

# Gravitational collapse with tangential pressure

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Using the general formalism for spherical gravitational collapse developed in [1], we investigate here the final fate of a spherical distribution of a matter cloud, where radial pressures vanish but tangential pressures are non-zero. Within this framework, firstly we examine the effect of introducing a generic small pressure in a well-known black hole formation process, which is that of an otherwise pressure-free dust cloud. The intriguing result we find is that a dust collapse that was going to a black hole final state could now go to a naked singularity final configuration, when arbitrarily small tangential pressures are introduced. The implications of such a scenario are discussed in some detail. Secondly, the approach here allows us to generalize the earlier results obtained on gravitational collapse with non-zero tangential pressure, in the presence of a non-zero cosmological constant. Finally, we discuss the genericity of black hole and naked singularity formation in collapse with non-zero tangential pressure. The treatment here gives a unified and complete picture on collapse final states, in terms of black hole and naked singularity formation, generalizing the earlier results obtained for this class of collapse models. Thus the role of tangential stresses towards determining collapse endstates emerges in a straightforward and transparent manner in our treatment.

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## I. INTRODUCTION

Since the first proposal of the Cosmic Censorship Conjecture (CCC) [2], a great effort has been devoted in understanding the mechanism by which black holes and naked singularities can form as the end state of the complete gravitational collapse of a massive object. However, no general proof, or even a suitable mathematically rigorous formulation of the CCC is available despite the efforts of many decades. Therefore it has become essential and inevitable that gravitational collapse within the framework of general relativity be studied in detail.

The current situation is that, despite the great amount of efforts devoted in the past years in studying the CCC and its implications, the issue of final fate of complete gravitational collapse of a massive body is far from being entirely resolved. It is clear now how naked singularities have to be considered as a general feature of general relativistic physics, and that they may appear as the end-state of collapse in a broad variety of situations (see e.g. [3] and references therein). Still, it is not yet fully understood how different kinds of matter lead to the formation of either a black hole or a naked singularity. To give an example, how the presence of pressure in the form of tangential stresses and radial stresses might direct the collapse towards the formation of black holes or naked singularities is still a matter of debate.

The first gravitational collapse scenario studied in detail within the framework of the general theory of relativity was the so called Oppenheimer-Snyder-Datt model [4], that represents a collapsing ball made of homogeneous pressureless dust. It is seen here that an event horizon forms before the appearance of the final spacetime singularity at the end of collapse, therefore hiding the singularity inside a black hole.

Further studies of inhomogeneous dust collapse though, such as those in the Lemaitre-Tolman-Bondi (LTB) models [5], showed that naked singularities can indeed form during collapse, thus violating the cosmic censorship [6]. However, the dust models are idealized scenarios that ignore pressures within the cloud. Therefore, collapse models with pressures, which are more physically realistic scenarios, were examined and it was soon discovered that for the collapse with non-zero pressures also, the end state is just as likely to be a naked singularity, as for models without pressures (see e.g. [7]).

In particular, collapse models with a vanishing radial pressure, but with non-vanishing tangential stresses, have been studied in some detail [8]. These models are of particular interest, because they provide a somewhat clear insight into the role of pressure towards determining the collapse final states.

Using the formalism developed in [1] to study here the class of collapse models with a non-zero tangential pressures, three main results are obtained here. Firstly, we examine the effect of introducing small tangential pressure perturbations, on the black hole formation process, for a collapsing inhomogeneous dust cloud which is otherwise pressure-free. It is seen that a matter cloud that was collapsing to a black hole final state could change its course, and instead result in a naked singularity final configuration, when small tangential pressure perturbations are allowed in an otherwise pressure-free cloud. It is seen that the opposite scenario occurs as well. Secondly, we generalize here earlier results on gravitational collapse with pressure, in the presence of a non-zero cosmological constant. Finally, from the analysis of some collapse models, some interesting insights can be obtained on the genericity or otherwise of naked singularity formation in gravitational collapse, in the presence of a non-zero tangential pressure. Using the approach developed here, it is clearly seen how the black hole and naked singularity final states in collapse are distinguished, depending on the initial configurations of the matter density, the tangential stresses profiles and the velocities of the collapsing shells. Given this initial data, the Einstein equations

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then fully determine the allowed evolutions that will lead to the formation either of a black hole or a naked singularity.

Various subcases of this class of collapse models have been studied in the past by different authors, as we point out below, but our treatment here gives a unified and complete picture on collapse final states in terms of black hole and naked singularity formation, thus generalizing the earlier results. Thus the role of tangential stresses towards determining the final state of a collapsing matter cloud emerges transparently in our treatment. This helps us to see in a collective manner the different results in this class of collapse models, which are otherwise somewhat scattered in a variety of approaches that do not always make it straightforward to see the overall conclusions in totality. Within this framework, the known results are easily recovered, and new scenarios and insights are obtained.

In section II we review as needed the general formalism for gravitational collapse in the presence of pressures, as restricted to the case of non-vanishing tangential pressure and vanishing radial pressure. The set of Einstein equations that governs the dynamical evolution of the collapsing cloud and the procedure to solve them is outlined. The regularity and physical reasonableness conditions that one would like to impose upon the matter cloud, in order to avoid any unphysical behaviour of the matter fields are described. The initial and boundary conditions that determine the dynamical evolution of the matter cloud are also discussed. Section III discusses the evolution of collapse, a function related to the tangent to the outgoing null geodesics at the singularity is evaluated, and it is the sign of this quantity that governs the final outcome of collapse in terms of either a black hole or naked singularity. The occurrence of trapped surfaces during collapse is discussed, and the visibility of the singularity is studied through the analysis of outgoing radial null geodesics.

The formalism is then applied in section IV to examine the effect of introducing small tangential pressures on the black hole formation process, for an otherwise pressure-free inhomogeneous dust cloud collapse. In the section V, we discuss the genericity of black holes and naked singularities obtained as collapse endstates, for this class of collapse models. The homogeneous and inhomogeneous dust cases are also briefly discussed, giving a review of known results for dust, obtained as a special case of the above treatment. In section VI, we generalize some earlier results on collapse with tangential pressure in the presence of a cosmological constant, which is asymptotically de-Sitter or anti-de Sitter spacetime. We study how the presence of a cosmological term in the equations affects the dynamical behaviour of the cloud. The study is then applied to the classes studied in section IV, by adding a cosmological term and analyzing how the dynamics is altered. This constitutes a new class of solutions whose behaviour was not studied earlier. Finally, in section VII, we outline the interesting features of the above treatment in connection with previous results obtained by other authors.

## II. EINSTEIN EQUATIONS AND REGULARITY CONDITIONS

The general treatment to obtain the evolution and final fate of a spherical gravitational collapse with non-zero pressures was developed in [1], and it constitutes the backbone upon which the present work is developed. One of the advantages of this approach is that it shows the final stages of collapse in a rigorous and yet transparent fashion, where all the crucial elements that enter towards the determination of the final outcome are enclosed in a single straightforward equation. It is clear that in order for the singularity to be visible, some outgoing nonspacelike geodesics must come out of it to reach the faraway observers in the universe. This visibility of outgoing geodesics is shown to be related to the tangent to the null geodesics near the singularity, which in turn is shown to be dependent on the mass distribution, the pressure and the velocity profiles for the collapsing cloud. Once these quantities are set at the initial time, the evolution, as governed by the Einstein equations, determines entirely the final outcome of collapse.

In spherical coordinates, the most general line element for a spherically symmetric collapsing cloud of matter depends upon three functions  $\nu$ ,  $\psi$  and  $R$  of the comoving radial coordinate  $r$  and the comoving time  $t$ , and it can be written as,

$$ds^2 = -e^{2\nu(t,r)} dt^2 + e^{2\psi(t,r)} dr^2 + R(t,r)^2 d\Omega^2 . \quad (\text{II.1})$$

We shall assume here through out that the radial pressure vanishes. The energy-momentum tensor, expressed in co-moving coordinates then takes the form,

$$T_t^t = -\rho; T_r^r = 0; T_\theta^\theta = T_\phi^\phi = p_\theta . \quad (\text{II.2})$$

The Einstein equations then relate the three functions  $\nu$ ,  $\psi$ ,  $R$  to the energy density and pressures of the system, and they can then be written as,

$$p_r = -\frac{\dot{F}}{R^2 \dot{R}} = 0 , \quad (\text{II.3})$$

$$\rho = \frac{F'}{R^2 R'} , \quad (\text{II.4})$$

$$\nu' = \frac{2p_\theta R'}{\rho R} , \quad (\text{II.5})$$

$$2\dot{R}' = R' \frac{\dot{G}}{G} + \dot{R} \frac{H'}{H} , \quad (\text{II.6})$$

$$\frac{F}{R} = 1 - G + H , \quad (\text{II.7})$$

where the function  $F$  is the Misner-Sharp mass of the system and we have introduced the functions  $H(r, t)$  and  $G(r, t)$ , which are defined as,

$$H = e^{-2\nu(r,v)} \dot{R}^2 , \quad (\text{II.8})$$

$$G = e^{-2\psi(r,v)} R'^2 . \quad (\text{II.9})$$

Since there is a scaling gauge freedom available for the radial coordinate  $r$ , we can define without any loss of generality a scale so that at the initial time we have  $R(r, t_i) = r$ . To this purpose, we introduce the scaling function  $v(r, t)$  such that,

$$v(r, t_i) = 1 .$$

The gravitational collapse condition is given here by  $\dot{R} < 0$ . The area function  $R(r, t)$  can then be expressed as,

$$R = rv \quad (\text{II.10})$$

and the requirement of collapse is fulfilled once  $\dot{v} < 0$ .

The singularity is then achieved by reaching  $v = 0$  (which corresponds to  $\dot{R} = 0$ ), and it is realized along the singularity time curve  $t_s(r)$ . Note that as we approach the regular center of coordinates  $r = 0$ , when  $v \neq 0$ , the energy density remains bounded, although the physical radius  $R$  goes to zero. Therefore, this indicates that the genuine curvature singularity, where the physical radii of all the matter shells reach a vanishing value and the energy density diverges, is reached only in the case when  $v = 0$ . This is reflected in the scaling function by the requirement that  $v(r, t_s(r)) = 0$  at the singularity. To avoid the presence of shell-crossing singularities (that can be shown to be gravitationally weak and thus possibly removable [9]), we further assume that,

$$R'(r, t) > 0. \quad (\text{II.11})$$

The shell focusing singularity at  $v = 0$  is a true curvature singularity, and therefore is not removable from the spacetime, and its visibility or otherwise is the main issue that concerns us here.

From equation (II.3), it follows that the Misner-Sharp mass  $F$  in this case does not depend on time ( $\dot{F} = 0$ ), and therefore it must be a function of  $r$  only. We are then left with four equations in the six unknown  $\rho(r, t)$ ,  $p_\theta(r, t)$ ,  $R(r, t)$ ,  $F(r)$ ,  $G(r, t)$ ,  $H(r, t)$ .

Collapse models with such a vanishing radial pressure, but non-zero tangential stresses were studied in the mass-area coordinates by Magli [10], where the analytic structure of these solutions was derived. The algebraic equation which governs the nature of the central singularity was written and its dependence upon the initial configuration and the equation of state for the matter field was analyzed. Still, given the non-trivial nature of the problem, the task to study the outcome of collapse in full generality proved to be non-attainable. Further work by Gonçalves, Jhingan and Magli [11] studied the final stages of collapse in the limited case of marginally bound collapse while the analysis of one of us [8] concentrated on models with some restrictions on the pressure profiles (namely, assuming for the metric function  $\nu = \nu(R)$ ). Also, the work by Harada, Nakao and Iguchi [12] analyzed the nakedness and strength of the central singularity obtaining some general conditions which were later specialized to the dust and ‘Einstein cluster’ scenarios. In this work the nakedness and curvature strength of the shell focusing singularity was studied in detail, providing a classification of the curvature strength of the singularity based on some conditions (namely, the strong curvature condition, gravity dominance condition, and the limiting focusing condition).

Another interesting model with vanishing radial stresses that has been studied is the so called ‘Einstein cluster’. It is a specific subclass of models with tangential stresses that is constituted by a collapsing cloud of counter rotating particles. Since the tangential stress is due to the particles’ angular

momentum, the ‘Einstein cluster’ is a particularly interesting model, because it reproduces the effects of rotation without departing from spherical symmetry. The analysis of the singularity forming in the ‘Einstein cluster’ [13] shows once more how departure from dust models to include pressures does not forbid the formation of naked singularities.

With the definitions (II.10), (II.8), we substitute the unknowns  $R, H$  with  $v, \nu$ . The definition of  $F(r)$  provides one of the two free functions and determines the energy density via equation (II.4). We can choose  $\nu$  as the second free function and therefore evaluate  $p_\theta$  through equation (II.5), which can be written in the form,

$$p_\theta = \frac{1}{2} \rho R \frac{\nu'}{R}. \quad (\text{II.12})$$

Without any loss of generality, we can obtain  $G$  from (II.6) once we rewrite it as,

$$\dot{G} = 2\nu' \frac{\dot{R}}{R'} G \quad (\text{II.13})$$

and we define a suitable function  $A(r, v)$  from

$$A_{,v}(r, v) = \nu' \frac{r}{R}. \quad (\text{II.14})$$

Then equation (II.13) can be integrated to obtain,

$$G(r, t) = b(r) e^{2A(r, v)}. \quad (\text{II.15})$$

The arbitrary function  $b(r)$  above that results from the integration can be interpreted following the analogy with dust LTB models. Then  $b(r)$  turns out to be related to the velocity of the collapsing shells (more precisely to the velocity of the infalling particles at the boundary of the cloud), and once we write it as

$$b(r) = 1 + r^2 b_0(r), \quad (\text{II.16})$$

we can see that the values of  $b_0 = \text{const.}$  in the dust limit correspond to the bound ( $b_0 < 0$ ), unbound ( $b_0 > 0$ ) or marginally bound ( $b_0 = 0$ ) Lemaitre-Tolman-Bondi models.

Writing equation (II.7) in the form of a potential,

$$\dot{R}^2 = e^{2\nu} \left( \frac{F}{R} + G - 1 \right) = U(r, t), \quad (\text{II.17})$$

it is possible to analyze the dynamical behaviour of different pressure models with the usual phase space diagrams of classical mechanics. Some examples will be discussed in section VI. It turns out that depending on the velocity profile and the equation describing the tangential stresses, collapsing scenarios and bouncing scenarios are both allowed [10]. This marks a considerable difference with the LTB dust case where the potential  $U(r, t)$  is always decreasing, thus allowing only collapse to occur.

Finally, from equation (II.7) we get the equation of motion:

$$\dot{v} = -\frac{e^\nu}{r} \sqrt{\frac{F}{R} + G - 1}, \quad (\text{II.18})$$

where the negative sign has been considered in order to describe gravitational collapse. In some cases equation (II.18) can be integrated to obtain the function  $v(r, t)$ , thus solving completely the system of Einstein equations. In other situations, considerations on the behaviour of  $v(r, t)$  near the center of the collapsing cloud suffice to provide the analysis of the dynamical behaviour of the system approaching the singularity.

A viable physical description of the gravitational collapse of a massive body must be subdued to certain constraints and regularity conditions in order to ensure that the physical quantities that appear in the model are well defined and regular everywhere.

Requiring regularity of the energy density at the initial surface from which the collapse commences, and at the center of the cloud throughout collapse, imposes some constraints on the classes of mass profiles that can be considered. Specifically, the mass profile  $F(r)$  must have the form,

$$F(r) = r^3 M(r), \quad (\text{II.19})$$

which immediately implies the regularity of the density function at the center of the cloud at all regular epochs. By this prescription, we ensure that the Misner-Sharp mass  $F(r)$  vanishes at the center of the cloud and that the energy density  $\rho$  away from the singularity has a finite value at the center. A further condition that may be imposed sometimes is that the energy density does not present a cusp at  $r = 0$ , and that it is a smooth function of the radial coordinate  $r$ . This can be achieved by requiring that the linear term of the energy density expansion near the center vanishes, which is reflected in the requirement that  $M'(r) = 0$  at  $r = 0$ . Of course, this is an additional requirement and in a general physically realistic scenario, this may or may not be realized. In any case, requiring smoothness brings in some additional mathematical simplification.

The requirement that there is no pressure gradient at  $r = 0$  implies that the tangential pressure must become equal to the radial pressure (i.e. zero in our case) at the center. This imposes the condition that  $p_\theta$  goes as  $r$  at least, near the center, and implies regularity for the metric function  $\nu$  via equation (II.5). This condition then implies that we have in general,

$$\nu(r, v) = r^2 g(r, v), \quad (\text{II.20})$$

with  $g$  being a regular function and  $\nu(r, t) = \nu(r, v(r, t))$ . Regularity of  $\nu$  implies regularity of the function  $A$  via equation (II.14), and we have,

$$A(r, v) = r^2 a(r, v), \quad (\text{II.21})$$

with  $a$  being again a regular function. The relation between  $a$  and  $g$  is given by the equation (II.14), which can be written as

$$a_{,v} = \frac{2g + rg'}{R'}. \quad (\text{II.22})$$

As we have seen before, a specific behavior of  $R$  is required in order to avoid shell crossing singularities. The choice of

$R$  satisfying  $R' > 0$  implies further regularity for the energy density. This in turn gives some constraints on the non-vanishing terms in the expansion of the energy density at a constant time surface. Namely, we find that in order to avoid the shell crossing singularities, we must require  $M' \leq 0$ , which, from a physical point of view, is a reasonable condition since we expect the density to be higher at the center of the cloud, and decreasing as we move away from the center. In the case when  $M'(0) = 0$ ,  $\rho$  turns out to be a decreasing function from the center outwards if  $M''(0) < 0$ .

As noted, from the regularity of  $\dot{v}$  at the center of the cloud, we get the behaviour of  $b$  as in equation (II.16), and this turns out to be related to the velocity profile, and thus the kinetic energy, of the infalling particles. Positivity of  $G$  is thus reflected in the condition that

$$b_0 \geq -\frac{1}{r^2}. \quad (\text{II.23})$$

Furthermore, the requirement that  $R' > 0$  implies that moving away from the center, the contribution of the kinetic energy to the total energy of the collapsing cloud decreases [9]. Therefore, the requirement of avoidance of shell crossing implies that some further condition must be imposed upon  $b_0(r)$ . In general, from the Misner-Sharp mass we get

$$F'(1 - G + H) > F(H' - G'), \quad (\text{II.24})$$

which, in the limit of the dust case, becomes,

$$b_0 M' > M b_0'. \quad (\text{II.25})$$

The marginally bound case ( $b_0 = 0$ ) is obviously allowed, and we see that considering only the constant values for the function  $b_0(r)$  (which correspond in the dust case to the bound and unbound LTB models), the bound case, i.e.  $b_0 < 0$  might imply the occurrence of shell crossing if  $M' > 0$ . The bound case is therefore allowed under the restrictions imposed as above. We note from equation (II.23) that the permitted values for  $b_0$  have a limited range, namely  $b_0 \in (-\frac{1}{r_b^2}, 0)$ , where  $r_b$  is the boundary of the cloud (an infinite cloud cannot therefore have a bound velocity profile). On the other hand, the unbound models, which are also allowed, describe a collapse scenario that is not uniquely gravitational. Some additional energy is impressed to the infalling particles from outside, causing the particles to have a non zero velocity at all times. We note that in the unbound case, solutions with  $\dot{R} = 0$  at the initial time are not possible.

In the more general situation where  $b_0 = b_0(r)$ , the fact that certain velocity profiles are allowed or not might depend on the boundary conditions imposed on the matter cloud. Therefore, for some specific cases, both positive and negative functions  $b_0(r)$  can be considered, without causing the presence of shell crossings.

At this point, we note that the requirement that the regularity conditions must be satisfied constrains the class of allowed models and the allowed equation of state. Regularity would be reasonable to demand from the point of view of the physical validity, since it guarantees that the metric functions and physical quantities are well behaved at any regular epochs of

collapse, and at the center. We see that this imposes certain non-trivial restrictions on the possible choice of the matter models. In fact, we notice how regularity and the choice of a linear equation of state of the type  $p_\theta = k\rho$ , as was assumed in some previous works (see for example, [14]), are mutually exclusive. This is immediately seen from equation (II.5) which now becomes,

$$\nu' = 2k \frac{R'}{R}, \quad (\text{II.26})$$

and once integrated and evaluated at the initial time  $t = t_i$  it gives,

$$\nu_i(r) = 2k \ln r + \text{const.} \quad (\text{II.27})$$

Thus the metric function diverges as  $r$  tends to zero, while on the other hand regularity demands that  $\nu \sim r^2$  at all times, near the center. We thus see that in comoving coordinates the assumption of a linear equation of state leads to a breakdown of the coordinate system at the center, or the metric functions not being regular. This would imply that the initial data set is not regular, or that a more suitable coordinate system is required. The inspection of the Kretschmann scalar  $K$  for this model shows that for any  $k \neq 0$ , the central shell presents a curvature singularity (where  $K$  diverges) at all times, and thus the initial data surface is not regular. This is confirmed also by the divergence at  $r = 0$ , of the energy density  $\rho$ , as evaluated from equation (II.4). It is seen that at the initial epoch also, the energy density diverges necessarily at the center. For the tangential pressure models, the choice  $p_\theta = k\rho$  turns out to be consistent with the regularity condition  $F(r) = r^3 M(r)$  only in the dust case, where  $k = 0$ .

In order for the matter model to be physically realistic, suitable energy conditions, in particular the weak energy conditions, must be satisfied. That is, the energy density as measured by local observers must be non-negative, and for any timelike vector  $V^a$  we require,

$$T_{ab} V^a V^b \geq 0, \quad (\text{II.28})$$

which reduces to

$$\rho \geq 0, \quad \rho + p_\theta \geq 0. \quad (\text{II.29})$$

Since the physical radius  $R$  is always positive, to ensure the positivity of energy density we must require  $F'$  and  $R'$  or their ratio to be positive. As we have seen, positivity of  $R'$  ensures that there are no shell crossing singularities occurring, while the fact that  $F'$  is positive is granted by the choice of positive  $M(r)$  with a suitable behaviour. From equation (II.12) we see that positivity of  $\rho + p_\theta$  is then given by

$$\frac{1}{2} \frac{R}{R'} \nu' \geq -1 \quad (\text{II.30})$$

and is satisfied automatically (and not only) by any increasing function  $\nu$ . The sign of  $\nu'$  is the only factor deciding positivity or negativity of the tangential pressure and it is worth noting that negative tangential stress profiles are also allowed and

they satisfy energy conditions provided that the relation expressed in equation (II.30) is satisfied. Furthermore, for small values of  $r$ , from equation (II.30) we see that regardless of the values taken by  $\nu$ , there will always be a neighborhood of  $r = 0$  for which  $|p_\theta| < \rho$  and therefore  $\rho + p_\theta \geq 0$  is automatically satisfied in a close neighborhood of the center.

Finally, specifying the initial conditions for the collapsing matter cloud consists in prescribing the values of the three metric functions and that of the density and stress profiles at the initial time  $t_i$ , given as functions of  $r$  [15]. These can be written as,

$$\begin{aligned} \rho(r, t_i) &= \rho_i(r), \quad p_\theta(r, t_i) = p_{\theta_i}(r), \\ R(r, t_i) &= R_i(r), \quad \nu(r, v(r, t_i)) = \nu_i(r), \quad \psi(r, t_i) = \psi_i(r). \end{aligned}$$

As noted earlier, at the initial time we have taken the scale such that  $R_i = r$ , which implies that the scaling function  $v = 1$  at the initial epoch. Furthermore, from  $R'_i = 1$  we get  $\nu'(r, t_i) = 0$ .

Not all of these initial value functions can be chosen arbitrarily. In fact, the scaling function, the choice of the mass profile, and the Einstein equations impose some constraints. From equations (II.4) and (II.19), we see that

$$\rho_i = \frac{F'(r)}{r^2} = 3M(r) + rM'(r), \quad (\text{II.31})$$

while from equation (II.12) we get,

$$p_{\theta_i} = \frac{1}{2} \nu'_i(r) \rho_i(r) r. \quad (\text{II.32})$$

Also, from equation (II.20) we can write,

$$\nu_i(r) = r^2 g(r, v(r, t_i)) = r^2 g_i(r). \quad (\text{II.33})$$

In turn,  $\nu_i$  can be related to the function  $a$ , defined by equation (II.21), at the initial time via equation (II.14),

$$a_{,v}(r, v(r, t_i)) = (a_{,v})_i = 2g_i + r g'_i. \quad (\text{II.34})$$

Finally, we can relate the initial condition for  $\psi$  to the initial value of the function  $A(r, t)$  through equation (II.9) and equation (II.14),

$$A(r, v(r, t_i)) = A_i(r) = -\psi_i - \frac{1}{2} \ln b. \quad (\text{II.35})$$

If we want the initial collapse configuration to be not trapped, in order to see how different initial matter configurations evolve forming trapped surfaces, we must further require,

$$\frac{F(r_b)}{R_i} = r_b^2 M(r_b) < 1, \quad (\text{II.36})$$

which will impose some restrictions on the possible choices of the radial boundary in accordance with  $M(r)$ . This condition reflects on the initial configuration for  $G$  and  $H$  since  $\frac{F}{R} < 1$  implies  $H < G$ . Therefore to avoid trapped surfaces at the initial time the velocity of the infalling shells must satisfy

$$-\dot{R} > \sqrt{b} e^{A+\nu}. \quad (\text{II.37})$$

This means that the scenario where collapse commences from an equilibrium configuration where  $\dot{R} = 0$ , can be taken only as a limiting case.

As we noted above, considering only the presence of the tangential pressures implies conservation of the Misner-Sharp mass during the collapse. Therefore, matching to an exterior spherically symmetric vacuum solution leads inevitably to consider the Schwarzschild metric [16]. The collapsing cloud has a compact support within the boundary taken at  $r = r_b$ , and the pressure of the matter composing the collapsing star is assumed to vanish at the boundary. Matching conditions imply continuity of the metric and its first derivatives across this surface, although it should be noted that eventual discontinuities in the first derivatives of the metric across the boundary may be interpreted as an additional distribution of matter acting as a layer separating the interior from the exterior. In this case, the jump of the second fundamental form across the boundary is related to an additional stress-energy tensor layer confined to the boundary. Such a matching is in principle always possible, and together with the regularity conditions and the energy conditions, might impose some further restrictions on the initial configurations allowed.

### III. COLLAPSE EVOLUTION AND FINAL STATE

As we show below, the possible future evolution of the collapsing matter cloud depends upon the choice made for the matter content, namely the two free functions that determine the mass profile and the tangential pressure profile, and the initial conditions.

In order to investigate the final outcome of collapse, we will study the time curves that lead the shell of matter labeled with a specific coordinate value  $r$  to fall into the singularity. As already mentioned, the spacetime singularity is achieved for  $v = 0$ , and the divergence of the energy density (and eventually of the tangential pressure) indicates that it must be a true curvature singularity.

Given the fact that the function  $v(r, t)$  is monotonically decreasing in time, we can invert it and consider the time  $t$  as a function of the variables  $r$  and  $v$ . This is equivalent to considering the area-radius coordinates  $(r, R)$  as done in some earlier works (see e.g. [10, 12]) with the relation between  $R$  and  $v$  given by equation (II.10). The function  $t(r, v)$  will identify the time at which the shell labeled by a specific value  $r$  reaches the event  $v$ . In this manner, the occurrence of the singularity is described by the time curve  $t_s(r) = t(r, 0)$ . The tangent of the singularity curve  $t_s(r)$  at  $r = 0$ , *i.e.* at the central singularity, is then seen to be related to the eventual existence of outgoing radial null geodesics escaping away from the singularity, and therefore to the possible visibility or otherwise of the singularity itself to faraway observers.

From equation (II.18) we can write,

$$dt = - \frac{e^{-\nu(r,v)}}{\sqrt{\frac{M(r)}{v} + \frac{b(r)e^{2A(r,v)} - 1}{r^2}}} dv, \quad (\text{III.1})$$

with the minus sign chosen in accordance with  $\dot{v} < 0$ , in order

to describe the collapse. Integrating the above equation, while treating the radial coordinate  $r$  as a constant, we get

$$t(r, \bar{v}) = t_i + \int_{\bar{v}}^1 \frac{e^{-\nu}}{\sqrt{\frac{M}{v} + \frac{be^{2A} - 1}{r^2}}} dv. \quad (\text{III.2})$$

It follows therefore that, the time taken by the shell at any given value of  $r$  to reach the singularity, located at  $v = 0$ , is given by,

$$t_s(r) = t_i + \int_0^1 \frac{e^{-\nu}}{\sqrt{\frac{M}{v} + \frac{be^{2A} - 1}{r^2}}} dv. \quad (\text{III.3})$$

We note that  $t(r, v)$  is in general a differentiable function at least outside the singularity, which would be the case because of the regularity of all the functions involved. That is the case in many of the known collapse models. In any case, whenever it is at least a  $C^2$  function, we can always write it as an expansion near the central shell  $r = 0$  in the form,

$$t(r, v) = t(0, v) + r\chi_1(v) + o(r^2), \quad (\text{III.4})$$

with

$$t(0, \bar{v}) = t_i + \int_{\bar{v}}^1 \frac{1}{\sqrt{\left(\frac{M(0)}{v} + b_0(0) + 2a(0, v)\right)}} dv \quad (\text{III.5})$$

and  $\chi_1(v) = \left. \frac{dt}{dr} \right|_{r=0}$ , given by,

$$\chi_1(\bar{v}) = - \frac{1}{2} \int_{\bar{v}}^1 \frac{\frac{M'}{v} + \frac{rb'e^A + rbA'e^A - 2(be^A - 1)}{r^3}}{\left(\frac{M}{v} + \frac{be^{2A} - 1}{r^2}\right)^{\frac{3}{2}}} \Bigg|_{r=0} dv. \quad (\text{III.6})$$

Note that the above quantities are defined only if  $\frac{M(0)}{v} + b_0(0) + 2a(0, v) > 0$ . This may be called the reality condition for the gravitational collapse to take place. The choices of  $b(r)$  and  $A(r, v)$ , made from (II.16) and (II.21) imply that as  $r \rightarrow 0$ , we must have  $\frac{be^A - 1}{r^2} \rightarrow b_0 + 2a(0, v)$ , and therefore we see how  $\nu$  cannot have constant terms or terms linear in  $r$  in a close neighborhood of  $r = 0$ . This justifies the regularity conditions for  $b(r)$  and  $\nu(r)$  that were deduced earlier. Furthermore, we can see how the regularity condition for  $A$ , that comes directly from that for  $\nu$ , ensures that  $t(0, v)$  and  $\chi_1$  do not diverge. Evaluating  $\chi_1$  explicitly we get

$$\chi_1(\bar{v}) = - \frac{1}{2} \int_{\bar{v}}^1 \frac{\frac{M'(0)}{v} + b'_0(0) + 2a'(0, v)}{\left(\frac{M(0)}{v} + b_0(0) + 2a(0, v)\right)^{\frac{3}{2}}} dv. \quad (\text{III.7})$$

By continuity, we can therefore take that the singularity curve  $t_s(r)$ , which corresponds to  $v = 0$ , is differentiable, or in any case even if it is only a  $C^2$  function, we can take it that it has at least a well-defined tangent at the singularity. This is the case in important known collapse models such as the Tolman-Bondi-Lemaître dust collapse, or the ‘Einstein cluster’ models, and also for the Oppenheimer-Snyder-Datt

homogeneous dust collapse. The singularity curve is computed as the time it takes for the shell located at  $r$  to reach the singularity and can then be written as,

$$t_s(r) = t_0 + r\chi_1(0) + o(r^2), \quad (\text{III.8})$$

where  $t_0 = t_s(0)$  is the time at which the central shell becomes singular. It can be shown that for any generic initial configuration there exist classes of dynamical collapse evolutions that have a well defined singularity curve.

While the collapse evolves, the trapped surfaces may form in the collapsing cloud. If the trapped surfaces form at a time anteceding the formation of singularity then the latter will be covered, thus giving rise to a black hole at the end of the collapse. If, on the other hand, the trapped surfaces do not form, or form at a later time after the singularity, then the singularity might be visible to faraway observer [17].

The boundary of the trapped region is delimited by the apparent horizon. The equation for the apparent horizon in a spherically symmetric spacetime is given by  $g^{ij}R_{,i}R_{,j} = 0$ , which for the metric (II.1) becomes  $G - H = 0$ . From equation (II.7) we can write it as,

$$\frac{F}{R} = 1. \quad (\text{III.9})$$

The behavior of the apparent horizon for the collapsing cloud with vanishing radial pressure is analogous to that of the dust case, since  $\frac{F}{R} = \frac{r^2 M(r)}{v}$  in both these cases. If we note that as  $v \rightarrow 0$  we must have  $r \rightarrow 0$  on the apparent horizon, it is immediately clear that the only singularity that can eventually be visible in the present case is that at the center of the collapsing cloud. In fact, since it is not possible to satisfy  $1 - \frac{F}{R} > 0$  near the singularity but away from the center, the region surrounding the singularity cannot be timelike, and therefore any singularity that might eventually form near the center with  $r > 0$  must not be visible.

The difference with the dust case is then given by the different possible behaviours of the function  $v(r, t)$ . It is this function, as given from equation (II.18), that determines the apparent horizon curve in dependence of the mass, velocity and the tangential stress profiles. We can see the apparent horizon as a curve  $r_{ah}(t)$  given by

$$r_{ah}^2 = \frac{v_{ah}}{M(r_{ah})}, \quad (\text{III.10})$$

with  $v_{ah} = v(r_{ah}(t), t)$ . Inversely, we can also write it as a curve  $t_{ah}(r)$ , which represents the time at which the shell labeled as  $r$  becomes trapped.

In order to understand when the central singularity may be visible to faraway observers in the spacetime, we shall analyze the time curve of the apparent horizon which is given by,

$$t_{ah}(r) = t_s(r) - \int_0^{v_{ah}(r)} \frac{e^{-\nu}}{\sqrt{\frac{M}{v} + \frac{be^{2A}-1}{r^2}}} dv, \quad (\text{III.11})$$

where  $t_s(r)$  is the singularity time curve, whose initial point is  $t_0 = t_s(0)$ . Near  $r = 0$ , equation (III.11) can be written in

the form,

$$\begin{aligned} t_{ah}(r) &= t_{ah}(0) + \left. \frac{dt_{ah}}{dr} \right|_{r=0} r + o(r^2) = \\ &= t_0 + \chi_1(0)r + o(r^2). \end{aligned} \quad (\text{III.12})$$

From this, it is easy to see how the presence of stresses affects the time of the formation of the apparent horizon. In fact, all the initial configurations that cause  $\chi_1(0)$  to be positive force the boundary of the trapped surfaces to form after the occurrence of the singularity, thus leaving the ‘door open’ for the null geodesics to possibly come out, away from the singularity.

We can verify the visibility of the central singularity through the behaviour of radial null geodesics. This means that in order to verify the visibility of the singularity we must check if there are any future directed null or non-spacelike geodesics that terminate in the past at the singularity (for a detailed analysis of outgoing geodesics in singular spacetimes originating from collapse see [18]). It can be shown that if the singularity is radially censored then it must be completely censored. On the other hand, if there are any future directed non-spacelike curves coming out from the singularity and reaching external observers then the singularity is naked. Further, in [1] it was proven that  $\chi_1(0) > 0$  is a necessary and sufficient condition for outgoing radial null geodesics to escape the singularity, at least locally. Outgoing radial null geodesics are given by the equation,

$$\frac{dt}{dr} = e^{\psi-\nu} = \frac{R'}{\sqrt{b}} e^{-\nu-A}, \quad (\text{III.13})$$

with the positive sign considered in order for  $t$  to be increasing moving away from the singularity.

Now, introducing the variable  $u = r^\alpha$ , we can write equation (III.13) as

$$\begin{aligned} \frac{dR}{du} &= \frac{1}{\alpha r^{\alpha-1}} (R' + \dot{R}e^{\psi-\nu}) = \\ &= \frac{R'}{\alpha r^{\alpha-1}} \left( 1 - \frac{e^{-A}}{\sqrt{b}} \sqrt{\frac{F}{R} + G - 1} \right). \end{aligned} \quad (\text{III.14})$$

We thus look for a radial null geodesic terminating in the past at the singularity, whose equation near  $r = 0$  takes the form

$$R = x_0 u, \quad (\text{III.15})$$

with  $x_0$  finite and positive. Such null geodesics terminate at the singularity with a definite tangent and correspond to the curve

$$t(r) = t_0 + x_0 r^\alpha. \quad (\text{III.16})$$

For a specific choice of  $\alpha = \frac{5}{3}$ , we can solve equation (III.14) and evaluate the value of the tangent to the null geodesics at the central singularity which is given by  $r = 0, t = t_0$ . Thus obtaining,

$$x_0 = \lim_{r \rightarrow 0} \lim_{t \rightarrow t_0} \frac{R}{u} = \left. \frac{dR}{du} \right|_{(0, t_0)} = \left( \frac{3}{2} \sqrt{M_0} \chi_1(0) \right)^{\frac{2}{3}}, \quad (\text{III.17})$$

which is positive and finite at the central singularity if  $\chi_1(0) > 0$ . It is now clear how  $\chi_1$  is directly related to the tangent to the null geodesics at the singularity. In this case we conclude that the future directed radial null geodesics do come out of the singularity to reach far away observers.

Let us stress that it is the sign of the quantity  $\chi_1(0)$  that uniquely determines the visibility of the singularity, as in fact  $\chi_1(0) > 0$  implies that the time curve of the apparent horizon is future directed increasing, and future directed null geodesics in that case do come out. Therefore the central singularity occurring at  $t_0$  is not covered.

All those initial configurations for which  $M'(0)$ ,  $b_0(0)$  and  $a_1(v)$  cause  $\chi_1(0)$  to be positive will force the apparent horizon to appear after the formation of the singularity, and will allow the null geodesics to come out of the singularity that forms at the center of the collapsing cloud. We conclude that the condition  $\chi_1(0) > 0$  not only implies that the trapped surfaces form at a later stage during collapse than the singularity, but it is also the sufficient condition for null geodesics from the singularity to be visible (at least locally) to external observers.

#### IV. SMALL PRESSURE PERTURBATIONS IN DUST COLLAPSE

We would now like to examine here, using the formalism above, how the final fate of the gravitational collapse is affected when pressure perturbations are introduced in a pressure-free inhomogeneous dust cloud, which was otherwise going to terminate in a black hole final state. This provides a useful insight into the role of pressure in gravitational collapse dynamics. Certain classes of collapse models with a non-zero tangential pressure are analyzed, their properties are discussed, and the collapse end-states are examined.

For the sake of definiteness, we consider the case when the Misner-Sharp mass is given by an expression near the center as,

$$M(r) = M_0 + M_1 r + M_2 r^2 + o(r^2). \quad (\text{IV.1})$$

Further expressing the function  $g(r, v)$  (and therefore the tangential pressure) in the form,

$$g(r, v) = g_0(v) + g_1(v)r + g_2(v)r^2 + o(r^2), \quad (\text{IV.2})$$

we have,

$$p_\theta = \frac{r^2}{vR'^2} \left( 3M_0 g_0 + 4M_1 g_0 r + \frac{9}{2} M_0 g_1 r + \dots \right). \quad (\text{IV.3})$$

With these expressions, we are then able to write the quantity  $\chi_1(0)$  in dependence of the physically relevant profiles  $M(r)$ ,  $b(r)$ ,  $a(r, v)$  as,

$$\chi_1(0) = -\frac{1}{2} \int_0^1 \frac{\frac{M_1}{v} + b'_0(0) + 2a_1(v)}{\left(\frac{M_0}{v} + b_0(0) + 2a_0(v)\right)^{\frac{3}{2}}} dv, \quad (\text{IV.4})$$

where the function  $a(r, v)$  (which is related to  $\nu$  via equation (II.14)) has been written as,

$$a(r, v) = a_0(v) + a_1(v)r + a_2(v)r^2 + o(r^2). \quad (\text{IV.5})$$

We see now immediately that the behaviour of the tangential pressure (be it positive, negative, increasing or decreasing) is here reflected in the terms  $a(0, v)$  and  $a'(0, v)$  in  $\chi_1(0)$ , and can influence the final outcome of the gravitational collapse. In fact, it is the choice of the coefficients of the initial density, velocity, and stress profiles that determines the quantities  $M$ ,  $b$  and  $a$ , which are responsible for the behaviour of  $\chi_1(0)$ , which in turn determines the tangent to the singularity curve at the origin. As we noted earlier, the sign of  $\chi_1(0)$  uniquely determines the final fate of collapse in terms of either a black hole or a naked singularity. As it was shown in [1], positivity of  $\chi_1(0)$  is a sufficient condition for outgoing null geodesics to come out from the singularity, thus making it visible to external observers. In some cases of physical interest it turns out that  $\chi_1(0) = 0$ . This is likely to happen when marginally bound collapse with only quadratic terms in the expansion of energy density and pressure is considered, but it is not the only possibility. In general, whenever  $\chi_1(0) = 0$ , the analysis of higher order terms must be carried out. In this case, we can write  $t_s(r)$  as

$$t_s(r) = t_0 + r^2 \chi_2(0) + o(r^3) \quad (\text{IV.6})$$

and evaluate  $\chi_2(0) = \left. \frac{d^2 t}{dr^2} \right|_{r=0}$  as

$$\begin{aligned} \chi_2(0) = \int_0^1 & \left[ \frac{3}{8} \frac{(2a_1 + b'_0(0))^2}{\left(\frac{M_0}{v} + b_0(0) + 2a_0\right)^{\frac{5}{2}}} + \right. \\ & \left. - \frac{1}{2} \frac{\frac{M_2}{v} + 2a_2 + 2a_0^2 + 2b_0(0)a_0 + b''_0(0)}{\left(\frac{M_0}{v} + b_0(0) + 2a_0\right)^{\frac{3}{2}}} + \right. \\ & \left. - \frac{g_0(v)}{\sqrt{\frac{M_0}{v} + b_0(0) + 2a_0}} \right] dv. \quad (\text{IV.7}) \end{aligned}$$

We also note here that the negative tangential pressures are not forbidden by the energy conditions. Here we observe, how the presence of negative pressures, that may be due to the presence of exotic matter in the form of dark matter or some other repulsive effects, may influence the final stages of collapse, since it will affect the value of  $\chi_1$  through the function  $a_1(v)$  given in (II.22).

The second order term in the equation for the singularity curve is given by equation (IV.7). It turns out that this term becomes relevant in many physically realistic scenarios which consider the mass profile and the density profile to be written with only quadratic terms in  $r$ . From the regularity conditions analyzed earlier, we can further see that  $M'(0) = 0$  (that is reflected in the requirement that  $M_1 = 0$ ) is a reasonable assumption. This is consistent with the analysis of energy density profiles, that, for physically realistic scenarios, have only quadratic terms in the expansion. The energy density  $\rho$ , can thus be written as  $\rho(r, t) = \rho_0(t) + \rho_2(t)r^2 + \rho_4(t)r^4 + \dots$

In the situation where mass and density profiles are given only in quadratic terms, we then have,

$$M_{2n+1} = a_{2n+1} = g_{2n+1} = 0. \quad (\text{IV.8})$$



We can now examine a few physically relevant classes of collapse models with such a property, together with the prescription for marginally bound collapse ( $b_0 = 0$ ). This choice of the mass, pressure profile and velocity profile causes  $\chi_1(0)$  to vanish and the final outcome of collapse is decided by the  $\chi_2(0)$  term, which in this case reads,

$$\chi_2(0) = -\frac{1}{2} \int_0^1 \frac{\frac{M_2}{v} + 2a_2 + 2a_0^2 + 2g_0 \left(\frac{M_0}{v} + 2a_0\right)}{\left(\frac{M_0}{v} + 2a_0\right)^{\frac{3}{2}}} dv. \quad (\text{IV.9})$$

The analysis is then carried out along the same lines described above.

Firstly, an important class of collapse models, also to be noted as the reference frame, is that of an inhomogeneous dust cloud. Inhomogeneous dust has been widely studied and the occurrence of naked singularities in such cases is well-known. From the framework given above, we recover the inhomogeneous dust when  $a(r, v) = g(r, v) = 0$ , and that together with the requirement that only quadratic terms appear in the density, yields

$$\chi_2^{dust}(0) = -\frac{1}{2} \int_0^1 \frac{M_2 \sqrt{v}}{M_0^{\frac{3}{2}}} dv = -\frac{1}{3} \frac{M_2}{M_0^{\frac{3}{2}}}. \quad (\text{IV.10})$$

The collapse will end in a naked singularity for all negative values of  $M_2$ . Further, in this case, the structure of the apparent horizon can be evaluated explicitly. Note that for  $M(r) = M_0$  the above case reduces to the Oppenheimer-Snyder-Datt collapse scenario, describing homogeneous dust, and all the terms  $\chi_i$ s vanish as the singularity is simultaneous.

Let us now consider the effect of introducing small tangential pressure perturbations in an otherwise pressure-free inhomogeneous dust cloud collapse, that was going to a black hole final state. To this aim, we choose a class of collapse models restricted by the assumption that the second order term in the mass profile vanishes. We see that if  $M_2 = 0$  then  $\chi_2^{dust}(0) = 0$ , and the final outcome of collapse for the inhomogeneous dust cloud is then decided by the next order term, namely  $\chi_4^{dust}(0) = -\frac{1}{3} \frac{M_4}{M_0^{\frac{3}{2}}}$ . Exactly in the same manner as

described above, all positive values of  $M_4$  will cause the collapse to end in a black hole. If now we introduce an arbitrarily small tangential pressure of the form (IV.3), we see that near  $r = 0$  this can cause the collapse to end in a naked singularity, whenever the function  $g_0(v)$  is chosen in such a way that  $\chi_2(0)$ , as given by equation (IV.9) with  $M_2 = 0$ , is positive. We therefore have constructed a class of models of small tangential pressure perturbations of the LTB collapse scenario that drastically changes the final outcome of collapse on introduction of small pressures.

A second illustrative class of models is that of a ‘quasi-Hookean’ pressures, which is given by a specific choice of the tangential stresses corresponding to

$$a_{2n-2} = (-1)^{n+1} \mu_0 \left(1 - \frac{1}{v^2}\right), \quad (\text{IV.11})$$

for  $n = 1, 2, 3, \dots$ . This choice leads to an energy function

$G(r, v)$  of the form

$$G(r, v) = \frac{1 + b_0 r^2}{\left[1 + \mu_0 \left(1 - \frac{1}{1+r^2}\right) \left(1 - \frac{1}{v^2}\right)\right]^2}. \quad (\text{IV.12})$$

In this case, putting again  $b_0 = 0$ , we get

$$\begin{aligned} \chi_2(0) = & -\frac{1}{2} \int_0^1 \frac{M_2 v^2 + 2M_0 \mu_0}{(M_0 v + 2\mu_0 v^2 - 2\mu_0)^{\frac{3}{2}}} dv + (\text{IV.13}) \\ & + \int_0^1 \frac{\mu_0 v (v^2 - 1) \left(1 - \mu_0 \left(1 + \frac{1}{v^2}\right)\right)}{(M_0 v + 2\mu_0 v^2 - 2\mu_0)^{\frac{3}{2}}} dv. \end{aligned}$$

Some more considerations on the dynamical behaviour of this particular case will be presented in section VI also. What we can see from the above is that the values of the constants  $M_2$  and  $\mu_0$  are responsible for the final outcome of collapse in terms of black hole or naked singularity, and that either of the outcomes are possible depending on the choices of these parameters.

Finally, it is interesting to study the behaviour of a subclass of the previous model, choosing the class of small tangential stresses in the form,

$$p_\theta = 6 \frac{M_0 g_2}{v^3} r^4 + o(r^6). \quad (\text{IV.14})$$

In this case we see that  $a_0 = a_1 = 0$ , and near  $r = 0$  we have  $A(r, v) = a_2(v)r^4 + o(r^6)$  and  $g_2 = \frac{v}{4} a_{2,v}$ . Then the final outcome of collapse is decided by the sign of

$$\begin{aligned} \chi_2(0) = & -\frac{1}{2} \int_0^1 \frac{\sqrt{v}(M_2 + 2a_2 v)}{M_0^{\frac{3}{2}}} dv = \\ = & \chi_2^{dust}(0) - \frac{1}{M_0^{\frac{3}{2}}} \int_0^1 a_2 v^{\frac{3}{2}} dv. \quad (\text{IV.15}) \end{aligned}$$

We see that in this case it is possible that the introduction of a suitable tangential stress in the inhomogeneous dust cloud will cause it to end in a naked singularity, while the same model with a vanishing pressure resulted in a black hole, and viceversa. The condition to be satisfied by the tangential stress is  $3 \int_0^1 a_2 v^{\frac{3}{2}} dv < -M_2$ .

To see this more explicitly, in analogy with the previous example, consider for instance

$$a_2 = -\mu_0 \left(1 - \frac{1}{v^2}\right), \quad (\text{IV.16})$$

then the tangential stress near the center of the cloud will be given by,

$$p_\theta = 3 \frac{M_0 \mu_0}{v^5} r^4 + o(r^6). \quad (\text{IV.17})$$

We note that the tangential pressure as well as the density diverge at the singularity. The collapse here ends in a naked singularity whenever

$$M_2 < -\frac{24}{5} \mu_0, \quad (\text{IV.18})$$

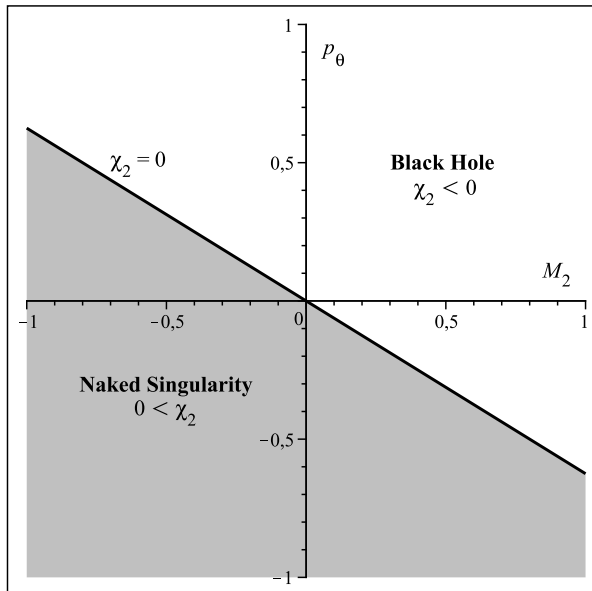


FIG. 1: The Naked singularity region and the black hole region in the parameter space  $\{M_2, p_\theta\}$  for fixed values of  $M_0, r$  and  $v$ .

as can easily be calculated from equation (IV.15).

In Fig. 1, the occurrence of black hole and naked singularity phases in the latest example are shown, depending on the possible values chosen in the parameters space of  $p_\theta$  and  $M_2$ , for fixed values of  $M_0, r$  and  $v$  (taken at the initial time). The line  $\chi_2 = 0$  corresponds to  $p_\theta = -\frac{5}{8} \frac{M_0 M_2}{v^5} r^4$ , and acts as a critical surface separating the two outcomes, in which case the evaluation of higher order  $\chi_i$  is necessary in order to determine the final fate of collapse. The axis  $p_\theta = 0$  corresponds to the LTB inhomogeneous dust case, with the origin being the Oppenheimer-Snyder-Datt model.

We see clearly that to have a naked singularity in the dust case, we must require  $M_2 < 0$ . This is not the case anymore in the more general situation with non-vanishing tangential stresses. In fact, such scenarios allow for positive values of  $M_2$  to lead to a naked singularity, provided that the tangential stress (and hence  $\mu_0$  in this example) is negative, as well as negative values of  $M_2$  leading to black holes whenever a suitable positive tangential pressure is introduced. The interesting feature here is that the introduction of tangential pressures does not rule out the possibility of formation of naked singularity.

This shows that the conjecture suggested in [10] does not hold (even in the more restrictive case of quadratic density and pressure profiles), and that naked singularities are a general feature of gravitational collapse with pressures. The conjecture stated that given a known Lemaitre-Tolman-Bondi collapse model ending in a black hole, all quadratic pressure perturbations to the model will necessarily originate a black hole. Now we see that adding a suitable generic pressure perturbation to the LTB model can in fact change the outcome of collapse.

From a physical point of view, the most relevant quadrant in the model shown in Fig. 1 is the first one (where  $p_\theta > 0$

and  $M_2 < 0$ ), since it corresponds to positive pressures and radially decreasing energy density profile. We see that both the black hole and naked singularity outcomes are possible in this case.

We shall stress however that the model illustrated above, while presenting a simple and intriguing structure, does not depict the only possible structure of collapse with tangential pressures. Other examples, with different choices of the functions  $M, b$  and  $a$ , might provide an entirely different picture for the final stages of collapse, thus indicating once more how rich and complex is the description of collapse resulting from general relativistic analysis.

This model shows also how the black hole formation process described in the well known homogeneous dust scenario is not ‘stable’ when arbitrarily small stress perturbations are introduced. Since, as shown in Fig. 1, it lies on the critical surface separating the ‘naked singularity configuration space’ from the ‘black hole configuration space’, any close neighborhood of the OSD model will contain initial data leading to both outcomes.

On the other hand, we see that in the context of this model, inhomogeneous dust collapse leading either to black hole or naked singularity (thus away from the critical surface where  $\chi_2 = 0$ ), is ‘stable’ under small pressure perturbations, in the sense that it is always possible to find a small neighborhood of the configuration space which will develop the same final outcome.

The above considerations illustrate the role pressures can play towards determining the final fate of collapse. Clearly, depending on their nature, the non-zero tangential pressures can help create a naked singularity, or a black hole. It follows that a mere introduction of pressure in the collapsing cloud, by itself, cannot help us restore the cosmic censorship. Finally we note that the structure illustrated for the above model extends naturally to more general pressure profiles, still leaving unchanged the general behaviour where both naked singularities and black holes might generate at the end of collapse.

## V. GENERICITY OF BLACK HOLE AND NAKED SINGULARITY FINAL STATES

The consideration above helps us gain an important insight into the nature and genericity of the black hole and naked singularity formation in gravitational collapse. Let us take, for example, a scenario where the term  $\chi_2$  is positive, thus leading the collapse to a naked singularity final fate. In that case, it is clear that there always exists a small enough perturbation in the initial data, which is in the form of small changes in the initial density, pressures, and velocities of the collapsing shells, which will still preserve by continuity the positive sign for the  $\chi_2$  term, thus preserving the naked singularity final fate of the collapse. A similar conclusion will hold for the black hole final state of collapse as well.

In this sense, if a collapse evolution was to develop in a naked singularity, there is a small neighborhood around that specific initial set in the configuration space of initial data, which also continues to take the collapse to a naked singular-

ity. In such a sense, the naked singularity formation (as well as the black hole formation) is generic in the gravitational collapse with non-zero tangential pressures case.

In the light of these results, we can see that the idea that the introduction of pressures must favor the black hole formation process is not correct, at least for non-vanishing tangential stresses. One may argue that this situation arises because we have neglected radial pressures, a choice that, while still not being the dust model, is not sufficiently ‘realistic’. Nevertheless the same arguments can be extended to the case of generic pressures (as it will be done in a future work), to show that collapse scenarios leading to a naked singularity have a small neighborhood in the configuration space of initial pressure profiles, that still leads to a naked singularity.

Since the parameters that determine the final fate of collapse cannot be chosen arbitrarily if one wishes to obtain a black hole, we can say that a specific final outcome of collapse requires some tuning in the initial data in contrast with the possible view that any initial configuration with pressures will necessarily destroy the naked singularity formation picture and lead to a black hole.

The fact that clearly emerges from the present analysis is that the scenarios describing the formation of naked singularities as the endstate of collapse are not mere counterexamples to cosmic censorship whose relevance is limited and confined. We can therefore say that the presence of tangential stresses can ‘undress’ a singularity that was covered in the absence of stresses, or viceversa, a naked singularity dust model can be covered by the introduction of a suitable positive pressure.

It is therefore useful to relate the scenario presented here with the known collapse models without pressures, namely the dust collapse scenarios. The first collapse model within the framework of general relativity was studied by Oppenheimer, Snyder and Datt, and it described the collapse of a massive sphere made of homogeneous dust. The Lemaitre-Tolman-Bondi models then considered the collapse where inhomogeneous dust is allowed. It turned out that when inhomogeneities in the density distributions are allowed, then the appearance of naked singularities takes place as possible endstate of gravitational collapse. Dust collapse models are well known and widely studied and the structure of the formation of trapped surfaces and of the singularity is well established, also in the case of higher spacetime dimensions [19].

The dust collapse is obtained from the above framework, when we impose  $p_r = p_\theta = 0$ . In that case, from equation (II.5) it follows that  $\nu' = 0$ , which together with the condition  $\nu(0) = 0$  gives  $\nu = 0$ . This in turns implies that  $G = b(r)$  and  $H = \dot{R}^2$ . For inhomogeneous dust in general we have,

$$\chi_1(0) = -\frac{1}{2} \int_0^1 \frac{\frac{M'(0)}{v} + b'_0(0)}{\left(\frac{M(0)}{v} + b_0(0)\right)^{\frac{3}{2}}} dv. \quad (\text{V.1})$$

We see that inhomogeneous dust can lead to the formation of naked singularities or black holes, depending on the behaviour of the mass profile and the velocity profile [20]. Furthermore, in the marginally bound case it is possible to evaluate explic-

itly the apparent horizon from equation (III.11) as

$$t_{ah}(r) = t_0 - \frac{1}{3} \frac{M_1}{M_0^{\frac{3}{2}}} r + \left( \frac{1}{4} \frac{M_1^2}{M_0^{\frac{5}{2}}} - \frac{1}{3} \frac{M_2}{M_0^{\frac{3}{2}}} \right) r^2 + o(r^3), \quad (\text{V.2})$$

while in the bound and unbound cases, the apparent horizon curve can be read from equation (III.11).

Collapse is said to be homogeneous when  $\rho = \rho(t)$ . Homogeneity, together with the initial condition  $v(r, t_i) = 1$ , implies some constraints on the possible matter models and velocity profiles. Now let us consider the following three conditions,

- (i)  $M = M_0$ ,
- (ii)  $v = v(t)$ ,
- (iii)  $b_0(r) = k$ ,

where the second condition implies a simultaneous collapse, in which all shells of matter fall in the singularity at the same time. We can see that homogeneity, dust and the above conditions are not all mutually independent. In fact, in the case of collapse with vanishing radial stresses, which is under consideration here, we can prove the following statements:

1. Homogeneity  $\Leftrightarrow$  (i) and (ii),
2. (ii)  $\Leftrightarrow$  (i), (iii) and dust.

We can see from this that homogeneous collapse implies necessarily dust, while vice versa is not necessarily true (note, however, that when radial stresses are included it is possible to have homogeneous collapse also in the case of a perfect fluid with homogeneous pressures). Therefore, the Oppenheimer-Snyder-Datt model implies necessarily that the three conditions above are satisfied. We note that the simultaneous collapse (which implies homogeneous collapse in the case when  $M = M_0$ ) implies that all shells fall into the singularity at the same time  $t = t_0$ , thus excluding the possibility of its visibility due to the structure of the apparent horizon in that case. The quantity  $\chi_i(0)$  vanishes identically in this case for every  $i$ . In order for the singularity arising in dust collapse scenarios to be visible, inhomogeneities in the density distribution must be allowed for and have to be taken into consideration, thus dropping the condition (i), which automatically implies the non-occurrence of a simultaneous collapse.

To see the first statement, it is sufficient to use equation (II.4) with the requirement of homogeneity. From that it follows that  $v(r, t) = 3CM(r)\alpha(t)$ , which together with the requirement that  $v(t_i) = 1$  implies (i) and (ii). To see the inverse implication, it is sufficient to put (i) and (ii) into equation (II.4). To prove the second statement, we make use of equation (II.7) written in the form,

$$\dot{v} = e^\nu \sqrt{\frac{M(r)}{v} + \frac{b(r)e^{2A} - 1}{r^2}} = h(t). \quad (\text{V.3})$$

Since  $h$  does not depend on  $r$ , it has the same value for any  $r$ , which is also that obtained for  $r = 0$ , namely,

$$h = \sqrt{\frac{M_0}{v} + b_0(0)}. \quad (\text{V.4})$$

From this it follows immediately,

$$M(r) = M_0, \quad g(r, v) = a(r, v) = 0, \quad b_0(r) = b_0(0). \quad (\text{V.5})$$

Vice versa, assuming dust, and conditions (i) and (iii) to hold, we see that  $\dot{v}$  does not depend on  $r$ , which implies  $v(r, t) = \alpha(t) + \beta(r)$ . That together with the requirement  $v(t_i) = 1$  implies  $v = v(t)$ .

## VI. GRAVITATIONAL COLLAPSE WITH COSMOLOGICAL CONSTANT

It is interesting and indeed important to see how the presence of a non-zero cosmological constant can affect the dynamical evolution of a gravitationally collapsing massive matter cloud. The interest in models with cosmological constant arises primarily from the observation of the accelerating speed in the expansion of the universe, an observed phenomenon that indicates that the universe might be filled with some repulsive force known as ‘dark energy’. One of the possible ways by which dark energy can be described is through the introduction of a non-zero cosmological term in Einstein equations.

The earlier works on the subject were constrained by some simplifying assumptions. Namely the collapsing matter was taken to be a dust cloud or the function  $\nu$  was chosen to have a specific dependence on  $t$  and  $R$  in order to simplify the integration of equation (II.13) (see [21]).

In the present section we extend the previous results to the most general case and test how the dynamics and the final stages of collapse of the models described above are affected by the introduction of the cosmological constant. In doing so, we consider an example of dynamical collapse in the presence of tangential stress and cosmological term, that connects directly with the models described in section IV.

The reason for the interest in the introduction of only tangential stresses in the collapsing cloud with cosmological constant resides in the fact that, from equation (II.19), we see that the matching with an exterior vacuum background must be done with the well-known Schwarzschild-deSitter or anti-deSitter spacetimes. On the other hand, the presence of radial pressures would require the matching with a more complicated generalized Vaidya metric.

Einstein equations (II.3) and (II.4), in the presence of a cosmological constant term take the form,

$$\Lambda = \frac{\dot{F}}{R^2 \dot{R}}, \quad (\text{VI.1})$$

$$\rho + \Lambda = \frac{F'}{R^2 R'}, \quad (\text{VI.2})$$

while the other equations remain unchanged. The first one above leads to a Misner-Sharp mass of the form

$$F(r, t) = r^3 M(r) + \frac{1}{3} \Lambda R^3, \quad (\text{VI.3})$$

while the energy density remains unchanged. The energy conditions here imply

$$\rho \geq 0, \quad \rho + \Lambda \geq 0, \quad \rho + p_\theta \geq 0, \quad (\text{VI.4})$$

however, all of the above need not be respected now, and we may have a weaker form of energy conditions holding, depending on what the sign of the cosmological constant is chosen to be. The matching with the exterior metric in this case needs to be done with a spacetime that is asymptotically either Schwarzschild-deSitter or Schwarzschild-anti-deSitter, depending on the sign of the  $\Lambda$  term.

The integration of Einstein equations proceeds exactly in the same way as described above, and we therefore obtain,

$$\dot{v}^2 = e^{2\nu} \left( \frac{M}{v} + \frac{1}{3} \Lambda v^2 + \frac{be^{2A} - 1}{r^2} \right) = \tilde{V}(r, v), \quad (\text{VI.5})$$

where the quantity  $\tilde{V}(r, v)$  can now be interpreted as an effective potential. The regions of allowed motion are those for which  $\tilde{V}(r, v) \geq 0$  (since  $\dot{v}^2 \geq 0$ ). The zeros of  $\tilde{V}$  correspond to turning points of the dynamics where the collapse is halted and the shell bounces back. Therefore, if  $\tilde{V}(\bar{r}, v) = 0$  for some  $\bar{r}$ , the cloud will bounce back and not collapse indefinitely towards the center. Of course, this situation is not allowed in the dust cases when  $\Lambda = 0$ ,  $r$  is small and where gravity is the only force acting on the collapsing particles. Still, bouncing behaviours were found to be possible in scenarios with tangential stresses and bound velocity profiles. In the same manner, whenever  $\Lambda \neq 0$ , it is possible to observe such bouncing behaviours in the dynamics of the collapsing shells.

Typically, such a scenario will become relevant for collapsing clouds of rather large sizes in the universe, for example, for the case of a large cluster of galaxies collapsing under its own gravitational attraction, or possibly for a large collapsing star cluster. In any case, for the sizes of a collapsing star of an average size, the effects of a non-zero  $\Lambda$  may not make any significant impact on the final outcome of its gravitational collapse.

Following a similar analysis of the time curves as that developed in section III, we can evaluate  $t(r, v)$  in this case as,

$$t(r, \bar{v}) = \int_{\bar{v}}^1 \frac{e^{-\nu}}{\sqrt{\frac{M}{v} + \frac{1}{3} \Lambda v^2 + \frac{be^{2A} - 1}{r^2}}} dv. \quad (\text{VI.6})$$

This leads the numerator of  $\chi_1(0)$  (which is the essential element in determining its sign) to have exactly the same value as in equation (IV.4):

$$\chi_1(0) = -\frac{1}{2} \int_0^1 \frac{\frac{M_1}{v} + b'_0(0) + 2a_1(v)}{\left( \frac{M_0}{v} + \frac{1}{3} v^2 + b_0(0) + 2a_0(v) \right)^{\frac{3}{2}}} dv. \quad (\text{VI.7})$$

We can thus see that the presence of the  $\Lambda$  term affects collapse in the sense that it might introduce some positive roots in the effective potentials, and therefore would give rise to some bouncing behaviours. But it will not affect the terms that are responsible for the sign of  $\chi_1(0)$ , which determines the final outcome of collapse to be a black hole or a naked singularity. This appears reasonable, when we think of the presence of the cosmological constant as related to a dark energy in the universe, because the effect that the dark energy will have

on collapse can be that of disrupting and preventing collapse from happening by causing a bounce. But it cannot influence the nature of the final fate of a collapse that is uniquely determined by the initial configurations for densities, pressures and velocities.

Furthermore, the presence of an added positive  $\Lambda$ -term can lead to  $\frac{M}{v} + \frac{1}{3}\Lambda v^2 + \frac{be^{2\Lambda}-1}{r^2} > 0$  while  $\frac{M}{v} + \frac{be^{2\Lambda}-1}{r^2} < 0$ , thus in fact allowing the evolution for some configurations that were forbidden in the case of vanishing  $\Lambda$ .

The visibility of the singularity in the case of collapse with a cosmological constant differs from the case studied above also in the structure and formation of the apparent horizon. From equation (VI.3) and equation (III.9) we get

$$r_{ah}^2 \left( \frac{M}{v_{ah}} + \frac{\Lambda}{3} v_{ah}^2 \right) = 1, \quad (\text{VI.8})$$

from which we see that in general the apparent horizon equation is a cubic equation in  $v$  that can have zero, one or more than one positive roots, thus affecting considerably the formation of trapped surfaces. Still it is possible to show (see [21]), that the presence of the cosmological constant does not prevent the possibility of formation of naked singularities, which continues to be a general feature of gravitational collapse.

It is interesting at this point to analyze some examples of dynamical behaviours based on the above considerations. We will thus make a specific, although reasonable, choice of the mass, pressure and velocity profiles and confront the effective potential given by equation (VI.5) written in the form,

$$V(r, v) = e^{-2\nu} \dot{v}^2 = \left( \frac{M(r)}{v} + \frac{1}{3}\Lambda v^2 + \frac{G(r, v) - 1}{r^2} \right) \quad (\text{VI.9})$$

for different values of  $\Lambda$ .

Following the examples provided in section IV we will choose only quadratic terms in the mass and stress profiles and study a ‘quasi-Hookean’ equation for the pressure of the form

$$e^{-A(r, v)} = 1 - \mu_0 \frac{r^4}{1+r^2} \left( 1 - \frac{1}{v^2} \right). \quad (\text{VI.10})$$

It is easy to check that in this case  $a_0 = 0$  and  $a_{2n-2} = \sum_{n=2}^{\infty} (-1)^n r^{2n}$  ( $n = 2, 3, \dots$ ), and we obtain a model that has the same singular behaviour of that analyzed in equation (IV.15). Therefore, for a suitable choice of the parameters, according to the inequality given by (IV.18), the final outcome of such a scenario can be either a black hole or a naked singularity. Furthermore, in order to keep the example as general as possible while maintaining the mathematics behind it manageable, we will consider bound, unbound and marginally bound collapse models only in the case of  $b_0$  constant. Finally, for simplicity, in the mass profile we will consider only one non-vanishing term in the expansion. This choice does not affect the dynamical behaviour very much, since the lower order terms in the expansion are the ones responsible for the general features of the mass profile in a neighborhood of the center.

We therefore choose:

$$M(r) = M_0 + M_2 r^2, \quad (\text{VI.11})$$

$$b(r) = 1 + k r^2, \quad (\text{VI.12})$$

$$G(r, v) = \frac{1 + k r^2}{\left[ 1 - \mu_0 \frac{r^4}{1+r^2} \left( 1 - \frac{1}{v^2} \right) \right]^2}. \quad (\text{VI.13})$$

As seen before the following model can lead to a black hole or a naked singularity, depending on the values chosen for  $M_2$ ,  $\mu_0$  and  $k$ . Some plots of the dynamics of this model for a specific choice of the parameters are given in the figures. The functions so defined correspond partially to the functions analyzed by Magli in [10] (where the cosmological term was not considered) and appear reasonable in the sense that the pressure is small compared to the energy density.

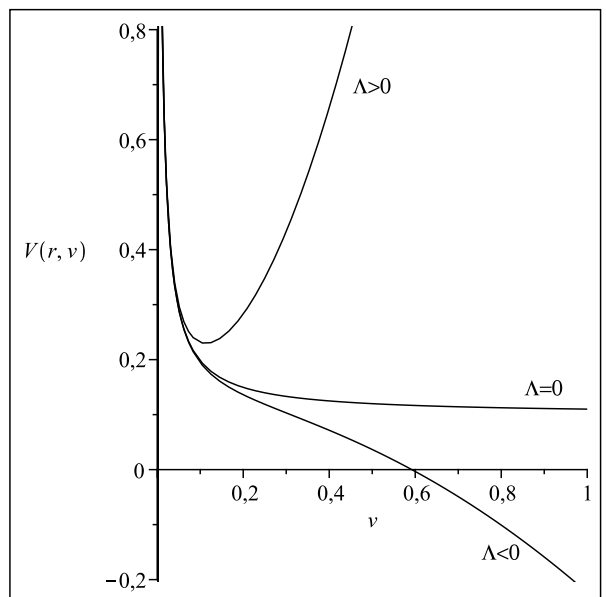


FIG. 2: Sample plot of  $V(r, v)$  in the unbound case, at a fixed value of  $r$  close to the center, with  $M_0 = 0.01$ ,  $M_2 = -0.001$ ,  $\mu_0 = 0.1$  and  $k = 1$ .

The first thing that can be noted is that the introduction of a positive cosmological constant can lead to unbound models even in cases that were bounded in the absence of the  $\Lambda$ -term, thus making the dynamics possible for configurations that were previously not allowed. On the contrary, a negative cosmological constant term might introduce zeroes in  $V(r, v)$ , thus altering and limiting the range of allowed configurations.

We can see that, since the dynamics happens in the region of positive  $V$ , when  $V$  has no zeroes then no bouncing behaviour can happen. Since in the region close to the center  $V$  diverges to plus infinity, we can infer that the core of the cloud will always collapse to the singularity. On the other hand, we note that at larger values of  $r$  oscillating behaviours between two fixed radii are possible. Furthermore at larger values of  $r$  bouncing behaviours can happen whenever the shell labeled with  $r$  reaches an event  $v$  for which  $V(r, v) = 0$ . In this case the collapse halts and the dynamics is reversed. In the figures

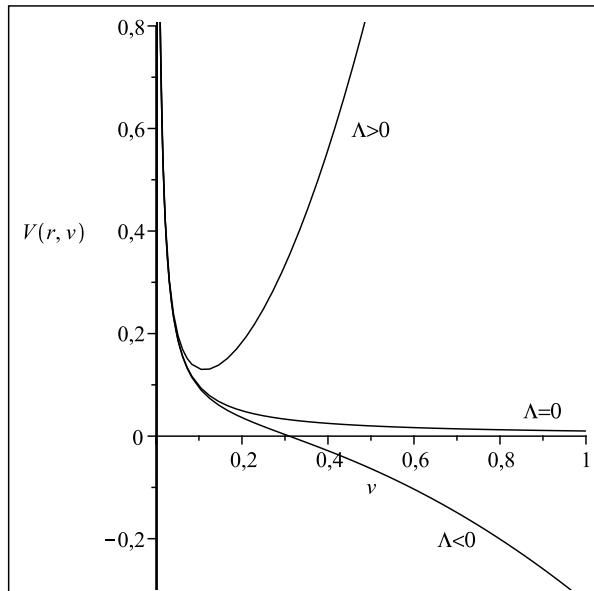


FIG. 3: Sample plot of  $V(r, v)$  in the marginally bound case, at a fixed value of  $r$  close to the center, with  $M_0 = 0.01$ ,  $M_2 = -0.001$ ,  $\mu_0 = 0.1$  and  $k = 0$ .

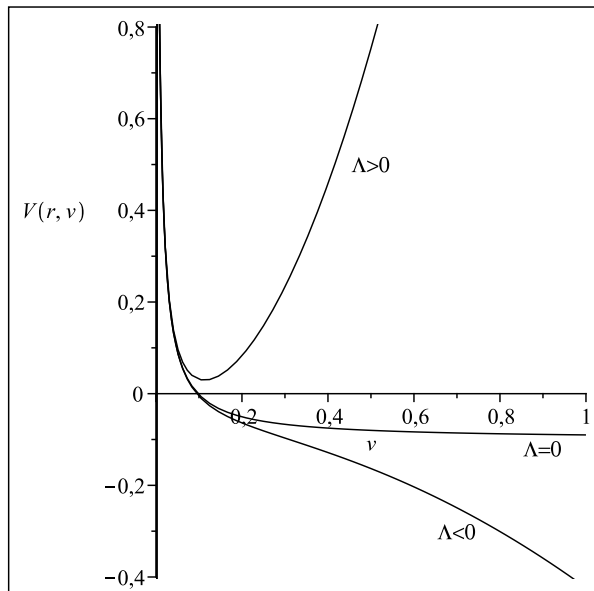


FIG. 4: Sample plot of  $V(r, v)$  in the bound case, at a fixed value of  $r$  close to the center, with  $M_0 = 0.01$ ,  $M_2 = -0.001$ ,  $\mu_0 = 0.1$  and  $k = -0.1$ .

we have restricted the analysis to a close neighborhood of the center of the cloud.

As we have noted earlier, the key physical interest for the bouncing behaviour comes for the collapsing clouds which are large enough in physical scales, but for stellar collapse, the cosmological constant plays no significant role. Of course, even without a cosmological term, a non-zero pressure can cause a bounce in the outer regions of the cloud. However, the inner region must still collapse to the singularity necessar-

ily, where the behaviour of the cloud is very much like the dust collapse. This is consistent with the regularity condition we mentioned earlier, namely that the tangential pressure must go to a vanishing value in the limit of approach to the center. A negative sign for the cosmological term preserves the same qualitative nature of the collapse as above, however, a positive cosmological constant brings in interesting changes in the collapse dynamics, as noted above.

## VII. CONCLUDING REMARKS

The framework to describe collapse of massive objects in general relativity is extremely rich and complex. Even in the limited case of spherical symmetry a wide array of different situations and collapse scenarios arises.

We have here presented a simple and straightforward analysis of the gravitational collapse of a spherical distribution of mass in the presence of tangential stresses. We have shown, how the occurrence of trapped surfaces and the eventual visibility of the singularity forming at the end of the collapse process is directly related to the nature and behaviour of the singularity curve.

This result further supports the conclusion that for a generic spherical object, the end-state of complete gravitational collapse will be either a black hole or a naked singularity. The initial configuration as well as the equations describing the properties of the matter cloud during collapse are the only features that, once coupled to Einstein equations for the dynamics of the model, will determine the final fate of collapse.

Since the radial and tangential pressures play different roles during the evolution of collapse dynamics, the present work helps understand how much the outcome is due to either of them. By investigating the role played by tangential pressures near the formation of the singularity in full generality, without any restrictions on the type of matter responsible for the pressure itself, we allowed for the possibility to consider negative stresses also. These, while still satisfying the weak energy condition (i.e. not considering phantom sources, and therefore constraining the pressure to a lower finite bound), are possible candidates for those exotic matter sources that have recently become of great interest in both cosmology and astrophysics.

In the past years many studies have been devoted to understand the physical processes involved in the complete gravitational collapse of a massive star. The treatment presented above offers a comprehensive tool to deal with the occurrence of naked singularities in the case of vanishing radial stresses and consequently provides a general picture in which earlier results are included. As an example, the work by Magli on collapse with tangential stresses [10], that provides a complete characterization of the outcome of collapse in terms of the root equation developed by Joshi and Dwivedi, can be translated in the present framework. As noted, the use of mass-area coordinates (or the area-radius coordinates) ultimately reduces to the change of variables  $(r, t) \rightarrow (r, v)$ , made here in order to study the equation of motion (II.18). Also, the analysis of the Joshi-dwivedi root equation, which governs the visibility of the singularity, is equivalent to the analysis of radial null

geodesics exposed in section III.

Furthermore, the results that give the conditions regarding the strength of the curvature singularity occurring at the center which were obtained by Nakao et al. [12] can be translated in the above formalism. Equations (2.13) and (2.14) in that paper are then found to be equivalent to equations (III.4) and (II.18) which determine the equation of motion and the expansion of  $t(r, v)$  near the center to first order in  $r$ , while equation (2.18) gives a condition for the occurrence of outgoing null geodesics equivalent to equation (III.17).

The models provided in section IV and section VI can be connected to the ones discussed in the previous papers and therefore provide a useful tool to broaden the spectrum of collapse scenarios studied in detail and expand our comprehension of the final stages of complete gravitational collapse in general relativity, from the analytical viewpoint.

The fact that positivity of the first non-vanishing  $\chi_i(0)$  is a necessary and sufficient condition for the occurrence of naked singularities may be possibly used in computer simulations, to analyze the endstate of collapse in a broader and more realistic range of situations. In fact, the early works in numerical relativity by Shapiro and Teukolsky [22] on naked singularity formation were hindered by the fact that the connection between the delay of trapped surfaces formation and the existence of outgoing null geodesics was still unclear. Shapiro

and Teukolsky considered models describing the collapse of prolate spheroids (thus in connection with the problem of axially symmetric collapse and the hoop conjecture), and models describing the collapse of counter rotating particles with vanishing angular momentum (the ‘Einstein cluster’ cited above). In both cases they found evidence for the non-occurrence of the apparent horizon before the time of the formation of the singularity. Still, Wald and Iyer showed that the fact that on certain slices trapped surfaces form at a later stage does not necessarily imply the visibility of the central singularity (they pointed out a time slicing of the Schwarzschild space-time whose evolution approaches arbitrarily the singularity without the appearance of an apparent horizon in any slice) [23].

We can argue here that in comoving coordinates the occurrence of trapped surfaces and the visibility of the singularity are related via the sign of  $\chi_i(0)$ , thus providing a firm footing for further numerical investigations. Hopefully, numerical simulations of naked singularity formation will help shed a better light on the final stages of realistic collapse and open new possibilities for the investigations of what happens in last moments of the life of a star. From an astrophysical point of view, this approach is clearly important, because if such processes indeed happen in the final stages of collapse of massive objects, it is then crucial to understand what kind of signature they bear in order to be able to eventually ‘observe’ them.

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