Representation of Convex Geometries by Convex Structures on a Plane

Weak Carousel rule

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Abstract

Convex geometries are closure systems satisfying anti-exchange axiom with combinatorial properties. Every convex geometry is represented by a convex geometry of points in n-dimensional space with a special closure operator. Some convex geometries are represented by circles on a plane. This paper proves that not all convex geometries are represented by circles on a plane by providing a counterexample. We introduce Weak n-Carousel rule and prove that it holds for configurations of circles on a plane.

1. Introduction

Convex geometries were studied from different perspectives and under different names since 1930. In particular, the theory of convex geometries was brought to attention by Edelman and Jamison's survey paper in 1985 ([7]). Convex geometries are interesting combinatorial objects which generalize a notion of convexity on Euclidean plane. There are many structures which share their properties. Some of them include convex objects in Euclidean space, convex sets in posets, subsemilattices in a semilattice, path-closed subgraphs of a graph ([7]). It is crucial to understand the driving example well and find structural connections between the general example and other convex geometries. The concept of convex geometries was generated by a geometrical example: a set of points on a plane with a special closure operator. This convex geometry is called *affine* convex geometry. It was proved in [8] that every convex geometry is a sub-geometry of some affine convex geometry in $\mathbb{R}^n$. This result answers a representation problem of convex geometries and proves an existence of some dimension of a space for a representation. Representation problem of convex geometries ([3]):

Problem 1.1. *To find a universal class C of finite convex geometries s.t. every convex geometry is a sub-geometry of some geometry in C*

We are interested in finding the smallest dimension of a space for a representation. Adaricheva in [1] proved that there is a property, namely n-Carousel rule, which restricts convex geometries not satisfying it from being embedded in a space of a particular dimension. Carousel rule was introduced in [4]. Using the result from [1], we know that for a particular dimension of a space, there are convex geometries that could not be represented by points in this space. Czedli in [5] modifies a construction and uses circles on a plane for a representation. It was shown in [5] that every convex geometry of convex dimension cdim=2 could be represented by circles on a plane. Convex dimension is a parameter associated with a convex geometry, a minimal number of chains needed to realize an alignment for a convex geometry (see [7] for more details about convex dimension). There is a natural question following from this result concerning whether every convex geometry could be represented by circles on a plane ([6]). This paper disproves this conjecture by providing an example of convex geometry which is not represented by circles.

We are introducing Weak n-Carousel rule (2.9) which is a weakening of existing n-Carousel rule. We prove that a geometry of circles on a plane satisfies 2-Weak Carousel rule. We borrow example from [1] which fails not only 2-Carousel rule but Weak 2-Carousel rule as well. Therefore, it is not represented by circles.

To prove that a geometry of circles satisfies 2-Weak Carousel rule, we start with proving Weak Carousel property for triangles, which is a slight simplification of Weak 2-Carousel rule. We consider two circles inside a triangle and model their location by their projections on sides of a triangle. Thus, we transform a problem from considering positions of circles to looking at configurations of segments. We find that there are 216 possible configurations of segments which we reduce to 38 cases up to isomorphism. We dismiss some of these cases by proving that a location of circles with these projections is not realizable. For this, we prove a number of lemmas. We show that all other cases are possible for realization and in all these cases the property holds.

Section 2 contains main concepts and definitions. Section 3 describes existing representations of convex geometries. Example of affine convex geometry that fails Weak 2-Carousel rule is described in Section 4. Weak Carousel property for triangles is proved in Section 5. The proof of Weak Carousel property for triangles requires a number of geometrical results which are stated and proved in Section 6. Weak 2-Carousel rule for circles is proved in Section 7.

2. Main concepts and definitions

Definition 2.1. Given any set X, a closure operator on X is a mapping $\varphi : 2^X \to 2^X$ with the following properties:
1) $Y \subseteq \varphi(Y)$ for every $Y \subseteq X$;
2) If $Y \subseteq Z$, then $\varphi(Y) \subseteq \varphi(Z)$ for $Y, Z \subseteq X$;
3) $\varphi(\varphi(Y)) = \varphi(Y)$ for $Y \subseteq X$.

Definition 2.2. Pair (X, φ) is called a convex geometry if φ is a closure operator on X with additional properties:
1) $\varphi(\emptyset) = \emptyset$;
2) if $Y = \varphi(Y)$ and $x, z \notin Y$, then $z \in \varphi(Y \cup x)$ implies that $x \notin \varphi(Y \cup z)$ (Anti-exchange property).

We could alternatively associate convex geometries with an alignment-a special family of subsets of a base set.

Definition 2.3. Given any (finite) set X, an *alignment* on X is a family $\mathcal{F}$ of subsets of X which satisfies two properties:
1) $X \in \mathcal{F}$;
2) If $Y, Z \in \mathcal{F}$, then $Y \cap Z \in \mathcal{F}$

The following relationships between a closure operator and an alignment could be easily checked ([7]):

Proposition 2.4. *If φ is a closure operator on a set X, then $\mathcal{F} = \{Y : \varphi(Y) = Y, Y \subseteq X\}$ is an alignment on X.*

Proposition 2.5. *Let $\mathcal{F}$ be an alignment on a set X. Define $\varphi(Y) = \cap\{Z \in \mathcal{F}, Y \subseteq Z\}$ for every $Y \subseteq X$. Then, φ is a closure operator on X.*

Convex geometries could be defined equivalently through an alignment:

Definition 2.6. Pair $(X, \mathcal{F})$ is called a *convex geometry* if $\mathcal{F}$ is an alignment on X with additional properties:
1) $\emptyset \in \mathcal{F}$;
2) if $Y \in \mathcal{F}$ and $Y \neq X$, then $\exists a \in X \setminus Y$ s.t. $Y \cup \{a\} \in \mathcal{F}$

Definition 2.7. Convex geometry $G_1 = (X, \mathcal{F}_1)$ is a sub-geometry of $G_2 = (Y, \mathcal{F}_2)$ if there is a one-to-one map $f : \mathcal{F}_1 \to \mathcal{F}_2$ s.t.
1) $f(A \wedge B) = f(A) \wedge f(B)$, $A, B \subseteq X$;
2) $f(A \vee B) = f(A) \vee f(B)$, $A, B \subseteq X$.

Definition 2.8. A convex geometry (A, φ) satisfies *n-Carousel rule* if $x, y \in \varphi(S), S \subseteq A$, implies $x \in \varphi\{y, a_1 ... a_n\}$ for some $a_1, ... a_n \in S$.

Definition 2.9. A convex geometry (A, φ) satisfies *Weak n-Carousel rule* if $x, y \in \varphi(S), S \subseteq A$, implies either $x \in \varphi\{y, a_1 ... a_n\}$ or $y \in \varphi\{x, a_1 ... a_n\}$ for some $a_1, ... a_n \in S$.

Definition 2.10. Consider $F = (X, ch_c)$, where X is a set of circles in $\mathbb{R}^2$ and ch_c is defined as follows: $ch_c(Y) = \{z \in X : \tilde{z} \subseteq CH(\cup \tilde{y}), y \in Y\}$ for $Y \subseteq X$, where CH is a usual convex hull operator and $\tilde{y}$ is a set of points in y ($\tilde{z}$ is a set of points in z, respectively). We call F a *geometry of circles on a plane.*

Weak n-Carousel rule defined in 2.9, now could be reformulated for a convex geometry of circles on a plane.

Definition 2.11. Consider $F = (A, ch_c)$ a convex geometry of circles on a plane with $A = \{a, b, c, x, y\}$. Then F satisfies *Weak 2-Carousel rule* if $x, y \subseteq ch_c(\{a, b, c\})$ implies either $x \in ch_c(y, i, j)$ or $y \in ch_c(x, i, j)$, where $i, j \in S$

To prove Weak 2-Carousel rule for circles, we simplify the rule and consider two circles and three points. We slightly modify the rule to have a geometrical formulation.

Definition 2.12. A configuration of two circles x and y and a set S of distinct points A, B, C is said to satisfy *Weak Carousel Property for Triangles* if x, y is in a triangle $\triangle ABC$ implies either
x is in a convex hull of y and any two points from S, or
y is in a convex hull of x and any two points from S.

3. Known representations of convex geometries

We say that a convex geometry is represented by another convex geometry when it is a sub-geometry of the second one, i.e. there exists a one-to-one mapping from alignment of the first geometry to alignment of the second geometry preserving operations of meet ($\wedge$) and join ($\vee$) (See Definition 2.7).

There are two types of representation. First, there is so called *Weak Representation*. If we want to represent a convex geometry by a particular class of convex geometries, we want to find a mapping from that convex geometry into some representative of the given class that is one-to-one and preserves closure operator (i.e. satisfies the condition in **??**). Second, there is *Strong Representation*. In this type of representation, a mapping must additionally be onto mapping.

The first universal class for a weak representation of convex geometries was presented in [3] by Adaricheva et al in 2003. The model for a representation was a class of infinite convex geometries defined on a Boolean structure of all subsets of some set X. Convex geometries from a universal class were of the form $(2^X, \varphi)$, where φ corresponds to an alignment comprising all algebraic subsets of 2^X. It was proved that every finite convex geometry could be represented by an infinite convex geometry in the form of a lattice of algebraic subsets of algebraic lattices. It is a nontrivial construction which appears in many fundamental concepts. However, structures from the representation class must be infinite. Although the result provides a representation for all convex geometries, the representation could not be done only for finite X. Therefore, the question of finding representation by a class of finite convex geometries was raised. Problem 1.1 asks whether it is possible to find a universal class of finite convex geometries for a representation of convex geometries.

In 2005 Kashiwabara et al. proved that every finite convex geometry could be represented as a sub-geometry of a finite convex geometry ([8]). Universal class for this weak representation was in the form of affine convex geometries.

Definition 3.1. *Affine convex geometry* is a convex geometry $C_0(\mathbb{R}^n, X) = (X, ch)$, where X is a set of points in $\mathbb{R}^n$ and ch is defined as follows: for $Y \subseteq X, ch(Y) = CH(Y) \cap X$, where CH is a usual convex hull operator.

Every finite convex geometry could be embedded in some affine convex geometry in $C_0(\mathbb{R}^n, X)$. Nevertheless, this representation may require a high dimension of a space. Therefore, it is worth investigating the smallest possible dimension for a representation.

Adaricheva in [2] showed that some special convex geometries have this type of representation but with a lower dimension of a space. Refer to [2] for details.

Apart from finding the smallest dimension of a space for a weak representation, a question of strong representation is also of a great interest in representations of convex geometries. A question of which convex geometries could have a strong representation by affine convex geometries in $\mathbb{R}^n$ was raised by Edelman and Jamison in 1985 (See Problem 1 in [7]). The authors in [7] noted that such characterization problem is considerably difficult. There is a partial result by Adaricheva and Wild in 2010 for a strong representation of convex geometries by affine convex geometries in $\mathbb{R}^2$ ([4]). Edelman and Jamison provide a representation of convex geometries by compatible orderings (See [7] for details). There is a recent result by Richter and Rogers in 2015 that provides a nice geometrical visualization of the representation by orderings ([9]). It was proved that every convex geometry could be represented by a special location of polygons on a plane ([9]). It was shown in [9] that every finite convex geometry with a convex dimension $cdim = k$ (a minimal number of chains needed to realize an alignment for a convex geometry) could be represented as k-gons on a plane. A geometry of k-gons with a special location and a closure operator was proved to be a convex geometry (See [9] for more details). Although this representation is a nice illustration of a strong representation, polygons may be non-convex, while we are interested in preserving the nature of convexity.

Czedli in 2014 generalizes a construction and instead of geometry of points defines a geometry of circles (2.10) and considers this geometry for a representation ([5]). It was proved in [5], that geometry of circles on a plane is a convex geometry. By working with a geometry of circles, Czedli relaxes the construction of a geometry on a set of points because geometry of circles is not atomistic.

Definition 3.2. A convex geometry $G = (X, \varphi)$ is called *atomistic* if $\varphi(\{x\}) = \{x\}$ for every $x \in X$

Affine convex geometries are atomistic since for any affine convex geometry $G = (X, ch)$, any $x \in X$ is just a point in $\mathbb{R}^n$ and $ch(\{x\}) = \{x\}$. Geometry of circles on a plane is not in general atomistic. For a convex geometry of circles $G = (X, ch_c)$, it is possible that $ch_c(\{x\}) = \{x, y\}$ for $x, y \in X$ that describes a case when a circle y is inside a circle x.

If we consider affine convex geometries as a class for a representation of convex geometries, then we could strongly represent only atomistic convex geometries. But since convex geometries are not in general atomistic, we do not have a strong representation by affine convex geometries for all convex geometries. Therefore, a construction of a convex geometry by circles on a plane in [5] could be investigated for a possibility of strong representation.

The main result of [5] shows that every convex geometry of convex dimension $cdim = 2$ could be strongly represented by a geometry of circles on a plane. With a relaxed construction, now there is a natural question whether any convex geometry has a strong representation by a geometry of circles on a plane.

In this paper, we disprove this conjecture by showing an example of convex geometry that is not represented by a convex geometry of circles on a plane. We show that all convex geometries of circles on a plane must satisfy Weak 2-Carousel Rule. We demonstrate an example of convex geometry that fails Weak 2-Carousel Rule. Therefore, this convex geometry could not be represented by a geometry of circles on a plane.

We prove that a strong representation of convex geometries by geometries of circles on a plane is not possible in general. Therefore, it is necessary to consider higher dimensions of a space. This leads to the following problem which was formulated by Adaricheva in [6]:

Problem 3.3. *Is every finite convex geometry could be represented by a convex geometry of balls in* $\mathbb{R}^n$

4. Example of convex geometry that is not representable by circles

We borrow examples of convex geometries in this section from [1].

Consider an example of affine convex geometry $G' = (X, \mathcal{F}')$, where $X = \{a_0, a_1, a_2, x, y\}$ is a set of points on a plane as shown in Figure 1. Then, $\mathcal{F}' = \mathcal{P}(X) \backslash \{a_0a_1a_2, a_0a_2x, a_0a_1y, a_0a_1a_2x, a_0a_1a_2y\}$.

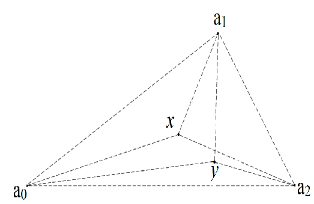

Figure 1

Consider $G = (X, \mathcal{F})$, where $\mathcal{F} = \mathcal{P}(X) \backslash \{a_0a_1a_2, a_0a_1a_2x, a_0a_1a_2y\}$, which we get by adding $\{a_0a_2x\}$ and $\{a_0a_1y\}$ to the alignment $\mathcal{F}'$ of affine convex geometry G'.

It could be directly checked that $\mathcal{F}$ for G satisfies the properties of alignments from Definition 2.3. Initial alignment $\mathcal{F}'$ satisfies the required properties since G' is affine convex geometry. We add $\{a_0a_2x\}$ and $\{a_0a_1y\}$ to $\mathcal{F}'$. All subsets of added sets are in $\mathcal{F}$ and their intersection is also in $\mathcal{F}$. So, G is indeed a convex geometry ([1]).

From [1], $G = (X, \mathcal{F})$ does not satisfy 2-Carousel rule. It was shown in [1] that convex geometries failing n-Carousel rule could not be weakly represented by affine convex geometries in $\mathbb{R}^n$. So, G could not be weakly represented by affine convex geometries in $\mathbb{R}^2$. Now, we show that G does not also satisfy Weak 2-Carousel rule.

Let $\varphi : X \to X$ be a corresponding closure operator to the alignment F of G.

Then, $x \in \varphi(a_0a_1a_2)$ since $\varphi(a_0a_1a_2) = X$.
$\varphi(a_ia_jx) = \{a_ia_jx\}$, for $i, j = 0, 1, 2$
$\varphi(a_ia_jy) = \{a_ia_jy\}$, for $i, j = 0, 1, 2$

Hence, G does not satisfy Weak 2-Carousel rule since x and y are in a closure of $\{a_0, a_1, a_2\}$ but x is not in a closure of any two from $\{a_0, a_1, a_2\}$ and y. Moreover, y is not in a closure of any two from $\{a_0, a_1, a_2\}$ and x.

We prove in Section 7 that a geometry of circles satisfies Weak 2-Carousel rule (Theorem 7.1). Since G does not satisfy Weak 2-Carousel rule, G could not be represented by circles on a plane.

5. Weak Carousel Property for Triangles

Theorem 5.1. *Every configuration of two circles and a set of three distinct points in $\mathbb{R}^2$ satisfies Weak Carousel Property for Triangles.*

Proof. Consider two circles x and y in a triangle $\triangle ABC$. Circles x and y are projected on segments on sides of $\triangle ABC$. In the example in Figure 1, we are using notation, say, x_B^{BC} and x_C^{BC}, for edges of a projection of a circle x on BC which are closest to B and C respectively.

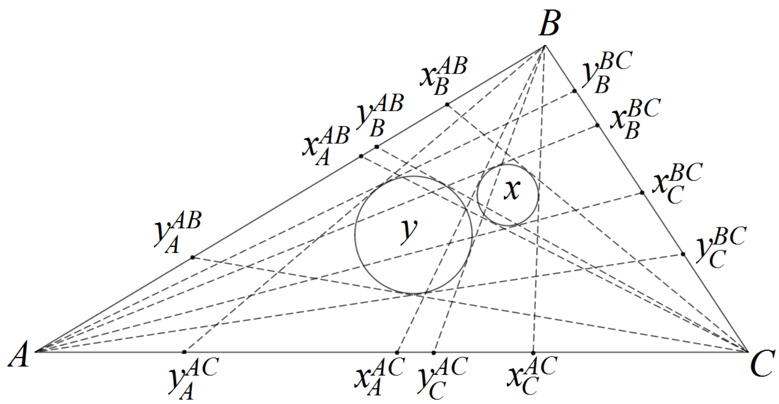

Figure 2

There are six possible configurations of projections of x and y on each side of a triangle. We assign a number i, that takes integer values from 1 to 6, to each configuration of segments on one side of $\triangle ABC$. We illustrate configurations corresponding to values of i in Figure 18 in Appendix A. We assume that x_1 and y_1 in Figure 18 are first in orders x_1x_2 and in y_1y_2 respectively in a clockwise walk around $\triangle ABC$, assuming that ABC is in a clockwise order.
We denote a configuration of projections of two circles inside a $\triangle ABC$ by C_{jkl}, where j, k, l are integer numbers that take values of i and denote a configuration of segments on sides AB, BC, AC respectively. Say, example from Figure 2 is C_{546}.

Six possible configurations for each side produce 216 possible configurations of projections of x and y for $\triangle ABC$. We reduce 216 configurations to 38 classes up to isomorphism. Say, C_{241} and C_{124} are isomorphic cases since they represent the same triangle with sides rotated clockwise. For every class S_n,where n is an number that takes integer values from 1 to 38, we associate configurations C_{jkl} out of 216 configurations that are isomorphic. Refer to Appendix B) for a full list of grouping 216 configurations in 38 classes.

Definition 5.2. We say a projection is *later (earlier)* than another, if in a clockwise walk around $\triangle ABC$ (assuming that ABC is a clockwise order of vertexes) it is second (first) in order on a side of $\triangle ABC$.

In the example in Figure 1, x_B^{BC} is later than y_B^{BC} and x_C^{AC} is earlier than y_A^{AC}. It is clear that, say, for x to be in a convex hull of y and B, C, x_A^{AC} must be earlier than y_A^{AC} and x_A^{AB} must be later than y_A^{AB}. On the other hand, x is not in a convex hull of A, B and y, since both x_C^{BC} and x_C^{AC} are earlier than y_C^{BC} and y_C^{AC} respectively.

In the proofs below, we will not distinguish whhether these projections are disjoint or overlapping.

1) Consider a class S_2 when in a clockwise walk around $\triangle ABC$, projections of circles x and y are in the same order, say, projections of y are "strictly later" than projections of x (for example, y_A^{AC} is later than x_A^{AC}, and y_C^{AC} is later than x_C^{AC} on AC). Figure 4 illustrates a configuration C_{222} as an example from a class S_2. Take tangent lines to x, connecting the first edge point of each x-projection, which we meet when we walk around $\triangle ABC$ clockwise, with the opposite vertex of $\triangle ABC$ (See Figure 3). Then circle x is inscribed into $\triangle A'B'C'$ formed by these three lines. It follows from assumption that y is inside $\triangle A'B'C'$. Moreover, y should have points in each of three disjoint areas of $\triangle A'B'C'$, whose union is $\triangle ABC \backslash x$. Then due to Lemma 6.4, the case is dismissed as impossible for realization.

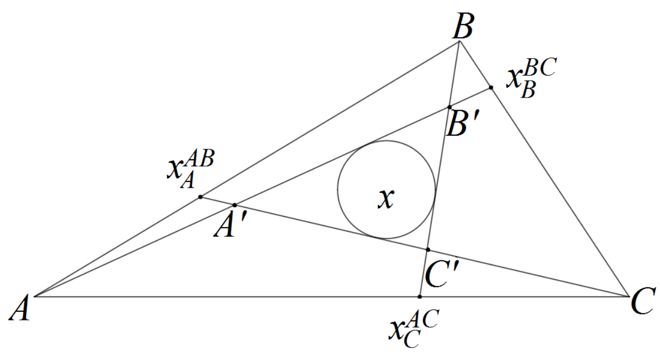

FIGURE 3

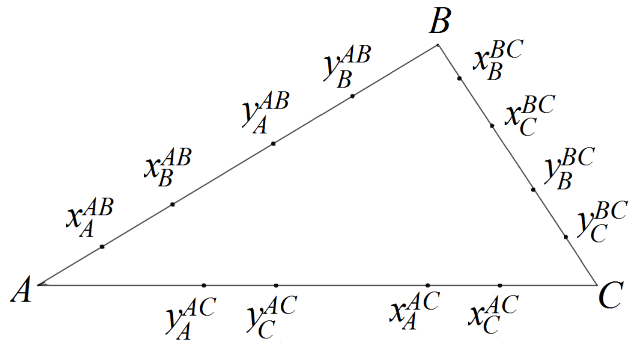

FIGURE 4

Configurations from classes S_9, S_{25}, S_{38} are dismissed using similar argument.

2) Consider a class S_{13} when a segment of a projection of y is inside a segment of a projection of x on one side of $\triangle ABC$ but a projection of x is inside a projection of y on another side of $\triangle ABC$. Say, $y_B^{BC} y_C^{BC}$ is inside $x_B^{BC} x_C^{BC}$ and $x_A^{AC} x_C^{AC}$ is inside $y_A^{AC} y_C^{AC}$. Figure 6 illustrates a configuration C_{234} as an example from class S_{13}. Take tangent lines to x, connecting A with x_B^{BC} and x_C^{BC} (See Figure 5). It follows from assumption that y is inside $\triangle A x_B^{BC} x_C^{BC}$. Moreover, y should have points in each of two disjoint areas of $\triangle A x_B^{BC} x_C^{BC}$, whose union is $\triangle A x_B^{BC} x_C^{BC} \setminus x$. Then due to Lemma 6.5, the case is dismissed as impossible for realization.

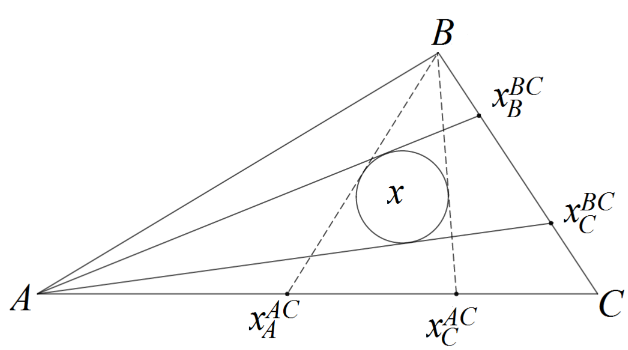

FIGURE 5

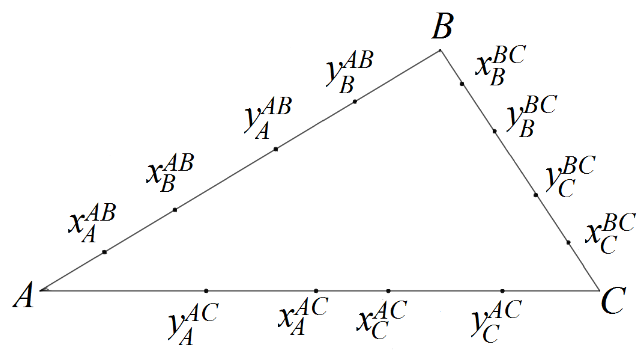

FIGURE 6

Configurations from classes $S_5, S_{12}, S_{28}, S_{31}, S_{32}$ are dismissed using similar argument.

3) Consider a class S_{15}. Figure 8 illustrates a configuration C_{135} as an example from class S_{15}. Take tangent lines to x, connecting A with x_B^{BC} and x_C^{BC} (See Figure 7). It follows from assumption that y should have points in $\triangle x_C^{AC}BC$ and $\triangle Ax_A^{AB}C$. Moreover, y is inside $\triangle Ax_B^{BC}x_C^{BC}$. So, y should have points in each of two disjoint areas of $\triangle Ax_B^{BC}x_C^{BC}$, whose union is $\triangle Ax_B^{BC}x_C^{BC}\backslash x$. Then due to Lemma 6.5, the case is dismissed as impossible for realization.

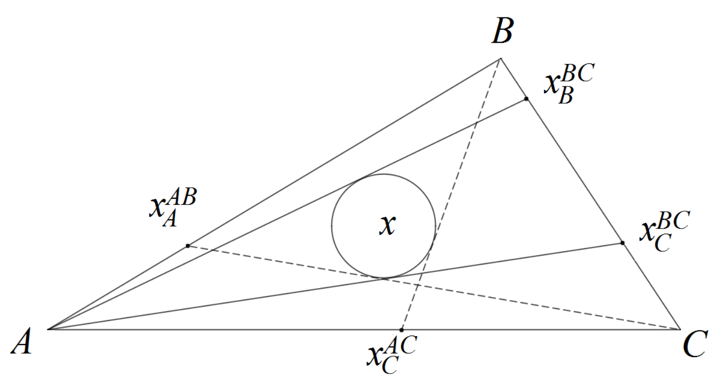

FIGURE 7

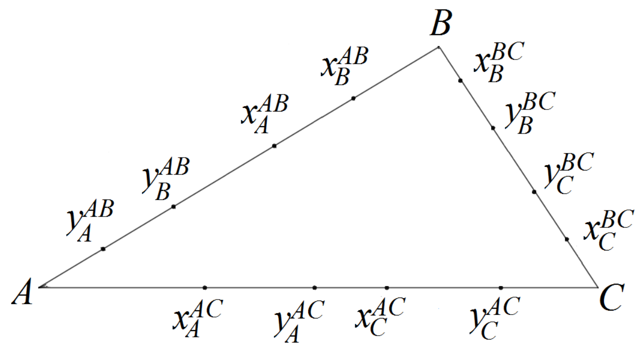

FIGURE 8

Configurations from classes $S_{17}, S_{21}, S_{22}, S_{34}, S_{35}$ are dismissed using similar argument.

4) Consider a class S_{10}. Figure 10 illustrates a configuration C_{225} as an example from class S_{10}. Take tangent lines to x, connecting B with x_A^{AC} and x_A^{AC}, A with x_C^{BC}, C with x_B^{AB} (See Figure 9). Let Ax_C^{BC} and Cx_B^{AB} intersect in a point O. It follows from assumption that y should be in $\triangle COx_B^{BC}$. Moreover, y should have a point in $\triangle x_A^{AC}Bx_C^{AC}$. Then, since O is not in $\triangle x_A^{AC}Bx_C^{AC}$, due to Lemma 6.6 the case is dismissed as impossible for realization.

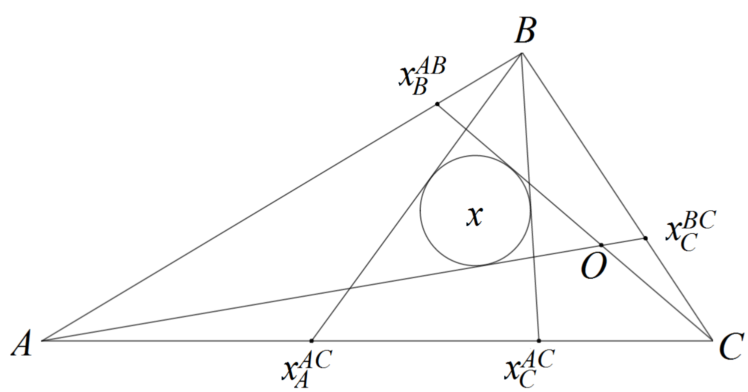

FIGURE 9

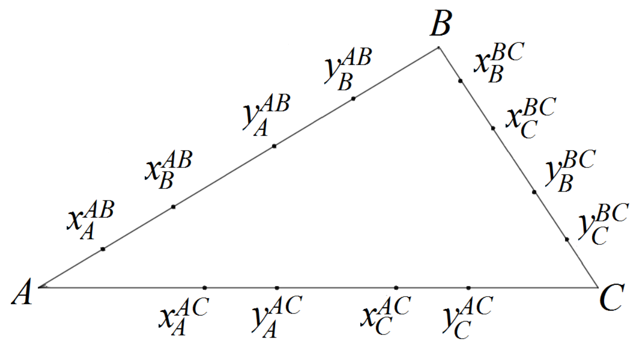

FIGURE 10

Configurations from a class S_6 is dismissed using similar argument.

All other configurations satisfy Weak Carousel property. We illustrate realizations of non-dismissed configurations by representative from each of classes $S_1, S_3, S_7, S_{11}, S_{16}, S_{19}, S_{23}, S_{27}, S_{29}, S_{33}, S_{37}$ in Appendix C. Configurations from classes S_7 and S_8 are symmetric , so we could consider $\{S_7, S_8\}$ as one symmetric class. Similarly, $\{S_3, S_4\}$, $\{S_{11}, S_{14}\}, \{S_{16}, S_{18}\}, \{S_{19}, S_{20}\}, \{S_{23}, S_{24}, S_{26}\}, \{S_{29}, S_{30}\}, \{S_{33}, S_{36}\}$ are symmetric classes.

Thus, cases that are not dismissed are all realizable and Weak Carousel property for Triangles holds in all of them.

□

6. LEMMAS

Definition 6.1. Let *a projection point* associated with a circle inside a triangle be an endpoint of a perpendicular from a center of the circle to a side of the triangle.

Consider arbitrary triangle $\triangle ABC$ on a plane. Let $g(r_g, O_g)$ be a circle inscribed in $\triangle ABC$ with a center O_g and a radius r_g. Let G_1, G_2 and G_3 be projection points of x on AB, AC and BC respectively.

Lemma 6.2. *If M_1 is a point on AG_1, then the largest circle associated with M_1 and that is completely inside $\triangle ABC$ is the circle inscribed in the angle $\angle BAC$ with a projection point M_1.*

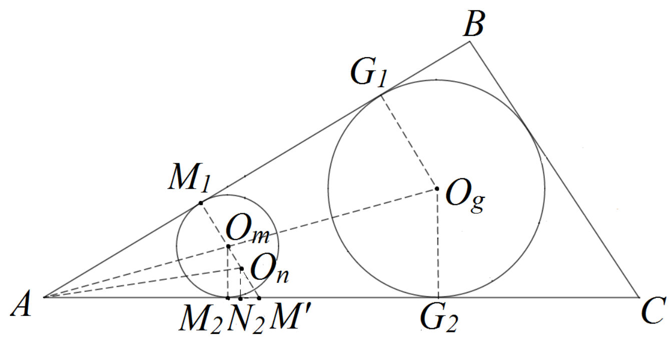

FIGURE 11

Proof. Let $M' \in AC$ s.t. $M_1M' \perp AB$. Centers of circles associated with a projection point M_1 are on M_1M'. Let $m(r_m, O_m)$ be a circle with a projection point M_1 inscribed in $\angle BAC$.

Suppose there exists a circle $n(r_n, O_n)$ with a projection point M_1 that is completely inside $\triangle ABC$ and $r_n > r_m$. Then O_n is in O_mM'. Let N_2 be a projection point of n on AC.

Let $\angle BAC = 2\alpha$. Then $\angle BAO_m = \angle CAO_m = \alpha$. Let $\angle O_mAO_n = \beta$. Then $AO_n = M_1O_n/sin(\alpha+\beta) = O_nN_2/sin(\alpha-\beta)$. $M_1O_n = O_nN_2 = r_n \Rightarrow sin(\alpha+\beta) = sin(\alpha-\beta) \Rightarrow \beta = 0$. Hence, $O_m = O_n$. But $r_n > r_m$ implies that n is not completely inside $\triangle ABC$ which contradicts our assumption about n. Therefore, there exists no such circle n.

Thus, m is the largest circle that is completely inside $\triangle ABC$ associated with the projection point M_1. □

Lemma 6.3. *Consider arbitrary triangle $\triangle ABC$ on the plane. Let Dr be the disc defined by inscribed circle with center O, whose touch point $G_1 \in AB$ and touch point $G_2 \in AC$. Consider point $E \in \triangle ABC$ that belongs to quadrilateral AG_1OG_2, but lays outside disk Dr. Let Dr^* be any disk with center F such that $Dr^* \subseteq \triangle ABC$ and $E \in Dr^*$. Then projection of F to AB belongs to AG_1 and projection to AC belongs to AG_2.*

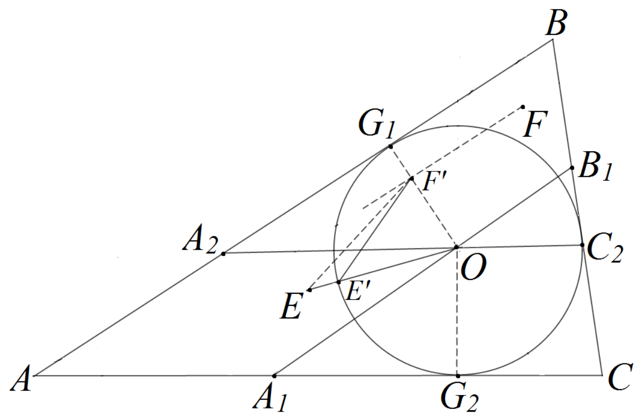

FIGURE 12

Proof. Suppose that radius of Dr is r. Then the radius of any disk $Dr^* \subseteq \triangle ABC$, distinct from Dr, is $r_1 < r$. We want to show that, for any point F outside quadrilateral AG_1OG_2, the distance $|FE|$ is either greater than r, or greater than r_1, which is the distance from F to one of the sides of triangle, hence, the disk with center at F containing point E will not be inside $\triangle ABC$. Thus, the only possible position for F will be inside quadrilateral AG_1OG_2, hence, the projections of F are as required.

Draw line (A_1B_1) parallel to (AB) and line (A_2C_2) parallel to (AC) through point O. If point $F \in \triangle ABC$ is outside quadrilateral AG_1OG_2, then it is either in OG_1BB_1, in OG_2CC_2, or in triangle $\triangle OB_1C_2$.

For any point $F \in OB_1C_2$ and any point E taken in smaller angle G_1OG_2, in particular, in quadrilateral AG_1OG_2, $\angle EOF > \pi/2$. Hence, by cosine theorem, $|FE| > |OE| > r$.

For two other possibilities of location of F, the argument is similar, so we consider only one. Let $F \in OG_1BB_1$. Take point $F' \in OG_1$, whose distance to line (AB) is the same as for F: in other words, line (FF') is parallel to (AB), and $|F'G_1| \leq r$. Then $\angle EF'F > \pi/2$, hence, $|FE| > |F'E|$. Connect a with O and take $E' \in EO$, which belongs to circle of disk Dr. Then $\angle EE'F' > \pi/2$, hence, $|F'E| > |F'E'|$.

So, it remains to show that $|F'E'| \geq |F'G_1| = r_1$, the latter being the radius of the largest circle centered at F', which is inside $\triangle ABC$. This would imply that any circle centered at F with radius $> r_1$ will not be inside triangle, while $|FE| > r_1$.

Cases $F' = O$ and $F' = G_1$ are obvious, so we assume $F' \in OG_1$ and $0 < r_1 < r$.

Denote $\alpha = \angle F'OE'$ and use cosine theorem:
$|F'E'|^2 = |OE'|^2 + |OF'|^2 - 2|OE'||OF'|\cos\alpha = r^2 + (r - r_1)^2 - 2r(r - r_1)\cos\alpha$
$= 2r(r - r_1)(1 - \cos\alpha) + r_1^2.$

Define $f(r_1) = |F'E'|^2 - r_1^2$, we need to show that $f(r_1) \geq 0$. Indeed, $f(r_1) = 2r(r - r_1)(1 - \cos\alpha) \geq 0$, and we are done.

□

Lemma 6.4. *Consider $w = \triangle ABC \backslash g = w_A \cup w_B \cup w_C$, the complement of g in $\triangle ABC$, which is disjoint union of three areas (See Figure 13). Then any circle y that contains a point in each of areas w_A, w_B, w_C, is not completely inside $\triangle ABC$.*

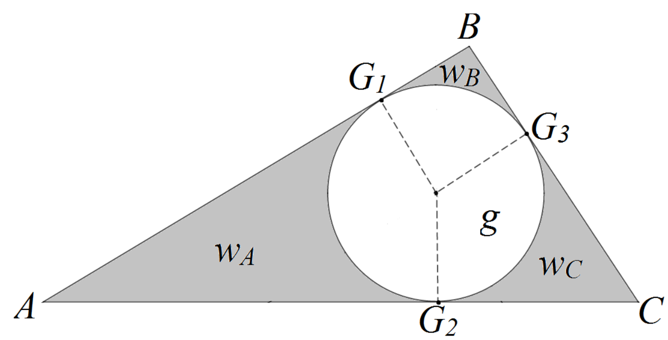

FIGURE 13

Proof. By 6.3, a position of projection points of circles that contain a point in w_A is restricted to AG_1 and AG_2. Similarly, in w_B to BG_1, BG_3 and in w_C to CG_2, CG_3. Suppose there exists a circle y that contains points in each of areas w_A, w_B, w_C and completely inside $\triangle ABC$. Then y has projection points to AB in AG_1, BG_1, to BC in BG_3, CG_3 abd to AC in BC, AG_2, CG_2. Then, y has projection points G_1, G_2 and G_3. Then, y is centered in O_g and have points in w_A, w_B, w_C that are disjoint from g. But x is the largest circle that could be inside $\triangle ABC$. Therefore there exists no such circle y. □

Lemma 6.5. *Suppose $p(r_p, O_p)$ is a circle inscribed in $\angle BAC$ of a triangle $\triangle ABC$. Consider $w = \triangle ABC \backslash p = w_1 \cup w_2$, the complement of p in $\triangle ABC$, which is disjoint union of two areas (See Figure 14). Then any circle y that contains a point in each of areas w_1 and w_2 is not completely inside $\triangle ABC$.*

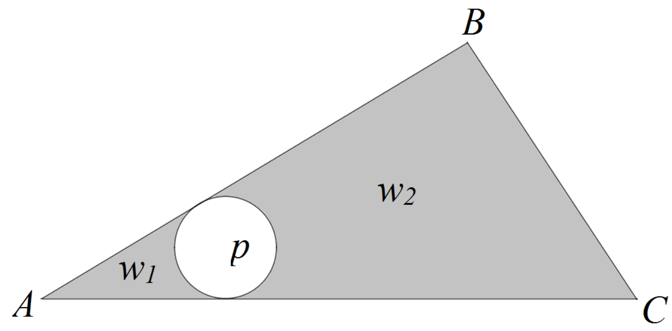

FIGURE 14

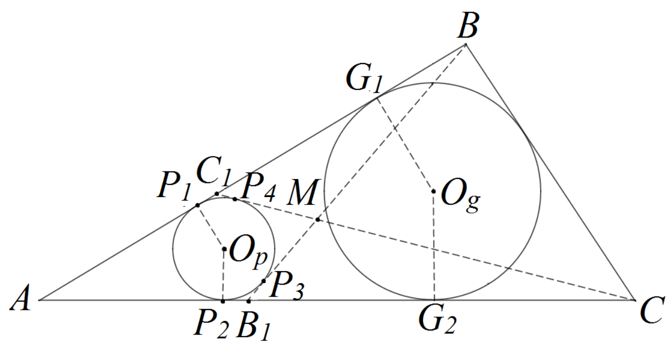

FIGURE 15

Proof. Let P_1 and P_2 be projection points of p to AB and AC respectively. Draw tangent lines CC_1 and BB_1 to p and let P_3 and P_4 be projection points of p to BB_1 and CC_1 respectively. Let M be a point of intersection of CC_1 and BB_1.

Suppose there exist a circle y that contains some points E_1 and E_2 from w_1 and w_2 respectively.

Case 1. E_2 is in $w_2 \cap \triangle ACC_1$.

Consider $\triangle ACC_1$: y contains E_1 from w_1, then due to 6.3, projection points of y to AB and AC are restricted to AP_1 and AP_2. If E_2 is in $C_1P_1P_4\backslash p$, then projection points of y to AB and CC_1 are restricted to C_1P_1 and C_1P_4. If E_2 is in $CP_2P_4\backslash p$, then projection points of y to AC and CC_1 are restricted to C_{P2} and C_{P4}. Then, from a similar argument made in 6.4, there exists no such circle y.

The argument is similar when E_2 is in $w_2 \cap \triangle ABB_1$.

Case 2. E_2 is in $\triangle BMC$.

Consider $g(r_g, O_g)$ that is inscribed in $\triangle ABC$. Then E_2 is either in $\triangle BMC \cap g$ or in area $\triangle BMC\backslash g$.

If E_2 is in area $\triangle BMC$ disjoint from g, then Then, from a similar

argument made in 6.4, there exists no such circle y.
If E_2 is in $\triangle BMC \cap g$, then y intersect p in more than two points. It contradicts to two circles intersecting in at most two points.

Therefore, there exists no such y.

□

Lemma 6.6. *Suppose s is a circle inscribed in $\angle A_1AA_2 < \pi$. Circle s divides $\angle A_1AA_2$ in two disjoint areas w_1 and w_2 (See Figure 16). Then tangent lines to any point of s bordering w_1 and to any point of s bordering w_2 intersect in a point outside the interior area of $\angle A_1AA_2$.*

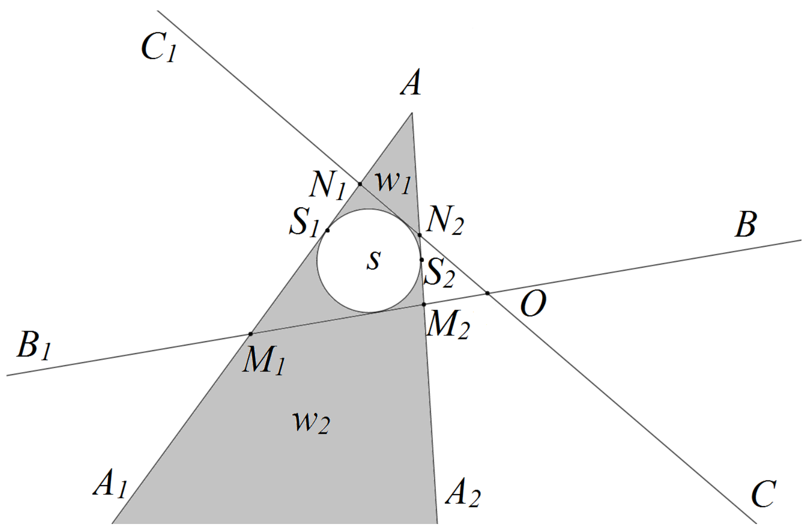

FIGURE 16

Proof. Let S_1 and S_2 be touch points of s with AA_1 and AA_2 respectively. Let P_1 be a point on s that coincides with S_1 at first. Let CC_1 be tangent line to s at P_1. If we move P_1 on part of s bordering w_1, then CC_1 moves from a position of AA_2 anticlockwise until it becomes AA_1.

Let P_2 be a point on s that coincides with S_2 at first. Let BB_1 be tangent line to s at P_2. If we move P_2 on part of s bordering w_2, then BB_1 moves from a position of AA_1 clockwise until it becomes AA_2.

Let points of intersections of CC_1 and BB_1, BB_1 and AA_1, BB_1 and AA_2, CC_1 and AA_1, CC_1 and AA_2 be O, M_1, M_2, N_1 and N_2 respectively. Then position of point O moves on the line BB_1 toward M_2 until CC_1 becomes AA_2. Similarly O moves on the line CC_1 toward N_2 until BB_1 becomes AA_2.

Then, O could not be inside interior area of $\angle A_1AA_2$.

□

7. Weak 2-Carousel rule for a geometry of circles on a plane

Theorem 7.1. *Every convex geometry of circles in $\mathbb{R}^2$ satisfies Weak 2-Carousel rule.*

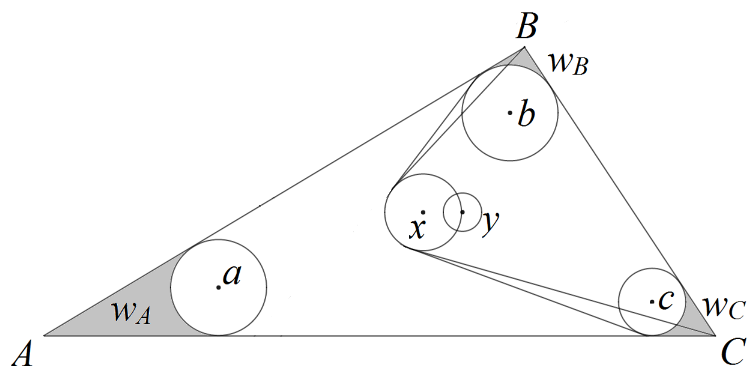

Figure 17

Proof. Consider circles x, y that are in a convex hull of circles a, b, c. be circles (See Figure 17). $x,\ y \in CH\{a, b, c\}$

Draw tangent lines to a, b, c and denote point of intersection by A, B and C (17). Let w_A, w_B, w_C denote disjoint areas as shown on the picture.

$CH\{a, b, c\} = CH\{A, B, C\} \backslash \{w_A \cup w_B \cup w_C\}$, so $x,\ y \in CH\{A, B, C\}$

We could use Theorem 5.1, for x, y, and a set of points $S = \{A, B, C\}$. By Theorem 5.1, either x is in a convex hull of two points from S and y, or y is in a convex hull of two points from S and x.

Say, $y \in CH\{x, B, C\}$. But $y \in CH\{x, B, C\} \Rightarrow y \in CH\{x, b, c\}$.

Therefore, either x is in a convex hull of two circles from $\{a, b, c\}$ and y, or y is in a convex hull of two circles from $\{a, b, c\}$ and x. □

8. Concluding Remarks

We demonstrated an example of convex geometry of $cdim = 6$ that fails Weak 2-Carousel property and therefore could not be represented by circles on a plane. This convex geometry fails 2-Carousel property, so it is not also weakly represented by affine convex geometries on a plane ([1]). Hence, we ask the following problem:

Problem 8.1. *Is a convex geometry of $cdim = 3, 4$ or 5 strongly represented by a geometry of circles on a plane?*

Since, we proved existence of geometries that are not strongly representable by circles in $\mathbb{R}^2$, we would like to consider higher dimensions of a space. Therefore, we ask the following question:

Problem 8.2. *Is every finite convex geometry could be strongly represented by balls in* $\mathbb{R}^n$

It is of interest to find some relationship, if such exists, between a convex dimension of a convex geometry and a dimension of a space for a representation.

Problem 8.3. *In which dimension of a space, a convex geometry with cdim* $= k, k \in \mathbb{N}$ *has a representation?*

9. Acknowledgements

I express sincere gratitude to my supervisor Dr. Kira Adaricheva for the continuous support and sharing extensive knowledge in the field of convex geometries. Thanks to Dr. Kira Adaricheva for providing a shorter proof of Lemma 6.3. I would also like to thank Dr. Rustem Takhanov for being a second reader and giving valuable comments regarding the paper. I thank my collaborator Zhanbota Myrzakul with whom we were learning a theory of convex geometries throughout a year.

10. Appendices

Appendix A

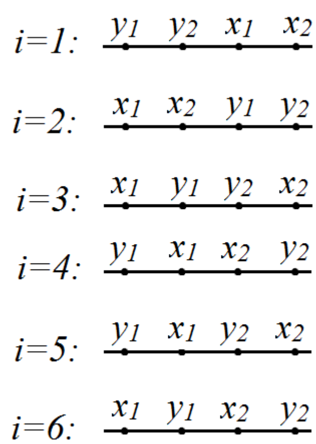

Figure 18

Appendix B

$S_1 = \{C_{121}, C_{122}, C_{112}, C_{221}, C_{211}, C_{212}\}$
$S_2 = \{C_{111}, C_{222}\}$

$S_3 = \{C_{123}, C_{142}, C_{214}, C_{231}, C_{312}, C_{421}\}$
$S_4 = \{C_{124}, C_{132}, C_{213}, C_{241}, C_{321}, C_{412}\}$
$S_5 = \{C_{113}, C_{131}, C_{224}, C_{242}, C_{311}, C_{422}\}$
$S_6 = \{C_{114}, C_{141}, C_{223}, C_{232}, C_{322}, C_{411}\}$

$S_7 = \{C_{125}, C_{162}, C_{216}, C_{251}, C_{512}, C_{621}\}$
$S_8 = \{C_{126}, C_{152}, C_{215}, C_{261}, C_{521}, C_{612}\}$
$S_9 = \{C_{115}, C_{151}, C_{226}, C_{262}, C_{511}, C_{622}\}$
$S_{10} = \{C_{116}, C_{161}, C_{225}, C_{252}, C_{522}, C_{611}\}$

$S_{11} = \{C_{133}, C_{244}, C_{313}, C_{331}, C_{424}, C_{442}\}$
$S_{12} = \{C_{134}, C_{243}, C_{324}, C_{341}, C_{413}, C_{432}\}$
$S_{13} = \{C_{143}, C_{234}, C_{314}, C_{342}, C_{423}, C_{431}\}$
$S_{14} = \{C_{144}, C_{233}, C_{323}, C_{332}, C_{414}, C_{441}\}$

$S_{15} = \{C_{135}, C_{246}, C_{351}, C_{462}, C_{513}, C_{624}\}$
$S_{16} = \{C_{136}, C_{245}, C_{361}, C_{452}, C_{524}, C_{613}\}$

$$S_{17} = \{C_{145}, C_{236}, C_{362}, C_{451}, C_{514}, C_{623}\}$$
$$S_{18} = \{C_{146}, C_{235}, C_{352}, C_{461}, C_{523}, C_{614}\}$$

$$S_{19} = \{C_{163}, C_{254}, C_{316}, C_{425}, C_{542}, C_{631}\}$$
$$S_{20} = \{C_{164}, C_{253}, C_{325}, C_{416}, C_{532}, C_{664}\}$$
$$S_{21} = \{C_{153}, C_{264}, C_{315}, C_{426}, C_{531}, C_{642}\}$$
$$S_{22} = \{C_{154}, C_{263}, C_{326}, C_{415}, C_{541}, C_{632}\}$$

$$S_{23} = \{C_{165}, C_{256}, C_{516}, C_{562}, C_{625}, C_{651}\}$$
$$S_{24} = \{C_{166}, C_{255}, C_{525}, C_{552}, C_{616}, C_{661}\}$$
$$S_{25} = \{C_{155}, C_{266}, C_{551}, C_{515}, C_{626}, C_{662}\}$$
$$S_{26} = \{C_{156}, C_{265}, C_{526}, C_{561}, C_{615}, C_{652}\}$$

$$S_{27} = \{C_{333}, C_{444}\}$$
$$S_{28} = \{C_{334}, C_{343}, C_{344}, C_{433}, C_{434}, C_{443}\}$$

$$S_{29} = \{C_{335}, C_{353}, C_{446}, C_{464}, C_{533}, C_{644}\}$$
$$S_{30} = \{C_{336}, C_{363}, C_{445}, C_{454}, C_{544}, C_{633}\}$$
$$S_{31} = \{C_{345}, C_{364}, C_{436}, C_{453}, C_{534}, C_{643}\}$$
$$S_{32} = \{C_{346}, C_{354}, C_{435}, C_{463}, C_{543}, C_{634}\}$$

$$S_{33} = \{C_{365}, C_{456}, C_{536}, C_{564}, C_{645}, C_{653}\}$$
$$S_{34} = \{C_{366}, C_{455}, C_{545}, C_{554}, C_{63}, C_{636}\}$$
$$S_{35} = \{C_{355}, C_{466}, C_{535}, C_{553}, C_{646}, C_{664}\}$$
$$S_{36} = \{C_{356}, C_{365}, C_{546}, C_{563}, C_{635}, C_{654}\}$$

$$S_{37} = \{C_{556}, C_{565}, C_{566}, C_{655}, C_{656}, C_{665}\}$$
$$S_{38} = \{C_{555}, C_{666}\}$$

Appendix C

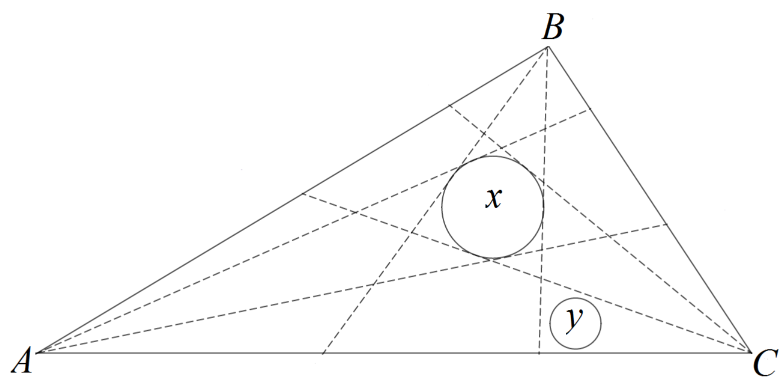

Figure 19. S_1

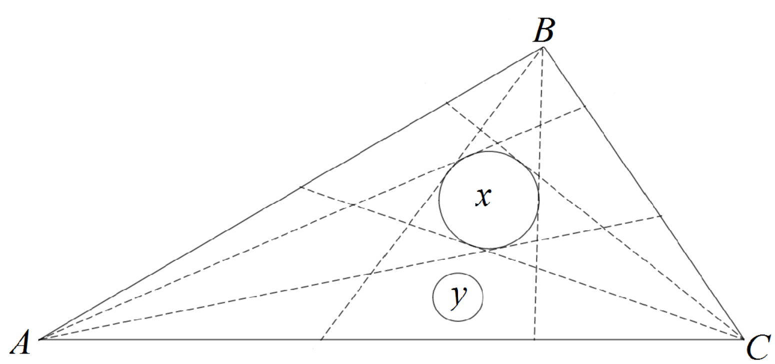

Figure 20. S_3

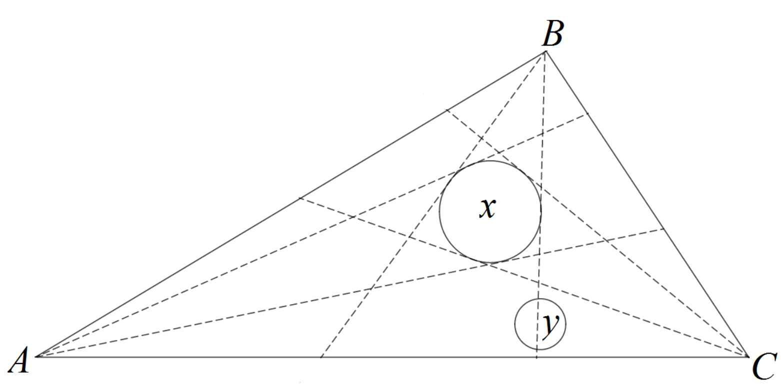

Figure 21. S_7

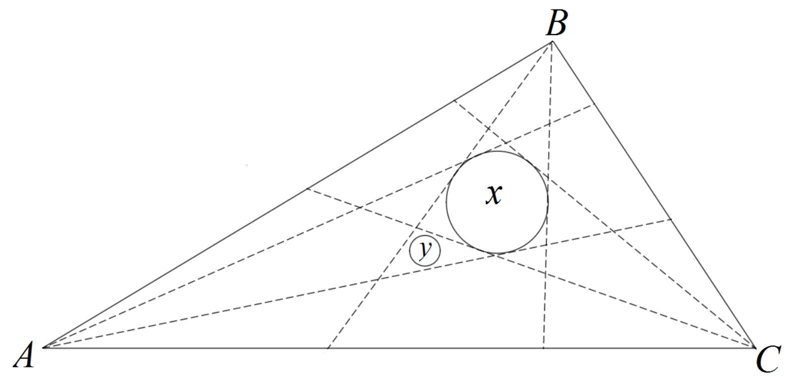

FIGURE 22. S_{11}

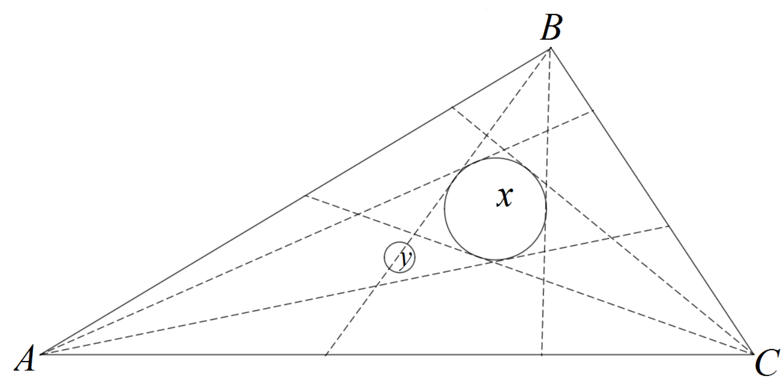

FIGURE 23. S_{16}

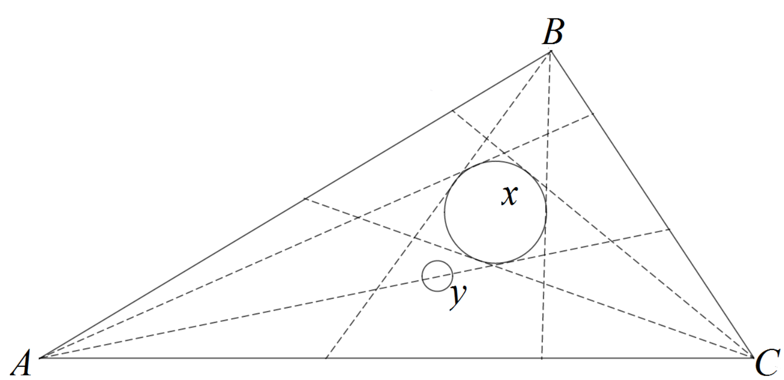

FIGURE 24. S_{19}

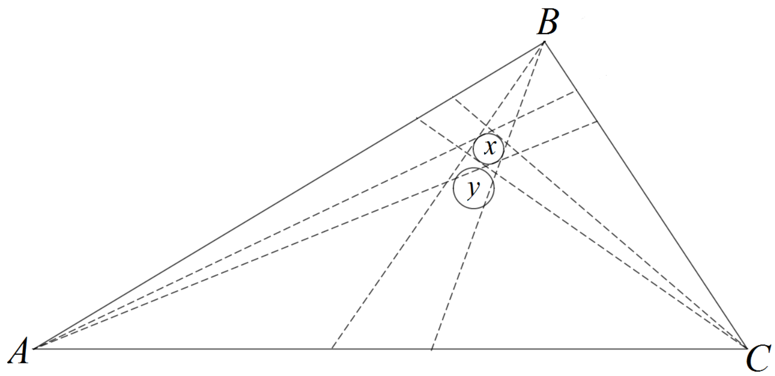

FIGURE 25. S_{23}

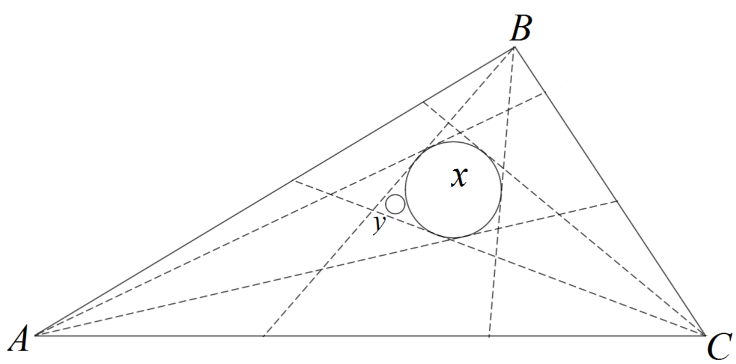

FIGURE 26. S_{27}

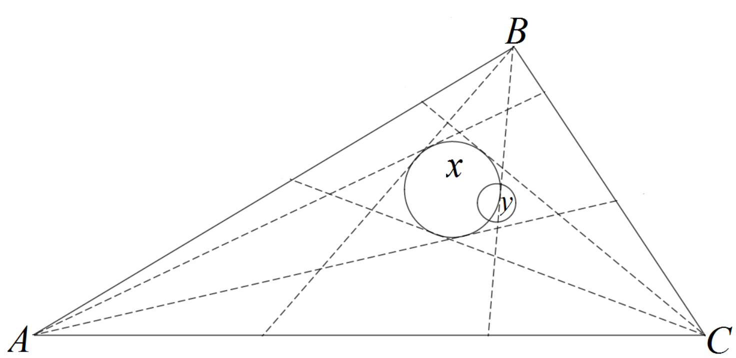

FIGURE 27. S_{29}

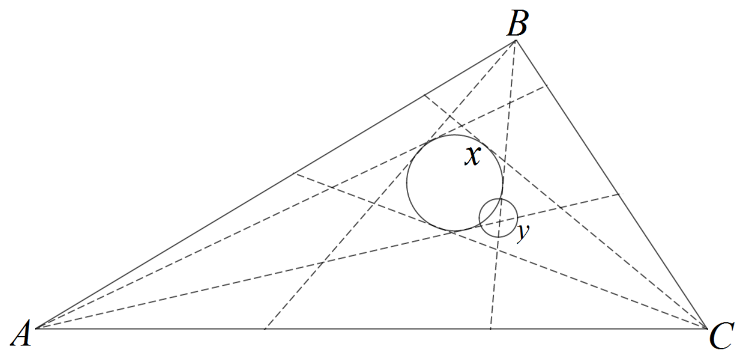

FIGURE 28. S_{33}

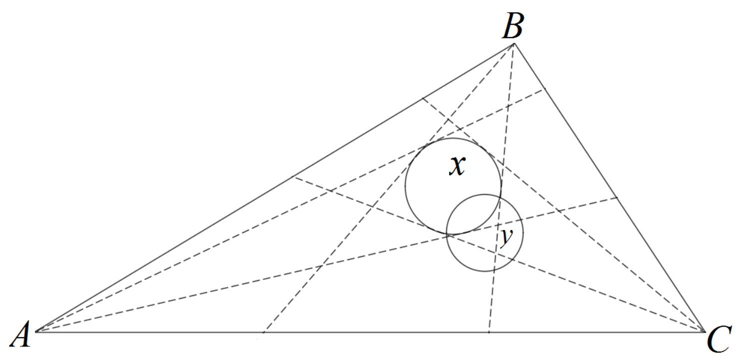

FIGURE 29. S_{37}

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