EMBEDDING FINITE LATTICES INTO FINITE BIATOMIC LATTICES

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Abstract. For a class $\mathcal{C}$ of finite lattices, the question arises whether any lattice in $\mathcal{C}$ can be embedded into some atomistic, *biatomic* lattice in $\mathcal{C}$. We provide answers to the question above for $\mathcal{C}$ being, respectively,

— The class of all finite lattices;
— The class of all finite *lower bounded* lattices (solved by the first author's earlier work).
— The class of all finite *join-semidistributive lattices* (this problem was, until now, open).

We solve the latter problem by finding a quasi-identity valid in all finite, atomistic, biatomic, join-semidistributive lattices but not in all finite join-semidistributive lattices.

1. Introduction

A lattice L is *biatomic*, if it is atomic (i.e., every element of $L \setminus \{0\}$ lies above some atom of L) and whenever p, a, and b are elements of L such that p is an atom, a and b are nonzero, and $p \leq a \vee b$, there are atoms $x \leq a$ and $y \leq b$ such that $p \leq x \vee y$, see Definition 2.1.

In our first result of this paper, Theorem 2.3, we prove that any finite lattice can be easily embedded atom-preservingly into a finite biatomic one.

Biatomicity arises naturally in geometric lattices such as lattices of subspaces of a vector space or, more generally, projective geometries. It was also noticed by M. K. Bennett [4] that in geometric lattices, biatomicity is equivalent to modularity.

Biatomicity is probably even more common among *convex geometries*. The lattice theoretical facet of these at first finite and purely combinatorial structures was studied in P. H. Edelman [6] and P. H. Edelman and R. Jamison [7]. In [3], these structures, now generalized to the infinite case, were considered as an antithesis of geometric lattices, in terms of the properties of the closure operators that define them. We mention the lattices of convex subsets of a given affine space and the lattice of subsemilattices of a given meet-semilattice as a few examples of biatomic convex geometries.

Still, not all convex geometries are biatomic, thus to describe the biatomic members within a given class of such structures would be of great interest.

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Convex geometries are closely connected with the class of join-semidistributive lattices. A lattice L is called *join-semidistributive*, if

$$x \vee y = x \vee z \text{ implies that } x \vee y = x \vee (y \wedge z), \text{ for all } x,\ y,\ z \in L.$$

It is proved in [3] that every finite join-semidistributive lattice can be embedded, in an atom-preserving way, into a finite, atomistic, join-semidistributive lattice, or, equivalently, into a finite atomistic convex geometry. This convex geometry is not generally biatomic.

The other construction in [3] embeds any finite join-semidistributive lattice into the lattice $\mathbf{S_p}(A)$ of algebraic subsets of some (generally infinite) algebraic and dually algebraic lattice A. This is also a convex geometry with the additional properties that it is atomistic, biatomic, and join-semidistributive. This, together with [3, Theorem 1.4], implies that *every join-semidistributive lattice L can be embedded into an atomistic, biatomic, join-semidistributive lattice L'*, see also the proof of [3, Theorem 3.26].

It is asked in Problem 4 of [3] whether L' can be taken to be finite whenever L is finite. In the present paper, we solve this problem in the negative, by showing a quasi-identity θ that is satisfied by all finite, atomistic, biatomic, join-semidistributive lattices but not by all finite atomistic join-semidistributive lattices, see Theorem 7.1. This result is inspired by a geometrical example of finite convex geometry that in general produces non-biatomic join-semidistributive lattices.

In contrast with this, we prove that every finite atomistic join-semidistributive lattice L can be $\langle \vee, \wedge, 0, 1\rangle$-embedded, in an atom-preserving way and with the congruence extension property, into a finite, atomistic, join-semidistributive lattice L' such that all biatomicity problems of L can be solved in L', see Theorem 6.1.

We also study the case of finite *lower bounded* lattices, an important subclass of join-semidistributive lattices. The first author's earlier work [1] provides an embedding of any finite lower bounded lattice into some finite biatomic convex geometry, which implies Corollary 4.2 (see also Theorem 6.1). Still, we do not know whether such an embedding can be done atom-preservingly.

This contributes to the list of open problems that concludes the paper.

2. Biatomic lattices

For a lattice L with zero (i.e., least element), we denote by $\operatorname{At} L$ the set of atoms of L. The following definition recalls classical notions, related to their counterparts in [4].

Definition 2.1. A lattice L with zero is

- *atomic*, if every element of L is above an atom of L;
- *atomistic*, if every element of L is a join of atoms of L;
- *biatomic*, if L is atomic and for every atom p of L and all nonzero a, $b \in L$, if $p \leq a \vee b$, then there are atoms $x \leq a$ and $y \leq b$ such that $p \leq x \vee y$.

We observe that every finite lattice is atomic, and L is atomistic iff for all a, $b \in L$ such that $a \nleq b$, there exists $p \in \operatorname{At} L$ such that $p \leq a$ and $p \nleq b$. The following lemma is trivial.

Lemma 2.2. *Let L be an atomic lattice. Then L is biatomic iff for every atom p of L and all a, $b \in L \setminus \{0\}$ such that $p \nleq a$, $p \nleq b$, and $p \leq a \vee b$, there exists an atom $q \leq a$ of L such that $p \leq q \vee b$.*

For a lattice L with zero and $x \in L$, let $\mathsf{at}(x)$ be the statement that x is an atom of L. We can prove right away the following easy embedding result.

Theorem 2.3. *Let L be a finite lattice. Then L has a $\langle \vee, \wedge, 0, 1, \mathsf{at} \rangle$-embedding into some finite, atomistic, biatomic lattice M.*

Proof. Put $A = L \setminus (\{0\} \cup \operatorname{At} L)$. For each $a \in A$, let p_a and q_a be new distinct elements, and put

$$M = L \cup \{p_a \mid a \in A\} \cup \{q_a \mid a \in A\}.$$

We define a partial ordering on M, extending the partial ordering of L, by making all the elements of $\{p_a \mid a \in A\} \cup \{q_a \mid a \in A\}$ mutually incomparable, and by saying that

$$x \leq p_a \text{ iff } x \leq q_a \text{ iff } x = 0,$$
$$p_a \leq x \text{ iff } q_a \leq x \text{ iff } a \leq x,$$

for all $a \in A$ and $x \in L$. Then it is straightforward to verify that $\leq$ is a lattice ordering on M and that the inclusion map from L into M is a $\langle \vee, \wedge, 0, 1, \mathsf{at} \rangle$-embedding. Since $a = p_a \vee q_a$ for all $a \in A$, the lattice M is atomistic.

To prove that M is biatomic, it is convenient to use Lemma 2.2. So let $p \in \operatorname{At} M$, let a, $b \in M$ such that $p \nleq a$, $p \nleq b$, and $p \leq a \vee b$ (in particular, a and b are incomparable, thus they are nonzero), we find an atom q of M such that $q \leq a$ and $p \leq q \vee b$. If a is an atom of M then $q = a$ works, so suppose from now on that $a \in A$. If $b \in L$, then $p \leq a \vee b = p_a \vee b$, thus $q = p_a$ is as required. If $b \notin L$, say, $b = p_c$ for some $c \in A$ such that $c \nleq a$, then $p \leq a \vee p_c = a \vee c = p_a \vee p_c$, so $q = p_a$ is as required again. □

3. Equivalence of definitions of embedding into finite biatomic join-semidistributive lattices

We say that a partially ordered set is *nœtherian*, if it has no infinite strictly increasing chain. Equivalently, every nonempty subset has a maximal element. Of course, every finite partially ordered set is nœtherian.

In this paragraph we will prove that if an atomistic lattice can be embedded into some nœtherian, join-semidistributive, biatomic lattice, then it also can be embedded into an *atomistic* such lattice. We will work toward the proof of this statement in Corollary 3.5.

An immediate application of nœtherianity gives the following result.

Lemma 3.1. *Let L be a nœtherian lattice and let G be a subset of L. Then every join of elements of G is a finite join of elements of G.*

We recall the following elementary property of join-semidistributive lattices that follows immediately from [8, Theorem 1.21].

Lemma 3.2. *Let K be a join-semidistributive lattice with zero, let $a \in K$ and let X and Y be finite sets of atoms of K. If $a \vee \bigvee X = a \vee \bigvee Y$, then $a \vee \bigvee X = a \vee \bigvee (X \cap Y)$.*

For a subset S of a join-semilattice L, we denote the set of all joins of nonempty finite subsets of S by $S^{\vee}$.

Proposition 3.3. *Let L be a nœtherian, biatomic, (join-semidistributive) lattice and let $a \in L$. We put $S = [0, a] \cap \mathrm{At}\, L$. Then $T = (\{0\} \cup S)^{\vee}$ is a nœtherian, biatomic, (join-semidistributive), atomistic lattice.*

Proof. By definition, T is a $\langle \vee, 0 \rangle$-subsemilattice of L. Since L is nœtherian, it follows from Lemma 3.1 that T is a lattice in its own right, and it is nœtherian. It is obviously atomistic, with $\mathrm{At}\, T = S$.

Let $p \in S$ and let x, $y \in T \setminus \{0\}$ such that $p \leq x \vee y$. Since L is biatomic, there are atoms $u \leq x$ and $v \leq y$ of L such that $p \leq u \vee v$. From x, $y \leq a$, it follows that u, $v \in S$. This shows that T is biatomic.

Now we prove that T is join-semidistributive provided L is. Let b, x, $y \in T$ such that $b \vee x = b \vee y$. By the definition of T, there are finite subsets X and Y of S such that $x = \bigvee X$ and $y = \bigvee Y$. It follows from Lemma 3.2 that $b \vee x = b \vee \bigvee (X \cap Y)$. Since $\bigvee (X \cap Y) \leq x \wedge_T y$, we get $b \vee x = b \vee (x \wedge_T y)$, thus completing the proof that T is join-semidistributive. □

For subsets P and Q in a lattice L, we say that P *separates the elements of* Q, if for all x, $y \in Q$ with $x \nleq y$, there exists $p \in P$ such that $p \leq x$ and $p \nleq y$. Observe, in particular, that L is atomistic iff $\mathrm{At}\, L$ separates the elements of L.

Corollary 3.4. *Let L be a lattice. Then the following are equivalent:*

(i) *There is a lattice embedding from L into some finite (resp., nœtherian), join-semidistributive, biatomic lattice M such that $\mathrm{At}\, M$ separates the elements of L.*

(ii) *There is a lattice embedding from L into some finite (resp., nœtherian), join-semidistributive, biatomic, atomistic lattice.*

The results above also hold for "0-lattice embedding" instead of "lattice embedding".

Proof. We prove the nontrivial direction (i)⇒(ii), for "nœtherian"—the proof for "finite" is similar. Let M be a nœtherian, biatomic, join-semidistributive lattice containing L as a sublattice such that $\mathrm{At}\, M$ separates the elements of L. We put

$$S = \{p \in \mathrm{At}\, M \mid p \leq 1_L\},$$

and we observe that S separates the elements of L. Now we put $T = (\{0_M\} \cup S)^{\vee}$. By Proposition 3.3, T is a nœtherian, atomistic, biatomic, join-semidistributive lattice, with $\mathrm{At}\, T = S$ and $0_T = 0_M$.

Now we shall investigate further the interaction of the lattices T, M, L.

Define a map $f \colon L \to T$ by the rule

$$f(x) = \bigvee\nolimits_T \{p \in S \mid p \leq x\}, \quad \text{for all } x \in L.$$

In view of Lemma 3.1, f is well-defined. We observe that $f(x) \leq x$, for all $x \in L$.

Claim 1. *The map f is a meet-embedding from L into T. If $0_M = 0_L$, then $f(0_L) = 0_T$.*

Proof of Claim. It is clear that f is order-preserving and that if $0_M = 0_L$, then $f(0_L) = 0_T$. Let x, $y \in L$ such that $x \nleq y$. Since S separates the elements of L, there exists $p \in S$ such that $p \leq x$ and $p \nleq y$. Thus $p \leq f(x)$ (by the definition of f) and $p \nleq f(y)$ (because $f(y) \leq y$), whence $f(x) \nleq f(y)$. So f is an order-embedding.

Now let x, $y \in L$. We prove that $f(x) \wedge_T f(y) \leq f(x \wedge_L y)$. Let $p \in S$ such that $p \leq f(x) \wedge_T f(y)$. Since $f(x) \leq x$ and $f(y) \leq y$, this implies that $p \leq x$ and $p \leq y$. Thus, since L is a sublattice of M, the inequality $p \leq x \wedge_L y$

holds, whence $p \leq f(x \wedge_L y)$. Therefore, since f is order-preserving, f is a meet-homomorphism. $\square$ Claim 1.

Claim 2. *The map f is a join-homomorphism from L to T.*

Proof of Claim. Let $x,\ y \in L$ and let $p \in S$ such that $p \leq f(x \vee_L y)$. Suppose that $p \nleq f(x) \vee_T f(y)$. In particular, x and y are nonzero in L, thus in M. By the definition of f, this means that $p \leq x \vee_L y = x \vee_M y$. Hence, since M is biatomic and x and y are nonzero in M, there are atoms $u \leq x$ and $v \leq y$ of M such that $p \leq u \vee_M v$. Observe that $u,\ v \in S$. Moreover, $u \leq f(x)$ and $v \leq f(y)$, whence $p \leq u \vee_M v = u \vee_T v \leq f(x) \vee_T f(y)$, a contradiction.

Therefore, we have proved that $f(x \vee_L y) \leq f(x) \vee_T f(y)$. Since f is order-preserving, the converse inequality holds, which concludes the proof of the claim. $\square$ Claim 2.

The proof of Corollary 3.4 is completed. $\square$

Corollary 3.5. *Let L be an atomistic lattice. Then the following are equivalent:*

(i) *L has a $\langle \vee, \wedge, 0\rangle$-embedding into some finite (resp., nœtherian), join-semidistributive, biatomic lattice.*

(ii) *L has a $\langle \vee, \wedge, 0\rangle$-embedding into some finite (resp., nœtherian), join-semidistributive, biatomic, atomistic lattice.*

Proof. We prove the nontrivial direction (i)⇒(ii), for "nœtherian" (the proof for "finite" is similar). Let M be a nœtherian, join-semidistributive, biatomic lattice such that L is a 0-sublattice of M. By Corollary 3.4, it is sufficient to prove that the atoms of M separate the elements of L. So let $x,\ y \in L$ such that $x \nleq y$. Since L is atomistic, there exists $p \in \operatorname{At} L$ such that $p \leq x$ and $p \nleq y$. Since M is atomic, there exists an atom q of M below p. Then $q \leq x$. Furthermore, $0 \leq q \wedge y \leq p \wedge y = 0$, and hence $q \wedge y = 0 < q$, that is, $q \nleq y$. This proves our assertion. $\square$

4. Embedding finite lower bounded lattices

For lattices K and L, a lattice homomorphism $f\colon K \to L$ is *lower bounded*, if the preimage under f of any principal dual ideal of L is either empty or has a least element. A lattice L is *lower bounded*, if every homomorphism from a finitely generated lattice to L is lower bounded. We refer the reader to [2, 8] for more details.

For a finite meet-semilattice P, we denote by $\mathbf{Sub}_\wedge(P)$ the lattice of all subsemilattices of P ($\varnothing$ included). We state here the following result from the first author's paper [1].

Theorem 4.1. *A finite lattice L is lower bounded iff it can be embedded into $\mathbf{Sub}_\wedge(P)$ for some finite meet-semilattice P.*

As $\mathbf{Sub}_\wedge(P)$ is lower bounded, atomistic, and biatomic, this implies immediately the following result.

Corollary 4.2. *Any finite lower bounded lattice can be embedded into some finite, atomistic, biatomic, lower bounded lattice.*

For a finite, atomistic, lower bounded lattice L, Theorem 4.1 says that there exists an embedding from L into $\mathbf{Sub}_\wedge(P)$ for some finite meet-semilattice P. This embedding can be chosen to preserve the zero, however, it may not preserve atoms.

The reason for this is that all lattices of the form $\mathbf{Sub}_\wedge(P)$ have the property that for all atoms x and y, there are at most three atoms below $x \vee y$, while there are finite, atomistic, lower bounded lattices that fail this property.

5. One-atom extensions of finite atomistic lattices

We start with the following definition.

Definition 5.1. Let L be a finite atomistic lattice. An *extension pair* of L is a pair $(a; M)$, where the following are satisfied:

(i) $a \in L \setminus (\{0\} \cup \operatorname{At} L)$;

(ii) M is a meet-subsemilattice of L that contains $\{0\} \cup [a, 1]$.

For any $a \in L$, we put $L_a = L \setminus [a, 1]$. For an extension pair $(a; M)$, we put

$$L(a; M) = (L_a \times \{0\}) \cup (M \times \{1\}),$$

endowed with the componentwise ordering.

By definition, a *closure operator* of L is a map $f \colon L \to L$ such that $f \circ f = f$, $f(x) \geq x$, and $x \leq y$ implies that $f(x) \leq f(y)$, for all $x, y \in L$. Fix an extension pair $(a; M)$ of a finite atomistic lattice L. Let f be the closure operator of L associated with M, that is, $f \colon L \to L$ is given by the rule

$$f(x) = \text{least } y \in M \text{ such that } x \leq y, \qquad \text{for all } x \in L.$$

We observe that $L(a; M)$ is a meet-subsemilattice of $L \times \mathbf{2}$ (where $\mathbf{2} = \{0, 1\}$) that contains both $(0_L, 0)$ and $(1_L, 1)$ as elements. Hence it is a lattice in its own right. For $(x, \varepsilon) \in L \times \mathbf{2}$, we denote by $\overline{(x, \varepsilon)}$ the least element of $L(a; M)$ above (x, ε). This element can easily be calculated, by the rule

$$\overline{(x,0)} = \begin{cases} (x, 0) & (\text{if } a \nleq x), \\ (x, 1) & (\text{if } a \leq x), \end{cases}$$

$$\overline{(x,1)} = (f(x), 1).$$

We leave to the reader the straightforward proof of the following lemma.

Lemma 5.2. *Let L be a finite atomistic lattice and let $(a; M)$ be an extension pair of L. Then the lattice $L(a; M)$ is finite atomistic, and the map $j \colon L \to L(a; M)$ defined by $j(x) = \overline{(x, 0)}$ for all $x \in L$ is a $\langle \vee, \wedge, 0, 1, \mathsf{at} \rangle$-embedding from L into $L(a; M)$. Furthermore,* $\operatorname{At}(L(a; M)) = (\operatorname{At} L \times \{0\}) \cup \{(0, 1)\}$.

Hence, $L(a; M)$ is an atomistic extension of L by exactly one atom, here $(0, 1)$. In the sequel, the only properties of $L' = L(a; M) = L[p^*]$, where $p^* = (0, 1)$ is the new atom, that will be used are the ones listed below:

$$L' = L \cup \{p^* \vee x \mid x \in M\} = L \cup \{p^* \vee x \mid x \in L\}, \tag{5.1}$$

$$p^* \leq x \Leftrightarrow a \leq x, \tag{5.2}$$

$$x \leq p^* \vee y \Leftrightarrow x \leq f(y), \tag{5.3}$$

for all x, $y \in L$. Furthermore, $\operatorname{At} L' = \operatorname{At} L \cup \{p^*\}$. From now on we shall use the more wieldy description of $L(a; M)$ given by (5.1), (5.2), and (5.3).

Remark 5.3. It is not difficult to verify that conversely, every $\langle \vee, 0, 1 \rangle$-extension of L by exactly one atom below 1 is, up to isomorphism above L, of the form $L(a; M)$ for exactly one extension pair $(a; M)$ of L. However, we shall not need this fact.

Our next result describes when $L(a;M)$ is join-semidistributive. For a subset X of L, we denote by $\operatorname{Max} X$ the set of all maximal elements of X.

Lemma 5.4. *Let L be a finite, atomistic, join-semidistributive lattice and let $(a;M)$ be an extension pair of L with associated closure operator f. Then $L(a;M)$ is join-semidistributive iff the following conditions are satisfied:*

(i) $\operatorname{Max} L_a \subseteq M$;
(ii) $f(x \vee u) = f(x \vee v)$ *implies that* $u \leq f(x)$, *for all* $x \in L$ *and all distinct atoms* u *and* v *of* L.

Proof. We put $L' = L(a;M)$. Suppose first that L' is join-semidistributive. Let $x \in \operatorname{Max} L_a$. Suppose that $x \notin M$. Since L is atomistic, there exists an atom u of L such that $u \leq f(x)$ while $u \nleq x$. From $x < x \vee u$ and the maximality of x in $L_a = L \setminus [a,1]$, it follows that $a \leq x \vee u$, whence $x \vee u \in M$. So, since $x \vee u \leq f(x)$, we obtain that $x \vee u = f(x)$. Moreover, $p^* \leq a \leq x \vee u = f(x)$, thus $x \vee u = f(x) \vee p^* = x \vee p^*$, and so, by the join-semidistributivity of L', $u \leq x$, a contradiction. Therefore, $\operatorname{Max} L_a \subseteq M$.

Now let $x \in L$ and u, v be distinct atoms of L such that $f(x \vee u) = f(x \vee v)$. It follows from (5.3) that $x \vee u \vee p^* = x \vee v \vee p^*$, whence, by the join-semidistributivity of L', $u \leq x \vee p^*$, and therefore, again by (5.3), $u \leq f(x)$.

Conversely, suppose that both conditions (i) and (ii) are satisfied. To prove the join-semidistributivity of L', if suffices to prove that $x \vee u = x \vee v > x$ cannot happen, for all $x \in L'$ and all distinct atoms u and v of L' (see [3, Lemma 1.2]). Since L is join-semidistributive, this holds if x, u, $v \in L$.

Now suppose that $x \in L$ and $v = p^*$, so $x \vee p^* = x \vee u > x$. Hence, by using (5.3), we obtain that $x \vee u \leq f(x) \leq f(x) \vee p^* = x \vee p^* = x \vee u$, thus $p^* \leq f(x)$, that is, by (5.2), $a \leq f(x)$. Moreover, $x \vee u = f(x) > x$, so $x \in L_a$. Hence there exists $y \in \operatorname{Max} L_a$ such that $x \leq y$. By assumption, $y \in M$, and consequently $f(x) \leq y$, a contradiction since $a \leq f(x)$ and $a \nleq y$.

Since $x \vee u = x \vee v > x$, the last case to consider is $x = y \vee p^*$ for some $y \in L$ (see (5.1)). It follows again from (5.3) that $f(y \vee u) = f(y \vee v)$, and consequently, by assumption, $u \leq f(y)$, and so $x \vee u = x$, a contradiction. □

6. Partially biatomic extensions

By definition, a "biatomicity problem" in a lattice L is a formal expression of the form $p \leq a \vee b$, where $p \in \operatorname{At} L$, a, $b \in L \setminus \{0\}$, and the inequality $p \leq a \vee b$ holds while $p \nleq a$, $p \nleq b$. A *solution* of the problem above in L consists of atoms $x \leq a$ and $y \leq b$ of L such that $p \leq x \vee y$. Recall that a lattice embedding $f\colon K \hookrightarrow L$ has the *congruence extension property*, if every congruence of K is the inverse image under f of some congruence of L.

The present section will be mainly devoted to proving the following results.

Theorem 6.1. *Every finite, atomistic, join-semidistributive* (*resp., lower bounded*) *lattice L admits a $\langle \vee, \wedge, 0, 1, \mathsf{at} \rangle$-embedding with the congruence extension property into some finite, atomistic, join-semidistributive* (*resp., lower bounded*) *lattice L' such that all biatomicity problems in L can be solved in L'.*

Remark 6.2. It will turn out that the embedding from L into L' in Theorem 6.1 preserves more than the congruences, it is in fact an embedding for the transitive closure $\lhd$ of the join-dependency relation D. This is equivalent to L' being a congruence-preserving extension of L in the finite, lower bounded case, but not in general.

The core of the difficulty underlying Theorem 6.1 consists of solving very special sorts of biatomicity problems. In Lemma 6.3 to Corollary 6.6, we let L be a finite, atomistic, join-semidistributive lattice, and p, q, $a \in L$ such that p and q are distinct atoms, $a \in L \setminus (\{0\} \cup \operatorname{At} L)$, $p \leq a \vee q$, and $p \nleq x \vee q$ for all $x < a$ in L. Furthermore, we let $f\colon L \to L$ be the map defined by the rule

$$f(x) = \begin{cases} x & (\text{if } q \nleq p \vee x), \\ p \vee x & (\text{if } q \leq p \vee x), \end{cases} \tag{6.1}$$

for all $x \in L$.

Lemma 6.3. *The following assertions hold.*

(i) *The map f is a closure operator of L.*
(ii) *If we denote by M the range of f, then $(a; M)$ is an extension pair of L.*
(iii) *$L(a; M)$ is join-semidistributive.*
(iv) *Denote by p^* the unique atom of $L(a; M) \setminus L$. Then $p < p^* \vee q$ and $p^* < a$.*

Hence, $L(a; M)$ is a join-semidistributive extension of L in which the biatomicity problem $p \leq a \vee q$ has a solution.

Proof. The assertion (i) is straightforward. Furthermore, it is obvious that $\{0, 1\}$ is contained in M. Now let $x \in [a, 1]$, we prove that $f(x) = x$. This is obvious if $q \nleq p \vee x$, so suppose that $q \leq p \vee x$. From $p \leq a \vee q$, $q \leq p \vee x$, and the join-semidistributivity of L, it follows that $p \leq x \vee a = x$, whence $f(x) = x \vee p = x$. This completes the proof of (ii).

Now let $x \in \operatorname{Max} L_a$. We prove that $f(x) = x$. This is trivial if $q \nleq p \vee x$, so suppose that $q \leq p \vee x$. If $p \nleq x$, then, by the maximality assumption on x, $a \leq p \vee x$. Thus, from $p \leq a \vee q$, $p \wedge a = 0$, and the join-semidistributivity of L, it follows that $p \leq x \vee q$. Thus, since $q \leq p \vee x$ and by the join-semidistributivity of L, we obtain that $p \leq x$, a contradiction. Therefore, $p \leq x$, so $f(x) = p \vee x = x$. This proves that $\operatorname{Max} L_a \subseteq M$.

Let $x \in L$ and u, v be distinct atoms of L such that $f(x \vee u) = f(x \vee v)$. We prove that $u \leq f(x)$.

If $q \nleq p \vee x \vee u$, then $q \nleq p \vee x \vee v$. Otherwise we would have

$$p \vee x \vee u = f(x \vee u) = f(x \vee v) = x \vee v,$$

thus $q \leq p \vee x \vee v = p \vee x \vee u$, a contradiction. Thus $x \vee u = f(x \vee u) = f(x \vee v) = x \vee v$, whence, by the join-semidistributivity of L, $u \leq x \leq f(x)$.

Suppose now that $q \leq p \vee x \vee u$. By the previous paragraph, $q \leq p \vee x \vee v$, and thus $p \vee x \vee u = f(x \vee u) = f(x \vee v) = p \vee x \vee v$. Hence, by the join-semidistributivity of L, we have $u \leq p \vee x$, and so $q \leq p \vee x \vee u = p \vee x$. Therefore, $u \leq p \vee x = f(x)$. By Lemma 5.4, this completes the proof of assertion (iii).

The assertion (iv) follows immediately from $f(q) = p \vee q > p$. □

From Lemma 6.4 to Corollary 6.6, we let M and p^* be as in the statement and proof of Lemma 6.3. For a finite lattice K, we let D_K denote the relation of join-dependency on the set of join-irreducible elements of K. Observe that for *atoms* x and y of K, the relation D_K takes the following simple form:

$$x \, D_K \, y \quad \text{iff} \quad x \neq y \text{ and } \exists u \in K \text{ such that } x \leq y \vee u \text{ and } x \nleq u.$$

Further, we denote by $\overline{D}_K$ the binary relation on $\mathrm{J}(K)$ defined by $x\,\overline{D}_K\,y$ iff either $x\,D_K\,y$ or $x = y$. Then we let $\triangleleft_K$ denote the transitive closure of D_K and $\trianglelefteq_K$ denote the reflexive, transitive closure of D_K.

Furthermore, since L is finite, atomistic, and join-semidistributive, it follows from Lemma 3.2 that every element a of L has a minimal decomposition, that is, a least (with respect to containment) subset X of $\mathrm{At}\,L$ such that $a = \bigvee X$. We denote this set of atoms by $\partial^L(a)$ ("extreme boundary of a"), or $\partial(a)$ if L is understood. Note that $\partial(a)$ is also the unique irredundant decomposition of a. Observe that $\partial(a)$ consists exactly of the elements which are join-prime in the interval $[0, a]$. First it is convenient to prove the following lemma.

Lemma 6.4. *For any $u \in \partial(a)$, the following relations hold:*

(i) $p\,D_L\,u$;
(ii) $p^*\,D_{L[p^*]}\,u$.

Proof. For any $u \in \partial(a)$, we put $a \smallsetminus u = \bigvee(\partial(a) \setminus \{u\})$. From the fact that $u \in \partial(a)$, it follows that $a \smallsetminus u < a$, and so $p \nleq (a \smallsetminus u) \vee q$ by the minimality assumption on a. However, $p \leq a \vee q = (a \smallsetminus u) \vee u \vee q$ while $p \neq u$ (because $p \nleq a$), whence $p\,D_L\,u$.

Furthermore, $p^* \leq a = (a \smallsetminus u) \vee u$ while, since $a \smallsetminus u < a$, we have by (5.2) that $p^* \nleq a \smallsetminus u$, and consequently $p^*\,D_{L[p^*]}\,u$. □

Lemma 6.5. *For all x, $y \in \mathrm{At}\,L$, the following assertions hold:*

(i) $x\,D_{L[p^*]}\,y$ *implies that* $x \triangleleft_L y$;
(ii) $x\,D_{L[p^*]}\,p^*$ *implies that* $x\,\overline{D}_L\,p$;
(iii) $p^*\,D_{L[p^*]}\,x$ *implies that there exists* $u \in \partial(a)$ *such that* $u\,\overline{D}_L\,x$;
(iv) $x \triangleleft_{L[p^*]} y$ *iff* $x \triangleleft_L y$;
(v) $p^* \triangleleft_{L[p^*]} p^*$ *iff there exists* $u \in \partial(a)$ *such that* $u \triangleleft_L p$.

Proof. (i) By assumption, $x \neq y$ and there exists $u \in L[p^*]$ such that $x \leq y \vee u$ and $x \nleq u$. Suppose that the relation $x\,D_L\,y$ does *not* hold. So $u \notin L$, and therefore there exists $v \in M$ such that $u = v \vee p^*$, hence, by (5.3), $x \leq f(y \vee v)$ and $x \nleq v$. Since the relation $x\,D_L\,y$ does not hold, we obtain that $x \nleq y \vee v$. Hence $f(y \vee v) > y \vee v$, from where we obtain $f(y \vee v) = p \vee y \vee v$ (so $q \leq p \vee y \vee v$). Then, since $x \leq p \vee y \vee v$ and $x \nleq y \vee v$, we obtain that

$$x\,\overline{D}_L\,p. \tag{6.2}$$

If $q \leq p \vee v$, then, since $v \in M$, the equalities $v = f(v) = p \vee v$ holds by the definition of f, whence $p \leq v$, and so $x \leq p \vee y \vee v = y \vee v$, a contradiction. Hence $q \nleq p \vee v$, but $q \leq p \vee y \vee v$, so we obtain the relation

$$q\,\overline{D}_L\,y. \tag{6.3}$$

Finally, since $p \leq a \vee q$ and $p \nleq a$, we obtain that $p\,D_L\,q$, therefore, from (6.2) and (6.3), it follows that $x \triangleleft_L y$.

(ii) There exists $u \in L[p^*]$ such that $x \leq p^* \vee u$ and $x \nleq u$. Thus $p^* \vee u \neq u$, so $u \in L$ and $x \leq f(u)$. From the relation $x \nleq u$, it follows that $f(u) = p \vee u$, so $x \leq p \vee u$ while $x \nleq u$, and so $x\,\overline{D}_L\,p$.

(iii) There exists $v \in L[p^*]$ such that $p^* \leq x \vee v$ and $p^* \nleq v$. Thus $v \in L$, and $a \leq x \vee v$ while $a \nleq v$. From the second relation, it follows that there exists $u \in \partial(a)$ such that $u \nleq v$. However, $u \leq a \leq x \vee v$, and so $u\,\overline{D}_L\,x$.

(iv) From the fact that the natural embedding from L into $L[p^*]$ is atom-preserving, it follows that $x \lhd_L y$ implies that $x \lhd_{L[p^*]} y$ for all x, $y \in \mathrm{At}\, L$. Conversely, for any x, $y \in \mathrm{At}\, L$, the relation $x \lhd_{L[p^*]} y$ means that there are a positive integer n and atoms $z_0 = x, z_1, \dots, z_n = y$ of $L[p^*]$ such that $z_i \, D_{L[p^*]} \, z_{i+1}$ for all $i < n$. We prove by induction on n that this implies that $x \lhd_L y$. For $n = 1$, the conclusion follows from item (i) above. Suppose that $n \geq 2$. If $z_{n-1} \neq p^*$, then it follows from the induction hypothesis that $x \lhd_L z_{n-1}$, while, by item (i) above, $z_{n-1} \lhd_L y$, so $x \lhd_L y$. Suppose now that $z_{n-1} = p^*$. Then $z_{n-2} \neq p^*$. Thus, by the induction hypothesis, $x \unlhd_L z_{n-2}$ (the equality may hold, *e.g*, for $n = 2$). Furthermore, it follows from items (ii) and (iii) above that $z_{n-2} \, \overline{D}_L \, p$ and $u \, \overline{D}_L \, y$ for some $u \in \partial(a)$. But from Lemma 6.4(i), it follows that $p \, D_L \, u$, and so $z_{n-2} \lhd_L y$. Therefore, $x \lhd_L y$.

(v) There exists $z \in \mathrm{At}\, L$ such that $p^* \lhd_{L[p^*]} z \lhd_{L[p^*]} p^*$. From (ii), (iii), and (iv), it follows that $u \unlhd_L z$, for some $u \in \partial(a)$, and $z \unlhd_L p$, whence $u \unlhd_L p$, but $u \neq p$ (because $p \nleq a$), and so $u \lhd_L p$.

Conversely, let $u \in \partial(a)$ such that $u \lhd_L p$. Thus we also have $u \lhd_{L[p^*]} p$. Since $p \leq p^* \vee q$ by Lemma 6.3 (iv) and since p, p^*, and q are distinct atoms, the relation $p \, D_{L[p^*]} \, p^*$ holds. From Lemma 6.4(ii), it follows that $p^* \, D_{L[p^*]} \, u$, so $p^* \, D_{L[p^*]} \, u \lhd_{L[p^*]} p \, D_{L[p^*]} \, p^*$, whence $p^* \lhd_{L[p^*]} p^*$. □

Corollary 6.6.

(i) *The canonical embedding from L into $L[p^*]$ has the congruence extension property; in fact, it is an embedding for the $\lhd$ relation on atoms.*
(ii) *If L is lower bounded, then $L[p^*]$ is lower bounded.*

Proof. (i) By Theorem 2.30 and Lemma 2.36 in [8], it is sufficient to prove that $x \unlhd_L y$ iff $x \unlhd_{L[p^*]} y$, for all atoms x and y of L, which follows immediately from the stronger statement Lemma 6.5(iv).

(ii) It is well-known that a finite lattice K is lower bounded iff it has no D_K-cycle, that is, the relation $\lhd_K$ is *irreflexive*, see [8, Corollary 2.39]. Suppose that L is lower bounded. It follows from Lemma 6.5(iv) that the relation $x \lhd_{L[p^*]} x$ holds for no $x \in \mathrm{At}\, L$. Suppose that $p^* \lhd_{L[p^*]} p^*$. It follows from Lemma 6.5(v) that there exists $u \in \partial(a)$ such that $u \lhd_L p$. By Lemma 6.4(i), $p \lhd_L u$, whence L has a D_L-cycle, a contradiction. Therefore, the relation $x \lhd_{L[p^*]} x$ holds for no atom x of $L[p^*]$. □

Proof of Theorem 6.1. We present the proof for "join-semidistributive", the proof for "lower bounded" is similar. Since L is finite, it suffices to prove that every biatomicity problem $p \leq a \vee b$ in L can be solved in some finite, atomistic, join-semidistributive $\langle \vee, \wedge, 0, 1, \mathsf{at} \rangle$-extension of L in which L has the congruence extension property. We argue by induction on $\ell_L(a) + \ell_L(b)$, where $\ell_L(x)$ denotes the minimal size of a subset X of $\mathrm{At}\, L$ such that $x = \bigvee X$, for all $x \in L$. If $\ell_L(a) = \ell_L(b) = 1$ then the biatomicity problem $p \leq a \vee b$ is already solved in L, by $x = a$ and $y = b$. Now suppose, for example, that $b = c \vee q$, for some $c \in L \setminus \{0\}$ and some atom q such that $\ell_L(c) < \ell_L(b)$. Let $\overline{a} \leq a \vee c$ be minimal such that $p \leq \overline{a} \vee q$. By Lemma 6.3, there exists a finite join-semidistributive $\langle \vee, \wedge, 0, 1, \mathsf{at} \rangle$-extension L_1 of L, in which L has the congruence extension property, such that there exists an atom $p' \leq \overline{a}$ with $p \leq p' \vee q$. So $p' \leq a \vee c$ in L_1 and $\ell_{L_1}(a) + \ell_{L_1}(c) \leq \ell_L(a) + \ell_L(c) < \ell_L(a) + \ell_L(b)$. Thus, arguing as above, we obtain a finite join-semidistributive $\langle \vee, \wedge, 0, 1, \mathsf{at} \rangle$-extension L_2 of L_1, in which L_1 has the congruence extension property, with atoms $x \leq a$

and $v \leq c$ such that $p' \leq x \vee v$. So $p \leq p' \vee q \leq x \vee (v \vee q)$. Thus, again by Lemma 6.3, there exists a finite join-semidistributive $\langle \vee, \wedge, 0, 1, \mathsf{at} \rangle$-extension L_3 of L_2, in which L_2 has the congruence extension property, with an atom $y \leq v \vee q$ such that $p \leq x \vee y$. Observe that $y \leq v \vee q \leq c \vee q = b$. $\square$

7. A quasi-identity for nœtherian biatomic join-semidistributive lattices

Let θ be the following quasi-identity in the language $\langle \vee, \wedge \rangle$ of lattice theory:

$$[u \leq a \vee b \vee v \ \&\ v \leq a \vee c \vee u \ \&\ (a \vee u) \wedge (b \vee c) \leq a \\ \&\ (a \vee b) \wedge (a \vee u) = (a \vee c) \wedge (a \vee v) = (a \vee u) \wedge (a \vee v) = a] \\ \Longrightarrow u \leq a.$$

The present section will be mainly devoted to proving the following result.

Theorem 7.1. *Every nœtherian, atomistic, biatomic join-semidistributive lattice with zero satisfies θ.*

Let M be a nœtherian, atomistic, biatomic join-semidistributive lattice with zero. Observe that M is a *complete* lattice. Let a, b, c, u, and v be elements of M satisfying the premise of θ, that is, the statement

$$u \leq a \vee b \vee v \ \&\ v \leq a \vee c \vee u \ \&\ (a \vee u) \wedge (b \vee c) \leq a \\ \&\ (a \vee b) \wedge (a \vee u) = (a \vee c) \wedge (a \vee v) = (a \vee u) \wedge (a \vee v) = a.$$

Suppose that $u \nleq a$, and put $S = \operatorname{At} M \setminus [0, a]$. Since M is atomistic, there exists $p \in S$ such that $p \leq u$.

Lemma 7.2. *There are elements u_0, v_0 of $S^{\vee}$ such that the following inequalities hold:*

$$\begin{aligned} &u_0 \leq u \text{ and } v_0 \leq v; \\ &u_0 \leq a \vee b \vee v_0 \text{ and } v_0 \leq a \vee c \vee u_0. \end{aligned} \tag{7.1}$$

Proof. Suppose that $v \leq a$. Then $u \leq a \vee b$, thus $u \leq (a \vee b) \wedge (a \vee u) = a$, a contradiction. Hence $v \nleq a$.

Put $x_0 = p$; observe that $x_0 \in S$. So $x_0 \leq u \leq a \vee b \vee v$, with v nonzero (because $v \nleq a$). Thus, since M is biatomic, there exists an atom y_0 of M such that $y_0 \leq v$ and $x_0 \leq a \vee b \vee y_0$. If $y_0 \leq a$, then $x_0 \leq a \vee b$, but $x_0 \leq u$, and so $x_0 \leq u \wedge (a \vee b) \leq a$, a contradiction. Hence $y_0 \in S$.

Proceeding the same way with the inequality $y_0 \leq a \vee c \vee u$ and then inductively, we obtain elements x_n and y_n, for $n < \omega$, of S such that $x_n \leq u$, $y_n \leq v$, $x_n \leq a \vee b \vee y_n$, and $y_n \leq a \vee c \vee x_{n+1}$, for all $n < \omega$. Then $u_0 = \bigvee_{n<\omega} x_n$ and $v_0 = \bigvee_{n<\omega} y_n$ (these are, by Lemma 3.1, finite joins) are as required. $\square$

Now, for $n < \omega$, suppose we have constructed u_n, $v_n \in S^{\vee}$ that satisfy the following inequalities:

$$\begin{aligned} u_n &\leq a \vee b \vee v_n; \\ v_n &\leq a \vee c \vee u_n; \\ u_n &\leq a \vee u; \\ v_n &\leq a \vee v. \end{aligned} \tag{7.2}$$

Since $u_n \leq b \vee (a \vee v_n)$ and M is biatomic, for every $x \in S \cap [0, u_n]$, there exists an atom $x^* \leq a \vee v_n$ of M such that $x \leq b \vee x^*$. If $x^* \leq a$, then $x \leq a \vee b$. However, $x \leq u_n \leq a \vee u$, and so $x \leq (a \vee u) \wedge (a \vee b) = a$, a contradiction since $x \in S$. Hence, $x^* \in S$, so that $v_{n+1} = \bigvee\{x^* \mid x \in S \cap [0, u_n]\}$ belongs to $S^\vee \cap [0, a \vee v_n]$ and $u_n \leq b \vee v_{n+1}$. Proceeding in a similar fashion with the inequality $v_n \leq c \vee (a \vee u_n)$, we obtain elements u_{n+1} and v_{n+1} of $S^\vee$ such that the following inequalities hold, see the right half of Figure 1:

$$\begin{aligned} u_{n+1} &\leq a \vee u_n; \\ v_{n+1} &\leq a \vee v_n; \\ u_n &\leq b \vee v_{n+1}; \\ v_n &\leq c \vee u_{n+1}. \end{aligned} \tag{7.3}$$

We verify that all the inequalities (7.2) are satisfied with n replaced by $n + 1$.

- $u_{n+1} \leq a \vee u_n \leq a \vee b \vee v_{n+1}$, and, similarly, $v_{n+1} \leq a \vee c \vee u_{n+1}$.
- $u_{n+1} \leq a \vee u_n \leq a \vee u$, and, similarly, $v_{n+1} \leq a \vee v$.

Therefore, the values u_0 and v_0 obtained in Lemma 7.2 can be extended to sequences $(u_n)_{n<\omega}$ and $(v_n)_{n<\omega}$ of elements of $S^\vee$ that satisfy the inequalities listed in (7.2) and (7.3) for all $n < \omega$.

A straightforward application of the last two inequalities in (7.3) yields immediately the following lemma.

Lemma 7.3. *The sequence* $(b \vee c \vee u_{2n})_{n<\omega}$ *is increasing.*

Since M is nœtherian, there exists $n < \omega$ such that $b \vee c \vee u_{2n} = b \vee c \vee u_{2n+2}$. Therefore, by using the last two inequalities in (7.3), we also obtain the following equality:

$$b \vee c \vee u_{2n} = b \vee c \vee v_{2n+1}. \tag{7.4}$$

For any $n < \omega$, we let U_n and V_n be finite subsets of S such that $u_n = \bigvee U_n$ and $v_n = \bigvee V_n$. The existence of such sets is ensured by Lemma 3.1.

Lemma 7.4. $U_k \cap V_l = \varnothing$, *for all* k, $l < \omega$.

Proof. Let $x \in U_k \cap V_l$. Then $x \leq u_k \leq a \vee u$ and $x \leq v_l \leq a \vee v$, thus $x \leq (a \vee u) \wedge (a \vee v) = a$, which contradicts the fact that $x \in S$. □

Now (7.4) can be written as $b \vee c \vee \bigvee U_{2n} = b \vee c \vee \bigvee V_{2n+1}$. But from Lemma 3.2 (applied to $K = M$) and Lemma 7.4, it follows that $U_{2n} \cap V_{2n+1} = \varnothing$, and consequently $u_{2n} = \bigvee U_{2n} \leq b \vee c$. However, $u_{2n} \leq a \vee u$, so $u_{2n} \leq (a \vee u) \wedge (b \vee c) \leq a$, a contradiction since $u_{2n} \in S^\vee$. This completes the proof of Theorem 7.1.

Corollary 7.5. *There exists a finite, atomistic, join-semidistributive lattice* L *that cannot be embedded into any finite* (*or even nœtherian*) *atomistic biatomic join-semidistributive lattice.*

Proof. Put $L = \mathbf{Co}(\mathbb{Q}^2, \{a, b, c, u, v\})$ where a, b, c, u, v are as on the left half of Figure 1, the lattice of all intersections with $\{a, b, c, u, v\}$ of all convex subsets of $\mathbb{Q}^2$, see [3]. It is well-known that all lattices of that form are join-semidistributive. This configuration is obtained, for example, with $a = \{(0, 3)\}$, $b = \{(-2, 0)\}$, $c = \{(2, 0)\}$, $u = \{(-1, 1)\}$, and $v = \{(1, 1)\}$. Then the premise of θ holds in L for these elements, although $u \nleq a$: hence L does not satisfy θ. □

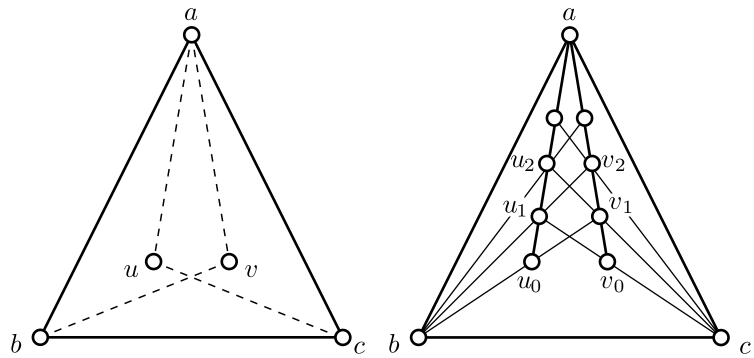

FIGURE 1.

Remark 7.6. A closer look at the proof of Theorem 7.1 shows that L cannot be embedded into any finite, biatomic, join-semidistributive lattice M such that there is an atom of M below either u or v but not below a.

Remark 7.7. The lattice L can be embedded into an algebraic, atomistic, biatomic convex geometry (see [3]), namely, the lattice $\mathbf{Co}(\mathbb{Q}^2)$ of all convex subsets of $\mathbb{Q}^2$. In fact, L can be embedded into a *join-semidistributive* atomistic biatomic sublattice of $\mathbf{Co}(\mathbb{Q}^2)$, namely, the lattice of convex polytopes of $\mathbb{Q}^2$, that is, finitely generated convex subsets of $\mathbb{Q}^2$. Of course, this lattice is neither nœtherian, nor complete.

Remark 7.8. We recall that a partially ordered set is *well-founded*, if every nonempty subset has a minimal element. Then one can prove that the quasi-identity θ is satisfied by every well-founded, biatomic, join-semidistributive lattice L, *under the additional assumption that u is an atom of L*. We do not know whether the latter assumption can be eliminated, see Problem 6.

Remark 7.9. It is proved in [3] that every finite join-semidistributive lattice has a zero-preserving lattice embedding into a lattice of the form $\mathbf{S_p}(A)$ (the lattice of all algebraic subsets of A), for a lattice A that is both algebraic and dually algebraic. In particular, $\mathbf{S_p}(A)$ is biatomic and lower continuous. It follows that θ is not satisfied by all lower continuous, atomistic, biatomic join-semidistributive lattices.

We observe the following immediate consequence of Corollary 3.5 and Theorem 7.1.

Corollary 7.10. *The lattice $L = \mathbf{Co}(\mathbb{Q}^2, \{a, b, c, u, v\})$ of Corollary* 7.5 *cannot be embedded into any nœtherian, biatomic, join-semidistributive lattice.*

8. THE QUASIVARIETY $\mathbf{Q}(\mathcal{BJ}_{\mathrm{f}})$

Notation. We denote by $\mathcal{BJ}$ the class of all atomistic, biatomic, join-semidistributive lattices, and by $\mathcal{LB}$ the class of all lower bounded lattices.

We use standard notation for the basic operators defined on $\langle\vee,\wedge\rangle$-structures, in particular, for a class $\mathcal{K}$ of $\langle\vee,\wedge\rangle$-structures, we define

- the class $\mathcal{K}_{\mathrm{f}}$ of all *finite* structures from $\mathcal{K}$,
- the class $\mathbf{S}(\mathcal{K})$ of all structures that are embeddable into some structure of $\mathcal{K}$,

- the class $\mathbf{P}_{\boldsymbol{\omega}}(\mathcal{K})$ of all *finite* direct products of structures of $\mathcal{K}$,
- the class $\mathbf{P}_{\mathbf{u}}(\mathcal{K})$ of all ultraproducts of structures of $\mathcal{K}$,
- the quasivariety $\mathbf{Q}(\mathcal{K})$ generated by $\mathcal{K}$.

It was proved in [2] that $\mathcal{LB} \subset \mathbf{Q}(\mathcal{LB}_{\mathrm{f}}) \subset \boldsymbol{SD}_{\vee}$, both containments being proper. The class of biatomic join-semidistributive lattices provides a new element in this hierarchy. Our interest in this section will be focused on the quasivariety generated by $\mathcal{BJ}_{\mathrm{f}}$.

First we state that the finite members of this quasivariety are those embeddable into lattices from $\mathcal{BJ}_{\mathrm{f}}$.

Proposition 8.1. *The finite members of* $\mathbf{Q}(\mathcal{BJ}_{\mathrm{f}})$ *are exactly the lattices that are embeddable into some finite, atomistic, biatomic, join-semidistributive lattice. In formula,* $\mathbf{Q}(\mathcal{BJ}_{\mathrm{f}})_{\mathrm{f}} = \mathbf{S}(\mathcal{BJ}_{\mathrm{f}})$.

Proof. It follows from results of the algebraic theory of quasivarieties that the equality $\mathbf{Q}(\mathcal{K}) = \mathbf{S}\mathbf{P}_{\mathbf{u}}\mathbf{P}_{\boldsymbol{\omega}}(\mathcal{K})$ holds for any class $\mathcal{K}$, see, for example, [9, Corollary 2.3.4(3)]. We wish to prove that any finite member L of $\mathbf{Q}(\mathcal{BJ}_{\mathrm{f}})$ is embeddable into some member of $\mathcal{BJ}_{\mathrm{f}}$. Since the class $\mathcal{BJ}_{\mathrm{f}}$ is closed under finite direct products, that is, $\mathbf{P}_{\boldsymbol{\omega}}(\mathcal{BJ}_{\mathrm{f}}) \subseteq \mathcal{BJ}_{\mathrm{f}}$, it follows from the formula above that there exists a lattice embedding $f \colon L \hookrightarrow L'$ where $L' \in \mathbf{P}_{\mathbf{u}}(\mathcal{BJ}_{\mathrm{f}})$, that is, L' is an ultraproduct of members of $\mathcal{BJ}_{\mathrm{f}}$. Since L is a finite system in a finite first-order language, a standard argument about ultraproducts shows that L can be embedded into some system from $\mathcal{BJ}_{\mathrm{f}}$. □

Evidently, the proof above can be extended to any finite first-order language, in particular the language $\langle \vee, \wedge, 0 \rangle$ if we want to deal with lattices with zero, and so on.

Proposition 8.2. *The following proper containments hold:*
$\mathbf{Q}(\mathcal{LB}_{\mathrm{f}}) \subset \mathbf{Q}(\mathcal{BJ}_{\mathrm{f}}) \subset \boldsymbol{SD}_{\vee}$.

Proof. It follows from Corollary 4.2 that $\mathbf{Q}(\mathcal{LB}_{\mathrm{f}}) \subseteq \mathbf{Q}(\mathcal{BJ}_{\mathrm{f}})$. Furthermore, the finite members of $\mathbf{Q}(\mathcal{LB}_{\mathrm{f}})$ are exactly the finite lower bounded lattices while the lattice $\mathbf{Co}(\mathbf{4})$ of all order-convex subsets of a four-element chain is finite, atomistic, biatomic, join-semidistributive, and not lower bounded, which shows that the containment above is proper. The containment $\mathbf{Q}(\mathcal{BJ}_{\mathrm{f}}) \subseteq \boldsymbol{SD}_{\vee}$ holds by definition, and Corollary 7.5 provides an example of a finite join-semidistributive lattice which, by Proposition 8.1, does not belong to $\mathbf{Q}(\mathcal{BJ}_{\mathrm{f}})$. □

9. Open problems

According to Corollary 7.5, there exists a finite atomistic join-semidistributive lattice that cannot be embedded into any finite, atomistic, biatomic, join-semidistributive lattice. However, it is not hard to prove that for all finite atomistic lattices K and L such that K has a $\langle \vee, 0, \mathsf{at} \rangle$-embedding into L, if L is join-semidistributive (resp. lower bounded), then so is K. Thus, in view of Theorem 6.1, the following question is natural.

Problem 1. Let L be a finite, atomistic, join-semidistributive (resp., lower bounded) lattice. Does L have a $\langle \vee, 0, \mathsf{at} \rangle$-embedding into some finite, atomistic, biatomic, join-semidistributive (resp., lower bounded) lattice?

Problem 2. For a finite join-semidistributive lattice L, is it *decidable* whether L can be embedded into some finite atomistic biatomic join-semidistributive lattice?

A variant of Problem 2 is the following.

Problem 3. Is the quasivariety $\mathbf{Q}(\mathcal{BJ}_{\mathrm{f}})$ (see Section 8) finitely based? That is, is the set of all quasi-identities satisfied by all finite atomistic biatomic join-semidistributive lattices equivalent to one of its finite subsets?

By Proposition 8.1, a positive answer to Problem 3 would imply a positive answer to Problem 2. Nevertheless we conjecture that Problem 3 has a negative solution.

Problem 4. Let L be a finite lattice. If L has a $\langle \vee, \wedge, 0 \rangle$-embedding into some finite, atomistic, biatomic, join-semidistributive lattice, then does L have an *atom-preserving* such embedding?

Problem 5. Does every finite lower bounded lattice L have an *atom-preserving* embedding into some finite, biatomic, lower bounded lattice?

We have seen that every finite lower bounded lattice admits a zero-preserving lattice embedding into some finite atomistic biatomic lower bounded lattice, see Corollary 4.2.

Our final problem asks for extensions of Theorem 7.1.

Problem 6. Does any complete, *upper continuous* (resp., *well-founded*), atomistic, biatomic, join-semidistributive lattice satisfy the quasi-identity θ defined in Section 7?

If we replace "join-semidistributive" by "convex geometry" then the answer to the corresponding problem is *no*, as, for example, $\mathbf{Co}(\mathbb{Q}^2)$ does not satisfy θ while it is a complete, algebraic (thus upper continuous), atomistic, biatomic convex geometry. However, $\mathbf{Co}(\mathbb{Q}^2)$ is not join-semidistributive, see [5, p. 234]. See also Remarks 7.8 and 7.9.

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