LATTICES OF QUASI-EQUATIONAL THEORIES AS CONGRUENCE LATTICES OF SEMILATTICES WITH OPERATORS, PART I

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ABSTRACT. We show that for every quasivariety $\mathcal{K}$ of structures (where both functions and relations are allowed) there is a semilattice $\mathbf{S}$ with operators such that the lattice of quasi-equational theories of $\mathcal{K}$ (the dual of the lattice of sub-quasivarieties of $\mathcal{K}$) is isomorphic to $\operatorname{Con}(\mathbf{S}, +, 0, \mathcal{F})$. As a consequence, new restrictions on the natural quasi-interior operator on lattices of quasi-equational theories are found.

1. MOTIVATION AND TERMINOLOGY

Our objective is to provide, for the lattice of quasivarieties contained in a given quasivariety (*Q-lattices* in short), a description similar to the one that characterizes the lattice of subvarieties of a given variety as the dual of the lattice of fully invariant congruences on a countably generated free algebra. Just as the result for varieties is more naturally expressed in terms of the lattice of equational theories, rather than the dual lattice of varieties, so it will be more natural to consider lattices of quasi-equational theories rather than lattices of quasivarieties.

The basic result is that the lattice of quasi-equational theories extending a given quasi-equational theory is isomorphic to the congruence lattice of a semilattice with operators preserving join and 0. These lattices support a natural quasi-interior operator, the properties of which lead to new restrictions on lattices of quasi-equational theories.

This is the first paper in a series of four. Part II shows that if $\mathbf{S}$ is a semilattice with both 0 and 1, and $\mathcal{G}$ is a *group* of operators on $\mathbf{S}$ such that each operator in $\mathcal{G}$ fixes both 0 and 1, then there is a quasi-equational theory $\mathcal{T}$ such that $\operatorname{Con}(\mathbf{S}, +, 0, \mathcal{G})$ is isomorphic to the lattice of quasi-equational theories extending $\mathcal{T}$. The third part [30] shows that if $\mathbf{S}$ is any semilattice with operators, then $\operatorname{Con} \mathbf{S}$ is isomorphic to the lattice of implicational theories extending some given implicational theory, but in a language that may

not include equality. The fourth paper [23], with T. Holmes, D. Kitsuwa and S. Tamagawa, concerns the structure of lattices of atomic theories in a language without equality.

The setting for varieties is traditionally *algebras*, i.e., sets with operations, whereas work on quasivarieties normally allows *structures*, i.e., sets with operations and relations. Some adjustments are required for the more general setting. Let us review the universal algebra of structures, following Section 1.4 of Gorbunov [17]; see also Gorbunov and Tumanov [19, 20] and Gorbunov [15].

The *type* of a structure is determined by its *signature* $\sigma = \langle \mathcal{F}, \mathcal{R}, \rho \rangle$ where $\mathcal{F}$ is a set of function symbols, $\mathcal{R}$ is a set of relation symbols, and $\rho : \mathcal{F} \cup \mathcal{R} \to \omega$ assigns arity. A *structure* is then $\mathbf{A} = \langle A, \mathcal{F}^{\mathbf{A}}, \mathcal{R}^{\mathbf{A}} \rangle$ where A is the carrier set, $\mathcal{F}^{\mathbf{A}}$ is the set of operations on A, and $\mathcal{R}^{\mathbf{A}}$ is the set of relations on A.

For structures $\mathbf{A}$ and $\mathbf{B}$ of the same type, a map $h : \mathbf{A} \to \mathbf{B}$ is a *homomorphism* if it preserves operations and $h(R^{\mathbf{A}}) \subseteq R^{\mathbf{B}}$ for each relation symbol R. An *endomorphism* of $\mathbf{A}$ is a homomorphism $\varepsilon : \mathbf{A} \to \mathbf{A}$.

The *kernel* $\ker h$ of a homomorphism h is a pair $\kappa = \langle \kappa_0, \kappa_1 \rangle$ where

- κ_0 is the equivalence relation on A induced by h, i.e., $(x, y) \in \kappa_0$ iff $h(x) = h(y)$,
- $\kappa_1 = \bigcup_{R \in \mathcal{R}} \kappa_1^R$ where $\kappa_1^R = h^{-1}(R^{\mathbf{B}}) = \{ \mathbf{s} \in A^{\rho(R)} : h(\mathbf{s}) \in R^{\mathbf{B}} \}$.

Equality is treated differently because, in standard logic, equality is assumed to be a congruence relation. Indeed, the statements that $\approx$ is reflexive, symmetric, transitive, and compatible with the functions of $\mathcal{F}$ and the relations of $\mathcal{R}$, are universal Horn sentences. Thus in normal quasi-equational logic we are working in the quasivariety given by these laws. This is not necessary: see Parts III and IV [30, 23].

A *congruence* on a structure $\mathbf{A} = \langle A, \mathcal{F}^{\mathbf{A}}, \mathcal{R}^{\mathbf{A}} \rangle$ is a pair $\theta = \langle \theta_0, \theta_1 \rangle$ where

- θ_0 is an equivalence relation on A that is compatible with the operations of $\mathcal{F}^{\mathbf{A}}$, and
- $\theta_1 = \bigcup_{R \in \mathcal{R}} \theta_1^R$ where each $\theta_1^R \subseteq A^{\rho(R)}$ and $R^{\mathbf{A}} \subseteq \theta_1^R$, i.e., the original relations of $\mathbf{A}$ are contained in those of θ_1, and for each $R \in \mathcal{R}$, if $\mathbf{a} \in \theta_1^R$ and $\mathbf{b} \in A^{\rho(R)}$ and $\mathbf{a}\,\theta_0\,\mathbf{b}$ componentwise, then $\mathbf{b} \in \theta_1^R$.

Note that if $h : \mathbf{A} \to \mathbf{B}$ is a homomorphism, then $\ker h$ is a congruence on $\mathbf{A}$. The collection of all congruences on $\mathbf{A}$ forms an algebraic lattice $\operatorname{Con} \mathbf{A}$ under set containment.

A subset $S \subseteq A$ is a *subuniverse* if it is closed under the operations of $\mathbf{A}$. A *substructure* of $\mathbf{A}$ is $\mathbf{S} = \langle S, \mathcal{F}^{\mathbf{S}}, \mathcal{R}^{\mathbf{S}} \rangle$ where S is a subuniverse of A, for each operation symbol $f \in \mathcal{F}$ the operation $f^{\mathbf{S}}$ is the restriction of $f^{\mathbf{A}}$ to $S^{\rho(f)}$, and for each relation symbol $R \in \mathcal{F}$ the relation $R^{\mathbf{S}}$ is $R^{\mathbf{A}} \cap S^{\rho(R)}$.

Given a congruence θ on a structure $\mathbf{A}$, we can form a *quotient structure* $\mathbf{A}/\theta$ by defining operations and relations on the θ_0-classes of A in the natural way. The isomorphism theorems carry over to this more general setting. In particular, if $h : \mathbf{A} \to \mathbf{B}$ is a homomorphism, then $h(\mathbf{A})$ is a substructure of $\mathbf{B}$, and $h(\mathbf{B})$ is isomorphic to $\mathbf{A}/\ker h$.

A congruence is *fully invariant* if, for every endomorphism ε of $\mathbf{A}$,

- $a\,\theta_0\,b$ implies $\varepsilon(a)\,\theta_0\,\varepsilon(b)$, and
- for each $R \in \mathfrak{R}$, $\mathbf{a} \in \theta_1^R$ implies $\varepsilon(\mathbf{a}) \in \theta_1^R$.

The lattice of fully invariant congruences is denoted Ficon $\mathbf{A}$.

The congruence generation theorems are straightforward to generalize. Let $C \subseteq A^2$ and let D be a set of formulae of the form $R(\mathbf{a})$ with $R \in \mathfrak{R}$ and $\mathbf{a} \in A^{\rho(R)}$. The congruence generated by $C \cup D$, denoted $\mathrm{con}(C \cup D)$, is the least congruence $\theta = \langle \theta_0, \theta_1 \rangle$ such that $C \subseteq \theta_0$ and $\mathbf{a} \in \theta_1^R$ for all $R(\mathbf{a}) \in D$. The equivalence relation θ_0 is given by the usual Mal'cev construction applied to C, and θ_1 is the closure of $D \cup \mathfrak{R}^{\mathbf{A}}$ with respect to θ_0, i.e., if $R(\mathbf{a}) \in D$ and $\mathbf{a}\,\theta_0\,\mathbf{b}$ componentwise, then $\mathbf{b} \in \theta_1^R$.

A *variety* is a class closed under homomorphic images, substructures and direct products. Varieties are determined by laws of the form $s \approx t$ and $R(\mathbf{s})$ where s, t and the components of $\mathbf{s}$ are terms. That is, a variety is the class of all similar structures satisfying a collection of atomic formulae. If $\mathcal{V}$ is a variety of structures and $\mathbf{F}$ is the countably generated free structure for $\mathcal{V}$, then the lattice $\mathrm{L}_v(\mathcal{V})$ of subvarieties of $\mathcal{V}$ is dually isomorphic to the lattice of fully invariant congruences of $\mathbf{F}$, i.e., $\mathrm{L}_v(\mathcal{V}) \cong^d$ Ficon $\mathbf{F}$. In the case of varieties of algebras (with no relational symbols in the language), this is equivalent to adding the endomorphisms of $\mathbf{F}$ to its operations and taking the usual congruence lattice, so that $\mathrm{L}_v(\mathcal{V}) \cong^d \mathrm{Con}\,(F, \mathcal{F} \cup \mathrm{End}\,\mathbf{F})$. For structures in general, this simplification does not work. (These standard results are based on Birkhoff [8].)

A *quasivariety* is a class of structures closed under substructures, direct products and ultraproducts (equivalently, substructures and reduced products). Quasivarieties are determined by laws that are *quasi-identities*, i.e., Horn sentences

$$\&_{1 \le i \le n}\alpha_i \implies \beta$$

where the α_i and β are atomic formulae of the form $s \approx t$ and/or $R(\mathbf{s})$.

If $\mathcal{K}$ is a quasivariety and $\mathbf{A}$ a structure, then a congruence θ on $\mathbf{A}$ is said to be a $\mathcal{K}$-congruence if $\mathbf{A}/\theta \in \mathcal{K}$. Since the largest congruence is a $\mathcal{K}$-congruence, and $\mathcal{K}$-congruences are closed under intersection, the set of $\mathcal{K}$-congruences on $\mathbf{A}$ forms a complete meet subsemilattice of Con $\mathbf{A}$, denoted $\mathrm{Con}_{\mathcal{K}}\,\mathbf{A}$. Moreover, $\mathrm{Con}_{\mathcal{K}}\,\mathbf{A}$ is itself an algebraic lattice.

Let us adopt some notation to reflect the standard duality between theories and models. For a variety $\mathcal{V}$, let $\mathrm{ATh}(\mathcal{V})$ denote the lattice of "equational" (really, atomic) theories extending the theory of $\mathcal{V}$, so that $\mathrm{ATh}(\mathcal{V}) \cong^d \mathrm{L}_v(\mathcal{V})$. Likewise, for a quasivariety $\mathcal{K}$, let $\mathrm{QTh}(\mathcal{K})$ denote the lattice of quasi-equational theories containing the theory of $\mathcal{K}$, so that $\mathrm{QTh}(\mathcal{K}) \cong^d \mathrm{L}_q(\mathcal{K})$.

Gorbunov and Tumanov described the lattice $\mathbf{L}_q(\mathcal{K})$ of quasivarieties contained in a given quasivariety $\mathcal{K}$ in terms of algebraic subsets. This description requires some definitions.

- Given $\mathcal{K}$, let $\mathbf{F} = \mathbf{F}_{\mathcal{K}}(\omega)$ be the countably generated $\mathcal{K}$-free structure. Then $\mathrm{Con}_{\mathcal{K}}\,\mathbf{F}$ denotes the lattice of all $\mathcal{K}$-congruences of $\mathbf{F}$.

- Define the isomorphism relation I and embedding relation E on $\mathrm{Con}_{\mathcal{K}}\,\mathbf{F}$ by

$$\varphi\,I\,\psi \quad \text{if} \quad \mathbf{F}/\psi \cong \mathbf{F}/\varphi$$
$$\varphi\,E\,\psi \quad \text{if} \quad \mathbf{F}/\psi \leq \mathbf{F}/\varphi.$$

- For a binary relation R on a complete lattice $\mathbf{L}$, let $\mathrm{Sp}(\mathbf{L}, R)$ denote the lattice of all R-closed algebraic subsets of $\mathbf{L}$. (Recall that $S \subseteq L$ is *algebraic* if it is closed under arbitrary meets and nonempty directed joins. The set S is *R-closed* if $s \in S$ and $s\,R\,t$ implies $t \in S$.)

The characterization theorem of Gorbunov and Tumanov [20] then says that

$$\mathrm{L}_q(\mathcal{K}) \cong \mathrm{Sp}(\mathrm{Con}_{\mathcal{K}}\,\mathbf{F}, I) \cong \mathrm{Sp}(\mathrm{Con}_{\mathcal{K}}\,\mathbf{F}, E).$$

See Section 5.2 of Gorbunov [17]; also cf. Hoehnke [21].

By way of comparison, we might say that the description of the lattice of subvarieties by $\mathrm{L}_v(\mathcal{V}) \cong^d$ Ficon $\mathbf{F}$ reflects equational logic, whereas the representation $\mathrm{L}_q(\mathcal{K}) \cong \mathrm{Sp}(\mathrm{Con}_{\mathcal{K}}\,\mathbf{F}, E)$ say reflects structural properties (closure under S, P and direct limits). We would like to find an analogue of the former for quasivarieties, ideally something of the form $\mathrm{L}_q(\mathcal{K}) \cong^d \mathrm{Con}\,\mathbf{S}$ for some semilattice $\mathbf{S}$ with operators, reflecting quasi-equational logic. This is done below. Indeed, while our emphasis is on the structure of Q-lattices, Bob Quackenbush has used the same general ideas to provide a nice algebraic proof of the completeness theorem for quasi-equational logic [33].

The lattice $\mathrm{QTh}(\mathcal{K})$ of theories of a quasivariety is algebraic and (completely) meet semidistributive. Most of the other known properties of these lattices can be described in terms of the natural equa-interior operator, which is the dual of an equational closure operator on $\mathrm{QTh}(\mathcal{K})$. See Appendix II or Section 5.3 of Gorbunov [17].

A.M. Nurakunov [31], building on earlier work of R. McKenzie [28] and R. Newrly [29], has recently provided a nice algebraic description of the lattices $\mathrm{ATh}(\mathcal{V})$, where $\mathcal{V}$ is a variety of algebras, as congruence lattices of monoids with two additional unary operations satisfying certain properties. See Appendix III.

Finally, let us note two (related) major differences between quasivarieties of structures *versus* algebras. Firstly, the greatest quasi-equational theory in $\mathrm{QTh}(\mathcal{K})$ need not be compact if the language of $\mathcal{K}$ has infinitely many relations. Secondly, many nice representation theorems for quasivarieties use one-element structures, whereas one-element algebras are trivial. Indeed, in light of Theorem 2 below, Theorem 5.2.8 of Gorbunov [17] (from Gorbunov and Tumanov [18]) can be stated as follows.

Theorem 1. *The following are equivalent for an algebraic lattice $\mathbf{L}$.*

(1) $\mathbf{L} \cong \mathrm{Con}(\mathbf{S}, +, 0)$ *for some semilattice $\mathbf{S}$.*
(2) $\mathbf{L} \cong \mathrm{QTh}(\mathcal{K})$ *for some quasivariety $\mathcal{K}$ of one-element structures.*

Congruence lattices of semilattices are coatomistic, i.e., every element is a meet of coatoms. Thus the Q-lattices for the special quasivarieties in the preceding theorem are correspondingly atomistic.

2. Congruence lattices of semilattices

Let $\mathrm{Sp}(\mathbf{L})$ denote the lattice of algebraic subsets of a complete lattice $\mathbf{L}$. If $\mathbf{L}$ is an algebraic lattice, let $\mathbf{L}_c$ denote its semilattice of compact elements. This is a join semilattice with zero. The following result of Fajtlowicz and Schmidt [11] directly generalizes the Freese-Nation theorem [13]. See also [12], [22], [34].

Theorem 2. *If $\mathbf{L}$ is an algebraic lattice, then $\mathrm{Sp}(\mathbf{L}) \cong^d \mathrm{Con}\, \mathbf{L}_c$.*

Proof. For an arbitrary join 0-semilattice $\mathbf{S} = \langle S, +, 0 \rangle$ we set up a Galois correspondence between congruences of $\mathbf{S}$ and algebraic subsets of the ideal lattice $\mathcal{I}(\mathbf{S})$ as follows.

For $\theta \in \mathrm{Con}\, \mathbf{S}$, let $h(\theta)$ be the set of all θ-closed ideals of $\mathbf{S}$.

For $\mathcal{H} \in \mathrm{Sp}(\mathcal{I}(\mathbf{S}))$, let $x\, \rho(\mathcal{H})\, y$ if $\{I \in \mathcal{H} : x \in I\} = \{J \in \mathcal{H} : y \in J\}$.

It is straightforward to check that h and ρ are order-reversing, that $h(\theta) \in \mathrm{Sp}(\mathcal{I}(\mathbf{S}))$ and $\rho(\mathcal{H}) \in \mathrm{Con}\, \mathbf{S}$.

To show that $\theta = \rho h(\theta)$, we note that if $x < y$ (w.l.o.g.) and $(x,y) \notin \theta$, then $\{z \in S : x + z\, \theta\, x\}$ is a θ-closed ideal containing x and not y. Hence $(x,y) \notin \rho h(\theta)$.

To show that $\mathcal{H} = h\rho(\mathcal{H})$, consider an ideal $J \notin \mathcal{H}$. For any $x \in S$, let $\hat{x} = \bigcap\{I \in \mathcal{H} : x \in I\}$, noting that $\hat{x} \in \mathcal{H}$. Then $\{\hat{x} : x \in J\}$ is up-directed, whence $\bigcup\{\hat{x} : x \in J\} \in \mathcal{H}$. Therefore the union properly contains J, so that there exist $x < y$ with $x \in J$ and $y \in \hat{x} - J$, and J is not $\rho(\mathcal{H})$-closed. Thus $J \notin \mathcal{H}$ implies $J \notin h\rho(\mathcal{H})$, as desired. $\qquad\square$

Compare this with the following result of Adaricheva, Gorbunov and Tumanov ([5] Theorem 2.4, also [17] Theorem 4.4.12).

Theorem 3. *Let $\mathbf{L}$ be a join semidistributive lattice that is finitely presented within the class $\mathbf{SD}_\vee$. Then $\mathbf{L} \leq \mathrm{Sp}(\mathbf{A})$ for some algebraic and dually algebraic lattice $\mathbf{A}$.*

On the other hand, Example 4.4.15 of Gorbunov [17] gives a 4-generated join semidistributive lattice that is not embeddable into any lower continuous lattice satisfying $\mathrm{SD}_\vee$.

Keith Kearnes points out that the class $\mathcal{ES}$ of lattices that are embeddable into congruence lattices of semilattices is not first order. Indeed, every finite meet semidistributive lattice is in $\mathcal{ES}$, and $\mathcal{ES}$ is closed under S and P. Now the quasivariety $\mathrm{SD}_\wedge$ is generated by its finite members (Tumanov [35], Theorem 4.1.7 in [17]), while $\mathcal{ES}$ is properly contained in $\mathrm{SD}_\wedge$. Hence $\mathcal{ES}$ is not a quasivariety, which means it must not be closed under ultraproducts. This result has been generalized in Kearnes and Nation [25].

3. Connection with Quasivarieties

In this section, we will show that for each quasivariety $\mathcal{K}$ of structures, the lattice of quasi-equational theories $\mathrm{Qth}(\mathcal{K})$ is isomorphic to the congruence lattice of a semilattice with operators.

Given a quasivariety $\mathcal{K}$, let $\mathbf{F} = \mathbf{F}_{\mathcal{K}}(\omega)$ be the $\mathcal{K}$-free algebra on ω generators, and let $\mathrm{Con}_{\mathcal{K}}\ \mathbf{F}$ be the lattice of $\mathcal{K}$-congruences of $\mathbf{F}$. For a set S of atomic formulae, recall that the $\mathcal{K}$-congruence generated by S is

$$\mathrm{con}_{\mathcal{K}}\ S = \bigcap \{\psi \in \mathrm{Con}\ \mathbf{F} : \mathbf{F}/\psi \in \mathcal{K} \text{ and } S \subseteq \psi\}.$$

Then let $\mathbf{T} = \mathbf{T}_{\mathcal{K}}$ denote the join semilattice of compact $\mathcal{K}$-congruences in $\mathrm{Con}_{\mathcal{K}}\ \mathbf{F}$. Thus $\mathbf{T} = (\mathrm{Con}_{\mathcal{K}}\ \mathbf{F}_{\mathcal{K}}(\omega))_c$ consists of finite joins of the form $\bigvee_j \varphi_j$, with each φ_j either $\mathrm{con}_{\mathcal{K}}\ (s, t)$ or $\mathrm{con}_{\mathcal{K}}\ R(\mathbf{s})$ for terms s, t, $s_i \in \mathbf{F}$ and a relation R.

Let X be a free generating set for $\mathbf{F}_{\mathcal{K}}(\omega)$. Any map $\sigma_0 : X \to \mathbf{F}$ can be extended to an endomorphism $\sigma : \mathbf{F} \to \mathbf{F}$ in the usual way. Since the image $\sigma(\mathbf{F})$ is a substructure of $\mathbf{F}$, the kernel of an endomorphism σ is a $\mathcal{K}$-congruence. The endomorphisms of $\mathbf{F}$ form a monoid $\mathrm{End}\ \mathbf{F}$.

The endomorphisms of $\mathbf{F}$ act naturally on $\mathbf{T}$. For $\varepsilon \in \mathrm{End}\ \mathbf{F}$, define

$$\widehat{\varepsilon}(\mathrm{con}_{\mathcal{K}}\ (s, t)) = \mathrm{con}_{\mathcal{K}}\ (\varepsilon s, \varepsilon t)$$
$$\widehat{\varepsilon}(\mathrm{con}_{\mathcal{K}}\ R(\mathbf{s})) = \mathrm{con}_{\mathcal{K}}\ R(\varepsilon \mathbf{s})$$
$$\widehat{\varepsilon}(\bigvee_j \varphi_j) = \bigvee_j \widehat{\varepsilon}\varphi_j.$$

The next lemma is used to check the crucial technical details that $\widehat{\varepsilon}$ is well-defined, and hence join-preserving.

Lemma 4. *Let $\mathcal{K}$ be a quasivariety, $\mathbf{F}$ a $\mathcal{K}$-free algebra, and $\varepsilon \in \mathrm{End}\ \mathbf{F}$. Let α, $\beta_1, \ldots, \beta_m$ be atomic formulae. In $\mathrm{Con}_{\mathcal{K}}\ \mathbf{F}$,*

$$\mathrm{con}_{\mathcal{K}}\ \alpha \leq \bigvee \mathrm{con}_{\mathcal{K}}\ \beta_j \qquad \text{implies} \qquad \widehat{\varepsilon}(\mathrm{con}_{\mathcal{K}}\ \alpha) \leq \bigvee \widehat{\varepsilon}(\mathrm{con}_{\mathcal{K}}\ \beta_j).$$

Proof. For an atomic formula α and a congruence θ, let us write $\alpha \in \theta$ to mean either (1) α is $s \approx t$ and $(s, t) \in \theta_0$, or (2) α is $R(\mathbf{s})$ and $\mathbf{s} \in \theta_1^R$. So for the lemma, we are given that if $\mathbf{F}/\psi \in \mathcal{K}$ and $\beta_1, \ldots, \beta_m \in \psi$, then $\alpha \in \psi$. We want to show that if $\mathbf{F}/\theta \in \mathcal{K}$ and $\varepsilon\beta_1, \ldots, \varepsilon\beta_m \in \theta$, then $\varepsilon\alpha \in \theta$.

Let $\theta \in \mathrm{Con}\ \mathbf{F}$ be a congruence such that $\mathbf{F}/\theta \in \mathcal{K}$, and let $h : \mathbf{F} \to \mathbf{F}/\theta$ be the natural map. Then $h\varepsilon : \mathbf{F} \to \mathbf{F}/\theta$, and since $h\varepsilon(\mathbf{F})$ is a substructure of $h(\mathbf{F})$, the image is in $\mathcal{K}$. Now $\beta_1, \ldots, \beta_m \in \ker h\varepsilon$, and so $\alpha \in \ker h\varepsilon$. Thus $\varepsilon\alpha \in \ker h = \theta$, as desired. $\square$

Now let ξ be a compact $\mathcal{K}$-congruence. Suppose that $\xi = \bigvee_i \varphi_i$ and $\xi = \bigvee_j \psi_j$ in $\mathbf{T}$, with each φ_i and ψ_j being a principal $\mathcal{K}$-congruence. Then for each i we have $\varphi_i \leq \bigvee_j \psi_j$, whence $\widehat{\varepsilon}\varphi_i \leq \bigvee_j \widehat{\varepsilon}\psi_j$ by Lemma 4. Thus $\bigvee_i \widehat{\varepsilon}\varphi_i \leq \bigvee_j \widehat{\varepsilon}\psi_j$. Symmetrically $\bigvee_j \widehat{\varepsilon}\psi_j \leq \bigvee_i \widehat{\varepsilon}\varphi_i$, and so $\widehat{\varepsilon}\xi = \bigvee_j \widehat{\varepsilon}\psi_j = \bigvee_i \widehat{\varepsilon}\varphi_i$ is well-defined.

It then follows from the definition of $\widehat{\varepsilon}$ that if $\varphi = \bigvee_i \varphi_i$ and $\psi = \bigvee_j \psi_j$ in $\mathbf{T}$, then

$$\widehat{\varepsilon}(\varphi \vee \psi) = \widehat{\varepsilon}(\bigvee_i \varphi_i \vee \bigvee_j \psi_j)$$

$$= \bigvee_i \widehat{\varepsilon}\varphi_i \vee \bigvee_j \widehat{\varepsilon}\psi_j)$$

$$= \widehat{\varepsilon}\varphi \vee \widehat{\varepsilon}\psi.$$

Thus $\widehat{\varepsilon}$ preserves joins. Also note that for the zero congruence we have $\widehat{\varepsilon}(0) = 0$.

Let $\widehat{\mathcal{E}} = \{\widehat{\varepsilon} : \varepsilon \in \text{End } \mathbf{F}\}$, and consider the algebra $\mathbf{S} = \mathbf{S}_{\mathcal{K}} = \langle \mathbf{T}, \vee, 0, \widehat{\mathcal{E}}\rangle$. By the preceding remarks, the operations of $\widehat{\mathcal{E}}$ are *operators* on $\mathbf{S}$, i.e., $(\vee, 0)$-homomorphisms, so $\mathbf{S}$ is a join semilattice with operators. With this setup, we can now state our main result.

Theorem 5. *For a quasivariety $\mathcal{K}$,*

$$\text{L}_q(\mathcal{K}) \cong^d \text{Con } \mathbf{S}$$

where $\mathbf{S} = \langle \mathbf{T}, \vee, 0, \widehat{\mathcal{E}}\rangle$ with $\mathbf{T}$ the semilattice of compact congruences of $\text{Con}_{\mathcal{K}} \mathbf{F}$, $\mathcal{E} = \text{End } \mathbf{F}$, and $\mathbf{F} = \mathbf{F}_{\mathcal{K}}(\omega)$.

In Part II, we will use this technical variation.

Theorem 6. *Let $\mathcal{K}$ be a quasivariety and let $n \geq 1$ be an integer. Then the lattice of all quasi-equational theories that*

(1) *contain the theory of $\mathcal{K}$, and*
(2) *are determined relative to $\mathcal{K}$ by quasi-identities in at most n variables,*

is isomorphic to $\text{Con } \mathbf{S}_n$, where $\mathbf{S}_n = \langle \mathbf{T}_n, \vee, 0, \widehat{\mathcal{E}}\rangle$ with $\mathbf{T}_n$ the semilattice of compact congruences of $\text{Con}_{\mathcal{K}} \mathbf{F}$, $\mathcal{E} = \text{End } \mathbf{F}$, and $\mathbf{F} = \mathbf{F}_{\mathcal{K}}(n)$.

We shall prove Theorem 5, and afterwards discuss the modification required for Theorem 6, which is essentially just replacing $\mathbf{F}_{\mathcal{K}}(\omega)$ by $\mathbf{F}_{\mathcal{K}}(n)$.

For the proof of Theorem 5, and for its application, it is natural to use two structures closely related to the congruence lattice instead. For an algebra $\mathbf{A}$ with a join semilattice reduct, let $\text{Don } \mathbf{A}$ be the lattice of all reflexive, transitive, compatible relations R such that $\geq \subseteq R$, i.e., $x \geq y$ implies $x \mathrel{R} y$. Let $\text{Eon } \mathbf{A}$ be the lattice of all reflexive, transitive, compatible relations R such that

(1) $R \subseteq \leq$, i.e., $x \mathrel{R} y$ implies $x \leq y$, and
(2) if $x \leq y \leq z$ and $x \mathrel{R} z$, then $x \mathrel{R} y$.

Lemma 7. *If $\mathbf{A} = \langle A, \vee, 0, \mathcal{F}\rangle$ is a semilattice with operators, then $\text{Con } \mathbf{A} \cong \text{Don } \mathbf{A} \cong \text{Eon } \mathbf{A}$.*

Proof. Let $\delta : \text{Con } \mathbf{A} \to \text{Don } \mathbf{A}$ *via* $\delta(\theta) = \theta \circ \geq$, so that

$$x \, \delta(\theta) \, y \quad \text{iff} \quad x \, \theta \, x \vee y$$

and let $\gamma : \mathrm{Don}\,\mathbf{A} \to \mathrm{Con}\,\mathbf{A}$ *via* $\gamma(R) = (R\cap\leq)\circ(R\cap\leq)^{\smile}$, so that

$$x\,\gamma(R)\,y \quad \text{iff} \quad x\,R\,x\vee y\,\&\,y\,R\,x\vee y.$$

Now we check that, for $\theta \in \mathrm{Con}\,\mathbf{A}$ and $R \in \mathrm{Don}\,\mathbf{A}$,

(1) $\delta(\theta) \in \mathrm{Don}\,\mathbf{A}$,
(2) $\gamma(R) \in \mathrm{Con}\,\mathbf{A}$,
(3) δ and γ are order-preserving,
(4) $\gamma\delta(\theta) = \theta$,
(5) $\delta\gamma(R) = R$.

This is straightforward and only slightly tedious.

Similarly, let $\varepsilon : \mathrm{Don}\,\mathbf{A} \to \mathrm{Eon}\,\mathbf{A}$ *via* $\varepsilon(R) = R\cap\leq$, and $\delta' : \mathrm{Eon}\,\mathbf{A} \to \mathrm{Don}\,\mathbf{A}$ *via* $\delta'(S) = S\circ\geq$, and check the analogous statements for this pair, which is again routine. Note that for a congruence relation θ the corresponding eon-relation is $\varepsilon\delta(\theta) = \theta\cap\leq$, while for $S \in \mathrm{Eon}\,\mathbf{A}$ we have the congruence $\gamma\delta'(S) = S\circ S^{\smile}$. □

Now we define a Galois connection between T^2 and structures $\mathbf{A} \in \mathcal{K}$. (The collection of structures $\mathbf{A} \in \mathcal{K}$ forms a proper class. However, every quasivariety is determined by its finitely generated members. So we could avoid any potential logical difficulties by restricting our attention to structures $\mathbf{A}$ defined on some fixed infinite set large enough to contain an isomorphic copy of each finitely generated member of $\mathcal{K}$.) For a pair $(\beta,\gamma) \in T^2$ and $\mathbf{A} \in \mathcal{K}$, let $(\beta,\gamma)\,\Xi\,\mathbf{A}$ if, whenever $h : \mathbf{F} \to \mathbf{A}$ is a homomorphism, $\beta \leq \ker h$ implies $\gamma \leq \ker h$.

Then, following the usual rubric for a Galois connection, for $X \subseteq T^2$ let

$$\kappa(X) = \{A \in \mathcal{K} : (\beta,\gamma)\,\Xi\,\mathbf{A} \text{ for all } (\beta,\gamma) \in X\}.$$

Likewise, for $Y \subseteq \mathcal{K}$, let

$$\Delta(Y) = \{(\beta,\gamma) \in T^2 : (\beta,\gamma)\,\Xi\,\mathbf{A} \text{ for all } \mathbf{A} \in Y\}.$$

We must check that the following hold for $X \subseteq T^2$ and $Y \subseteq \mathcal{K}$.

(1) $\kappa(X) \in \mathrm{L_q}(\mathcal{K})$,
(2) $\Delta(Y) \in \mathrm{Don}\,\mathbf{S}$,
(3) $\Delta\kappa(X) = X$ if $X \in \mathrm{Don}\,\mathbf{S}$,
(4) $\kappa\Delta(Y) = Y$ if $Y \in \mathrm{L_q}(\mathcal{K})$.

To prove (1), we show that $\kappa(X)$ is closed under substructures, direct products and ultraproducts. Closure under substructures is immediate, and closure under direct products follows from the observation that if $h : \mathbf{F} \to \prod_i \mathbf{A}_i$ then $\ker h = \bigcap \ker \pi_i h$. So let $\mathbf{A}_i \in \kappa(X)$ for $i \in I$, let U be an ultrafilter on I, and let $h : \mathbf{F} \to \prod \mathbf{A}_i/U$ be a homomorphism. Since $\mathbf{F}$ is free, we can find $f : \mathbf{F} \to \prod \mathbf{A}_i$ such that $h = gf$ where $g : \prod \mathbf{A}_i \to \prod \mathbf{A}_i/U$ is the standard map. Let $(\beta,\gamma) \in X$ with $\beta = \bigvee \varphi_j$ and $\gamma = \bigvee \psi_k$, where these are finite joins and each φ and ψ is of the form $\mathrm{con}_{\mathcal{K}}\,\alpha$ for an atomic formula α. Each α in turn is of the form either $s \approx t$ or $R(\mathbf{s})$.

Assume $\beta \leq \ker h$. Then $h(\alpha_j)$ holds for each j, so that for each j we have $\{i \in I : \pi_i f(\alpha_j)\} \in U$. Taking the intersection, $\{i \in I : \forall j \ \pi_i f(\alpha_j)\} \in U$. In other words, $\{i \in I : \beta \leq \ker \pi_i f\} \in U$, and so the same thing holds for γ. Now we reverse the steps to obtain $\gamma \leq \ker h$, as desired. Thus $\kappa(X)$ is also closed under ultraproducts, and it is a quasivariety.

To prove (2), let $Y \subseteq \mathcal{K}$. It is straightforward that $\Delta(Y) \subseteq T^2$ is a relation that is reflexive, transitive, and contains $\geq$. Moreover, if $(\beta, \gamma) \in \Delta(Y)$ and $\beta \vee \tau \leq \ker h$ for an appropriate h, then $\gamma \vee \tau \leq \ker h$, so $\Delta(Y)$ respects joins.

Again let $(\beta, \gamma) \in \Delta(Y)$ and $h : \mathbf{F} \to \mathbf{A}$ with $A \in Y$. Let $\widehat{\varepsilon} \in \widehat{\mathcal{E}}$ and assume that $\widehat{\varepsilon}\beta \leq \ker h$. This is equivalent to $\beta \leq \ker h\varepsilon$, as both mean that $h\varepsilon(\alpha_j)$ holds for all j, where $\beta = \bigvee \mathrm{con}_{\mathcal{K}} \alpha_j$. Hence $\gamma \leq \ker h\varepsilon$, yielding $\widehat{\varepsilon}\gamma \leq \ker h$. Thus $\Delta(Y)$ is compatible with the operations of $\widehat{\mathcal{E}}$. We conclude that $\Delta(Y) \in \mathrm{Don}\ \mathbf{S}$.

Next consider (4). Given that Y is a quasivariety, we want to show that $\kappa\Delta(Y) \subseteq Y$. Let $\mathbf{A} \in \kappa\Delta(Y)$, and let $\&_j \alpha_j \implies \zeta$ be any quasi-identity holding in Y. Set $\beta = \bigvee \mathrm{con}_{\mathcal{K}} \alpha_j$ and $\gamma = \mathrm{con}_{\mathcal{K}} \zeta$, and let $h : \mathbf{F} \to \mathbf{A}$ be a homomorphism. Then $(\beta, \gamma) \in \Delta(Y)$, whence as $\mathbf{A} \in \kappa\Delta(Y)$ we have $\beta \leq \ker h$ implies $\gamma \leq \ker h$. Thus $\mathbf{A}$ satisfies the quasi-identity in question, which shows that $\kappa\Delta(Y) \subseteq Y$, as desired.

Part (3) requires the most care (we must show that relations in Don $\mathbf{S}$ correspond to theories of quasivarieties). Given $X \in \mathrm{Don}\ \mathbf{S}$, we want to prove that $\Delta\kappa(X) \subseteq X$.

Let $(\mu, \nu) \in T^2 - X$. Define a congruence θ on $\mathbf{F}$ as follows.

$$\theta_0 = \mu$$
$$\theta_{k+1} = \theta_k \vee \bigvee \{\gamma | (\beta, \gamma) \in X \text{ and } \beta \leq \theta_k\}$$
$$\theta = \bigvee_k \theta_k.$$

Let $\mathbf{C} = \mathbf{F}/\theta$. We want to show that $\mathbf{C} \in \kappa(X)$ and that $\nu \nleq \theta$.

Claim a. If ψ *is compact and* $\psi \leq \theta$, *then* $(\mu, \psi) \in X$. We prove by induction that if compact $\psi \leq \theta_k$, then $(\mu, \psi) \in X$. For $k = 0$ this is trivial.

Assume the statement holds for k. Suppose we have a finite collection of $(\beta_i, \gamma_i) \in X$ with each $\beta_i \leq \theta_k$. Let $\xi = \bigvee \beta_i$, so that ξ is compact and $\beta_i \leq \xi \leq \theta_k$. Then $(\xi, \beta_i) \in X$, so by transitivity $(\xi, \gamma_i) \in X$ for all i. Hence $(\xi, \bigvee \gamma_i) \in X$. Now inductively $(\mu, \xi) \in X$, and so $(\mu, \bigvee \gamma_i) \in X$.

Claim b. If $(\beta, \gamma) \in X$ *and* $\beta \leq \theta$, *then* $\gamma \leq \theta$. This holds by construction and compactness.

Claim c. $\mathbf{F}/\theta \in \kappa(X)$. Suppose $h : \mathbf{F} \to \mathbf{F}/\theta$, $(\beta, \gamma) \in X$ and $\beta \leq \ker h$. Let $f : \mathbf{F} \to \mathbf{F}/\theta$ be the standard map with $\ker f = \theta$. There exists an endmorphism ε of $\mathbf{F}$ such that $h = f\varepsilon$. Then, using Claim b and an argument

above,

$$\beta \leq \ker h = \ker f\varepsilon \implies \widehat{\varepsilon}\beta \leq \ker f = \theta$$
$$\implies \widehat{\varepsilon}\gamma \leq \theta = \ker f$$
$$\implies \gamma \leq \ker f\varepsilon = \ker h.$$

Claim d. $(\mu, \nu) \notin \Delta\kappa(X)$. This is because $\mathbf{C} \in \kappa(X)$ by Claim c and $\mu \leq \theta = \ker f$, while $\nu \not\leq \theta$ by Claim a.

This completes the proof of (3), and hence Theorem 5.

Only a slight modification is required for Theorem 6. Consider the collection of quasivarieties $\mathcal{C}$ satisfying the conditions of the theorem:

(1) $\mathcal{C} \subseteq \mathcal{K}$, and
(2) $\mathcal{C}$ is determined relative to $\mathcal{K}$ by quasi-identities in at most n variables.

These properties mean that a structure $\mathbf{C}$ is in $\mathcal{C}$ if and only if

(1)′ Every map $f_0 : \omega \to \mathbf{C}$ extends to a homomorphism $f : \mathbf{F}_{\mathcal{K}}(\omega) \to \mathbf{C}$, and
(2)′ Every map $g_0 : n \to \mathbf{C}$ extends to a homomorphism $g : \mathbf{F}_{\mathcal{C}}(\omega) \to \mathbf{C}$.

Quasivarieties satisfying conditions (1) and (2) are closed under arbitrary joins, and thus under containment they form a lattice which we will denote by $\mathbf{L}_q^n(\mathcal{K})$. This is a complete join subsemilattice of $\mathbf{L}_q(\mathcal{K})$; the corresponding dual lattice of theories is a complete meet subsemilattice $\mathrm{QTh}^n(\mathcal{K})$ of $\mathrm{QTh}(\mathcal{K})$. The proof of Theorem 5 gives us $\mathrm{QTh}(\mathcal{K})$ as the congruence lattice of a semilattice with operators obtained from $\mathbf{F}_{\mathcal{K}}(\omega)$. In view of condition (2)′, the same construction with $\mathbf{F}_{\mathcal{K}}(\omega)$ replaced throughout by $\mathbf{F}_{\mathcal{K}}(n)$ yields $\mathrm{QTh}^n(\mathcal{K})$.

4. Interpretation

The foregoing analysis is rather structural and omits the motivation, which we supply here. Let β and γ be elements of $\mathbf{T}$, i.e., compact $\mathcal{K}$-congruences on the free structure $\mathbf{F}$. Then these are finite joins in $\mathrm{Con}_{\mathcal{K}} \mathbf{F}$ of principal congruences, say $\beta = \bigvee \mathrm{con}_{\mathcal{K}} \alpha_j$ and $\gamma = \bigvee \mathrm{con}_{\mathcal{K}} \zeta_k$, where each α and ζ is an atomic formula of the form $s \approx t$ or $R(\mathbf{s})$. The basic idea is that the congruence $\mathrm{con}(\beta, \beta \vee \gamma)$, on the semilattice $\mathbf{S}$ of compact $\mathcal{K}$-congruences of $\mathbf{F}$ with the endomorphisms as operators, should correspond to the conjunction over the indices k of the quasi-identities $\&_j \alpha_j \implies \zeta_k$, and that furthermore the quasi-equational consequences of combining implications (modulo the theory of $\mathcal{K}$) behaves like the join operation in $\mathrm{Con} \mathbf{S}$. But $\beta \geq \gamma$ should mean that $\beta \implies \gamma$, so it is really Don $\mathbf{S}$ that we want. On the other hand, all the nontrivial information is contained already in Eon $\mathbf{S}$, and these three lattices are isomorphic.

Let $H(\beta, \gamma)$ denote the set of all quasi-identities $\&_j \alpha_j \implies \zeta_k$ where the atomic formulae α_j and ζ_k come from join representations $\beta = \bigvee \mathrm{con}_{\mathcal{K}} \alpha_j$

and $\gamma = \bigvee \mathrm{con}_{\mathcal{K}}\ \zeta_k$. Let Δ and κ be the mappings from the Galois connection in the proof of Theorem 5. The semantic versions of the structural results of the preceding section then take the following form.

Lemma 8. *Let $\mathcal{Q}$ be a quasivariety contained in $\mathcal{K}$. The set of all pairs (β, γ) such that $\mathcal{Q}$ satisfies each of the sentences in $H(\beta, \gamma)$ is in* Don **S***, where* $\mathbf{S} = \langle \mathbf{T}, \vee, 0, \widehat{\mathcal{E}} \rangle$ *with* $\mathbf{T}$ *the semilattice of compact congruences of* $\mathrm{Con}_{\mathcal{K}}\ \mathbf{F}$, $\mathcal{E} = \mathrm{End}\ \mathbf{F}$, *and* $\mathbf{F} = \mathbf{F}_{\mathcal{K}}(\omega)$.

Lemma 9. *Let Y be a collection of structures contained in $\mathcal{K}$. The following are equivalent.*

    (1) $(\beta, \gamma) \in \Delta(Y)$.
    (2) *Every* $\mathbf{A} \in Y$ *satisfies all the implications in* $H(\beta, \gamma)$.
    (3) *The quasivariety* $\mathrm{SPU}(Y)$ *satisfies all the implications in* $H(\beta, \gamma)$.

Lemma 10. *Let $X \subseteq T^2$, where $\mathbf{T}$ is as in Lemma 8. The following are equivalent for a structure $\mathbf{A}$.*

    (1) $\mathbf{A} \in \kappa(X)$.
    (2) *For every pair $(\beta, \gamma) \in X$, $\mathbf{A}$ satisfies all the quasi-identities of $H(\beta, \gamma)$.*

As always, it is good to understand both the semantic and logical viewpoint.

5. Congruence lattices of semilattices with operators

Let us examine more closely lattices of the form $\mathrm{Con}(\mathbf{S}, +, 0, \mathcal{F})$. The following theorem summarizes some fundamental facts about their structure.

Theorem 11. *Let $(\mathbf{S}, +, 0, \mathcal{F})$ be a semilattice with operators.*

    (1) *An ideal I of $\mathbf{S}$ is the 0-class of some congruence relation if and only if $f(I) \subseteq I$ for every $f \in \mathcal{F}$.*
    (2) *If the ideal I is $\mathcal{F}$-closed, then the least congruence with 0-class I is $\eta(I)$, the semilattice congruence generated by I. It is characterized by*

$$x\ \eta(I)\ y \qquad \textit{iff} \qquad x + i = y + i \textit{ for some } i \in I.$$

    (3) *There is also a greatest congruence with 0-class I, which we will denote by $\tau(I)$. To describe this, let $\mathcal{F}^{\dagger}$ denote the monoid generated by $\mathcal{F}$, including the identity function. Then*

$$x\ \tau(I)\ y \qquad \textit{iff} \qquad (\forall h \in \mathcal{F}^{\dagger})\ h(x) \in I \Longleftrightarrow h(y) \in I.$$

The proof of each part of the theorem is straightforward. As a sample application, it follows that if $\mathbf{S}$ is a *simple* semigroup with one operator, then $|\mathbf{S}| = 2$.

The maps η and τ from Theorem 11 induce operations on the entire congruence lattice $\mathrm{Con}(\mathbf{S}, +, 0, \mathcal{F})$. If θ is a congruence with 0-class I, define $\eta(\theta) = \eta(I)$ and $\tau(\theta) = \tau(I)$. The map η is known as the *natural*

equa-interior operator on $\mathrm{Con}(\mathbf{S}, +, 0, \mathcal{F})$. This terminology will be justified below.

The natural equa-interior operator induces a partition of $\mathrm{Con}(\mathbf{S}, +, 0, \mathcal{F})$.

Theorem 12. *Let* $\mathbf{S} = \langle S, +, 0, \mathcal{F} \rangle$ *be a semilattice with operators. The natural equa-interior operator partitions* $\mathrm{Con}(\mathbf{S})$ *into intervals* $[\eta(\theta), \tau(\theta)]$ *consisting of all the congruences with the same* 0*-class (which is an* $\mathcal{F}$*-closed ideal).*

The natural equa-interior operator on the congruence lattice of a semilattice with operators plays a role dual to that of the equaclosure operator for lattices of quasivarieties.

Adaricheva and Gorbunov [4], building on Dziobiak [9], described the natural equational closure operator on Q-lattices. In the dual language of theories, the restriction of quasi-equational theories to atomic formulae gives rise to an equa-interior operator (defined below) on $\mathrm{QTh}(\mathcal{K})$. Finitely based subvarieties of a quasi-variety $\mathcal{K}$ are given by quasi-identities that can be written as $x \approx x \implies \&_k \beta_k$ for some atomic formulae β_k. By Lemma 9, the corresponding congruences are of the form $\mathrm{con}(0, \theta)$ where θ is a compact $\mathcal{K}$-congruence on the free algebra $\mathbf{F}_{\mathcal{K}}(\omega)$. More generally, subvarieties of $\mathcal{K}$ correspond to joins of these, i.e., to congruences of the form $\bigvee_{\theta \in I} \mathrm{con}(0, \theta)$ for some ideal I of the semilattice of compact $\mathcal{K}$-congruences. Thus we should expect the map η to be the analogous interior operator on congruence lattices of semilattices with operators.

We now define an equa-interior operator abstractly to have those properties that we know to hold for the natural equa-interior operator on the lattice of theories of a quasivariety. One of our main goals, in this section and the next two, is to extend this list of known properties using the representation of the lattice of theories as the congruence lattice of a semilattice with operators.

An *equa-interior operator* on an algebraic lattice $\mathbf{L}$ is a map $\eta : L \to L$ satisfying the following properties.

(I1) $\eta(x) \leq x$

(I2) $x \geq y$ implies $\eta(x) \geq \eta(y)$

(I3) $\eta^2(x) = \eta(x)$

(I4) $\eta(1) = 1$

(I5) $\eta(x) = u$ for all $x \in X$ implies $\eta(\bigvee X) = u$

(I6) $\eta(x) \vee (y \wedge z) = (\eta(x) \vee y) \wedge (\eta(x) \vee z)$

(I7) The image $\eta(\mathbf{L})$ is the complete join subsemilattice of $\mathbf{L}$ generated by $\eta(\mathbf{L}) \cap \mathbf{L}_c$.

(I8) There is a compact element $w \in \mathbf{L}$ such that $\eta(w) = w$ and the interval $[w, 1]$ is isomorphic to the congruence lattice of a semilattice. (Thus the interval $[w, 1]$ is coatomistic.)

Property (I5) means that the operation τ is implicitly defined by η, *via*

$$\tau(x) = \bigvee \{z \in \mathbf{L} : \eta(z) = \eta(x)\}.$$

Thus $\tau(x)$ is the largest element z such that $\eta(z) = \eta(x)$. Likewise, properties (I1) and (I3) insure that $\eta(x)$ is the least element z' such that $\eta(z') = \eta(x)$. By (I2), if $\eta(x) \leq y \leq \tau(x)$, then $\eta(y) = \eta(x)$. Thus the kernel of η, defined by $x \approx y$ iff $\eta(x) = \eta(y)$, is an equivalence relation that partitions **L** into disjoint intervals of the form $[\eta(x), \tau(x)]$. We will refer to this as the *equa-partition* of **L**.

Now τ is not order-preserving in general. However, it does satisfy a weak order property that can be useful.

Lemma 13. *Let **L** be an algebraic lattice, and assume that η satisfies conditions (I1)–(I5). Define τ as above. Then for any subset $\{x_j : j \in J\} \subseteq L$,*

$$\tau(\bigwedge_{j \in J} x_j) \geq \bigwedge_{j \in J} \tau(x_j).$$

Proof. We have

$$\eta(\bigwedge \tau x_j) \leq \bigwedge \eta \tau x_j \leq \bigwedge x_j \leq \bigwedge \tau x_j$$

and that's all in one block of the equa-partition, while $\bigwedge x_j \leq \tau(\bigwedge x_j)$, which is the top of the same block. Thus $\bigwedge \tau x_j \leq \tau(\bigwedge x_j)$. $\qquad \square$

Property ($I7$) has some nice consequences.

Lemma 14. *Let η be an equa-interior operator on an algebraic lattice **L**.*

(1) *The image $\eta(\mathbf{L})$ is an algebraic lattice, and x is compact in $\eta(\mathbf{L})$ iff $x \in \eta(\mathbf{L})$ and x is compact in **L**.*
(2) *If X is up-directed, then $\eta(\bigvee X) = \bigvee \eta(X)$.*

For any quasivariety $\mathcal{K}$, the natural equa-interior operator on the lattice of theories of $\mathcal{K}$ satisfies the eight listed basic properties. Congruence lattices of semilattices with operators come close. For an ideal I in a semilattice with operators, let $\mathrm{con}_{\mathrm{SL}}(I)$ denote the semilattice congruence generated by collapsing all the elements of I to 0.

Theorem 15. *If $\mathbf{S} = \langle S, +, 0, \mathcal{F} \rangle$ is a semilattice with operators, then the map η on Con $\mathbf{S}$ given by $\eta(\theta) = \mathrm{con}_{\mathrm{SL}}(0/\theta)$ satisfies properties (I1)–(I7).*

Proof. Property (I6) is the hard one to verify. Let α, β, $\gamma \in$ Con $\mathbf{S}$ and let $\xi = \eta(\alpha)$. Then $x \xi y$ if and only if there exists $z \in S$ such that $z \alpha 0$ and $x + z = y + z$. (This is the semilattice congruence but it's compatible with $\mathcal{F}$.) We want to show that

$$(\xi \vee \beta) \wedge (\xi \vee \gamma) \leq \xi \vee (\beta \wedge \gamma).$$

Let a, $b \in$ LHS. Then there exist elements such that

$$a \beta c_1 \xi c_2 \beta c_3 \ldots b$$
$$a \gamma d_1 \xi d_2 \gamma d_3 \ldots b.$$

Let z be the join of the elements witnessing the above ξ-relations. Then

$$a \xi a + z \beta c_1 + z = c_2 + z \beta c_3 + z = \ldots b + z \xi b$$

so that $a \, \xi \, a + z \, \beta \, b + z \, \xi \, b$, and similarly $a \, \xi \, a + z \, \gamma \, b + z \, \xi \, b$. Thus $a, b \in \text{RHS}$, as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Property (I8), on the other hand, need not hold in the congruence lattice of a semilattice with operators. The element w of (I8), called the *pseudo-one*, in lattices of quasi-equational theories corresponds to the identity $x \approx y$. For an equa-interior operator on a lattice **L** with 1 compact, we can take $w = 1$; in particular, this applies when the semilattice has a top element, in which case we can take $w = \text{con}(0, 1)$. But in general, there may be no candidate for the pseudo-one.

Note that property (I8) implies that a lattice is dually atomic (or *coatomic*). Let $x < 1$ in **L**. If $x \vee w < 1$ then it is below a coatom, while if $x \vee w = 1$ then by the compactness of w there is a coatom above x that is not above w. In particular, the lattice of theories of a quasivariety is coatomic (Corollary 5.1.2 of Gorbunov [17]).

Consider the semilattice $\mathbf{\Omega} = (\omega, \vee, 0, p)$ with $p(0) = 0$ and $p(x) = x - 1$ for $x > 0$. Then Con $\mathbf{\Omega} \cong \omega + 1$, which has no pseudo-one (regardless of how η is defined). Thus Con $\mathbf{\Omega}$ is not the dual of a Q-lattice. Likewise, Con $\mathbf{\Omega}$ fails to be dually atomic.

In each of the next two sections we will discuss an additional property of the natural equa-interior operator on semilattices with operators. The point of this is that an algebraic lattice cannot be the dual of a Q-lattice unless it admits an equa-interior operator satisfying all these conditions. Indeed, we should really consider the representation problem in the context of pairs $(\mathbf{L}, \eta)$, rather than just the representation of a lattice with an unspecified equa-interior operator.

For the sake of clarity, let us agree that the term *equa-interior operator* refers to conditions (I1)–(I8) for the remainder of the paper, even though we are proposing that henceforth a ninth condition should be included in the definition.

6. A NEW PROPERTY OF NATURAL EQUA-INTERIOR OPERATORS

The next theorem gives a property of the natural equa-partition on congruence lattices of semilattices with operators that need not hold in all lattices with an equa-interior operator.

Theorem 16. *Let* $\mathbf{S} = \langle S, +, 0, \mathcal{F} \rangle$ *be a semilattice with operators, and let* η, τ *denote the bounds of the natural equa-partition on* Con $\mathbf{S}$. *If the congruences* ζ, γ, χ *satisfy* $\eta(\zeta) \leq \eta(\gamma)$ *and* $\tau(\chi) \leq \tau(\gamma)$, *then*

$$\eta(\eta(\zeta) \vee \tau(\zeta \wedge \chi)) \leq \eta(\gamma).$$

Proof. Assume that ζ, γ, χ satisfy the hypotheses, and let $0/\zeta = Z$, $0/\gamma = C$ and $0/\chi = X$ be the corresponding ideals. So $Z \subseteq C$ and $\tau(X) \subseteq \tau(C)$. For notation, let $\alpha = \tau(Z \cap X)$.

We want to show that $0/(\eta(Z) \vee \alpha) \subseteq C$, so let $w \in$ LHS. For any $z \in Z$ we have $(z, w) \in \eta(Z) \vee \alpha$. Fix an element $z_0 \in Z$. We claim that there exist elements $z^* \in Z$ and $w^* \in S$ such that $z_0 \leq z^* \leq w^*$, $w \leq w^*$ and $z^* \, \alpha \, w^*$.

There is a sequence

$$z_0 = s_0 \; \eta(Z) \; s_1 \; \alpha \; s_2 \; \eta(Z) \; s_3 \; \ldots \; s_k = w.$$

Let $t_j = s_0 + \cdots + s_j$ for $0 \leq j \leq k$. Thus we obtain

$$z_0 = t_0 \; \eta(Z) \; t_1 \; \alpha \; t_2 \; \eta(Z) \; t_3 \; \ldots \; t_k$$

with

$$t_0 \leq t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_k.$$

Put $z' = t_1$ and $w' = t_k$, so that with $z_0 \leq z' \in Z$ and $w \leq w'$. Moreover, we may assume that k is minimal for such a sequence.

If $k > 2$, then $z' = t_1 \; \alpha \; t_2 \; \eta(Z) \; t_3 \; \alpha \; t_4$. By the definition of $\eta(Z)$, there exists $u \in Z$ such that $t_2 + u = t_3 + u$. Joining with u yields the shorter sequence

$$z'' = t_1 + u \; \alpha \; t_2 + u = t_3 + u \; \alpha \; t_4 + u \; \ldots$$

contradicting the minimality of k. Thus $k \leq 2$, which yields the conclusion of the claim with $z^* = t_1$ and $w^* = t_2$.

Next, we claim that $(z^*, w^*) \in \tau(X)$. This follows from the sequence of implications:

$$
\begin{aligned}
f(z^*) \in X &\implies f(z^*) \in X \cap Z \\
&\implies f(w^*) \in X \cap Z \\
&\implies f(w^*) \in X \\
&\implies f(z^*) \in X
\end{aligned}
$$

which hold for any $f \in \mathcal{F}$, using the $\mathcal{F}$-closure of Z, $(z^*, w^*) \in \tau(X \cap Z)$ and $z^* \leq w^*$.

Thus $(z^*, w^*) \in \tau(X) \subseteq \tau(C)$. But $z^* \in Z \subseteq C = 0/\tau(C)$, whence $w^* \in C$ and $w \in C$, as desired. $\square$

For an application of this condition, consider the lattice $\mathbf{K}$ in Figure 1. It is straightforward to show that $\mathbf{K}$ has a unique equa-interior operator, with $h(t) = 0$ if $t \leq a$ and $h(t) = t$ otherwise. Indeed, any equa-interior operator on $\mathbf{K}$ must have $h(a) \vee (x \wedge z) = (h(a) \vee x) \wedge (h(a) \vee z)$, from which it follows easily that $h(a) = 0$. But then we cannot have $h(x) = 0$, else $h(1) = h(a \vee x) = 0$, a contradiction. Thus $h(x) = x$ and symmetrically $h(z) = z$. This in turn yields that $h(c) = c$.

But $\mathbf{K}$ is not the congruence lattice of a semilattice with operators. The only candidate for the equa-interior operator fails the condition of Theorem 16 with the substitution $\zeta \mapsto z$, $\gamma \mapsto c$, $\chi \mapsto x$. Therefore $\mathbf{K}$ is not the lattice of theories of a quasivariety. We could have also derived this latter fact by noting that $\mathbf{K}$ is not dually biatomic: in $\mathbf{K}$ we have $a \geq x \wedge z$ which is not refinable to a meet of coatoms.
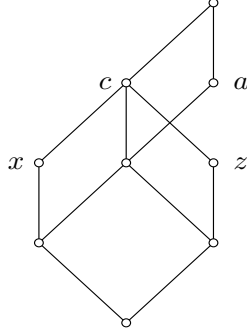
FIGURE 1. **K**

On the other hand, **K** can be represented as a filterable sublattice of $\mathrm{Con}(\mathbf{B}_3, +, 0)$, where $\mathbf{B}_3$ is the Boolean lattice on three atoms. (See Appendix II for this terminology.) Indeed, if the atoms of $\mathbf{B}_3$ are p, q, r then we can take

$$a \mapsto [0]\ [p, q, r, p \vee q, p \vee r, q \vee r, 1]$$
$$c \mapsto \mathrm{con}(0, p \vee q)$$
$$x \mapsto \mathrm{con}(0, p)$$
$$z \mapsto \mathrm{con}(0, q).$$

We will pursue the comparison of congruence lattices and lattices of algebraic sets in the appendices.

Taking a cue from this example, we continue investigating the consequences of the condition of Theorem 16. Recall that, whenever η satisfies (I1)–(I5), we have $\eta(y) = \eta(x)$ iff $\eta(x) \le y \le \tau(x)$. The condition can be written as follows, where we use the fact that $\eta(u) \le c$ iff $\eta(u) \le \eta(c)$.

$$(\dagger) \quad \tau(x) \le \tau(c)\ \&\ \eta(z) \le c \implies \eta(\eta(z) \vee \tau(x \wedge z)) \le c$$

This holds for the natural equa-interior operator on congruence lattices of semilattices with operators, and we want to see how it applies to pairs $(\mathbf{L}, h)$ where h is an arbitrary equa-interior operator on $\mathbf{L}$.

There is a two-variable version of the condition, which is obtained by putting $c = \eta(z) \vee \tau(x)$.

$$(\ddagger) \qquad \eta(\eta(z) \vee \tau(x \wedge z)) \le \eta(z) \vee \tau(x)$$

This appears to be slightly weaker than $(\dagger)$.

Consider the Boolean lattice $\mathbf{B}_3$ with atoms x, y, z and the equa-interior operator with $h(y) = 0$ and $h(t) = t$ otherwise. Then $(\mathbf{B}_3, h)$ fails the condition $(\ddagger)$, though $\mathbf{B}_3$ is a dual Q-lattice with another equa-interior operator by Theorem 1.

There are two additional conditions on equa-interior operators that are known to hold in the duals of Q-lattices: bicoatomicity and the four-coatom condition. (See Section 5.3 of Gorbunov [17].) Unfortunately, congruence lattices of semilattices with operators need not be coatomic (there is an example in the discussion of property (I8) in Section 5), but duals of Q-lattices are, so we will impose this as an extra condition. In that case, we will see that (†) implies both of these properties.

A lattice $\mathbf{L}$ is *bicoatomic* (or *dually biatomic*) if whenever p is a coatom of $\mathbf{L}$ and $p \geq u \wedge v$ properly, then there exist coatoms $c \geq u$ and $d \geq v$ such that $p \geq c \wedge d$.

Theorem 17. *Let $\mathbf{L}$ be a coatomic lattice and let h be an equa-interior operator on $\mathbf{L}$. If $(\mathbf{L}, h)$ satisfies property (†), then $\mathbf{L}$ is bicoatomic.*

Proof. Assume $1 \succ p \geq u \wedge v$ properly in $\mathbf{L}$. We want to find elements c, z with $1 \succ c \geq u$, $z \geq v$, and $c \wedge z \leq p$. (Then apply the argument a second time.)

Note that $p \geq \eta(p) \vee (u \wedge v) = (\eta(p) \vee u) \wedge (\eta(p) \vee v)$. Put $x = \eta(p) \vee u$ and $z = \eta(p) \vee v$. Let $1 \succ c \geq \tau(x)$ and note $\tau(x) \geq x \geq u$.

Suppose $c \wedge z \not\leq p$. Put $z' = c \wedge z$. Then $\eta(z') \not\leq p$, for else since $\eta(p) \leq z'$ we would have $\eta(z') = \eta(p) = \eta(z' \vee p) = \eta(1) = 1$, a contradiction. Now we apply (†). Surely $\tau(x) \leq c$ and $\eta(z') \leq z' \leq c$. Moreover $\eta(p) \leq z' \wedge x \leq z \wedge x \leq p$ whence $\eta(z' \wedge x) = \eta(p)$, and thus $\tau(z' \wedge x) = p$. But then $\eta(\eta(z') \vee \tau(x \wedge z')) = \eta(\eta(z') \vee p) = \eta(1) = 1$, again a contradiction. Therefore $c \wedge z \leq p$, as desired. □

The dual of the four-coatom condition played a significant role in the characterization of the atomistic, algebraic Q-lattices. This too is a consequence of property (†). For coatoms a, d we write $a \sim d$ to indicate that $|\uparrow (a \wedge d)| = 4$, in which case the filter $\uparrow (a \wedge d)$ is exactly $\{1, a, d, a \wedge d\}$. A lattice $\mathbf{L}$ with an equa-interior operator η satisfies the *four-coatom condition* if, whenever a, b, c, d are coatoms of $\mathbf{L}$ such that $a \sim d$, $\eta(a) \not\leq d$, $\eta(c) \leq d$ and $\eta(c) = \eta(a \wedge b)$, then $\eta(c) = \eta(b \wedge d)$.

Theorem 18. *The four-coatom condition holds in a lattice with an equa-interior operator η satisfying (†).*

Proof. As $\eta(c) \leq b$, d is given, we need that $\eta(b \wedge d) \leq c$. Supposing not, substitute $x = a \wedge d$, $z = \eta(b \wedge d)$, and the element d into (†). Note that $\tau(a \wedge d) \neq a$ has $\eta(a) \not\leq d$. Thus $\tau(a \wedge d) \leq d$, and of course $\eta(b \wedge d) \leq d$. But we also have $\eta(c) \leq a \wedge b \wedge d \leq a \wedge b$ and $\eta(a \wedge b) = \eta(c)$, so $\eta(\eta(b \wedge d) \vee \tau(a \wedge b \wedge d)) = \eta(\eta(b \wedge d) \vee c) = \eta(1) = 1$, a contradiction. Thus $\eta(b \wedge d) \leq c$, as desired. □

7. Coatomistic congruence lattices and a stronger property

One of the most intriguing hypotheses about lattices of quasivarieties is formulated for atomistic lattices. Dually, it can be expressed as follows:

> *Can every coatomistic lattice of quasi-equational theories be represented as* $\mathrm{Con}(\mathbf{S}, +, 0)$, *i.e., without operators?*

This hypothesis is shown to be valid in the case when the lattice of quasi-equational theories is dually algebraic [3]. The problem provides a motivation for investigating which coatomistic lattices can be represented as lattices of equational theories, or congruence lattices of semilattices, with or without operators.

Consider the class $\mathcal{M}$ of lattices dual to $\mathrm{Sub}_f \mathbf{M}$, where $\mathbf{M}$ is an infinite semilattice with 0, and $\mathrm{Sub}_f \mathbf{M}$ is the lattice of finite subsemilattices of $\mathbf{M}$, topped by the semilattice $\mathbf{M}$ itself.

Evidently, lattices in $\mathcal{M}$ are coatomistic, and they are algebraic but not dually algebraic. Besides, it is straightforward to show that they cannot be presented as $\mathrm{Con}(\mathbf{S}, +, 0)$. Thus, it would be natural to ask whether such lattices can be presented as $\mathrm{Con}(\mathbf{S}, +, 0, \mathcal{F})$, for a non-empty set of operators on $\mathbf{S}$. In many cases the answer is "no" simply because there might be no equa-interior operator. For example, let $\mathbf{M}$ be a meet semilattice such that the dual of $\mathrm{Sub}_f \mathbf{M}$ admits an equa-interior operator. If a is an element of $\mathbf{M}$ that can be expressed as a meet in infinitely many ways, then $\eta(a) = 0$ by Lemma 22 below. Hence $\mathbf{M}$ can contain at most one such element.

It turns out to be feasible to show that certain lattices from $\mathcal{M}$, that *do* admit an equa-interior operator, still cannot be represented as $\mathrm{Con}(\mathbf{S}, +, 0, \mathcal{F})$. The crucial factor here is to understand the behavior of infinite meets of coatoms, or more generally infinite meets of elements $\tau(x)$, in the congruence lattice of a semilattice with operators. The restriction given by Theorem 19 can be expressed as a ninth basic property of the natural equa-interior operator (as it implies (†)).

Aside: Coatoms arise naturally in another context, that does not make the lattice coatomistic. Suppose $\mathbf{S} = \langle S, +, 0, \mathcal{F} \rangle$ has the property that for each $\mathcal{F}$-closed ideal I, every $f \in \mathcal{F}$, and every $x \in S$,

$$f(x) \in I \implies x \in I.$$

Then the congruence $\tau(I)$ partitions S into I and $S - I$, and hence is a coatom. In particular, this property holds whenever

- $\mathcal{F}$ is empty, or
- $\mathcal{F}$ is a group, or
- every $f \in \mathcal{F}$ is increasing, i.e., $x \leq f(x)$ for all $x \in S$.

In all these cases, $\tau(\theta)$ is a coatom for every $\theta \in \mathrm{Con}\,\mathbf{S}$. We will be particularly concerned with the case when $\mathcal{F}$ is a group in Part II [7].

Theorem 19. *Let* $\mathbf{S} = \langle S, +, 0, \mathcal{F} \rangle$ *be a semilattice with operators, I an arbitrary index set, and χ, γ, and ζ_i for $i \in I$ congruences on* $\mathbf{S}$. *The natural equa-interior operator on* $\mathrm{Con}\,\mathbf{S}$ *has the following property: if $\eta(\chi) \leq \gamma$ and $\bigwedge_{i \in I} \tau(\zeta_i) \leq \tau(\gamma)$, then*

$$\eta\left(\eta(\chi) \vee \bigwedge_{i \in I} \tau(\chi \wedge \zeta_i)\right) \leq \gamma.$$

For the proof, it is useful to write down abstractly the two parts of the argument of the proof of Theorem 16.

Lemma 20. *Let α, χ, $\zeta \in \mathrm{Con}(\mathbf{S}, +, 0, \mathcal{F})$ and let X be the 0-class of χ.*

(1) *If $u \in X$ and $(u, v) \in \chi \vee \alpha$, then there exist elements u^*, v^* with $u \leq u^* \in X$, $v \leq v^*$, $u^* \leq v^*$, and $(u^*, v^*) \in \alpha$.*

(2) *If $u \in X$, $u \leq v$ and $(u, v) \in \tau(\chi \wedge \zeta)$, then $(u, v) \in \tau(\zeta)$.*

Now, under the assumptions of the theorem, let $u \in X$ and $(u, v) \in \eta(\chi) \vee \bigwedge \tau(\chi \wedge \zeta_i)$, so that v is in the 0-class of the LHS. Then by Lemma 20(1), there exist u^*, v^* with $u \leq u^* \in X$, $v \leq v^*$, $u^* \leq v^*$ and $(u^*, v^*) \in \bigwedge \tau(\chi \wedge \zeta_i)$. Then $(u^*, v^*) \in \tau(\chi \wedge \zeta_i)$ for every i, whence by Lemma 20(2) $(u^*, v^*) \in \tau(\zeta_i)$ for every i, so that $(u^*, v^*) \in \bigwedge \tau(\zeta_i)$.

Let X and C denote the 0-classes of χ and γ, respectively. By assumption, we have $u^* \in X \subseteq C$, and $(u^*, v^*) \in \bigwedge \tau(\zeta_i) \leq \tau(\gamma)$, so $v^* \in C$ as well. A *fortiori*, $v \in C$, as desired.

This proves Theorem 19. Thus we obtain the ninth fundamental property of the natural equa-interior operator on the congruence lattice of a semilattice with operators.

(I9) For any index set I, if $\eta(x) \leq c$ and $\bigwedge \tau(z_i) \leq \tau(c)$, then $\eta(\eta(x) \vee \bigwedge_{i \in I} \tau(x \wedge z_i)) \leq c$.

As before, there is also a slightly simpler (and weaker) variation:

$$(I9') \qquad \eta(\eta(x) \vee \bigwedge_{i \in I} \tau(x \wedge z_i)) \leq \eta(x) \vee \bigwedge \tau(z_i).$$

Clearly, if $|I| = 1$ then property (I9) reduces to property (†). In fact, for I finite, (†) implies (I9). But for I infinite, property (I9) seems to carry a rather different sort of information, as we shall see below.

Consider the case when $|I| = 2$; the argument for the general finite case is similar. Assume that $\eta(x) \leq c$ and $\tau(y) \wedge \tau(z) \leq \tau(c)$. Using (I6), (†), and the fact that $\eta(u \wedge v) = \eta(\eta(u) \wedge \eta(v))$, we calculate

$$\eta(\eta(x) \vee (\tau(x \wedge y) \wedge \tau(x \wedge z))) = \eta((\eta(x) \vee (\tau(x \wedge y)) \wedge (\eta(x) \vee \tau(x \wedge z))))$$
$$\leq \eta((\eta(x) \vee (\tau(y)) \wedge (\eta(x) \vee \tau(z))))$$
$$= \eta(\eta(x) \vee (\tau(y) \wedge \tau(z)))$$
$$\leq c$$

as desired.

With property (I9) as a tool-in-hand, we turn to a thorough investigation of the (dual) dependence relation for coatoms of $\mathrm{Con}(\mathbf{S}, +, 0, \mathcal{F})$; see Theorems 23 and 24 below. Throughout the remainder of this section, χ, ζ and α will denote distinct coatoms of the congruence lattice. Repeatedly, we use the basic property of equa-interior operators that $\eta x \vee (y \wedge z) = (\eta x \vee y) \wedge (\eta x \vee z)$. Our goal is to generalize (to whatever extent possible) the following property of finite sets of coatoms.

Theorem 21. *Let* **L** *be a lattice with an equa-interior operator. If for coatoms* $x, z_1, \ldots, z_k, a_1, \ldots, a_k$ *of* **L** *we have* $x \wedge z_i \leq a_i$ *properly, then* $\eta x \vee \bigwedge_{i=1}^{k} z_i = 1$.

The proof uses the next lemma.

Lemma 22. *Suppose* $x \wedge z \leq a$ *properly for coatoms in a lattice with an equa-interior operator. Then* $\eta a \leq x \wedge z$, *and thus*

(1) $\tau(x \wedge z) = a$,
(2) $\eta x \not\leq a$,
(3) $\eta x \not\leq z$.

Proof. If say $\eta a \not\leq x$, then $\eta a \vee x = 1$, and using (I6) we would have

$$\eta a \vee z = (\eta a \vee x) \wedge (\eta a \vee z)$$
$$= \eta a \vee (x \wedge z) \leq a$$

whence $z \leq a$, a contradiction. So $\eta a \leq x$, and symmetrically $\eta a \leq z$. Since $\eta a \leq x \wedge z \leq a = \tau a$, we have $\tau(x \wedge z) = a$.

It follows that we cannot have $\eta x \leq a$, else

$$\eta a = \eta(x \wedge z) \leq \eta x \leq a,$$

implying that $\eta x = \eta a$, and thus $\eta a = \eta(x \vee a) = \eta 1 = 1$ by (I5) and (I4), a contradiction. Therefore also $\eta x \not\leq z$, else $\eta x \leq x \wedge z \leq a$. $\qquad\square$

The theorem now follows immediately, because

$$\eta x \vee \bigwedge_{i=1}^{k} z_i = \bigwedge_{i=1}^{k} (\eta x \vee z_i) = 1.$$

The property of Theorem 21 can fail when there are infinitely many z_i's, even in the congruence lattice of a semilattice. Let **Q** be the join semilattice in Figure 2. Consider the ideals

$$X = \{0, u_1, u_2, u_3, \ldots\}$$
$$Z_i = \downarrow v_i$$
$$A_i = \downarrow u_i$$

for $i \in \omega$, and let $\chi = \tau(X)$, $\zeta_i = \tau(Z_i)$ and $\alpha_i = \tau(A_i)$. Then an easy calculation shows that $\bigwedge \zeta_i = 0$, and the infinite version of the property of the theorem fails.

Nonetheless, we shall show that a couple of infinite versions do hold.

Theorem 23. *Let* **L** *be a lattice with an equa-interior operator satisfying property* (I9). *If for coatoms* a, x *and* z_i ($i \in I$) *of* **L** *we have* $x \wedge z_i \leq a$ *properly, then* $\eta x \vee \bigwedge_{i \in I} z_i = 1$.

Proof. By Lemma 22, we have $\tau(x \wedge z_i) = a$ for every i, and $\eta x \not\leq a$. Hence $\eta x \vee \bigwedge \tau(x \wedge z_i) = 1$. Then property $(I9')$ gives the conclusion immediately. $\qquad\square$
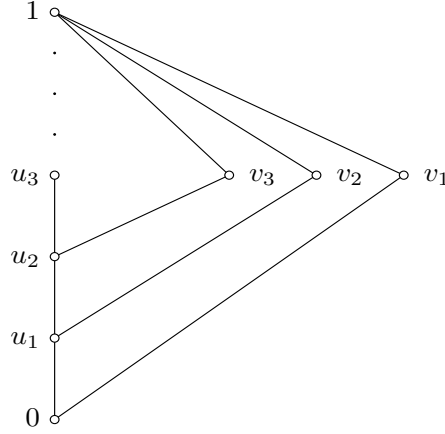
FIGURE 2. $\mathrm{Con}(\mathbf{S}, +, 0)$ does not satisfy the infinite analogue of Theorem 21.

Theorem 24. *Let $\mathbf{L}$ be a lattice with an equa-interior operator satisfying property (I9). Let x, a_i and z_i be coatoms of $\mathbf{L}$ with $x \wedge z_i \leq a_i$ properly for all $i \in I$. If $\bigwedge_{i \in I} a_i \not\leq x$, then $\bigwedge_{i \in I} z_i \not\leq x$.*

Proof. Again, by Lemma 22, we have $\tau(x \wedge z_i) = a_i$ for every i. Now apply (I9) directly with $c = x$. $\qquad\square$

Let us now use these results to show that certain coatomistic lattices are not lattices of quasi-equational theories. Call an infinite $(\wedge)$-semilattice $\mathbf{M}$ *cute* if it has an element a and different elements $m, m_j \in M \backslash \{a\}$, $j \in \omega$, with $m \wedge m_j = a$.

Examples of cute semilattices are $\mathbf{M}_\infty$: countably many m_i covering the least element a, or $\mathbf{M}_2$: a chain $\{m_j, j \in \omega\}$ in addition to elements m, a, satisfying $m \wedge m_j = a$ for all j. It was asked in [2] (p. 175), in connection with the hypothesis about the atomistic Q-lattices mentioned above in the dual form, whether $\mathrm{Sub}_f \mathbf{M}_\infty$ is a Q-lattice. The following result, an immediate application of Theorem 23, answers this question in the negative.

Theorem 25. *If $\mathbf{M}$ is a cute semilattice, then the dual of $\mathrm{Sub}_f \mathbf{M}$ is not representable as $\mathrm{Con}(\mathbf{S}, +, 0, \mathcal{F})$. Hence $\mathrm{Sub}_f \mathbf{M}$ is not a Q-lattice.*

It would be desirable to extend Theorem 25 to all lattices from $\mathcal{M}$. In particular, we may ask about possibility to represent $\mathbf{L} = (\mathrm{Sub}_f \mathbf{P_1})^d$, where the semilattice $\mathbf{P_1}$ consists of two descending chains $\{b_i, i \in \omega\}$, $\{a_i, i \in \omega\}$ with defining relations $a_{i+1} = a_i \wedge b_{i+1}$, $b_0 > a_0$.

Every equa-interior operator η on $\mathbf{L}$ would satisfy: $\eta(\{a_i\}) = [a_i, b_0]$, $\eta(\{b_i\}) \geq [b_i, b_0]$. In particular, $\eta(c) = 0$, $c \in \mathbf{L}$, implies $c = 0$ (equivalently,

$\tau(0) = 0$). This makes $\mathbf{P_1}$ drastically different from cute semilattices. *Is the dual of* $\mathrm{Sub}_f\ \mathbf{P_1}$ *representable as* $\mathrm{Con}(\mathbf{S}, +, 0, \mathcal{F})$?

Another interesting case to consider would be $\mathrm{Sub}_f\ \mathbf{C}$ where $\mathbf{C}$ is an infinite chain, so that every finite subset of C is a subsemilattice.

8. Appendix I: Complete sublattices of subalgebras

In the first two appendices, we analyze conditions that were used in older descriptions of lattices of quasivarieties; see Gorbunov [17].

Note that $\mathrm{Con}(\mathbf{S}, +, 0, \mathcal{F})$ is a complete sublattice of $\mathrm{Con}(\mathbf{S}, +, 0)$, which is dually isomorphic to $\mathrm{Sp}(\mathcal{I}(\mathbf{S}))$, which is the lattice of subalgebras of an infinitary algebra. (Joins of non-directed sets can be set to 1.) In this context we are considering complete sublattices of $\mathrm{Sub}\ \mathbf{A}$ where $\mathbf{A}$ is a semilattice, or a complete semilattice, or a complete algebra of algebraic subsets.

Let ε be a binary relation on a set S. A subset $X \subseteq S$ is said to be ε-*closed* if $c \in X$ and $c\,\varepsilon\,d$ implies $d \in X$.

Recall that a quasi-order ε on a semilattice $\mathbf{S} = \langle S, \wedge, 1 \rangle$ is *distributive* if it satisfies the following conditions.

(1) If $c_1 \wedge c_2\,\varepsilon\,d$ then there exist elements d_1, d_2 such that $c_i\,\varepsilon\,d_i$ and $d = d_1 \wedge d_2$.

(2) If $1\,\varepsilon\,d$ then $d = 1$.

The effect of the next result is that for a semilattice $\mathbf{S}$, *any* complete sublattice of $\mathrm{Sub}\ \mathbf{S}$ can be represented as the lattice of all ρ-closed subsemilattices, for some distributive quasi-order ρ.

Theorem 26. *Let* $\mathbf{S} = \langle S, \wedge, 1 \rangle$ *be a semilattice with* 1, *and let* ε *be a distributive quasi-order on* $\mathbf{S}$. *Then* $\mathrm{Sub}\,(\mathbf{S}, \varepsilon)$, *the lattice of all* ε-*closed subsemilattices (with* 1*), is a complete sublattice of* $\mathrm{Sub}\ \mathbf{S}$.

Conversely, let $\mathbf{T}$ *be a complete sublattice of* $\mathrm{Sub}\ \mathbf{S}$. *Define a relation* ρ *on* $\mathbf{S}$ *by* $c\,\rho\,d$ *if for all* $\mathbf{X} \in \mathbf{T}$ *we have* $c \in X \implies d \in X$. *Then* ρ *is a distributive quasi-order, and* $\mathbf{T}$ *consists precisely of the* ρ-*closed subsemilattices of* $\mathbf{S}$. *Furthermore,* ρ *satisfies the following conditions.*

(3) *If* $c\,\rho\,d_1$, d_2 *then* $c\,\rho\,d_1 \wedge d_2$.

(4) *For all* $c \in S$, $c\,\rho\,1$.

The correspondence between complete sublattices of $\mathrm{Sub}\ \mathbf{S}$ *and distributive quasi-orders satisfying (3) and (4) is a dual isomorphism.*

The proof is relatively straightforward.

The description of all complete sublattices of $\mathrm{Sub}\ \mathbf{S}$, the lattice of all complete subsemilattices of a complete semilattice $\mathbf{S}$, is almost identical, except that complete meets appear in the conditions.

(1)′ If $\bigwedge c_i\,\varepsilon\,d$ then there exist elements d_i such that $c_i\,\varepsilon\,d_i$ and $d = \bigwedge d_i$.

(3)′ If $c\,\varepsilon\,c_i$ for all i, then $c\,\varepsilon\,\bigwedge c_i$.

Complete semilattices satisfying (1)′ are called *Brouwerian* by Gorbunov [17]. The results can be summarized thusly.

Theorem 27. *Let* $\mathbf{S} = \langle S, \wedge, 1 \rangle$ *be a complete semilattice. Then there is a dual isomorphism between complete sublattices of* $\mathrm{Sub}\,\mathbf{S}$ *and quasi-orders satisfying conditions (1)′, (2), (3)′ and (4).*

For complete sublattices of $\mathrm{Sp}(\mathbf{A})$, the lattice of algebraic subsets of an algebraic lattice $\mathbf{A}$, we must also deal with joins of nonempty up-directed subsets, and once $\mathbf{A}$ fails the ACC matters get more complicated. A quasi-order ε on $\mathbf{A}$ is said to be *continuous* if it has the following property.

(5) If C is a directed set and $\bigvee C \, \varepsilon \, d$, then there exists a directed set D such that $d = \bigvee D$ and for each $d \in D$ there exists $c \in C$ with $c \, \varepsilon \, d$.

This is a very slight weakening of Gorbunov's definition [17]. As above, we have this result of Gorbunov.

Theorem 28. *Let* ε *be a continuous Brouwerian quasi-order on a complete lattice* $\mathbf{A}$*. Then* $\mathrm{Sp}(\mathbf{A})$*, the lattice of* ε*-closed algebraic subsets, is a complete sublattice of* $\mathrm{Sp}(\mathbf{A})$*.*

Now for any algebra $\mathbf{B}$ we can define the *embedding relation* E on $\mathrm{Con}\,\mathbf{B}$ by $\theta \, E \, \psi$ if $\mathbf{B}/\psi \leq \mathbf{B}/\theta$. A fundamental result of Gorbunov characterizes Q-lattices in terms of the embedding relations (Corollaries 5.2.2 and 5.6.8 of [17]).

Theorem 29. *Let* $\mathcal{K}$ *be a quasivariety and let* $\mathbf{F} = \mathbf{F}_{\mathcal{K}}(\omega)$*. The embedding relation is a continuous Brouwerian quasi-order on* $\mathrm{Con}_{\mathcal{K}}\,\mathbf{F}$*, and* $\mathbf{L}_q(\mathcal{K}) \cong \mathrm{Sp}(\mathrm{Con}_{\mathcal{K}}(\mathbf{F}, E))$*.*

For comparison, we note that the isomorphism relation need not be continuous; see Gorbunov [17], Example 5.6.6.

We do not know (and doubt) that the relation ρ corresponding to a complete sublattice of $\mathrm{Sp}(\mathbf{A})$ need always be continuous. However, our representation of $\mathrm{Con}(\mathbf{S}, +, 0, \mathcal{F})$ as dually isomorphic to a complete sublattice of $\mathrm{Sp}(\mathcal{I}(\mathbf{S}))$ could be unraveled to give the ρ relation explicitly in that case. *Are these particular relations always continuous?*

9. Appendix II: Filterability and equaclosure operators

The natural equational closure operator on $L_q(\mathcal{K})$ is given by the map $h(\mathcal{Q}) = \mathrm{H}(\mathcal{Q}) \cap \mathcal{K}$ for quasivarieties $\mathcal{Q} \subseteq \mathcal{K}$. That is, $h(\mathcal{Q})$ consists of all members of $\mathcal{K}$ that are in the variety generated by $\mathcal{Q}$, or equivalently, that are homomorphic images of $\mathbf{F}_{\mathcal{Q}}(X)$ for some set X. For the corresponding map on $\mathrm{Sp}(\mathrm{Con}\,\mathbf{F}_{\mathcal{K}}(\omega))$, let X be the algebraic subset of all $\mathcal{Q}$-congruences of $\mathrm{Con}\,\mathbf{F}_{\mathcal{K}}(\omega)$. Then $\varphi = \bigwedge X$ is the natural congruence with $\mathbf{F}/\varphi \cong \mathbf{F}_{\mathcal{Q}}(\omega)$, and the filter $\uparrow \varphi$ is the algebraic subset associated with $h(\mathcal{Q})$, that is, all $h(\mathcal{Q})$-congruences of $\mathrm{Con}\,\mathbf{F}_{\mathcal{K}}(\omega)$.

Abstractly, let ε be a distributive quasi-order on an algebraic lattice $\mathbf{A}$. Then it is not hard to see that the map $h(X) = \uparrow \bigwedge X$ on $\mathrm{Sp}(\mathbf{A}, \varepsilon)$ will satisfy the duals of conditions (I1)–(I7) so long as $\uparrow \bigwedge X$ is ε-closed for every

$X \in \mathrm{Sp}(\mathbf{A}, \varepsilon)$. A quasi-order that satisfies this crucial condition,

$$c \geq \bigwedge X \ \& \ c \, \varepsilon \, d \implies d \geq \bigwedge X$$

is said to be *filterable*. If the quasi-order ε is filterable, then the closure operator $h(X) = \uparrow \bigwedge X$ on $\mathrm{Sp}(\mathbf{A}, \varepsilon)$ is again called the *natural* closure operator determined by ε. We can also speak of a complete sublattice of $\mathrm{Sp}(\mathbf{A})$ as being filterable if the quasi-order it induces *via* Theorem 26 is so.

Dually, a sublattice $\mathbf{T} \leq \mathrm{Con}(\mathbf{S}, +, 0)$ is filterable if, for each $\theta \in \mathbf{T}$, the semilattice congruence generated by the 0-class of θ is in $\mathbf{T}$. As we have observed, this is the case when $\mathbf{T} = \mathrm{Con}(\mathbf{S}, +, 0, \mathcal{F})$ for some set of operators $\mathcal{F}$. Thus we obtain a slightly different perspective on Theorem 15.

Theorem 30. *For a semilattice* $\mathbf{S}$ *with operators,* $\mathbf{T} = \mathrm{Con}(\mathbf{S}, +, 0, \mathcal{F})$ *is a filterable complete sublattice of* $\mathrm{Con}(\mathbf{S}, +, 0)$. *Thus* $\mathbf{T}$ *supports the natural interior operator* $h(\theta) = \mathrm{con}(0/\theta)$, *which satisfies conditions (I1)–(I7).*

In fact, the natural interior operator on $\mathrm{Con}(\mathbf{S}, +, 0, \mathcal{F})$ also satisfies condition (I9). However, as we saw in Section 6, a filterable sublattice of $\mathrm{Con}(\mathbf{S}, +, 0)$ may fail condition (†), which is the finite index case of (I9), even with $\mathbf{S}$ finite. Thus being a congruence lattice of a semilattice with operators is a stronger property than just being a filterable sublattice of $\mathrm{Con}(\mathbf{S}, +, 0)$.

10. Appendix III: Lattices of equational theories

In this appendix, we summarize what is known about lattices of equational theories. Throughout the section, $\mathcal{V}$ will denote a variety of algebras, with no relation symbols in the signature. For this situation, atomic theories really are equational theories. The lattice of equational theories is, of course, dual to the lattice of subvarieties of $\mathcal{V}$.

From the basic representation $\mathrm{ATh}(\mathcal{V}) \cong \mathrm{Ficon}\,\mathbf{F}_{\mathcal{V}}(\omega)$, we see that the lattice is algebraic. Its top element 1 has the basis $x \approx y$, and thus 1 is compact. On the other hand, J. Ježek proved that any algebraic lattice with countably many compact elements is isomorphic to an interval in some lattice of equational theories [24].

R. McKenzie showed that every lattice of equational theories is isomorphic to the congruence lattice of a groupoid with left unit and right zero [28]. N. Newrly refined these ideas, showing that a lattice of equational theories is isomorphic to the congruence lattice of a monoid with a right zero and one additional unary operation [29]. A. Nurakunov added a second unary operation and proved a converse: a lattice is a lattice of equational theories if and only if it is the congruence lattice of a monoid with a right zero and two unary operations satisfying certain properties [31].

Nurakunov's conditions are rather technical, but they just codify the properties of the natural operations on the free algebra $\mathbf{F}_{\mathcal{V}}(X)$ that they model. If $X = \{x_0, x_1, x_2, \dots\}$ and s, t are terms, then

$$s \cdot t = t(s, x_1, x_2, \dots).$$

The two unary operations are the endomorphism φ_+ and φ_-, where $\varphi_+(x_i) = x_{i+1}$ for all i, while $\varphi_-(x_0) = x_0$ and $\varphi_-(x_i) = x_{i-1}$ for $i > 0$.

W.A. Lampe used McKenzie's representation to prove that lattices of equational theories satisfy a form of meet semidistributivity at 1, the so-called Zipper Condition [26]:

$$\text{If } a_i \wedge c = z \text{ for all } i \in I \text{ and } \bigvee_{i \in I} a_i = 1, \text{ then } c = z.$$

A similar but stronger condition was found by M. Erné [10] and G. Tardos (independently), which was refined yet further by Lampe [27]. These results show that the structure of lattices of equational theories is quite constrained at the top, whereas Ježek's theorem shows that this is not the case globally. Confirming this heuristic, D. Pigozzi and G. Tardos proved that every algebraic lattice with a completely join irreducible greatest element 1 is isomorphic to a lattice of equational theories [32].

Again, we propose that one should investigate $\mathrm{ATh}(\mathcal{V})$ for varieties of structures.

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