이학박사 학위논문

Topics in the singularities of plurisubharmonic functions

(다중버금조화함수의 특이성에 관한 연구)

2021년 8월

서울대학교 대학원
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이 논문을 이학박사 학위논문으로 제출함

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안종봉의 이학박사 학위논문을 인준함

2021년 6월

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Topics in the singularities of plurisubharmonic functions

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
to the faculty of the Graduate School of
Seoul National University

by

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August, 2021

Abstract

Topics in the singularities of plurisubharmonic functions

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Plurisubharmonic functions are fundamental objects in complex analysis with many applications in complex geometry and even in algebraic geometry. Their singularities can be extremely complicated : some of the most important tools one can use to study the singularities include multiplier ideals and approximation theorems.

In the first part, based on joint work with Hoseob Seo, we study problems on equisingular approximation. Recently Guan gave a criterion for the existence of decreasing equisingular approximations with analytic singularities, in the case of diagonal type plurisubharmonic functions. We generalize a weaker version of this to arbitrary toric plurisubharmonic functions.

In the second part, we study plurisubharmonic singularities on singular varieties. Our main result in this part is a generalization of the Rashkovskii-Guenancia theorem on multiplier ideals of toric plurisubharmonic functions to the normal $\mathbb{Q}$-Gorenstein case. This also generalizes an algebraic result of Blickle to analytic multiplier ideals.

Key words: Plurisubharmonic functions, Multiplier ideal sheaves, Toric plurisubharmonic functions, Equisingular approximations
Student Number: 2014-21200

Contents

Chapter 1

Introduction

A *plurisubharmonic function* is one of the most important objects in complex analysis for connecting algebraic geometry and analytic geometry. The notion of plurisubharmonic (psh for short) functions was first independently developed by [Le42] and [O42] to characterize the pseudoconvexity of domains in $\mathbb{C}^n$. Plurisubharmonic functions are not only used for characterization of convexities but also used in many areas of complex geometry. For example, plurisubharmonic functions are local weights of singular Hermitian metrics with semipositive curvature (cf. [D10]).

The singularities of plurisubharmonic functions can be extremely complicated. Some of the most important tools one can use to study the singularities include multiplier ideals and approximation theorems. In this thesis, we present two main results (in Chapter 3 and in Chapter 4, respectively) from our study of multiplier ideals and approximation theorems.

After setting up preliminaries in Chapter 2, in Chapter 3, based on joint work with Hoseob Seo, we study problems on equisingular approximation. Recently Guan gave a criterion for the existence of decreasing equisingular approximations with analytic singularities, in the case of diagonal type plurisubharmonic functions. We generalize a weaker version of this to arbitrary toric plurisubharmonic functions.

In Chapter 4, we study plurisubharmonic singularities on singular varieties. Our main result in this part is a generalization of the Rashkovskii-Guenancia theorem on multiplier ideals of toric plurisubharmonic functions to the normal $\mathbb{Q}$-Gorenstein case. This also generalizes an algebraic result of Blickle to analytic multiplier ideals.

In the following Sections 1.1 and 1.2, we have more description of the two main results in Chapter 3 and Chapter 4, respectively.

1.1 Equisingular approximations of plurisubharmonic functions

Since singularities of psh functions are highly complicated in general, one frequently approximates a psh function by other psh singularities which are easier to handle.

In the fundamental work [D92a], Demailly gave a crucial method of approximating a general psh function φ by ones easier to understand, namely those given by multiplier ideals $\mathcal{J}(m\varphi)$ for $m \geqslant 1$. Since then, the Demailly approximation has had far-reaching developments and applications, see e.g. [DK01], [DPS01], [D10], [D13], [R12], [K14], [K16], [G16], [G20], [GL20].

In [DPS01, Theorem 2.3], an important variant of Demailly approximation was given so that one can approximate φ by a decreasing *equisingular* sequence $\varphi_m \to \varphi$ which means that the multiplier ideals are all equal : $\mathcal{J}(\varphi_m) = \mathcal{J}(\varphi)$. Such decreasing equisingular approximation was applied in the proof of the hard Lefschetz theorem [DPS01, Theorem 2.1]. However, the key property of analytic singularities could not be preserved in [DPS01, Theorem 2.3].

Indeed, Guan later showed by an example [G16] that one cannot in general expect all three of 'decreasing', 'equisingular' and 'analytic singularities' to hold simultaneously for an approximation of psh functions. On the other hand, it is known (from [D92a], [DPS01] and [D13]) that any two of the three can be made to hold in an approximation.

Moreover in a later paper [G20], for the special case of *diagonal* psh functions, Guan gave the following criterion for the existence of decreasing equisingular approximations with analytic singularities.

Theorem 1.1.1 (Qi'an Guan)**.** [G20, Theorem 1.1] Let $1 \leqslant m < n$ be integers. Let $a_1, \ldots, a_m$ be positive real numbers. The psh function $\varphi = \log \sum_{i=1}^{m} |z_i|^{a_i}$ on $\mathbb{C}^n$ has a decreasing equisingular approximation with analytic singularities near 0 if and only if one of the following conditions holds:

1. The psh function φ itself has analytic singularities near 0, i.e., there exists $c \in \mathbb{R}_{>0}$ such that $\frac{a_i}{c} \in \mathbb{Q}_{>0}$ for each $1 \leqslant i \leqslant m$.
2. The equation $\sum_{i=1}^{m} \frac{x_i}{a_i} = 1$ has no positive integer solutions.

Note that the function φ in Theorem 1.1.1 does not necessarily have analytic singularities when a_i's are irrational, cf. [K16, Example 4.1]. For example, $\varphi(z_1, z_2) := \log(|z_1|^{\sqrt{2}} + |z_2|^{\sqrt{3}})$ in $\mathbb{C}^2$ does not have analytic singularities but satisfies (2) in Theorem 1.1.1. Therefore φ has a decreasing equisingular approximation with analytic singularities near 0. In Chapter 3, we will generalize a weaker version of Theorem 1.1.1 for arbitrary toric psh functions. This is our first main result of this thesis, obtained from joint work with Hoseob Seo.

Theorem 1.1.2. Let φ be a toric psh function defined on $D(0, r) \subset \mathbb{C}^n$. Then the following are equivalent.

1. φ admits a decreasing, equisingular approximation (φ_m) by toric psh functions which have analytic singularities.
2. There exists a polyhedron P in $\mathbb{R}^n$ satisfying the following three conditions:
    - (i) $(2/c)P$ is a rational polyhedron for some $c > 0$,

(ii) $P(\varphi) \subseteq P$ and $P + \mathbb{R}^n_+ \subseteq P$,

(iii) $(\operatorname{int} P) \cap \mathbb{Z}^n_+ = (\operatorname{int} P(\varphi)) \cap \mathbb{Z}^n_+$.

This is a weaker version of Theorem 1.1.1 since in (1), the approximant φ_m itself is assumed toric. Here, $r = (r_1, \dots, r_n)$ is a polyradius of a polydisk in $\mathbb{C}^n$ and a *polyhedron* is a finite intersection of upper hyperplanes in $\mathbb{R}^n$ (see Definition 3.2.10, Definition 3.2.12 and Theorem 3.2.16). In particular, if all equations of hyperplanes are represented by rational coefficients and rational constant, we say that the polyhedron is rational.

Our main strategy is to consider convex conjugates of toric psh functions. We will present an explicit characterization for convex functions associated to toric psh functions with analytic singularities. Then we will show the relation between convex functions and their conjugates when convex functions are from toric psh functions with analytic singularities. Using this we will prove the main theorem using convergence of convex conjugates.

1.2 Multiplier ideal sheaves on singular varieties

On a complex manifold, plurisubharmonic functions already have complicated singularities. On a (reduced) singular variety or on a (reduced) complex space, plurisubharmonic functions are still defined. Study of their singularities becomes certainly much harder in this setting of a singular variety.

As a first guide, we need to look at the study of singularities in algebraic geometry, in the context of the minimal model program and singularity of pairs, cf. [KM98]. Let (X, Δ) be a pair and let $\mathfrak{a}$ be an ideal sheaf defined on X. Then the *multiplier ideal sheaf* $\mathcal{J}(X, \Delta)$ of $\mathfrak{a}$ on (X, Δ) is defined as

$$\mathcal{J}((X, \Delta), \mathfrak{a}) = \mu_\star \mathcal{O}_{X'}(K_{X'} - \lfloor \mu^\star (K_X + \Delta) + cF \rfloor).$$

Here μ is a log resolution of Δ and $\mathfrak{a}$. Also F is the inverse image sheaf of $\mathfrak{a}$ by μ. For more on definitions and properties of the multiplier ideal sheaves on normal varieties, we refer to [L04], [FH09], [BFFU15].

In this thesis, as a first step toward plurisubharmonic singularities on a singular variety, we study toric psh functions. By a *toric psh function*, we mean a psh function which is invariant under the torus action. These ideas are based on convex geometry related to toric psh functions and monomial ideals. For related topics, we refer to [Ho01], [Gu11], [R11], [Bl04] for the concepts of *Newton polyhedron*(or *Newton convex body*) of monomial ideals and toric psh functions for computations of multiplier ideal sheaves.

As the second main result of this thesis, we generalize the Rashkovskii-Guenancia theorem ([R11], [Gu11]) to toric psh functions on a singular toric variety.

Theorem 1.2.1. [Theorem 4.4.1] Let X be a normal $\mathbf{Q}$-Gorenstein affine toric variety given by the cone $\sigma \subset N_{\mathbb{R}}$ whose dimension is set to be $n = \dim N_{\mathbb{R}}$. Let φ be a toric psh function on X. Then the multiplier ideal $\mathcal{J}(\varphi) := \mathcal{J}(\varphi)(X)$ of φ on X is a monomial ideal and given by the following condition

$$\chi^v \in \mathcal{J}(\varphi) \iff v - \operatorname{div}(K_X) \in \operatorname{int}(P(\varphi))$$

where χ^v is a monomial in the affine coordinate ring $\mathbb{C}[X]$ of X and $\operatorname{div}(K_X)$ is the point associated to a canonical divisor of X in the vector space $M_{\mathbb{R}}$, the dual space of the vector space $N_{\mathbb{R}}$.

In fact, Theorem 1.2.1 generalizes results in [Gu11], [R11], [Ho01] and [Bl04]. We also have the following corollary.

Corollary 1.2.2. Let X be a $\mathbb{Q}$-Gorenstein affine toric variety and let φ be a toric psh function defined on X. Then the openness property holds, i.e.,

$$\mathcal{J}(\varphi) = \mathcal{J}((1+\epsilon)\varphi) \text{ for } \epsilon \ll 1.$$

Corollary 1.2.2 says that the openness property hold for toric psh functions defined on affine toric varieties, as a partial generalization in this special case of the openness theorem of Guan and Zhou [GZ15].

Chapter 2

Preliminaries

In Chapter 2, we prepare preliminaries needed for our main results in the following two chapters. In Section 2.1, we introduce the notion of psh functions and their properties. In Section 2.2, we introduce psh singularities and multiplier ideal sheaves of psh functions. In Section 2.3, we introduce toric psh funtions together with their properties and some examples.

2.1 Plurisubharmonic functions

In this section we will introduce psh functions. These objects appear to characterize the convexity of domains in $\mathbb{C}^n$. However, psh functions do not play an important role in several complex variables merely. Plurisubharmonic functions are used in complex geometry vastly with notion of singularities in complex geometry. Now, let $\Omega \subset \mathbb{C}^n$ be an open set. Most of materials are included in [B], [DX].

Definition 2.1.1. A function $\varphi : \Omega \longrightarrow [-\infty, \infty)$ is said to be *psh* if φ is upper-semicontinuous, locally in L^1, not identically $-\infty$ on any component of Ω, and the restriction of φ to each complex line is subharmonic, i.e.,

$$\varphi(z_0) \leqslant \frac{1}{2\pi}\int_0^{2\pi} \varphi(z_0+\xi e^{i\theta})d\theta$$

for all $z_0 \in \Omega$ and $\xi \in \mathbb{C}^n$ such that $\{z_0 + z\xi \mid z \in \mathbb{C}, |z| \leqslant 1\}$.

Remark 2.1.2. Some authors do not impose a condition $\varphi \in L^1_{loc}(\Omega)$ nor being identically $-\infty$. In our paper, we add $\varphi \in L^1_{loc}$ to assure the well-definedness of $\sqrt{-1}\partial\bar{\partial}\varphi$ as a current and to exclude the trivial case being identically $-\infty$.

The set of psh functions on Ω is denoted by $Psh(\Omega)$. We mention some properties of psh functions. Some authors include psh functions that identically equal to $-\infty$ on some component. But for the sake of convenience, we do not include them.

Proposition 2.1.3. Plurisubharmonic functions have the following properties.

i. If $\varphi \in Psh(\Omega)$, then it is also subharmonic as $2n$-variables.

ii. If $(\varphi_k) \subset Psh(\Omega)$ is a decreasing sequence of psh functions and if $\varphi := \lim\limits_{k\to\infty} \varphi_k$ is not identically $-\infty$, then φ is also psh.

iii. If $\varphi \in Psh(\Omega)$ is psh and (ρ_ϵ) is a family of smoothing kernel, then the convolution $(\varphi_\epsilon) := (\varphi \star \rho_\epsilon)$ is smooth, defined on Ω_ϵ. Moreover, the family (φ_ϵ) is non-decreasing in ϵ and $\lim\limits_{\epsilon\to 0}\varphi_\epsilon = \varphi$.

iv. Let $\varphi_1, \ldots, \varphi_k \in Psh(\Omega)$ and let $\chi : \mathbb{R}^k \to \mathbb{R}$ be a convex function which is non-decreasing in each variable. Then the composition $\chi(\varphi_1, \ldots, \varphi_k)$ is also in $Psh(\Omega)$. In particular $\varphi_1 + \ldots + \varphi_k$, $\max\{\varphi_1, \ldots, \varphi_k\}$, $\log(e^{\varphi_1} + \ldots + e^{\varphi_k})$ are psh.

v. Let $(\varphi_\alpha) \subset Psh(\Omega)$ be locally uniformly bounded above and $\varphi = \sup \varphi_\alpha$. Then the regularized upper envelope

$$\varphi^\star := \lim_{\epsilon \to 0} \sup_{B(z,\epsilon)} \varphi$$

is psh and is equal to φ a.e..

vi. Let $\varphi \in C^2(\Omega)$. Then φ is psh iff its complex Hessian

$$\left(\frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} \right)_{1 \leqslant j,k \leqslant n}$$

is pointwise semi-positive definite. Equivalently, $\sqrt{-1}\partial\bar{\partial}\varphi \geqslant 0$.

vii. Let $f : \Omega \to \Omega'$ be a holomorphic mapping between domains. Then if $\varphi \in Psh(\Omega')$, $f^\star\varphi \in Psh(\Omega)$ as a distribution.

Note that most properties in Proposition 2.1.3 follow from the properties of subharmonic functions. Also, we can use Proposition 2.1.3 (vii) to define psh functions on complex manifolds.

Definition 2.1.4. Let X be a complex manifold of dimension n and let $\varphi : X \to [-\infty, \infty)$. Then φ is said to be *psh* on X if for any local trivialization $U \subset \mathbb{C}^n$, $g : U \to X$, $g^\star\varphi \in Psh(U)$. If φ is locally equal to the sum of a psh function and a smooth function, we say that φ is *quasi-psh*.

Note that the above definition is well-defined, since every transition function is holomorphic and do not affect pshity of φ. Sometimes we have to deal with psh singularities on compact complex manifolds. However we know that the only possible psh functions defined on compact complex manifolds should be constant. Therefore instead of considering psh functions, we sometimes consider the class of quasi-psh functions. Next, we define an important class of quasi-psh functions, namely quasi-psh functions with analytic singularities.

Definition 2.1.5. Let X be a complex manifold of dimension n and φ a quasi-psh function on X. Then φ is said to have *analytic singularities* if for any $x \in X$, there is a neighborhood of $x \in U$ and holomorphic functions $f_1, \ldots, f_k \in \mathcal{O}(U)$ such that φ can be represented as $\varphi = \frac{c}{2}\log(|f_1|^2 + \ldots + |f_k|^2) + \mathrm{O}(1)$ for some $c \geqslant 0$ on U.

We sometimes denote $\varphi = c\log|\mathfrak{a}| + \mathrm{O}(1)$ where $\mathfrak{a} = (f_1, \ldots, f_k)$ is an ideal in $\mathcal{O}(U)$. In this notation, we say φ has analytic singularities of type $\mathfrak{a}^c$. Note that we did not define the function $c\log|\mathfrak{a}|$ and we cannot even define $\log|\mathfrak{a}|$ as a function, since we cannot choose a canonical set of generators for $\mathfrak{a}$. But the notation as above makes sense because change of generators only affects by $\mathrm{O}(1)$ term.

2.2 Plurisubharmonic singularities

For a given psh function φ, there are several ways to measure how φ is singular. We begin with the Lelong number of psh functions.

2.2.1 Lelong numbers of psh functions

In this subsection, we will define the Lelong number and variants of Lelong numbers for φ. Then we will interpret them by an algebraic language. We will fix X being a complex manifold of dimension n and Ω being a domain in $\mathbb{C}^n$ in this subsection.

Let $\varphi \in Psh(\Omega)$. Pick $x \in \Omega$ such that $D(x, r) \subset\subset \Omega$. Due to subharmonicity of φ, we know that $f(t) := \sup_{B(x,e^t)} \varphi$ is convex increasing function defined on $(-\infty, \log r]$. Thus, we have $\dfrac{f(t) - f(\log r)}{t - \log r}$ is non-decreasing function of t. Letting $t \to -\infty$, we obtain the following limit.

Definition 2.2.1. Let φ, f as above. Then we define the *Lelong number* of φ at x by

$$\nu_x(\varphi) = \lim_{t \to -\infty} \frac{f(t)}{t}.$$

Note that the convexity of f implies

$$\frac{f(t) - f(\log r)}{t - \log r} \geqslant \lim_{t \to -\infty} \frac{f(t) - f(\log r)}{t - \log r} = \nu_x(\varphi)$$

for $t \leqslant \log r$. In other words, $\varphi(z) \leqslant \nu_x(\varphi) \log \frac{|z-x|}{r} + \sup_{B(x,r)} \varphi$. Since r is fixed, we can arrange $\sup_{B(x,r)} \varphi$ to be $\mathrm{O}(1)$. It follows that

$$\nu_x(\varphi) = \max\{\gamma \in \mathbb{R}_+ \mid \varphi(z) \leqslant \gamma \log \frac{|z-x|}{r} + \mathrm{O}(1) \text{ near } x\}.$$

This type of inequality is in particular valid in the case when φ has analytic singularities. Suppose that φ has the singularity of type $\mathfrak{a}^c$ at x. Then the Lelong number of φ at x is the product of c and the multiplicity of $\mathfrak{a}$ at x.

2.2.2 Multiplier ideal sheaves of psh functions

The notion of multiplier ideal sheaf was introduced in [N89] (cf. [D93b]) (while related ideas had already existed).

Definition 2.2.2. Let φ be a psh function defined on an open subset $\Omega \subset \mathbb{C}^n$. The *multiplier ideal sheaf* $\mathcal{J}(\varphi)$ of φ is the ideal sheaf of $\mathcal{O}_\Omega$ such that each germ satisfies the following integrability condition:

$$\mathcal{J}(\varphi)_x = \{f \in \mathcal{O}_{\Omega,x} \mid |f|^2 e^{-2\varphi} \text{ is locally integrable at } x\}.$$

Here, the measure is taken to be the Lebesgue measure on (Ω, z) where z is a local holomorphic coordinate.

If φ has analytic singularities, the definition of $\mathcal{J}(\varphi)$ is related to algebraic multiplier ideal sheaf. Explicitly, if φ is locally equal to $\frac{c}{2}\log(|f_1|^2 + \ldots + |f_k|) + \mathrm{O}(1)$, then the multiplier ideal sheaf is equal to $\mathcal{J}(\mathfrak{a}^c)$ where $\mathfrak{a}$ is an ideal on $\mathcal{O}(U)$ generated by $f_1, \ldots, f_k$. For the proof of this, we need the following basic functorial property.

Proposition 2.2.3 ([D10, Proposition 5.8], [L04, Proposition 9.3.43])**.** Let $\mu : X' \to X$ be a modification of complex manifolds and let φ be a psh function defined on X. Then

$$\mu_\star(\mathcal{O}(K_{X'} \otimes \mathcal{J}(\varphi \circ \mu)) = \mathcal{O}(K_X) \otimes \mathcal{J}(\varphi).$$

Before discussing the well-definedness of multiplier ideal sheaves, recall the definition of algebraic multiplier ideal sheaf. Let $\mathfrak{a}$ be an ideal sheaf and let $c > 0$ be a positive number. Let $\mu : X' \to X$ be a log resolution of $\mathfrak{a}$. Then we define the *(algebraic) multiplier ideal sheaf* $\mathcal{J}(\mathfrak{a}^c)$ associated to c and $\mathfrak{a}$ by

$$\mathcal{J}(\mathfrak{a}^c) = \mu_\star \mathcal{O}_{X'}(K_{X'/X} - \lfloor c \cdot D \rfloor).$$

Here, $K_{X'/X}$ is the relative canonical divisor of X' over X and $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-D)$.

Proposition 2.2.4. [L04, Theorem 9.3.42] Let X be a complex manifold and let $\varphi = \frac{c}{2}\log(|f_1|^2 + \ldots + |f_k|^2) + \mathrm{O}(1)$ be a psh function with analytic singularities defined on an open subset $U \subset X$, then $\mathcal{J}(\varphi) = \mathcal{J}(\mathfrak{a}^c)$ where $\mathfrak{a}$ is an ideal generated by $f_1, \ldots, f_k \in \mathcal{O}(U)$.

We end this section with relating the Lelong number with multiplier ideal sheaf.

Lemma 2.2.5. [Sk72] Let φ be a psh function defined on an open subset $\Omega \subset \mathbb{C}^n$ and let $x \in \Omega$.

i. If $\nu_x(\varphi) < 1$, $e^{-2\varphi}$ is locally integrable near x, i.e., $\mathcal{J}(\varphi)_x = \mathcal{O}_{\Omega,x}$.

ii. If $\nu_x(\varphi) \geqslant n+s$ for some integer $s \geqslant 0$, then $e^{-2\varphi} \geqslant C|z-x|^{-2n-2s}$ in a neighborhood of x and $\mathcal{J}(\varphi)_x \subset \mathfrak{m}_x^{s+1}$ where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{\Omega,x}$.

iii. The zero variety $V(\mathcal{J}(\varphi))$ satisfies $E_n(\varphi) \subset V(\mathcal{J}(\varphi)) \subset E_1(\varphi)$ where $E_c(\varphi)$ is the c-upperlevel set of Lelong numbers of φ.

2.3 Toric plurisubharmonic functions

In this section, we briefly introduce what is toric psh function and related properties of toric psh functions. We begin with the definition.

Definition 2.3.1. Let $D(0,\mathbf{r})$ be a polydisk in $\mathbb{C}^n$ with a polyradius $\mathbf{r} = (r_1, \ldots, r_n)$. A psh function defined on $D(0,\mathbf{r})$ is said to be *toric*(or *multi-circled* in [R11]) if its value is invariant under torus action, i.e., $\varphi(z_1, \ldots, z_n) = \varphi(e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n)$ where $\theta_1, \ldots, \theta_n$ are elements in $\mathbb{R}$.

In case psh function φ is toric, φ has a nice property by following.

Proposition 2.3.2. Let φ be a toric psh function defined on $D(0,\mathbf{r})$. Then one can associate the increasing convex function g defined on $(-\infty, \log r_1) \times \ldots \times (-\infty, \log r_n)$ which satisfies $g(\log|z_1|, \ldots, \log|z_n|) = \varphi(z_1, \ldots, z_n)$.

Sketch of the Proof. Fix $z \in D(0,r)$ and let the radius of each component be given by t_i. Then $\varphi(z)$ is equal to $\sup\limits_{w \in D(0,t)} \varphi(w)$ where $t = (t_1, \ldots, t_n)$. Note that $t \mapsto \sup\limits_{w \in D(0,t)} \varphi(w)$ is already increasing and convex by convexity properties of psh functions. See [DX, §1.5, 5.13, 5.14] for the convexity properties of psh functions. □

Also, there is a very nice description of multiplier ideal sheaf when φ is toric. Before characterization, we need the following preliminary tools in convex analysis. Let us begin with the definition.

Definition 2.3.3. Let $g : \mathbb{R}^n \longrightarrow (-\infty, +\infty]$ be a convex function which is not trivial in the sense of being g is not identically neither $-\infty$ nor ∞. Then define the *convex conjugate* $g^* : \mathbb{R}^n \longrightarrow (-\infty, +\infty]$ by $g^*(x) \overset{\text{def}}{=} \sup_{y \in \mathbb{R}^n} (\langle x, y \rangle - g(y))$. Also, the domain of g^* is called the *Newton convex body* of g and we denote the Newton convex body of g by $P(g)$.

Remark 2.3.4. We mention some properties of g^*. Let g be a convex function defined on $\mathbb{R}^n$.

1. g^* is also a convex function.
2. If g is increasing in each variable, then g^* is decreasing in each variable, and vice versa.
3. $g^{**} \leqslant g$ and g^{**} is lower semicontinuous. $g^{**} = g$ if and only if g is convex and lower semicontinuous.

The first two statements are straightforward and see [H07, Chapter 2] for the last.

We define $P(\varphi)$ by $P(g)$ where g is the increasing convex function associated with φ.

Remark 2.3.5. We mention some properties of $P(\varphi)$.

1. The Newton convex body $P(g)$ is closed under the operation of translation by $v \in \mathbb{R}^n_{\geqslant 0}$. Indeed, if $x \in P(g)$ and $v \in \mathbb{R}^n_{\geqslant 0}$, then $x + v \in P(g)$. Later, we will generalize this property to toric psh functions defined on arbitrary affine toric variety.
2. If φ has analytic singularities of monomial ideal $\mathfrak{a}$ and $c > 0$, then $P(\varphi)$ is a convex hull of the union of $c \cdot v + \mathbb{R}^n_{\geqslant 0}$ where $z^v \in \mathfrak{a}$.

Remark 2.3.6. Let g be a convex function on $\mathbb{R}^n_- := \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n < 0\}$ which is increasing in each variable. For positive real numbers $r_1, \ldots, r_n$ and $x \in \mathbb{R}^n$, we have the following inequality:

$$\begin{aligned}\sup_{y \in I_{\mathbf{r}}} (\langle x, y\rangle - g(y)) &\leqslant \sup_{y \in \mathbb{R}^n_-} (\langle x, y\rangle - g(y)) \\ &= \sup_{y \in \mathbb{R}^n_-} (\langle x, y - \mathbf{r}\rangle - g(y - \mathbf{r}) + \langle x, \mathbf{r}\rangle - g(y) + g(y - \mathbf{r})) \\ &\leqslant \sup_{y \in I_{\mathbf{r}}} (\langle x, y\rangle - g(y)) + \langle x, \mathbf{r}\rangle\end{aligned}$$

where $I_{\mathbf{r}} = (-\infty, -r_1) \times \ldots \times (-\infty, -r_n)$ is a product of open intervals in $\mathbb{R}$ and $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n$. This shows that shrinking the domain of a toric psh function near the origin does not affect its Newton convex body. Also, the structure of Newton convex body determines L^2-integrability of monomials with respect to $e^{-2\varphi}$, so this observation gives the integrability of a function on a bounded open subset containing 0 is independent of a choice of a bounded open subset.

Remark 2.3.7. In [Gu11, Definition 1.7], [R11, Section 3.1], authors independently define the notion of Newton convex bodies([R11] used the term *indicator diagram* instead) of φ to characterize multiplier ideal sheaves of toric psh functions. In this paper, I mainly use terms used in [Gu11].

Using these definitions and notions we can describe the following characterization of multiplier ideal sheaf in toric psh functions

Theorem 2.3.8. [Gu11, Theorem 1.13], [R11, Proposition 3.1] Let φ be a toric psh function defined on $D(0, \mathbf{r})$. Then the multiplier ideal $\mathcal{J}(\varphi) := \mathcal{J}(\varphi)(D(0, \mathbf{r}))$ is a monomial ideal and we have:

$$z^\alpha \in \mathcal{J}(\varphi) \iff \alpha + \mathbf{1} \in \operatorname{int}(P(\varphi)).$$

Chapter 3

Equisingular approximations of plurisubharmonic functions

In Chapter 3, we will discuss results on the approximation of psh functions by psh functions with analytic singularities. We will describe sufficent and necessary conditions for admitting decreasing equisingular with analytic singularities approximation for toric psh functions. Among other things, we use convex analysis for our main theorem in this chapter.

3.1 Equisingular approximations

In this section, we introduce some preliminaries for our main theorem. Let us begin with the Demailly approximation theorem.

Theorem 3.1.1. [D92a] Let φ be a psh function on a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$. For every $m > 0$, let $\mathscr{H}_\Omega(m\varphi)$ be the Hilbert space of holomorphic functions f on Ω such that $\int_\Omega |f|^2 e^{-2m\varphi} d\lambda < \infty$ and let $\varphi_m = \frac{1}{2m} \log \sum |\sigma_l|^2$ where (σ_l) is an orthonormal basis of $\mathscr{H}_\Omega(m\varphi)$. Then there are constants $C_1, C_2 > 0$ independent of m such that

i. $\varphi(z) - \dfrac{C_1}{m} \leqslant \varphi_m(z) \leqslant \sup\limits_{|\zeta - z| < r} \varphi(\zeta) + \dfrac{1}{m} \log \dfrac{C_2}{r^n}$ for every $z \in \Omega$ and

$r < d(z, \partial\Omega)$. In particular, φ_m converges to φ pointwise and in L^1_{loc} topology on Ω as $m \to \infty$ and

ii. $\nu(\varphi, z) - \dfrac{n}{m} \leqslant \nu(\varphi_m, z) \leqslant \nu(\varphi, z)$ for every $z \in \Omega$.

The proof of Theorem 3.1.1 uses L^2 extension of holomorphic functions from points. Theorem 3.1.1 connects some results in algebraic geometry into the analytic geometry. For example, any positive singular metric of singular Hermitian line bundle can be approximated by psh functions with logarithmic poles. These functions can be transformed into a metric associated with simple normal crossing divisors via techniques in algebraic geometry such as log resolution of ideal sheaf. We refer to [D93b], [DK01] for applications to algebraic geometry.

Example 3.1.2. [K14, Theorem 2.1] proved that there exists a decreasing subsequence of (φ_n) in Theorem 3.1.1 in sense of adding some constants. Explicitly, $(\varphi_{(k_n)})$ with $(k_n) = (2^n)$ is decreasing if we add some constant to each psh approximant. The proof uses the subadditivity of multiplier ideal sheaves. (For the subadditivity theorem, see [DEL00].) However we do not expect that the approximation in Theorem 3.1.1 being decreasing in general. Let $X = \mathbb{C}^2$ with coordinates (x, y) and let $D = \sum_{i=1}^{3} \frac{2}{3} D_i$ where $D_1 = \{x = 0\}$, $D_2 = \{y = 0\}$, $D_3 = \{x + y = 0\}$. Then there is no (C_n) such that makes $\{\varphi_n + C_n\}$ decreasing.

Now we will introduce some preliminary results on equisingular approximations and examples. We mainly follow [DPS01], [G16], [G20].

Before introducing the fundamental result from [DPS01], we define the equisingularities of two psh functions.

Definition 3.1.3. Let φ, ψ be two (quasi-)psh functions defined on complex manifold X. Then φ, ψ are said to be *equisingular* if their two multiplier ideal sheaves coincide.

Theorem 3.1.4. [DPS01] Let $T = \alpha + \sqrt{-1}\partial\bar{\partial}\varphi$ be a closed $(1,1)$-current on a compact Hermitian manifold (X, ω), where α is a smooth $(1,1)$-closed form and φ a quasi-psh function. Let γ be a continuous real $(1,1)$-form such that $T \geqslant \gamma$. Then there is a sequence (φ_ν) converging to φ pointwise where

1. φ_ν is smooth in $X \backslash Z_\nu$ of an analytic set $Z_\nu \subset X$.
2. $\{\varphi_\nu\}$ is a decreasing sequence and $Z_\nu \subset Z_{\nu+1}$ for all ν.
3. $\int_X (e^{-2\varphi} - e^{-2\varphi_\nu}) dV_\omega$ is finite for every ν and converges to 0 as $\nu \to \infty$.
4. $\mathcal{J}(\varphi_\nu) = \mathcal{J}(\varphi)$ for all ν.
5. $T_\nu = \alpha + \sqrt{-1}\partial\bar{\partial}\varphi_\nu$ satisfies $T_\nu \geqslant \gamma - \epsilon_\nu \omega$, where $\lim\limits_{\nu\to\infty} \epsilon_\nu = 0$.

We have two remarks.

Remark 3.1.5. Condition 3 in Theorem 3.1.4 is stronger than condition 4 in Theorem 3.1.4. Indeed, since we know $\varphi_\nu \geqslant \varphi$, $\mathcal{J}(\varphi) \subseteq \mathcal{J}(\varphi_\nu)$. Then if $f \in \mathcal{J}(\varphi_\nu)$, condition 3 directly tells us that f is also in $\mathcal{J}(\varphi)$.

Remark 3.1.6. Condition 1 in Theorem 3.1.4 gives a very intriguing problem. The problem is whether φ_ν can have analytic singularities whose poles are along Z_ν. Unfortunately, the answer is negative, because approximants φ_ν in the proof of Theorem 3.1.4 may be locally equal to φ itself near some singular point.

Now the following is a specific example due to Guan of psh singularities that cannot admit a decreasing equsingular approximation with analytic singularities.

Example 3.1.7. [G16] Let $n \geqslant 2$ and $(\mathbb{C}^n, (z_1, \ldots, z_n))$ be coordinates defined on $\mathbb{C}^n$. Let

$$\varphi_1(z) := \log(\max\{|z_1|, \ldots, |z_{n-1}|, |z_n|^a\}),$$

where $1 < a < \frac{3}{2}$ is irrational. Let

$$\varphi_2 := \max\{\varphi_1 - 18n, 6\log(|z_1|^2 + \cdots + |z_n|^2) - 6n\}.$$

Let

$$\varphi := -M_\eta(-\varphi_2, 0),$$

where $\eta = (\frac{1}{1000}, \frac{1}{1000})$ and M_η is as in [DX, Lemma I.5.18]. Then there exists a $c > 0$ that $c\varphi$ is psh and does not admit decreasing, equisingular approximation with analytic singularities.

We remark that the proof that φ is psh needs some cumbersome computations based on definition of M_η. Also it can be shown that φ is equal to φ_1 near 0.

The work of Guan [G20] generalized this example in more broader category. He presented a criterion whether psh function in certain class admits the decreasing equisingular approximation with anlaytic singularities or not. First, we briefly introduce what class of psh functions we will deal with.

Let $(z_1, \ldots, z_n)$ be the coordinates on $\mathbb{C}^n$. We will consider the following class of psh weights:

$$\{\log \sum_{i=1}^{m} |z_i|^{a_i} \mid m \leqslant n, a_i > 0 \text{ for any } 1 \leqslant i \leqslant m\}.$$

Here, a psh weight means a germ of psh function at 0.

Theorem 3.1.8. [G20] The weight $\varphi = \log \sum_{i=1}^{m} |z_i|^{a_i}$ has decreasing equisingular approximations with analytic singularities near 0 if and only if one of the following statements holds:

1. φ has analytic singularity near 0, i.e., there exists $c \in \mathbb{R}_{>0}$ such that $\frac{a_i}{c} \in \mathbb{Q}_{>0}$ for any $1 \leqslant i \leqslant m$.
2. The equation $\sum_{i=1}^{m} \frac{x_i}{a_i} = 1$ has no positive integer solutions.

In this subsection, we will give ideas used for the proof of Theorem 3.1.8. Before we prepare the proof of theorem, we add a simple remark.

Remark 3.1.9. Theorem 3.1.8 contains the result of Example 3.1.7. Indeed, if we set a_i's in φ_1 to be $a_1, \ldots, a_{m-1} = 1$, $a_m = a$ where a is an irrational between $(1, \frac{3}{2})$. Then φ_1 satisfies the condition 2 of Theorem 3.1.8. Thus φ_1 does not admit a decreasing equisingular approximation with analytic singularities.

Note that the function φ in Theorem 3.1.8 does not necessarily have analytic singularities when a_i's are irrational, cf. [K16, Example 4.1]. For example, $\varphi(z_1, z_2) := \log\left(|z_1|^{\sqrt{2}} + |z_2|^{\sqrt{3}}\right)$ in $\mathbb{C}^2$ does not have analytic singularities but satisfies (2) in Theorem 3.1.8.

Remark 3.1.10. When $\varphi = \log \sum_{i=1}^{m} |z_i|^{a_i}$, one can easily compute the multiplier ideal of φ at $0 \in \mathbb{C}^n$ using the Rashkovskii-Guenancia theorem 2.3.8. Note that the Newton convex body of φ is given by intersection of $\mathbb{R}^n_{\geqslant 0}$ and $\{x \in \mathbb{R}^n \mid \sum_{i=1}^{m} \frac{x_i}{a_i} \geqslant 1\}$. Then the multiplier ideal sheaf of φ at 0 is monomial and described as

$$\{z^{\mathbf{m}} \in \mathcal{J}(\varphi)_0 \mid \mathbf{m} + \mathbf{1} \in \mathrm{int}(P(\varphi))\}.$$

Here, $\mathbf{m} = (m_1, \ldots, m_n)$ is a multi-index for exponent of monomial and $\mathbf{1} = (1, \ldots, 1)$.

3.2 Equisingular approximation of toric psh functions

So far, we introduced a series of examples related to nonexistence of decreasing equisingular approximation with analytic singularities. In particular, examples of [G16] and [G20] are both toric psh functions. Inspired by methods and proofs of these counterexamples, we would like to present a criterion of existence of toric decreasing equisingular approximation with analytic singularities. Our main objective is following.

Theorem 3.2.1. Let φ be a toric psh function defined on $D(0,r)$. The followings are equivalent.

1. φ admits a decreasing, equisingular approximation (φ_m) by toric psh functions which have analytic singularities.

2. There exists a polyhedron P satisfying the following three conditions:

   (i) $(2/c)P$ is a rational polyhedron for some $c > 0$,

   (ii) $P(\varphi) \subseteq P$ and $P + \mathbb{R}^n_{\geqslant 0} \subseteq P$,

   (iii) $(\operatorname{int} P) \cap \mathbb{Z}^n_{\geqslant 0} = (\operatorname{int} P(\varphi)) \cap \mathbb{Z}^n_{\geqslant 0}$.

Here, $r = (r_1, \ldots, r_n)$ is a polyradius of a polydisk in $\mathbb{C}^n$ and a *polyhedron* is a finite intersection of upper hyperplanes in $\mathbb{R}^n$ (see Definition 3.2.10, Definition 3.2.12 and Theorem 3.2.16). In particular, if all equations of hyperplanes are represented by rational coefficients and rational constant, we say that the polyhedron is rational.

For the proof of Theorem 3.2.1, we delineate the behavior of toric plurisubahrmonic functions with analytic singularities. Also, we will present the relations between convergence of sequence of convex functions and convergence of its conjugates. Most of preliminaries are found in Section 2.3.

Section 3.2 is organized as follows. In Subsection 3.2.1, we characterize how toric psh functions with analytic singularities and their Newton convex bodies look like. We also interpret the result of Guan [G20] using convex analysis related to toric psh functions. In Subsection 3.2.2, we observe how convex conjugates of toric psh functions with analytic singularities should behave and demonstrate relationships between the convergence of convex functions and the convergence of their conjugates. Finally, in Subsection 3.3, we prove the main theorem and present some relevant examples.

3.2.1 Newton convex bodies for analytic singularities

In this subsection, we will prove the following characterization of psh function with analytic singularities and what convex conjugate of toric psh functions with analytic singularities looks like.

Proposition 3.2.2. Let φ be a toric psh funtion with analytic singularities on a unit polydisk $D(0,1) \subseteq \mathbf{C}^n$. Then φ is associated to a monomial ideal with weight $c \in \mathbb{R}_+$, i.e., $\varphi \simeq \frac{c}{2}\log(|z|^{2\alpha_1} + \cdots + |z|^{2\alpha_m})$ near 0 where $\alpha_1, \ldots, \alpha_m$ are multi-indices and $\simeq$ means that their difference is in $\mathrm{O}(1)$.

Remark 3.2.3. For a toric psh function φ with analytic singularities, if we write

$$\varphi = \frac{c}{2}\log(|g_1|^2 + \cdots + |g_r|^2) + \mathrm{O}(1)$$

near 0, then it is hard to say that the value of $|g_1|^2 + \cdots + |g_r|^2$ is independent of torus action. Notwithstanding the failure above, we can say vanishing of $|g_1|^2 + \cdots + |g_r|^2$ is invariant under torus actions.

Proof. We will show by the induction on dimension of domain. Write

$$\varphi = \frac{c}{2}\log(|g_1|^2 + \cdots + |g_r|^2) + \mathrm{O}(1)$$

near $z = 0$.

(Induction on n) Let $n = 1$. Let $g_1, \ldots g_r$ have a common zero at 0 with multiplicity k. Then we may assume $|g_1|^2 + \cdots + |g_r|^2$ is nonvanishing at 0 by extracting $|z|^{2k}$. If $|g_1|^2 + \cdots + |g_r|^2$ vanish at some point $z_0 \neq 0$, by Remark 3.2.3, it vanishes on the circle $|z| = |z_0|$. By the maximum principle $|g_1|^2 + \cdots + |g_r|^2$ vanishes on the disk $\mathbf{D}(0, |z_0|)$, contradiction. Thus $|g_1|^2 + \cdots + |g_r|^2$ is nowhere vanishing. In particular, it is bounded below by some positive number C on some locally compact neighborhood of 0. So, we can always write $\varphi = \frac{1}{2}\log|z|^{2k} + \mathrm{O}(1)$ near 0.

Now, suppose $n \geqslant 2$. We introduce some auxiliary notations for convenience: H_j is the hyperplane defined by z_j and $z(i)^{\alpha(i)}$ is a monomial of

$z_1, \cdots, \widehat{z_i}, \cdots z_n$ with multi-index exponent $\alpha(i)$. If the common zero set of $g_1, \ldots, g_r$ contains all H_j, $1 \leqslant j \leqslant n$, then similarly, one can extract z^α where α is a multi-index from all $g_1, \ldots, g_r$ so that $|g_1|^2 + \cdots + |g_r|^2$ vanish identically on none of H_j, $1 \leqslant j \leqslant n$. So, we may assume that $\{g_1 \ldots, g_r\}$ has no common factor which is a nontrivial monomial. Now if we restrict φ on H_j, by the induction hypothesis,

$$\begin{aligned}\varphi|_{H_j} &= \frac{c}{2}\log\left(|g_1|_{H_j}|^2 + \cdots + |g_r|_{H_j}|^2\right) + \mathrm{O}(1)\\ &\simeq \frac{c}{2}\log\left(\left|z(j)^{\alpha(j,1)}\right|^2 + \cdots + \left|z(j)^{\alpha(j,m_j)}\right|^2\right).\end{aligned}$$

If we put

$$h_j(z(j)) = \frac{|g_1(j)|^2 + \cdots + |g_r(j)|^2}{\left|z(j)^{\alpha(j,1)}\right|^2 + \cdots + \left|z(j)^{\alpha(j,m_j)}\right|^2},$$

then it is nowhere vanishing, well-defined positive-valued function on H_j. In particular, it is bounded below by some positive number $C_j > 0$. Let C' be the minimal number among $C_1, \ldots, C_n$.

We can argue as above procedure for all j and obtain the set S by joining $z(j)^{\alpha(j,i_j)}$. Here, $1 \leqslant j \leqslant n$ and $1 \leqslant i_j \leqslant m_j$. We may regard such $\alpha(i, i_j)$ as a multi-index in n variables inserting 0 for i-th component which is the excluded index while we were restricting to the hyperplane H_i. So, we may re-index such messy notations by $z^{\beta_1}, \cdots, z^{\beta_l}$. Now, we are enough to show the following equality:

$$\varphi = \frac{c}{2}\log\left(\left|z^{\beta_1}\right|^2 + \cdots + \left|z^{\beta_l}\right|^2\right) \tag{*}$$

up to O(1).

Proof of (*): Since every torus-invariant subvariety of $D(0,1)$ is given by an intersection of hyperplanes and $Z(g_1, \ldots, g_r)$ does not have any codimension 1 irreducible components, we know that φ itself has a pole set of codimension

$\geqslant 2$. We now observe the function

$$h(z)=\frac{|g_1|^2+\cdots+|g_r|^2}{|z^{\beta_1}|^2+\cdots+|z^{\beta_l}|^2}.$$

If it has a pole at some point η, then η should be in some H_j. But on H_j, $h(z)\leqslant h_j(z)$ and $h_j(z)$ cannot blow up at η. Thus it is well-defined. Again, using similar argument with $(n=1)$-case, depending upon the maximum principle and Remark 3.2.3, we know that h cannot vanish at w where all w_i are nonzero. Now we are enough to check that if some $w_i=0$, say $w_n=0$, then $h(w)\geqslant \frac{C}{n}\min_{w_j=0}(h_j(w))\geqslant \frac{CC'}{n}$ for some $C>0$ by Lemma 3.2.4. Therefore, h is bounded below by some positive lower bound near 0. □

□

Lemma 3.2.4. Let a, b_i, $1\leqslant i\leqslant n$, $b_1\leqslant\cdots\leqslant b_n$ are positive real numbers, then

$$\frac{a}{b_1+\cdots+b_n}\geqslant\frac{C}{n}\min_{1\leqslant i\leqslant n}\frac{a}{b_i},\quad\text{where}\ \ C=\left(\frac{b_1}{b_n}+\cdots+\frac{b_n}{b_1}\right)^{-1}.$$

Proof. It is straightforward from the rearrangement inequality :

$$\left(\frac{b_n}{b_1}+\cdots+\frac{b_1}{b_n}\right)na\geqslant(b_1+\cdots+b_n)\left(\frac{a}{b_1}+\cdots+\frac{a}{b_n}\right).$$

In fact, we can take two increasing sequences by $x_i=b_i$ and $y_i=\frac{a}{b_{n+1-i}}$ for $1\leqslant i\leqslant n$. Then $n(x_ny_n+\cdots+x_1y_1)\geqslant(x_1+\ldots x_n)(y_1+\ldots y_n)$. □

Using Proposition 3.2.2, we have a useful characterization for toric psh with analytic singularities.

Corollary 3.2.5. If φ is a toric psh with analytic singularities, written as

$$\varphi=\frac{c}{2}\log\left(|z|^{2b_1}+\cdots+|z|^{2b_r}\right)+\mathrm{O}(1),$$

and g is a convex increasing function associated to φ defined on $\mathbb{R}^n_-$, then g is of a form $c \max_{1 \leqslant i \leqslant r} \langle b_i, x \rangle$ upto $\mathrm{O}(1)$.

Proof. Since we know that $\log \max_{1 \leqslant i \leqslant r} |z|^{b_i} \leqslant \log(|z|^{b_1} + \cdots + |z|^{b_r}) \leqslant \log \max_{1 \leqslant i \leqslant r} r \cdot |z|^{b_i}$, φ can be written as $\frac{c}{2} \log \max_{1 \leqslant i \leqslant r} |z|^{2b_i} + \mathrm{O}(1)$. This concludes the proof ☐

Using this, we can associate the Newton convex body associated with toric psh with analytic singularities. For a set of finite points $b = \{b_1, \ldots, b_r\}$ in $\mathbb{R}^n_+$, let $P(b)$ be the Minkowski addition of the convex hull of b and $\mathbb{R}^n_+$. We call $P(b)$ the closed polytope determined by b.

Proposition 3.2.6. Let φ be of a form as in Corollary 3.2.5 and let g be the associated convex function defined on $\mathbb{R}^n_{\leqslant 0}$. Then $P(\varphi)$ is the closed subset in $\mathbb{R}^n_+$ represented as $cP(b) + \mathbb{R}^n_{\geqslant 0}$, where b is the set of exponents in a representation of φ and $P(b)$ is the convex hull of $\{b_1, \ldots, b_r\}$.

Remark 3.2.7. In Proposition 3.2.6, we need not assume that φ itself is of analytic singularities. Indeed, no conditions of b_i are imposed.

Proof. Since $P(c\varphi) = cP(\varphi)$, we may assume that $c = 1$. It is just from writing conditions of being in $P(\varphi)$. Denote $P(b) + \mathbb{R}^n_{\geqslant 0}$ by Q. Then since each $b_i \in P(\varphi)$, $Q \subseteq P(\varphi)$, due to the minimality of convex hull $P(b)$ and each $b_i \in P(\varphi)$.

If $t \in P(\varphi)$ is not in Q, then there is a unique vector v that determines the distance $d(t, P(b) + \mathbb{R}^n_+) = d(t, t + v) = |v| > 0$. Here, the uniqueness of v follows from the convexity of set Q. Also v should be in $\mathbb{R}^n_{\geqslant 0}$. This v determines a unique region A_v defined as:

$$A_v = \{y \in \mathbb{R}^n \mid \langle -v, y \rangle \leqslant \langle -v, t + v \rangle\}.$$

In fact, A_v is the lower half-space of a supporting hyperplane of Q at $t + v$ which is perpendicular to $-v$. Since $b_j \in A$ for every $1 \leqslant j \leqslant r$, we know that

$$\langle t - b_j + v, -v \rangle \geqslant 0.$$

Now. let $y_k = -kv - \epsilon \in \mathbb{R}^n_-$ be a sequence of points in $\mathbb{R}^n_-$ where k is a positive integer and ϵ is a small vector in $\mathbb{R}^n_-$. By the definition of $P(\varphi)$, the following sequence $t_k = \langle t, y_k\rangle - \max\limits_{1\leqslant i\leqslant r}\langle b_i, y_k\rangle$ should be bounded above. It is equivalent to $\langle t, y_k\rangle - \max\limits_{1\leqslant i\leqslant r}\langle b_i, y_k\rangle = \min\limits_{1\leqslant i\leqslant r}\langle t - b_i, y_k\rangle$ is bounded above. We can reformulate this again by

$$\begin{aligned}
\min_{1\leqslant i\leqslant r}\langle t - b_i, y_k\rangle &= \min_{1\leqslant i\leqslant r}\langle t - b_i, -kv - \epsilon\rangle \\
&= \min_{1\leqslant i\leqslant r}\langle t - b_i, -kv\rangle + \langle t - b_i, -\epsilon\rangle \\
&\geqslant \min_{1\leqslant i\leqslant r}\left(\langle t - b_i + v, -kv\rangle + \langle -v, -kv\rangle + C\right) \\
&\geqslant \min_{1\leqslant i\leqslant r} k|v|^2 + C.
\end{aligned}$$

Here, C is a bounded constant coming from $\langle t - b_i, -\epsilon\rangle$ and the last inequality comes from our observation $\langle t - b_j + v, -v\rangle \geqslant 0$ discussed above. But this goes to ∞ as $k \to \infty$ and contradicts our definition of $P(\varphi)$. □

Remark 3.2.8. The above proposition also demonstrates that the definition of Newton convex body of a psh function is a generalization of the definition of Newton convex body of a monomial ideal. In fact, a Newton convex body of a monomial ideal $\mathfrak{a} = (z^{b_1}, \dots, z^{b_m})$ where $b_1, \dots, b_m$ are exponents of generators of $\mathfrak{a}$ is defined by $\mathrm{Conv}(b_1, \dots, b_m) + \mathbb{R}^n_{\geqslant 0}$. This is coherent with the Newton convex body of a toric psh function with anlytic singularities determined by a monomial ideal $\mathfrak{a}$. See [Bl04, Ho01] for details of the definition.

Combining these results with the Rashkovskii-Guenancia's theorem Theorem 2.3.8, we can interpret the result of [G20] in the category of toric psh functions.

Example 3.2.9 ([G20])**.** If $\varphi = \max\limits_{1\leqslant i\leqslant m} \log|z_i|^{a_i}$ defined as a germ of a toric psh function at $(\mathbf{C}^n, 0)$, then by Proposition 3.2.6, its Newton convex body can be computed concretely, $P(\varphi) = \left(H \cap \mathbb{R}^n_{\geqslant 0}\right) + \mathbb{R}^n_{\geqslant 0}$, where H is the hyperplane defined by a linear equation $\sum_{i=1}^{m} \frac{x_i}{a_i} = 1$.

3.2.2 Convex conjugate of analytic singularities

In this section, we will characterize the convex conjugate of toric psh functions with analytic singularities and prove the relevance between convergence of convex functions and convergence of their convex conjugates.

Definition 3.2.10. A closed subset $P \subseteq \mathbb{R}^n$ is a $\mathcal{H}$-*polyhedron* if P is given by the intersection of finite numbers of half-spaces. More explicitly, there exist p vectors $a_1, \ldots, a_p$ and p real numbers $b_1, \ldots, b_p$ such that P is given by $P = \{x \in \mathbb{R}^n : \langle a_i, x\rangle \leqslant b_i \text{ for all } i = 1, \ldots, p\}$.

By normal vectors in this paper, we mean *outward* normal vectors. If all a_i and b_i can be taken to be in $\mathbb{Q}^n$ and $\mathbb{Q}$ respectively, P is said to be *rational.*

Theorem 3.2.11. [S, Theorem 2.4.9] Let P be a $\mathcal{H}$-polyhedron in $\mathbb{R}^n$ and p a point in the boundary of P. If $F_1, \ldots, F_m$ are the facet of P containing p and $a_1, \ldots, a_m$ are normal vectors for $F_1, \ldots, F_m$ respectively, then every normal vector a of a supporting hyperplane of P at p is in the conical hull of $a_1, \ldots, a_m$, that is, there are nonnegative real numbers $\lambda_1, \ldots, \lambda_m$ such that

$$a = \lambda_1 a_1 + \cdots + \lambda_m a_m$$

Definition 3.2.12. A closed subset $P \subseteq \mathbb{R}^n$ is a $\mathcal{V}$-*polyhedron* if there exist a finite set of points Y and a finite set of vectors V such that P is the sum of the convex hull of Y and the conical hull of V, that is,

$$P = \operatorname{conv}(Y) + \operatorname{cone}(V).$$

As in the case of a $\mathcal{H}$-polyhedron, a $\mathcal{V}$-polyhedron is said to be rational if one can take all points in Y and all vectors in V from $\mathbb{Q}^n$.

Lemma 3.2.13. Let Q be a rational $\mathcal{H}$-polyhedron in $\mathbb{R}^n_-$ such that $Q+\mathbb{R}^n_- \subseteq Q$ and let g be a convex function with $\mathrm{dom}(g) = Q$, increasing in each variable. Then the followings are equivalent.

1. The epigraph of g is a rational $\mathcal{H}$-polyhedron.
2. There are a finite set of vectors $\{s_1, \ldots, s_N\}$ in $\mathbb{Q}^n_+$ and a finite set of rational numbers $\{a_1, \ldots, a_N\}$ such that

$$g(x) = \max_{1 \leqslant i \leqslant N} (\langle s_i, x\rangle + a_i)$$

on Q.

Symmetrically, if we set P as a rational $\mathcal{H}$-polyhedron in $\mathbb{R}^n_+$ such that $P + \mathbb{R}^n_+ \subseteq P$ and let h be a convex function with $\mathrm{dom}(h) = P$, decreasing in each variable. Then the followings are equivalent.

1. The epigraph of h is a rational $\mathcal{H}$-polyhedron.
2. There are a finite set of vectors $\{t_1, \ldots, t_N\}$ in $\mathbb{Q}^n_-$ and a finite set of rational numbers $\{b_1, \ldots, b_N\}$ such that

$$h(x) = \max_{1 \leqslant i \leqslant N} (\langle t_i, x\rangle + b_i)$$

on P.

Proof. Suppose that $\mathrm{epi}(g)$ is a rational $\mathcal{H}$-polyhedron. Let $S^t x \leqslant a$ be a system of essential inequalities for $\mathrm{epi}(g)$, where S is an $(n+1) \times (p+q)$ matrix $[s_1 \cdots s_{p+q}]$ with $s_i, a \in \mathbb{Q}^{n+1}$. We may assume that $s_{p+1}, \ldots, s_{p+q}$ corresponds with essential inequalities for Q, that is, the $(n+1)$-th coordinate of s_k is nonzero if and only if $k = 1, \ldots, p$. Thus we can normalize $s_1, \ldots, s_p$ so that their $(n+1)$-th coordinates are all -1. Set $s_k = (s'_k, -1) \in \mathbb{R}^n \times \mathbb{R}^1 (k = 1, \ldots, p)$. Now we shall prove that g can be written as the form

$$g(x') = \max_{1 \leqslant i \leqslant p} (\langle s'_i, x'\rangle - a_i),$$

where a_i is an i-th coordinate of a. Then $x = (x', x_{n+1}) \in \operatorname{epi}(g)$ if and only if $x' \in Q$ and x satisfies the *nonvertical* inequaliteis in $S^t x \leqslant a$:

$$\begin{gathered}\langle s'_1, x'\rangle - a_1 \leqslant x_{n+1},\\ \vdots \\ \langle s'_p, x'\rangle - a_p \leqslant x_{n+1}.\end{gathered}$$

Equivalently, $x = (x', x_{n+1}) \in \operatorname{epi}(g)$ if and only if $x' \in Q$ and

$$\max_{1 \leqslant i \leqslant p} (\langle s'_i, x'\rangle - a_i) \leqslant x_{n+1}. \tag{$*$}$$

Observing that $g(x') = \inf\{x_{n+1} : (x', x_{n+1}) \in \operatorname{epi}(g)\}$, we have

$$g(x') = \max(\langle s'_i, x'\rangle - a_i).$$

Note that every s'_i should be in $\mathbb{Q}^n_+$, because all s'_i are essential and g is increasing.

The converse is immediate from the observation $(*)$. □

Remark 3.2.14. In Lemma 3.2.13, if Q is a (not necessarily rational) $\mathcal{H}$-polyhedron, the followings are equivalent (with the same proof).

1. The epigraph of h is a $\mathcal{H}$-polyhedron.
2. There are a finite set of vectors $\{t_1, \ldots, t_N\}$ in $\mathbb{R}^n_-$ and a finite set of real numbers $\{b_1, \ldots, b_N\}$ such that

   $$h(x) = \max_{1 \leqslant i \leqslant N} (\langle t_i, x\rangle + b_i)$$

   on Q.

Theorem 3.2.15. Let g and Q be as in Lemma 3.2.13 and assume that g satisfies one of the equivalent condition in Lemma 3.2.13. If h is the convex conjugate of g, then $\operatorname{epi}(h)$ is a rational $\mathcal{V}$-polyhedron.

Proof. Assume that g can be written as

$$g(x') = \max_{1 \leqslant i \leqslant p} (\langle s'_i, x' \rangle - a_i)$$

on Q with $s'_i \in \mathbb{Q}^n_+$ and $a_i \in \mathbb{Q}$. Observe that $h(s'_i) = \sup_{y'}(\langle s'_i, y' \rangle - g(y'))$ attains its supremum at any y' such that $(y', g(y'))$ is on a facet F_i of $\operatorname{epi}(g)$ which is given by the equation $\langle (s'_i, -1), x \rangle = a_i$. Thus we have $g(y') = \langle s'_i, y' \rangle - a_i$ for such y' and thus $h(s'_i) = a_i$. Observe that in general s' is contained in $P = \operatorname{dom}(h)$ and $h(s') = k$ if and only if $\langle (s', -1), x \rangle = k$ is a supporting hyperplane of $\operatorname{epi}(g)$ and meets $\operatorname{epi}(g)$. For notational convenience, write $s_i = (s'_i, -1)$. Let V be the set of points in $\mathbb{R}^{n+1}$ given by

$$\begin{aligned} V = \{(u', b) \in \mathbb{R}^n_+ \times \mathbb{R} : \langle u', x' \rangle = b \text{ is a supporting hyperplane } H' \\ \text{such that } H' \cap Q \text{ is a facet of } Q. \} \end{aligned}$$

We will prove

$$\operatorname{epi}(h) = \operatorname{conv}((s'_1, a_1), \ldots, (s'_p, a_p)) + \operatorname{cone}(V \cup \{e_{n+1}\}), \tag{3.2.1}$$

where $e_{n+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$.

Let s' be a point in P. Since $\langle s', y' \rangle - g(y')$ is a piecewise-affine concave function in y' on Q, it attains the supremum, say at $y'_0 \in Q$. By the above observation, $\langle (s', -1), x \rangle = h(s')$ is a supporting hyperplane of $\operatorname{epi}(g)$ at y_0. If y'_0 is in the interior of Q, then $(s', -1)$ is a positive combination of the normal vectors of the *nonvertical* facets of $\operatorname{epi}(g)$ containing y_0. Here, by a nonvertical facet, we mean that its normal vector has nonzero $(n+1)$-th component. Without loss of generality, suppose that $F_1, \ldots, F_m$ are the facets of $\operatorname{epi}(g)$ containing y_0. Then by Theorem 3.2.11, there exist $\lambda_1, \ldots, \lambda_m \geqslant 0$ such that

$$(s', -1) = \lambda_1 s_1 + \cdots + \lambda_m s_m.$$

Comparing the $(n+1)$-th component of both sides of this, we know that s' is given by the convex combination of $s_1, \dots, s_m$ with coefficients $\lambda_1, \dots, \lambda_m$. Furthermore, y_0 satisfies the equation $\langle s_i, x\rangle = a_i$ for all $i = 1, \dots, m$, the convex combination of these m equations with coefficients $\lambda_1, \dots, \lambda_m$ also holds at y_0. Therefore,

$$h(s') = \langle (s', -1), y_0\rangle = \sum_{i=1}^{m} \lambda_i \langle s_i, y_0\rangle = \sum_{i=1}^{m} \lambda_i a_i$$

holds and thus $(s', h(s'))$ is contained in $\operatorname{conv}(s_1, \dots, s_m)$.

Now, assume that y_0' is on the boundary of Q and cannot be taken to be in the interior of Q. Let $u_1', \dots, u_l'$ be normal vectors of the facets of Q at y_0' with $\langle u_i', y_0'\rangle = b_i$. Write $u_i = (u_i', 0)$ for $i = 1, \dots, l$. By Theorem 3.2.11 again, we obtain

$$(s', -1) = \sum_{i=1}^{m} \lambda_i s_i + \sum_{j=1}^{l} \mu_j u_j, \tag{3.2.2}$$

where $\sum_i \lambda_i = 1$ and $\mu_j \geqslant 0$ for all j. Applying $\langle \bullet, (y_0', g(y_0'))\rangle$ on both sides of (3.2.2), we have

$$h(s') = \sum_{i=1}^{m} \lambda_i h(s_i') + \sum_{j=1}^{l} \mu_j b_i,$$

which implies

$$(s', h(s')) = \sum_{i=1}^{m} \lambda_i (s_i', a_i) + \sum_{j=1}^{l} \mu_j (u_j', b_j) \in \operatorname{conv}((s_1', a_1), \dots, (s_p', a_p)) + \operatorname{cone}(V).$$

This shows that $\operatorname{epi}(h)$ is contained in the sum of the convex hull of (s_1', a_1) $, \dots, (s_p', a_p)$ and the conical hull of $V \cup \{e_{n+1}\}$. The converse follows immediately from the definition of supporting hyperplanes. Because the image of a $\mathcal{V}$-polyhedron under a projection is again a $\mathcal{V}$-polyhedron, we conclude that P and $\operatorname{epi}(h)$ are $\mathcal{V}$-polyhedron. Since we can take (s_i', a_i) and (u_i', b_i) to be rational, P and $\operatorname{epi}(h)$ are also rational. $\square$

Theorem 3.2.16 ([M], [Z, Theorem 1.2])**.** Every $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron. Also every $\mathcal{V}$-polyhedron is a $\mathcal{H}$-polyhedron.

Thanks to Theorem 3.2.16, we can drop $\mathcal{H}$ or $\mathcal{V}$ from $\mathcal{H}$-polyhedrons or $\mathcal{V}$-polyhedrons and just call them polyhedrons. Now we have the following characterization for toric psh functions with analytic singularities.

Theorem 3.2.17. Let φ be a toric psh function on $D(0, r) \subseteq \mathbb{C}^n$ with analytic singularities and let g be the convex function associated to φ. Then the domain P of g^* is a polyhedron such that $P + \mathbb{R}^n_+ \subseteq P$ and $(2/c)P$ is rational. Furthermore, g^* can be written as

$$g^*(y) = \frac{c}{2} \max_{1 \leqslant i \leqslant N} (\langle t_i, y \rangle + b_i) + \mathrm{O}(1) \tag{3.2.3}$$

where $t_i \in \mathbb{Q}^n_-$ and $b_i \in \mathbb{Q}$.

Conversely, let $P \subseteq \mathbb{R}^n_+$ be a polyhedron such that $(2/c)P$ is rational and $P + \mathbb{R}^n_+ \subseteq P$ and let h be a function on P defined by

$$h(y) = \frac{c}{2} \max_{1 \leqslant i \leqslant N} (\langle s_i, y \rangle + a_i) + v(y) \tag{3.2.4}$$

where $s_i \in \mathbb{Q}^n_+$, $a_i \in \mathbb{Q}$ and v is a bounded function such that h is convex and decreasing in each variable. Then $\varphi(z_1, \ldots, z_n) := h^*(\log |z_1|, \ldots, \log |z_n|)$ is a toric psh function with analytic singularities on $D(0, r) \subseteq \mathbb{C}^n$ for some $r > 0$.

Proof. This is an immediate consequence of Lemma 3.2.13, Remark 3.2.14 and Theorem 3.2.15. □

Remark 3.2.18. In the converse part of Theorem 3.2.17, r could be any positive real number such that

$$(-\infty, -r)^n \subseteq \mathrm{dom}(h^*).$$

For the proof of main theorem, we want to describe a relationship between the convergence of a sequence of convex functions with the convergence of its conjugate. Start with the following simple lemma.

Lemma 3.2.19. Let (f_k) be a decreasing sequence of convex functions defined on an open subset in $\mathbb{R}^n$. Then $\lim\limits_{k\to\infty} f_k$ is also convex.

Proof. We can prove the convexity directly.

$$\begin{aligned}\lim_{k\to\infty} f_k(\lambda x + (1-\lambda)y) &\leqslant \lambda f_n(x) + (1-\lambda) f_n(y)\\ &\leqslant \lambda f_m(x) + (1-\lambda) f_n(y)\end{aligned}$$

Here $m \leqslant n$ are arbitrary positive integers. Letting $n \to \infty$ and then letting $m \to \infty$, we obtain the result. □

For the sake of our argument, we introduce a notion of lower semicontinuous regularization. For a family of lower semicontinuous functions (f_α) which is locally uniformly bounded below, its infimum $f = \inf\limits_{\alpha} f_\alpha$ is not lower semicontinuous in general. To resolve this we define the *lower semicontinuous regularization* by:

$$f^{\triangle}(x) = \lim_{\epsilon\to 0} \inf_{y\in B(x,\epsilon)} f(y) \leqslant f(x).$$

Then it is easy to check that $f^{\triangle}$ is the largest lower semicontinuous which is $\leqslant f$. Also note that $f^{\triangle}(x)$ is equal to $f(x)$ whenever f is lower semicontinuous at x. Using this notion, we are now ready to prove the following lemma.

Lemma 3.2.20. Let (g_m) be an increasing sequence of lower semicontinuous convex functions defined on $\mathbb{R}^n$ converging to a convex function g pointwise. Then (g_m^*) is a decreasing sequence converging to g^* pointwise on the relative interior of $\operatorname{dom} g^*$

Proof. First, we know that convex conjugate operation is order-reversing, so (g_m^*) is decreasing. Also, we know that for each m, $g_m^{**} = g_m$ by lower semicontinuity of g_m. Then using the well-known fact of convex conjugate $(\inf_\alpha f_\alpha)^*(x^*) = \sup_\alpha f_\alpha^*(x^*)$, we obtain

$$\begin{aligned}(\inf_m g_m^*)^*(x) &= \sup_m g_m^{**}(x)\\ &= \sup_m g_m(x) = g(x).\end{aligned}$$

Taking convex conjugate to both side again, we have $(\inf_m g_m^*)^{**}(x) = g^*(x)$. We observe that $\inf_m g_m^*$ is convex by the previous lemma. Using the property from Remark 2.3.4 (3) to $(\inf_m g_m^*)(x)$, $g^*(x) \leqslant (\inf_m g_m^*)(x)$. In general, we can not say about the lower semicontinuity of $\inf_m g_m^*$. Since $g_m \leqslant g$, we know (g_m^*) is locally uniformly bounded below by g^* and we can think about the lower semicontinuous regularization of $(\inf_m g_m^*)^\triangle \leqslant \inf_m g_m^*$. Taking both sides to **, which is order-preserving and we know that $(\inf_m g_m^*)^{**}$ is equal to g^*. Also, $(\inf_m g_m^*)^\triangle$ is lower semicontinuous and convex so the double conjugate of the left side is equal to itself $(\inf_m g_m^*)^\triangle$. For convexity of $(\inf_m g_m^*)^\triangle$, we refer to [H07, Proposition 2.2.2]. What we have shown is $(\inf_m g_m^*)^\triangle \leqslant g^*$. Combining this with $g^* \leqslant (\inf_m g_m^*)$, we obtain

$$(\inf_m g_m^*)^\triangle \leqslant g^* \leqslant (\inf_m g_m^*).$$

Since $(\inf_m g_m^*)$ is convex, it is continuous in the relative interior of $\operatorname{dom} g^*$. This implies that $(\inf_m g_m^*)$ in fact coincides with $(\inf_m g_m^*)^\triangle$ in the realtive interior of $\operatorname{dom} g^*$. This concludes the proof. □

3.3 Proof of Theorem 3.2.1 and some examples

Now we are ready to prove the main theorem.

Proof of Theorem 3.2.1. If (φ_m) is a decreasing sequence of toric psh functions with analytic singularities converging to to φ and $\mathcal{J}(\varphi_m) = \mathcal{J}(\varphi)$ for all $n \geqslant 1$, then $P := P(\varphi_1)$ satisfies the three conditions of the statement 2.

Now assume that there exists a polyhedron P satisfying the three conditions in the statement 2. Let g be the convex function associated to φ. Then we can find a sequence of points $(u_i, \alpha_i)_{i=1}^{\infty}$ in $\mathbb{Q}_{\geqslant 0}^{n+1} \times \mathbb{Q}$ such that

$$\operatorname{epi}(g^*) = \frac{c}{2} \cdot \bigcap_{i=1}^{\infty} H^+_{u_i, \alpha_i} \tag{3.3.1}$$

where $H^+_{u,\alpha}$ is the closed half-space defined by $\{x \in \mathbb{R}^{n+1} : \langle u, x \rangle \geqslant \alpha\}$. Indeed, let q be a point in $\mathbb{Q}_{\geqslant 0}^{n+1} \cap (\mathbb{R}_{\geqslant 0}^{n+1} \backslash \operatorname{epi}(g^*))$. Since $\operatorname{epi}(g^*)$ is a closed convex set and $d(q, \operatorname{epi}(g^*)) > 0$, there exists $(u', \alpha') \in \mathbb{R}_{\geqslant 0}^{n+1} \times \mathbb{R}$ such that $H_{u',\alpha'}$ separates q and $\operatorname{epi}(g^*)$ strongly, that is, there exists $\epsilon > 0$ such that $q + \epsilon B(0,1) \subset \operatorname{int}(H^-_{u',\alpha'})$ and $\operatorname{epi}(g^*) + \epsilon B(0,1) \subset \operatorname{int}(H^+_{u',\alpha'})$, where $B(0,1)$ is the unit ball in $\mathbb{R}^{n+1}$. We can choose $(u, \alpha) \in \mathbb{Q}_{\geqslant 0}^{n+1} \times \mathbb{Q}$ which is sufficiently close to (u', α') so that the hyperplane $H_{u,\alpha}$ also separates q and $\operatorname{epi}(g^*)$ strongly. Enumerating all points in $\mathbb{Q}_{\geqslant 0}^{n+1} \cap (\mathbb{R}_{\geqslant 0}^{n+1} \backslash \operatorname{epi}(g^*))$ by positive integers gives (3.3.1). Let g_i^* be the convex function on $\mathbb{R}^n$ whose epigraph is given by

$$(P \times \mathbb{R}) \cap \left(\frac{c}{2} \cdot \bigcap_{j=1}^{i} H^+_{u_j, \alpha_j} \right).$$

It is obvious that g_i^* is increasing in each variable and lower semicontinuous. Let φ_i be the psh function associated to the convex conjugate of g_i^*. Then all φ_i have analytic singularities by Theorem 3.2.17. Furthermore, φ_i is equisingular to φ since the Newton convex body $P(\varphi_i)$ of φ_i lies between P and $P(\varphi)$. Note that each g_i^* is of the form (3.2.4) without a bounded function, we may assume that each φ_i is defined on $D(0,r)$. Since (g_i^*) is an increasing sequence of convex functions, (φ_i) is a decreasing sequence converging to φ on $D(0,r)$ by Lemma 3.2.20. □

Remark 3.3.1. Assuming 2 in Theorem 3.2.1 with φ being diagonal, we can show that the condition of our main theorem implies the condition of

Theorem 3.1.8. If $P = P(\varphi)$ satisfies 2 in Theorem 3.2.1, it is nothing but 1 in Theorem 1.1.1. Assume now that φ does not have analytic singularities and $t = (t_1, \ldots, t_n)$ is a positive integer solution of $\sum\limits_{i=1}^{m} \frac{x_i}{a_i} = 1$. A vector $c \cdot (a_1^{-1}, \ldots, a_m^{-1}, 0, \ldots, 0)$ cannot be rational for every $c > 0$. This implies that t should be contained in $\operatorname{int} P$, which contradicts (iii) of Theorem 3.2.1 2.

We can create fruitful examples with this theorem. For this, given a closed convex set $P \subset \mathbb{R}^n_+$ satisfying $P + \mathbb{R}^n_+ \subset P$, we can construct a psh function defined in $D(0, r) \subset \mathbb{C}^n$ for some polyradius r whose Newton convex body is equal to P. To elaborate the statement, we introduce the following related notions.

Definition 3.3.2. cf. [Si98], [K15] Let $\mathfrak{a}_\bullet = (\mathfrak{a}_k)$ be a graded sequence of ideals in $\mathbb{C}[z_1, \ldots, z_n]$, i.e., $\mathfrak{a}_p \cdot \mathfrak{a}_q \subset \mathfrak{a}_{p+q}$ for any $p, q \in \mathbb{Z}_{\geqslant 0}$. Then a *Siu psh function* associated to $\mathfrak{a}_\bullet$ is defined as

$$\varphi = \varphi_{\mathfrak{a}_\bullet} = \log\left(\sum_{k \geqslant 1} \epsilon_k |\mathfrak{a}_k|^{1/k}\right)$$

where ϵ_k is a choice of nonnegative coefficients that make the series converge.

In [KS20], it was proved that for any given convex set $P \in \mathbb{R}^n_{\geqslant 0}$ satisfying $P + \mathbb{R}^n_{\geqslant 0} \subset P$, there exists a graded sequence of ideals $\mathfrak{a}_\bullet$ and a Siu psh function associated to $\mathfrak{a}_\bullet$ whose Newton convex body is exactly equal to P(See [KS20, Proposition 2.9]). As a result, for an arbirtrary convex subset $P \in \mathbb{R}^n_{\geqslant 0}$ satisfying $P + \mathbb{R}^n_{\geqslant 0} \subset P$, we can construct a toric psh function φ whose Newton convex body is equal to P.

Next, we introduce a notion of extreme point. Let K be a convex set.

Definition 3.3.3. [H07, Definition 2.1.8] A point x in K is called *extreme* if

$$x = \lambda_1 x_1 + \lambda_2 x_2,\ x_1, x_2 \in K \Rightarrow x_1 = x_2 = x$$

where $\lambda_1, \lambda_2 > 0, \lambda_1 + \lambda_2 = 1$.

Example 3.3.4. Let $P = \{(x, y) \in \mathbb{R}^2_{\geqslant 0} \mid xy \geqslant 1\} \cap \mathbb{R}^2_{\geqslant 0}$. Then we can construct a Siu psh function φ associated to a graded sequence of monomial ideals whose Newton convex body is equal to P. We will show that for every $c > 0$, $c\varphi$ satisfies the condition (2) in Theorem 3.2.1. There are two cases of sets of lattices we need to consider. First, let $A_1, \dots, A_N$ be lattice points in $\mathbb{R}_{>0} \backslash P(c\varphi)$. Then for each A_j for $1 \leqslant j \leqslant N$, there exists a unique point B_j on $\partial P(c\varphi)$ such that the distance between A_j and B_j is the distance between A_j and $P(c\varphi)$. Let $H_j = \{a_j x + b_j y + c_j = 0\}$ be the equation of tangent line of $xy = c$ at B_j. Then, by changing a_j, b_j, c_j slightly, we can take H_j having following properties.

1. For each $1 \leqslant j \leqslant N$, H_j separates A_j and $P(c\varphi)$.
2. For each $1 \leqslant j \leqslant N$, H_j is rational.

Secondly, there are lattice points $B'_1, \dots, B'_M$ on the $\partial P(c\varphi)$. Then for each $1 \leqslant j \leqslant M$, let H'_j be the tangent line of $xy = c$ at B'_j. Now, if we take the polyhedron defined as

$$P = \bigcap_{j=1}^{N} H_j^+ \cap \bigcap_{j=1}^{M} {H'}_j^+,$$

then this P exactly satisfies the condition (2) in Theorem 3.2.1. Here, H_j^+ and ${H'}_j^+$ are upper hyperplanes such that contains $P(c\varphi)$.

Remark 3.3.5. In particular, we would like to emphasize that there exist a toric psh function φ whose boundary has a lattice point on its interior, but admits a decreasing, equisingular approximation with analytic singularities. Note that such φ does not exist when we only consider in the category of psh functions $\log \max |z_i|^{a_i}$ without analytic singularities, because it neither holds (1) nor (2) in Theorem 3.1.8.

Chapter 4

Multiplier ideal sheaves on singular varieties

In Chapter 4, we will discuss the notion of multiplier ideal sheaves on singular varieties and related properties. Since most of analytic multiplier ideal sheaves are infeasible to compute in singular cases, so we describe a combinatoric characterization when psh functions are toric. The results contain a generalization of Rashkovskii-Guenancia's theorem Theorem 2.3.8. We begin by preliminary notions.

4.1 Singularities of normal varieties

In this section, we introduce definitions and notions related to our main results. All varieties in this section is of field $\mathbf{k} = \mathbb{C}$. Also, we mean varieties by irreducible varieties. Most of materials in Section 4.1 come from [KM98], [K97].

4.1.1 Canonical sheaves on normal varieties

Let X be a normal variety. For a divisor D(formal finite sum of irreducible closed subvarieties of codimension 1), we define the *divisorial sheaf* $\mathcal{O}(D)$ associated to D by $\mathcal{O}(D)(U) = \{f \in \mathbf{k}(X) \mid \operatorname{div}(f) + D|_U \geqslant 0\}$. Here $\mathbf{k}(X)$ is the function field of X. In general, $\mathcal{O}(D)$ is coherent of rank 1, i.e., the vector space $\mathcal{O}(D)_\eta \otimes \mathbf{k}(X)$ is a $\mathbf{k}(X)$-vector space of rank 1, but not necessarily invertible.

Example 4.1.1. Let $X = \{xy = z^2\} \subset \mathbb{C}^3$ be a normal variety and let $D = \{x = z = 0\}$ and let $E = \{y = z = 0\}$ and take $U = X \backslash E$.
Take $h = \dfrac{1}{x}$. Then $h \in \Gamma(U, \mathcal{O}(D))$. Also, if we take $h = \dfrac{1}{z}$, then $\operatorname{div}(h) + D|_U = ((-D - E) + D)|_U = E|_{X\backslash E} = 0$. Since U meets D, neither of $\dfrac{x}{z}, \dfrac{z}{x}$ can be regular in U. Hence $\mathcal{O}(D)$ is not invertible.

Definition 4.1.2. Let X be a normal variety and D be a divisor. If $\mathcal{O}(D)$ happens to be locally free of rank 1, we say D is a *Cartier divisor*. Otherwise, D is called a *Weil divisor*. A $\mathbb{Q}$-*divisor* is a linear combination of prime divisors with rational coefficients. A $\mathbb{Q}$-divisor is said to be $\mathbb{Q}$-*Cartier* if there exists an integer $m \in \mathbb{Z}_{>0}$ such that mD is a Cartier divisor.

Next, we are going to define a canonical divisor, which is closely related to sheaf of holomorphic $(n,0)$-form in complex manifold. Let X be a normal variety of dimension n. As we said, when X is smooth, we define the *canonical line bundle* to be $\omega_X := \det(\Omega^1_{X/\mathbf{k}})$, i.e., the n-th exterior power of the cotangent bundle of X over $\mathbf{k}$. When X is not smooth, let $U = X \backslash X_{sing}$ and consider $\omega_U := \det(\Omega^1_{U/\mathbf{k}})$. Let θ_U be a rational section of ω_U, i.e. locally, θ_U can be written as

$$\theta_U = \frac{g_2(z)}{g_1(z)} dz_1 \wedge \ldots \wedge dz_n.$$

Take the divisor of θ_U defined by $\operatorname{div}(\theta_U) \overset{\text{loc}}{:=} \operatorname{div}(\frac{g_2}{g_1})$ is well-defined on U.

Since X is normal, $X \backslash U$ is of codim $\geqslant 2$ and we know that the natural restriction map $Z^1(X) \xrightarrow{\rho} Z^1(U)$ is an isomorphism. Here, $Z^1(X), Z^1(U)$ are abelian groups of Weil divisors on X, U respectively. Now, we define K_X by the inverse image of ρ by $\operatorname{div}(\theta_U)$, which is a $\mathbb{Z}$-Weil divisor on X and depending on the choice of θ_U.

The divisorial sheaf $\mathcal{O}_X(K_X)$ is well-defined, i.e., independent of choice of θ_U. Indeed, for any two rational forms θ_U and θ'_U on U, they are linearly equivalent on U and can be extended to linear equivalence on $Z^1(X)$. It is well-known that two divisorial sheaves $\mathcal{O}_X(D_1), \mathcal{O}_X(D_2)$ are isomorphic when two divisors D_1, D_2 are linearly equivalent. We call $\mathcal{O}_X(K_X)$ by the *canonical sheaf* of X.

Definition 4.1.3. Let X be a normal variety. Then X is said to be *Gorenstein* if the canonical sheaf is invertible(or, a canonical divisor is Cartier). X is said to be $\mathbb{Q}$-*Gorenstein* if there is an integer $m \in \mathbb{Z}_{>0}$ such that the sheaf associated to a multiple of canonical sheaf is invertible(or a canonical divisor is $\mathbb{Q}$-Cartier.

Remark 4.1.4. Since one can pull-back rational function by morphism between normal varieties, we can naturally consider the pull-back of K_X whenever K_X is ($\mathbb{Q}$-)Cartier. In general, we need some supplementary divisor to make K_X being $\mathbb{Q}$-Cartier. We will discuss this notion in the following section.

4.1.2 Singularities of pairs

Let $f : Y \to X$ be a birational morphism between normal varieties. Since K_X is not $\mathbb{Q}$-Cartier in general, we take a $\mathbb{Q}$-Weil divisor B on X such that $K_X + B$ is $\mathbb{Q}$-Cartier. Thus we can take pull-back $m(K_X + B)$ for some $m \in \mathbb{Z}_{>0}$. Define

$$f^*(K_X + B) := \frac{1}{m} f^*(m(K_X + B)).$$

Then singularities of pair (X, B) are measured by the difference between K_Y and $f^*(K_X + B)$. Note that for K_Y, one can choose a rational differential form

θ_V defined on $V = Y \backslash Y_{sing}$ well-behaved under the choice of θ_U which determines K_X. So, the difference between K_Y and $f^*(K_X + B)$ is independent of the choice of θ_U.

Example 4.1.5. Let X be a smooth surface and let $Y \to X$ be the blow-up of a point $p \in X$. Then locally, blow-up can be described by the following monomial morphism

$$(u, v) \mapsto (uv, v) = (s, t).$$

In particular, $ds \wedge dt = (vdu + udv) \wedge dv = vdu \wedge dv$. Since $\{v = 0\}$ is a local defining equation for the exceptional divisor E and we know that $\mathcal{O}_X(K_X)$ is a sheaf of holomorphic $(n, 0)$-forms if X is smooth, we obtain

$$K_Y = f^*K_X + E.$$

Definition 4.1.6. Let X be a normal variety such that the canonical divisor K_X is $\mathbb{Q}$-Cartier. Let $m \in \mathbb{Z}_{>0}$ be an index of K_X where the divisorial sheaf $\mathcal{O}(mK_X)$ is locally free. We say that X has *terminal* (resp. *canonical*) singularities if there is a log resolution of singularities for $(X, B = \phi)$ $f : Y \to X$ such that $K_Y = f^*K_X + \sum_{i \in I} a_i E_i$ such that $a_i > 0$(resp. $a_i \geqslant 0$) where $\mathrm{Exc}(f) = \bigcup_{i \in I} U_i$.

Remark 4.1.7. If X is smooth, then X has terminal singularities.

Example 4.1.8. Let $X = \{xy - z^2 = 0\} \subset \mathbb{C}^3$. If we blow up X at $p = (0, 0, 0)$, then $K_Y = f^*K_X + 0 \cdot E$. Thus X is not terminal, but canonical(There is a well-known fact that terminal surface is smooth).

We define log singularities of pair (X, B).

Definition 4.1.9. The pair (X, B) is *klt (Kawamata log terminal)* if $a_i > -1$ and also the coefficients of $B = \sum b_j B_j$ with $bj \in (0, 1)$. The pair (X, B) is *lc (log canonical)* if $a_i \geqslant -1$ and also the coefficients of $B = \sum b_j B_j$ with $bj \in [0, 1]$.

Remark 4.1.10. 1. In [K97], [KM98], being klt allows $b_j \in (-\infty, 1)$.

2. In the analytic setting, suppose X is smooth. Let $B = \sum b_j B_j = \sum b_j \operatorname{div}(h_j)$ where h_j are local holomorphic functions. Then

$$(X, B) : klt \iff h = \prod \frac{1}{|h_j|^{2b_j}} \text{ is locally integrable.}$$

4.2 Toric geometry

In this section, we review some basic facts from toric geometry which are necessary and intuitive. Most of materials are from [F93]. Again, we fix our base field $\mathbf{k} = \mathbb{C}$.

4.2.1 Convex Polyhedral Cones

From now on, we denote N for the lattice(which is isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{N}$). For given N, $N \otimes \mathbb{R}$ becomes the n-dimensional vector space. Denote it V unless we note for it specifically. Now, a *convex polyhedral cone* generated by $v_1, \ldots, v_k$ is a set

$$\sigma = \{c_1 v_1 + \ldots + c_k v_k | c_1, \ldots, c_k \geqslant 0\}.$$

Such vectors $v_1, \ldots, v_k$ are called the generators for σ. The dimension **dim** σ of σ is defined by the dimension of the vector space spanned by σ. The dual $\sigma^\vee$ of any subset σ is defined by the set of equations of supporting hyperplanes, i.e.,

$$\sigma^\vee = \{u \in V^\star | \langle u, v \rangle \geqslant 0 \text{ for any } v \in \sigma\}.$$

A *face* τ of σ is the intersection of σ with any supporting hyperplane:

$$\tau = \sigma \cap u^\perp = \{v \in \sigma : \langle u, v \rangle = 0\}$$

for some u in $\sigma^\vee$. A cone itself is regarded as a face, while others are called proper faces. In particular, a face τ is called a *facet* if it is of codimension one.

We present the properties of convex polyhedral cones and their dual cones. See [F93, §1] for the proofs.

Proposition 4.2.1. Let σ, $\sigma^\vee$, V be as above.

i $(\sigma^\vee)^\vee = \sigma$.

ii Any face is also a convex polyhedral cone and any intersection of faces is a face. A face of face is also a face.

iii Any proper face is contained in some facet.

iv The topological boundary of a cone that spans V is the union of its proper faces(or facets).

v If σ spans V and $\sigma \neq V$, then σ is the intersection of the half-spaces $H_\tau = \{v \in V : \langle u_\tau, v\rangle \geqslant 0\}$, as τ ranges over the facets of σ. Here, u_τ is a vector(may not be unique) that satisfies a relation $\tau = \sigma \cap {u_\tau}^\perp$ for a facet τ of σ.

vi The dual of a convex polyhedral cone is a convex polyhedral cone.

This demonstrates that polyhedral cones also can be defined as the intersection of half-spaces: for generators $u_1, \dots u_t$ of $\sigma^\vee$,

$$\sigma = \{v \in V : \langle u_1, v\rangle \geqslant 0, \dots \langle u_t, v\rangle \geqslant 0\}$$

We say that σ is *rational* if all of its generators can be taken from N. From the above procedure, we can check that $\sigma^\vee$ is also rational. Indeed, the form of u_τ is a solution for linear system of equations which have coefficients as integers.

Proposition 4.2.2. Let $\sigma^\vee$ be the dual cone of σ. Then followings hold.

- i (Gordan's Lemma) If σ is a rational convex polyhedral cone, then $S_\sigma = \sigma^\vee \cap M$ is a finitely generated semigroup.
- ii If τ is a face of σ, then $\sigma^\vee \cap \tau^\perp$ is a face of $\sigma^\vee$, with $\dim(\tau)+\dim(\sigma^\vee \cap \tau^\perp) = n = \dim(V)$. This sets up a 1-1 correspondence between the faces of σ and the faces of $\sigma^\vee$. The smallest face of σ is $\sigma \cap (-\sigma)$.
- iii If $u \in \sigma^\vee$, and $\tau = \sigma \cap u^\perp$, then $\tau^\vee = \sigma^\vee + \mathbb{R}_{\geqslant 0} \cdot (-u)$.
- iv Let σ be a rational convex polyhedral cone, and let u be in $S_\sigma = \sigma^\vee \cap M$. Then $\tau = \sigma \cap u^\perp$ is a rational convex polyhedral cone. All faces of σ have this form, and $S_\tau = S_\sigma + \mathbb{Z}_{\geqslant 0} \cdot (-u)$.
- v If σ and σ' are rational convex polyhedral cones whose intersection τ is a face of each, then $S_\tau = S_\sigma + S_{\sigma'}$.

We end up this subsection by charaterizing cones of our main interest.

Proposition 4.2.3. For a convex polyhedral cone σ, the followings are equivalent:

1. $\sigma \cap (-\sigma) = \{0\}$;
2. σ contains no nonzero linear subspace;
3. there is a u in $\sigma^\vee$ with $\sigma \cap u^\perp = \{0\}$;
4. $\sigma^\vee$ spans $V^\star$.

Remark 4.2.4. A cone satisfying the above conditions is called *strongly convex.* If the cone is strongly convex, then the rays generated by a minimal set of generators are exactly the one-dimensional faces of σ. We will write "$\tau \prec \sigma$" to mean that τ is a face of σ. A cone is called *simplicial*, or a *simplex*, if it is generated by independent generators.

4.2.2 Affine toric varieties

We have seen S_σ is a finitely generated semigroup if σ is a strongly convex rational polyhedral cone. Any additive semigroup S determines a "group ring" $\mathbb{C}[S]$, which is a commutative $\mathbb{C}$-algebra. As a vector space, it has a basis χ^u, as u varies over S, with multiplication determined by addition in S:

$$\chi^u \cdot \chi^{u'} = \chi^{u+u'}.$$

The unit 1 is χ^0. Generators $\{u_i\}$ for the semigroup S determine generators $\{\chi^{u_i}\}$ for the $\mathbb{C}$-algebra $\mathbb{C}[S]$.

Any finitely generated $\mathbb{C}$-algebra A determines a complex affine variety, which we denote by Spec$A$. In our applications, A will be a domain, so Spec$A$ will be an irreducible variety. We will speak of a point of Spec$A$ for an ordinary closed point unless we specify otherwise.

For $A = \mathbb{C}[S]$ constructed from a semigroup, the points are easy to describe: they correspond to homomorphisms of semigroups from S to $\mathbb{C}$, where $\mathbb{C}$ is regarded as an abelian semigroup via multiplication:

$$\text{Specm } \mathbb{C}[S] = \text{Hom}_{sg}(S, \mathbb{C}).$$

For a semigroup homomorphism x from S to $\mathbb{C}$ and u in S, the value of the corresponding function χ^u at the corresponding point of Specm$\mathbb{C}[S]$ is the image of u by the map $x : \chi^u(x) = x(u)$.

When $S = S_\sigma$ arises from a strongly convex rational polyhedral cone, we set $A_\sigma = \mathbb{C}[S_\sigma]$ and

$$U_\sigma = \text{Spec } \mathbb{C}[S_\sigma] = \text{Spec } A_\sigma,$$

the corresponding *affine toric variety.* All of these semigroups will be subsemigroups of the group $M = S_{\{0\}}$. If $e_1, \dots, e_n$ is a basis for N, and $e_1^\star, \dots, e_n^\star$

is the dual basis for M, write

$$X_i = \chi^{e_i^\star} \in \mathbb{C}[M].$$

As a semigroup, M has generators, $\pm e_1^\star, \ldots, \pm e_n^\star$, so

$$\begin{aligned} \mathbb{C}[M] &= \mathbb{C}[X_1, \frac{1}{X_1}, \ldots, X_n, \frac{1}{X_n}] \\ &= \mathbb{C}[X_1, \ldots, X_n]_{X_1 \cdot \ldots \cdot X_n} \end{aligned}$$

which is the ring of *Laurent polynomials* in n variables. So

$$U_{\{0\}} = \text{Spec } \mathbb{C}[M] \cong \mathbb{C}^\star \times \ldots \times \mathbb{C}^\star = (\mathbb{C}^\star)^n$$

is an affine algebraic *torus*. All of our semigroups S will be subsemigroups of a lattice M, so $\mathbb{C}[S]$ will be a subalgebra of $\mathbb{C}[M]$; in particular, it is a domain. When a basis for M is chosen as above, we usually write elements of $\mathbb{C}[S]$ as Laurent polynomials in the corresponding variables X_i. Note that all of these algebras are generated by *monomials* in the variables X_i.

The torus $T = T_N$ corresponding to M or N can be written intrinsically:

$$T_N = \text{Spec } \mathbb{C}[M] = \text{Hom}(M, \mathbb{C}^\star) = N \otimes_{\mathbb{Z}} \mathbb{C}^\star.$$

For a basic example, let σ be the cone with generators $e_1, \ldots, e_k$ for some k, $1 \leqslant k \leqslant n$. Then

$$S_\sigma = \mathbb{Z}_{\geqslant 0} \cdot e_1^\star + \ldots + \mathbb{Z}_{\geqslant 0} \cdot e_k^\star + \mathbb{Z} \cdot e_{k+1}^\star + \ldots + \mathbb{Z} \cdot e_n^\star$$

Hence $A_\sigma = \mathbb{C}[X_1, \ldots X_k, X_{k+1}, \frac{1}{X_{k+1}}, \ldots, X_n, \frac{1}{X_n}]$, and

$$U_\sigma = \mathbb{C} \times \ldots \times \mathbb{C} \times \mathbb{C}^\star \times \ldots \times \mathbb{C}^\star = \mathbb{C}^k \times \mathbb{C}^{n-k}.$$

It follows from that if σ is generated by k elements that can be completed to a basis for N, then U_σ is a product of affine k-space and an $(n-k)$-dimensional torus. In particular, such affine toric varieties are nonsingular.

Example 4.2.5. Let N be a lattice of rank 3, and let σ be the cone generated by four vectors v_1, v_2, v_3, and v_4 that generate N and satisfy $v_1+v_3 = v_2+v_4$. The variety U_σ is a "cone over a quadric surface". If we take $N = \mathbb{Z}^3$ and $v_i = e_i$ for $i = 1, 2, 3$, so $v_4 = e_1+e_3-e_2$, then S_σ is generated by $e_1^\star, e_3^\star, e_1^\star+e_2^\star$, and $e_2^\star + e_3^\star$, so

$$A_\sigma = \mathbb{C}[X_1, X_3, X_1X_2, X_2X_3] = \mathbb{C}[W, X, Y, Z]/(WZ - XY).$$

Therefore U_σ is the hypersurface defined by $WZ = XY$ in $\mathbb{C}^4$.

If σ is a cone in N, the torus T_N acts on U_σ,

$$T_N \times U_\sigma \to U_\sigma,$$

as follows. A point in $t \in T_N$ can be identified with a map $M \to \mathbb{C}^\star$ of groups, and a point $x \in U_\sigma$ with a map $S_\sigma \to \mathbb{C}$ of semigroups; the product $t \cdot x$ is the map of semigroups $S_\sigma \to \mathbb{C}$ given by

$$u \mapsto t(u)x(u).$$

The dual map on algebras, $\mathbb{C}[S_\sigma] \to \mathbb{C}[S_\sigma] \otimes \mathbb{C}[M]$, is given by mapping χ^u to $\chi^u \otimes \chi^u$ for $u \in S_\sigma$. When $\sigma = \{0\}$, this is the usual product in T_N. These maps are compatible with inclusions of open subsets corresponding to faces of σ. In particular, they extend the action of T_N on itself.

4.2.3 Singularities in toric geometry

In this section, we will discuss the criterion for being U_σ nonsingular and the resolution of singularities for toric varieties.

Proposition 4.2.6. An affine toric variety U_σ is nonsingular if and only if σ is generated by part of a basis for the lattice N, in which case

$$U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^\star)^{n-k}, \quad k = \dim(\sigma).$$

We therefore call a cone *nonsingular* if it is generated by part of a basis for the lattice, and we call a fan nonsingular if all of its cones are nonsingular. Although a toric variety may be singular, every toric variety is normal:

Proposition 4.2.7. Each ring $A_\sigma = \mathbb{C}[S_\sigma]$ is integrally closed.

To define the multiplier ideal sheaves on toric varieties X, we need a resolution of singularities and log resolution of an ideal sheaf. In toric varieties, there is a combinatoric characterization for resolution of singularities. Let Σ be a fan defined in the lattice N.

Theorem 4.2.8. [CLS11, Theorem 11.1.9, 11.2.2] Every fan Σ has a refinement Σ' with the following properties:

1. Σ' is smooth.
2. Σ' contains every smooth cone of Σ.
3. Σ' is obtained from Σ by a sequence of star subdivisions.
4. The toric morphism $\phi : X_{\Sigma'} \to X_\Sigma$ is a projective resolution of singularities.

Furthermore we can set ϕ as an SNC resolution of singularities.

Theorem 4.2.9. [CLS11, Theorem 11.3.10] Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be a monomial ideal. Then there is a toric morphism $\phi : X_\Sigma \to \mathbb{C}^n$ that is a log resolution of $\mathfrak{a}$.

4.3 Multiplier ideal sheaves on singular varieties

In this section, we will discuss the definition of multiplier ideal sheaves on singular varieties and its subtleties. Let us begin with the definition of psh functions defined on normal variety X.

Definition 4.3.1. Let X be a normal variety of dimension n. Let φ be an upper semicontinuous function defined on X. Then we say φ is *psh* if there is a local embedding $U \hookrightarrow V$ into a complex manifold V and a psh function Φ on V such that $\varphi = \Phi|_U$. Here, U is an open subset of X. A psh function φ is said to have *analytic singularities* if there is an ideal sheaf $\mathfrak{a}$ and an exponent c such that φ can be locally written as $\frac{c}{2}\log(|g_1|^2 + \ldots + |g_m|^2) + \mathrm{O}(1)$ where $g_1, \ldots, g_m$ are local generators of $\mathfrak{a}$.

We also have to define the analytic multiplier ideal sheaf for psh function defined on X. To do this, we first define how volume forms are defined on singular varieties.

Let X be a normal $\mathbf{Q}$-Gorenstein variety and ω_X be its canonical sheaf of index m, i.e., ω_X^m is an invertible sheaf. Choose a local generator β of ω_X^m at $x \in U \subset X$ where U is open in X. Then $\alpha = \beta^{\frac{1}{m}}$ defines an $(n, 0)$-form on U_{reg}.

Using this α, we are able to define the analytic multiplier ideal sheaf of psh functions. Let $\nu = c_n \alpha \wedge \bar{\alpha}$ be a volume form on U_{reg} determined by α.

Definition 4.3.2. Let X be a normal $\mathbf{Q}$-Gorenstein variety and φ be a psh function defined on X. Then the *multiplier ideal sheaf* of φ is the ideal sheaf of holomorphic functions $\mathcal{J}(\varphi)$ whose each ring of sections satisfies the following L^2-integrability condition. Indeed, for an open subset U of X,

$$\mathcal{J}(\varphi)(U) = \left\{ f \in \mathcal{O}(U) \middle| \int_V |f|^2 e^{-2\varphi} \nu < \infty \text{ for any } V \subset\subset U_{reg} \right\}.$$

Note that $\mathcal{J}(\varphi)(U)$ is well-defined, i.e., it is independent of a choice of local generator β.

Remark 4.3.3. Let $\pi : X' \to X$ be a log resolution of singularities of a pair $(X, \Delta = 0)$. Then the integrability condition on Definition 4.3.2 can be rephrased as

$$\int_{\pi^{-1}(V)} |\pi \circ f|^2 e^{-2\pi^\star \varphi} \prod |z_i|^{2b_i} dz \wedge d\bar{z} < \infty.$$

Here (z_i) is a local coordinate chart for $\pi^{-1}(V)$ and b_i are coefficients of exceptional divisors come from the log resolution.

First of all, we prove the coherence of $\mathcal{J}(\varphi)$.

Proposition 4.3.4. $\mathcal{J}(\varphi)$ is coherent.

Proof. Note that the direct image sheaf of coherent sheaf by proper morphism is coherent. Let $\pi : X' \to X$ be a log resolution of $(X, 0)$ and let $K_{X'} = \pi^\star K_X + \sum b_i E_i$. Define an ideal sheaf $\mathcal{I}$ on X' whose local section is defined by

$$\mathcal{I}(W) = \left\{ g \in \mathcal{O}(W) \mid \int_W |g|^2 e^{-2\pi^\star \varphi} \prod |z_i|^{2b_i} dz \wedge d\bar{z} < \infty \right\}.$$

Here W is a locally bounded open subset of X' and (z_i) is a local coordinate for W such that $E_i = \{z_i = 0\}$. Since the multiplier ideal sheaf $\mathcal{J}(\varphi)$ is a direct image sheaf of $\mathcal{I}$, we are enough to show the coherence of $\mathcal{I}$.

The proof of the coherence of this ideal is analogous to the proof of coherence of multiplier ideal sheaves in complex manifold. Let $\mathcal{H}^2(W, \varphi)$ be the family of ideal sheaves on W generated by finite subsets of holomorphic functions satisfying the integral condition in $\mathcal{I}$. Then $\mathcal{H}^2(W, \varphi)$ has a maximal element which is a coherent ideal sheaf on W. Since the result is local, we are enough to check that $\mathcal{I}|_W$ is coherent. Let $\mathscr{I}$ be a maximal element in $\mathcal{H}^2(W, \varphi)$ and we are going to show $\mathscr{I} = \mathcal{I}$. Note that $\mathscr{I} \subset \mathcal{I}$ is obvious.

To prove the equality, fix $x \in W$ and let $E_1, \dots, E_k$ be exceptional divisors containing x. Note that if $k = 0$, $\mathcal{I}$ is locally coherent and there is nothing to prove. Thus, we only consider $k > 0$. Using change of coordinates, we may assume $x = 0$ and $E_i = \{z_i = 0\}$. We will show then $\mathcal{I}_x = \mathscr{I}_x$. By the viewpoint of Krull's intersection theorem(See [E13, §5.4]), we are enough to check that $\mathscr{I}_x + \mathcal{I}_x \cap \mathfrak{m}_x^{s+1} = \mathcal{I}_x$ for every integer $s \geqslant 0$. Here let x be in some proper intersection of E_i. Now for $f \in \mathcal{I}_x$ and let θ be a cut-off function such that $\theta = 1$ near x. Solve the equation $\bar{\partial} u = g := \bar{\partial}(\theta f)$ using Theorem **??** where the weight is given by

$$\tilde{\varphi} := \pi^\star \varphi - \sum b_i \log|z_i| + \sum (\frac{n}{k} + b_i + \frac{s}{k}) \log|z_i| + |z|^2.$$

Then the Lelong number of $\tilde{\varphi}$ at x is $\nu_x(\tilde{\varphi}) \geqslant (n+s)$ and by Lemma 2.2.5, we have $F := u - \theta f$ is holomorphic and $F \in \mathscr{I}$. Now, we have $f_x - F_x = u_x \in \mathcal{I}_x \cap \mathfrak{m}^{s+1}$. This concludes the proof. $\square$

Next, the definition of analytic multiplier ideal sheaf is coherent with the definition of algebraic multiplier ideal sheaf. For this, we define the algebraic multiplier ideal sheaf in singular case.

Definition 4.3.5. Let X be a normal variety and let (X, Δ) be a pair. Let $\mathfrak{a}$ be an ideal sheaf and $c > 0$ a rational number. Fix a log resolution $\mu : X' \to X$ of $\mathfrak{a}$ that also resolves the pair (X, Δ). Suppose that $K_{X'} = \mu^*(K_X + \Delta) + \sum a(E)E$ and $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$ where $-F = \sum b(E)E$. Then define the *(algebraic) multiplier ideal sheaf* associated to $\mathfrak{a}$ and c by

$$\begin{aligned} \mathcal{J}((X,\Delta),\mathfrak{a}) &= \mu_\star \mathcal{O}_{X'}(K_{X'} - \lfloor \mu^\star(K_X + \Delta) + cF \rfloor) \\ &= \mu_\star \mathcal{O}_{X'}(\sum \lceil a(E) + cb(E) \rceil E). \end{aligned}$$

Proposition 4.3.6. Let φ be a psh function of analytic singularities represented by $\mathfrak{a}^c$. Then $\mathcal{J}(\varphi) = \mathcal{J}_{alg}(\mathfrak{a}^c)$. Here, $\mathcal{J}_{alg}$ means the definition of algebraic multiplier ideal sheaf.

Proof. Let U be an open subset in X and let $f \in \mathcal{J}(\varphi)(U)$. Let $\pi : Y \to X$ be a log resolution of an ideal $\mathfrak{a}$ so that $\pi^\star\mathfrak{a}$ is an invertible sheaf $\mathcal{O}(-E)$ associated with a simple normal crossing divisor $E = \sum b_i E_i$ where E_i is defined to be $\{z_i = 0\}$ on some local coordinate chart $(V, (z_i)) \subset \pi^{-1}(U)$.

We are enough to check the local integrability of f on U, so by the change of coordinates, we are enough to check the integrability on local coordinate chart of $\pi^{-1}(U)$. Let $K_Y = \pi^\star K_X + \sum a_i E_i$. Then since $\pi^\star\nu$ is equal to $\prod |z_i|^{2a_i} d\lambda$ where $d\lambda$ is the Lebesgue measure on V, the integrability is equivalent to

$$\int_V |\pi^\star f|^2 e^{-2\pi^\star\varphi} \prod |z_i|^{2a_i} d\lambda < \infty$$

for all coordinate charts $V \subset \pi^{-1}(U)$.

Since $e^{\pi^\star\varphi}$ can be represented as a product of $|z_i|^{b_i}$ upto O(1) function, the above integrability condition can be reformulated as $\int_V |\pi^\star f|^2 \prod |z_i|^{2(a_i - cb_i)} d\lambda < +\infty$. Thus, for the integrability, we need to check that the multiplicity of f with respect to z_i is greater than $cb_i - a_i - 1$ for each i, i.e., whether $\pi^\star f$ divides $z_i^{\lfloor cb_i - a_i \rfloor}$ or not should be checked. Explicitly,

$$\begin{aligned} f \in \mathcal{J}(\varphi) &\iff f \in \pi_\star\mathcal{O}_Y(-\textstyle\sum \lfloor cb_i - a_i \rfloor E_i) \\ &\iff f \in \pi_\star\mathcal{O}_Y(\textstyle\sum \lceil a_i - cb_i \rceil E_i) \\ &\iff f \in \pi_\star\mathcal{O}_Y(K_Y - \lfloor \pi^\star K_X + cE \rfloor) \\ &\iff f \in \mathcal{J}_{alg}(\mathfrak{a}^c). \end{aligned}$$

This concludes the proof.

□

Here, there should be limits on defining multiplier ideal sheaf on general normal variety X, since singularities are assumed to be $\mathbb{Q}$-Gorenstein. So, we would like to mention a variant of multiplier ideal sheaf, so-called the multiplier module. Let X be a normal variety which is not necessary to be $\mathbb{Q}$-Gorenstein.

Definition 4.3.7. [Bl04, Definition 2] Let X be a normal variety and let $\mathfrak{a}$ be a sheaf of ideals on X. Let $\mu : Y \to X$ be a log resolution of $\mathfrak{a}$. Then we define the *multiplier module* by $\mathcal{J}_\omega(\mathfrak{a}^c) := \mu_\star \mathcal{O}_Y(K_Y - \lfloor cA \rfloor) \subseteq \omega_X$ where $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-A)$ and $c > 0$.

First of all, it is well-defined, i.e., it is independent of a choice of a log resolution. If we obtain two multiplier modules of an ideal from two different log resolutions, we can take a common log resolution which dominates both of them.

Note that it may not be an ideal sheaf indeed. However, in some specific cases such as affine toric varieties, we can consider multiplier module as an ideal sheaf. We will discuss this on later section.

For general case, we use a language of differential geometry. Let X be a normal variety and φ be a psh function defined on X and let U be an open subset of X. Then one can define a submodule $\mathcal{J}_\omega(\varphi)$ of ω_X which consists of elements satisfying the integrability in U:

$$\beta \in \mathcal{J}_\omega(\varphi)(U) \iff \sqrt{-1}^{n^2} f \wedge \overline{f} e^{-2\varphi} \in L^1_{loc}(U_{reg})$$

where f is restriction of β in U_{reg}.

Unlike multiplier ideal sheaf cases, we have the functorial property. Indeed, if $\mu : X' \to X$ is a modification, $\mu_\star(\mathcal{J}_{\omega_{X'}}(\varphi \circ \mu)) = \mathcal{J}_{\omega_X}(\varphi)$. It is straightforward due to change of variables, see [D10, Proposition 5.8] for the proof. Using this, we can prove that the definition above is indeed a generalization of algebraic definition of multiplier module. Let us distinguish between algebraic definition and analytic definition by denoting $\mathcal{J}_{\omega,an}$ and $\mathcal{J}_{\omega,al}$ for a moment.

Proposition 4.3.8. Let φ be a psh function with analytic singularities related to $\mathfrak{a}^c$. Then $\mathcal{J}_{\omega,an}(\varphi) = \mathcal{J}_{\omega,al}(\mathfrak{a}^c)$.

Proof. Let U be a fixed relatively compact open subset of X and let $\mu : Y \to X$ be a log resolution of $\mathfrak{a}$. Write $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-\sum b_i E_i) = \mathcal{O}_Y(-E)$. Take a relatively compact coordinate chart $(V, (y_i)) \subset \mu^{-1}(U)$ and $E_i = \{y_i = 0\}$ locally. Here we may check integrability condition on V instead of $\mu^{-1}(U)$, since the integrability condition in multiplier module is local. By the change of variables, we have the following integrability condition which is equivalent to $f \in \mathcal{J}_{\omega,an}(U)$

$$\int_V \frac{1}{|y|^{2cb_i}} c_n \mu^\star f \wedge \overline{\mu^\star f} < +\infty.$$

Write $\mu^\star f = g dy_1 \wedge \cdots \wedge dy_n$. Then the integrabilty is equivalent to $\operatorname{div}(g) - c\mathbf{b} > -\mathbf{1}$. Here $\mathbf{b} = \operatorname{div}(\prod y_i^{b_i})$ and $\mathbf{1} = \operatorname{div}(y_1 \cdots y_n)$. Thus $\operatorname{div}(g) \geqslant \lfloor cE \rfloor$. Since a holomorphic n-form $dy_1 \wedge \cdots \wedge dy_n$ corresponds to a basis of $\mathcal{O}_Y(K_Y)(V)$, we get $g dy_1 \wedge \cdots \wedge dy_n \in \mathcal{O}_Y(K_Y - \lfloor cE \rfloor)(V)$. Hence $\mu^\star f \in \mathcal{O}_Y(K_Y - \lfloor cE \rfloor)(\mu^{-1}(U))$.

□

4.4 Multiplier ideal sheaves on toric varieties

In general, computation of a volume form in Definition 4.3.2 seems quite difficult. So, we can not obtain any satisfactory example for multiplier ideal sheaves of psh functions on singular varieties. Instead, if we restrict our case in toric psh functions, we can get a combinatoric characterization of multiplier ideal sheaves whose computations are feaasible.

In this section, we will define notion of toric psh functions on toric varieties and related objects for computing multiplier ideal sheaves of toric psh functions. Explicitly, we will prove the following theorem.

Theorem 4.4.1. Let X be a normal, $\mathbf{Q}$-Gorenstein affine toric variety given by the cone $\sigma \subset N_{\mathbb{R}}$ whose dimension is set to be $n = \dim N_{\mathbb{R}}$. Let φ be a toric psh function defined on X. Then the multiplier ideal $\mathcal{J}(\varphi) := \mathcal{J}(\varphi)(X)$ of φ on X is a monomial ideal and given by:

$$\chi^v \in \mathcal{J}(\varphi) \iff v - \operatorname{div}(K_X) \in \operatorname{int}(P(\varphi)).$$

Here, $\operatorname{div}(K_X)$ is a point in vector space $N_{\mathbb{R}}$ whose point is related to the $\mathbb{Q}$-Cartier divisor K_X. We will explicitly define $\operatorname{div}(K_X)$ in later section.

4.4.1 Newton convex bodies of toric psh functions on $\mathbb{C}^n$

In this subsection, we will introduce the definition of Newton convex body of a psh function defined on $\mathbb{C}^n$ and define the Newton convex body of psh functions φ on general toric varieties and prove its well-definedness. Note that we already know how the Newton convex body of psh functions defined on polydisk $D(0, \mathbf{r})$. We can observe that the Newton convex body is actullay irrelevant to choice of $\mathbf{r}$. So we can use this simple observation to enlarge our domains of definition for $P(\varphi)$. We start with the definition of a toric psh function φ on a toric variety X.

Definition 4.4.2. Let X be a toric variety equipped with the torus action $\mathbf{T} \times X \to X$ and let φ be a psh function on X. Then φ is said to be toric if it is invariant under the torus action, i.e., $\varphi(gx) = \varphi(x)$ for every pair $(g, x) \in \mathbf{T} \times X$.

Remark 4.4.3. Here, we note that being φ toric is invariant under composites with toric morphisms. In fact, let $\pi : Y \to X$ be a toric morphism between two toric varieties and φ be a toric psh function. Then since π is a holomophic mapping, being φ psh is obvious. For φ being toric, we are enough to check $\varphi \circ \pi(h \cdot y) = \varphi \circ \pi(y)$ where h is an element of the torus acts on Y. Since π is equivariant under the group actions on X and Y, we know that $\pi(h \cdot y) = \pi(h) \cdot \pi(y)$ and $\pi(h)$ is the element in the torus of X. Hence, $\varphi \circ \pi(h \cdot y) = \varphi(\pi(h) \cdot \pi(y)) = \varphi(\pi(y)) = \varphi \circ \pi(y)$.

Recall the definition of Newton convex body of toric psh functions defined on $D(0, r)$(Definition 2.3.1) and Remark 2.3.6. Using these properties, we will define the notion of Newton convex body of toric psh functions defined on $\mathbb{C}^n$.

Definition 4.4.4. Let φ be a toric psh function defined on $\mathbb{C}^n$ and g be a convex function associated with φ on $\mathbb{R}^n$. Define $P(\varphi)$ by the Newton convex body of $\varphi|_{D(0,\mathbf{1})}$. Here, $\mathbf{1}$ is a polyradius whose each radius is 1.

Remark 4.4.5. I would like to emphasize that this definition does not generalize our notion of Newton convex body of convex function. Indeed, if we set $\varphi(z) = \log|z|$ defined on $\mathbb{C}$, then the Newton convex body of associated convex function is just $\{1\}$, which is totally different from the Newton convex body of $\tilde{\varphi}(z) = \log|z|$ defined on $D(0,1) \subset \mathbb{C}$. We define this new notion because we only focus on the local L^2 -integrability of holomorphic functions with respect to the weight $e^{-2\varphi}$.

Remark 4.4.6. This definition also generalizes the Newton polygon of monomial ideals in $\mathbb{C}^n$ in toric geometry. See [Ho01] for the definition of Newton polygon. For the sake of terminology, we just refer Newton convex body for dealing with analytic objects.

If we define the Newton convex body of toric psh function φ defined on $\mathbb{C}^n$, we can check that the $\mathcal{J}(\varphi)(\mathbb{C}^n)$ can be computed in exactly the same way as the Rashkovskii-Guenancia theorem. Explicitly,

Theorem 4.4.7. Let φ be a toric psh function defined on $\mathbb{C}^n$. Then the multiplier ideal $\mathcal{J}(\varphi) := \mathcal{J}(\varphi)(\mathbb{C}^n)$ is a monomial ideal and we have:

$$z^\alpha \in \mathcal{J}(\varphi) \iff \alpha + \mathbf{1} \in \operatorname{int}(P(\varphi)).$$

4.4.2 Newton convex bodies of toric psh functions on affine toric variety

For the definition of $P(\varphi)$ on a general affine normal variety X, we begin with a desingularization with a star-subdivision procedure. Say k subcones $\sigma_1, \dots, \sigma_k$ are created during this procedure and μ_i, π_i $1 \leqslant i \leqslant k$ are corresponding dual lattice maps, morphisms induced by inclusion maps of σ_i,

$1 \leqslant i \leqslant k$. Furthermore, by change of coordinates, we may assume that μ_i maps $\sigma^\vee$ to $Cone(e_1^\star, \dots, e_n^\star)$ and domain of π_i looks like a neighborhood of $0 \in \mathbb{C}^n$.

Definition 4.4.8. Let φ be a toric psh function defined on X. Define the Newton convex body of φ $P(\varphi)$ by $\bigcap_{i=1}^{k} \mu_i^{-1}(P(\varphi \circ \pi_i))$.

Since a star-subdivision may not be unique, we should clarify the well-definedness of $P(\varphi)$, i.e., it is independent of choice of subdivisions. For the proof, we will use the following lemma.

Lemma 4.4.9. Suppose that there are two resolution of singularities $\widetilde{X}_1, \widetilde{X}_2 \xrightarrow{\pi_1, \pi_2} X$, both of which are obtained by star-subdivisions. Then there is a common resolution of singularities $\widetilde{X} \xrightarrow{\pi} X$ which dominates both π_1 and π_2 and also is obtained by star-subdivision.

Proof. Let Σ be the fan realted with X and let Σ_1, Σ_2 be fans in $N_{\mathbb{R}}$ representing $\widetilde{X}_1, \widetilde{X}_2$. Then let $\widetilde{\Sigma} = \Sigma_1 \cup \Sigma_2$. Note that this union can be interpretted as a subdivision of each Σ_i so that is proper and birational. We can subdivide this $\widetilde{\Sigma}$ in sense of Theorem 4.2.8. Using the abuse of notation, we let $\widetilde{\Sigma}$ be a subdivided fan of union. Then $\widetilde{\Sigma} \to \Sigma_i \to \Sigma, i = 1, 2$ where both the first and second map are given by the identity maps on $N_{\mathbb{R}}$. This gives the result. ☐

For discussing the well-definedness, by the above lemma, we may assume that two resolutions are related with domination, i.e., π_2 dominates π_1. So our problem is reduced to the following proposition.

Proposition 4.4.10. Let $X = \mathbb{C}^n$ be the affine toric variety and $\pi : Y \to X$ be a modification of X where Y be a smooth toric variety obtained by star-subdivision of σ into k subcones $\sigma_1, \dots, \sigma_k$. Being similar as above, we let μ_i and π_i be corresponding dual lattice inclusion and morphism from smooth coordinate chart of Y for each $1 \leqslant i \leqslant k$. Let φ be a toric psh function defined on X. Then

$$v \in P(\varphi) \iff \mu_i(v) \in P(\varphi \circ \pi_i) \text{ for all } 1 \leqslant i \leqslant k.$$

Proof. We know that $\mu_1, \ldots, \mu_k$ are just the identity mapping which embeds subcone σ_i into the cone σ. For $P(\varphi \circ \pi_i)$ we need to consider $\varphi \circ \pi_i$ as a psh function defined on $\mathbb{C}^n$. Indeed, for a moment, transform π_i so that the dual cone $\sigma_i^\vee$ of σ_i becomes cone $\tau = \langle e_1^\star, \ldots, e_n^\star \rangle$ and let this map be given by the matrix B_i, i.e., $e_j^\star \mapsto b_j$ where b_j is the j-th column of B_i. This is just a composite of π_i by some toric isomorphism with $\mathbb{C}^n$. Denote this composite transformed morphism by π_i' and its related linear transformation μ_i'. Then we can consider $\varphi \circ \pi_i'$ as a function defined on $\mathbb{C}^n$. Let denote the associated convex function to $\varphi \circ \pi_i'$ by g_i. Then $g_i(y_W) = g(B_i^t \cdot y_W)$, here y_W is a coordinate on $\mathbb{R}^n$ which comes from taking $\log|\cdot|$ to the standard coordinate chart of U_τ and B_i^t is the transpose of B_i. Hence, we can represent by $P(\varphi \circ \pi_i')$ using B_i:

$$\sup_{y_W \in \mathbb{R}^n_-} (\langle x, y_W \rangle - g_i(y_W)) \leqslant \mathrm{O}(1) \iff \sup_{y_W \in \mathbb{R}^n_-} (\langle x, y_W \rangle - g(B_i^t \cdot y_W)) \leqslant \mathrm{O}(1)$$

Now, letting $x = B_i \cdot x_i$ and $y_i = B_i^t \cdot y_W$, we know that the characterization of $P(\varphi \circ \pi_i)$ is then equivalent to

$$\sup_{y_i \in B_i^t \mathbb{R}^n_-} (\langle x_i, y_i \rangle - g(y_i)) \leqslant \mathrm{O}(1).$$

This characterizes how $(\mu_i')^{-1}(P(\varphi \circ \pi_i')) = \mu_i^{-1}(P(\varphi \circ \pi_i)) = P(\varphi \circ \pi_i)$ looks like. Hence, from the viewpoint of the above characterization, the intersection of all $P(\varphi \circ \pi_i)$ is in fact,

$$\left\{ x \in M_\mathbb{R} \middle| \sup_{y_i \in B_i^t \mathbb{R}^n_-} (\langle x, y_i \rangle - g(y_i)) \leqslant \mathrm{O}(1) \text{ for all } 1 \leqslant i \leqslant k \right\}.$$

So, we are now enough to check that $\bigcup_{i=1}^{k} B_i^t \mathbb{R}^n_-$ is equal to $\mathbb{R}^n_-$, or equivalently,

$\bigcup_{i=1}^{k} B_i^t\mathbb{R}^n_+ = \mathbb{R}^n_+$. Indeed, recall that B_i is a linear transformation in $M_{\mathbb{R}}$ that sends $\sigma_i^\vee$ into $\tau = \langle e_1, \ldots, e_n\rangle^\vee$. We can consider the $\mathbb{R}^n_+$ in the left hand side as a cone $\langle e_1, \ldots, e_n\rangle$. Since the transpose of a linear map is just a dual mapping of original linear map, it maps $\langle e_1, \ldots, e_n\rangle$ into exactly σ_i. So, we conclude the proof. $\square$

Remark 4.4.11. If we set subdivision as trivial subdivision, i.e., no subdivision, we know that $P(\varphi)$'s definition is nothing but Definition 4.4.4. Also, if we set φ to have analytic singularities of type $\mathfrak{a}^c$, then this definition coincides with the original definition of the Newton convex body $P(\mathfrak{a}^c)$. Let the monomials $\chi^{v_1}, \ldots, \chi^{v_r}$ be generators of $\mathfrak{a}$. Then $P(\mathfrak{a}^c)$ is equal to $\mathrm{Conv}(cv_1, \ldots, cv_r) + \sigma^\vee$ and each $P(\varphi \circ \pi_i)$ is given as $\mathrm{Conv}(cv_1, \ldots, cv_r) + \sigma_i^\vee$. Since $\sigma^\vee = \bigcap \sigma_i^\vee$, their intersection for all $1 \leqslant i \leqslant k$ should be equal to $\mathrm{Conv}(cv_1, \ldots, cv_r) + \sigma^\vee$. For technicality, we refer Section 3.2 for related topics.

We end up this subsection by the description of the canonical representation for the canonical divisor of X. Let $u_1, \ldots, u_r$ be minimal edges of σ, i.e., generators of one-dimensional face of σ. Then the closure of orbit of each edge u_i defines prime divisor D_i of X which is torus-invariant and also there is the fact that every torus-invariant divisor can be written as a linear combination of such D_i's. From the viewpoint of this description, a divisor $\sum_{i=1}^{r} a_i D_i$ is $\mathbf{Q}$-Cartier if and only if there is a $\mathbf{Q}$-valued vector $m \in M_{\mathbf{Q}}$ such that $\langle m, u_i\rangle = a_i$ for every i.

Now, it is well-known fact that there is a canonical choice of divisor K_X which is torus-invariant, explicitly, $-\sum_{i=1}^{r} D_i$. Thus we can check whether X is $\mathbf{Q}$-Gorenstein or not easily. Also, this implies that $\mathcal{O}_X(K_X)$ can be naturally embedded into $\mathcal{O}_X$ as a monomial ideal when X is affine toric variety. In particular, we can view the multiplier module as an ideal sheaf of $\mathcal{O}_X(K_X)$.

4.4.3 Proof of the Theorem 4.4.1

We will present the original Rashkovskii-Guenancia theorem which relates the integrability of monomials with respect to a toric psh weight φ with its Newton convex body $P(\varphi)$. Before presentation, we simply demonstrate lemma which is a sufficient condition for being monomial ideals. In this lemma, the condition is slightly different with the original paper. But the proof is exactly the same. So, I did not include the proof.

Lemma 4.4.12. [Gu11, Lemma 1.12] If J is an ideal of $\mathbb{C}[S_\sigma]$ such that for every $f \in J$, monomials appear in f are also in J, then J is a monomial ideal.

Now we prove the main theorem Theorem 4.4.1.

Proof. (Proof of Theorem 4.4.1) Let $X \subset \mathbb{C}^N$ be a closed torus equivariant embedding, i.e., $\mathbf{T}_X \hookrightarrow \mathbf{T}_{\mathbb{C}^N}$ is a group homomorphism. Then for local integrability, we are enough to show that $\chi^v e^{-\varphi}$ is L^2-integrable on $D(0, \mathbf{r}) \cap X$ for arbitrary $\mathbf{r} \in \mathbb{R}^N_{>0}$.Here 0 is the unique fixed point which is invariant under the torus action by $\mathbf{T}_X$. Assume that X is determined by the cone $\sigma \subset N_{\mathbb{R}}$.

From the viewpoint above, we can consider $\mathcal{J}(\varphi)$ as $\mathcal{J}(\varphi) = \{f | |f|^2 e^{-2\varphi}$ is integrable with respect to a measure defined by a volume form on $D(0, \mathbf{r}) \cap X_{reg}\}$. Rewrite this integrability condition using a toric desingularization $\pi : \widetilde{X} \to X$ which is also a log resolution. Assume that there are r smooth coordinate charts $U_1, \ldots, U_r$ such that cover $\pi^{-1}(X_{sing})$ and come from the subdivided cones $\sigma_1, \ldots, \sigma_r$. Also, we may assume, by change of coordinate via lattice mapping, that $U_i = \mathbb{C}^n$ and $\pi_i : \mathbb{C}^n \to X$ is a toric morphism for each i. Since we restricted our domain to integrate by a relatively compact subset of X near 0, we may assume that $U_i = D(0, \mathbf{r}_i)$ for some $\mathbf{r}_i$. Then by change of coordinate again, we may assume all $\mathbf{r}_i$ are equal to $\mathbf{1}$.

First, we will verify that $\mathcal{J}(\varphi)$ is indeed a monomial ideal. Consider $f \in \mathcal{J}(\varphi)$ can be written as $\sum a_v x^v$ where v are elements of S_σ. Then pulling back, integrability condition is written in nonsingular model. In fact, for each i,

$$\int_{D(0,\mathbf{1})} |f \circ \pi_i|^2 e^{-2\varphi\circ\pi_i} |z^i|^{2a^i} < +\infty.$$

Since each v is mapped bijectively to a lattice point in $M_{\mathbb{R}}$ which represents a monomial of U_i, use the Parseval's theorem so that

$$\sum \int_{D(0,\mathbf{1})} |a_v|^2 |x^v \circ \pi_i|^2 e^{-2\varphi\circ\pi_i} |z^i|^{2a^i} < +\infty.$$

This implies x^v should be in $\mathcal{J}(\varphi)$ for all v with $a_v \neq 0$. Hence, by Lemma 4.4.12, we conclude that $\mathcal{J}(\varphi)$ is a monomial ideal.

Let μ_i be the corresponding dual lattice morphism of π_i for each i. The integrability condition near 0 is then reformulated as follows:

$$\begin{aligned}
&\int_{D(0,\mathbf{1})} |\chi^v \circ \pi_i|^2 e^{-2\varphi\circ\pi_i} |z^i|^{2a^i} < +\infty \\
\iff\ & |\chi^v \circ \pi_i|^2 |z^i|^{2a^i} \text{ is integrable w.r.t. the weight } e^{-2\varphi\circ\pi_i} \\
\iff\ & \mu_i(v) + a^i + \mathbf{1} \in \text{ int } P(\varphi\circ\pi_i) \\
\iff\ & \mu_i(v) - \mu_i(\operatorname{div}(K_X)) \in \text{ int } P(\varphi\circ\pi_i) = \text{int } \mu_i(P(\varphi)).
\end{aligned}$$

Here, for each i, $(z_1^i, \dots, z_n^i)$ is the coordinate chart of $U_i = D(0, \mathbf{1})$ and $(a_1^i, \dots, a_n^i)$ is n-tuple of coefficients of simple normal crossing divisors coming from the relative canonical divisor. The second $\iff$ follows from Theorem 4.4.7.

Now, take both sides to μ_i^{-1} and we obtain the result. □

We conclude this section by the explanation how our main theorem generalizes the Rashkovskii-Guenancia theorem.

Remark 4.4.13. The proof also shows the case if φ is defined on $X \cap D(0, \mathbf{r})$ where $D(0, \mathbf{r})$ is a polydisk in $\mathbb{C}^N$ which embeds in X. So, we have the following corollary. Here, we define $P(\varphi)$ in the sense of Definition 4.4.8.

Corollary 4.4.14. Let X be a normal, $\mathbb{Q}$-Gorenstein affine toric variety given by the cone $\sigma \subset N_{\mathbb{R}}$ whose dimension is set to be $n = \dim N_{\mathbb{R}}$. Let φ be a toric psh function defined on $X \cap D(0, \mathbf{r})$. Then the multiplier ideal of φ on X $\mathcal{J}(\varphi) := \mathcal{J}(\varphi)(X \cap D(0, \mathbf{r}))$ is monomial and $\mathcal{J}(\varphi)$ is given by:

$$\chi^v \in \mathcal{J}(\varphi) \iff v - \operatorname{div}(K_X) \in \operatorname{int}(P(\varphi)).$$

Corollary 4.4.14 also generalizes the original Rashkovskii-Guenancia's theorem when we set X to be the affine space $\mathbb{C}^n$.

Corollary 4.4.15. Let X be a normal, $\mathbb{Q}$-Gorenstein affine toric variety and let φ be a toric psh function defined on X. Then the openness property holds, i.e.,

$$\mathcal{J}(\varphi) = \mathcal{J}((1+\epsilon)\varphi) \text{ for } \epsilon \ll 1.$$

Proof. It follows from the fact that $(1+\epsilon)P(\varphi) = P((1+\epsilon)\varphi)$. Explicitly, we can view this convex body in nonsingular model(with a desingularization $\mu : \widetilde{X} \to X$) of X. Then we can choose the smallest ϵ among ϵ's that satisfy openness property of $P(\varphi \circ \mu_i)$. Here, $\mu_i : \mathbb{C}^n \to X$ is a composite of toric coordinate chart map and desingularization μ. $\square$

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국문초록

다중조화버금함수는 복소해석학 뿐 아니라 복소기하학, 나아가 대수기하학에서 중요한 연구 대상입니다. 다중조화버금함수의 특이점들은 굉장히 복잡하고 어렵고 직접적인 관찰 대신 이를 연구하기 위한 도구로 승수 아이디얼과 근사 정리를 이용하곤 합니다.

첫번째 결과로 서울대학교 수학연구소 소속인 서호섭 박사후 연구원과 equisingular 근사 정리에 대해서 소개하려고 합니다. 최근에 Qi'an Guan에 의해 발표된 해석적 특이점을 갖는 decreasing, equisingular 근사 정리라는 주제를 다중조화버금함수가 toric일 때 부분적으로 일반화할 수 있음을 설명합니다.

두번째 결과는 특이 다양체 위에서의 다중조화버금함수입니다. 기존의 다양체에서와 달리 특이 다양체에서 다중조화버금함수 및 승수 아이디얼이 어떻게 정의되는지 소개합니다. 또한 주요 결과로서, toric 다중조화버금함수의 경우, 승수 아이디얼을 계산하는데 주요 공식 중 하나인 Rashkovskii-Guenancia의 일반화를 제시합니다. 이 결과는 Blickle의 대수적 승수 아이디얼 공식을 해석적으로 일반화한 것이기도 합니다.

주요어휘: 다중조화버금함수, 승수 아이디얼, Toric 다중조화버금함수, Equisingular 근사 정리
학번: 2014-21200

감사의 글

먼저 학업적으로 미숙한 저를 올바른 길로 인도해주시고 부족한 점을 채워주신 김다노 교수님께 진심으로 감사드립니다. 앞으로 어디에서든 교수님께서 학문을 대하는 태도와 자세를 본받도록 노력하겠습니다. 또한, 김영훈 교수님을 비롯해 이훈희 교수님, 현동훈 교수님, 인하대학교 안태용 교수님께 박사논문 발표를 세심하고 꼼꼼하게 조언해주셔서 감사하다는 말씀을 드리고 싶습니다. 마지막으로 진로에 관한 고민에 대해 많은 조언을 해주시고 격려를 해주신 류경석 교수님께 깊은 감사를 전합니다.

대학원에서 함께 공부하며 동고동락한 벗들에게도 감사의 말씀을 전하고 싶습니다. 가장 먼저 연구의 진행 방향을 같이 고민하고 연구 방향에 대해 누구보다 아낌없이 조언해주신 서호섭 박사님께 깊은 감사를 표합니다. 또한, 같이 입학한 14전기 동료들, 14후기 동료들, 축구동아리 동아리원들에게 학업적으로나 정서적으로 큰 도움을 받았습니다. 개인적인 고민을 들어주고 격려해 준 유상훈 형, 박성하 형, 정남호 형, 그리고 많은 시간을 함께 보낸 서동균 형, 김지승 형, 민찬호 형, 이준석 형, 서방남 형, 김민현 형, 최정우에게 감사하다는 말씀을 드리고 싶습니다. 그 밖에 많은 사람들에게도 말로 다 할 수 없는 감사를 전합니다.

마지막으로 학업적 성취를 이루는 동안 뒤에서 아낌없이 믿어주시고 응원해주신 부모님과 누나에게 감사하다는 말씀을 드리고 싶습니다. 가족들은 제가 연구 활동을 할 수 있는 가장 큰 원동력이자 동기 부여였습니다. 그들의 격려와 조언이 없었다면 저는 박사 과정을 잘 마무리 지을 수 없었을 것입니다.

지금까지 저를 도와주시고 지도해주신 분들의 가르침 아로 새겨서 모든 일들을 잘 헤쳐나갈 수 있도록 하겠습니다. 다시 한 번 진심으로 감사드립니다.