Notes on the description of join-distributive lattices by permutations

Kira Adaricheva and Gábor Czédli

Abstract. Let L be a join-distributive lattice with length n and width $(\mathrm{Ji}\, L) \leq k$. There are two ways to describe L by $k-1$ permutations acting on an n-element set: a combinatorial way given by P. H. Edelman and R. E. Jamison in 1985 and a recent lattice theoretical way of the second author. We prove that these two approaches are equivalent. Also, we characterize join-distributive lattices by trajectories.

Introduction. For $x \neq 1$ in a finite lattice L, let x^* denote the join of upper covers of x. A finite lattice L is *join-distributive* if the interval $[x, x^*]$ is distributive for all $x \in L \setminus \{1\}$. For other definitions, see K. Adaricheva [2], K. Adaricheva, V.A. Gorbunov and V.I. Tumanov [3], and N. Caspard and B. Monjardet [6], see G. Czédli [7, Proposition 2.1 and Remark 2.2] for a recent survey, and see (3) before the proof of Corollary 6 later for a particularly useful variant. The study of (the duals of) join-distributive lattices goes back to R. P. Dilworth [11], 1940. There were a lot of discoveries and rediscoveries of these lattices and equivalent combinatorial structures; see [3], [7], B. Monjardet [17], and M. Stern [18] for surveys. Note that join-distributivity implies semimodularity; the origin of this result is the combination of M. Ward [19] (see also R. P. Dilworth [11, page 771], where [19] is cited) and S. P. Avann [5] (see also P. H. Edelman [13, Theorem 1.1(E,H)], when [5] is recalled).

The *join-width* of L, denoted by $\mathrm{width}\,(\mathrm{Ji}\, L)$, is the largest k such that there is a k-element antichain of join-irreducible elements of L. As usual, S_n stands for the set of permutations acting on the set $\{1, \dots, n\}$. There are two known ways to describe a join-distributive lattice with join-width k and length n by $k-1$ permutations; our goal is to enlighten their connection. This connection exemplifies that Lattice Theory can be applied in Combinatorics and vice versa. We also give a new characterization of join-distributive lattices.

Two constructions. For $n \in \mathbb{N} = \{1, 2, \dots\}$ and $k \in \{2, 3, \dots\}$, let $\vec{\sigma} = \langle \sigma_2, \dots, \sigma_k \rangle \in S_n^{k-1}$. For convenience, $\sigma_1 \in S_n$ will denote the identity permutation. In the powerset join-semilattice $\langle P(\{1, \dots, n\}); \cup \rangle$, consider the subsemilattice $L_{\mathrm{EJ}}(\vec{\sigma})$ generated by

$$\big\{\{\sigma_i(1), \dots, \sigma_i(j)\} : i \in \{1, \dots, k\},\ j \in \{0, \dots, n\}\big\}. \tag{1}$$

2010 *Mathematics Subject Classification*: Primary 06C10; secondary 05E99.

Key words and phrases: Join-distributive lattice, semimodular lattice, diamond-free lattice, trajectory, permutation.

This research was supported by the NFSR of Hungary (OTKA), grant numbers K77432 and K83219, and by TÁMOP-4.2.1/B-09/1/KONV-2010-0005.

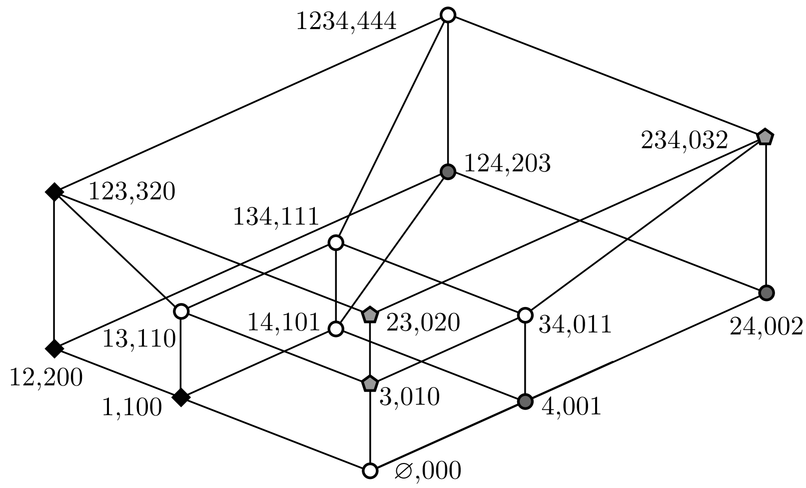

FIGURE 1. An example of $L_{\mathrm{EJ}}(\vec\sigma)$ and $L_{\mathrm{C}}(\vec\pi)$

Since it contains $\varnothing$, $L_{\mathrm{EJ}}(\vec\sigma)$ is a lattice, the *Edelman-Jamison lattice* determined by $\vec\sigma$. Its definition above is a straightforward translation from the combinatorial language of P. H. Edelman and R. E. Jamison [14, Theorem 5.2] to Lattice Theory. Actually, the original version in [14] describes a decomposition of convex geometries.

To present an example, let $\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$. Then $\vec\sigma = \langle \sigma_2, \sigma_3\rangle \in S_4^2$, and $L_{\mathrm{EJ}}(\vec\sigma)$ is depicted in Figure 1. In the label of an element in $L_{\mathrm{EJ}}(\vec\sigma)$, only the part before the comma is relevant; to save space, subsets are denoted by listing their elements without commas. For example, 134,111 in the figure stands for the subset $\{1,3,4\}$ of $\{1,2,3,4\}$. The chain defined in (1), apart from its top $\{1,2,3,4\}$ and bottom $\varnothing$, corresponds to the black-filled small squares for $i = 1$, the light grey-filled pentagons for $i = 2$, and the dark grey-filled circles for $i = 3$. Note that $L_{\mathrm{EJ}}(\vec\sigma)$ consists of all subsets of $\{1,2,3,4\}$ but $\{2\}$.

Next, we recall a related construction from G. Czédli [7]. Given $\vec\pi = \langle \pi_{12}, \dots, \pi_{1k}\rangle \in S_n^k$, we let $\pi_{ij} = \pi_{1j} \circ \pi_{1i}^{-1}$ for $i, j \in \{1, \dots, k\}$. Here we compose permutations from right to left, that is, $(\pi_{1j} \circ \pi_{1i}^{-1})(x) = \pi_{1j}(\pi_{1i}^{-1}(x))$. Note that $\pi_{ii} = \mathrm{id}$, $\pi_{ij} = \pi_{ji}^{-1}$, and $\pi_{jt} \circ \pi_{ij} = \pi_{it}$ hold for all $i, j, t \in \{1, \dots, k\}$. By an *eligible $\vec\pi$-tuple* we mean a k-tuple $\vec x = \langle x_1, \dots, x_k\rangle \in \{0, 1, \dots, n\}^k$ such that $\pi_{ij}(x_i + 1) \ge x_j + 1$ holds for all $i, j \in \{1, \dots, k\}$ such that $x_i < n$. Note that an eligible $\vec\pi$-tuple belongs to $\{0, 1, \dots, n-1\}^k \cup \{\langle n, \dots, n\rangle\}$ since $x_j = n$ implies $x_i = n$. The set of eligible $\vec\pi$-tuples is denoted by $L_{\mathrm{C}}(\vec\pi)$. It is a poset with respect to the componentwise order: $\vec x \le \vec y$ means that $x_i \le y_i$ for all $i \in \{1, \dots, k\}$. It is trivial to check that $\langle n, \dots, n\rangle \in L_{\mathrm{C}}(\vec\pi)$ and that $L_{\mathrm{C}}(\vec\pi)$ is a meet-subsemilattice of the k-th direct power of the chain $\{0 \prec 1 \prec \dots \prec n\}$.

Therefore, $L_{\mathrm{C}}(\vec{\pi})$ is a lattice, the *$\vec{\pi}$-coordinatized lattice*. Its construction is motivated by G. Czédli and E. T. Schmidt [8, Theorem 1], see also M. Stern [18], which asserts that there is a surjective cover-preserving join-homomorphism $\varphi\colon \{0 \prec \cdots \prec n\}^k \to L$, provided L is semimodular. Then, as it is easy to verify, $u \mapsto \bigvee\{x : \varphi(x) = u\}$ is a meet-embedding of L into $\{0 \prec \cdots \prec n\}^k$.

To give an example, let $\pi_{12} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$, $\pi_{13} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$, and let $\vec{\pi} = \langle \pi_{12}, \pi_{13} \rangle \in S_4^2$. Then Figure 1 also gives $L_{\mathrm{C}}(\vec{\pi})$; the eligible $\vec{\pi}$-tuples are given after the commas in the labels. For example, 23,020 in the figure corresponds to $\langle 0, 2, 0 \rangle$. Note that if $\mu_{12} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$ and $\mu_{13} = \pi_{13}$, then $L_{\mathrm{C}}(\vec{\pi}) \cong L_{\mathrm{C}}(\vec{\mu})$. Furthermore, the problem of characterizing those pairs of members of S_n^k that determine the same lattice is not solved yet if $k \geq 3$. For $k = 2$ the solution is given in G. Czédli and E. T. Schmidt [10]; besides $L_{\mathrm{C}}(\vec{\pi}) \cong L_{\mathrm{C}}(\vec{\mu})$ above, see also G. Czédli [7, Example 5.3] to see the difficulty.

The connection of join-distributivity to $L_{\mathrm{EJ}}(\vec{\pi})$ and $L_{\mathrm{C}}(\vec{\pi})$ will be given soon.

The two constructions are equivalent. For $\langle \gamma_2, \ldots, \gamma_k \rangle \in S_n^{k-1}$, we let $\langle \gamma_2, \ldots, \gamma_k \rangle^{-1} = \langle \gamma_2^{-1}, \ldots, \gamma_k^{-1} \rangle$.

Proposition 1. *For every $\vec{\sigma} \in S_n^{k-1}$, $L_{\mathrm{EJ}}(\vec{\sigma})$ is isomorphic to $L_{\mathrm{C}}(\vec{\sigma}^{-1})$.*

In some vague sense, Figure 1 reveals why $L_{\mathrm{EJ}}(\vec{\sigma})$ could be of the form $L_{\mathrm{C}}(\vec{\pi})$ for some $\vec{\pi}$. Namely, for $x \in L_{\mathrm{EJ}}(\vec{\sigma})$ and $i \in \{1, \ldots, k\}$, we can define the i-th coordinate of x as the length of the intersection of the ideal $\{y \in L_{\mathrm{EJ}}(\vec{\sigma}) : y \leq x\}$ and the chain given in (1). However, the proof is more complex than this initial idea.

Proof. Denote $\vec{\sigma}^{-1}$ by $\vec{\pi} = \langle \pi_{12}, \ldots, \pi_{1k} \rangle$. Note that $\pi_{11} = \sigma_1^{-1} = \mathrm{id} \in S_n$. For $U \in L_{\mathrm{EJ}}(\vec{\sigma})$ and $i \in \{1, \ldots, k\}$, let $U(i) = \max\bigl\{j : \{\sigma_i(1), \ldots, \sigma_i(j)\} \subseteq U\bigr\}$, where $\max \varnothing$ is defined to be 0. We assert that the map

$$\varphi\colon L_{\mathrm{EJ}}(\vec{\sigma}) \to L_{\mathrm{C}}(\vec{\pi}), \text{ defined by } U \mapsto \langle U(1), \ldots, U(k) \rangle,$$

is a lattice isomorphism. To prove that $\varphi(U)$ is an eligible $\vec{\pi}$-tuple, assume that $i, j \in \{1, \ldots, k\}$ such that $U(i) < n$. Then $\sigma_i(U(i) + 1) \notin U$ yields $\sigma_i(U(i) + 1) \notin \{\sigma_j(1), \ldots, \sigma_j(U(j))\}$. However, $\sigma_i(U(i) + 1) \in \{1, \ldots, n\} = \{\sigma_j(1), \ldots, \sigma_j(n)\}$, and we conclude that $\sigma_i(U(i) + 1) = \sigma_j(t)$ holds for some $t \in \{U(j) + 1, \ldots, n\}$. Hence

$$\begin{aligned} \pi_{ij}(U(i) + 1) &= (\pi_{1j} \circ \pi_{i1})(U(i) + 1) = \pi_{1j}\bigl(\pi_{i1}(U(i) + 1)\bigr) \\ &= \pi_{1j}\bigl(\pi_{1i}^{-1}(U(i) + 1)\bigr) = \sigma_j^{-1}\bigl(\sigma_i(U(i) + 1)\bigr) \\ &= \sigma_j^{-1}\bigl(\sigma_j(t)\bigr) = t \geq U(j) + 1. \end{aligned}$$

This proves that $\varphi(U)$ is an eligible $\vec{\pi}$-tuple, and φ is a map from $L_{\mathrm{EJ}}(\vec{\sigma})$ to $L_{\mathrm{C}}(\vec{\pi})$. Since $L_{\mathrm{EJ}}(\vec{\sigma})$ is generated by the set given in (1), we conclude

$$U = \bigcup_{i=1}^{k} \{\sigma_i(1), \dots, \sigma_i(U(i))\}.$$

This implies that U is determined by $\langle U(1), \dots, U(k)\rangle = \varphi(U)$, that is, φ is injective. To prove that φ is surjective, let $\vec{x} = \langle x_1, \dots, x_k\rangle$ be a $\vec{\pi}$-eligible tuple, that is, $\vec{x} \in L_{\mathrm{C}}(\vec{\pi})$. Define

$$V = \bigcup_{i=1}^{k} \{\sigma_i(1), \dots, \sigma_i(x_i)\}. \tag{2}$$

(Note that if $x_i = 0$, then $\{\sigma_i(1), \dots, \sigma_i(x_i)\}$ denotes the empty set.) For the sake of contradiction, suppose $\varphi(V) \neq \vec{x}$. Then, by the definition of φ, there exists an $i \in \{1, \dots, k\}$ such that $\sigma_i(x_i + 1) \in V$. Hence, there is a $j \in \{1, \dots, k\}$ such that $\sigma_i(x_i+1) \in \{\sigma_j(1), \dots, \sigma_j(x_j)\}$. That is, $\sigma_i(x_i+1) = \sigma_j(t)$ for some $t \in \{1, \dots, x_j\}$. Therefore,

$$\begin{aligned}\pi_{ij}(x_i+1) = \pi_{1j}(\pi_{i1}(x_i+1)) = \pi_{1j}(\pi_{1i}^{-1}(x_i+1) = \sigma_j^{-1}(\sigma_i(x_i+1))\\ = \sigma_j^{-1}(\sigma_j(t)) = t \leq x_j,\end{aligned}$$

which contradicts the $\vec{\pi}$-eligibility of $\vec{x}$. Thus $\varphi(V) = \vec{x}$ and φ is surjective.

We have shown that φ is bijective. For $\vec{x} \in L_{\mathrm{C}}(\vec{\pi})$, $\varphi^{-1}(\vec{x})$ is the set V given in (2). Thus φ and φ^{-1} are monotone, and φ is a lattice isomorphism. $\square$

Two descriptions. The following theorem is a straightforward consequence of Theorems 5.1 and 5.2 in P. H. Edelman and R. E. Jamison [14], which were formulated and proved within Combinatorics.

Theorem 2. *Up to isomorphism, join-distributive lattices of length n and join-width at most k are characterized as lattices $L_{\mathrm{EJ}}(\vec{\sigma})$ with $\vec{\sigma} \in S_n^{k-1}$.*

The next theorem was motivated and proved by the second author [7] in a purely lattice theoretical way.

Theorem 3. *Up to isomorphism, join-distributive lattices of length n and join-width at most k are characterized as the $\vec{\pi}$-coordinatized lattices $L_{\mathrm{C}}(\vec{\pi})$ with $\vec{\pi} \in S_n^{k-1}$.*

Remark 4. Since there is no restriction on $(n, k) \in \mathbb{N} \times \{2, 3, \dots\}$ in Theorems 2 and 3, one might have the feeling that, for a given n, the join-width of a join-distributive lattice of length n can be arbitrarily large. This is not so since, up to isomorphism, there are only finitely many join-distributive lattices of length n.

The statement of Remark 4 follows from the fact that each join-distributive lattice of length n is dually isomorphic to the lattice of closed sets of a convex geometry on the set $\{1, \dots, n\}$, see P. H. Edelman [12, Theorem 3.3] together with the sixteenth line in the proof of Theorem 1.9 in K. Adaricheva, V.A. Gorbunov and V.I. Tumanov [3]; see also [7, Lemma 7.4], where this is surveyed. The statement also follows, in a different way, from [7, Corollary 4.4].

Remark 5. Obviously, Proposition 1 and Theorem 2 imply Theorem 3 and, similarly, Proposition 1 and Theorem 3 imply Theorem 2. Thus we obtain a new, combinatorial proof of Theorem 3 and a new, lattice theoretical proof of Theorem 2.

Comparison. We can compare Theorems 2 and 3, and the corresponding original approaches, as follows.

In case of Theorem 2, the construction of the lattice $L_{\mathrm{EJ}}(\vec{\sigma})$ is very simple, and a join-generating subset is also given.

In case of Theorem 3, the elements of the lattice $L_{\mathrm{C}}(\vec{\pi})$ are exactly given by their coordinates, the eligible $\vec{\pi}$-tuples. Moreover, the meet operation is easy, and we have a satisfactory description of the optimal meet-generating subset since it was proved in [7, Lemma 6.5] that

$$\mathrm{Mi}(L_{\mathrm{C}}(\vec{\pi})) = \big\{\langle \pi_{11}(i)-1,\dots,\pi_{1k}(i)-1\rangle : i\in\{1,\dots,n\}\big\}.$$

Characterization by trajectories. For a lattice L of finite length, the set $\big\{[a,b] : a\prec b,\ a,b\in L\big\}$ of prime intervals of L will be denoted by $\mathrm{PrInt}(L)$. For $[a,b],[c,d]\in\mathrm{PrInt}(L)$, we say that $[a,b]$ and $[c,d]$ are *consecutive* if $\{a,b,c,d\}$ is a covering square, that is, a 4-element cover-preserving boolean sublattice of L. The transitive reflexive closure of the consecutiveness relation on $\mathrm{PrInt}(L)$ is an equivalence, and the blocks of this equivalence relation are called the *trajectories* of L; this concept was introduced for some particular semimodular lattices in G. Czédli and E. T. Schmidt [9]. For distinct $[a,b],[c,d]\in\mathrm{PrInt}(L)$, these two prime intervals are *comparable* if either $b\leq c$, or $d\leq a$. Before formulating the last statement of the paper, it is reasonable to mention that, for any finite lattice L,

$$L \text{ is join-distributive iff it is semimodular and meet-semidistributive.} \tag{3}$$

This follows from K. Adaricheva, V.A. Gorbunov and V.I. Tumanov [3, Theorems 1.7 and 1.9]; see also D. Armstrong [4, Theorem 2.7] for the present formulation.

Corollary 6. *For a semimodular lattice L, the following three conditions are equivalent.*

(i) *L is join-distributive.*
(ii) *L is of finite length, and for every trajectory T of L and every maximal chain C of L, $|\mathrm{PrInt}(C)\cap T| = 1$.*
(iii) *L is of finite length, and no two distinct comparable prime intervals of L belong to the same trajectory.*

As an interesting consequence, note that each of (ii) and (iii) above, together with semimodularity, implies that L is finite.

Proof of Corollary 6. Since any two comparable prime intervals belong to the set of prime intervals of an appropriate maximal chain C, (ii) implies (iii). So we have to prove that (i) $\Rightarrow$ (iii) and that (iii) $\Rightarrow$ (i); we give two alternative arguments for each of these two implications. Let $n = \operatorname{length} L$.

Assume (i). Then L is semimodular by (3), and it contains no cover-preserving diamond by the definition of join-distributivity. Thus G. Czédli [7, Lemma 3.3] implies (ii).

For a second argument, assume (i) again. Let $\operatorname{Pow}(\{1,\dots,n\})$ denote the set of all subsets of $\{1,\dots,n\}$. It is known that L is isomorphic to an appropriate join-subsemilattice $\mathfrak{F}$ of the powerset $\bigl(\operatorname{Pow}(\{1,\dots,n\});\cup\bigr)$ such that $\varnothing\in\mathfrak{F}$ and each $X\in\mathfrak{F}\setminus\{\varnothing\}$ contains an element a with the property $X\setminus\{a\}\in\mathfrak{F}$. The structure $\langle\{1,\dots,n\};\mathfrak{F}\rangle$ is an *antimatroid* on the base set $\{1,\dots,n\}$ (this concept is due to R. E. Jamison-Waldner [16]), and the existence of an appropriate $\mathfrak{F}$ follows from P. H. Edelman [12, Theorem 3.3] and D. Armstrong [4, Lemma 2.5]; see also K. Adaricheva, V.A. Gorbunov and V.I. Tumanov [3, Subsection 3.1] and G. Czédli [7, Section 7]. Now, we can assume that $L=\mathfrak{F}$. We assert that, for any $X,Y\in\mathfrak{F}$,

$$X\prec Y \quad\text{iff}\quad X\subset Y \text{ and } |Y\setminus X|=1. \tag{4}$$

The "if" part is obvious. For the sake of contradiction, suppose $X\prec Y$ and x and y are distinct elements in $Y\setminus X$. Pick a sequence $Y=Y_0\supset Y_1\supset\cdots\supset Y_t=\varnothing$ in $\mathfrak{F}$ such that $|Y_{i-1}\setminus Y_i|=1$ for $i\in\{1,\dots,t\}$. Then there is a j such that $|Y_j\cap\{x,y\}|=1$. This gives the desired contradiction since $X\cup Y_j\in\mathfrak{F}$ but $X\subset X\cup Y_j\subset Y$.

Armed with (4), assume that $\{A=B\wedge C,B,C,D=B\cup C\}$ is a covering square in $\mathfrak{F}$. Note that A and $B\cap C$ can be different; however, $A\subseteq B\cap C$. By (4), there exist $u,x\in D$ such that $B=D\setminus\{u\}$ and $C=D\setminus\{x\}$. These elements are distinct since $B\neq C$. Hence $x\in B$ and, by $A\subseteq C$, $x\notin A$. Using (4) again, we obtain $A=B\setminus\{x\}$. We have seen that whenever $[A,B]$ and $[C,D]$ are consecutive prime intervals, then there is a common x such that $A=B\setminus\{x\}$ and $C=D\setminus\{x\}$. This implies that for each trajectory T of $\mathfrak{F}$, there exists an $x_T\in\{1,\dots,n\}$ such that $X=Y\setminus\{x_T\}$ holds for all $[X,Y]\in T$. Clearly, this implies that (iii) holds for $\mathfrak{F}$, and also for L.

Next, assume (iii). Since any two prime intervals of a cover-preserving diamond would belong to the same trajectory, L contains no such diamond. Again, there are two ways to conclude (i).

First, by [7, Proposition 6.1], L is isomorphic to $L_{\mathrm{C}}(\vec{\pi})$ for some k and $\vec{\pi}\in S_n^{k-1}$, and we obtain from Theorem 3 that (i) holds.

Second, H. Abels [1, Theorem 3.9(a$\Rightarrow$b)] implies that L is a cover-preserving join-subsemilattice of a finite distributive lattice D. Thus if $x\in L\setminus\{1\}$, then the interval $[x,x^*]_L$ of L is a cover-preserving join-subsemilattice of D. Let $a_1,\dots,a_t$ be the covers of x in L, that is, the atoms of $[x,x^*]_L$. If we had, say, $a_1\le a_2\vee\cdots\vee a_t$, then we would get a contradiction in D as follows: $a_1=a_1\wedge(a_2\vee\cdots\vee a_t)=(a_1\wedge a_2)\vee\cdots\vee(a_1\wedge a_t)=x\wedge\cdots\wedge x=x$.

Thus $a_1, \ldots, a_t$ are independent atoms in $[x, x^*]_L$. Therefore, it follows from G. Grätzer [15, Theorem 380] and the semimodularity of $[x, x^*]_L$ that the sublattice S generated by $\{a_1, \ldots, a_t\}$ in L is the 2^t-element boolean lattice. In particular, length $S = t =$ length $\big([x, x^*]_L\big)$ since $\{x, x^*\} \subseteq S \subseteq [x, x^*]_L$. Since the embedding is cover-preserving, the length of the interval $[x, x^*]_D$ in D is also t. Hence $|\mathrm{Ji}\,([x, x^*]_D)| = t$ by [15, Corollary 112], which clearly implies $|[x, x^*]_D| \leq 2^t$. Now from $[x, x^*]_L \subseteq [x, x^*]_D$ and $2^t = |S| \leq |[x, x^*]_L| \leq |[x, x^*]_D| \leq 2^t$ we conclude $[x, x^*]_L = [x, x^*]_D$. This implies that $[x, x^*]_L$ is distributive. Thus (i) holds. $\square$

References

[1] Abels, H.: The geometry of the chamber system of a semimodular lattice. Order **8**, 143–158 (1991)

[2] Adaricheva, K. Representing finite convex geometries by relatively convex sets. European Journal of Combinatorics, to appear; `http://arxiv.org/abs/1101.1539`

[3] Adaricheva, K., Gorbunov, V.A., Tumanov, V.I.: Join-semidistributive lattices and convex geometries. Advances in Math. 173, 1–49 (2003)

[4] Armstrong, D.: The Sorting Order on a Coxeter Group. Journal of Combinatorial Theory Series A **116**, 1285–1305 (2009)

[5] Avann, S.P.: Application of the join-irreducible excess function to semimodular lattices. Math. Annalen **142**, 345–354 (1961)

[6] Caspard, N., Monjardet, B.: Some lattices of closure systems on a finite set. Discrete Mathematics and Theoretical Computer Science **6**, 163–190 (2004)

[7] Czédli, G.: Coordinatization of join-distributive lattices. `http://arxiv.org/abs/1208.3517`

[8] Czédli, G., Schmidt, E.T.: How to derive finite semimodular lattices from distributive lattices?. Acta Mathematica Hungarica **121**, 277–282 (2008)

[9] Czédli, G., Schmidt, E.T.: The Jordan-Hölder theorem with uniqueness for groups and semimodular lattices. Algebra Universalis **66**, 69–79 (2011)

[10] Czédli, G., Schmidt, E.T.: Intersections of composition series in groups and slim semimodular lattices by permutations. `http://arxiv.org/abs/1208.4749v1`

[11] Dilworth, R.P.: Lattices with unique irreducible decompositions. Annals of Mathematics (2) **41**, 771–777 (1940)

[12] Edelman, P.H.: Meet-distributive lattices and the anti-exchange closure. Algebra Universalis **10**, 290–299 (1980)

[13] Edelman, P.H.: Abstract convexity and meet-distributive lattices. In: Proc. Conf. Combinatorics and ordered sets (Arcata, Calif., 1985), Contemp. Math. **57**, 127–150, Amer. Math. Soc., Providence, RI (1986).

[14] Edelman, P.H., Jamison, R.E.: The theory of convex geometries. Geom. Dedicata **19**, 247–271 (1985)

[15] Grätzer, G.: Lattice Theory: Foundation. Birkhäuser Verlag, Basel (2011)

[16] Jamison-Waldner, R.E.: Copoints in antimatroids. In: Combinatorics, graph theory and computing, Proc. 11th southeast. Conf., Boca Raton/Florida 1980, Vol. II, Congr. Numerantium 29, 535–544 (1980)

[17] Monjardet, B.: A use for frequently rediscovering a concept. Order **1**, 415–417 (1985)

[18] Stern, M.: Semimodular Lattices. Theory and Applications, Encyclopedia of Mathematics and its Applications **73**. Cambridge University Press (1999)

[19] Ward, M.: Structure Residuation. Annals of Mathematics (2) **39**, 558-568 (1938)

Kira Adaricheva

Department of Mathematical Sciences, Yeshiva University, New York, 245 Lexington ave., New York, NY 10016, USA
e-mail: adariche@yu.edu

Gábor Czédli
University of Szeged, Bolyai Institute. Szeged, Aradi vértanúk tere 1, HUNGARY 6720
e-mail: czedli@math.u-szeged.hu
URL: http://www.math.u-szeged.hu/~czedli/