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RECENT DEVELOPMENTS IN GRAVITATIONAL COLLAPSE AND SPACETIME SINGULARITIES

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It is now known that when a massive star collapses under the force of its own gravity, the final fate of such a continual gravitational collapse will be either a black hole or a naked singularity under a wide variety of physically reasonable circumstances within the framework of general theory of relativity. The research of recent years has provided considerable clarity and insight on stellar collapse, black holes and the nature and structure of spacetime singularities. We discuss several of these developments here. There are also important fundamental questions that remain unanswered on the final fate of collapse of a massive matter cloud in gravitation theory, especially on naked singularities which are hypothetical astrophysical objects and on the nature of cosmic censorship hypothesis. These issues have key implications for our understanding on black hole physics today, its astrophysical applications, and for certain basic questions in cosmology and possible quantum theories of gravity. We consider these issues here and summarize recent results and current progress in these directions. The emerging astrophysical and observational perspectives and implications are dicussed, with particular reference to the properties of accretion discs around black holes and naked singularities, which may provide characteristic signatures and could help distinguish these objects.

Keywords: Gravitational Collapse; Black Holes; Naked Singularities

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1. Introduction

What is the final fate of a massive star that has exhausted its internal nuclear fuel and which then undergoes a catastrophic gravitational collapse under the force of its own gravity? What are the implication of such a phenomenon on our basic understanding of gravitation theory, and what will be the observational implications as far as very high energy astrophysical phenomena are concerned?

This is the arena where we in fact come face to face with the regime of extreme and ultra-strong gravity fields, and the answer must be sought within the framework of a gravitation theory such as the Einstein's theory of gravity. This is where the new and unfamiliar universe encompassing the ultra-strong gravity effects actually reveals itself and we must encounter exotic astrophysical objects such as the spacetime singularities, black holes, and other ultra-compact entities in nature.

¿From such a perspective, considerable work has been done in past years to understand the dynamical gravitational collapse in general relativity, and many new insights have been obtained. At the same time, several basic issues also remain unclear or unanswered, and there is a general consensus that the nature of cosmic censorship, black holes and naked singularities remain one of the most important unresolved issues in gravitation theory and black hole physics today. These issues would necessarily have far-reaching implications and applications for our understanding on fundamental aspects of gravity theories and applications to high energy astrophysics.

Our purpose here is to discuss several key problems and questions on gravitational collapse, formation of black holes and naked singularities as final state for a continual collapse, and the related issues regarding the nature and structure of spacetime singularities. These problems are closely related to the nature of cosmic censorship hypothesis and our current understanding on black hole physics. The understanding of such questions is also basic to the theoretical developments as well as the modern astrophysical applications of black holes which are being vigorously pursued today.

Further, we discuss and consider here certain interesting astrophysical implications emerging from the current work on gravitational collapse within the framework of general relativity. An important question would be, if naked singularities, which are hypothetical astrophysical objects, actually formed in gravitational collapse of massive stars, how these would look observationally different from black holes. A lead that is emerging from the recent work is that the accretion discs around these ultra-high gravity objects, namely black holes and naked singularities, would be significantly different from each other providing characteristic signatures. Therefore there exists a possibility to distinguish these two different outcomes of collapse observationally. We discuss some of these current developments here.

We thus consider and take up below a series of outstanding questions on these topics, where we attempt to clarify what is already known, while separating the issues which remain unanswered as yet, and on which more work still remains to be done. It is our hope that such a treatment will clarify where the research frontiers are moving on these problems and what remain the major outstanding questions where more work is needed. We intend to review here in this manner some of the major challenges in black hole physics today, and the current progress on the same. It is emphasized that to secure a concrete foundation for the basic theory of black hole physics as well as to understand the high-energy astrophysical phenomena, it is essential to gain a suitable insight into these questions. This will be of course from a perspective of what we think are the important problems, mainly within an analytical treatment of the Einstein's theory of gravity in the framework of the general relativity field equations, and no claim to completeness is made.

2. Stellar collapse

The stars have a life cycle wherein they are born in gigantic clouds of dust and galactic material, they then evolve and shine for millions of years, and eventually enter the phase of dissolution and extinction. Stars shine by burning their nuclear fuel within, which is mainly hydrogen, fusing it into helium and later into other heavier elements. Eventually, when all matter is converted to iron, no more nuclear processes capable of producing energy are possible and no new internal energy is produced within the star. The all-pervasive force of gravity then takes over to determine the final fate and evolution of such a star. Earlier there was a balance between the force of gravity that pulled matter of the star towards its center and the outwards pressures generated by internal fusion processes. This balance kept the star stable and going, while it lived its normal life span of shining and radiating light and energy produced within. Once the internal pressures subside, gravity takes over, and the star begins to contract and collapses onto itself.

A star as massive as ten or twenty times the sun would burn much faster and live only a few million years, as compared to the lifetime of several billion years for a smaller star such as the sun. When the sun runs out of internal fuel, its core will contract under its own gravity, but it will then be eventually supported by a new force within, created by fast moving electrons, called the electron degeneracy pressure. Such an object is called a white dwarf. Similarly, stars up to three to five times the mass of the sun would settle to the final state, which is a neutron star, after an initial collapse and losing some of their original mass. These are pure neutron objects created in the collapse under the strong crush of gravity which collapses atoms too. The quantum pressure of these neutrons then support the star, which is barely some ten to twenty kilometers in size. The final outcome of collapse thus depends on the initial mass of the star, which again stabilizes at a much smaller radius due to the balancing pressures generated by either electrons or neutrons.

The more massive stars cannot, however, settle to a white dwarf or neutron star state because these quantum pressures are then just not sufficient to balance gravity and stabilize the collapsing star beyond the neutron star mass limit. Then a continual gravitational collapse, which no known physical forces are able to halt, becomes inevitable once the star exhausted its internal fuel. So the life-history of a star of large mass is essentially different from the small mass stars, and as the astrophysicist Subrahmanyan Chandrasekhar pointed out way back in 1934, "...one is left speculating on other possibilities." [1] What will be the final fate of such a continual gravitational collapse of a massive star? The answer must be determined by the Einstein theory of gravitation, as gravity now is the sole force deciding the future evolution of the star.

Gravitation theory and relativistic astrophysics have gone through extensive developments in past decades, further to the discovery of quasars in 1960s, and also other very high energy phenomena in the universe such as the gamma-ray bursts. For compact objects such as neutron stars and for situations involving very high energy densities and masses, strong gravity fields governed by the general theory of relativity play an important role and dictate the observed high energy phenomena having intriguing physical properties.

Several models to explain the gamma-ray bursts, which emit in a few seconds energy of the sun's entire lifetime, have been proposed in terms of a collapsar, invoking collapse of a single massive star as the mechanism required to produce the extreme burst of ultra-high energies. The gravitational collapse of a massive star or much larger matter clouds, lie at the heart of astrophysics of such phenomena. It is the key physical process which is basic to the formation of a star itself from interstellar clouds, in formation of galaxies and galaxy clusters, and in a variety of cosmic happenings including structure formation in the universe.

3. Final fate of gravitational collapse

What is the final fate of a massive star towards the end of its life cycle, when it exhausted its internal nuclear fuel and started shrinking and collapsing under the pull of its own gravity? This is one of the most important and outstanding unresolved problems in astrophysics and cosmology today.

When the massive star runs out of its nuclear fuel, the force of gravity takes over and a catastrophic gravitational collapse of the star takes place. The star that lived for millions of years and which stretched to millions of kilometers in size, now collapses catastrophically within a matter of seconds. According to the general theory of relativity, the outcome of such a continual collapse will be a spacetime singularity where all physical quantities such as densities and spacetime curvatures diverge.

The fundamental question of the fate of a massive star, when it collapses continually under the force of its own gravity, was highlighted by Chandrasekhar [1], when he pointed out: 'Finally, it is necessary to emphasize one major result of the whole investigation, namely, that the life history of a star of small mass must be essentially different from the life-history of a star of large mass. For a star of small mass the natural white-dwarf stage is an initial step towards complete extinction. A star of large mass ($> M_c$) cannot pass into the white-dwarf stage, and one is left speculating on other possibilities.'

It is possible to see the seeds of modern black hole physics already present in the above inquiry made on the final fate of massive stars. The issue of endstate of large mass stars has, however, remained unresolved and elusive for a long time of many decades after that. In fact, a review of the status of the subject many decades later notes, 'Any stellar core with a mass exceeding the upper limit that undergoes gravitational collapse must collapse to indefinitely high central density... to form a (spacetime) singularity'. [2] The reference above is to the prediction by general relativity, that under reasonable physical conditions, the gravitationally collapsing massive star must terminate into a spacetime singularity. [3]

While Chandra's work pointed out the stable configuration limit for the formation of a white dwarf, the issue of final fate of a star which is much more massive with tens of solar masses, remains very much open even today. Such a star cannot settle either as a white dwarf or a neutron star final state.

The issue is clearly important both in high energy astrophysics as well as cosmology. For example, our observations today on the existence of dark energy in the universe and the cosmic acceleration it produces are intimately connected to the observations of supernovae in the universe, which are the product of collapsing stars. It is the observational evidence coming from supernovae, that are exploding in the faraway universe, which tells us how the universe may be accelerating away and the rate at which this acceleration takes place. At the heart of such a supernova underlies the phenomenon of catastrophic gravitational collapse of the massive star, wherein a powerful shockwave is generated, blowing off the outer layers of the star.

If such a star is able to throw away enough of its matter in such an explosion, it might eventually settle as a neutron star. But in the case otherwise, or if further matter accreted onto the neutron star, there will be a continual collapse again, and we shall have to then explore and investigate the question of final fate of such a massive collapsing star. But the stars, which are more massive and well above the normal supernovae mass limits must straightaway enter a continual collapse mode at the end of their life cycles, without any intermediate neutron star stage. The final fate of the star in this case must be then decided by the general theory of relativity alone.

The important point here is, more massive stars which are tens of times the mass of the sun burn much faster and are far more luminous. Such stars then cannot endure more than about ten to twenty million years, which is a much shorter span of life as compared to stars such as the sun, which live much longer. Therefore, the question of final fate of such short lived massive stars is of central importance in astronomy and astrophysics.

What needs to be investigated then is what happens in terms of the final outcome, when such a massive star dies on exhausting its internal nuclear fuel. As we indicate here, the general theory of relativity then predicts that the collapsing massive star must terminate into a spacetime singularity, where the matter energy densities, spacetime curvatures and other physical quantities blow up. It then becomes crucial to know whether such super-ultra-dense regions, forming in the stellar collapse, are visible to an external observer in the universe, or whether they will be always hidden within a black hole and an event horizon of gravity that could form as the star collapses. This is one of the most important issues in the physics of black holes today.

3.1. *A black hole is born: The Oppenheimer-Snyder-Datt model*

To understand the final state of collapse for a massive star, we need to trace the time evolution of the system or its dynamical progression using the Einstein equations of gravity. The star shrinks under the force of its own gravity, which comes to dominate other basic interactions of nature such as the weak and strong nuclear forces that typically provided the outwards pressure to balance the pull of gravity.

This problem was considered for the first time by Oppenheimer and Snyder, and independently by Datt, in the late 1930s. [4,5] In order to deal with the rather complex Einstein equations, they assumed the density to be homogeneous and the same everywhere within the spherical star. They also neglected gas pressure, taking it to be zero. Their calculations then showed that an event horizon develops as collapse progresses, such that no material particles or photons from the region escape to faraway observers. Once the star collapses to a radius smaller than the horizon, it enters the black hole, finally collapsing to a spacetime singularity with extreme densities that is hidden inside the black hole and invisible to any external observers. For the collapsing star to create a black hole, an event horizon must develop prior to the time of the final singularity formation.

Very considerable amount of research and astrophysical applications have been developed on such black holes in past decades, which occupy a major role in astrophysics and cosmology today. To understand how an event horizon and black hole can form when a massive star collapses, let us consider the model for an homogeneous star that eventually collapses to a singularity of infinite density and spacetime curvatures. The relativistic calculations imply that as the star collapses the force of gravity on its surface keeps growing and eventually a stage is reached when no light emitted from its surface is able to escape away to faraway observers. This is the epoch when an event horizon formed, and the star then enters the black hole region of spacetime. The infalling emitter does not feel any thing special while entering the horizon, but any faraway observer stops seeing the light from him. The strong gravity of the star causes this one-way membrane, that is the event horizon to form. Within the horizon collapse continues further to crush the star into a singularity. Such black holes can suck in more matter from surroundings and grow bigger and bigger.

The physics that is accepted today as the backbone of the general mechanism describing the formation of black holes as the endstate of collapse relies on this very simple and widely studied Oppenheimer-Snyder-Datt (OSD) dust model, which describes the collapse of a spherical cloud of homogeneous dust. In the OSD case, all matter falls into the spacetime singularity at the same comoving time, while the event horizon forms earlier than the singularity, thus covering it. A black hole region in the spacetime results as the endstate of collapse.

It is of course clear that the homogeneous and pressureless dust is a rather highly idealized and unphysical model of matter. Taking into account inhomogeneities in the initial density profile it is possible to show that the behaviour of the horizon can in fact change drastically, thus leaving two different kinds of outcomes as the possible result of generic dust collapse: the black hole, in which the horizon forms at a time anteceding the singularity, and the naked singularity, in which the horizon is delayed, thus allowing the null geodesics or light rays to escape the central singularity where the density and curvatures diverge, to reach faraway observers. [6,7,8] It is clear that once the light rays escape, then the material particles or the timelike geodesics will also escape from the singularity.

The issue of such a collapse has to be probed necessarily within the framework of a suitable theory of gravity, because the ultra-strong gravity effects will be necessarily important in such a scenario. This purpose was achieved by the OSD model that used the general theory of relativity to examine the final fate of an idealized massive matter cloud, namely a spatially homogeneous ball with no rotation or internal pressure, and assumed to be spherically symmetric. As said, the dynamical collapse created the spacetime singularity, preceded by an event horizon, thus developing a black hole in the spacetime. The singularity would be hidden inside such a black hole, and the collapse eventually settled to a final state which is the Schwarzschild geometry (see Fig. 1).

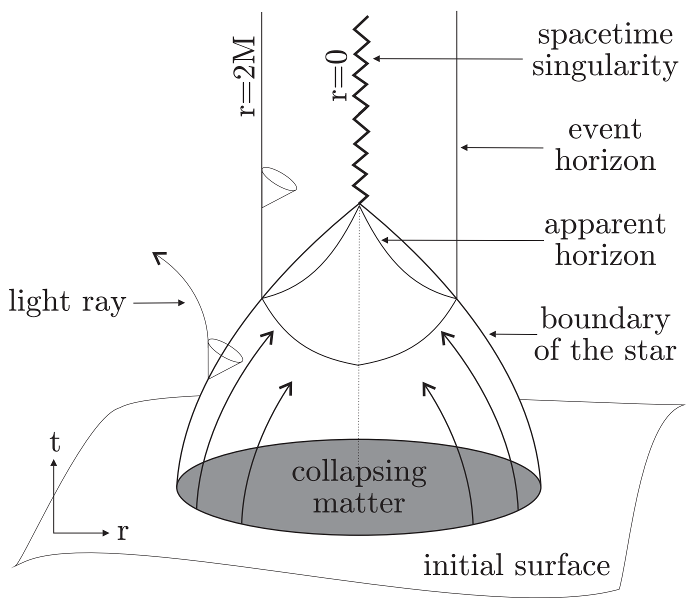

Fig. 1. Dynamical evolution of a homogeneous spherical dust cloud collapse, as described by the Oppenheimer-Snyder-Datt solution.

Interestingly, there was not much attention paid to this model at that time, and it was widely thought by gravitation theorists and astronomers that it would be absurd for a star to reach such a final ultra-dense state during its evolution. It was in fact as late as 1960s only, that a resurgence of interest took place in such black holes, their dynamical formation and physical properties that they would exhibit for the surrounding regions of spacetime. This was mostly due to some important observational developments in astronomy and astrophysics that happened around that time, such as the discovery of several very high energy phenomena in the universe, like quasars, radio galaxies and such others, where no known laws of physics were able to explain the observations that were related to such extremely high energy phenomena in the cosmos. Attention was drawn then to the dynamical gravitational collapse and its final fate, and in fact the term 'black hole' was coined just around the same time in 1969, by John Wheeler.

3.2. *Predictions of General Relativity*

According to Einstein's general theory of relativity, the collapse must proceed to create a space-time singularity. This is a region where the physical parameters such as the energy density and the space-time curvatures blow up taking arbitrarily large values. Thus the usual laws of physics break down near such a singularity. This is the regime of ultra-strong gravity fields, with other basic forces of nature playing only a secondary role. Quantum effects must also become important in such strong fields at ultra-small length scales near the singularity, and eventually what is really needed would be a quantum theory of gravity which would possibly resolve the singularity. [9]

Specifically, the outcome of a continual collapse in a fully general scenario of a spacetime with an evolving matter field is described by the singularity theorems in general relativity. Subject to the following conditions, namely, (i) An energy condition requiring the positivity of energy density for matter fields, (ii) A reasonable causal structure of the spacetime in terms of a causality condition such as chronology or strong causality condition, and (iii) A condition that trapped surfaces exist or develop in the spacetime, which is a sufficient condition ensuring that sufficient mass is packed in a small enough region, the singularity theorems in general relativity [3] imply that the spacetime must contain a singularity in the form of geodesic incompleteness.

It follows that for any general relativistic gravitational collapse developing from regular initial data, in a spacetime without any symmetry conditions such as spherical symmetry necessarily holding, if the above physically reasonable conditions are satisfied then the collapse must create a spacetime singularity necessarily. In all physically reasonable scenarios, the densities, curvatures and all other physical quantities would typically blow up in the limit of approach to such a spacetime singularity.

The OSD collapse scenario discussed above would be a special case of such a general gravitational collapse, which terminates into a final simultaneous spacetime singularity, which is the final endstate of the collapsing matter cloud.

In view of the generality of the singularity theorems, it would be expected that any physically realistic gravitational collapse, where the massive star collapses continually towards the end of its life cycle, must terminate into a spacetime singularity of ultra-high densities and extreme spacetime curvatures.

3.3. *What the singularity theorems do not predict*

It must be noted, however, that the singularity theorems in general relativity predict only the existence of spacetime singularities, under a set of physically reasonable conditions. The above theoretical result on the existence of singularities is of a rather general nature, and provides no information at all of any kind on the nature and structure of such singularities.

In particular, these theorems give us no information as to whether such spacetime singularities, whenever they form especially in a gravitational collapse, will be necessarily covered in the event horizons of gravity and thus hidden from us, or whether these could also be visible to external observers in the universe.

Specifically, the possibility remains very much open that a spacetime singularity develops in gravitational collapse, however, which is no longer covered by an event horizon and may be causally connected to faraway observers in the universe. In such a case, the ultra-high density and curvature regions would be able to communicate with and send out signals to exterior faraway observers. In this sense, gravity predicts exciting outcomes for the final fate of a massive collapsing star, with profound implications for fundamental physics.

Therefore, whether a strong curvature singularity that formed in a realistic collapse would be visible or hidden from a faraway observer in the universe remains very much an open question in the Einstein's theory of gravity. The key physical feature that decides the visibility or otherwise of the singularity is the interplay between the structure and time-curve of the singularity and that of the trapped surface formation in the spacetime.

3.4. *The cosmic censorship conjecture*

Unlike the idealized and rather special model that the OSD homogeneous collapse scenario described above, real stars have an inhomogeneous density (namely, higher at their centers), and they also have non-zero pressures within them as they collapse. Moreover, the stars also rotate. Would every massive star collapsing towards the end of its life cycle turn into a black hole necessarily, just as the OSD case? The cosmic censorship conjecture supposes that the answer to this question is in the affirmative, namely, that the singularity forming in collapse always hides within an event horizon, never to be seen by external observers.

Theorists generally believed that in such circumstances, a black hole will always form covering the singularity, which will then be always hidden from external observers. Such a black hole is a region of spacetime from which no light or particles can escape. The assumption that spacetime singularities of collapse would be always covered by black holes is called the Cosmic Censorship Conjecture (CCC).[10,11] Thus, whatever the physical conditions and forces within the massive stars may be, their continual collapse must yield a black hole. Censorship assumption amounts to a constraint on the nature of allowed dynamical evolutions for collapsing clouds in general relativity.

As of today, we do not have any proof or any specific mathematically rigorous formulation of the CCC available within the framework of gravitation theory. Therefore, one of the key questions in black hole physics today is, is it possible that such singularities of collapse, which are super-ultra-dense regions forming in spacetime, be visible to external observers in the universe? This is one of the most important unresolved issues in gravitation theory currently.

If the singularities were always covered in horizons and if CCC were true, that would provide a much needed basis for the theory and astrophysical applications of black holes. On the other hand, if the spacetime singularities which result from a continual collapse of a massive star were visible to external observers in the universe, we would then have the opportunity to observe and investigate the super-ultra-dense regions in the universe, which form due to gravitational collapse and where extreme high energy physics and also the quantum gravity effects will be at work.

The crucial physical question here is, can we really see and observe such super ultra-dense regions which develop when massive stars collapse, in violation to the CCC? If CCC is to hold then the singularity must be necessarily enveloped and hidden in a region of spacetime from which no particles or light can escape to faraway observers, and which cannot communicate with external universe. Such a region is called a black hole, and its boundary is termed the event horizon. The event horizon is, by definition, the boundary of the region of spacetime that is accessible to an observer at infinity, and is a one-way membrane that allows no matter or light to escape away but lets these fall in through it and into the black hole.

In other words, this conjecture means that collapse of a massive star must necessarily produce a black hole, which hides within an event horizon the spacetime singularity of gravitational collapse. But since past many decades it is only a conjecture. At times it is taken to be true, but the inevitability of event horizons covering the singularities of collapsing stars has yet to be demonstrated.

The point here is that, as discussed above, general relativity predicts the necessary existence and formation of a spacetime singularity from gravitational collapse, but it does not tell that the singularity must be within the black hole region only. In other words, amongst the two, namely the horizon and singularity, which one must come first as the star collapses is not answered by the general relativity. Therefore the hypothesis, which is the foundation of all of black hole physics today, remains of crucial importance for all of the basic theory and astrophysical applications of black holes which are extensively pursued in recent years.

To decide on the validity or otherwise of censorship, one must therefore study the dynamical collapse of massive stars to determine their final fate. In recent years, detailed analytical research on gravitational collapse has given insights into when black holes form and when they would not, in fact, form. Some models suggest that visible ultra-dense regions, or naked singularities, may arise naturally and generically as an outcome of collapse. If so, the implications would be enormous and would touch on nearly every aspect of astrophysics and fundamental physics. They might account for extreme high-energy phenomena such as the gamma-rays bursts that have defied explanation. They might also offer a laboratory to explore quantum gravity effects that are otherwise extremely difficult to observe.

The CCC came into existence in 1969, when Penrose suggested and assumed that what happens in the OSD picture of the gravitational collapse as discussed above, would be the generic final fate of a realistic collapsing massive star in general. In other words, it was assumed that the collapse of a realistic massive star will terminate into a black hole only, which hides the singularity, and thus no visible or naked singularities must develop in gravitational collapse.

To be precise, the key elements that are crucial to the definition of the CCC are three:

- Dynamics: meaning that singularities must arise from regular matter fields through the dynamical time evolution that is governed by Einstein equations.
- Physical validity: meaning that the models must describe some physically realistic matter fields, typically those obeying to some energy conditions.
- Generic: meaning that the collapse outcome must not be different, or change suddenly, whenever a slight change is introduced in the model (in the form of slightly different matter fields or perturbing the system's symmetries).

It is easy to see that the last two conditions are subject to a rather broad interpretation, leaving some room for different formulations and interpretations of the cosmic censorship conjecture.

Further to the CCC formulation, many important developments then took place in the black hole physics which started in the earnest, and several important theoretical aspects as well as astrophysical applications of black holes started developing. The classical as well as quantum aspects of black holes were then explored and interesting thermodynamic analogies for black holes were also developed. Many astrophysical applications for the real universe then started developing for black hole, as for example in models using black holes for the description of phenomena such as jets emitted from the centers of galaxies, extremely energetic gamma rays burst, and such others.

The key issue raised by the CCC, however, still remained very much open, namely whether a real star will necessarily go the OSD way only as for its final state of collapse, and whether the final singularity will be always necessarily covered within an event horizon of gravity. This is because, real stars are inhomogeneous, have internal pressure forces and so on, as opposed to the idealized OSD assumptions. Thus this remains an unanswered question, which is one of the most important issues in gravitation physics and black hole physics today.

A spacetime singularity that is visible to faraway observers in the universe is called a naked singularity (see Fig. 2). The point here is, while the general relativity predicts the existence of singularity as the endstate for collapse, it gives no information at all on the nature or structure of such singularities, and whether they will be covered by event horizons, or would be visible to external observers in the universe.

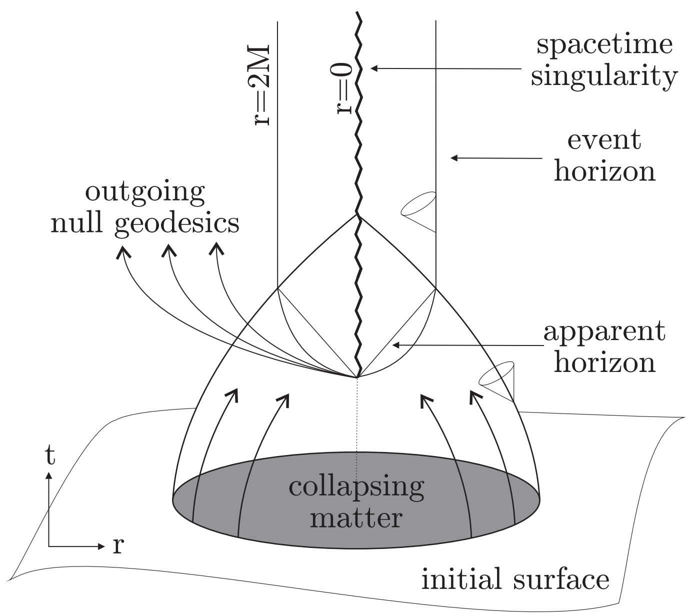

Fig. 2. A spacetime singularity of gravitational collapse which is visible to external observers in the universe, in violation to the cosmic censorship conjecture.

As there is no proof, or even any mathematically rigorous statement available for CCC after many decades of serious effort, what is really needed to resolve the issue is a meticulous study of gravitational collapse models for a realistic collapse configuration, with inhomogeneities and pressures included. These scenarios need to be worked out and analyzed in detail within the framework of Einstein gravity. Only such considerations will allow us to determine the final fate of collapse in terms of either a black hole or naked singularity final state.

It was hoped, as theorists kept developing black hole physics, that a derivation from basic physical laws, of censorship would soon arrive. However, despite numerous attempts over past four decades, this is still not realized, and we do not even have any rigorous mathematical formulation of the hypothesis. The reasons for this are becoming clear now, as we discuss below. According to the principles as laid out by the singularity theorems, singularities are inevitable, but no such principles apply to the event horizon as to when it exists or covers the singularity. In fact, the initial singularity in cosmology that created the observed universe is not hidden inside a horizon, it is visible in principle. Whether this is also true for stars has been an elusive question because the Einstein equations are highly non-linear and complex.

The censorship hypothesis nevertheless provided a major impetus to developments in black hole physics. Assuming, provisionally, that it holds, physicists have investigated the properties of black holes and created detailed laws of black hole dynamics and related aspects. Meanwhile, they have also applied the concept of black holes to explain various ultra-high energy processes observed in the universe, such as quasars and X-ray-emitted binary star systems.

4. Recent developments on collapse

It is clear that in view of the lack of any theoretical progress on CCC, the only important and meaningful way to make any progress on this problem is to make a detailed and extensive study of gravitational collapse in general relativity. Some recent progress in this direction is summarized below. While we seem to have now a good understanding of the black hole and naked singularity formations as final fate of collapse in very many gravitational collapse models, the key point now is to understand the genericity and stability of these outcomes, as viewed in a suitable framework. We discuss these issues here in some detail. Recent developments on throwing matter into a black hole and the effect it may have on its horizon and certain quantum aspects will be discussed later. The issue of predictability or its breakdown in gravitational collapse is of interest as we shall point out later.

In particular, we discuss below the recent results investigating the final fate of a massive star within the framework of the Einstein gravity, and the stability and genericity aspects of the gravitational collapse outcomes in terms of black holes and naked singularities. It is pointed out that some of the new results obtained in recent years in the theory of gravitational collapse imply interesting possibilities and understanding for the theoretical advances in gravity as well as towards new astrophysical applications.

Over the past years, many gravitational collapse models have been worked out and studied in detail at first considering simplified models such as dust or self similar collapse (see e.g. Refs. 13–17). and later refining to more elaborated models with the inclusion of pressures. The generic conclusion that has emerged out of these studies is that, both the black holes and naked singularity final states do develop as collapse endstates, for a realistic gravitational collapse that incorporates inhomogeneities as well as non-zero pressures within the interior of the collapsing matter cloud. Subject to various regularity and energy conditions to ensure the physical reasonableness of the model, it is the initial data, in terms of the initial density, pressures, and velocity profiles for the collapsing shells, that determine the final fate of collapse as either a naked singularity or a black hole (for further detail and references see e.g. Ref. 12).

4.1. *Collapse studies*

As we noted, while the censorship conjecture is at the heart of the modern day black hole physics and its astrophysical applications, not to speak of the proof, even any rigorous mathematical formulation of the same is also far from the sight today. How does one resolve this profound issue at the heart of physics and astrophysics of black holes? Because a direct proof that applies under all conditions is so difficult, our group and several other teams have considered a variety of specific collapse scenarios, which provide many useful insights on the final fate of a massive star. In essence, we have conducted a series of computational experiments to compile a list of conditions under which gravitational collapse leads to black hole or a naked singularity.

The basic strategy here is to examine all possible courses of evolution for a massive star with a given set of physically reasonable properties and regularity conditions, which are basic consistency requirements such as having a positive energy, regularity of the initial data from which the collapse evolves as the cloud collapses under self-gravity, and such others. Mathematically, these are the allowed dynamical solutions to Einstein equations. We supply the initial data for collapse in terms of the initial density and pressure profiles of matter within the star, and the velocities of the collapsing concentric shells of gas. We then compute the causal structure within the collapsing cloud, which reveals how the light and matter are trapped as gravity grows stronger as collapse progresses. This decides whether the final singularity of collapse would be visible or covered by a horizon.

It should be noted that gravitational collapse studies have in fact a long history beginning with the Oppenheimer-Snyder-Datt models that we mentioned above. In particular, strict homogeneity is only an idealization and we must allow for inhomogeneities of density in a matter cloud. One such model was considered by Seifert and collaborators [18,19], wherein neighboring shells of matter intersected to create momentary singularities that were not covered by horizons. However, these were not taken seriously in that, an observer would not be destroyed and crushed to a zero volume while passing through the same, which is the true sign for a genuine singularity. The observer would just sail through undamaged to another region of spacetime while going through such a weak singularity.

A realistic star has typically a higher density at its center, slowly decreasing as one moves outwards. In 1979 Eardley and Smarr [20] performed a numerical simulation of such a model with zero pressure, and an exact mathematical treatment by Christodoulou [21] followed in 1984. The model revealed a naked singularity at the center where the physical radius of the cloud becomes zero. However, Newman soon showed [22] that this singularity is again gravitationally weak. Many researchers, including Newman and Joshi in 1988 (see e.g. Ref. 23), unsuccessfully tried to formulate a rigorous theorem that naked singularities are always gravitationally weak. What we realized then was that we simply did not know enough about gravitational collapse to formulate a censorship theorem that holds generally. We had to continue on the longer road of slowly building up our knowledge by considering case studies involving realistic collapse models.

Subsequently, researchers found many scenarios of inhomogeneous pressureless collapse where strong-curvature naked singularities, the ones that are genuine crushing singularities, developed from regular initial conditions. A general treatment for such a scenario was developed by Joshi and Dwivedi in 1993 (see Ref. 6). In particular, it became clear that while the homogeneous pressureless collapse considered by Oppenheimer and Snyder produced a black hole, a more realistic density profile with density higher at the center and decreasing as one moves away can give rise to a naked singularity, which is an intriguing situation indeed. While these dust collapse models ignored pressure, the general techniques developed in the above work to understand the dynamical evolution of collapse did find applications later when collapse models with pressure were considered.

4.2. *Collapse formalism*

As we know, when nuclear processes end at the core of the star, the transition from the equilibrium phase to the collapsing phase happens very rapidly. Therefore, in analytical models describing collapse it is reasonable to assume that collapse starts from equilibrium with only gravity acting on the particles of the star and without considering effects due to other forces.

In this section we summarize and review the key mathematical features of gravitational collapse in spherical symmetry and review some of the basic assumptions and equations. The reader who is already familiar with the formalism or not interested in the technical details may skip this section.

The most general metric describing a spherically symmetric collapsing cloud in the comoving coordinates r and t depends upon three functions $\nu(r,t)$, $\psi(r,t)$ and $R(r,t)$, and is written as,

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\psi(t,r)}dr^2 + R(t,r)^2 d\Omega^2 \ . \tag{1}$$

The energy-momentum tensor for a fully general anisotropic fluid is then written as,

$$T_t^t = -\rho;\ T_r^r = p_r;\ T_\theta^\theta = T_\phi^\phi = p_\theta \ , \tag{2}$$

where ρ is the energy density and p_r and p_θ are the radial and tangential pressures. We note that the widely studied dust collapse is obtained for $p_r = p_\theta = 0$, while a perfect fluid collapse is given by $p_r = p_\theta$. The metric functions ν, ψ and R are related to the energy-momentum tensor via the Einstein equations that can be given in the form:

$$p_r = -\frac{\dot{F}}{R^2\dot{R}} \,, \tag{3}$$

$$\rho = \frac{F'}{R^2R'} \,, \tag{4}$$

$$\nu' = 2\frac{p_\theta - p_r}{\rho + p_r}\frac{R'}{R} - \frac{p_r'}{\rho + p_r} \,, \tag{5}$$

$$2\dot{R}' = R'\frac{\dot{G}}{G} + \dot{R}\frac{H'}{H} \,, \tag{6}$$

$$F = R(1 - G + H) \,, \tag{7}$$

where we have defined the functions H and G as,

$$H = e^{-2\nu(r,t)}\dot{R}^2, \; G = e^{-2\psi(r,t)}R'^2, \tag{8}$$

and where F is the Misner-Sharp mass of the system which is related to the total amount of matter enclosed in the comoving shell labeled by r at the time t.

Gravitational collapse is obtained by requiring that $\dot{R} < 0$ and the central shell-focusing singularity is reached when $R = 0$. At the singularity the energy density and spacetime curvatures blow up. Divergence of ρ is obtained also whenever $R' = 0$, thus indicating the presence of a 'shell-crossing' singularity. Such singularities are generally found to be gravitationally weak and do not correspond to a required divergence of the curvature scalars. Therefore this indicates a breakdown of the coordinate system being used, rather than a true and genuine physical singularity. [24] Since we are interested only in the occurrence of strong curvature shell-focusing singularities, in the following discussion we will always assume that $R' > 0$ throughout the collapse.

The system presents an additional degree of freedom due to scale invariance. We can then choose the initial time in such a way so that $R(r, t_i) = r$, and we introduce the scaling function $v(r, t)$ defined by,

$$R = rv \,, \tag{9}$$

with $v(r, t_i) = 1$ (so that the collapse will be described by $\dot{v} < 0$ and the singularity will be obtained at the value $v = 0$). This is a better definition for the singularity since at $v \neq 0$ the energy density does not diverge anywhere on the spacelike surfaces, including at the center $r = 0$. In this manner, for a regular mass function F, the divergence of ρ is reached only at the singularity. We note that the function $v(t, r)$ is monotonically decreasing in t close to the formation of the singularity and therefore can be used as a 'time' coordinate replacing t itself. We can then perform a change of coordinates from (r, t) to (r, v) and thus we have $t = t(r, v)$. In this case the derivative of v with respect to r in the (r, t) coordinates shall be considered as a function of the new coordinates, $v' = w(r, v)$. Then the Misner-Sharp mass can be taken to be a function of the comoving radius r and the comoving time t, expressed via the 'temporal' label v as

$$F = F(r, v(r,t)) \ . \tag{10}$$

To solve the system of Einstein equations we then proceed as follows:

- If p_r is related to ρ via an equation of state, the integration of the same, once we substitute equations (3) and (4), will give F as a function of R and its derivatives. If no equation of state is provided then F is a fully free function.
- ρ and p_r are given from equations (3) and (4) as functions of F and R, substituting F leaves ρ and p_r as functions of R and its derivatives.
- Integration of the equation (5) gives ν as a function of p_θ and R and its derivatives. If no relation is given for p_θ (as in the perfect fluid case) then we can take p_θ as a free function of r and v.
- Integration of equation (6) will generally give $G(r, v) = b(r)e^{2A(r,v)}$ with $A(r, v)$ given by $A_{,v} = \nu' \frac{r}{R'}$ and $b(r)$ being a free function coming from the integration which describes the velocity profile of the particles.
- Finally, from equation (7) we are left with one differential equation in R and its derivatives that acts as an equation of motion for the system. Once this is solved the whole system of Einstein equations is solved.

The equation of motion is given by

$$\dot{R}^2 = e^{2\nu}\left(\frac{F}{R} + G - 1\right) \ , \tag{11}$$

or in terms of the scaling factor v as,

$$\dot{v} = -e^{\nu}\sqrt{\frac{F}{r^3 v} + \frac{be^{2A} - 1}{r^2}} \ , \tag{12}$$

where the minus sign has been taken since we are dealing with collapse. In order to have a solution the following 'reality condition' must be satisfied:

$$\left(\frac{F}{r^3 v} + \frac{be^{2A} - 1}{r^2}\right) > 0. \tag{13}$$

As is well-known, under completely general conditions it is not possible to fully solve the system of Einstein equations globally, the main reason being that these are a complicated set of non-linear partial differential equations. Nevertheless, the relevant information about the final fate of collapse in the general form of matter fields and curvature invariants can be extracted by an analysis of the behaviour of the solutions near the singularity and near the center of the cloud. As we point out below, this can be done by integrating the Einstein equations by one order, and by reducing them to a first order system so as to obtain a time evolution equation for the collapsing cloud which decides the nature of the final singularity curve in the case of a continual collapse. This will help us decide the local visibility or otherwise of the central singularity.

4.2.1. *Regularity and energy conditions*

Einstein equations by themselves provide only the relations between geometry and matter distribution within the collapsing matter cloud, actually without giving any statement about what kind of matter we are dealing with, that is responsible for the given spacetime geometry. ¿From a physical perspective, not every type of matter distribution can be allowed, and some restrictions on possible matter models must be made based on physical reasonableness. These restrictions typically come in the form of energy conditions ensuring the positivity of mass-energy density and the sum of pressure and energy density. For example the weak energy conditions can be written as:

$$\rho \geq 0,\ \rho + p_r \geq 0,\ \rho + p_\theta \geq 0. \tag{14}$$

These typically give some constraints on the behaviour of the mass function F. For example, it is easy to see that from the first one, if we impose the condition of no 'shell-crossing' singularities, it follows that $F' > 0$. Also some conditions arise on the allowed pressures, though positive pressures are obviously always allowed, imposing the dominant energy condition would further require that pressures cannot exceed the energy density. At the same time some negative pressures can be allowed as well.

Further to these, certain regularity conditions must be fulfilled in order for the matter fields to be well-behaved at the center of the cloud at the initial epoch from which collapse evolves. These are the finiteness of the energy density at all times anteceding the singularity and regularity of the Misner-Sharp mass in $r = 0$, and they imply that we must have generically,

$$F(r,t) = r^3 M(r, v(r,t)) \ , \tag{15}$$

where M is a regular function going to a finite value M_0 at $r = 0$. Also, requiring that the energy density has no cusps at the center is reflected in the condition,

$$M'(0, v) = 0 \ . \tag{16}$$

In Ref. 25 it was shown that vanishing of the pressure gradients near the center of the cloud implies that the radial and tangential stresses must assume the same value in the limit of approach to the center. This requirement comes from the fact that the metric functions should be at least $\mathcal{C}^2$ at the center of the cloud and is a consequence of the fact that the Einstein equation (5) for a general fluid contains a term in $p_r - p_\theta$ that therefore must vanish at $r = 0$ thus indicating that in the final stages of collapse of the core of a star the cloud behaves like a perfect fluid in proximity of $r = 0$. Further, since the gradient of the pressures must vanish at $r = 0$, we see that $p' \simeq r$ near $r = 0$, which for the metric function ν implies that near the center,

$$\nu(r,t) = r^2 g(r, v(r,t)) + \bar{g}(t) \ , \tag{17}$$

where the function $\bar{g}(t)$ can be absorbed in a redefinition of the time coordinate t. From the above, via the definition for A we can write $A_{,v} = \frac{2g+rg'}{R'}r^2$.

Regularity at the center implies some requirements for the velocity profile $b(r)$ as well. Then b can be written as,

$$b(r) = 1 + r^2 b_0(r) \ , \tag{18}$$

near $r = 0$. We can now interpret the free function $b(r)$ in relation with the known Lemaitre-Tolman-Bondi (LTB) dust models. [26,27,28] In fact, the cases with b_0 constant are equivalent to the bound ($b_0 < 0$), unbound ($b_0 > 0$) and marginally bound ($b_0 = 0$) LTB collapse models, depending on the sign of the quantity b_0.

4.2.2. *Initial data and matching*

Specifying the initial conditions from which collapse evolves consists in prescribing the values of the three metric functions and of the density and pressure profiles as functions of r on the initial time surface given by $t = t_i$. Once this is done and the eventual free functions are provided, the dynamical evolution of the collapsing cloud is entirely determined. Specifying initial conditions reduces to defining the following functions:

$$\rho(r,t_i) = \rho_i(r), \ p_r(r,t_i) = p_{r_i}(r), \ p_\theta(r,t_i) = p_{\theta_i}(r),$$
$$R(r,t_i) = R_i(r), \ \nu(r,v(r,t_i)) = \nu_i(r), \ \psi(r,t_i) = \psi_i(r).$$

Since the initial data must obey Einstein equations it follows that not all of the initial value functions can be chosen arbitrarily.

As said before the choice of the scale function v is such that $R_i = r$. Then, for example in the case of perfect fluids, from equations (3), (4) and (15), writing $M_i(r) = M(r,v(r,t_i)) = M(r,1)$ we get

$$\rho_i = 3M_i(r) + rM_i'(r) \ , \ p_i = -(M_{,v})_i \ . \tag{19}$$

¿From equation (5) we can write,

$$\nu_i(r) = r^2 g(r,v(r,t_i)) = r^2 g_i(r) \ , \tag{20}$$

which in turn can be related to the function A at the initial time,

$$A_{,v}(r,v(r,t_i)) = (A_{,v})_i = 2g_i r^2 + g_i' r^3 \ . \tag{21}$$

Finally, the initial condition for ψ can be related to the initial value of the function $A(r,t)$ from equation (6) and equation (21),

$$A(r,v(r,t_i)) = A_i(r) = -\psi_i - \frac{1}{2}\ln b(r) \ . \tag{22}$$

Further to this we must require that the initial configuration is not trapped, since we intend to study the formation of trapped surfaces during collapse. We therefore must require

$$\frac{F_i(r)}{R_i} = r^2 M_i(r) < 1 \ , \tag{23}$$

from which we see how the choice of the initial matter configuration M_i is related to the initial boundary of the collapsing cloud. Some restrictions on the choices of the radial boundary must be made in order not to have trapped surfaces at the initial time.

Finally we mention how the spacetime describing the collapsing cloud can be matched to a known exterior metric. In the case of dust (or in the case of collapse with only tangential pressures) the Misner-Sharp mass is conserved during collapse and therefore it is possible to have a matching with a Schwarzschild exterior. On the other hand if we consider an anisotropic fluid model or a perfect fluid model (thus allowing for the presence of radial pressures) the Misner-Sharp mass is in general not conserved during collapse. In this case matching with an exterior spherically symmetric solution leads to consider the generalized Vaidya spacetime, where the mass that is not conserved during collapse is described as outgoing radiation in the exterior spacetime. It can be proven that matching to a generalized Vaidya exterior is always possible when the collapsing cloud is taken to have a compact support within the boundary taken at $r = r_b$, and the pressure of the matter is assumed to vanish at the boundary. [29,30,31]

4.2.3. *Collapse final states*

In order to study the final outcome of collapse we turn our attention to the equation of motion (12) which can be inverted to give the function $t(r, v)$ that represents the time at which the comoving shell labeled r reaches the event v,

$$t(r,v) = t_i + \int_v^1 \frac{e^{-\nu}}{\sqrt{\frac{M}{v} + \frac{be^{2A}-1}{r^2}}} dv \ . \tag{24}$$

Then the time at which the shell labeled by r becomes singular, called the singularity curve, can be written as

$$t_s(r) = t(r,0) = t_i + \int_0^1 \frac{e^{-\nu}}{\sqrt{\frac{M}{v} + \frac{be^{2A}-1}{r^2}}} dv \ . \tag{25}$$

Regularity ensures that, in general, $t(r, v)$ is at least $\mathcal{C}^2$ near the singularity and therefore can be expanded as,

$$t(r,v) = t(0,v) + \chi_1(v)r + \chi_2(v)r^2 + o(r^3) \ . \tag{26}$$

The coefficients $\chi_1 = \frac{dt}{dr}|_{r=0}$ and $\chi_2 = \frac{1}{2}\frac{d^2t}{dr^2}|_{r=0}$ are determined by the expansions of the functions M, A and b close to the center and it is their values that determine the visibility of the singularity. The singularity curve in fact becomes

$$t_s(r) = t_0 + r\chi_1(0) + r^2\chi_2(0) + o(r^3) \ , \tag{27}$$

where $t_0 = t(0, 0)$ is the time at which the central shell becomes singular.

For physical purposes, or to study a special case, we can assume that all the involved quantities can be expanded in a power series in a close neighborhood of

$r = 0$ and that energy density and pressures present only quadratic terms in the expansions. In this case we will obtain $\chi_1 = 0$ and the final fate of collapse will be decided by the term χ_2. Of course, the situations where discontinuities are allowed to be present can be analyzed also with some small technical modifications to the above formalism.

4.2.4. *Trapped surfaces and outgoing null geodesics*

The crucial elements that are necessary in order to understand if within a certain collapse model there are future directed null geodesics that are emanating from the singularity and reaching external observers in the universe are two: the apparent horizon, which marks the limit of trapped surfaces, and the null geodesic curves. If the outgoing light rays coming from the singularity do not meet the apparent horizon in their journey within the cloud, then they can reach the boundary of the cloud and propagate freely in the outside universe, thus reaching any future observer.

This means that the boundary of the cloud also plays an important role towards the local or global visibility of the singularity. Nevertheless within analytical models that, for the time being, do not necessarily rely on observational data, it is always possible to choose the boundary suitably in order for the singularity to be globally visible. The apparent horizon is given by the condition

$$1 - \frac{F}{R} = 0 \, , \tag{28}$$

which describes a curve $v_{ah}(r)$ given implicitly by

$$r^2 M(r, v_{ah}) - v_{ah} = 0 \, . \tag{29}$$

Inversely, the apparent horizon curve can be expressed as the curve $t_{ah}(r)$ which gives the time at which the shell labeled by r becomes trapped.

¿From the above equation it is clear that in general the apparent horizon curve can have any behaviour, that is, it could be increasing or decreasing. In order for the singularity to be visible it is clearly necessary that the apparent horizon does not form before the singularity. We shall consider here the most common situation where the time at which the central shell becomes singular coincides with the time at which the same becomes trapped. In this case if the apparent horizon curve is constant or decreasing there is no possibility for the central singularity to be visible. On the other hand an increasing apparent horizon curve can allow for future directed null geodesics to escape the singularity. All shells with $r > 0$, when they become singular, are instead necessarily trapped since in this case $t_s(r) > t_{ah}(r)$.

In order to understand what are the features that determine the visibility of the singularity to external observers we can evaluate the time curve of the apparent horizon in these models as

$$t_{ah}(r) = t_s(r) - \int_0^{v_{ah}(r)} \frac{e^{-\nu}}{\sqrt{\frac{M}{v} + \frac{be^{2A}-1}{r^2}}} dv \, , \tag{30}$$

where $t_s(r)$ is the singularity time curve, whose initial point is $t_0 = t_s(0)$. Close to the center the apparent horizon curve behaves like the singularity curve and therefore a necessary condition for visibility of the central singularity is that $t_{ah}(r)$ be increasing. ¿From the above we see how the presence of pressures affects the time of the formation of the apparent horizon. In fact, it is easy to see that all the configurations for which $\chi_1 > 0$ (or $\chi_2 > 0$ in case that $\chi_1 = 0$) will cause the trapped surfaces to form at a later stage than the singularity, thus leaving the door open for the null geodesics to escape the central singularity.

At this point we turn our attention to radial outgoing null geodesics. ¿From the metric (1) we can write

$$\frac{dt_{ng}}{dr} = \pm e^{\psi-\nu} , \tag{31}$$

from which we can get the general expression for the null geodesic curve $t_{ng}(r)$. If we impose the initial condition $t_{ng}(0) = t_0$, that means that the light ray is originated at the singularity, and the condition for the geodesic to be future directed (which determines the sign) we can, in principle, integrate the above equation. At this point to see if the singularity is visible we just need to check under which conditions we have $t_{ng}(r) < t_{ah}(r)$ for $r \in (0, r_b)$.

In Ref. 25 it was shown that the condition that t_{ah} is increasing, which is equivalent to the positivity of the first non-vanishing χ, is not only necessary but also sufficient for the local visibility of the central singularity (see also Refs. 32–34). Furthermore it can be proven that if there is one outgoing radial null geodesic that escapes the singularity then there is necessarily a whole family of such geodesics. 35

On a more general ground, the scenario of collapse of a cloud composed of a generic fluid offers a wider spectrum of possibilities. In fact we can see from equation (28) that whenever the mass function F goes to zero as collapse evolves it is possible to delay the formation of trapped surfaces in such a way that a portion of the singularity curve $t_s(r)$ becomes timelike. This leads to the possibility that shells different from the central one be visible when they become singular. [36]

Nevertheless it must be noted that the pressure of the fluid must turn negative at some point before the formation of the singularity in order for the mass function to be entirely radiated away during collapse. This might seem an unphysical artificial feature but we note that negative pressures are worth investigating as they could provide a means to represent classically, within general relativity, some effects of a quantum theory of gravity that become apparent close to the formation of the singularity. [38]

4.2.5. *An example: Dust collapse*

If we apply the above formalism to the case of dust collapse where pressures are set to zero (as for example in the OSD model mentioned above), the whole structure of

Einstein equations simplifies drastically. We note of course that the density could be inhomogeneous in space, even when pressures are vanishing.

The spacetime metric in this case is described by the well-known and widely studied Lemaitre-Tolman-Bondi (LTB) collapse model. In this case, $p_r = 0$ implies the condition $M = M(r)$ and from $\nu' = 0$ together with the condition $\nu(0) = 0$ we get $\nu = 0$ identically. Then we must have $G = b(r) = 1 + r^2 b_0(r)$ and the equation of motion becomes $\dot{v} = -\sqrt{\frac{M}{v} + b_0}$ which can be explicitly integrated. The metric in this case takes the form

$$ds^2 = -dt^2 + \frac{(v + rv')^2}{1 + b_0 r^2} dr^2 + r^2 v^2 d\Omega^2, \tag{32}$$

and the singularity and the apparent horizon curves can be evaluated. It is easy to see that $t_{ah}(0) = t_s(0)$ and therefore only the central singularity can eventually be visible. If we assume that $M(r)$ can be expanded in a series near the center as $M(r) = M_0 + M_1 r + M_2 r^2 + o(r^3)$ (where from regularity we shall impose $M_1 = 0$), we can easily evaluate $\chi_1(0)$ and $\chi_2(0)$ which turn out to be

$$\chi_1(0) = -\frac{1}{2}\int_0^1 \frac{M_1 + b_{01}v}{(M_0 + vb_{00})^{\frac{3}{2}}}\sqrt{v}dv \ , \tag{33}$$

$$\chi_2(0) = \frac{3}{8}\int_0^1 \frac{(M_1 + b_{01}v)^2}{(M_0 + vb_{00})^{\frac{5}{2}}}\sqrt{v}dv - \frac{1}{2}\int_0^1 \frac{M_2 + b_{02}v}{(M_0 + vb_{00})^{\frac{3}{2}}}\sqrt{v}dv \ , \tag{34}$$

where we have expanded b_0 in a series as $b_0 = b_{00} + b_{01}r + b_{02}r^2 + o(r^3)$. We then see that it is the values of the first non-vanishing terms M_i and b_{0i} (with $i > 0$) which determines the visibility or otherwise of the central singularity.

Simplifying even further we obtain the homogeneous dust case that was discussed in section 3.1. The OSD model is achieved when $M = M_0$ and $b_0 = k$, then we have $F' = 3M_0 r^2$, $R' = v(t)$, and the energy density is homogeneous throughout the collapse and is given by $\rho = \rho(t) = 3M_0/v^3$. The spacetime geometry then becomes the Oppenheimer-Snyder-Datt metric, which is given by,

$$ds^2 = -dt^2 + \frac{v^2}{1 + kr^2} dr^2 + r^2 v^2 d\Omega^2 \ . \tag{35}$$

In this case all the quantities χ_i identically vanish thus showing that the singularity curve reduces to that of a simultaneous singularity, namely $t_s(r) = t_0$.

In the OSD homogeneous collapse case, the trapped surfaces and the apparent horizon develop much earlier than the formation of the singularity, thus creating a black hole in the spacetime as the collapse final state. But when some inhomogeneities are allowed in the initial density profile, such as a higher density at the center of the star, the trapped surface formation is delayed in a natural manner, thus leaving the final singularity of collapse to be visible to faraway observers in the universe (see e.g. Ref. 39).

4.3. *Collapse with non-zero pressure*

The next step, in order to study more realistic collapse scenarios, is that of introducing pressures in the model. Gravitational collapse with non-zero pressures included has been discussed and investigated in past years in detail by many researchers (see for example Ref. 40 for a study of the initial data leading to different outcomes, Ref. 41 for a discussion of the Einstein cluster model, Ref. 42 for collapse with only radial pressures, or Refs. 43, 44 for perfect fluid models). Given the difficulties arising from Einstein equations the full integration of the system is not possible even in the simplest cases when pressures are included. Therefore the general line of inquiry follows a path which can be broadly outlined as follows: Firstly, the general structure for Einstein equations to study the spherical collapse is considered. We describe how the equations can be integrated up to first order, thus obtaining the equation of motion for the system. The regularity conditions and energy conditions that give physically reasonable models are considered and imposed on the models. The final stages of collapse are then discussed, evaluating key elements that determine when the outcome will be a black hole or a naked singularity.

It is seen that there is a specific function which is related to the tangent of outgoing geodesics at the singularity, whose sign solely determines the time of formation of trapped surfaces in relation with the time of formation of the singularity. We also analyze the occurrence of trapped surfaces during collapse and the possibility that radial null geodesics do escape, thus making it visible. We can show how both features are related to the sign of the above mentioned function, thus obtaining a necessary and sufficient condition for the visibility of the singularity.

Collapse models with non-zero pressures were discussed in detail by various researchers (see Refs. 45–47 for the tangential pressure case, or Refs. 48, 49). Considering a form of pressure generated by rotation of particles within the collapsing cloud, they showed how the naked singularities develop as collapse end states. Several models with a realistic equation of state, which specifies how the density and pressure within the cloud are related, were also investigated, including a model by Ori and Piran and by our group (see Refs. 50, 51).

Today, we have a general formalism to treat spherical collapse from initial data. General matter fields can include realistic features such as pressure and inhomogeneities in density distribution, and reasonable equations of state of matter are incorporated. Physicists are even beginning to consider situations where matter takes on some other form, such as a fundamental quantum field, or is converted to radiation in a sudden phase transition in the very late stages of collapse. What these works show in a generic manner is that the collapse with non-zero pressures can lead to either the formation of a black hole or a naked singularity as the final state. The outcome is decided by the initial data and the dynamical evolutions as allowed by the Einstein equations. It turns out that the collapse ends in a naked singularity in a wide variety of situations.

These conclusions and models are by now fairly widely accepted, and it is now clear that physically reasonable gravitational collapse can create naked singularity. However, in its original statement, Penrose included a condition, namely that 'generic' gravitational collapse would always produce black holes. Are the above models generic? In fact, they are in the sense of the initial data for matter being fully generic as required. Though the key difficulty for such a requirement, as such and in general, is that there is no mathematically well formulated definition of genericity available in general relativity and gravitation theory, which one could meaningfully use to answer this question completely one way or the other. Stability and genericity are extremely difficult concepts to formulate precisely in general relativity, within the framework of a general spacetime with a Lorentzian metric with an indefinite signature, with no definite and clear criteria available today. In fact, there are formidable mathematical difficulties in achieving the same as we shall discuss later. Therefore, the only way to proceed is by asking further questions, such as is there any perturbation of spherical symmetry that would remove these naked singularities, or would naked singularities form in non-spherical collapse, and so on.

4.4. *Are naked singularities stable and generic?*

Firstly, as we have noted the concepts such as genericity and stability are at present not well-defined in general relativity, as they are in the Newtonian gravity. The major difficulty towards obtaining a unique and well posed definition comes from the non-uniqueness of topology, or the concept of 'nearness' itself in a given spacetime manifold. [3]

For example, in order to quantify when two spacetime geometries are nearby, we can define a certain topology on a space of all spacetime metrics by requiring that the values of the metric components are 'nearby'. On the other hand, we can also additionally require that the first n derivatives of the metric functions are also nearby. The problem lies in the fact that the resulting topologies will be different. It is easy to see how this fact is connected to the basic problem of arriving at a well-formulated definition of the cosmic censorship itself. [52]

¿From this perspective, certain studies have been developed on the concept of critical collapse (see e.g. Ref. 53 and references therein) and certain spacetime geometries with naked singularities have been studied in the past with the aim to better understand their genericity and stability. For example in Ref. 54 it was shown that if we consider self-similar massless scalar field collapse (which is a somewhat special model) the initial data leading to naked singularity has a positive codimension, in a certain space of initial data, and this led to the conclusion that the occurrence of naked singularity is not generic in that case. On the other hand in Refs. 55, 56 it was shown that the naked singularity occurrence is generic for certain types of matter fields such as inhomogeneous dust and some others. Therefore, the issue of genericity and stability of naked singularities in collapse remains wide open for spherically symmetric models (and even more so if we wish to consider departure from spherical symmetry) with different forms of matter field (see Ref. 57 for stability of the Cauchy horizon or Ref. 58 for stability under linear perturbations).

To begin with, one needs to adopt some definition for what is meant when we talk about stability and genericity of some outcome of gravitational collapse. To this purpose the definitions of stability and genericity will always need to be referred to the specific space of initial data set that is being considered. In fact what is stable or generic in a certain space of initial data (as for example that of inhomogeneous dust) need not be stable in another, bigger space (as for example that of perfect fluid collapse). We say that a certain outcome of collapse in terms of either black hole or naked singularity from a certain set of initial data is stable if there exist a whole neighborhood of that initial data set which leads to the same outcome. The neighborhood, as said, being defined within the specific space of initial data that we are considering.

Furthermore, we will say that a certain outcome of collapse is generic, within a specific space of initial data, if the measure of the subset of initial data leading to that outcome in the original initial data space is non-zero.

This definition, though being related to the usual definition of genericity for dynamical systems, does not coincide with it in the sense that typically an outcome can be said to be generic in the sense of dynamical systems if the set of initial data leading to that outcome is open and dense in the set of all initial data. With this definition we could prove that both black holes and naked singularities are 'non-generic' already in the space of initial data for Lemaitre-Tolman-Bondi collapse.

Therefore, with the definitions referred to as above, we investigated the initial data sets leading to black holes and naked singularities in the space of all initial data sets for different type I matter models (including dust and perfect fluid). We showed that the initial data set leading the collapse to a naked singularity, just like the initial data set leading to a black hole, forms an open subset of a suitable function space comprising of the initial data, with respect to an appropriate norm which makes the function space an infinite dimensional Banach space. Furthermore considering the possible measure theoretic aspects of this open set it can be shown that a suitable well-defined measure of this set can be given and it must be strictly positive. [55]

4.4.1. *The genericity and stability of collapse outcomes*

We investigate here the genericity and stability aspects for naked singularities and black holes that arise as the final states for a complete gravitational collapse of a spherical massive matter cloud. The form of the matter considered is a general *Type I* matter field, which includes most of the physically reasonable matter fields such as dust, perfect fluids, massless scalar fields and such other physically interesting forms of matter widely used in gravitation theory (for a complete definition of different matter fields, see for example, 3, section 4.3). Our aim is to examine and characterize the above two possible outcomes in terms of stability of the initial data and genericity. We show that both black holes and naked singularities are generic outcomes of a complete collapse, once genericity is defined in a suitable sense in an appropriate space of functions.

While general relativity may predict the existence of both black holes and naked singularities as collapse outcomes, an important question that needs to be answered is, what would a realistic continual gravitational collapse of a massive star in nature would end up with. Thus the key issue under an active debate now is the following: Even if naked singularities did develop as collapse end states, would they be generic or stable in some suitably well-defined sense, as permitted by the gravitation theory? The point here is, if naked singularity formation in collapse was necessarily 'non-generic' in some appropriately well-defined sense, then for all practical purposes, a realistic physical collapse in nature might always end up in a black hole, whenever a massive star ended its life.

In fact, such a genericity requirement has been always discussed and desired for any possible mathematical formulation of CCC right from its inception. However, the main difficulty here has again been that, there is no well-defined or precise notion of genericity available in gravitation theory and the general theory of relativity as we discussed above. Again, it is only various gravitational collapse studies that can provide us with more insight into this genericity aspect.

A result that is relevant here is the following [40]: For a spherical gravitational collapse of a rather general (type I) matter field, satisfying the energy and regularity conditions, given any regular density and pressure profiles at the initial epoch, there always exist classes of velocity profiles for the collapsing shells and dynamical evolutions as determined by the Einstein equations, that, depending on the choice made, take the collapse to either a black hole or naked singularity final state (see e.g. Fig. 3 for a schematic illustration of such a scenario).

Such a distribution of final states of collapse in terms of the black holes and naked singularities can be seen much more transparently when we consider a general inhomogeneous dust collapse, for example, as discussed in Ref. 60 (see Fig. 4).

What determines fully the final fate of collapse here are the initial density and velocity profiles for the collapsing shells of matter. One can see here clearly how the different choices of these profiles for the collapsing cloud distinguish and determine between the two final states of collapse, and how each of the black hole and naked singularity states appear to be 'generic' in terms of their being distributed in the space of final states. Typically, the result we have here is, given any regular initial density profile for the collapsing dust cloud, there are velocity profiles that take the collapse to a black hole final state, and there are other velocity profiles that take it to naked singularity final state. In other words, the overall available velocity profiles are divided into two distinct classes, namely the ones which take the given density profile into black holes, and the other ones that take the collapse evolution to a naked singularity final state. The same holds conversely also, namely if we choose a specific velocity profile, then the overall density profile space is divided into two segments, one taking the collapse to black hole final states and the other taking it to naked singularity final states. The clarity of results here give us much

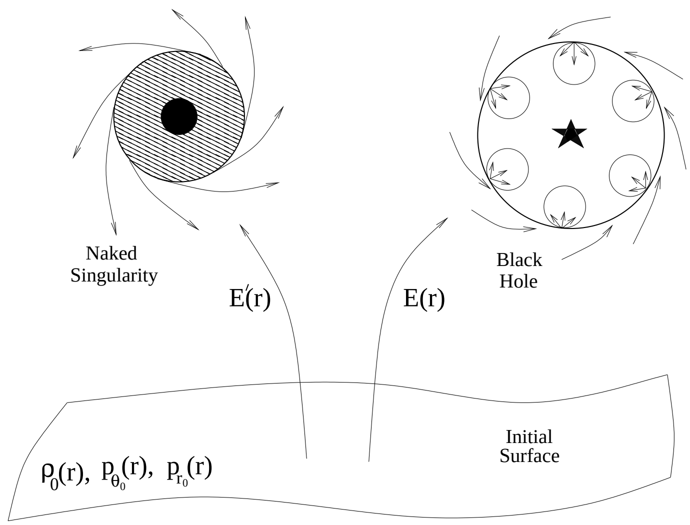

Fig. 3. Evolution of spherical collapse for a general matter field with inhomogeneities and non-zero pressures included where ρ_0, p_{θ_0} and p_{r_0} specify the initial data set and $E(r)$, $E'(r)$ are possible velocity profiles.

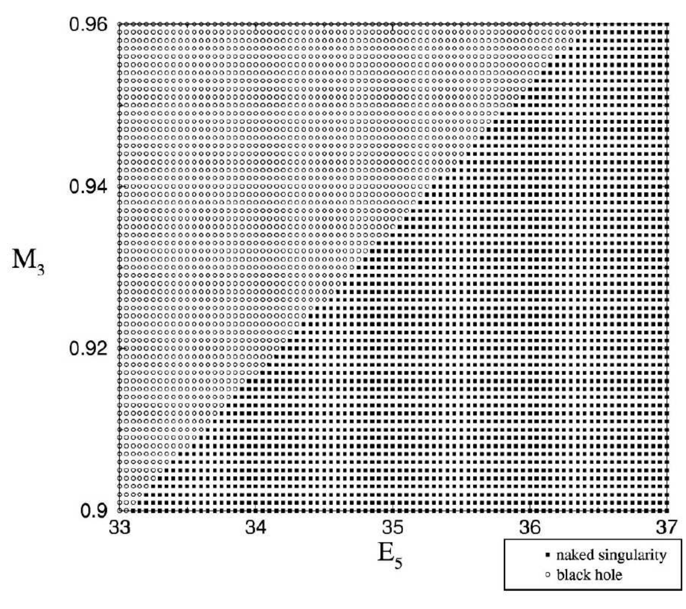

Fig. 4. Collapse final states for inhomogeneous dust in terms of the initial mass and velocity profiles for the collapsing shells, where the mass profile is given by $M = \Sigma_{i=3}^{\infty} M_i r^i$ and the velocity profile is given by $E = -E_2 r^2 - \Sigma_{j=5}^{\infty} E_j r^j$ with E_2 kept fixed.

understanding on the final fate of a collapsing matter cloud.

Typically, all stars have a higher density at the center, which slowly decreases as one moves away. So it is very useful to incorporate inhomogeneity into dynamical collapse considerations. However, much more interesting is the collapse with non-zero pressures which are very important physical forces within a collapsing star. We briefly consider below a typical scenario of collapse with a non-zero pressure component, and for further details we refer to Ref. 61.

For a possible insight into genericity of naked singularity formation in collapse it is interesting to study the final outcomes of collapse once some kinds of pressures are introduced in the collapsing cloud. To this aim, we investigated the effect of introducing small pressure perturbations, in the form of tangential stresses or perfect fluids, in the collapse dynamics of the classic Oppenheimer-Snyder-Datt (OSD) gravitational collapse, which is an idealized model assuming zero pressure, and which terminates in a black hole final fate as discussed above. Thus we study the stability of the OSD black hole under introduction of small stresses or pressure perturbations.

It is seen explicitly that there exist classes of stress perturbations such that the introduction of a smallest tangential pressure within the collapsing OSD cloud changes the endstate of collapse to formation of a naked singularity, rather than a black hole. What follows is that small stress perturbations within the collapsing cloud change the final fate of collapse from being a black hole to a naked singularity. The same model can also be viewed as a first order perturbation of the known spacetime metric describing the dust cloud. Thus we can understand here the role played by pressures in a well-known gravitational collapse scenario. A specific and physically reasonable but generic enough class of perturbations is considered so as to provide a good insight into the genericity of naked singularity formation in collapse when the OSD collapse model is perturbed by introduction of small pressures, either the tangential ones or more realistic perfect fluids. ¿From this analysis we gain some useful and important clarification into the structure of the censorship principle which as yet remains to be properly understood.

4.4.2. *Instability of the OSD black hole*

With the aim of understanding the stability properties of known dust collapse models leading to the formation of a black hole, we studied the effects of introducing small pressure perturbations in an otherwise pressure-free gravitational collapse scenario which was to terminate in a black hole final state.

We followed the collapse formalism in spherical symmetry developed in section 4.2. The analysis of pressure perturbations in known collapse models, inhomogeneous but otherwise pressure-free, shows how collapse final states in terms of black hole or naked singularity are affected and altered. This allows us to examine how stable these outcomes are and to understand within the initial data space the set of conditions that lead the collapse to a naked singularity or to a black hole in order to evaluate how abundant are these final states.

Of course many examples of spacetimes with naked singularity can be found, but their relevance in models describing physically viable scenarios is still a matter of much debate, as it has been the case since the first formulation of the CCC. The reason for this being that many of these collapse scenarios are, however, restricted by some simplifying assumptions such as the absence of pressures, as is the case for dust models. It is well-known that the pressures cannot be neglected in realistic models describing stars in equilibrium. It seems natural therefore that if one wishes to study analytically what happens during the last stages of the life of a massive star, when its core collapses under its own gravity thus forming a compact object as a remnant, pressures must be taken into account.

Therefore, given the fact that naked singularities do arise as collapse endstates under physically viable conditions, this further analysis helps us understand in a clear manner whether these examples are a mere curiosity with no impact on the general structure of collapse of massive stars or whether they might be regarded seriously towards possible observations.

Hence, further to early works that showed the occurrence of naked singularities in dust collapse, much effort has been devoted to understanding the role played by pressures. [62]

The presence of pressures is a crucial element towards the description of realistic sources as we know that stars and compact objects are generally sustained by matter with strong stresses (either isotropic or anisotropic). At first it was believed that the naked singularity scenario could be removed by the introduction of pressures, thus implying that more realistic matter models would lead only to the formation of a black hole. We now know that this is not the case. The final outcome of collapse with pressure is entirely decided by its initial configuration and allowed dynamical evolutions and it can be either a black hole or a naked singularity. Furthermore it is now clear that within each model (be it dust, tangential pressure or others) the data set leading to naked singularities is not a subset of ‘zero measure’ of the set of all possible initial data. Despite all this work we can still say that much more is to be understood about the role that general pressures play during the final stages of collapse. Perfect fluid collapse has been studied mostly under some simplifying assumptions and restrictions in order to gain an understanding, but a general formalism for perfect fluids described by a physically valid equation of state is still lacking due to the intrinsic difficulties arising from Einstein equations. Considering both radial and tangential pressures is a fundamental step in order to better understand what happens in the ultra-dense regions that form at the center of the collapsing cloud prior to the formation of the singularity. For this reason, perfect fluids appear as a natural choice since these are the models that are commonly used to describe gravitating stars in equilibrium and since it can be shown that near the center of the cloud regularity implies that matter must behave like a perfect fluid.

At first let us consider the case of tangential pressure perturbations. Towards understanding the stability or otherwise of the OSD collapse model under the injection of small tangential pressure perturbations, we consider the dynamical development of the collapsing cloud, as governed by the Einstein equations. The visibility or otherwise of the final singularity that develops in collapse is determined by the behaviour of the singularity curve and of the apparent horizon, as shown in section 4.2.

In the tangential pressure case we have four equations (since the equation for the radial pressure (3) reduces to $p_r = 0$ which implies that $F = F(r)$) in six unknowns, namely ρ, p_θ, R, F, G, H. With the definitions of R, H and F, provided earlier we substitute these unknowns with v, ν and M.

We then have the freedom to specify two free functions, namely the mass profile $M(r)$ and the tangential pressure $p_\theta(r, v)$.

In this model we can integrate the Einstein equations, at least up to first order, to reduce them to a first order system, to obtain the function $v(t, r)$, or equivalently $t(r, v)$, which in general will depend upon $F(r)$, $b(r)$ and $A(r, v)$. The function $A(r, v)$ here depends on the nature of tangential pressure chosen.

Again, since $t(r, v)$ is at least C^2 we can write it as in equation (26), and the singularity curve is given by equation (27). This means that the singularity curve should have a well-defined tangent at the center.

¿From regularity we can write ν as $\nu(r, v) = r^2 g(r, v)$, which, from Einstein equation (5) with $p_r = 0$ gives

$$p_\theta = \left(g + \frac{1}{2} r g'\right) r\rho \frac{R}{R'} \; . \tag{36}$$

Therefore we can take $g(r, v)$ as the free function and since it must be a regular function it can be written near $r = 0$ as,

$$g(r, v) = g_0(v) + g_1(v) r + g_2(v) r^2 + ... \tag{37}$$

We can now study how the OSD gravitational collapse scenario gets altered when small pressure perturbations are introduced in the dynamical evolution of collapse.

If the collapse outcome is not a black hole, the final singularity of collapse cannot be simultaneous. We cannot have $v = v(t)$ as was the case for OSD. Allowing for $v = v(t, r)$ is enough to ensure a pressure perturbation, therefore, for simplicity, we can maintain unchanged the other conditions that gave the OSD model (namely $M(r, v) = M_0$ and $b(r) = b_0$). This way we ensure that we do not depart too much from the OSD scenario, thus bringing out with more clarity the role played by the pressure perturbations. Note that this choice also automatically forces the choice of the pressure perturbation to be a tangential pressure, since having $M = M_0$ immediately implies $p_r = 0$.

We know that the metric function $\nu(t, r)$ must identically vanish for the dust case. On the other hand, the above perturbation amounts to allowing for ν, or equivalently g to be non-zero and small. Proceeding as above we obtain that the tangential pressure can be written as $p_\theta = p_1 r^2 + p_2 r^3 + ...$, where p_1, p_2 are evaluated in terms of the coefficients of M, g, and R and its derivatives. In fact we get,

$$p_\theta = 3\frac{M_0 g_0}{v R'^2} r^2 + \frac{9}{2}\frac{M_0 g_1}{v R'^2} r^3 + ... \tag{38}$$

Considering small pressures in a close neighborhood of the center we can see how it is the choice of the sign of the function g_0 which ensures positivity or negativity of p_θ.

Then the first order coefficient χ_1 in the expansion of the singularity curve $t_s(r)$ is obtained as,

$$\chi_1(0) = -\int_0^1 \frac{v^{\frac{3}{2}} g_1(v)}{(M_0 + v b_0 + 2 v g_0(v))^{\frac{3}{2}}} dv \ . \tag{39}$$

As we have noted above, it is this quantity $\chi_1(0)$ that governs the nature of the singularity curve, the apparent horizon curve and the visibility of the singularity.

In general it can be seen that it is the matter initial data in terms of the density and pressure profiles, and the velocity of the collapsing shells, that decides the value of $\chi_1(0)$. In this case, having fixed M and b_0 from the OSD model the final outcome is entirely decided by the function g_1 in the expansion of the tangential pressure.

Of course, if one wishes to consider more realistic models where the pressure has only quadratic terms in r, the same analysis can be done following a similar method. In that case one would have to impose that $g_1 = 0$ and at the end we would find that $\chi_1(0) = 0$ and it will be the second term χ_2, which would be depending on g_2 to decide the final outcome of collapse.

¿From these considerations, it is possible to see how pressure perturbations can affect the time of formation of the apparent horizon, and therefore the formation of a black hole or naked singularity.

As can be seen from above, for all functions $g_1(v)$ for which $\chi_1(0)$ is positive the apparent horizon appears after the formation of the central singularity. In this case, the apparent horizon curve originates at the central singularity at $t = t_0$ and increases with increasing r, moving to the future. Thus we have $t_{ah} > t_0$ for $r > 0$ near the center. The behaviour of outgoing families of null geodesics has been analyzed in detail in such a case when $\chi_1(0) > 0$ and we can see that the geodesics terminate at the singularity in the past. Thus timelike and null geodesics come out from the singularity, making it visible to external observers. [40]

One thus sees that it is the term g_1 in the stresses p_θ which decides either the black hole or naked singularity final fate for the collapse. Since we have the freedom to choose g to be arbitrarily small it is then easy to see how introducing a generic small tangential pressure perturbation in the OSD model can change drastically the final outcome. In general it is a subspace of the space of all functions g_1 that makes $\chi_1(0)$ positive (which includes all those that are strictly negative), and that causes the collapse to end in a naked singularity. In fact, if we wish to give an example, we see that for the class of all non-vanishing tangential stresses with $g_0 = 0$ and $g_1 < 0$, even the slightest perturbation of the Oppenheimer-Snyder-Datt scenario would result in a naked singularity. Of course while this is an explicit example, by no means it reduces to be the only class. The important feature of this class is that it corresponds to a collapse model for a simple and straightforward perturbation of the Oppenheimer-Snyder-Datt spacetime metric.

In such a case, the geometry near the center can be written as,

$$ds^2 = -(1-2g_1r^3)dt^2 + \frac{(v+rv')^2}{1+b_0r^2-2g_1r^3}dr^2 + r^2v^2d\Omega^2 \ . \tag{40}$$

The metric above satisfies Einstein equations in the neighborhood of $r=0$ when the function $g_1(v)$ is small and bounded. For example, we can take $0<|g_1(v)|<\epsilon$, so that the smaller we take the parameter ϵ the bigger results the radius where the approximation is valid.

We can consider now the requirement that a realistic matter model should satisfy some energy conditions ensuring the positivity of mass and energy density. As seen earlier the weak energy condition would imply restrictions on the density and pressure profiles. In fact, the energy density as given by the Einstein equation must be positive. And since R is positive, to ensure positivity of ρ we require $R'>0$ (no shell crossing singularities) and $F'>0$ (which near the center is satisfied if $M_0>0$). The choice of positive M is physically reasonable and ensures also positivity of the Misner-Sharp mass. Positivity of $\rho+p_\theta$ is then obtained for small values of r throughout the collapse for any form of p_θ (including negative pressures). In fact, regardless of the values taken by M and g, there will always be a neighborhood of the center $r=0$ for which $|p_\theta|<\rho$ and therefore $\rho+p_\theta\geq 0$. Therefore we see that greater and greater negative pressures are allowed by the weak energy condition as we get closer and closer to the center of the cloud, or conversely, once we fixed a negative pressure given by g_1, the weak energy condition will provide the small boundary within which such a pressure is allowed.

What we see here is, in the space of initial data in terms of the initial matter densities and velocity profiles, any arbitrarily small neighborhood of the OSD collapse model, which is going to a black hole final fate, contains collapse evolutions that go to a naked singularity final fate. Such an existence of subspaces of collapse solutions, that go to a naked singularity final state rather than a black hole, in the arbitrary vicinity of the OSD black hole, presents an intriguing situation. It gives an idea of the richness of structure present in the gravitation theory, and indicates the complex solution space of the Einstein equations which are a complicated set of highly non-linear partial differential equations. What we see here is the existence of classes of stress perturbations such that an arbitrarily small change from the OSD model is a solution going to naked singularity.

This then provides an intriguing insight into the nature of cosmic censorship, namely that the collapse must be necessarily properly fine-tuned if it is to produce a simultaneous black hole only, just like the OSD case, as the collapse final endstate. Traditionally it was believed that the presence of stresses or pressures in the collapsing matter cloud would increase the chance of black hole formation, thereby ruling out dust models that were found to lead to a naked singularity as collapse endstate. It now becomes clear that this is actually not the case. The model described here shows how the bifurcation line that separates the phase space of 'black hole formation' from that of the 'naked singularity formation' runs directly over the simplest and most studied of black hole scenarios such as the OSD model.

Of course within all matter models with pressures the case of only tangential stresses is also somehow artificial and it could be objected that conclusions drawn from the analysis of only tangential pressure perturbations need not be definitive since the overall picture might be very different when general radial pressures are also considered. For this reason, it is useful to enlarge and extend the previous analysis to include the case of perfect fluid pressures. Perfect fluids are widely accepted as rather much more realistic matter sources, and the inclusion of small perfect fluid pressures in collapse seems not only reasonable but also necessary, if we wish to derive some conclusion from this whole analysis. We therefore asked the question: 'How is the OSD collapse scenario affected by the inclusion of small perfect fluid pressures in the collapsing cloud?'

If we wish to consider perfect fluid collapse we need to use the formalism presented in section 4.2 together with the assumption

$$p_r = p_\theta = p \; . \tag{41}$$

The procedure is similar to the one outlined above for the case of only non-vanishing tangential pressures but the whole mathematical structure describing the collapsing cloud is now different. In fact in the case of perfect fluid pressures we have five equations for the six unknowns $\rho,\ p,\ v,\ M,\ G,\ \nu$, where now M is a function of both r and v. Therefore by considering collapse of a perfect fluid we must match the collapsing cloud with an exterior generalized Vaidya spacetime, which is always possible.

If we provide an equation of state that links the density to the pressure the whole system becomes closed and we have no freedom whatsoever to choose any free function. The evolution is then uniquely determined by the initial data.

If, on the other hand, we do not provide an equation of state, we are left with the freedom to choose one free function, typically the mass profile $M(r, v)$, globally. This means that, in contrast with the tangential pressure case where the mass profile was conserved throughout the collapse, here we must specify how the mass of the cloud evolves during collapse.

The mathematical freedom coming from the absence of an equation of state has therefore a physical counterpart in that some models might not be physically viable. Still, imposing regularity and energy conditions is sufficient (even though this may be not entirely satisfactory) to study the physical reasonableness of the models.

¿From the investigation of perfect fluid collapse it was found that not only naked singularities are not ruled out in this case but also that the separation between the regions in the space of all possible evolutions that lead to black holes and to naked singularities has some interesting features. Here again we see how the introduction of small pressures can drastically change the final fate of the OSD model. Still we note that the result is more general, in the sense that adding a small pressure perturbation to a dust model (not necessarily homogeneous) leading to a black hole can be enough to change the outcome of collapse to a naked singularity, and viceversa.

In order to perturb the OSD matter cloud by introducing the radial pressure perturbations, we have chosen a suitable mass function M, motivated by physical reasons, as the free function. Such a mass profile is taken to be arbitrarily close to the OSD scenario, in the sense that we considered an arbitrarily small pressure, where by 'small' we mean that the pressure remains much smaller than the energy density at all times as the collapse develops and evolves in time.

The class of perfect fluid pressure perturbations considered is given by the choice of a mass profile of the form

$$M = M_0 + M_2(v)r^2 \ , \tag{42}$$

where M_0 is a constant and we have considered $M_1 = 0$ in accordance with most physical models that require only quadratic terms in the expansions of density and pressure. Then we consider the class,

$$M_2(v) = C + \epsilon(v) \ . \tag{43}$$

The pressure perturbation is small if we require that $M_0 \gg\mid M_2 \mid$ at all times.

We note that taking $\epsilon = 0$ reduces the model to inhomogeneous dust, and therefore further imposing $C = 0$ gives the Oppenheimer-Snyder-Datt case. If we wish to perturb the OSD model we need to take $C = 0$ and $\epsilon \neq 0$. The initial condition $M_2(1) = 0$, corresponding to $\epsilon(1) = 0$, ensures that at the initial epoch we have an homogeneous dust cloud and that the pressures are triggered only after collapse commences.

The freedom to chose the free function is here reflected in the freedom to choose $\epsilon(v)$ which is related to the pressure perturbation. In fact, from Einstein equations with the present choice of the mass profile the density and pressure are given by

$$p = -\frac{\epsilon_{,v}}{v^2}r^2 \ , \ \rho = \rho_{dust} - p + \frac{5\epsilon - \epsilon_{,v}v}{v^2(v+rv')}r^2 \ . \tag{44}$$

Furthermore, for the sake of clarity, we have considered a constant velocity profile given by $b_0(r) = 0$, in analogy with the marginally bound case in LTB models.

Evaluating the singularity curve we obtained $\chi_1(0) = 0$, as expected, and therefore the final fate of collapse is decided by $\chi_2(0)$ which turns out to be,

$$\chi_2(0) = -\frac{1}{2}\int_0^1 (C + Y(v))Z(v)dv - \frac{4}{9M_0^2}\int_0^1 W(v)Z(v)dv \ , \tag{45}$$

where we have defined the functions

$$Y(v) = \left(\epsilon + \frac{2}{3}\epsilon_{,v}v\right) \ , \tag{46}$$

$$W(v) = \epsilon v(\epsilon + \epsilon_{,v}v) \ , \tag{47}$$

$$Z(v) = v\left(M_0 + \frac{4}{3}\frac{\epsilon v}{M_0}\right)^{-3/2} \ . \tag{48}$$

Given this structure there are two possible behaviours for ϵ:

- $\epsilon > 0$, which implies $\epsilon_{,v} < 0$ and positive pressure,
- $\epsilon < 0$ giving $\epsilon_{,v} > 0$ and negative pressure.

It is easy to see that negative pressures more easily allow for the formation of naked singularities, nevertheless for positive pressures also it is possible to construct physically valid evolutions that terminate in a naked singularity. Requiring $\chi_2(0) > 0$ leads to a set of conditions to be satisfied by Y, W and Z and therefore by ϵ and M_0. Imposing these conditions is therefore sufficient to obtain a naked singularity as the endstate of collapse.

Now coming to the stability of the OSD collapse scenario let us call $\mathcal{D}$ the set of physically valid initial data for collapse, which will in general be split into two subsets given by the possible final outcomes as $\mathcal{D} = \mathcal{D}^{BH} \bigcup \mathcal{D}^{NS}$. Then every point $x \in \mathcal{D}$ is characterized as $x = \{M_i(r), p_{ri}(r), p_{\theta i}, b(r)\}$, where for example the OSD initial configuration is given by $x_{OSD} = \{M_0, 0, 0, b_0\} \in \mathcal{D}^{BH}$.

What we have shown is that for every neighborhood $\mathcal{U}(x_{OSD}) \subset \mathcal{D}$, however small, there exist physically valid pressure profiles with initial data $x \in \mathcal{U}$ such that $x \in \mathcal{D}^{NS}$ (see Fig. 5). This provides a strong indication that the OSD dust model is 'unstable' in that the initial configurations for its endstates lie on the critical surface separating the two possible outcomes of collapse discussed above. In this sense we can say that the homogeneous dust collapse model leading to a black hole is not stable under the introduction of small pressure perturbations.

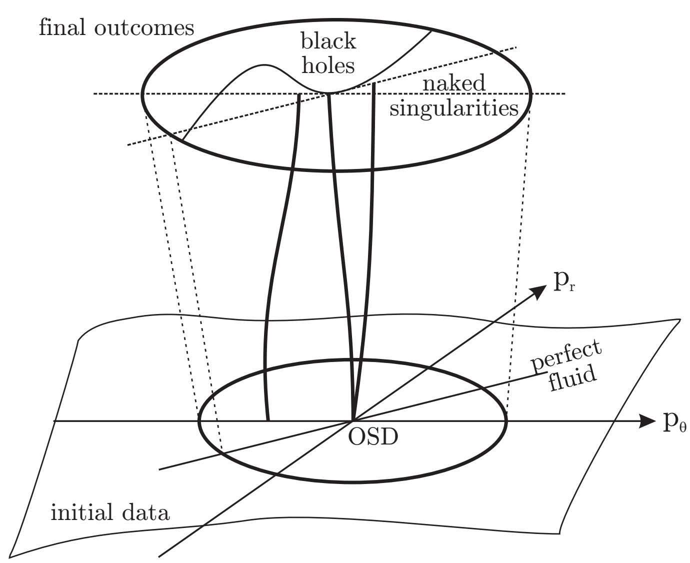

Fig. 5. Schematic view of how small pressure perturbations of an homogeneous dust cloud, both in the form of tangential stresses and perfect fluids, can lead to the formation of a naked singularity.

It has to be noted of course that the general issue of stability and genericity of collapse outcomes has been a deep problem in gravitation theory, and requires mathematical breakthroughs as well as evolving further physical understanding of the collapse phenomena. As noted above, this is again basically connected with the main difficulty of the cosmic censorship itself, which is the issue of how to effectively formulate the same. However, it is also clear from the discussion above, that a consideration of various collapse models along the lines as discussed here does yield considerable insight and inputs in understanding the gravitational collapse in general relativity and its final outcomes.

4.5. *The equation of state*

Here we give a brief discussion on possible equations of state that the collapse may permit. It is known that the presence of an equation of state introduces a differential relation for the previously considered mass function which was as such free, and that closes the system of Einstein equations. Examples of simple, astrophysically relevant, linear and polytropic equations of states have been considered, for which the collapse has been analyzed to understand its final fate in terms of black hole or a naked singularity. This treatment outlines the key features of the above approach and its advantages, and point to possible future uses of the same for astrophysical and numerical applications.

An equation of state is a constitutive relation that provides the link between some state quantities describing the system. Typically for a collapsing matter cloud an equation of state is provided once the pressure can be expressed as a function of the energy density (this could very well mean two different relations for both the pressures, in the case of anisotropic collapse). Stars during their equilibrium phase can be described by such equations of state, both at the level of classical mechanics [63], as well as within the context of general relativity [64,65]. It is reasonable to suppose then that once the nuclear fuel that maintains the star in equilibrium is exhausted collapse commences in a very short time from an initial configuration that is described by the same physical quantities related by the same equation of state. Therefore the physical values for p and ρ at the initial time can be taken from such models at equilibrium. The same holds true for the other quantities appearing in the equation of state. All these can be expressed in terms of the thermodynamical variables of the system such as the temperature and the molecular weight of the gas.

If we consider collapse of a perfect fluid (thus neglecting anisotropies that, as we said, must reduce to zero as r goes to zero), it is easy to see that a barotropic equation of state of the form $p = Y(\rho)$ introduces a further differential equation in the system of Einstein equations. This differential equation must be satisfied by the mass function $M(r, v)$, thus providing the connection between the equation for p (equation (3)) and that for ρ (equation (4)) and making them dependent on $R(r, v)$ and its derivatives only. The dynamics is then entirely fixed by the initial conditions and therefore we see how solving the equation of motion is equivalent to solve the whole system of equations. In the case of anisotropic collapse two equations of state (or one equation of state and one relation specifying the anisotropy), one for p_r and one for p_θ, are required in order to close the system.

The equations of state that are typically considered for perfect fluids at equilibrium are two. A linear relation of the type

$$p = k\rho \ , \tag{49}$$

where k is a constant, and a polytropic relation of the type

$$p = k\rho^\gamma \ , \tag{50}$$

where the exponent γ is typically written as $\gamma = 1+1/n$, with n being the polytropic index of the system. It can be shown that $n \leq 5$ (for $n > 5$ the cloud has no boundary at equilibrium).

As we said the introduction of an equation of state closes the system of Einstein equations and therefore no freedom to specify any functions remains. Gravitational collapse of a perfect fluid with a linear equation of state was studied in Ref. 66. The differential equation for M becomes

$$3\lambda M + \lambda r M_{,r} + [v + (\lambda+1) r v'] M_{,v} = 0 \ , \tag{51}$$

and it is possible to show that solutions exist and that both black holes and naked singularities are possible outcomes of collapse depending on the initial data and the velocity distribution of the particles. [37]

The study of collapse of a perfect fluid with polytropic equation of state is more complicated. In both cases solving the differential equation for M might prove to be unattainable. Nevertheless with the assumption that M can be expanded in a power series near $r = 0$, and expanding p and ρ accordingly, we can obtain a series of differential equations for each order M_i, by equating term by term from

$$\begin{aligned} p(r,v) &= p_0(v) + p_2(v) r^2 + ... = Y_0(v) + Y_2(v) r^2 + ... \ , \\ \rho(r,v) &= \rho_0(v) + \rho_2(v) r^2 + ... \ , \end{aligned} \tag{52}$$

with

$$Y_0 = Y(\rho_0) = -\frac{M_{0,v}}{v^2} \ , \ Y_2(v) = Y_{,\rho}\rho_2 = -\frac{M_{2,v}}{v^2} \ , \ ... \tag{53}$$

$$\rho_0 = \frac{3M_0}{v^3} \ , \ \rho_2 = \frac{5M_2}{v^3} - \frac{3M_0}{v^4} w_{,r}(0,v) \ , \ ... \tag{54}$$

This series of differential equations, if they can be satisfied by all M_i, and if the solutions converge to a finite mass function M, solves the problem, thus giving the explicit form of M. One thing that is interesting to note is that if pressures and density can be expanded in a power series near the center the behaviour close to $r = 0$ approaches that of an homogeneous perfect fluid.

Considering the whole process it is reasonable to assume that during collapse, which goes from a nearly Newtonian initial state, to a final state where a strong gravitational field is present, the equation of state describing the matter will vary, maybe even abruptly as in some phase transitions (as it happens when the nuclear saturation limit for the neutron star matter is reached).

Recent developments suggest that close to the formation of the singularity strong negative pressures might develop and the gravitational field might act repulsively thus disrupting the collapsing cloud and dispersing away the infalling matter. This kind of effect can be inferred already at a classical level but it is more evident when quantum corrections are considered. [38] This evidence suggests that the equation of state relating pressure and density (which must be always positive) evolves in a non-trivial manner during collapse.

Typically we can expect an adiabatic behaviour with small adiabatic index at the beginning of collapse when the energy density is much lower than the nuclear saturation energy. As collapse evolves it is possible that the equation of state will present a sharp transition when matter passes from one regime to another, as for example in the case when the nuclear saturation energy limit is exceeded. Furthermore towards the end of collapse repulsive forces become relevant thus giving rise to strong negative pressures and the speed of sound in the matter cloud approaches the speed of light (see Ref. 67).

It is clear now that to account for such extreme situations the usual linear and polytropic equations of state are not enough and the equations of state describing ultracompact objects are not well understood. [68] Therefore one is left with two options: either dropping the equation of state altogether, thus allowing for k and γ to vary with r and t (one can always choose the functions in such a way that they tend to become constant in the limit of weak gravitational field thus connecting the description close to equilibrium with the description close to the formation of the singularity), or to consider more exotic equations of state. Among the latter possibility gases with exotic properties (such as the Chaplygin gas, see Ref. 69) have been also considered in order to account for possible sources of dark energy and dark matter. Of course dark energy effects are not likely to be relevant for collapse at stellar scales but they might become important for bigger objects such as the compact ones dwelling at the center of galaxies that have masses of the order of many million solar masses.

4.6. *Non-spherical collapse*

It is true that most of the collapse models analyzed so far are spherical. However, many physicists believe that (as Roy Kerr pointed out to PSJ) if cosmic censorship is to hold as a basic principle of nature, it better holds in spherical class too, which has wide astrophysical significance. Thus, analyzing these models could be of great value from the perspective of censorship, for example to isolate the physical features that cause naked singularities. On the other hand, there are researchers who believe that the current classes analyzed so far are already good enough to begin investigating the physics and astrophysical implications of naked singularities.

While we have a good understanding now of spherical collapse for a generic matter field, non-spherical collapse remains a major uncharted territory (regardless of the fact that attempts have been made since many decades [70,71]). Several recent studies have found non-spherical models that also give rise to naked singularities. Also, certain analytical solutions in cylindrical symmetry provided some new examples of exact solutions describing collapse away from spherical symmetry (see e.g. Ref. 72, 73). Though the physical validity of such models remains doubtful, it is nevertheless one of the few instances in which we have analytical dynamical solutions that are not spherical.

The question now is whether these situations are contrived. Fast-developing numerical core collapse models in general relativity could be of help here. The results so far also show that naked singularities are, in fact, stable to small perturbations in the initial data of matter fields, to the introduction of non-zero pressures in the cloud, and so on. Therefore we have yet to find and isolate precisely the kind of perturbation that would make a given naked singularity go away. These situations are what physicists call 'generic' that is, they are not contrived. A tiny deviation in the initial data leads to much the same outcome. However, we should emphasize the general 'stability' proof for the naked singularity yet remains to be achieved.

The key issue when we are dealing with non-spherical collapse regards what happens to the 'deformations' that mark the departure from spherical symmetry. This is a major problem analytically since the set of Einstein equations describing the evolution of the cloud becomes immensely complicated when deformations and rotation are taken into account. Though there have been some attempts to study quasi-spherical and cylindrical collapse in full generality, the physical significance of such models remains for now somehow obscure and we can say that the study of non-spherical collapse within exact solutions of Einstein field equations is a field where most of the work still needs to be done.

Since a comprehensive analytical treatment of collapse away from spherical symmetry is still missing, the few insights that we have in the final fate of collapse of a non-spherical cloud come from numerical relativity. The study of non-spherical collapse is related to cosmic censorship through the so called 'no-hair theorem' which, roughly speaking, says that any asymptotically flat black hole vacuum solution of Einstein equations must be identified by only three quantities, namely mass, charge and angular momentum. This means that other vacuum solutions which are characterized by some other quantity, such as higher multipole moments for an axially symmetric spacetime, can have a naked singularity (as is the case for the class of Weyl metrics). Therefore one crucial issue with non-spherical collapse regards what happens to those higher multipole moments during the collapsing phase.

If the collapsing cloud retains its non-spherical shape throughout collapse, then the final vacuum configuration would not be represented by the Schwarzschild or Kerr [74] metrics. In this case naked singularities like those present in axially symmetric vacuum spacetimes (such as the Weyl class or the Tomimatsu-Sato solution) could be present (see e.g. 75). On the other hand, the presence of deformations might provide a modification in the density distribution of the collapsing source that opposes the pull of gravity, thus suggesting that non-spherical sources might produce some equilibrium configuration before complete collapse settles to a singularity (see for example, 76).

Recently there have been investigations to look for the observational signatures that such vacuum metrics would possess (see section 7), and the possibility that the Kerr metric might not be the best metric to describe the exterior of a rotating star has been suggested.

If, on the other hand, higher order multipole moments are radiated away during collapse then the final configuration must settle to a Schwarzschild or Kerr-Newman spacetime. Of course, if this happens to be the case, there has to be a mechanism, within the evolution of Einstein equations themselves, which explains why and how the higher multipole moments are radiated away. Such a mechanism, at present, is still a matter of speculation. Furthermore in this case one is still left with the possibility that the event-like naked singularities, like those that form in collapse with spherical symmetry, might form. In this sense the study of small perturbations of spherical symmetry shows that the final outcomes, whether they are black holes or naked singularities, are most likely stable.

Some numerical studies of the collapse of a rotating gas cloud showed that a final configuration with over-spinning angular momentum is unlikely to form under general conditions and therefore a Kerr black hole might be a most probable final outcome than a Kerr naked singularity. [77]

Nevertheless people have started to study the observational features of such singular spacetimes (as we discuss in section 7) in the hope to gain a better understanding of what trace they could leave in the universe if they were allowed to happen. Of course we still have no proof that such configurations might arise from collapse of a cloud with regular initial data, but just as well we have no proof that such configurations must be forbidden.

4.7. *Numerical simulations*

Since the important work by Shapiro and Teukolsky [78], who showed that axially symmetric collapse could lead to the formation of naked singularities, the investigations on gravitational collapse have been carried out both on the analytical side as well as on the numerical side. As we have noted, a lot of questions regarding complete collapse of massive bodies still remain unanswered from the theoretical point of view (like the issue of cosmic censorship), as well as from the observational perspective (like the mechanism behind supernovae explosions), and it is to answer these questions that computer simulations might prove very useful. [79,80] An exhaustive treatment of numerical simulations for gravitational collapse is beyond the scope of the discussion here and so we will give here a brief summary of results and open problems, including some references for the reader who may like to delve deeper into this growing subject. For many developments and references see e.g.

Ref. 81.

Due to the difficulties arising from the structure of Einstein equations when non-trivial situations are considered (such as non-spherical collapse with rotation, for example), numerical simulations can provide several insights into problems that are presently not addressable analytically. Among these the most important are the final fate of the complete collapse of a massive cloud to a compact object or the merger of two compact objects, two phenomena that are related to cosmic censorship as well as to astrophysics. Furthermore since numerical simulations allow to consider effects that are generally neglected in the analytical treatment, but are of much importance in the astrophysical context, they can be used to study the mechanism behind supernova explosions, core collapse, and high energy phenomena that occur when a star dies. All these scenarios are expected to produce gravitational waves, the holy grail of gravitational physics today, and therefore numerical modeling of how these gravitational waves can be produced is of much importance to the success of experiments such as LIGO and VIRGO. For example, the presence of electromagnetic fields and accretion disks are crucial elements that will affect the process by which the core of a star implodes and at present they can be studied only with numerical simulations, which are therefore the only mean to provide indications on how some of these high energy phenomena, such as relativistic jets or gamma-ray bursts, that are observed in the universe can occur.

As we know the complete collapse of a massive cloud produces a singularity, that can be naked or visible depending on the initial conditions. Since diverging quantities cannot be obviously handled numerically, a crucial issue of numerical simulations is how these singularities are treated or excised (see Ref. 82 and references therein). A numerical model where the singularity region is removed from the simulation regardless of the presence of an apparent horizon will not provide much insight on the issue of cosmic censorship for those scenarios, while it could still prove to be extremely useful towards the understanding of the issues of rotation and dissipation of higher multipole moments that might settle the collapse to a Kerr black hole or an axially symmetric naked singularity. On the other hand, numerical models can be used to trace the apparent horizon in more complicated matter models thus giving some hints to what happens to trapped surfaces in realistic collapse scenarios (see Ref. 83).

There are two main collapse scenarios that have been widely investigated and that are of crucial relevance from the astrophysical point of view:

- The complete collapse of rotating bodies.
- The merger of two compact objects.

It is important to notice that when one takes into account enough elements to make the simulation somehow closer to reality the time of computation grows enormously thus requiring the use of very powerful computers. For this reason very often such simulations had to restrict the space of parameters considered. Typically simulations were carried out in one or two spatial dimensions only and it was only recently that full 3D simulations of core collapse supernovae and of binary mergers allowed to study these phenomena in more detail. Furthermore for the above reason simulations that consider the microscopic structure of matter, including neutrino emissions and electroweak interactions, are generally constrained to be non-relativistic while fully general relativistic simulations typically neglect the microscopic details of the matter cloud.

Within the problem of the complete collapse of a rotating body the main features that have been investigated through numerical simulations are the influence of rotation and magnetic fields (see Ref. 84 and ref therein). These affect greatly the last stages of collapse and are responsible for the waveform of the emission of gravitational waves and the structure of the accretion disk that surrounds the final compact object. The way that the outer layers are ejected from the collapsing spinning body can help us understand some questions that are crucial to the cosmic censorship such as: Does the collapse of a rotating object always end in a Kerr spacetime? Or is it possible that a rapidly rotating object collapses to a super-spinning Kerr solution?

Further, these simulations provide valuable physical insights into astrophysical questions such as how the matter ejected during collapse due to rotation falls back on the compact object in the form of an accretion disk, or what kind of features these accretion disks will present in terms of thickness, angular momentum and light emission. Or also, how the presence of a magnetic field will affect the infall of particles and the formation of high energy jets.

Recently the attention has been posed towards the issue of binary mergers. These models describe the merger of two inspiraling compact objects that can be taken to be two black holes, two neutron stars or one black hole and a neutron stars. The effect of unequal masses for the initial objects on the final configurations have been studied and a variety of scenario have been proposed in which the system settles to a final Kerr black hole with an accretion disk. These mergers are believed to occur frequently in the universe and are a major source of gravitational waves. Therefore the simulation of the waveforms emitted during the merger is one of the crucial results of these simulations, though it is not the only one. In fact accretion disks and the mechanism by which the matter from the disk accretes onto the final compact object is thought to be one possible mechanism for the production of gamma-ray bursts (see Ref. 85). Furthermore, crucial to the cosmic censorship conjecture, the possible formation of superspinning Kerr spacetimes have been studied. In principle such a spacetime could result from the merger of two rapidly rotating compact objects (see Ref. 86), from the increase in angular momentum due to the inflow of particles from the accretion disk, or from the complete collapse of a star with high angular momentum (see Ref. 77). In practice there are no definitive results on the possibility of the formation of superspinning Kerr spacetimes, though there are arguments that suggest that overspinning a Kerr black hole might not be possible. The discussion on these issues is still very much open. What has become clear is that the mechanism by which such a final configuration could in principle be produced would be very different from that of producing a Kerr black hole.

Finally, we mention here some work that has been developed towards the goal of a viable description of the processes that lead to type II supernovae explosions. Many different settings of numerical simulations have been carried out considering the microscopic effects, in order to provide a viable model for the production of the shockwaves that create these supernovae. These simulations describe the core collapse of a star from the microscopic point of view, taking into account nuclear interactions and neutrino productions. Within this field a lot of progress has been made in the last few years (with the computational power that finally enabled for full 3D simulations) and a lot of attention has been put on the influence of neutrino production during the explosion (see Ref. 87 and ref therein). Nevertheless these simulations still face one crucial problem in the fact that the efficiency of the explosion, even taking 'neutrino heating' into account, is not enough to produce a supernova. What typically happens in the simulations is that the explosion dies out after a few hundred kilometers, and thus no known mechanism so far has been able to revive the shockwave that would produce the supernova. [88]

Almost all models for type II supernovae explosions consider a shockwave that is generated when the infalling matter coming from the outer shells reaches the inner compact core. The compact object at the center constitutes a barrier onto which the matter from the outer layers bounces, thus creating the shockwave. The energy of the shockwave is related to the size of the core that creates it. Nevertheless, during the process of complete collapse, it is still possible that another wall, a quantum-gravity limit, exists at a shorter scale. This, as we have seen in previous sections, is where general relativity breaks down predicting a singularity. It is possible that another shockwave be created once this limit is reached. Therefore if the structure of collapse is such that no horizon exists at that time it is also possible that such a shockwave propagates outwards, providing the missing energy for the explosion to occur. As far as we are aware no simulations has been carried out taking these possible effects into account and therefore at present we do not know if these constitute a viable solution to the problem of the missing energy for type II supernovae explosions.

5. Structure of naked singularities

As such the collapse models studied so far and the resulting singularities exhibit a wide variety of interesting structure. The thing that emerges clearly is that not all singularities are the same in their properties and therefore their physical implications as well as their connection to the CCC will also be different from case to case. In some models, only a part of the singularity is visible, rest covered in the horizon eventually, but elsewhere they can remain visible forever. This depends on the form of matter collapsing, the equation of state used, and such other properties. Basically they seem to come in all varieties, depending on different models of collapse considered. Typically, the naked singularity develops in the geometric center of collapse to begin with, but later it can spread to other regions, or get covered, as the collapse progresses.

5.1. *Are they always massless or with negative mass?*

A question is asked sometimes on the mass of naked singularities, namely if they are always 'massless', because typically in spherical collapse the naked singularities forming at the center of the cloud has the function F tending to a vanishing value in the limit of approach to the singularity.

We note, however, that this need not always be the case. For example, in higher dimensional gravity, timelike naked singularities do develop which are massive both within Einstein's theory (see e.g. Refs. 89, 90) as well as in alternatives theory of gravity (see Ref. [91]). Massive singularities are present in solutions of Einstein field equations in classical relativity as well, the most famous of them probably being the one appearing in Kerr's solution, though at present we do not know if such a naked singularity can form via some physically viable dynamical process.

In any case, in our opinion what really matters when a singularity is visible is that super-ultra-dense regions where densities and curvatures really blow up are visible to external observers in the universe. That is of actual physical consequence, rather than the 'mass' of the singularity itself, which is actually not an object or part of the spacetime, and in fact it may even be resolved when quantum gravity effects are taken into account.

Given the examples of massive timelike singularities and others such as above, and also other examples of collapse evolutions available so far, it is clear that naked singularities, when they form in collapse come in many types of varieties and properties depending on the physical situations as well as the form of the matter considered and such other factors, and it is not possible to actually rule out any of these by some kind of a theorem. Therefore, in our view, imposing such restrictions will not help preserve the cosmic censorship conjecture.

Also, there is some discussion in the literature on the negative mass Schwarzschild singularities. It is asked sometimes if naked singularities, whenever they form, must always have a negative mass similarly to the specific Schwarzschild case mentioned. As we have clarified above, that is not the case. Generically, if we take the case of a spherically symmetric collapse, the mass function has a vanishing or positive value in the limit of approach to the singularity which is visible.

In fact, one would like to impose all possible physical reasonability conditions, such as the positivity of the mass-energy density, regularity of the initial data from which the collapse evolves, and such other regularity conditions. We note that constructing models with naked singularities and negative mass is of course possible, but the physical validity of the same would be quite unclear. Then the idea is to see whether naked singularities still develop in gravitational collapse that is developing from regular initial data and under physically viable conditions. The answer that follows is in affirmative as shown by many studies we discussed above. These singularities have no negative mass. In fact, the negative mass Schwarzschild singularity is not obtained as a result of any dynamical collapse evolution, and to that extent it is not physical.

5.2. *An object or an event?*

Are naked singularities always pointlike in time, or also extended? This is related to the question of whether they are like an object or an event. In typical spherically symmetric collapse models when considered in comoving coordinates, the first point of the singularity curve is visible, from which families of non-spacelike curves come out. The later points in comoving time, of the singularity curve, get hidden under the horizon.

Thus one can ask whether naked singularities, whenever they form, are always momentary or they could also be extended in time. We point out here that depending on the collapse scenario and the form of matter and the equation of state considered, the naked singularity could be pointlike, or it could be also extended in the comoving time. In particular, the timelike singularities, when they form in collapse are extended typically (see e.g. Ref. 36).

Even when they are pointlike or null singularities, the structure of the families of geodesics coming out from the singularity have been analyzed in detail (see for example Refs. 92–96 It is found that there are non-zero measure of non-spacelike curves coming out even from a pointlike naked singularity, and as far as the faraway external observer is concerned, once he gets to see the first ray from the singularity, for all times to come the null or timelike paths from singularity will keep reaching the observer.

This issue is also related to whether the naked singularities are always null, when they form in collapse. As we pointed out, they could be very much timelike, as well as extended in the space, rather than being just null and pointlike.

We note that as opposed to the naked singularities developing in collapse which are sometimes like an event, those occurring in the super-spinning Kerr geometry or many other vacuum models are ever-lasting, and in that sense they are like an 'object'. While such solutions do occur in general relativity, till recently what was not clear is whether such object-like naked singularities do arise from dynamical physical processes in gravity physics. In this connection, it is relevant to note that in Section 7 we shall discuss a class of gravitational collapse models which give rise to the final configuration which contains an ever-lasting naked singularity which develops in collapse from a regular initial data.

5.3. *Do naked singularities violate causality?*

One of the issues that is often considered in connection to the occurrence of naked singularities is that of the possible violation of causality that can arise in certain spacetimes.

Roughly speaking, causality violations occur in those spacetimes which contain closed timelike curves. An observer moving on such a curve would eventually find himself at the initial event in space and time even though locally his clock never went backwards. The most well known examples of such spacetimes are the Godel universe [97] and the solution found by Tipler for a spacetime outside a rotating cylinder. [98] Also, closed timelike curves are present in familiar solutions such as the Kerr metric. The singularity theorems by Penrose and Hawking show that singularities form generically from Einstein equations even when causality is protected, but these theorems are not very helpful when exploring the connection between causality violations and singularities.

While closed timelike curves are present in the Kerr spacetime, they are confined inside the horizon for the Kerr black hole. Therefore an observer entering the horizon would be able to travel back in time but would not be able to return to the outside universe, thus preserving the causal structure of the same. In some cases, closed timelike curves can be accessed by any observer in the Kerr metric, allowing therefore for the theoretical possibility of time travel. [3] ¿From such an example, it could be tempting to think that naked singularities would typically allow for causality violations to occur in the universe, thus making them undesirable. It is possible that the presence of the horizon would hide the closed timelike curves thus disconnecting them from the rest of the universe. Although this is true for some manifolds, there are also dynamical spacetimes such as the LTB models and others, which admit naked singularities but which have no closed timelike curves. It follows that in general there is no direct connection between these two phenomena of causality violation in a spacetime and the occurrence of naked singularities.

We note that even when singularity theorems assume some causality condition, the spacetime singularities and causality violation are independent phenomena. Therefore, in general, the connection if any between the spacetime singularities, and in particular naked singularities and causality violations is far from clear and could be much more subtle. One of the main reasons could reside in the fact that different kinds of naked singularities need to be treated differently. For example, what holds true for the Kerr solution need not be true for the Lemaitre-Tolman-Bondi metric. A spacetime without rotation, such as the LTB collapse model, could not have closed timelike curves and would therefore respect causality regardless of the presence of a naked singularity.

Furthermore it is clear that causality violations need to be better defined and studied in order to understand what is really undesirable about them (see for example Ref. 99). The possibility of closed timelike curves by itself could not be enough to make a spacetime 'physically unrealistic'. In fact if we want to summarize we can in principle divide causality violations into three main categories:

- Microscopic causality violations: in the case that closed timelike curves can occur at a microscopic level, thus being resolved or included by an eventual theory of quantum gravity.
- Local scale causality violations: in the case that closed timelike curves can occur at planetary or galactic level, thus giving rise to the possibility of

time travel with all the connected paradoxes.

- Cosmological causality violations: in the case that these can occur only on time scales comparable with the life of the universe and thus having no bearing whatsoever on our local picture of the cosmos. These closed timelike curves cannot be ruled out in principle.

As noted by many authors, global causality requirements for the whole universe might be too restrictive since we are able to experience only a limited, local, portion of the universe. From the fact that causality holds here and now it might be far fetched to conclude that causality holds globally for the entire universe. Furthermore non-local correlations have been studied within the framework of quantum mechanics since many years and it seems not impossible that at a microscopic level a full theory of quantum mechanics coupled to gravity might allow for some kind of causality violations to occur. Therefore we are left only with the second class of causality violations, namely the local scale violations, to be considered as undesirable in principle. Within this class (which for example excludes the Godel solution), can we say that naked singularities and closed timelike curves are linked in some way? Examples of spacetimes without singularities but with closed timelike curves are known since the early days of general relativity (see Ref. 100 and references therein), but such solutions cannot be obtained from the evolution of a regular matter cloud. In fact it was shown that the occurrence of singularities is a necessary condition for closed timelike curves to evolve from regular initial data [101,102], therefore suggesting that in dynamical configurations the formation of singularities is connected to the appearance of closed timelike curves. Nevertheless the possible relation that might link the occurrence of closed timelike curves with the behaviour of the horizon that could eventually cover the singularity has not been investigated so far, leaving at present the issue of causality violation separated from the fate of cosmic censorship.

5.4. *Local versus global visibility*

A singularity is said to be locally naked if there exist outgoing null or timelike trajectories that reach some observer in the spacetime. In this sense, for example, the singularity in the Reissner-Nordstrom spacetime is locally naked since observers located within the radius of the event horizon, but not outside it, can be reached by nonspacelike geodesics coming from the singularity. On the other hand a singularity is said to be globally visible if there exist outgoing nonspacelike geodesics that reach observers at future spatial or null infinity. In this case no horizon is present before the singularity and the light rays coming from the singularity can be seen by any future observer.

Global and local visibility are of course related to the cosmic censorship conjecture itself. As said, cosmic censorship comes in various forms and formulations and over the years some forms have been proposed that allow for the existence of locally visible singularities. The main distinction that is usually made is between two formulations of the conjecture that are called the Strong Cosmic Censorship and Weak Cosmic Censorship. The form of the conjecture that one assumes in the analysis has implications for the local and global properties of the spacetime containing the singularity that one is going to study.

In fact, the weak form of the CCC postulates that a singularity cannot be seen by any observers at null infinity, thus allowing for locally naked singularities to occur. In this case when we study the formation of the singularity and of the apparent horizon, we need not worry about the global structure of the horizon itself. Actually, proving that there are future directed outgoing null geodesics emanating from the singularity is enough to ensure local visibility.

As an example, one could consider the LTB collapse scenario. As we have shown earlier it is the first term in the expansion of the mass profile that determines the local visibility of the singularity. This holds regardless of where the boundary of the cloud is taken. On the other hand, a careful analysis of the behaviour of the apparent horizon shows that for some of these matter models the horizon curve, while being increasing in a neighborhood of the center, can become decreasing afterwards, thus covering the singularity eventually from observers at future infinity (see Fig. 6).

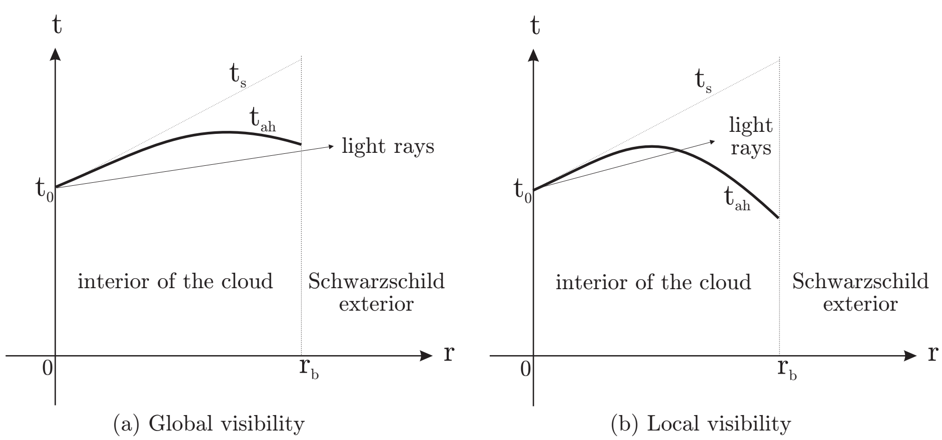

Fig. 6. (a) Globally visible singularity: light rays emitted from the central singularity can propagate inside the cloud until they reach the boundary, from which they can travel freely to reach observers at future infinity. (b) Locally visible singularity: light rays emitted from the central singularity can propagate inside the cloud but must fall into the apparent horizon before they reach the boundary.

It was shown (see Ref. 6, 103, 104) that for a fixed value of the boundary there is a range of values of the first non-vanishing term in the expansion of the mass profile for which the singularity is only locally naked and a range of values for which it is globally naked. ¿From a mathematical point of view, this is not relevant since in dust models we always have the freedom to choose the boundary as we like. Therefore for any given matter profile we can always ensure global visibility by taking r_b such that $t_{ah}(r)' > 0$ for all $r \in [0, r_b]$ (see Fig. 6).

The issue is somewhat different when we apply the formalism to the description of a collapsing star. First of all, the presence of pressures (that must vanish at the boundary) might affect the way the boundary and the mass profile are related. Furthermore, in this case the total mass and the radius of the boundary are fixed by realistic physical values for a star, depending on the stellar model chosen. It can then very well happen that even though the singularity that forms at the end of collapse is locally visible it might not be globally visible. Recently, Jinghan et al. [105] pointed out that if we idealize a star as composed only by dust such a scenario could happen.

On the other hand, as we have seen, the LTB model is very restrictive as far as the matter content is concerned, since dust is not a very realistic form of matter for a dense object as the core of a star. Furthermore, as indicated previously, assuming that the same equation of state (in this case dust) holds throughout the whole star is also unrealistic, since outer layers will be less dense and made of different matter constituents as compared to the inner layers. So if we apply the same reasoning as above, adopting the LTB model not to describe the whole star but only for its core we are once again left with some degree of freedom as to where to take the boundary of the core.

Therefore it is easy to see that for more elaborate and realistic matter models, which include pressures and different layers, there is still no concluding evidence in one direction or the other, and the issue of global visibility for a given collapse model remains very much open.

5.5. *Can energy come out of a naked singularity?*

We note here that even if a naked singularity forms in collapse, the singularity by itself does not necessarily have to radiate matter or energy thus bearing a signature that is detectable from far away in the universe. Such singularities forming at the end of collapse, unlike naked singularities such as the ones in the superspinning Kerr metric, are more like events rather than an object, being a moment when collapsing matter reaches its final doom, something like the big bang in reverse. Questions such as what will come out of a naked singularity are then not really meaningful; 'things' do not have to come out of it. What we really see is not the singularity itself but the signature of processes that occur in the extreme conditions of matter near this epoch, such as the shockwaves due to inhomogeneities in this ultra-dense medium, or quantum gravity effects in its vicinity.

Generally, when considering a complete gravitational collapse, it is often assumed that the boundaries set by the electromagnetic and nuclear forces are surpassed and nothing can halt the collapse. Therefore, in the ultra high density regions that surround the singularity, the pull of gravity is thought to be so strong that nothing can escape it. One would be inclined to think then that even if such naked singularities exists they would bear no impact on the rest of the universe since no signal can escape their gravitational field. Nevertheless, there are other ways by which naked singularities could in principle leave a trace in the outside universe, and from the study of exact solutions of the field equations we can see that there are scenarios in which the process of formation of the singularity can be accompanied by the emission of energy.

Firstly, we note that if we regard infinities as the mathematical signature of the breakdown of a theory then it is reasonable to suppose that some other physical effects will account for those divergencies thus resolving them. Therefore general relativity might not be the best tool to describe the final instants of collapse when the singularity forms. Generally it is believed that it will take some quantum theory of gravity to resolve the singularity, but there is no reason at present to rule out also some other classical gravitational effects coming from some corrections to general relativity (see e.g. the Eddington gravity alternative in Ref. 106 for an example of an alternative theory without matter singularities). Hence, once a suitable theory of gravity is able to resolve the divergence of the energy density that originates the singularity, it is very well possible that some new kind of barrier arises at short scales (such as the Planck scale for an eventual theory of quantum gravity), thus disrupting the singularity formation and creating a shockwave through which the matter-energy that was collapsing is ejected away. Such a shockwave would be immensely energetic and it would bear a clear signature of the level at which it has arisen.

This point raises the issue of the possible observational features that a naked singularity could have. Did we observe similar phenomena in the universe already? The answer is we do not know as yet. Since the stellar structure in realistic cases is much more complicated than in the idealized analytical models and it involves many layers of different materials with different properties, it is obvious that whatever might come from such a Planck-scale event would be scattered, absorbed and emitted many times before an actual signal comes out of the surface of the star. Therefore, of the many highly energetic events that are known to happen when a star collapses under its own gravity, we do not know if any of them bears the signature of the Planck scale physics that is happening very close to the center. For example, we still do not have as of now a comprehensive model that explains how supernovae explode. Computer simulations describing what happens at the core of the star when it explodes have found difficulties in producing the amount of energy necessary for the shockwaves to propagate through all the layers of the star, thus generating the explosion (see section 4.7 for more details). It seems not unreasonable to suppose that a barrier at a scale smaller than the Schwarzschild radius might provide this missing energy, nevertheless there are still no studies in this direction. A similar reasoning might apply also to gamma-ray bursts, a very energetic phenomenon created from the core collapse of a star that is still far away from being well-understood. [107] It could be that the gamma-ray bursts are created in the exploding outer layers of the star, and therefore it is not a direct effect of some possible quantum barrier. Nevertheless the mechanism by which the explosion is related to the collapse of the inner core is still not well-understood and it is indeed possible that certain kind of phenomena are linked to certain types of collapse.

On the other hand, even without calling for some modified theory of gravity or quantum gravity, it is still possible that some energy comes out from the singularity by classical effects only. In fact, if we allow for negative pressures to occur during collapse at a purely classical level, we find immediately that the mass profile of the collapsing star must be radiated away during the process.

Negative pressures close to the formation of the singularity could expel the inner shells whose particles would then collide with the infalling outer shells with very high energies. Particle collisions would then occur close to the Cauchy horizon with arbitrarily high center of mass energy thus turning the collapsing cloud into an immense particle accelerator (see Fig. 7).

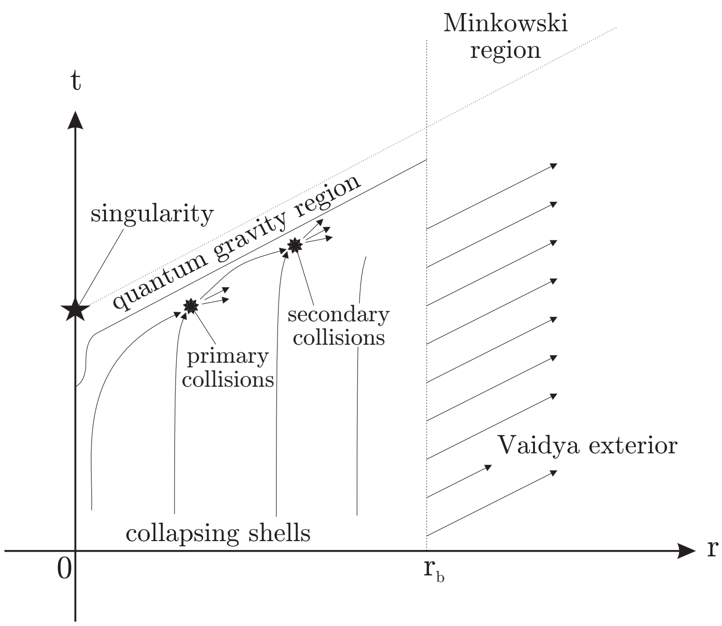

Fig. 7. The collapsing shells are repelled by the quantum gravity region. When the highly energetic particles traveling outwards collide with the particles from outer collapsing shells, fountains of collisions are created at arbitrarily high center of mass energies.

As an example we discuss here a model that was presented in Ref. 108 where a perfect fluid collapse was considered in spherical symmetry with a matter function given by a perturbation of an homogeneous perfect fluid close to the center as:

$$M(r,v) = M_0(v) + M_2(v)r^2; \tag{55}$$

This corresponds to a variable relation between energy and pressure:

$$\frac{p(r,v)}{\rho(r,v)} = k(r,v)\ . \tag{56}$$

Recall that in perfect fluid collapse if the equation of state is not specified we have the freedom to choose one function globally to solve the system of Einstein equations. In this case, considering equation (56) as the equivalent of an equation of state we can take $k(r,v)$ as the free function that once chosen will specify $M(r,v)$ via the differential equation (51). Writing k as

$$k(r,v) = k_0(v) + k_2(v)r^2 \ , \tag{57}$$

and taking $k_0(v)$ and $k_2(v)$ as the free functions for the system from the expansion of equation (51) we obtain the two differential equations

$$0 = 3k_0(v)M_0(v) + M_{0,v}v \ , \tag{58}$$

$$0 = 3k_2M_0 + 5k_0M_2 + M_{2,v}v + (1+k_0)M_{0,v}w_{,r}(0,v) \ , \tag{59}$$

where $w(r,v) = v'(r,v(r,t))$. The first one can be solved for $M_0(v)$ once $k_0(v)$ is specified and this gives

$$M_0(v) = C_1 e^{-3\int_v^1 \frac{k_0}{v}dv} \ . \tag{60}$$

In the second equation we use the freedom to specify $k_2(v)$ and choose a class that allows us to integrate the differential equation. Thus we define

$$k_2 = (1+k_0)k_0\frac{w_{,r}(0,v)}{v} \ , \tag{61}$$

and obtain

$$M_2(v) = \tilde{C}_2 e^{-5\int_v^1 \frac{k_0}{v}dv} \ . \tag{62}$$

This solves the system up to second order in r and $M(r,v)$ results as follows,

$$M(r,v) = C_1 e^{-3\Phi(v)}\left[1 + r^2C_2e^{-2\Phi(v)}\right] \ , \tag{63}$$

where we have set $\Phi(v) = \int_v^1 \frac{k_0(v)}{v}dv$ and $C_2 = \frac{\tilde{C}_2}{C_1}$.

We see that positivity of C_1 is enough to ensure positivity of M near the center, while $C_2 < 0$ would imply that the density is decreasing radially outwards thus imposing some conditions on the boundary of the cloud once C_2 is set.

We can choose k in such a way that at the beginning of collapse it is positive and almost constant, thus resembling a perfect fluid with linear equation of state. Then negative pressures are triggered close to the singularity by k turning negative. This can easily be done by suitably choosing the terms in the expansion of k.

As an example consider $k_0(v)$ to be expanded also near the singularity as

$$k_0(v) = k_{00} + k_{01}v + k_{02}v^2 + ... \ , \tag{64}$$

and for the sake of clarity all higher order terms are assumed to vanish.

Since the formation of trapped surfaces is governed by equation (28) and with the present choice of k_0 we get

$$\frac{F}{R} = r^2C_1v^{-3k_{00}-1}e^{-3k_{01}v-\frac{3}{2}k_{02}v^2}\left[1 + r^2C_2v^{-2k_{00}}e^{-2k_{01}v-\frac{2}{2}k_{02}v^2}\right] \ , \tag{65}$$

it is easy to see that the requirement that $\frac{F}{R} < 1$ near the singularity imposes that $k_{00} \leq -\frac{1}{3}$, which is a condition for avoidance of trapped surfaces. In this case not only the central shell but also all other shells are not trapped when they become singular. In fact the special case where $k_{00} = -\frac{1}{3}$ in the limit of approach of singularity gives $\frac{F}{R} \longrightarrow r^2 C_1$ which implies a finite radius for the trapped surfaces at all times. Choosing the boundary so that $r_b < \frac{1}{\sqrt{C_1}}$ is enough to avoid trapped surfaces during the whole collapse. On the other hand when $k_{00} < -\frac{1}{3}$ the radius of the apparent horizon diverges at the singularity thus indicating that to avoid trapped surfaces no restrictions are imposed on the boundary. Finally, as seen in section 4.2, for values of $k_{00} > -\frac{1}{3}$ only the central shell could be visible when becoming singular, depending on the sign of $\chi_2(0)$, which in turn would depend upon C_2.

The coefficients k_{01} and k_{02} can then be chosen so that the pressure is positive at the initial time (and for v close to 1), while it decreases during collapse and becomes negative at the latest stages of collapse eventually, thus leading to the formation of a naked singularity.

The presence of negative pressure requires that the function M must vanish at the singularity which suggests that the whole mass of the core gets radiated away during the last stages of collapse. Therefore a finite amount of energy escapes from the collapsing process. Furthermore the repulsive effects near the singularity would cause the infalling shells to be expelled and thus creating the conditions for particles escaping the central region to collide with particles of the infalling outer shells.

Particles escaping the vicinity of the singularity by this mechanism would have to be intrinsically very energetic since they originate in the region where quantum gravity is dominant. These particles, when traveling close to the Cauchy horizon would retain their energy even at larger distances from the singularity as they travel in the region of spacetime which can be described by a metric which is Minkowskian with small perturbations.

6. Current Status of Cosmic Censorship

In view of the developments such as those discussed above, it is natural to ask here what is the present status of the cosmic censorship hypothesis, as implied by the current work on gravitational collapse that has taken place in recent years.

6.1. *Does Cosmic Censorship hold?*

Firstly, we are compelled to ask at this stage the following question: Is the cosmic censorship conjecture correct or not as a basic principle of nature? Clearly, the answer either way has profound implications for fundamental physics and cosmology.

What comes out unambiguously from the work so far is, censorship is certainly not correct in an unqualified form, as it is sometimes taken to be. It is quite clear by now from the work so far that singularities appear in a wide variety of forms and from a wide array of scenarios in general relativity and to rule out all naked singularities at once with a single theorem seems at present not feasible. Therefore if cosmic censorship holds, that will be in a highly refined and fine-tuned form only, with suitable conditions and a formulation that is yet to be achieved. The work on gravitational collapse so far will play an important role here to achieve such a formulation. On the other hand, investigating the quantum and astrophysical processes in the vicinity of such visible ultra-dense regions predicted by dynamical collapse models can give rise to intriguing physical consequences as we point out and discuss in the next section.

The implications of these developments and the gravitational collapse studies for censorship are certainly important. We can now say with confidence that one cannot formulate censorship in a rather general way such as, 'Collapse of any massive star makes a black hole only', or, 'Any physically realistic gravitational collapse must end in a black hole only', as there are now many counter-examples to such statements. It follows that any formulation for cosmic censorship must carefully specify when a black hole will develop in collapse. Specifically, one must examine the collapse scenarios carefully and isolate the features that cause a naked singularity to arise. Physicists believed for a long time that spherical collapse must yield a black hole only, and naked singularities arise if at all in non-spherical collapse only. We now know this is clearly not the case. This explains why earlier efforts to formulate a general theorem for CCC failed.

The point, in other words is the following. If one is considering the gravitational collapse of a massive matter cloud, there will be specific and fine-tuned conditions which one must set on the initial data such as the matter densities and pressure profiles, and the allowed dynamical evolutions of the Einstein equations, so that the collapse endstate will be a black hole only. The key aim such conditions or fine-tuning will achieve is that these will ensure that in a continual general relativistic collapse, when a spacetime singularity develops as collapse endstate, the event horizon will develop necessarily prior to the epoch of the occurrence of the singularity.

As an example of such a fine-tuning, we noted the example of dust collapse. If we require the density to be not increasing from the center of the cloud outwards and we must get the black hole as the collapse endstate, then the density profile must be so fine-tuned so that it is fully homogeneous at the initial epoch from where the collapse develops, and then the velocity profile for the collapsing shells is to be so tuned, so that the density remains necessarily homogeneous at all later epochs. Only then one gets a black hole final state. In all other cases, a naked singularity develops as the final outcome of collapse. Similar fine-tunings will be necessary for more general collapse of different forms of matter, depending on the nature of the stress-energy tensor and the equation of state it follows.

Under such a scenario, a general statement for cosmic censorship conjecture seems very difficult or nearly impossible to formulate. The best one can do, probably is to specify a set of conditions for a given collapse scenario so that it terminates into a black hole. It would be then natural and appropriate to accept that at the level of theory, both black holes and naked singularities do occur as the final fate of a continual gravitational collapse of a massive matter cloud within the framework of general relativity.

6.2. *Why naked singularities form?*

It is natural then to ask here, what is really the physics that causes a naked singularity to develop in collapse, rather than a black hole. We need to know how particles and energy are allowed to escape from extremely strong gravity fields. We have examined this issue in some detail to bring out the role of inhomogeneities and non-zero pressures.

In Newtonian gravity, it is only the matter density that determines the gravitational field. In Einstein theory, however, density is just one attribute of the overall gravitational field, and the various quantities such as the space-time curvatures play an equally important role in dictating what the overall nature of the field is.

What our results show is that an inhomogeneous density profile and the resulting shearing effects in the matter cloud, as implied by the evolution of the cloud governed by the Einstein equations, could delay the trapping of light and matter which can then escape away even from very close to the singularity (For a definition of shearing effects in matter fields, see for example 39).

This is a general relativistic effect wherein even if the densities are very high, inhomogeneity creates paths for light or matter to escape, leading to a naked singularity rather than a black hole. If the amount of inhomogeneity is very small, below a critical limit, a black hole will form, but with sufficient inhomogeneity, trapping of material is delayed and a naked singularity arises.

While such a criticality can be clearly seen in the dust models, it actually comes out very transparently in the models where matter converts fully to radiation, originally constructed in 1940s by Vaidya [109,110] to model a radiating star. In these models, it is the rate of collapse, that is how fast or slow the cloud is collapsing, that decides between the formation of black hole and naked singularity. For further details on these models we refer to Ref. 111.

Let us consider the physics that lead to naked singularity formation in gravitational collapse. For transparency and clarity, we will discuss a specific model depicting collapse of a dust cloud in the absence of pressure, based on our general treatment of this case (see Refs. 6, 39). The cloud begins collapse from a position of rest, corresponding to the phase in a massive star's life when it has exhausted its internal fuel and the gravity takes over. This is a classical system governed by general relativity with all physical regularity conditions being satisfied, such as positivity of energy density and regularity of density and curvatures at the initial epoch when the collapse begins. If the density were taken to be completely homogeneous at the initial time, this would be exactly the Oppenheimer-Snyder model, with the collapse giving rise to a black hole. Let us now consider the physically realistic situation, where the density of the star is higher at the center and decreases as one moves away.

To determine the collapse end state, the trapping of light and matter is to be understood as gravity fields become more and more powerful, and the equation of the trapped region in spacetime is determined, which decides the formation of the event horizon developing dynamically as the collapse evolves. The key factor is the timing of the horizon during collapse. If the horizon forms well before the final singularity, the outcome is a black hole, but if it is delayed as the collapse evolves, we have visible ultra-dense regions forming in the universe. It turns out that sufficient inhomogeneity in the density distribution at the initial time delays the horizon. If the density decreases fast enough away from center of the star, the final outcome is a naked singularity, but in a slow decrease or nearly homogeneous case, a black hole results.

To understand the collapse evolution, see Figs. 1 and 2, which correspond to the homogeneous and inhomogeneous cases respectively. The arms of the cones denote paths of the ingoing and outgoing light rays. In the homogeneous case, the collapse begins with regular initial conditions where densities and curvatures are finite. As the collapse progresses, these increase and the focusing effect on light and matter grows. Then, there comes a time when a region in the spacetime starts developing such that the light from the same is simply unable to escape to any faraway observer in the universe, but stays confined and trapped only in a finite extent. Once light is trapped, the same always happens for matter, namely for the timelike geodesics as well, with its trajectory only within the cones. This corresponds to the light rays from surface of the star at a phase when the radius of the star has contracted to a distance proportional to its mass. As the collapse progresses, since the density is only time dependent, the entire cloud finally is crushed simultaneously to a singularity in future. The trapping of light and matter, however, occurs well before the final singularity developed, and therefore the singularity is well-hidden inside the black hole that thus formed, with no light signals or matter escaping from the ultra-dense regions near the spacetime singularity.

The collapse, however, develops quite differently once density is no longer homogeneous, as shown in Fig. 2. The light paths are shown as collapse progresses and as the star gets denser and denser. The entire cloud now no longer collapses simultaneously to the singularity as density is not homogeneous. The different matter shells arrive at different times at the singularity, one after the other with shells at larger radius coming later. As a result, at no stage before the epoch of formation of singularity are the light rays ever trapped, and rays and particles can come out in principle from the super ultra-dense regions which are arbitrarily near to the spacetime singularity. This lack of trapping happens due to a deficit in the focusing effect of gravity on light and matter. Due to inhomogeneity and decreasing density of the star away from center, there is never enough total mass at any given epoch to cause full light trapping, prior to the epoch of singularity formation. We analyzed this general relativistic effect of delay of horizon formation and related it to the spacetime shear. Physically, it may correspond to creation of general relativistic shocks due to inhomogeneity in such ultra-dense regions which may allow for escape and ejection of light and matter despite very high matter densities.

The general result that we have for this system is that two parameters fully determine the evolution and the final fate of collapse. These are the mass and velocity functions of the star, specifying the total mass within a given area radius and the velocities of the collapsing shells respectively at the initial epoch. Generically, given any initial density distribution, there is a non-zero-measure collection of velocity functions that take the star to either a naked singularity or a black hole final state depending on the choice made, which is freely available with all regularity conditions being satisfied. This shows the genericity and stability of the naked singularity within the given framework. Similar features arise also when pressures and other physical equations of state are incorporated for general collapsing matter fields, as we noted earlier.

When a naked singularity develops, it is characterized by three important attributes.

- First, there are families of infinitely many light and particle trajectories coming out from the singularity region.
- Second, it is a genuine curvature singularity in that the densities and curvatures become infinite and grow very powerfully in the limit of approach to the singularity along these paths.
- Finally, the overall spacetime satisfies all physical regularity conditions, so this becomes an interesting framework to study actual physical processes in the ultra-strong gravity regions.

6.3. *Reformulate Cosmic Censorship?*

It is obvious that naked singularities are a general feature arising in general relativistic gravitational collapse if we do not impose any restrictions on the structure of the spacetime or the energy-momentum. Nevertheless it has become clear now that even standard physical requirements such as regularity of the initial data or energy conditions are not enough to guarantee the absence of formation of strong curvature naked singularities as endstates of collapse.

Therefore, as a result of the gravitational collapse studies carried out in the past years, there have been some efforts to reformulate the cosmic censorship conjecture. As we noted earlier, the CCC does not hold in an unqualified form within the framework of the Einstein gravity.

One would wish then to investigate the possibility that there exist an alternative formulation of CCC that, while restricting to a narrower array of scenarios (thus allowing for certain types of naked singularities to occur), still can be proved mathematically.

One way here would be to specify a set of conditions so that the evolving collapse from regular initial data for a given matter field has the horizon developing necessarily prior to the development of the singularity. While some general indications are available here for spherical collapse [112] a general formulation in this direction is not clear as yet. It is clear that from an astrophysical perspective, such an effort will be crucial to ensure the validity and applicability of the black hole physics. Another idea would be to consider cosmic censorship only for certain specific matter models. [113]

A still more radical proposal is to consider the cosmic censorship in vacuum general relativity only where the spacetime contains no matter fields at all. Alternatively, one may allow for only selective 'suitable' matter fields, such a Maxwell field or massless scalar fields only, and then ask whether the censorship will hold [114]. One would like to ask whether in a pure gravity framework, without any matter fields, whether the cosmic censorship holds. If this can be achieved, then the idea would be to disregard any naked singularities arising in the gravitational collapse of matter clouds as 'singularities arising from matter' rather than the pure gravity itself, which would obey the CCC.

There is, however, no essential progress in this direction also so far as to how to formulate properly such a statement in a mathematically rigorous manner, and then to proceed to any possible proof for the same.

A similar idea, also suggested by Wald [115] is placed somehow in the middle. This proposal claims that fluid models such as perfect fluids and dust be disregarded because they fail to give an account of the microscopic properties of matter. The naked singularities occurring in these collapse models would then be due to the lack of a description of fundamental interactions that would resolve them either at a classical or at a quantum level.

This idea is much in line with what we mentioned before when we suggested that either classical effects or quantum correction would intervene to resolve the singularity. Nevertheless, considering only classical general relativity, the only fundamental fields that can be studied in this respect are the electromagnetic field and maybe scalar fields (and naked singularities have been shown to occur in massless scalar field collapse, see Ref. 116, 117, however, it is claimed that these are 'non-generic').

On the other hand, a full quantum treatment of the last stages of collapse is missing at present, leaving the possible reformulation of CCC only at the stage of a proposal. The fundamental point here is that this formulation lacks any connection with the issue of horizon formation.

In fact, if we agree to the idea that the singularities must be resolved when we take into account the microscopic structure of matter, then there is no need to conjecture that they can arise in some cases, being hidden within an horizon, and not in other cases, and the whole issue of cosmic censorship becomes rather futile.

Another important point, from an astrophysical perspective, is that, disregarding all the matter forms such as dust and perfect fluids with different reasonable and well-motivated equations of state, which have been widely used and studied in astrophysical contexts will also be far from being widely acceptable. In fact, collapse models with matter fields such as these and others have been studied and applied in astrophysics for decades, and to rule them all out just for the sake of possible formulation of a hypothesis would not seem to be reasonable.

The key point here is, in the end the physical problem we need to study and investigate is that of the final fate of a massive collapsing star when it shrinks gravitationally at the end of its life cycle. The massless scalar fields that are claimed to be 'suitable' for the purpose of a cosmic censorship statement are not observed in nature, and it will be far from reasonable to assume or claim that very massive stars are composed of or are dominated by massless scalar fields only, even in the later stages of their gravitational collapse. In fact, matter fields such as perfect fluids, radiation collapse models, and even dust collapse would be considered quite useful and important to study this physical problem of collapse of a star, rather than ruling them all out. This is one of the main reasons why so many studies have been already conducted for past many years for collapse of matter clouds with forms of matter such as above in the gravitation theory.

7. Astrophysical and observational perspectives

We have pointed out in the considerations here that the final fate of gravitational collapse of a massive star continues to be an exciting research frontier in the black hole physics and gravitation theory today. The outcomes here will be fundamentally important to the basic theory and astrophysical applications of black hole physics, and for modern gravitation physics. We highlighted certain key challenges in the field, and also several recent interesting developments were reviewed. Of course, by no means the issues and the list given here are complete or exhaustive in any manner, and there are several other interesting problems in the field as well.

We like to mention here a few points which we think require most immediate attention, and which will have possibly a maximum impact on the future development in the field:

1. The genericity of the collapse outcomes, in terms of black holes and naked singularities needs to be understood very carefully and in further detail. It is by and large well-accepted now, that the general theory of relativity does allow and gives rise to both black holes and naked singularities as final fate of a continual gravitational collapse, evolving from a regular initial data, and under reasonable physical conditions. What is not fully clear as yet is the distribution of these outcomes in the space of all allowed outcomes of collapse. The collapse models discussed above and considerations we gave here would be of some help in this direction, and may throw some light on the distribution of black holes and naked singularity solutions in the initial data space.

2. Many of the models of gravitational collapse analyzed so far are mainly of spherical symmetric collapse. Therefore, the non-spherical collapse needs to be understood in a much better manner. While there are some models which illustrate what the departures from spherical symmetry could do (see e.g. Ref. 118), some other analytical models of collapse for matter clouds with cylindrical symmetry

have also been studied (see Ref. 72). Though we note that on the whole, not very much is known for non-spherical collapse. Probably numerical relativity could be of help in this direction See for example Ref. 82 for a discussion on the evolving developments as related to applications of numerical methods to gravitational collapse issues. Also, other alternative would be to use global methods to deal with the spacetime geometry involved, as used in the case of singularity theorems in general relativity.

3. At the very least, the collapse models studied so far do help us gain much insight into the structure of the cosmic censorship, whatever final form it may have.

But on the other hand, there have also been attempts where researchers have explored physical applications and implications of the naked singularities investigated so far (see e.g. Refs. 49, 119, 120 and also references in there). If we could find astrophysical applications of the models that predict naked singularities as collapse final fate, and possibly try to test the same through observational methods and the signatures predicted, that could offer a very interesting avenue to get further insight into this problem as a whole.

4. An attractive recent possibility in that connection is to explore the naked singularities as possible particle accelerators (see Refs. 108, 142, 145, 146).

Also, the accretion discs around a naked singularity, wherein the matter particles are attracted towards or repulsed away from the singularities with great velocities could provide an excellent venue to test such effects and may lead to predictions of important observational signatures to distinguish the black holes and naked singularities in astrophysical phenomena (see, e.g. Refs. 121, 122).

5. Finally, further considerations on quantum gravity effects in the vicinity of naked singularities, which are super-ultra-strong gravity regions, could yield intriguing theoretical insights into the phenomena of gravitational collapse (see Ref. 38), as we shall discuss below in Section 9 .

7.1. *Observable signatures of naked singularities*

As we have seen naked singularities that appear in exact solutions of Einstein equations can take various different forms. It is then reasonable to suppose that their observational features, if present, might be various as well. If such objects do exist in the universe it is therefore crucial to any astrophysical endeavour to study how they might interact with the surrounding environment in order to understand whether they can be observed and how. [123] The key questions when considering observational features of naked singularities therefore are:

- Is there any observational signature coming out from the vicinity of a singularity?
- If there is, can we distinguish it from that of some other astrophysical objects?

As said, the singularity itself is not part of the spacetime, therefore here we are considering the ultra-dense region surrounding the singularity where relativity breaks down and quantum gravitational effects (or effects coming from modified gravity) become relevant. It would seem then that a theory of quantum gravity is needed in order to resolve the quantities that diverge and to make predictions on the observational signature of such objects. Nevertheless it is possible that a naked singularity presents some observable features already at a classical level, as it is also possible to study some eventual expected effects in a classical relativistic framework. Furthermore it is also possible that some modifications to general relativity, though not including quantum effects, could provide a framework where to understand better these extreme regimes and make predictions on their observational features.

Roughly speaking then the observability of a naked singularity might at present be studied by the use of three separate strategies:

- Considering relativistic effects at extremely high densities.
- Studying effects due to modified theories of gravity for interiors or very strong gravitational fields.
- Modeling possible quantum-gravity effects (either in the semiclassical approximation or by adapting some theory of quantum gravity, such as loop quantum gravity or string gravity, to some toy models).

So where could the observational signatures of naked singularities lie? If we look for the sign of singularities such as the ones that appear at the end of collapse, we have to consider explosive and high energy events. In fact such models expose the ultra-high density region at the time of formation of the singularity while the outer shells are still falling towards the center. In such a case, shockwaves emanating from the superdense region at scales smaller than the Schwarzschild radius (that could be due to quantum effects or repulsive classical effects) and collisions of particles near the Cauchy horizon could have effects on the outer layers. These would be considerably different from those appearing during the formation of a black hole, where the most dense regions are confined within the horizon and thus unable to communicate with the exterior.

If, on the other hand, we consider singularities such as the superspinning Kerr solution we can look for different kinds of observational signatures. Among these the most prominent features deal with the way the singularity could affect incoming particles, either in the form of light bending (such as in gravitational lensing), particle collisions close to the singularity, or properties of accretion disks (as it will be shown in next section).

Within this class of 'long lived' singularities there is another kind which presents intriguing possibilities, namely interior solutions that describe a regular source with a singular center. This is an entirely different kind of final state, as opposed to the vacuum solutions such as the Kerr metric, in the fact that it involves a finite matter cloud with a boundary larger than the horizon and which presents a singularity at its center. Interior solutions of Einstein field equations have been considered for decades as sources of the gravitational field. One key requirement in building such interiors used to be that the matter density should be regular all the way to the center of the source. Nevertheless there is no reason to impose that a stable configuration cannot evolve which presents a singularity at the center. These singularities, once again, will model the region of arbitrarily high density that develops at the center of the object and where gravity exhibits extreme properties as outlined before. It is then possible that such a source could be observationally different from a corresponding source with regular interior or from a vacuum solution such as a black hole. Furthermore if the overall density of the cloud is low enough, as it is the case for supermassive objects like the ones residing at the center of galaxies, then processes happening close to the center, such as particle collisions and lensing, could be visible to faraway observers.

In fact one key point when dealing with these models is understanding what kind of phenomena we are trying to describe. Namely, what kind of singularity are we looking at? Within gravitational collapse many different phenomena can be described. Of course there are the smaller, explosive, short lived phenomena like the collapse of the core of a star, which has been the main topic of this review. These will typically involve very high energies and very short time scales. On the other hand, the fairly recent discovery of active galactic nuclei has led astronomers to suppose that objects like supermassive black holes exist at the center of galaxies. These objects are still not very well understood today and it is indeed very plausible that processes such as gravitational collapse, on a much larger scale and on much longer time scales, are crucial to their development.

7.2. *Can we test censorship using Astronomical Observations?*

With so many high technology power missions to observe the cosmos, can we not just observe the skies carefully to determine the validity or otherwise of the cosmic censorship?

To answer this question, first of all we need once again to distinguish the two main candidates for naked singularities, which are very different in nature and which would bear very different observational signatures. The first candidate, that has been widely discussed above, is the naked singularity that results at the end of spherical collapse. When we are dealing with collapse of a star such a naked singularity will happen at the center of the collapsing cloud and it will most likely eventually be covered by the event horizon when collapse ends. Therefore the observational signature of such an event must be in the form of a short-lived explosive event. Similar phenomena are observed in the universe, for example gamma-ray bursts that are believed to originate from core collapse of massive stars. But a link between such events and the possible existence of a naked singularity at the core of the collapse has never been thoroughly investigated.

On the other side, we have scenarios such as the super-spinning Kerr solution, or some axially symmetric vacuum metrics, that are derived from exact solutions of

Einstein's field equations and which generally present naked singularities. We still do not know if such configurations could arise from a dynamical process such as collapse. Therefore it is legitimate to ask whether these metrics could represent some real object existing in the universe. The question of how these kind of objects could form from the collapse of an initially regular star is still unanswered and very little is known on the behaviour of a realistic source during the final stages of collapse. Therefore the fact that such exact solutions could or could not arise from the evolution of a regular matter source remains in the domain of speculation. Nevertheless recently scientists have turned their attention to these vacuum solutions in order to understand what observational properties they would have and in order to see if they could be detected by current observations.

The Kerr metric is the monopole solution for a source with angular momentum. In this case the key quantity that decides the nature of a compact object or the super massive central object at the center in a galaxy, is the ratio of its mass and the spin angular momentum per unit mass for the same, $a = J/M^2$. For a rotating object described by the Kerr metric, if the angular momentum to mass ratio is smaller than one, final state of collapse is a black hole, but for a larger ratio it is a naked singularity. In principle such a Kerr naked singularity could form from three different processes. The first one is the complete collapse of a rotating star with mass exceeding the neutron star limit. The possibility that collapse of such an object forms a superspinning Kerr has been suggested in Ref. 124 and investigated in some numerical simulations (see Ref. 84). The other possibilities by which a Kerr naked singularity can form are given by the inflow of angular momentum due to accreting particles and the merger of rapidly spinning compact objects (see section 4.7 for some references on numerical simulations of these processes).

A number of proposals to measure the mass and spin ratio for compact objects and for the galactic center have been made by different authors. Krolak (see Ref. 123), suggested using pulsar observations, gravitational waves and the spectra of X-rays binaries to test a for the center of our galaxy, Maeda [125], proposed using the shadow cast by the compact object to test a in stellar mass objects, Takahashi and others [126], have suggested using the X-ray energy spectrum emitted by the accretion disk, and finally Werner and Petters [127] suggested using certain observable properties of gravitational lensing that depend upon a.

So could such naked singularities form in some realistic physical processes? ¿From the measurement of X-ray binary systems there are indications that black holes with a mass to spin ratio very close to unity exist in the universe (see for example Ref. 128). Furthermore compact objects such as neutron stars with a greater or smaller than one can exist. Therefore, given the observational information that we have on the mass to angular momentum ratio for these compact objects, it is interesting to study how the parameter a which determines the separation of the black hole Kerr spacetime from the Kerr naked singularity, is affected when the object undergoes an inflow of mass and angular momentum due to an accretion disk.

The inflow of angular momentum due to accretion disks has been studied in order to understand if the particles falling onto a black hole can contribute to speed up the angular momentum, thus reaching the critical limit and removing the horizon. The idea of overspinning a black hole traces back to a thought experiment proposed by Wald [129], and it has important implications for cosmic censorship (for details see section 9).

¿From the astrophysical perspective these examples represent one possible way by which a Kerr naked singularity could be obtained. If this process proved to be physically viable then the investigation of observational properties of the Kerr naked singularities would become crucial for astrophysics. On the same line, the inverse procedure has also been investigated. This is the suggestion that the process of slowing down a Kerr naked singularity to a black hole, by means of infalling counter-rotating particles has a greater efficiency than the opposite process. [130]

Furthermore some studies have been carried out in order to understand how the process of angular momentum transfer works during the merger of two compact objects (such as neutron stars) with high spin. If the angular momentum is not dissipated away during the merger, these events could give rise to a final configuration in the form of a super-spinning Kerr naked singularity. Numerical simulations have shown that the 'sub-Kerr' models are more likely to be unstable and merge into a Kerr black hole than the corresponding super-spinning ones. [77]

Of course the Kerr metric is not the only vacuum exact solution with angular momentum. The study of exact solutions with higher multipole moments is connected with the issue of the no hair theorems and the formation of similar final configurations is speculative just as much as that of the Kerr solution (see Refs. 131, 132). There have been some attempts to study observational properties of metrics with higher multipole moments (see e.g. Ref. 133), but at present the fact that rotating compact objects in the universe are represented by the Kerr metric or by some more complicated axially symmetric spacetime remains unclear (see Ref. 134 for galactic nuclei and Ref. 135 for stellar mass objects). Nevertheless such metrics have received some attention in recent times and some attempt at answering the question whether they might bear an observational signature different from that of a Kerr metric has been made (see Ref. 136).

The basic issue here is that of sensitivity, namely how accurately and precisely can we measure and determine these parameters. A number of present and future astronomical missions could be of help. One of these is the Square-Kilometer Array (SKA) radio telescope, which will offer a possibility here, with a collecting area exceeding a factor of hundred compared to existing ones. The SKA astronomers point out they will have the sensitivity desired to measure the required quantities very precisely to determine the vital fundamental issues in gravitation physics such as the cosmic censorship, and to decide on its validity or otherwise.

Other missions that could in principle provide a huge amount of observational data are those that are currently hunting for the gravitational waves. Gravitational wave astronomy has yet to claim its first detection of waves, nevertheless in the coming years it is very likely that the first observations will be made by the experiments such as LIGO and VIRGO that are currently still below the threshold for observation. Then gravitational wave astronomy will become an active field with possibly large amounts of data to be checked against theoretical predictions and it appears almost certain that this will have a strong impact on open theoretical issues such as the Cosmic Censorship problem [137].

If we see objects with sufficiently high angular momentum compared to their mass, than the motivation for expecting the existence of an event horizon in collapse will reduce drastically, and we may be able to answer if naked singularities exist in the astrophysical reality. The prime survey targets here will include the innermost regions of our Galaxy, and globular clusters which are vast collections of oldest stars.

7.3. *Distinguishing black holes and naked singularities*

While gravitational collapse has been investigated extensively over past decades within the framework of gravitation theory, not much is known today about the observable signatures that would distinguish the black holes from the naked singularities, which are hypothetical astrophysical objects in nature predicted by the gravitation theory. We discussed and reviewed above first the existence, and then the genericity and stability of occurrence of naked singularities in a gravitational collapse.

The question of what observational signatures would then emerge and distinguish the black holes from naked singularities is then necessary to be investigated, and we must explore what special astrophysical consequences the latter may have.

With respect to the singularities that result at the end of a gravitational collapse, and which might have some observational signatures in the form of an explosive event, at present, we do not have the tools to say, even in principle, if they would look different from an explosive event originated from a collapse scenario where the central ultra-high dense region is trapped within an horizon.

One can argue that since gravity is attractive and nothing opposes the gravity at the very last stages of collapse, therefore nothing could in principle come out of the singular central region that develops at the end of collapse. Furthermore the core of the collapsing star would be so immensely dense that the mean free path of a particle trying to escape would be extremely short, thus making it unlikely that anything can escape from that region. On the other hand, we have seen that particle collisions close to the Cauchy horizon can happen under suitable conditions and that these can bear the signature of the singularity.

Furthermore at present we are not able to consider quantum effects that will appear at the core when the density reaches critical values. There are indications that these corrections will give rise to strong negative pressures that can disrupt collapse and create a shockwave that might propagate to the outer shells, thus dissipating away the mass of the star.

Finally, we have noted that simulations of core collapse supernovae so far have not been able to duplicate what happens at the last stages of collapse, and that in the computer simulation models some energy is missing in such a way that the explosion cannot in fact take place. Therefore it is possible, in principle, that this energy could be provided by some shockwave emanating from the central non-trapped ultra dense region, and that naked singularities would be in fact the fuel of type II supernovae.

At present this is a matter of speculation and if we wish to find a way to distinguish the black holes from naked singularities, it might help to consider axially symmetric naked singular solutions such as the Kerr spacetime. However, we do not know at present if they can actually be obtained from physical dynamical situations such as collapse, accretion, or merger.

Therefore, assuming that a Kerr naked singularity or some other object with a visible singularity exists in the universe, the question is: How can we distinguish it from a black hole or another compact object of the same mass?

In theory there are three different kinds of observations that one could devise in order to distinguish a naked singularity from a black hole. The first one relies on the study of accretion disks. It has been shown that the accretion properties of particles falling onto a naked singularity would be very different from those of black hole of the same mass (see for example Refs. 122, 138), and the resulting accretion disks would also be observationally different. In fact the properties of accretion disks have been studied in terms of the radiant energy, flux and luminosity, in a Kerr-like geometry with a naked singularity (see Ref. 121), and the differences from a black hole accretion disk have been investigated. Also, the presence of a naked singularity gives rise to powerful repulsive forces that create an outflow of particles from the accretion disk on the equatorial plane. This outflow that is otherwise not present in the black hole case, could be in principle distinguished from the jets of particles that are thought to be ejected from black hole's polar region and which are due to strong electromagnetic fields (see Ref. 139, 140). Furthermore, when charged test particles are considered the accretion disk's properties for the naked singularity present in the Reissner-Nordstrom spacetime have been shown to be observationally different from those of black holes (see Refs. 141, 142).

The second way of distinguishing black holes from naked singularities relies on gravitational lensing. It has been argued that when the spacetime does not possess a photon sphere, then the lensing features of light passing close to the singularity will be observationally different from those of a black hole (see Ref. 143). This method, however, does not appear to be very effective when a photon sphere is present in the spacetime. Assuming that a Kerr-like solution of Einstein equations with massless scalar field exists at the center of galaxies, its lensing properties were studied and it was found that there are effects due to the presence of both the rotation and scalar field that would affect the behavior of the bending angle of the light ray, thus making those objects observationally different from black holes (see Ref. 144).

Finally, a third way of distinguishing black holes from naked singularities comes from particle collisions and particle acceleration in the vicinity of the singularity. In fact, it is possible that the repulsive effects due to the singularity can deviate a class of infalling particles, making these outgoing eventually. These could then collide with some ingoing particle, and the energy of collision could be arbitrarily high, depending on the impact parameter of the outgoing particle with respect to the singularity. The net effect is thus creating a very high energy collision that resembles that of an immense particle accelerator and that would be impossible in the vicinity of a Kerr black hole. [145,146]

Black hole candidates from observations belong to two classes. There are stellar mass black holes that are thought to form either from collapse of very massive stars or from merger and accretion processes. In the cases when these candidates have an orbiting companion, we can measure their mass from the orbital periods of the companion. Similar measurement can be made when the black hole candidate is surrounded by a gas disk. Then there are supermassive black holes that are thought to exist at the center of galaxies and whose total mass, that can be estimated from the orbital period of nearby stars, is measured in millions of solar masses.

For both kinds of black hole candidates the angular momentum can be inferred from the analysis of the observed X-ray emission, though this measurement relies on some assumptions and it is not clear whether Kerr naked singularities could present a similar spectrum (see Ref. 126).

The theoretical models described above could then provide some framework where to test observations coming from stellar and supermassive black hole candidates. In fact, some observations in the millimeter wavelength of the supermassive object at the galactic center already suggests that the Kerr limit might be broken (see for example Ref. 147). ¿From investigations such as these, we see that naked singular spacetimes cannot be ruled out with the present knowledge that we have of these sources.

Of course, the Kerr naked singularity is not the only possible alternative to a black hole. As we mentioned before, naked singularities with non-vanishing higher multipole moments can be considered, but also other more exotic objects can provide some insight on the nature of these compact objects. In this connection, compact exotic objects such as boson stars or gravastars have been also considered (see Ref. 148 and references therein), though they could be unlikely candidates due either to the absence of evidence that the matter required to create such configurations can exist, or to the possibility that some dynamical instability against perturbations can arise for certain matter models, though these could be stable to perturbations for certain physical choices of parameters (see Ref. 149 and references therein for a further discussion). In general, the issues of such stabilities are complex and need much more careful attention.

Nevertheless, equilibrium configurations of regular matter sustained by pressures are predicted by general relativity, and these could also provide an alternative to model black hole candidates such as the galactic center, as we discuss next.

7.4. *Equilibrium configuration from collapse*

Here we will review a class of solutions of Einstein equations with naked singularities that arise asymptotically for suitable matter models when the pressures opposing collapse manage to halt the process, thus originating an equilibrium configuration (for details see Ref. 138).

This class of models describes how an initially regular matter cloud can evolve to form an equilibrium configuration with a finite radius slightly larger than the Schwarzschild radius and with a naked singularity at the center. This process can require arbitrarily long times for the equilibrium configuration to be finally achieved, and the average density of the cloud towards the final equilibrium can remain small, therefore suggesting that such a model could describe the formation of supermassive objects other than a black hole that could delve at the center of galaxies. Such an object, if it actually existed, would bear some different properties for the surrounding accretion disk as opposed to those of the accretion disk around a black hole, thus making it in principle observationally distinguishable.

As seen in section 4.2, the system of Einstein equations for a collapsing cloud can be reduced to solution of one first order partial differential equation acting as an equation of motion. We can then rewrite the equation in the form of an effective potential for any fixed shell r as,

$$V(r,v) = -\dot{v}^2 = -e^{2\nu}\left(\frac{M}{v} + \frac{G-1}{r^2}\right) . \tag{66}$$

As mentioned earlier, complete gravitational collapse is obtained when we require $\dot{v} < 0$ for all shells at all times. However, starting with a collapsing configuration, every shell will have in general three possible evolutions:

(1) $\dot{v} < 0$ at all times. Then the shell collapses to the singularity as described earlier.
(2) $\dot{v} = 0$ at a certain $\bar{t}$, while $\ddot{v} \neq 0$. In this case the shell labeled by r halts its collapse and bounces back re-expanding.
(3) $\dot{v} = \ddot{v} = 0$ at a certain $\bar{t}$. In this case collapse slows down until the shell reaches an equilibrium configuration.

The most general scenario for a collapsing object therefore has the inner core subject to continuous collapse while the outer shell can halt and bounce back (see Fig. 8).

It is easy to see that the parameters describing the matter content of the cloud can be suitably chosen in such a way that every shell stops collapsing, reaching an equilibrium state given by $\dot{v} = \ddot{v} = 0$ as in the third situation mentioned above. This is in fact the case when the effective potential $V(r,v)$ has an extremum at zero for every value of r. What is found is that in this case the cloud slows down and collapse halts giving rise to an equilibrium configuration that is described by an interior metric for the Schwarzschild spacetime. The final equilibrium configuration is given by $v_e(r)$ such that $V(r,v_e(r)) = 0$ (see Fig. 9).

¿From the usual analysis of dynamical systems close to equilibrium we can see that the time needed to stop collapse turns out to be infinite and therefore the

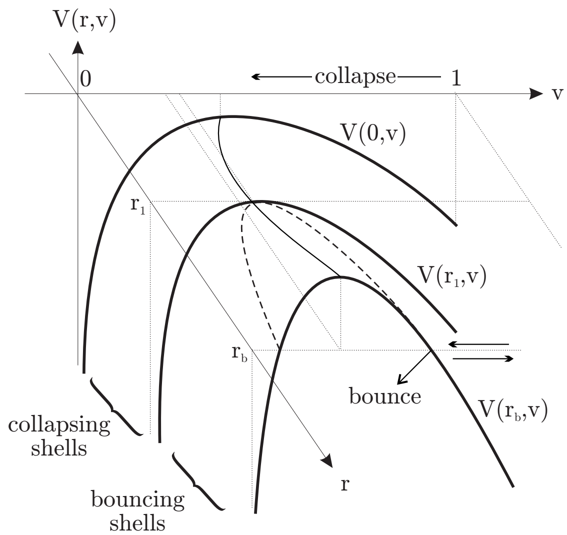

Fig. 8. The effective potential for a general matter cloud. The inner shells collapse to the final singularity which may be covered by an horizon or not. The outer shells halt and bounce back at a finite time. There is one limiting shell (r_1) for which collapse halts without bouncing back.

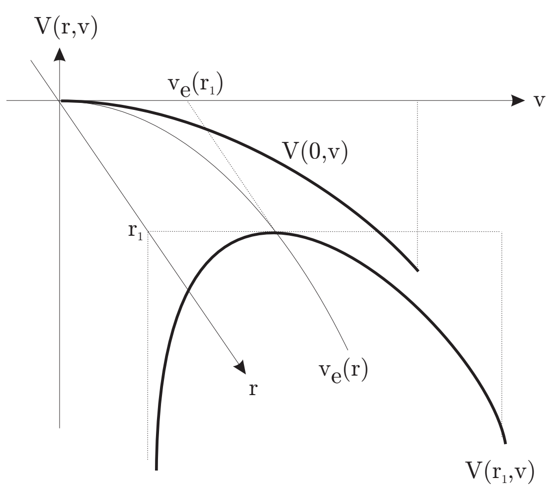

Fig. 9. The effective potential for a matter cloud originating an equilibrium configuration. All shells reach a stop where collapse halts (like for the limiting shell in the general case) and the cloud settles to a static configuration.

equilibrium configuration is achieved only asymptotically. This fact, together with the fine tuning necessary to halt the collapse for every shell, means that such a configuration must not be taken as a serious candidate for a realistic object. Nevertheless we note that any collapsing configuration arbitrarily close to the equilibrium solution will have a continuous collapse that is arbitrarily slow. In this sense the asymptotic limit of the equilibrium configuration represents an approximation of this physically valid, slowly evolving scenario. It is then interesting to study the properties of such static interiors that approximate a dynamical scenario. Since we know that collapse of the core of a massive star happens within a very short time, these models probably cannot be used to describe a similar situation. Nevertheless we know that immensely massive compact objects exist at the center of galaxies. These objects are long lived and compact enough to be possibly represented by such models.

Further, we note that as the equilibrium configuration is achieved from a matter distribution with regular initial data, we need not impose that the central shell be regular (as it is often done when looking for interior solutions for Schwarzschild). A singular central shell, where the density diverges, can also arise from such a collapse. In this case the central shell would represent a region of the compact object where quantum gravity takes hold and the observational features of such an object might be considerably different from those or a regular compact source or a black hole.

Models of these kind, where a collapsing configuration approaches a static limit, are possible for any kind of matter cloud with pressures. It is easy to see that within the framework of dust collapse there is no way for the particles to counteract the pull of gravity thus halting collapse, and therefore a static configuration is not possible. On the other hand if we include pressures, for example in the form of tangential pressures or perfect fluids, it is possible for such pressures to balance the attraction of gravity, thus generating a static configuration.

As an illustrative example we shall consider here the case of a cloud sustained only by tangential pressures. This choice will simplify considerably the structure of Einstein equations and it has its own interesting physical features. In fact a cloud composed of counter-rotating particles that move on circular orbits around the center is well approximated by the models with only tangential pressures. The choice of such a matter model simplifies considerably the field equations, since the amount of matter contained within any shell labeled by r is conserved, thus implying that $M = M(r)$. Furthermore from Einstein equations we know that we have the freedom to choose two free functions, namely the mass profile and the pressure profile (see section 4.2).

Therefore we can choose $M(r)$ and p_θ at equilibrium by imposing the equilibrium conditions,

$$V = V_{,v} = 0, \tag{67}$$

with

$$V_{,v} = e^{2\nu}\left(\frac{M}{v^2} - \frac{G_{,v}}{r^2}\right) - 2\nu_{,v}e^{2\nu}\left(\frac{M}{v} + \frac{G-1}{r^2}\right) . \tag{68}$$

In fact from these two equations, via Einstein equation (6) we get the two equations that fix the G and $G_{,v}$ at equilibrium in terms of the equilibrium solution $v_e(r)$ as,

$$G_e(r) = G(r, v_e(r)) = 1 - \frac{r^2 M(r)}{v_e(r)} , \tag{69}$$

$$(G_{,v})_e = G_{,v}(r, v_e(r)) = \frac{M(r) r^2}{v_e^2} , \tag{70}$$

Then from equations (4) and (5) evaluated at equilibrium, we obtain the energy density and tangential pressure at equilibrium as

$$\rho_e(r) = \frac{3M + rM'}{v_e^2(v_e + rv_e')} , \tag{71}$$

$$p_{\theta e}(r) = \frac{1}{4}\rho_e v_e \frac{(G_{,v})_e}{G_e} = \frac{1}{4}\frac{r^2 M(3M + rM')}{v_e^2(v_e + rv_e')(v_e - r^2 M)} . \tag{72}$$

On the other hand, once we have chosen $M(r)$ and $v_e(r)$, these two functions entirely determine the other quantities at equilibrium, namely $\rho_e(r), G_e(r), (G_{,v})_e(r)$ and $p_{\theta e}(r)$. Though we still have the freedom of two free functions during the evolution, we can see that one, namely the mass function $M(r)$, we have already chosen. Therefore, in order to achieve the desired final configuration, the class of allowed pressures $p_\theta(r, v)$ needs to be chosen in such a way that $p_\theta(r, v) \to p_{\theta e}(r)$ as $t \to \infty$, where the final pressure $p_{\theta e}(r)$ is determined by the choice of $v_e(r)$ as explained above. For this entire class of pressures the dynamical gravitational collapse will necessarily tend asymptotically to the static geometry specified by $M(r)$ and $v_e(r)$. As said the above equilibrium configuration need not be necessarily regular at the center. In fact since M is finite at $r = 0$ we see that whenever we have $v_e(0) = 0$, the energy density at equilibrium diverges at $r = 0$ and the final state presents a central singularity which has been obtained as the result of collapse from regular initial data.

Static interior solutions of the Schwarzschild spacetime sustained only by tangential pressures were studied in past [150], and the most general metric in this case is easily written. Therefore the class of equilibrium configurations described above must belong to this family of static metrics. The physical importance of these equilibrium configurations comes from the fact that for sufficiently large values of t the static models present an arbitrarily close approximation of the slowly collapsing cloud (see Fig. 10). Furthermore the ultra-high density region that develops at the center of the cloud is always visible. It is in this region that classical general relativity may eventually break down when densities high enough are reached. These densities are always obtained in a very large but finite time, before the actual mathematical singularity occurs, and it is the visibility of such regions during collapse which is the main reason for the study of the physical properties of such objects.

With the aim of studying the physical properties of these objects, we studied a specific equilibrium configuration given by $F(r) = M_0 r^3$ and $v_e(r) = cr^\alpha$, where, for the sake of clarity, we have imposed the scaling of the boundary in such a way that $c = 1$ and we have taken $\alpha > 0$. The case $\alpha = 0$ corresponds to a regular solution

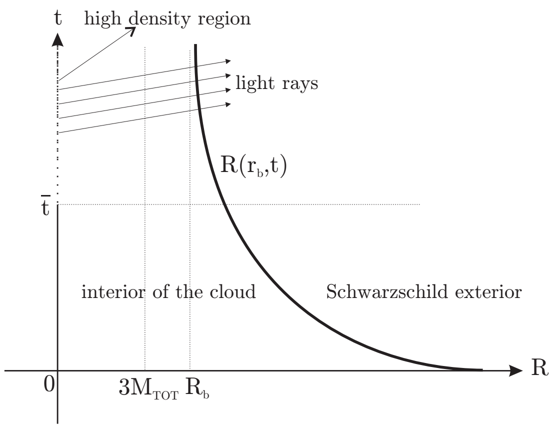

Fig. 10. The static configuration is achieved asymptotically and settles to a final radius greater than the photon sphere for the Schwarzschild black hole. The central density becomes arbitrarily large and approaches a singularity.

with positive constant density. Density and pressure at equilibrium become

$$\rho_e = \frac{3M_0}{(\alpha+1)}\frac{1}{R^{\frac{3\alpha}{\alpha+1}}} \ , \tag{73}$$

$$p_{\theta e} = \frac{3M_0^2}{4(\alpha+1)}\frac{R^{\frac{2-4\alpha}{\alpha+1}}}{\left(1-M_0R^{\frac{2-\alpha}{\alpha+1}}\right)} \ . \tag{74}$$

¿From the above we see that different values of α lead to different profiles for the pressure as $r \to 0$. In fact for $\alpha < 1/2$ we have $p_{\theta e} \to 0$, while for $\alpha > 1/2$ we have $p_{\theta e} \to +\infty$.

As an example we then considered the case $\alpha = 2$ and studied the motion of test particles in circular orbits in the equatorial plane and the properties of accretion disks. The energy density in this case is given by $\rho_e = M_0/R^2$ while the pressure satisfies a linear equation of state, $p_{\theta e} = k\rho_e$, with $4k = M_0/(1-M_0)$. ¿From the Misner-Sharp mass at the boundary we get the condition $2\mathbf{M_{TOT}}/R_b = M_0$ and the condition to avoid an event horizon is given by $M_0 < 1$. Furthermore, the weak energy condition holds when $k \geq -1$, (corresponding to $M_0 \leq 4/3$) and the effective sound speed c_θ, given by $c_\theta^2 = p_{\theta e}/\rho_e = k$, is less than unity if $M_0 < 4/5$.

Outgoing radial null geodesics in this spacetime are given by,

$$\frac{dR}{dt} = (1-M_0)\left(\frac{R}{R_b}\right)^{\frac{M_0}{2(1-M_0)}} \ . \tag{75}$$

and there are light rays escaping the singularity for all values of $M_0 < 2/3$ since the comoving time required by a photon to reach the boundary can be evaluated as $t_b = 2R_b/(2-3M_0) < +\infty$. Furthermore the Kretschmann scalar for this naked singularity model is given by

$$K = \frac{1}{4}\frac{M_0^2(28 - 60M_0 + 33M_0^2)}{(M_0 - 1)^2 R^4} \ . \tag{76}$$

thus indicating the presence of a strong central singularity.

Evaluating the two conserved quantities, we got energy per unit mass, $E = u_t = e^{2\nu}(dt/d\tau)$ and angular momentum per unit mass, $\ell = u_\phi = R^2(d\phi/d\tau)$. For circular orbits, we set $dR/d\tau = 0$, so we require the remaining terms in the above equation to add up to zero. In addition, their sum should achieve an extremum at radius R. This gives the two conditions

$$E^2 = \frac{2(1 - M_0)^2}{(2 - 3M_0)}\left(\frac{R}{R_b}\right)^{M_0/(1-M_0)} , \tag{77}$$

$$\frac{\ell^2}{R_b^2} = \frac{M_0}{(2 - 3M_0)}\left(\frac{R}{R_b}\right)^2 . \tag{78}$$

The normalization condition $u^\alpha u_\alpha = -1$ then gives

$$\frac{1}{G}\left(\frac{dR}{d\tau}\right)^2 - E^2 e^{-2\nu} + \left(1 + \frac{\ell^2}{R^2}\right) = 0 \ . \tag{79}$$

where R_b is the physical radius corresponding to the boundary of the matter cloud in the final equilibrium state. Note that both the energy per unit mass E and the angular momentum per unit mass ℓ vanish in the limit of $R = 0$, meaning that no mass or rotation is added to the central singularity by the accretion disk. The process of accretion does not affect the naked singularity, which can be considered stable in this respect. If we want the circular orbit to be stable, we must require that the term involving E^2 be less divergent as $R \to 0$ as compared to the term involving ℓ^2. This then gives the following results for $R < R_b$:

$$\text{Stable circular orbits :} \qquad M_0 \leq 2/3, \tag{80}$$

$$\text{Unstable circular orbits :} \qquad M_0 > 2/3. \tag{81}$$

Depending on the value of M_0, either all circular orbits in the interior of this object are stable, or all are unstable. Note that, apart from having unstable circular orbits, models with $M_0 > 2/3$ also give negative values of E^2 and ℓ^2.

It is known that a standard thin accretion disk can exist only at those radii where stable circular orbits are available. [151,152] Therefore for a Schwarzschild black hole of mass $\mathbf{M_{TOT}} = M_0 R_b/2$, an accretion disk will have its inner edge at $R = 6\mathbf{M_{TOT}}$, and inside this radius the gas plunges or free-falls.

Therefore depending on where we perform the matching with the exterior Schwarzschild solution we have two possible scenarios:

1- $M_0 \leq 1/3$: In this case, the external Schwarzschild metric has stable circular orbits all the way to the boundary $R = R_b$ where it meets the interior solution which again allows for stable circular orbits down to $R = 0$. The accretion disk will continue into the interior extending to $R = 0$, without an inner edge.

Assuming the matter cloud is transparent to radiation (as it is reasonable since in order to have an accretion disk the matter in the cloud must be weakly interacting), faraway observers will receive radiation coming from all radii until the center. The observed spectrum will obviously be very different from that of a disk surrounding a black hole of the same mass.

2- $1/3 < M_0 \leq 2/3$: In this case, an accretion disk will follow the Novikov-Thorne solution until $R = 6\mathbf{M_{TOT}}$ where the gas will plunge towards smaller radii. Once the gas reaches the boundary of the interior solution at $R = R_b$, circular orbits are once again allowed and the gas will shock and circularize to continue accreting on circular orbits all the way to $R = 0$. Again, since the accretion disk in this model consists of two distinct segments with a radial gap in between, we expect it to be observationally distinguishable from the previous case and from a black hole.

The interesting point is that these naked singularity models are easily distinguishable from a black hole of the same mass. Therefore it could be useful and interesting to use observational data on astrophysical black hole candidates to test if the presence of a similar naked singularity is possible.

8. Cosmic puzzles and the new perspective

We thus find from the detailed collapse calculations of recent years that the final fate of a collapsing star could be a naked singularity when the initial data is appropriate. Apart from its physical relevance, this violation of censorship also has profound philosophical aspects, such as the issue of predictability in the universe, the questions related to the nature and structure of singularities, and on possible validity or otherwise of classical gravity description in the vicinity of a naked singularity.

It is sometimes argued that breakdown of censorship means violation of predictability in spacetime, since the presence of naked singularities, together with geodesic incompleteness does not allow to uniquely predict the future evolution of the spacetime. Therefore we have no direct handle to know what a naked singularity may radiate and emit unless we study the physics in such ultra-dense regions. One would be able then to predict only partly but not fully the universe in the future of a given epoch of time. We consider some of these issues below.

A concern usually expressed is that if naked singularities occurred as the final fate of gravitational collapse, that would break the predictability in the spacetime, because the naked singularity is characterized by the existence of light rays and particles that emerge from the same. Typically, in all the collapse models discussed above, there is a family of future directed non-spacelike curves that reach external observers, and when extended in the past these meet the singularity. The first light ray that comes out from the singularity marks the boundary of the region that can be predicted from a regular initial Cauchy surface in the spacetime, and that is called the Cauchy horizon for the spacetime. The causal structure of the spacetime would differ significantly in the two cases, when there is a Cauchy horizon and when there is none. A typical gravitational collapse to a naked singularity, with the Cauchy horizon forming is shown in Fig 11.

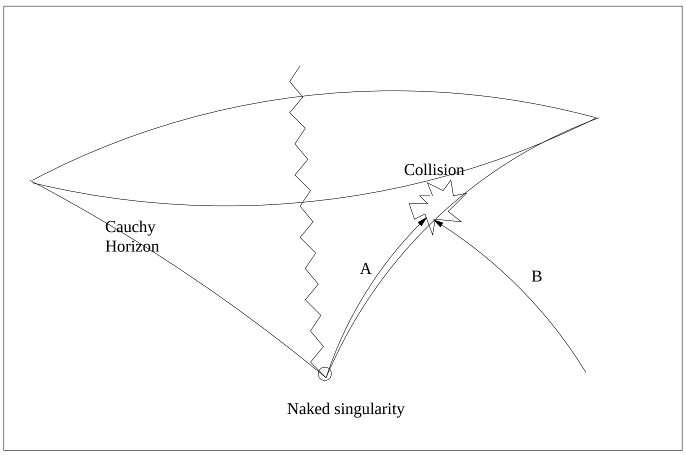

Fig. 11. The existence of a naked singularity is typically characterized by existence of a Cauchy horizon in the spacetime. Very high energy particle collisions can occur close to such a Cauchy horizon.

Here we would like to mention certain recent intriguing results in connection to the existence of a Cauchy horizon in a spacetime when there is a naked singularity resulting as final fate of collapse. Let us suppose the collapse resulted in a naked singularity. In that case, there are classes of models where there will be an outflow of energy and radiations of high velocity particles close to the Cauchy horizon, which is a null hypersurface in the spacetime. Such particles, when they collide with incoming particles, would give rise to a very high center of mass energy of collisions. The closer we are to the Cauchy horizon, the higher is the center of mass energy of collisions. In the limit of approach to the Cauchy horizon, these energies approach arbitrarily high values and could reach the Planck scale energies (see for example Refs. 108, 142, 145, 146).

The point here is, given a regular initial data on a spacelike hypersurface, one would like to predict the future and past evolutions in the spacetime for all times (see for example Ref. 3 for a discussion). Such a requirement is termed as the global hyperbolicity of the spacetime. A globally hyperbolic spacetime is a fully predictable universe, it admits a Cauchy surface, the data on which can be evolved for all times in the past as well as in future. Simple enough spacetimes such as the Minkowski or Schwarzschild are globally hyperbolic, but the Reissner-Nordstrom or Kerr geometries are not globally hyperbolic. For further details on these issues, we refer to Refs. 3, 12.

The key role that the event horizon of a black hole plays is that it hides the super-ultra-dense region formed in collapse from us. So the fact that we do not understand such regions has no effect on our ability to predict what happens in the universe at large. But if no such horizon exists, then the ultra-dense region might, in fact, play an important and even decisive role in the rest of the universe, and our ignorance of such regions would become of more than merely academic interest.

Yet such an unpredictability is common in general relativity, and not always directly related to censorship violation. Even black holes themselves need not fully respect predictability when they rotate or have some charge. For example, if we drop an electric charge into an uncharged black hole, the spacetime geometry radically changes and is no longer predictable from a regular initial epoch of time. A charged black hole admits a naked singularity which is visible to an observer within the horizon, and similar situation holds when the black hole is rotating. There is an important debate going on now for many years, if one could over-charge or over-rotate a black hole so that the singularity visible to observers within the horizon becomes visible to external far away observers too. We discuss this in some detail below. Another point is, if such a black hole was big enough on a cosmological scale, the observer within the horizon could survive in principle for millions of years happily without actually falling into the singularity, and would thus be able to observe the naked singularity for a long time. Thus, only purest of pure black holes with no charge or rotation at all respect the full predictability, and all other physically realistic ones with charge or rotation actually do not. As such, there are very many models of the universe in cosmology and relativity that are not totally predictable from a given spacelike hypersurface in the past. In these universes, the spacetime cannot be neatly separated into space and time foliation so as to allow initial data at a given moment of time to fully determine the future.

In our view, the real breakdown of predictability is the occurrence of spacetime singularity itself, which indicates the true limitation of the classical gravity theory. It does not matter really whether it is hidden within an event horizon or not. The real solution of the problem would then be the resolution of singularity itself, through either a quantum theory of gravity or in some way at the classical level itself. [153]

Actually, the cosmic censorship way to predictability, that of 'hiding the singularity within a black hole', and then thinking that we restored the spacetime predictability may not be the real solution, or at best it may be only a partial solution to the key issue of predictability in spacetime universes. In fact, it may be just shifting the problem elsewhere, and some of the current major paradoxes faced by the black hole physics such as the information paradox, the various puzzles regarding the nature of the Hawking radiation, and other issues could as well be a manifestation of the same.

Another issue is that censorship has been the foundation for the laws of black holes such as the area theorem and others, and their astrophysical applications. But these are not free of major paradoxes. First, all the matter entering a black hole must of necessity collapse into a spacetime singularity of infinite density and curvatures, where all known laws of physics break down. This was a reason why many gravitation theorists of 1940s and 1950s objected to black hole formation, and Einstein himself repeatedly argued against such a final fate of a collapsing star, writing a paper in 1939 to this effect. Second, as is well-known and has been widely discussed in the past few years, a black hole, by potentially destroying information, appears to contradict the basic principles of quantum theory. In that sense, the very formation of a black hole itself with a singularity within it appears to come laden with inherent problems. It is far from clear how one would resolve these basic troubles even if censorship were correct.

In view of such problems with the black hole paradigm, a possibility worth considering is the delay or avoidance of horizon formation as the star collapses under gravity. This happens when collapse to a naked singularity takes place, namely, where the horizon does not form early enough or is avoided. In such a case, if the star could radiate away most of its mass in the late stages of collapse, this may offer a way out of the black hole conundrum, while also resolving the singularity issue, because now there is no mass left to form the singularity. [108]

It has been observed recently that in the vicinity of the event horizon for an extreme Kerr black hole, if the test particles arrive with fine tuned velocities, they could undergo very high energy collisions with other incoming particles. In that case, as in the similar case with a naked singularity mentioned above, the possibility arises that one could see the Planck scale physics or ultra-high energy physics effects near the event horizon, given suitable circumstances (see for example Refs. 154–159).

What we mentioned above related to the particle collisions near Cauchy horizon is a similar scenario where the background geometry is that of a naked singularity. These results could mean that in strong gravity regimes, such as those of black holes or naked singularities developing in gravitational collapse, there may be a possibility of observation for ultra-high energy physics effects, which would be very difficult to see in near future in terrestrial laboratories.

While such a phenomena gives rise to the prospect of observing the Planck scale physics near the Cauchy horizon in the gravitational collapse framework, it also raises the following intriguing question. If extreme high energy collisions do take place very close to the null surface which is the Cauchy horizon, and if the idea of a singularity being the limit of the classical theory where quantum effects become relevant is valid then in a certain sense these collision models are essentially equivalent to creating a singularity at the Cauchy horizon itself. In that case, all or at least some of the Cauchy horizon would be converted into a spacetime singularity, and would effectively mark the end of the spacetime itself. In such a case, the spacetime manifold terminates at the Cauchy horizon, whenever a naked singularity is created in gravitational collapse. Since the Cauchy horizon marks in this case the boundary of the spacetime itself, the predictability is then restored for the spacetime, because the rest of the spacetime below and in the past of such a horizon is any way predictable before the Cauchy horizon formed.

There is another important aspect to the cosmic censorship problem that we discuss below now briefly. Let us suppose a black hole did form as the final end state of the gravitational collapse of a massive star. There is a constraint in this case for the horizon to survive, namely that the black hole must not contain too much of charge or should not spin too fast. In the case otherwise, the horizon cannot be sustained, it will breakdown and the singularity within will be visible.

Even if the black hole formed with a small enough charge and angular momentum to begin with, there is this key astrophysical process in its surroundings, namely that of accretion of matter around it which is the lot of debris and outer layers of the collapsing star. This matter around the black hole will as such fall into the same with great velocity, which could be classical or quantized, and with charge and angular momentum. Such in-falling particles could 'charge-up' or 'over-spin' the black hole, thus eliminating the event horizon. Thus, the very fundamental characteristic of a black hole, namely its trait of gobbling up the matter all around it and keep growing could become its own nemesis and a cause of its destruction.

Thus, even if a massive star collapsed into a black hole rather than a naked singularity, important issues remain such as the stability of the same to throwing in particles with charge or large angular momentum, and whether that can convert the black hole into a naked singularity by eliminating its event horizon. Many researchers have claimed this is possible, and gave models to create naked singularities this way. But there are others who claim there are physical effects which would save the black hole from over-spinning this way to destroy itself, and the issue is very much out to the jury. The point is, in general, stability of the event horizon and black holes continues to be an important issue for black holes formed in gravitational collapse. For a recent discussion on some of these issues, we refer to Refs. 160–167 and references therein.

The primary concern of the censorship hypothesis is of course formation of black holes only as collapse endstates, and their stability as above is a secondary issue. Therefore, what this means for cosmic censorship is, the collapsing massive star should not retain or carry too much of charge or spin, otherwise it will necessarily end up as a naked singularity, rather than a black hole final state.

It is clear that the black hole and naked singularity outcomes of a complete gravitational collapse for a massive star are very different from each other physically, and would have quite different observational signatures. In the naked singularity case, if it occurs in nature, we have the possibility to observe the physical effects happening in the vicinity of the ultra dense regions that form in the very final stages of collapse. However, in a black hole scenario, such regions are necessarily hidden within the event horizon of gravity. The fact that a slightest stress perturbation of the OSD collapse could change the collapse final outcome drastically, as we noted in section 4.4, changing it from black hole formation to a naked singularity, means that the naked singularity final state for a collapsing star must be studied very carefully to deduce its physical consequences, which are not well understood so far.

It is, however, widely believed that when we have a reasonable and complete quantum theory of gravity available, all spacetime singularities, whether naked or those hidden inside black holes, will be resolved away. As of now, it remains an open question if the quantum gravity will remove naked singularities. After all, the occurrence of spacetime singularities could be a purely classical phenomenon, and whether they are naked or covered should not be relevant, because quantum gravity will possibly remove them all any way. But one may argue that looking at the problem this way is missing the real issue. It is possible that in a suitable quantum gravity theory the singularities will be smeared out, though this has been not realized so far. Also there are indications that in quantum gravity also the singularities may not after all go away.

In any case, the important and real issue is, whether the extreme strong gravity regions formed due to gravitational collapse are visible to faraway observers or not. It is quite clear that the gravitational collapse would certainly proceed classically, at least till the quantum gravity starts governing and dominating the dynamical evolution at the scales of the order of the Planck length, *i.e.* till the extreme gravity configurations have been already developed due to collapse. The key point is, it is the visibility or otherwise of such ultra-dense regions that is under discussion, whether they are classical or quantum (see Fig 12).

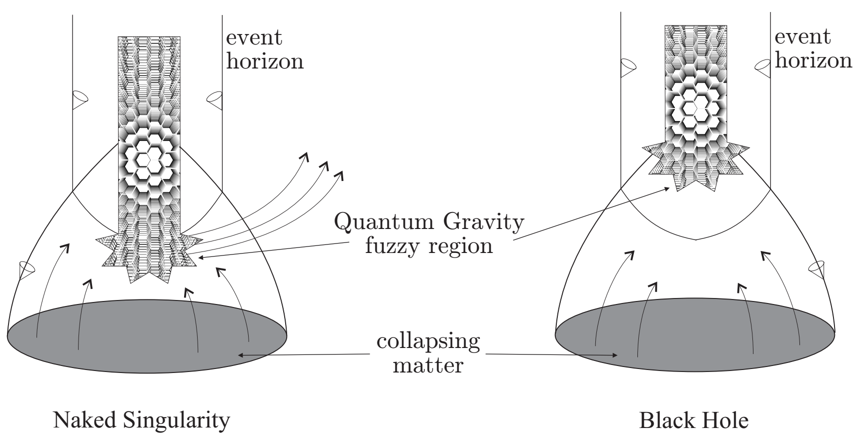

Fig. 12. Even if the naked singularity is resolved by the quantum gravity effects, the ultra-strong gravity region that developed in gravitational collapse will still be visible to external observers in the universe.

What is important is, classical gravity implies necessarily the existence of ultra-strong gravity regions, where both classical and quantum gravity come into their own. In fact, if naked singularities do develop in gravitational collapse, then in a literal sense we come face-to-face with the laws of quantum gravity, whenever such an event occurs in the universe. [114]

In this way, the gravitational collapse phenomenon has the potential to provide us with a possibility of actually testing the laws of quantum gravity. In the case of a black hole developing in the collapse of a finite sized object such as a massive star, such strong gravity regions are necessarily hidden behind an event horizon of gravity, and this would be well before the physical conditions became extreme near the spacetime singularity. In that case, the quantum effects, even if they caused qualitative changes closer to singularity, will be of no physical consequences. This is because no causal communications are then allowed from such regions. On the other hand, if the causal structure were that of a naked singularity, then the communications from such a quantum gravity dominated extreme curvature ball would be visible in principle. This will be so either through direct physical processes near a strong curvature naked singularity, or via the secondary effects, such as the shocks produced in the surrounding medium.

9. Star collapse: A Lab for quantum gravity?

At present, we have no mechanism or a complete theory to deal with both quantum effects and the intense force of gravity together. However, a quantum gravity theory should take over from the purely classical theory that is general relativity, in the very advanced stages of a gravitational collapse, when densities and spacetime curvatures assume extreme values. It is possible that a spacetime singularity basically represents the incompleteness of the classical theory and when quantum effects are combined with the gravitational force, the classical singularity may be resolved.

Therefore, more than the existence of a naked singularity, the important physical issue then is whether the extreme gravity regions formed in the gravitational collapse of a massive star are visible to external observers in the universe. An affirmative answer here would mean that such a collapse provides a good laboratory to study quantum gravity effects in the cosmos, which may possibly generate clues for an as yet unknown theory of quantum gravity. Quantum gravity theories in the making, such as the string theory or loop quantum gravity in fact are badly in need of some kind of an observational input, without which it is nearly impossible to constrain the plethora of possibilities.

We could say quite realistically that a laboratory similar to that provided by the early universe is created in the collapse of a massive star. However, the big bang, which is also a naked singularity in that it is in principle visible to all observers, happened only once in the life of the universe and is therefore a unique event. But a naked singularity of gravitational collapse could offer an opportunity to explore and observe the quantum gravity effects every time a massive star in the universe ends its life.

The important questions one could ask are: If in realistic astrophysical situations the star terminates as a naked singularity, would there be any observable consequences which reflect the quantum gravity signatures in the ultra-strong gravity region? Do naked singularities have physical properties different from those of a black hole? Such questions underlie our study of gravitational collapse.

¿From this perspective, the recent results on collapse indicate certain exciting implications for astrophysics, fundamental physics and quantum gravity. Consider the scenario when a collapsing star terminates into a naked singularity, making the ultra-strong super gravity region visible to external observers. In this context, we considered a cloud that collapsed to a naked singularity final state, and introduced loop quantum gravity effects. [38] It turned out that the quantum effects generated an extremely powerful repulsive force within the cloud. Classically the cloud would have terminated into a naked singularity, but quantum effects caused a burstlike emission of matter in the very last phases of collapse, thus dispersing the star and dissolving the naked singularity. The density remained finite and the spacetime singularity was eventually avoided. One could expect this to be a fundamental feature of other quantum gravity theories as well, but more work would be required to confirm such a conjecture.

For a realistic star, its final catastrophic collapse takes place in matter of seconds. A star that lived millions of years thus collapses in only tens of seconds. In the very last fraction of a microsecond, almost a quarter of its total mass must be emitted due to quantum effects, and therefore this would appear like a massive, abrupt burst to an external observer far away. Typically, such a burst will also carry with it specific signatures of quantum effects taking place in such ultra-dense regions. In our case, these included a sudden dip in the intensity of emission just before the final burstlike evaporation due to quantum gravity.

The question is, whether such unique astrophysical signatures can be detected by modern experiments, and if so, what they tell on quantum gravity, and if there are any new insights into other aspects of cosmology and fundamental theories such as string theory.

The stage at which the burst occurs in the gravitational collapse and the energy which would be emitted depend on a quantization parameter in the theory. For all theoretically favored choices of this parameter, the energy emitted in the radiation would be extreme. The emission mechanism of the burst has to be further investigated. The constituents of the burst may include extreme energy gamma-rays, cosmic rays, and neutrinos which raise an interesting possibility to use upcoming experiments such as Extreme Universe Space Observatory (EUSO) which may have needed sensitivity, and which is expected to be operational soon, to provide us test of this prediction. If tested these experiments would provide a proof of quantum gravity. Future astronomical experiments could then constrain the parameters in quantum gravity in the same way as particle accelerators at CERN and Fermilab constrain the parameters for the Standard Model.

Interestingly, these experiments which may constrain the value of the quantization parameter in the theory would also have consequences for cosmology, because loop quantum cosmology on which our work is based also changes the picture of cosmological dynamics in the very early Universe. For example, it can change the way inflation occurs and it has been shown that such a change has observable signatures in the Cosmic Microwave Background Radiation (CMBR), which is the suppression of power at large scales in CMBR as observed by the Wilkinson Microwave Anisotropy Probe (WMAP). Observable effects of quantum gravity in CMBR are also controlled by the same quantization parameter which determines the details of energy emission in the burst we considered, and the way a dying star would dim before a burst occurs. Thus any constraints on the quantization parameters would have direct consequence for cosmology as well as astrophysics.

The key point is, because the very final ultra-dense regions of the star are no longer hidden within a horizon as in the black hole case, the exciting possibility of observing these quantum effects arises now, independently of the quantum gravity theory used. An astrophysical connection to extreme high energy phenomena in the universe, such as the gamma-rays bursts that defy any explanations so far, may not be ruled out. Some researchers have also examined the possible generation of gravity waves from such ultra-strong gravity regions. [168]

Such a resolution of naked singularity through quantum gravity would be a philosophically satisfying possibility. Then, whenever a massive star undergoes a gravitational collapse, this might create a laboratory for quantum gravity in the form of a *Quantum Star* [169], that we may be able to possibly access. This would also suggest intriguing connections to high energy astrophysical phenomena. The present situation poses one of the most interesting challenges which have emerged through the recent work on gravitational collapse.

10. Concluding remarks

We hope the considerations here have shown that gravitational collapse, which essentially is the investigation of dynamical evolutions of matter fields under the force of gravity in the spacetime, provides one of the most exciting research frontiers in gravitation physics and high energy astrophysics.

There are issues here which have deep relevance both for theory as well as observational aspects in astrophysics and cosmology. Also these problems are of relevance for basics of gravitation theory and quantum gravity, and these inspire a philosophical interest and inquiry into the nature and structure of spacetime, causality, and predictability in the universe.

Active research is already happening in many of these areas as the discussion here has pointed out. Some of the most interesting areas from our own personal perspective are: Genericity and stability of collapse outcomes, examining the quantum gravity effects near singularities, observational and astrophysical signatures of the collapse outcomes, and several other related issues.

In particular, one of the interesting and important questions would be, if naked singularities which are hypothetical astrophysical objects, did actually form in nature, what distinct observational signatures would these present. In other words, how one would distinguish the black holes from naked singularities would be an important issue. There have already been some developments and efforts on this issue in recent years as we have indicated above. The point here is, there are already very high energy astrophysical phenomena being observed today, with several observational missions working both from ground and space. The black holes and naked singularities, which are logical consequences of the general theory of relativity as we consider the gravitational collapse of a massive star, would appear to be the leading candidates to explain these phenomena. Thus the observational signatures that each of these would present, and their astrophysical consequences would be naturally of much interest for the future theoretical as well as computational research, and for their applications.

In our view, there is a scope therefore for both theoretical as well as numerical investigations in these frontier areas, which may have much to tell for our quest on basic issues in quantum gravity, fundamental physics and gravity theories, and towards the expanding frontiers of modern high energy astrophysical observations.

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