate, where the TU-game is constructed from standard TU-games on cones. The third point indicates that the combination of the axiom of δ-reducing agent with proper cone efficiency yields a payoff distribution such that the heads $h(K)$ and $h(K^c)$ of two complementary proper cones K and K^c receive opposite payoffs when the underlying TU-game is the K-standard TU-game or the K^c-standard TU-game. A consequence of the second lemma is that, for each tree (N,L) and each discount rate $\delta\in[0,1]$, the set of solutions $\mathcal{E}_{(N,L),\delta}$ forms a linear space of dimension at most n.

Lemma 2 *Let Φ be a solution on $\mathcal{C}$ satisfying the axiom of δ-reducing agent. Consider any tree (N,L) and any discount rate $\delta\in[0,1]$. Then, we have:*

1. *For each $i\in N$ and each link $\{j,l\}\in L$ such that $i\notin\{j,l\}$, $\Phi_i(N,1_{K_{(j,l)}},L,\delta)=0$.*
2. *For each $i\in N$ and each $j\in L_i$, $\Phi_i(N,1_{K_{(j,i)}}+\delta\sum_{l\in L_i\setminus\{j\}}1_{K_{(i,l)}}+\delta 1_N,L,\delta)=0$.*

If, moreover, Φ satisfies proper cone efficiency, then it holds that:

3. *For each $i\in N$ and each $j\in L_i$, $\Phi_i(N,1_{K_{(i,j)}},L,\delta)+\Phi_j(N,1_{K_{(i,j)}},L,\delta)=0$.*

Proof. Let Φ be as hypothesized, and consider any tree (N,L) and $\delta\in[0,1]$.

Point 1. Consider any $i\in N$ and any link $\{j,l\}\in L$ such that $i\notin\{j,l\}$. For each proper cone $K\in\Delta^0_{(N,L)}$ with head $h(K)=i$, we have $1_{K_{(j,l)}}(K)=0$. Since, $i\neq l$, there is no successor $T\in s(K)$ such that $T=K_{(j,l)}$, and so, for each $T\in s(K)$, we have $1_{K_{(j,l)}}(T)=0$. Thus, $1_{K_{(j,l)}}(K)=\delta\sum_{T\in s(K)}1_{K_{(j,l)}}(T)$. Next, since $K_{(j,l)}$ is a proper cone, we have $1_{K_{(j,l)}}(N)=0$. Finally, consider a proper cone $K\in\Delta^0_{(N,L)}$ with head $h(K)\in L_i$ and such that $i\in N\setminus K$, i.e. the head of the complementary cone K^c is i. Since $i\neq l$, we have $K\neq K_{(j,l)}$, and so $1_{K_{(j,l)}}(K)=0$. It follows that:

$$1_{K_{(j,l)}}(N)=\delta\sum_{\{K\in\Delta_{(N,L)}:h(K)\in L_i,i\in N\setminus K\}}1_{K_{(j,l)}}(K)=0.$$

We conclude that i is δ-reducing agent in $(N,1_{K_{(j,l)}},L,\delta)\in\mathcal{C}$. By the axiom of δ-reducing agent, $\Phi_i(N,1_{K_{(j,l)}},L,\delta)=0$, as asserted.

Point 2. Consider any $i\in N$, any $j\in L_i$, and the TU-game given by:

$$1_{K_{(j,i)}}+\delta\sum_{l\in L_i\setminus\{j\}}1_{K_{(i,l)}}+\delta 1_N=:w.$$

Pick any proper cone $K \in \Delta^0_{(N,L)}$ with head $h(K) = i$. First, assume that $h(K^c) = j$ so that $K = K_{(i,j)}$. We have, $w(K_{(i,j)}) = 0$. We also deduce that, for each $l \in L_i \setminus \{j\}$, $K_{(i,l)} \notin s(K_{(i,j)})$, and, of course, $K_{(j,i)} \notin s(K_{(i,j)})$. Thus, for each $T \in s(K_{(i,j)})$, $w(K) = 0$, and so $w(K) = \delta \sum_{T\in s(K)} w(T)$. Next, we have $w(N) = \delta$. From the following equality $\{K \in \Delta_{(N,L)} : h(K) \in L_i, i \in N\backslash K\} = \{K_{(l,i)} : l \in L_i\}$, it holds that: for the chosen neighbor $j \in L_i$, $w(K_{(j,i)}) = 1$, and, for each other $l \in L_i \setminus \{j\}$, $w(K_{(l,i)}) = 0$. Therefore,

$$\begin{aligned} w(N) &= \delta w(K_{(j,i)}) \\ &= \delta \sum_{\{K\in\Delta_{(N,L)}:h(K)\in L_i, i\in N\backslash K\}} w(K). \end{aligned}$$

We conclude that i is a δ-reducing agent in $(N, w, L, \delta) \in \mathcal{C}$. By the axiom of δ-reducing agent, $\Phi_i(N, w, L, \delta) = 0$, as asserted.

Point 3. Assume that Φ satisfies proper cone efficiency and the axiom of δ-reducing agent. For any link $\{i, j\} \in L$, consider the discounted tree TU-games $(N, 1_{K_{(i,j)}}, L, \delta) \in \mathcal{C}$. By point 1, we deduce that, for each $l \notin \{i, j\}$, $\Phi_l(N, 1_{K_{(i,j)}}, L, \delta) = 0$. By proper cone efficiency, we obtain the desired result:

$$\begin{aligned} 0 &= 1_{K_{(i,j)}}(N) \\ &= \sum_{l\in N} \Phi_i(N, 1_{K_{(i,j)}}, L, \delta) \\ &= \Phi_i(N, 1_{K_{(i,j)}}, L, \delta) + \Phi_j(N, 1_{K_{(i,j)}}, L, \delta). \end{aligned}$$

■

Lemma 3 *Given the tree (N, L) and the discount rate $\delta \in [0, 1]$, the set of solutions $\mathcal{E}_{(N,L),\delta}$ on $\mathcal{C}_{(N,L),\delta}$ constitutes a (real) linear space, and the linear function*

$$F : \mathcal{E}_{(N,L),\delta} \longrightarrow \mathbb{R}^n, \quad \Phi(N, \cdot, L, \delta) \longmapsto F(\Phi(N, \cdot, L, \delta)) = \Phi(N, 1_N, L, \delta),$$

is injective.

Proof. Consider any tree (N, L) and any discount rate $\delta \in [0, 1]$. By the axiom of linearity, the solutions $\Phi(N, \cdot, L, \delta)$ on $\mathcal{C}_{(N,L),\delta}$ are linear, from which it follows that $\mathcal{E}_{(N,L),\delta}$ can be identified with the linear space of all $n \times (2^n - 1)$ real matrices. It remains to show that $F : \mathcal{E}_{(N,L),\delta} \longrightarrow \mathbb{R}^n$, $\Phi(N, \cdot, L, \delta) \longmapsto F(\Phi(N, \cdot, L, \delta)) = \Phi(N, 1_N, L, \delta)$ is injective, i.e. that the kernel of F, Ker(F), is reduced to the null solution $\mathbf{0}(N, \cdot, L, \delta)$.

Pick any solution $\Phi(N, \cdot, L, \delta)$ in $\mathcal{E}_{(N,L),\delta}$ such that $\Phi(N, \cdot, L, \delta) \in$ Ker(F), i.e. $\Phi(N, 1_N, L, \delta) = (0, \dots, 0)$. To show that $\Phi(N, \cdot, L, \delta) = \mathbf{0}(N, \cdot, L, \delta)$, we use the following two facts.

Fact 1: For each $\Phi(N, \cdot, L, \delta) \in \mathcal{E}_{(N,L),\delta}$, $\Phi(N, v, L, \delta) = \sum_{\{K\in\Delta_{(N,L)}:K\neq\emptyset\}} v(K)\Phi(N, 1_K, L, \delta)$.

To see this, consider any $\Phi(N,\cdot,L,\delta) \in \mathcal{E}_{(N,L),\delta}$. Let us recall first that for each TU-game (N,v), $v = \sum_{\{T\in 2^N:T\neq\emptyset\}} v(T)1_T$. Because $\Phi(N,\cdot,L,\delta)$ is linear, we have:

$$\forall (N,v,L,\delta) \in \mathcal{C}_{(N,L),\delta}, \quad \Phi(N,v,L,\delta) = \sum_{\{T\in 2^N:T\neq\emptyset\}} v(T)\Phi(N,1_T,L,\delta).$$

Next, consider any $(N,v,L,\delta) \in \mathcal{C}_{(N,L),\delta}$, and define $(N,w,L,\delta) \in \mathcal{C}_{(N,L),\delta}$ as follows: $v(T) = w(T)$ when $T \in \Delta_{(N,L)}$, and $w(T) = 0$ otherwise. By construction, (N,v,L,δ) and (N,w,L,δ) are cone equivalent. On the one hand, by Lemma 1, Φ satisfies the axiom of cone equivalence, so that:

$$\Phi(N,v,L,\delta) = \Phi(N,w,L,\delta).$$

On the other hand, by definition of (N,w,L,δ), we have:

$$\Phi(N,w,L,\delta) = \sum_{\{T\in 2^N:T\neq\emptyset\}} w(T)\Phi(N,1_T,L,\delta) = \sum_{\{T\in \Delta_{(N,L)}:T\neq\emptyset\}} v(T)\Phi(N,1_T,L,\delta).$$

Thus, we have:

$$\Phi(N,v,L,\delta) = \sum_{\{T\in \Delta_{(N,L)}:T\neq\emptyset\}} v(T)\Phi(N,1_T,L,\delta),$$

from which **Fact 1** follows.

Fact 2: $\forall \Phi(N,\cdot,L,\delta) \in \mathrm{Ker}(F), \forall \{i,j\} \in L, \quad \Phi_i(N,1_{K_{(i,j)}},L,\delta) = 0.$

Before proving **Fact 2**, first note that by point 2 of Lemma 2 and linearity of $\Phi(N,\cdot,L,\delta)$, we have:

$$\begin{aligned}
& \Phi_j(N,1_{K_{(i,j)}} + \delta \sum_{l\in L_j\setminus\{i\}} 1_{K_{(j,l)}} + \delta 1_N, L, \delta) \\
= \;& \Phi_j(N,1_{K_{(i,j)}},L,\delta) + \delta \sum_{l\in L_j\setminus\{i\}} \Phi_j(N,1_{K_{(j,l)}},L,\delta) + \delta\Phi_j(N,1_N,L,\delta) \\
= \;& 0,
\end{aligned}$$

and so, by using point 3 of Lemma 2 and the fact that $\Phi \in \mathrm{Ker}(F)$, we have:

$$\Phi_i(N,1_{K_{(i,j)}},L,\delta) = \delta \sum_{l\in L_j\setminus\{i\}} \Phi_j(N,1_{K_{(j,l)}},L,\delta). \tag{12}$$

Fact 2 is proven by induction on the number of elements $K_{(j,i)}$ of the complementary cone $K_{(j,i)}$ of $K_{(i,j)}$.

Initial step. Assume that $|K_{(j,i)}| = 1$. This means that $K_{(j,i)} = \{j\}$, and j is a leaf of (N,L). Equation (12) becomes $\Phi_i(N,1_{K_{(i,j)}},L,\delta) = 0$, as desired.

Induction hypothesis. Let p be a natural number strictly inferior to n. Assume that **Fact 2** is true for each proper cone $K \in \Delta^0_{(N,L)}$ such that the cardinality of the complementary cone K^c of K is at most p.

Induction step. Consider any proper cone $K_{(i,j)} \in \Delta^0_{(N,L)}$ such that $K_{(j,i)} = p+1$. Consider equation (12). By construction, for each $l \in L_j \setminus \{i\}$, each proper cone $K_{(j,l)}$ contains $K_{(i,j)} \cup \{j\}$. Since, $K_{(j,i)} = p+1$, $K_{(i,j)} = n-p-1$, which implies $K_{(j,l)} \geq n-p$ and $k_{lj} \leq p$. Thus, by the induction hypothesis, for each $l \in L_j \setminus \{i\}$, we have $\Phi_j(N, 1_{K_{(j,l)}}, L, \delta) = 0$. By equation (12), it follows that $\Phi_i(N, 1_{K_{(i,j)}}, L, \delta) = 0$, which completes the proof of **Fact 2**.

It follows from **Fact 2** and point 3 of Lemma 2 that:

$$\forall \{i,j\} \in L, \quad \Phi_i(N, 1_{K_{(i,j)}}, L, \delta) = 0, \text{ and } \Phi_j(N, 1_{K_{(i,j)}}, L, \delta) = 0.$$

By point 1 of Lemma 2, we have:

$$\forall l \in N \setminus \{i,j\}, \quad \Phi_l(N, 1_{K_{(i,j)}}, L, \delta) = 0, \tag{13}$$

Therefore, we conclude that:

$$\forall \{i,j\} \in L, \quad \Phi(N, 1_{K_{(i,j)}}, L, \delta) = (0, \ldots, 0), \text{ and by hypothesis } \Phi(N, 1_N, L, \delta) = (0, \ldots, 0),$$

Equivalently,

$$\forall K \in \Delta_{(N,L)}, K \neq \emptyset, \quad \Phi(N, 1_K, L, \delta) = (0, \ldots, 0). \tag{14}$$

Combining (14) and **Fact 1**, we obtain that for each coalition function v on N, $\Phi(N, v, L, \delta) = \mathbf{0}(N, \cdot, L, \delta)$. Conclude that the function F is injective. ■

We now have the material to prove Proposition 5.

Proof. (of Proposition 5). In a first step, we verify that for each tree (N, L) and each discount rate $\delta \in [0,1]$, the discounted hierarchical outcomes $h^r(N, \cdot, L, \delta)$, $r \in N$, belong to $\mathcal{E}_{(N,L),\delta}$. Obviously, each discounted hierarchical outcome satisfies linearity on $\mathcal{C}_{(N,L),\delta}$. By (3), and keeping in mind that $S_r(r) = N$, for each proper cone $K \in \Delta^0_{N,L}$, we have:

$$\sum_{j \in S_r(r)} h^r_i(N, v, L, \delta) = \delta^{d(r,r)} 1_K(N) = 0,$$

which proves that $h^r(N, 1_K, L, \delta)$ satisfies proper cone efficiency on $\mathcal{C}_{(N,L),\delta}$. Finally, by definition of a δ-reducing agent and by definition (2) of the discounted hierarchical outcomes, we easily conclude that the latter satisfies the axiom of δ-reducing agent on $\mathcal{C}_{(N,L),\delta}$. It follows that the discounted hierarchical outcomes $h^r(N, \cdot, L, \delta)$, $r \in N$, belong to $\mathcal{E}_{(N,L),\delta}$.

To show that the dimension of $\mathcal{E}_{(N,L),\delta}$ is equal to n, we first prove that the hierarchical outcomes $h^r(N, \cdot, L, \delta)$, $r \in N$, are linearly independent, and then we apply Lemma 3 to show that they generate $\mathcal{E}_{(N,L),\delta}$. Denote by $e^r = (e^r_i)_{i\in N}$, $r \in N$, the canonical vectors in $\mathbb{R}^n$, i.e. $e^r_i = 1$ if $i = r$, and $e^r_i = 0$ otherwise.

To show that the discounted hierarchical outcomes $h^r(N, \cdot, L, \delta)$, $r \in N$, are linearly independent, consider any linear combination of these discounted hierarchical outcomes equal to the null solution $\mathbf{0}(N, \cdot, L, \delta)$:

$$\sum_{r\in N} \alpha(r)h^r(N, \cdot, L, \delta) = \mathbf{0}(N, \cdot, L, \delta).$$

By applying the linear function F, as defined in the statement of Lemma 3, on both sides of the above equality, we have:

$$F\Bigg(\sum_{r\in N} \alpha(r)h^r(N, \cdot, L, \delta)\Bigg) = \sum_{r\in N} \alpha(r)F(h^r(N, \cdot, L, \delta)) = F(\mathbf{0}(N, \cdot, L, \delta)) = (0, \dots, 0). \tag{15}$$

By definition of F and (2), for each $r \in N$, $F(h^r(N, \cdot, L, \delta)) = h^r(N, 1_N, \delta) = e^r$. From this, equation (15) becomes:

$$\sum_{r\in N} \alpha(r)e^r = (0, \dots, 0),$$

which means that, for each $r \in N$, $\alpha(r) = 0$. We conclude that the discounted hierarchical outcomes are linearly independent on $\mathcal{E}_{(N,L),\delta}$.

It remains to show that $\mathcal{E}_{(N,L),\delta}$ is generated by the discounted hierarchical outcomes. Consider any $\Phi(N, \cdot, L, \delta) \in \mathcal{E}_{(N,L),\delta}$. By using the linearity of F and the fact that, for each $r \in N$, $F(h^r(N, \cdot, L, \delta)) = e^r$, we have:

$$\begin{aligned}
F(\Phi(N, \cdot, L, \delta)) &= \Phi(N, 1_N, L, \delta) \\
&= \sum_{r\in N} \Phi_r(N, 1_N, L, \delta)e^r \\
&= \sum_{r\in N} \Phi_r(N, 1_N, L, \delta)F(h^r(N, \cdot, L, \delta)) \\
&= F\Bigg(\sum_{r\in N} \Phi_r(N, 1_N, L, \delta)h^r(N, \cdot, L, \delta)\Bigg).
\end{aligned}$$

By Lemma 3, F is injective, from which we conclude that:

$$\Phi(N, \cdot, L, \delta) = \sum_{r\in N} \Phi_r(N, 1_N, L, \delta)h^r(N, \cdot, L, \delta).$$

Because $\Phi(N, \cdot, L, \delta)$ has been chosen arbitrarily in $\mathcal{E}_{(N,L),\delta}$, it follows that $\mathcal{E}_{(N,L),\delta}$ is generated by the discounted hierarchical outcomes. Therefore, the discounted hierarchical outcomes $h^r(N, \cdot, L, \delta)$, $r \in N$, form a basis for $\mathcal{E}_{(N,L),\delta}$ whose dimension is equal to n. Because this result remains true whatever the chosen tree (N, L) and discount rate $\delta \in [0, 1]$, the set of solutions satisfying linearity, proper cone efficiency and the axiom of δ-reducing agent on $\mathcal{C}$ coincides with the set of solutions Ψ given by (11). ■

From Proposition 5, we can derive a characterization of the average discounted tree solution. To this end, we invoke an extra axiom based on the following definition. A tree TU-game augmented by a discount rate $(N, v, L, \delta) \in \mathcal{C}$ is called *point unanimous* if there is $a \in \mathbb{R}$ such that $v = a1_N$. This condition means that a coalition enjoys a zero if it does not contain all the (connected) agents.

Communication ability If $(N, v, L, \delta) \in \mathcal{C}$ is point unanimous, then exists $c \in \mathbb{R}$ such that $\Phi_i(N, v, L, \delta) = c$ for each $i \in N$.

Proposition 6 *The average discounted tree solution Ψ defined as:*

$$\forall (N, v, L, \delta) \in \mathcal{C}, \quad \Psi(N, v, L, \delta) = \sum_{r \in N} n^{-1} h^r(N, v, L, \delta), \tag{16}$$

is the unique solution on $\mathcal{C}$ satisfying linearity, efficiency, the axiom of δ-reducing agent, and communication ability.

Proof. By (16) and Proposition 5, the average discounted tree solution satisfies linearity and δ-reducing agent. By (3), each discounted hierarchical outcome satisfies efficiency, so the average of the discounted hierarchical outcomes satisfies this axiom as well. Communication ability is the direct consequence of the linearity of the average discounted tree solution and the fact that, for each $(N, 1_N, L, \delta) \in \mathcal{C}$, $h^r(N, 1_N, \delta) = e^r$ as noted in the proof of Proposition 5.
Assume now that a solution on $\mathcal{C}$ satisfies linearity, efficiency, communication ability and the axiom of δ-reducing agent. Since efficiency implies proper cone efficiency, Proposition 5 implies that this solution is a discounted tree solution Ψ. Thus, by (11) we have:

$$\forall (N, v, L, \delta) \in \mathcal{C}, \quad \Psi(N, v, L, \delta) = \sum_{r \in N} \Psi_r(N, 1_N, L, \delta) h_r(N, v, L).$$

By communication ability, there is $c_{(N,L),\delta} \in \mathbb{R}$ such that:

$$\forall r \in N, \quad \Psi_r(N, 1_N, L, \delta) = c_{(N,L),\delta},$$

and by efficiency,

$$\sum_{r \in N} \Psi_r(N, 1_N, L, \delta) = 1,$$

which forces $c_{(N,L),\delta} = n^{-1}$, as desired. ■

Proposition 6 calls upon some comments. Firstly, Proposition 6 characterizes the average tree solution and the equal division in the extreme cases where $\delta = 1$ and $\delta = 0$, respectively. As a consequence, parameter δ allows for a trade-off between marginalism and egalitarianism. Secondly, if $\delta = 0$, then the underlying graph has no impact in the allocation process, while it is not the case when $\delta = 1$, which means that parameter δ also allows for taking into account the communication structure among the agents in several different ways.

Since efficiency implies proper cone efficiency, the logical independence of the axioms invoked in Propositions 5 and 6 can be demonstrated by exhibiting the following solutions:

- For any $(N, v, L, \delta) \in \mathcal{C}$, denote by $n(v, \delta)$ the number of agents who are not δ-reducing in (N, v, L, δ). The solution Φ on $\mathcal{C}$ which assigns to each $(N, v, L, \delta) \in \mathcal{C}$ a payoff $\Phi_i(N, v, L, \delta) = 0$ if i is a δ-reducing agent and a payoff $\Phi_i(N, v, L, \delta) = v(N)/n(v, \delta)$ otherwise satisfies efficiency, the axiom of δ-reducing agent and communication ability, but violates linearity. To see why this solution satisfies communication ability, note that if (N, v, L, δ) is point unanimous, then there exists at least one δ-reducing agent only if $v(N) = 0$.

- The solution Φ on $\mathcal{C}$ which assigns to each $(N, v, L, \delta) \in \mathcal{C}$ and each $i \in N$ a payoff $\Phi_i(N, v, L, \delta) = h_i^i(N, v, L, \delta)$ satisfies linearity, the axiom of δ-reducing agent and communication ability (every agent obtains $v(N)$ in case (N, v, L, δ) is point unanimous), but violates both efficiency and proper cone efficiency.

- The solution Φ on $\mathcal{C}$ which assigns to each $(N, v, L, \delta) \in \mathcal{C}$ and each $i \in N$ a payoff $\Phi_i(N, v, L, \delta) = v(\{i\}) + (v(N) - \sum_{j \in N} v(\{j\})/n$ satisfies linearity, efficiency and communication ability, but violates the axiom of δ-reducing agent for all $\delta \in [0, 1]$.

- Any discounted tree solution different from the average discounted tree solution satisfies linearity, efficiency and the axiom of δ-reducing agent, but violates communication ability.

Acknowledgments

Financial support by the National Agency for Research (ANR) – research program "DynaMITE: Dynamic Matching and Interactions: Theory and Experiments", contract ANR-13-BSHS1-0010 – and the "Mathématiques de la décision pour l'ingénierie physique et sociale" (MODMAD) project is gratefully acknowledged.

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