Epidemiology Biostatistics and Public Health - 2015, Volume 12, Number 2

ORIGINAL ARTICLES

Introducing the Harmonic Mean Solving a Tourist's Problem

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ABSTRACT

This exercise can be given to a group of students with basic knowledge of mathematics and physics at the beginning of a lesson. We can imagine that there will be some students that will solve the exercise using the "common sense" solution recalling basic notions of physics and some students that will solve the exercise recalling basic notion of physics and computing a mean velocity using the arithmetic mean, the most common mean. At the end of the exercise, the teacher will compare the two solutions and will present the harmonic mean as the fastest solution for the students that solved the problem using the "common sense" solution and as the correct mean to be used for the students that solved the exercise computing a mean velocity.

Key words: statistics, teaching, harmonic mean

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*DOI: **10.2427/11289***

Accepted on May 12, 2015

INTRODUCTION

This exercise is thought for students at the beginning of the introductory lesson on the measures of location, in particular on the harmonic mean. Presenting the exercise, the teacher can anticipate that a "common sense" as well as a "statistical" solution could be considered.

A tourist's problem

A sailing boat is working on the Lake Como; it sails downwind at 245 m/minute and upwind at 105 m/minute. A travel agency offers a round trip on the lake starting from the Como station and returning to it in 3 hours. A tourist is wondering: *"Will be possible to see George Clooney's villa in Laglio?"*.

Laglio is a small village near Como in Lombardy. It is situated on the western shore of the south-western branch of Lake Como, 14.5 km from the town of Como.

Solutions

"Common sense" solution: right but complex

First of all recall that **V**(elocity) = **D**(istance) / **T**(ime).

Let's define: d_g, t_g and v_g the distance, the time and the velocity to go (forward), respectively;

d_r, t_r and v_r the distance, the time and the velocity to return.

The known total time of the trip is $T = t_g + t_r$ so that $t_r = T - t_g$, and the unknown total distance the boat can run is $D = d_g + d_r$ with $d_g = d_r$.

Consequently: $v_g \cdot t_g = v_r \cdot t_r$ so that

$$t_r = \frac{v_g \times t_g}{v_r}$$

As $t_g + t_r = T$, one obtains:

$$t_g + \frac{v_g}{v_r} \times t_g = t_g \times \left(\frac{v_g + v_r}{v_r} \right) = T$$

and finally:

$$t_g = T \times \left(\frac{v_r}{v_g + v_r} \right)$$

Substituting the known terms with the corresponding values given by the problem

$$t_g = 180 \times \left(\frac{105}{105 + 245} \right) = 180 \times 0.3 = 54 \text{ minutes}$$

and $d_g = v_g \cdot t_g$ = 245 m/minute · 54 minutes = 13230 m = 13.23 km

George's villa cannot be seen!

"Statistical wrong" solution

It is convenient to write the equation $D = V_m \cdot T$ where V_m is the mean velocity of the whole trip, and D and T have the same meaning as before.

As we know, T = 180 minute; thus to compute D nothing else is to be done than to compute the mean velocity (V_m): the most common mean is the **arithmetic mean** of the two velocities: $(v_g + v_r)/2 = (245+105)/2 = 175$ m/minute.

In this case we obtain D = 175 · 180 = 31500 m therefore d_g = 15.75 km.

George's villa can be seen!

But this result differs from the result obtained in the "common sense" solution of the problem: 13.23 km. For the students that used the first solution the George's villa cannot be seen, instead, for the students that used the second solution the villa can be seen. Who is right?

"Statistical right" solution

Remembering that: $D=d_g+d_r$, $d_g=v_g \cdot t_g$, and $d_r=v_r \cdot t_r$, we can write $D = (v_g \cdot t_g + v_r \cdot t_r)$; but $D = V_m \cdot T = V_m \cdot (t_g + t_r)$ then

$$V_m = \frac{v_g \times t_g + v_r \times t_r}{t_g + t_r}$$

As previously, we can write t_r as a function of t_g. Namely,

$$t_r = \frac{v_g \times t_g}{v_r}$$

Therefore the mean velocity we are looking for is:

$$V_m = \frac{v_g t_g + v_r \times \frac{v_g t_g}{v_r}}{t_g + \frac{v_g t_g}{v_r}} = \frac{v_g t_g + v_g t_g}{\frac{v_r t_g + v_g t_g}{v_r}} = \frac{2 v_g t_g}{\frac{t_g (v_r + v_g)}{v_r}} =$$

$$2 v_g t_g \times \frac{v_r}{t_g (v_r + v_g)} = 2 \times \frac{v_g v_r}{v_g + v_r}$$

Substituting the numerical values given by the problem we have:

$$V_m = 2 \times \frac{245 \times 105}{245 + 105} = 147$$

D = 147 · 180 = 26460 m and d_g = 13.23 km

Unfortunately for the tourist, **George's villa cannot be seen!**

The formula

$$2 \frac{v_g v_r}{v_g + v_r}$$

gives the **harmonic mean** of the two velocities v_g and v_r. By definition the harmonic mean is the reciprocal of the arithmetic mean of the reciprocals; in the present context

$$V_{\text{harmonic mean}} = \frac{1}{\left(\frac{1}{v_g} + \frac{1}{v_r} \right) \times \frac{1}{2}} = \frac{2}{\frac{1}{v_g} + \frac{1}{v_r}} = 2 \times \frac{v_g v_r}{v_g + v_r}$$

CONCLUSION

As reported by Sheldon [1], the generalized mean

$$M_k = \left[\frac{1}{n}\left(x_1^k + x_2^k + \ldots + x_n^k\right)\right]^{\frac{1}{k}}$$

is the arithmetic mean if k=1, the harmonic mean if k=-1, the root mean square if k=2 and the geometric mean if k→0. Therefore, as said by Matejaš and Bahovec [2], *"… we can obtain a variety of means according to different practical situation"*. How can we choose the right mean in different practical situations?

The exercise that we proposed introduces the harmonic mean instead of the most common arithmetic mean.

The teacher can observe that the students that used the first solution get the right result but they do not recognise that the formula they used can be easily generalised as shown by the third solution.

To choose the correct mean value (V_m) (arithmetic, geometric, harmonic, ...), it should be noted that it is appropriate if, replaced in each of the n values on which it is calculated, it keeps unchanged the appearance of interest of the problem (in this case, the travel time).

We assume that the total distance D (=26460 m) covered by the boat in 3 h, with velocity v_g to go and v_r to return, is the same as the total distance covered with constant velocity (mean velocity V_m) to go forward and backward.

If there are n velocity (v_1, v_2, …, v_n) and all the covered distances (d_1, d_2, … , d_n) are equal to 1 (for simplicity), we have

$$T = \frac{1}{v_1} + \frac{1}{v_2} + - - - + \frac{1}{v_n}$$

Therefore

$$T = \frac{1}{v_1} + \frac{1}{v_2} + - - - + \frac{1}{v_n} = \frac{1}{V_m} + \frac{1}{V_m} + - - - + \frac{1}{V_m} = \frac{n}{V_m}$$

$$V_m = \frac{n}{T} = \frac{n}{1/v_1 + 1/v_2 + \ldots + 1/v_n}$$

That is the reciprocal of the arithmetic mean of reciprocals of velocity: ***the harmonic mean***.

References

[1] Sheldon N. The generalized mean, Teaching Statistics 2004; 26(1), 24–5. http://onlinelibrary.wiley.com/doi/10.1111/j.1467-9639.2004.0141a.x/full

[2] Mateja J and Bahovec V. Another Approach to Generalizing the Mean, Teaching Statistics 2008; 30(2), 49-52. http://onlinelibrary.wiley.com/doi/10.1111/j.1467-9639.2008.00310.x/abstract

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