이학 박사 학위논문

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오재성의 이학 박사 학위논문을 인준함

2021년 6월

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Orbit harmonics and cyclic sieving phenomena

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
to the faculty of the Graduate School of
Seoul National University

by

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August 2021

Abstract

Orbit harmonics and cyclic sieving phenomena

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Orbit harmonics is a tool in combinatorial representation theory which promotes the (ungraded) action of a linear group G on a finite set X to a graded action of G on a polynomial ring quotient by viewing X as a G-stable point locus in a complex space $\mathbb{C}^n$.

The cyclic sieving phenomenon is a notion in enumerative combinatorics which encapsulates the fixed-point structure of the action of a finite cyclic group C on a finite set X in terms of root-of-unity evaluations of an auxiliary polynomial $X(q)$.

In this thesis, we apply orbit harmonics to prove a variety of cyclic sieving results. This includes cyclic sieving results involving enumerations of combinatorial objects such as words, graphs or matrices, and symmetric functions such as Hall–Littlewood polynomials or Macdonald polynomials.

Key words: Cyclic sieving phenomena, Orbit harmonics, Point locus, Deformation, complex reflection group

Student Number: 2015-20268

Contents

Chapter 1

Introduction

Let X be a finite set with an action of a finite cyclic group $C = \langle c \rangle$ and let $\omega = \exp(2\pi i/|C|)$. Let $X(q) \in \mathbb{Z}_{\geq 0}[q]$ be a polynomial with nonnegative integer coefficients. We say that the triple $(X, C, X(q))$ *exhibits the cyclic sieving phenomenon* [RSW04] if for all $r \geq 0$ we have

$$|X^{c^r}| = |\{x \in X \,:\, c^r \cdot x = x\}| = X(\omega^r) = [X(q)]_{q=\omega^r}.$$

More generally, if X carries an action of a product $C_1 \times C_2 = \langle c_1 \rangle \times \langle c_2 \rangle$ of two finite cyclic groups and $X(q,t) \in \mathbb{Z}_{\geq 0}[q,t]$, the triple $(X, C_1 \times C_2, X(q,t))$ exhibits the *bicyclic sieving phenomenon* [BRS08] if for all $r, s \geq 0$ we have

$$|X^{(c_1^r, c_2^s)}| = |\{x \in X \,:\, (c_1^r, c_2^s) \cdot x = x| = X(\omega_1^r, \omega_2^s)$$

where $\omega_1 = \exp(2\pi i/|C_1|)$ and $\omega_2 = \exp(2\pi i/|C_2|)$. In typical sieving results, X is a set of combinatorial objects, the operators c, c' act on X by natural combinatorial actions, and $X(q)$ or $X(q,t)$ are generating functions for natural (bi)statistics on the set X.

Although ostensibly in the domain of enumerative combinatorics, the most desired proofs CSPs are algebraic. One seeks a representation-theoretic model for the action of C on X by finding a $\mathbb{C}$-vector space V carrying an action of a group G and possessing a distinguished basis $\{e_x : x \in X\}$ indexed by elements of the set X. The action of the generator $c \in C$ on X is modeled by a group element $g \in G$ which satisfies $g \cdot e_x = e_{c \cdot x}$ for all $x \in X$. If $\chi : G \to \mathbb{C}$ is the character of the G-module V, then

$$|X^{c^r}| = \mathrm{trace}_V(g^r) = \chi(g^r)$$

for all $r \geq 0$, transferring the enumerative problem of counting $|X^{c^r}|$ to the algebraic problem of calculating $\chi(g^r)$. These algebraic proofs are desired over brute force enumerative proofs because they give representation-theoretic insight about why a sieving result should hold.

In this thesis, we use the orbit harmonics method of zero-dimensional algebraic geometry to prove CSPs. The results proven in this fashion will have the 'nice' representation-theoretic proofs as outlined above. This approach unifies various CSPs coming from actions on word-like objects and 'quotients' thereof. The idea is to model the set X geometrically as a finite point locus in $\mathbb{C}^n$. The relevant algebra has roots in (at least) the work of Kostant [Kos63] and goes as follows.

The polynomial ring $\mathbb{C}[\mathbf{x}_n] := \mathbb{C}[x_1, \ldots, x_n]$ may be naturally viewed as the coordinate ring of polynomial functions $f : \mathbb{C}^n \to \mathbb{C}$. This identification gives rise to an action of the general linear group $\mathrm{GL}_n(\mathbb{C})$ on $\mathbb{C}[\mathbf{x}_n]$ by linear substitutions:

$$g \cdot f(v) := f(g^{-1} \cdot v) \text{ for all } g \in \mathrm{GL}_n(\mathbb{C}), f \in \mathbb{C}[\mathbf{x}_n], \text{ and } v \in \mathbb{C}^n.$$

By restriction, any subgroup of $\mathrm{GL}_n(\mathbb{C})$ also acts on $\mathbb{C}[\mathbf{x}_n]$.

Let $X \subseteq \mathbb{C}^n$ be a finite set of points which is closed under the action of $W \times C$ where

- $W \subseteq \mathrm{GL}_n(\mathbb{C})$ is a (finite) complex reflection group and
- C is a finite cyclic group acting on $\mathbb{C}^n$ by root-of-unity scaling.

Let

$$\mathbf{I}(X) := \{f \in \mathbb{C}[\mathbf{x}_n] \,:\, f(v) = 0 \text{ for all } v \in X\}$$

be the ideal of polynomials in $\mathbb{C}[\mathbf{x}_n]$ which vanish on X. Since X is finite, Lagrange Interpolation affords a $\mathbb{C}$-algebra isomorphism

$$\mathbb{C}[X] \cong \mathbb{C}[\mathbf{x}_n]/\mathbf{I}(X) \tag{1.0.1}$$

where $\mathbb{C}[X]$ is the algebra of all functions $X \to \mathbb{C}$. Since X is $W \times C$-stable, isomorphism (1.0.1) is also an isomorphism of ungraded $W \times C$-modules.

For any nonzero polynomial $f \in \mathbb{C}[\mathbf{x}_n]$, let $\tau(f)$ be the highest degree component of f. That is, if $f = f_d + \cdots + f_1 + f_0$ with f_i homogeneous of degree i and $f_d \neq 0$, we set $\tau(f) := f_d$. Given our locus X with ideal $\mathbf{I}(X)$ as above, we define a homogeneous ideal $\mathbf{T}(X)$ by

$$\mathbf{T}(X) := \langle \tau(f) \,:\, f \in \mathbf{I}(X),\ f \neq 0 \rangle \subseteq \mathbb{C}[\mathbf{x}_n].$$

The ideal $\mathbf{T}(X)$ is the *associated graded* ideal of $\mathbf{I}(X)$ and is often denoted $\mathrm{gr}\,\mathbf{I}(X)$. By construction $\mathbf{T}(X)$ is homogeneous and stable under $W \times C$. The isomorphism (1.0.1) extends to an isomorphism of $W \times C$-modules

$$\mathbb{C}[X] \cong \mathbb{C}[\mathbf{x}_n]/\mathbf{I}(X) \cong \mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X) \tag{1.0.2}$$

where $\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X)$ has the additional structure of a graded $W \times C$-module on which C acts by scaling in each fixed degree.

The action of $W \times C$ on X coincides [1] with the $W \times C$-action on the natural basis $\{e_x : x \in X\}$ of $\mathbb{C}[X]$ of indicator functions $e_x : X \to \mathbb{C}$ given by

$$e_x(y) = \begin{cases} 1 & x = y, \\ 0 & \text{otherwise.} \end{cases}$$

If the graded W-isomorphism type of $\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X)$ is known, the isomorphism (1.0.2) and Springer's theorem on regular elements [Spr74] (see Theorem 3.1.1) give bicyclic sieving results for the set X under product groups of the form $C' \times C$ where $C' \subseteq W$ is the subgroup generated by a *regular element* in W. Furthermore, if $G \subseteq W$ is any subgroup, the set X/G of G-orbits in X carries a residual action of C, and the Hilbert series of the G-invariant subspace $(\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X))^G$ is a cyclic sieving polynomial for the action of C on X/G. By varying

- the choice of point locus X and
- the choice of subgroup G of W

a variety of CSPs can be obtained. This method has been implicitly used [ARR15, Dou18] to prove CSPs before; the purpose of this thesis is to make its approach and utility explicit.

The procedure $X \rightsquigarrow \mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X)$ which promotes the (ungraded) locus X to the graded module $\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X)$ is known as *orbit harmonics*. Generators for the ideal $\mathbf{T}(X)$ may be found by computer from the point set X as follows. The ideal $\mathbf{I}(X)$ may be expressed as either an intersection or a

[1] up to duality, but permutation representations are self-dual

product

$$\mathbf{I}(X) = \bigcap_{(\alpha_1,\ldots,\alpha_n)\in X} \langle x_1-\alpha_1,\ldots,x_n-\alpha_n\rangle = \prod_{(\alpha_1,\ldots,\alpha_n)\in X} \langle x_1-\alpha_1,\ldots,x_n-\alpha_n\rangle$$

of ideals corresponding to the points $(\alpha_1,\ldots,\alpha_n)$ belonging to $X \subseteq \mathbb{C}^n$. If $\{g_1, g_2, \ldots, g_r\}$ is a Gröbner basis for $\mathbf{I}(X)$ with respect to any graded monomial order $\prec$ (i.e. we have $m \prec m'$ whenever m, m' are monomials in $\mathbb{C}[\mathbf{x}_n]$ with $\deg m < \deg m'$), then $\mathbf{T}(X)$ will be generated by

$$\{\tau(g_1), \tau(g_2), \ldots, \tau(g_r)\}.$$

While this facilitates the investigation by computer of a graded quotient $\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X)$ corresponding to a point locus X, a generating set of $\mathbf{T}(X)$ obtained in this way can be messy and not so enlightening. Finding a nice generating set of $\mathbf{T}(X)$ is often necessary to understand the graded W-isomorphism type of $\mathbb{C}[X]/\mathbf{T}(X)$.

In this thesis we will focus mostly on the reflection group $W = \mathfrak{S}_n$ acting by coordinate permutation on $\mathbb{C}^n$. This setting has received substantial attention in algebraic combinatorics. By making an appropriate choice of an $\mathfrak{S}_n$-stable locus X, orbit harmonics has produced graded $\mathfrak{S}_n$-modules which give algebraic models for various intricate objects in symmetric function theory [GP92, Gri21, HRS18]. We hope that this thesis inspires future connections between orbit harmonics and cyclic sieving.

We use orbit harmonics to reprove and unify a variety of known cyclic sieving results [ARR15, BRS08, BER11, RSW04, Rho10], prove some cyclic sieving results that seem to have escaped notice, and give new proofs of some results [Spr74, MN06] which are not *per se* in the field of cyclic sieving. It would be interesting to see if our methods apply to the notion of *dihedral sieving* due to Swanson [Swa21] (see also [RS20]).

The remainder of this thesis is organized as follows. In **Chapter 2** we give background on complex reflection groups and the representation theory of $\mathfrak{S}_n$. In **Chapter 3** we describe how orbit harmonics gives a new perspective on classical results of Springer and Morita-Nakajima. We also state our main tool for proving sieving results from point loci (Theorem 3.3.1). In **Chapter 4** we apply Theorem 3.3.1 to point loci corresponding to arbitrary, injective, and surjective functions between finite sets. In **Chapter 5** we apply Theorem 3.3.1 to other combinatorial loci. In **Chapter 6** we further investigate orbit harmonics in 'diagonal setting' to obtain cyclic sieving results involving Macdonald polynomials. We conclude in **Chapter 7** with possible future directions.

Chapter 2

Preliminaries

2.1 Combinatorics

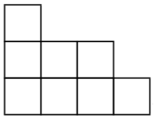
A *weak composition* of n is a sequence of non-negative integers which sum to n. A *composition* is a weak composition which consists of positive integers. A *partition* of n is a composition of n which is weakly decreasing. We denote $\mu \models n$ and $\lambda \vdash n$ for a composition μ and a partition λ of n.For a weak composition $\mu = (\mu_1, \dots, \mu_k)$, the *length* $l(\mu)$ of μ is k.

For a partition $\lambda \vdash n$ we abuse our notation so that a partition λ also denotes for its *Young diagram* (or Ferres diagram). We draw Young diagrams in French style:

$$\lambda = \{(i, j) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : i < \lambda_{j+1}\}.$$

The elements of a Young diagram are called cells. For example,

$$\lambda \quad = \quad (4, 3, 1) \quad = \{(0,0), (1,0), (2,0), (3,0), (0,1), (1,1), (2,1), (0,2)\} =$$



We define the *conjugate* λ' to be the partition obtained by reflecting λ with respect to the diagonal line $x = y$ in the plane.

A *tableau* of a partition λ is a filling $T : \lambda \to \mathbb{Z}_{>0}$. For a tableau T of shape λ, we denote $\mathrm{shape}(T) = \lambda$. The *content* of a tableau T of λ is a weak composition $(T_1, T_2, \dots)$ of n, where T_i is the number of i's appearing in T. A tableau T is called *semistandard* if each row is weakly increasing (left to right) and each column are strictly increasing (bottom to top). The *Kostka number* $K_{\lambda,\mu}$ is the number of semistandard tableaux of shape λ and content μ. A semistandard tableau is called *standard* if its content is $(1, 1, \dots)$. The set of standard tableaux of shape λ is denoted by $SYT(\lambda)$. Examples of semistandard tableau and standard tableau of shape $(4, 3, 1)$ are shown in the left and the right below, respectively.

$$\begin{array}{|c|c|c|c|}\cline{1-1} 4 \\ \cline{1-3} 2 & 2 & 4 \\ \hline 1 & 1 & 2 & 3 \\ \hline\end{array} \qquad \begin{array}{|c|c|c|c|}\cline{1-1} 5 \\ \cline{1-3} 2 & \mathbf{4} & 7 \\ \hline \mathbf{1} & \mathbf{3} & \mathbf{6} & 8 \\ \hline\end{array}$$

For a standard tableau T, a *descent* is an index i such that $i + 1$ is in the upper row than i. The *descent* $\mathrm{dex}(T)$ of T is defined as the number of descents of T and *major index* $\mathrm{maj}(T)$ of T is defined as the sum of all descents in T. For the standard tableau given right above, $1, 3, 4$ and 6 are descents (descents of the tableau is written in bold), so the major index of the tableau is $1 + 3 + 4 + 6 = 14$.

The *fake degree polynomial* of a partition λ is defined by the major index generating function for the standard tableaux of shape λ, i.e.,

$$f^{\lambda}(q) := \sum_{T \in SYT(\lambda)} q^{\mathrm{maj}(T)}.$$

The polynomial $f^{\lambda}(q)$ may be efficiently computed using the q-hook formula

$$f^{\lambda}(q) = q^{b(\lambda)} \frac{[n]!_q}{\prod_{c \in \lambda} [h_c]_q}$$

where the product is over the cells c in the Young diagram of λ and h_c is the hook length at the cell c. Here and throughout we use the standard q-analogs of numbers, factorials, binomial, and multinomial coefficients:

$$\begin{cases} [n]_q := 1 + q + \cdots + q^{n-1} \\ [n]!_q := [n]_q [n-1]_q \cdots [1]_q \end{cases} \qquad \begin{cases} {n \brack k}_q := \frac{[n]!_q}{[k]!_q \cdot [n-k]!_q} \\ {n \brack \mu_1, \ldots, \mu_r}_q := \frac{[n]!_q}{[\mu_1]!_q \cdots [\mu_r]!_q} \end{cases}$$

where $\mu = (\mu_1, \ldots, \mu_r)$ is a weak composition of n.

2.2 Symmetric functions

We denote by $\Lambda = \bigoplus_{d \geq 0} \Lambda_d$ the graded ring of symmetric functions in an infinite variable set $\mathbf{x} = (x_1, x_2, \ldots)$ over the ground field $\mathbb{C}(q)$. Here Λ_d denotes the subspace of Λ consisting of homogeneous symmetric functions of degree d. Two important elements of Λ_d are the *homogeneous* and *elementary* symmetric functions

$$h_d(\mathbf{x}) := \sum_{i_1 \leq \cdots \leq i_d} x_{i_1} \cdots x_{i_d} \qquad \text{and} \qquad e_d(\mathbf{x}) := \sum_{i_1 < \cdots < i_d} x_{i_1} \cdots x_{i_d}.$$

By restricting $h_d(\mathbf{x})$ and $e_d(\mathbf{x})$ to a finite variable set $\mathbf{x}_n = \{x_1, \ldots, x_n\}$, we obtain the homogeneous and elementary symmetric polynomials $h_d(\mathbf{x}_n)$ and $e_d(\mathbf{x}_n)$.

Bases of Λ_n are indexed by partitions of n. For a partition $\lambda \vdash n$, we let

$$h_\lambda(\mathbf{x}), \quad e_\lambda(\mathbf{x}), \quad s_\lambda(\mathbf{x}), \quad \widetilde{Q}_\lambda(\mathbf{x}; q) \quad \text{and} \quad \widetilde{H}_\lambda(\mathbf{x}; q, t)$$

denote the associated *homogeneous symmetric function*, *elementary symmetric function, Schur function, Hall-Littlewood symmetric function* and *Macdonald polynomial*. For any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ the h- and e-functions are defined by

$$h_\lambda(\mathbf{x}) := h_{\lambda_1}(\mathbf{x}) h_{\lambda_2}(\mathbf{x}) \cdots \qquad \text{and} \qquad e_\lambda(\mathbf{x}) := e_{\lambda_1}(\mathbf{x}) e_{\lambda_2}(\mathbf{x}) \cdots$$

and the Schur function is given by

$$s_\lambda(\mathbf{x}) := \sum_T \mathbf{x}^T$$

where the sum is over all semistandard tableaux T of shape λ and $\mathbf{x}^T$ is a shorthand for the monomial $x_1^{T_1} x_2^{T_2} \cdots$ where T_i is the number of i's in the tableau T.

The (modified) *Macdonald polynomials* $\widetilde{H}_\lambda(\mathbf{x}; q, t)$ indexed by partitions $\lambda \vdash n$ form another basis of Λ_n and they are defined by the unique family satisfying the following *triangulation* and *normalization* axioms [HHL05].

- $\widetilde{H}_\lambda[\mathbf{x}(1-q); q, t] = \sum_{\lambda \geq \mu} a_{\lambda,\mu}(q, t) s_\lambda$,
- $\widetilde{H}_\lambda[\mathbf{x}(1-t); q, t] = \sum_{\lambda \geq \mu'} b_{\lambda,\mu}(q, t) s_\lambda$,

- $\langle \widetilde{H}_\mu, s_{(n)} \rangle = 1$,

for suitable coefficients $a_{\lambda,\mu}, b_{\lambda,\mu'} \in \mathbb{Q}(q,t)$. Here, a partial order $\leq$ is called *dominance order* of partitions of n is defined by

$$\lambda \leq \mu \text{ if } \lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k \text{ for all } k,$$

Here, $[\cdot]$ denotes the plethystic substitution, and $\langle \cdot, \cdot \rangle$ denotes the Hall inner product defined by

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu},$$

where $\delta_{\lambda,\mu}$ is the Kronecker delta. These axioms are equivalent with Macdonald's triangularity and orthogonality axioms.

Expanding the Macdonald polynoimal with Schur functions, we may write

$$\widetilde{H}_\mu(\mathbf{x}; q, t) = \sum_\lambda \widetilde{K}_{\lambda,\mu}(q,t) s_\lambda(\mathbf{x}),$$

where the sum is over all partitions λ of n. The coefficients $\widetilde{K}_{\lambda,\mu}(q,t)$ is called the (modified) (q,t)-*Kostka polynomials*. A combinatorial description of general (q,t)-Kostka polynomials as a generating function for the standard tabelaux is unknown, and it is one of the major open problems in algebraic combinatorics.

The (modified) *Hall–Littlewood polynomial* $\widetilde{Q}_\mu(X;q)$ and q-*Kostka polynomial* $\widetilde{K}_{\lambda,\mu}(q)$ can be obtained by specializing $q = 0, t = q$ to the Macdonald polynomial $\widetilde{H}_\mu(X;q,t)$ and the (q,t)-Kostka polynomial $\widetilde{K}_{\lambda,\mu}(q,t)$. The q-Kostka polynomial $\widetilde{K}_{\lambda,\mu}(q)$ can also be defined as the generating function of the cocharge statistics for the semistandard tableaux of shape λ and content μ (see [Rho10] for definition of cocharge).

2.3 Representation theory of the symmetric group

A *class function* on a finite group G is a map $\varphi : G \to \mathbb{C}$ which is constant on conjugacy classes. The set $R(G)$ of all class functions $G \to \mathbb{C}$ forms a $\mathbb{C}$-algebra under pointwise addition and multiplication. We let $\langle -, - \rangle_G$ be the standard inner product on these class functions:

$$\langle \varphi, \psi \rangle_G := \frac{1}{|G|} \sum_{g \in G} \varphi(g)\overline{\psi(g)}.$$

If V, W are finite-dimensional G-modules with characters $\chi_V, \chi_W : G \to \mathbb{C}$, we extend this notation by setting $\langle V, W \rangle_G := \langle \chi_V, \chi_W \rangle_G$.

Irreducible representations of the symmetric group $\mathfrak{S}_n$ are in one to one correspondence with partitions λ of n. We let S^λ be the corresponding irreducible representation. If V is a finite dimensional $\mathfrak{S}_n$-module, there is a unique way of decomposing V into irreducibles as $V = \bigoplus_{\lambda \vdash n} c_\lambda S^\lambda$. The *Frobenius image* of V is the symmetric function defined by

$$\mathrm{Frob}(V) := \sum_{\lambda \vdash n} c_\lambda s_\lambda.$$

For example, if $\mu \vdash n$ and $\mathfrak{S}_\mu := \mathfrak{S}_{\mu_1} \times \mathfrak{S}_{\mu_2} \times \cdots$ is the corresponding parabolic subgroup of $\mathfrak{S}_n$, the *coset representation*

$$M^\mu := \mathbb{C}[\mathfrak{S}_n / \mathfrak{S}_\mu] = \mathbf{1} \uparrow_{\mathfrak{S}_\mu}^{\mathfrak{S}_n}$$

has Frobenius image $h_\mu(\mathbf{x})$.

If V is graded (or bigraded) $\mathfrak{S}_n$-module as $V = \bigoplus_{d \geq 0} V_d$ (or $V = \bigoplus_{d,e \geq 0} V_{d,e}$) the graded Frobenius image is the symmetric function over

$\mathbb{C}(q,t)$ given by

$$\mathrm{grFrob}(V;q) := \sum_{d\geq 0} \mathrm{Frob}(V_d)q^d$$

$$(\text{or } \mathrm{grFrob}(V;q,t) := \sum_{d,e\geq 0} \mathrm{Frob}(V_{d,e})q^d t^e).$$

We end this section with a fact about Kronecker coefficients. For $\mathfrak{S}_n$-modulues V and W, define the *inner tensor product* $V \otimes W$ to be the the usual tensor product of vector spaces with $\mathfrak{S}_n$-module structure given by

$$\sigma \cdot (v \otimes w) = (\sigma \cdot v) \otimes (\sigma \cdot w).$$

For given partitions λ, μ and ν of n, the *Kronecker coefficients* $g^{\lambda}_{\mu,\nu}$ is the multiplicity of S^λ in the inner tensor product $S^\mu \otimes S^\nu$, i.e.

$$S^\mu \otimes S^\nu \cong \bigoplus g^{\lambda}_{\mu,\nu} S^\lambda.$$

Although giving an explicit combinatorial description of general Kronecker coefficients is difficult in general (it is one of the major open problems in algebraic combinatorics), we have the following identity for special cases.

Proposition 2.3.1. *For partitions μ, ν of n, the Kronecker coefficient $g^{(n)}_{\mu,\nu} = \delta_{\mu,\nu}$, where $\delta_{\mu,\nu}$ is the Kronecker delta.*

2.4 Complex reflection groups

The general linear group $\mathrm{GL}_n(\mathbb{C})$ acts naturally on $V := \mathbb{C}^n$. An element $t \in \mathrm{GL}_n(\mathbb{C})$ is called a *reflection* if its fixed space $V^t := \{v \in V : t \cdot v = v\}$ has codimension one in V. A finite subgroup $W \subseteq \mathrm{GL}_n(\mathbb{C})$ is a *reflection* *group* if it is generated by reflections.

Let $W \subseteq \mathrm{GL}_n(\mathbb{C})$ be a complex reflection group. The full general linear group $\mathrm{GL}_n(\mathbb{C})$ acts on the polynomial ring $\mathbb{C}[\mathbf{x}_n] := \mathbb{C}[x_1, \ldots, x_n]$ by linear substitutions, and by restriction $\mathbb{C}[\mathbf{x}_n]$ is a graded W-module. Let $\mathbb{C}[\mathbf{x}_n]^W_+ \subseteq \mathbb{C}[\mathbf{x}_n]$ be the subspace of W-invariants with vanishing constant term and let $\langle \mathbb{C}[\mathbf{x}_n]^W_+ \rangle \subseteq \mathbb{C}[\mathbf{x}_n]$ be the ideal generated by this subspace. The *coinvariant ring* attached to W is the quotient

$$R_W := \mathbb{C}[\mathbf{x}_n]/\langle \mathbb{C}[\mathbf{x}_n]^W_+ \rangle.$$

This is a graded W-module.

For any irreducible W-module U, the *fake degree polynomial* $f^U(q)$ is the graded multiplicity of U in the coinvariant ring. That is, we have

$$f^U(q) := \sum_{d \geq 0} m_{U,d} q^d$$

where $m_{U,d}$ is the multiplicity of U in the W-module given by the degree d piece $(R_W)_d$ of R_W. When $W = \mathfrak{S}_n$ is the symmetric group of $n \times n$ permutation matrices, the irreducible representations of W correspond to partitions $\lambda \vdash n$, and $f^{S^\lambda}(q) = f^\lambda(q)$ specializes to our earlier definition.

Let $W \subseteq \mathrm{GL}_n(\mathbb{C})$ be a complex reflection group acting on $V = \mathbb{C}^n$. An element $c \in W$ is a *regular element* if it possesses an eigenvector $v \in V$ which has full W-orbit, or equivalently the stabilizer $W_v := \{w \in W : w \cdot v = v\}$ consists of the identity $e \in W$ alone. Such an eigenvector v is called a *regular eigenvector* and if $\omega \in \mathbb{C}^\times$ is such that $w \cdot v = \omega v$, the order of $w \in W$ will equal the order of ω in the multiplicative group $\mathbb{C}^\times$. For example, when $W = \mathfrak{S}_n$ a permutation $w \in W$ is a regular element if and only if it is a power of an n-cycle or an $(n-1)$-cycle.

2.5 Regular sequences

A length n sequence of polynomials $f_1, \ldots, f_n \in \mathbb{C}[\mathbf{x}_n]$ is a *regular sequence* if for all $1 \leq i \leq n$, the map $(-) \times f_i : \mathbb{C}[\mathbf{x}_n]/\langle f_1, \ldots f_{i-1}\rangle \to \mathbb{C}[\mathbf{x}_n]/\langle f_1, \ldots, f_{i-1}\rangle$ of multiplication by f_i is injective. If $f_1, \ldots, f_n$ is a regular sequence, we have a short exact sequence

$$\begin{aligned} 0 \quad \to \quad \mathbb{C}[\mathbf{x}_n]/\langle f_1, \ldots, f_{i-1}\rangle \quad &\xrightarrow{(-)\times f_i} \quad \mathbb{C}[\mathbf{x}_n]/\langle f_1, \ldots, f_{i-1}\rangle \\ \twoheadrightarrow \quad \mathbb{C}[\mathbf{x}_n]/\langle f_1, \ldots, f_{i-1}, f_i\rangle \quad &\to \quad 0 \end{aligned}$$

for all i. If the f_i are homogeneous, this implies that

$$\mathrm{Hilb}(\mathbb{C}[\mathbf{x}_n]/\langle f_1, \ldots, f_n\rangle; q) = [\deg f_1]_q \cdots [\deg f_n]_q$$

and in particular $\dim(\mathbb{C}[\mathbf{x}_n]/\langle f_1, \ldots, f_n\rangle) = \deg f_1 \cdots \deg f_n$.

A useful criterion for deciding whether a sequence of polynomials is regular is as follows. Let $f_1, \ldots, f_n \in \mathbb{C}[\mathbf{x}_n]$ be a length n sequence of homogeneous polynomials of positive degree. This sequence is regular if and only if the locus in $\mathbb{C}^n$ cut out by $f_1 = \cdots = f_n = 0$ consists of the origin $\{0\}$ alone.

Chapter 3

Classical results and sieving generating theorem

3.1 Springer's theorem on regular elements

Before applying orbit harmonics to prove sieving results, we state representation-theoretic results of Springer [Spr74] and Morita-Nakajima [MN06] which will be useful in our combinatorial work. We explain how orbit harmonics may be used to prove these results.

Let $W \subseteq \mathrm{GL}_n(\mathbb{C})$ be a complex reflection group and let $c \in W$ be a regular element with regular eigenvector $v \in \mathbb{C}^n$ whose eigenvalue is $\omega \in \mathbb{C}^\times$. Let $C = \langle c \rangle$ be the cyclic subgroup of W generated by c. We regard the coinvariant ring R_W as a graded $W \times C$-module, where W acts by linear substitutions and the generator $c \in C$ sends each variable x_i to ωx_i.

Theorem 3.1.1. *(Springer [Spr74]) Consider the action of $W \times C$ on W given by $(u, c^r) \cdot w := uwc^{-r}$. The corresponding permutation representation*

$\mathbb{C}[W]$ is isomorphic to R_W as an ungraded $W \times C$-module.

Remark 3.1.1. Theorem 3.1.1 gives a way to compute the irreducible characters of W on the subgroup C generated by c. As explained in [Spr74, Prop. 4.5], if U is an irreducible W-module with character $\chi : W \to \mathbb{C}$, then

$$\chi(c^r) = \mathrm{trace}_U(c^r) = f^{U^*}(\omega^r) = [f^{U^*}(q)]_{\omega^r}$$

where $f^{U^*}(q) \in \mathbb{C}[q]$ is the fake degree polynomial attached to the dual $U^* = \mathrm{Hom}_{\mathbb{C}}(U, \mathbb{C})$ of the representation U.

We describe an argument of Kostant [Kos63] which proves Theorem 3.1.1 using orbit harmonics. This argument will be used to give an orbit harmonics proof of a result [BRS08, Thm. 1.4] of Barcelo, Reiner, and Stanton (see Theorem 5.2.1).

We let C act on $\mathbb{C}^n$ by the rule $c \circ v' := \omega^{-1} v'$ for all $v' \in V$. The corresponding action of C on $\mathbb{C}[\mathbf{x}_n]$ by linear substitutions sends x_i to ωx_i for all i, just like the C-action on R_W in Theorem 3.1.1. Furthermore, this action of C on $\mathbb{C}^n$ commutes with the natural action $(w, v') \mapsto w \cdot v'$ of W, so we may regard $\mathbb{C}^n$ as a $W \times C$-module in this way.

Define the *Springer locus* to be the W-orbit of the regular eigenvector v of c:

$$W \cdot v := \{w \cdot v \,:\, w \in W\} \subseteq \mathbb{C}^n$$

The locus $W \cdot v$ is certainly closed under the action of W; we claim that X is also closed under the $\circ$-action of C. Indeed, for any $w \in W$ we have

$$c \circ (w \cdot v) = \omega^{-1}(w \cdot v) = w \cdot (\omega^{-1} v) = w \cdot (c^{-1} \cdot v) = (wc^{-1}) \cdot v \in W \cdot v.$$

Since the $\circ$ action of C and the $\cdot$ action of W on $\mathbb{C}^n$ commute, we may regard X as a $W \times C$-set. An inspection of the above chain of equalities and the regularity of v shows that the map $w \mapsto w \cdot v$ furnishes a $W \times C$-equivariant bijection

$$W \xrightarrow{\sim} W \cdot v \tag{3.1.1}$$

where the action of $W \times C$ on W is as in Theorem 3.1.1.

Chevalley [Che55] proved that there exist algebraically independent W-invariant polynomials $f_1, \ldots, f_n$ of homogeneous positive degree such that $\mathbb{C}[\mathbf{x}_n]^W = \mathbb{C}[f_1, \ldots, f_n]$. Furthermore, we have isomorphisms of ungraded W-modules

$$R_W = \mathbb{C}[\mathbf{x}_n]/\langle f_1, \ldots, f_n \rangle \cong \mathbb{C}[W].$$

We claim that the invariant polynomials $f_1, \ldots, f_n$ generate the ideal $\mathbf{T}(X)$. Indeed, for any $1 \leq i \leq n$, let $f_i(v) \in \mathbb{C}$ be the value of f_i on the regular eigenvector $v \in \mathbb{C}^n$. The W-invariance of f_i implies that $f_i - f_i(v) \in \mathbf{I}(W \cdot v)$, and taking the top degree component gives $f_i \in \mathbf{T}(W \cdot v)$, so that $\langle f_1, \ldots, f_n \rangle \subseteq \mathbf{T}(W \cdot v)$. On the other hand,

$$\dim(\mathbb{C}[\mathbf{x}_n]/\langle f_1, \ldots, f_n \rangle) = \dim \mathbb{C}[W] = |W| = |W \cdot v| = \dim(\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(W \cdot v)),$$

so that

$$\langle f_1, \ldots, f_n \rangle = \mathbf{T}(W \cdot v).$$

We use the ideal equality (3.1) to prove Theorem 3.1.1. Since the defining ideal $\langle \mathbb{C}[\mathbf{x}_n]_+^W \rangle$ of the coinvariant ring R_W is generated by $f_1, \ldots, f_n$, orbit harmonics furnishes isomorphisms of ungraded $W \times C$-modules

$$R_W = \mathbb{C}[\mathbf{x}_n]/\langle f_1, \ldots, f_n \rangle = \mathbb{C}[\mathbf{x}_n]/\mathbf{T}(W \cdot v) \cong \mathbb{C}[W \cdot v] \cong \mathbb{C}[W]$$

where the last isomorphism used the $W \times C$-equivariant bijection (3.1.1).

This completes the orbit harmonics proof of Theorem 3.1.1.

3.2 A theorem of Morita-Nakajima via Orbit Harmonics

In this section we consider the case of the symmetric group $W = \mathfrak{S}_n$. Throughout this section, we fix a weak composition $\mu = (\mu_1, \ldots, \mu_k)$ of n with k parts which has cyclic symmetry of order a for some $a \mid k$. That is, we have $\mu_i = \mu_{i+a}$ for all i with subscripts interpreted modulo k. Let c be an arbitrary but fixed generator of the cyclic group $\mathbb{Z}_{k/a}$. Morita and Nakajima proved [MN06] a variant of Springer's Theorem 3.1.1 as follows.

Let W_μ be the family of length n words $w_1 \ldots w_n$ over the alphabet $[k]$ in which the letter i appears μ_i times. The set W_μ carries an action of $\mathfrak{S}_n \times \mathbb{Z}_{k/a}$ where $\mathfrak{S}_n$ acts by subscript permutation and $\mathbb{Z}_{k/a}$ acts by $c : w_1 \ldots w_n \mapsto (w_1 + a) \cdots (w_n + a)$ where letter values are interpreted modulo k. Extending by linearity, the space $\mathbb{C}[W_\mu]$ is an $\mathfrak{S}_n \times \mathbb{Z}_{k/a}$-module.

Let $I_\mu \subseteq \mathbb{C}[\mathbf{x}_n]$ be the *Tanisaki ideal* attached to the composition μ and let $R_\mu := \mathbb{C}[\mathbf{x}_n]/I_\mu$ be the corresponding *Tanisaki quotient ring*. The ring R_μ is a graded $\mathfrak{S}_n$-module which may be described in three equivalent ways.

1. Let $\mathcal{F}\ell_n$ be the variety of complete flags $V_\bullet = (0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n)$ of subspaces of $\mathbb{C}^n$. If $X_\mu \in \mathrm{GL}_n(\mathbb{C})$ is a unipotent operator of Jordan type μ, The *Springer fiber* $\mathcal{B}_\mu$ is the closed subvariety of $\mathcal{F}\ell_n$ consisting of flags $V_\bullet$ which satisfy $X_\mu V_i = V_i$ for all i. The cohomology of $\mathcal{B}_\mu$ may be presented [Tan82] as

$$H^\bullet(\mathcal{B}_\mu; \mathbb{C}) = R_\mu.$$

Although the variety $\mathcal{B}_\mu$ is not stable under the action of $\mathfrak{S}_n \subseteq \mathrm{GL}_n(\mathbb{C})$, there is a *Springer representation* of $\mathfrak{S}_n$ on the cohomology ring $H^\bullet(\mathcal{B}_\mu; \mathbb{C})$.

2. Tanisaki [Tan82] and Garsia-Procesi [GP92] gave explicit generators for the defining ideal I_μ of R_μ. These generators are certain elementary symmetric polynomials $e_d(S)$ in a subset S of the variable set $\{x_1, \dots, x_n\}$ which depend on μ. This presentation makes it clear that R_μ is closed under the action of $\mathfrak{S}_n$; we will make no explicit use of it here. Garsia and Procesi used this presentation to show that the graded Frobenius image of R_μ is a Hall-Littlewood function:

$$\mathrm{grFrob}(R_\mu; q) = \widetilde{Q}_\mu(\mathbf{x}; q). \tag{3.2.1}$$

Here we interpret $\widetilde{Q}_\mu(\mathbf{x}; q) := \widetilde{Q}_{\mathrm{sort}(\mu)}(\mathbf{x}; q)$ where $\mathrm{sort}(\mu)$ is the partition obtained by sorting μ into weakly decreasing order.

3. Let $\alpha_1, \dots, \alpha_k \in \mathbb{C}$ be distinct complex numbers. We may consider the set $W_\mu \subseteq \mathbb{C}^n$ as a point locus by identifying

$$w_1 \dots w_n \leftrightarrow (\alpha_{w_1}, \dots, \alpha_{w_n}).$$

We have $I_\mu = \mathbf{T}(W_\mu)$ as ideals in $\mathbb{C}[\mathbf{x}_n]$. Since W_μ is closed under the coordinate permuting action of $\mathfrak{S}_n$, this makes it clear that $R_\mu = \mathbb{C}[\mathbf{x}_n]/I_\mu = \mathbb{C}[\mathbf{x}_n]/\mathbf{T}(W_\mu)$ is $\mathfrak{S}_n$-stable.

The orbit harmonics interpretation (3) of R_μ was used by Garsia and Procesi [GP92] to derive Equation (3.2.1).

Let $\omega := \exp(2a\pi i/k)$ be a primitive $(k/a)^{th}$ root-of-unity. We extend the graded $\mathfrak{S}_n$-action on R_μ to a graded $\mathfrak{S}_n \times \mathbb{Z}_{k/a}$-action by letting the distinguished generator $c \in \mathbb{Z}_{k/a}$ scale by ω^d in homogeneous degree d.

Theorem 3.2.1. *(Morita-Nakajima [MN06, Theorem 13]) We have an isomorphism of ungraded $\mathfrak{S}_n \times \mathbb{Z}_{k/a}$-modules*

$$\mathbb{C}[W_\mu] \cong R_\mu.$$

When $\mu = (1^n)$, Theorem 3.2.1 reduces to Theorem 3.1.1 when $W = \mathfrak{S}_n$ at the regular element $(1, 2, \ldots, n) \in \mathfrak{S}_n$. The proof of Theorem 3.2.1 given in [MN06] involves tricky symmetric function manipulations involving the Hall-Littlewood polynomials $\widetilde{Q}_\mu(\mathbf{x}; q)$ when q is specialized to a root of unity and relies on further intricate symmetric function identities due to Lascoux-Leclerc-Thibon [LLT94]. Orbit harmonics gives an easier and more conceptual proof.

Proof. Let $\zeta := \exp(2\pi i/k)$ be a primitive k^{th} root-of-unity with $\zeta^a = \omega$. In the interpretation (3) of R_μ described above, take the parameters $\alpha_1, \ldots, \alpha_k$ defining the point locus $W_\mu \subseteq \mathbb{C}^n$ to be $\alpha_j := \zeta^j$. If we let our distinguished generator c of $\mathbb{Z}_{k/a}$ act on $\mathbb{C}^n$ as scaling by ω, the set W_μ is closed under the action of the linear group $\mathfrak{S}_n \times \mathbb{Z}_{k/a}$. As discussed above, Garsia and Procesi [GP92] used orbit harmonics to give an isomorphism

$$\mathbb{C}[W_\mu] \cong \mathbb{C}[\mathbf{x}_n]/\mathbf{T}(W_\mu) = R_\mu$$

of ungraded $\mathfrak{S}_n \times \mathbb{Z}_{k/a}$-modules. $\square$

3.3 Loci and Sieving

Our main 'generating theorem' for sieving results is as follows. For any group W, write $\mathrm{Irr}(W)$ for the family of (isomorphism classes of) irreducible W-modules.

Theorem 3.3.1. *Let $W \subseteq \mathrm{GL}_n(\mathbb{C})$ be a complex reflection group, let $c' \in W$ be a regular element, let $C' = \langle c' \rangle$ be the subgroup of W generated by c', and let $\omega := \exp(2\pi i/k) \in \mathbb{C}^\times$. Let $C = \langle c \rangle \cong \mathbb{Z}_k$ be a cyclic group of order k and consider the action of $W \times C$ on $\mathbb{C}^n$ where c scales by ω and W acts by left multiplication. Let $X \subseteq \mathbb{C}^n$ be a finite point set which is closed under the action of $W \times C$.*

(1) Suppose that for $d \geq 0$, the isomorphism type of the degree d piece of $\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X)$ is given by

$$(\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X))_d \cong \bigoplus_{U \in \mathrm{Irr}(W)} U^{\oplus m_{U,d}}.$$

The triple $(X, C \times C', X(q,t))$ exhibits the bicyclic sieving phenomenon where

$$X(q,t) = \sum_{U \in \mathrm{Irr}(W)} m_U(q) f^{U^*}(t).$$

where $m_U(q) := \sum_{d \geq 0} m_{U,d} q^d$.

(2) Let $G \subseteq W$ be a subgroup. The set X/G of G-orbits in X carries a natural C-action and the triple $(X/G, C, X(q))$ exhibits the cyclic sieving phenomenon where

$$X(q) = \mathrm{Hilb}((\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X))^G; q).$$

Proof. Applying orbit harmonics to the action of $W \times C$ on X yields an isomorphism of ungraded $W \times C$-modules

$$\mathbb{C}[X] \cong \mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X). \tag{3.3.1}$$

Let $\zeta := \exp(2\pi i/n)$. To prove (1), apply Theorem 3.1.1 (and Remark 3.1.1) to get that for any $r, s \geq 0$, the trace of $(c^r, (c')^s) \in C \times C'$ acting on $\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X)$ is $\sum_{U \in \mathrm{Irr}(W)} m_U(\omega^r) f^{U^*}(\zeta^s) = X(\omega^r, \zeta^s)$. By the isomorphism (3.3.1), this coincides with the number of fixed points of $(c^r, (c')^s)$ acting on X, completing the proof of (1).

For (2), we take G-invariants of both sides of the isomorphism (3.3.1) to get an isomorphism of C-modules

$$\mathbb{C}[X/G] \cong (\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X))^G. \tag{3.3.2}$$

Since C acts on the graded vector space $(\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X))^G$ by root-of-unity scaling, we see that the trace of c^r on (3.3.2) is

$$[\mathrm{Hilb}((\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X))^G; q)]_{q=\omega^r} = X(\omega^r)$$

for the right-hand side, and the number of orbits in X/G fixed by c^s for the left-hand side, finishing the proof. □

Remark 3.3.1. We will mostly apply Theorem 3.3.1 in the case $W = \mathfrak{S}_n$. In this context $\mathrm{Irr}(W)$ coincides with the family of partitions $\lambda \vdash n$ and the $f^{U^*}(t)$ appearing in Theorem 3.3.1 (1) may be replaced by $f^\lambda(t)$.

In order to use Theorem 3.3.1 to prove a sieving result involving a set X, we must

- realize the relevant action on X in terms of an action on a point locus in $\mathbb{C}^n$, or a quotient thereof, and
- calculate the graded isomorphism type of $\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X)$, or the Hilbert series of $(\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X))^G$.

As we shall see, this program varies in difficulty depending on the combinatorial action in question.

Chapter 4

The Functional Loci

Rota's *Twelvefold Way* is a foundational result in enumeration and involves counting functions between finite sets which are arbitrary, injective, or surjective. Inspired by this, the loci considered in this section correspond to arbitrary, injective, and surjective functions between finite sets.

Definition 4.0.1. Given integers n and k, set $\omega := \exp(2\pi i/k)$. We define the following three point sets in $\mathbb{C}^n$:

$$\begin{aligned} X_{n,k} &:= \{(a_1, \ldots, a_n) \,:\, a_i \in \{\omega, \omega^2, \ldots, \omega^k\}\} \\ Y_{n,k} &:= \{(a_1, \ldots, a_n) \in X_{n,k} \,:\, a_1, \ldots, a_n \text{ are distinct}\} \\ Z_{n,k} &:= \{(a_1, \ldots, a_n) \in X_{n,k} \,:\, \{a_1, \ldots, a_n\} = \{\omega, \omega^2, \ldots, \omega^k\}\} \end{aligned}$$

Observe that $Y_{n,k} = \varnothing$ if $k < n$ and $Z_{n,k} = \varnothing$ if $n < k$. When $n = k$, we have the identification of $Y_{n,k} = Z_{n,k}$ with permutations in $\mathfrak{S}_n$. Each of these sets is closed under the action of $\mathfrak{S}_n \times \mathbb{Z}_k$, where $\mathbb{Z}_k$ scales by ω.

4.1 The Quotient Rings

In this section we describe generating sets for the homogeneous ideals $\mathbf{T}(X)$ corresponding to the functional loci defined in Section 4.0.1 and describe the graded $\mathfrak{S}_n$-isomorphism types of the quotients $\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X)$. This is easiest for the case of $X_{n,k}$ corresponding to arbitrary functions $\{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}$.

Proposition 4.1.1. *Let $n, k \geq 0$. The quotient ring $\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X_{n,k})$ has presentation*

$$\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X_{n,k}) = \mathbb{C}[\mathbf{x}_n]/\langle x_1^k, \ldots, x_n^k \rangle$$

and we have

$$\mathrm{grFrob}(\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X_{n,k}); q) = \sum_{\substack{\lambda_1 < k \\ \ell(\lambda) \leq n}} q^{|\lambda|} h_{m(\lambda)}$$

where the sum is over all partitions λ with largest part size $< k$ and length $\leq n$. Here $m(\lambda)$ is the partition obtained by rearranging the nonzero elements of $(n - \ell(\lambda), m_1(\lambda), m_2(\lambda), \ldots, m_{k-1}(\lambda))$ into weakly decreasing order.

Proof. Let $1 \leq i \leq n$. For any $(a_1, \ldots, a_n) \in X_{n,k}$ we have $a_i \in \{\omega, \omega^2, \ldots, \omega^k\}$ which means that $(x_i - \omega)(x_i - \omega^2) \cdots (x_i - \omega^k) \in \mathbf{I}(X_{n,k})$. Taking the highest degree component, we have $x_i^k \in \mathbf{T}(X_{n,k})$. We conclude that $\langle x_1^k, \ldots, x_n^k \rangle \subseteq \mathbf{T}(X_{n,k})$. On the other hand, we know that

$$\dim \mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X_{n,k}) = |X_{n,k}| = k^n$$

and since $\dim \mathbb{C}[\mathbf{x}_n]/\langle x_1^k, \ldots, x_n^k \rangle = k^n$, this proves the first assertion.

For the second assertion, observe that $\{x_1^{b_1} \cdots x_n^{b_n} \,:\, 0 \leq b_i < k\}$ is a basis for the quotient $\mathbb{C}[\mathbf{x}_n]/\langle x_1^k, \ldots, x_n^k\rangle$. The action of $\mathfrak{S}_n$ on these monomials decomposes this set into orbits indexed by the partitions appearing in the sum. For each partition λ, the corresponding orbit lies in degree $|\lambda|$ and has Frobenius image $h_{m(\lambda)}$. □

We turn our attention to the locus $Y_{n,k}$ corresponding to injective functions $[n] \to [k]$. The idea is to use regular sequences to reduce to the case where $k = n$.

Proposition 4.1.2. *For any $n \leq k$ we have*

$$\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(Y_{n,k}) = \mathbb{C}[\mathbf{x}_n]/\langle h_{k-n+1}(\mathbf{x}_n), \ldots, h_{k-1}(\mathbf{x}_n), h_k(\mathbf{x}_n)\rangle$$

and

$$\mathrm{grFrob}(\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(Y_{n,k}); q) = \begin{bmatrix} k \\ n \end{bmatrix}_q \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}(T)} s_{\mathrm{shape}(T)}$$

where the sum is over all standard tableaux T with n boxes.

Proof. We first show that $h_d(\mathbf{x}_n) \in \mathbf{T}(Y_{n,k})$ for any $d > k - n$. Indeed, introduce a new variable t and consider the quotient

$$\frac{(1-\omega t)(1-\omega^2 t)\cdots(1-\omega^k t)}{(1-x_1 t)(1-x_2 t)\cdots(1-x_n t)} = \sum_{d \geq 0} \sum_{a+b=d} (-1)^a e_a(\omega, \omega^2, \ldots, \omega^k) h_b(\mathbf{x}_n) t^d.$$

Whenever $(x_1, \ldots, x_n) \in Y_{n,k}$, the n factors in the denominator will cancel with n factors in the numerator, yielding a polynomial in t of degree $k - n$. If $d > k - n$, taking the coefficient of t^d on both sides yields $\sum_{a+b=d} (-1)^a e_a(\omega, \omega^2, \ldots, \omega^k) h_b(\mathbf{x}_n) \in \mathbf{I}(Y_{n,k})$ so that $h_d(\mathbf{x}_n) \in \mathbf{T}(Y_{n,k})$.

It is known that the sequence $h_{k-n+1}(\mathbf{x}_n), \ldots, h_{k-1}(\mathbf{x}_n), h_k(\mathbf{x}_n)$ of polynomials in $\mathbb{C}[\mathbf{x}_n]$ is regular. One way to see this is to check that the locus in $\mathbb{C}^n$ cut out by these polynomials consists only of the origin $\{0\}$. Indeed, the identities

$$h_d(x_1, \ldots, x_i) = x_i h_{d-1}(x_1, \ldots, x_i) + h_d(x_1, \ldots, x_{i-1})$$

mean that the system

$$\begin{aligned} h_{k-n+1}(x_1, \ldots, x_n) \quad &= \quad h_{k-n+2}(x_1, \ldots, x_n) \quad = \quad \cdots \\ &= \quad h_{k-1}(x_1, \ldots, x_n) \quad = \quad h_k(x_1, \ldots, x_n) = 0 \end{aligned}$$

is equivalent to the system

$$h_{k-n+1}(x_1, \ldots, x_n) = h_{k-n+2}(x_1, \ldots, x_{n-1}) = \cdots = h_{k-1}(x_1, x_2) = h_k(x_1) = 0.$$

The latter system is triangular and may be solved to yield the solution set $\{0\}$.

Since $h_{k-n+1}(\mathbf{x}_n), \ldots, h_{k-1}(\mathbf{x}_n), h_k(\mathbf{x}_n)$ is a regular sequence, we see that

$$\begin{aligned} &\dim \mathbb{C}[\mathbf{x}_n]/\langle h_{k-n+1}(\mathbf{x}_n), \ldots, h_{k-1}(\mathbf{x}_n), h_k(\mathbf{x}_n)\rangle \\ &\qquad = (k-n+1)\cdots(k-1)k = |Y_{n,k}| = \dim \mathbb{C}[\mathbf{x}_n]/\mathbf{T}(Y_{n,k}) \end{aligned}$$

which forces $\langle h_{k-n+1}(\mathbf{x}_n), \ldots, h_{k-1}(\mathbf{x}_n), h_k(\mathbf{x}_n)\rangle = \mathbf{T}(Y_{n,k})$. Furthermore, since the exact sequences

$$\begin{aligned} 0 \to \mathbb{C}[\mathbf{x}_n]/\langle h_{k-n+1}(\mathbf{x}_n), \ldots, h_{k-n+i-1}(\mathbf{x}_n)\rangle &\xrightarrow{(-)\times h_{k-n+i}(\mathbf{x}_n)} \\ \mathbb{C}[\mathbf{x}_n]/\langle h_{k-n+1}(\mathbf{x}_n), \ldots, h_{k-n+i-1}(\mathbf{x}_n)\rangle &\twoheadrightarrow \\ \mathbb{C}[\mathbf{x}_n]/\langle h_{k-n+1}(\mathbf{x}_n), \ldots, h_{k-n+i-1}(\mathbf{x}_n), h_{k-n+i}(\mathbf{x}_n)\rangle &\to 0 \end{aligned}$$

involve $\mathfrak{S}_n$-equivariant maps, we see that

$$\mathrm{grFrob}(\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(Y_{n,k});q) = \mathrm{grFrob}(\mathbb{C}[\mathbf{x}_n];q)\times(1-q^{k-n+1})\cdots(1-q^{k-1})(1-q^k)$$

for all $k \geq n$ and in particular

$$\begin{aligned}\mathrm{grFrob}(\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(Y_{n,k});q) &= \begin{bmatrix} k \\ n \end{bmatrix}_q \mathrm{grFrob}(\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(Y_{n,n});q) \\ = \begin{bmatrix} k \\ n \end{bmatrix}_q \mathrm{grFrob}(\mathbb{C}[\mathbf{x}_n]/\langle h_1(\mathbf{x}_n),\ldots,h_n(\mathbf{x}_n)\rangle;q) &= \begin{bmatrix} k \\ n \end{bmatrix}_q \sum_{T\in\mathrm{SYT}(n)} q^{\mathrm{maj}(T)} s_{\mathrm{shape}(T)}\end{aligned} \tag{4.1.1}$$

where the final equality is due to Lusztig (unpublished) and Stanley [Sta79]. □

The surjective locus $Z_{n,k}$ was studied by Haglund, Rhoades, and Shimozono [HRS18] and is the most difficult functional locus among the functional loci to analyze. We quote their results here.

Proposition 4.1.3. *(Haglund-Rhoades-Shimozono [HRS18]) For any $k \leq n$ we have*

$$\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(Z_{n,k}) = \mathbb{C}[\mathbf{x}_n]/\langle x_1^k, x_2^k, \ldots, x_n^k, e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \ldots, e_{n-k+1}(\mathbf{x}_n)\rangle$$

and

$$\mathrm{grFrob}(\mathbb{C}[x_1,\ldots,x_n]/\mathbf{T}(Z_{n,k});q) = \sum_{T\in\mathrm{SYT}(n)} q^{\mathrm{maj}(T)} \begin{bmatrix} n-\mathrm{des}(T)-1 \\ n-k \end{bmatrix}_q s_{\mathrm{shape}(T)}.$$

All known proofs of Proposition 4.1.3 involve intricate arguments such as Gröbner theory, a variant of Lehmer codes attached to points in $Z_{n,k}$, and somewhat involved symmetric function theory.

4.2 Bicyclic Sieving

We describe the bicyclic sieving phenomena obtained by applying Theorem 3.3.1 (1) to the loci $X_{n,k}, Y_{n,k}$, and $Z_{n,k}$. Rather than actions on point sets, we phrase the result in terms of actions on words.

Theorem 4.2.1. *Let n and k be positive integers. The following triples exhibit the bicyclic sieving phenomenon.*

1. *The triple $(X_{n,k}, \mathbb{Z}_n \times \mathbb{Z}_k, X_{n,k}(q,t))$ where $X_{n,k}$ is the set of all length n words $w_1 w_2 \dots w_n$ over the alphabet $[k]$, the cyclic group $\mathbb{Z}_n$ acts by rotating positions $w_1 w_2 \dots w_n \mapsto w_2 \dots w_n w_1$, the cyclic group $\mathbb{Z}_k$ acts by rotating values $w_1 w_2 \dots w_n \mapsto (w_1+1)(w_2+1)\cdots(w_n+1)$ (interpreted modulo k), and*

$$X_{n,k}(q,t) = \sum_{\substack{\mu \text{ a partition} \\ \ell(\mu) \leq n,\ \mu_1 \leq k}} q^{|\mu|} \begin{bmatrix} n \\ n-\ell(\mu), m_1(\mu), m_2(\mu), \dots \end{bmatrix}_t .$$

2. *The triple $(Y_{n,k}, \mathbb{Z}_n \times \mathbb{Z}_k, Y_{n,k}(q,t))$ where $Y_{n,k} \subseteq X_{n,k}$ is the subset of words with distinct letters, the group $\mathbb{Z}_n \times \mathbb{Z}_k$ acts by restricting its action on $X_{n,k}$, and*

$$Y_{n,k}(q,t) = \begin{bmatrix} k \\ n \end{bmatrix}_q \sum_{\lambda \vdash n} f^\lambda(q) f^\lambda(t).$$

3. *The triple $(Z_{n,k}, \mathbb{Z}_n \times \mathbb{Z}_k, Z_{n,k}(q,t))$ where $Z_{n,k} \subseteq X_{n,k}$ is the subset of words in which every letter in $[k]$ appears, the group $\mathbb{Z}_n \times \mathbb{Z}_k$ acts*

by restricting its action on $X_{n,k}$, and

$$Z_{n,k}(q,t) = \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}(T)} \begin{bmatrix} n - \mathrm{des}(T) - 1 \\ n-k \end{bmatrix}_q f^{\mathrm{shape}(T)}(t).$$

Proof. We prove these items in reverse order: (3), then (2), then (1). Item (3) follows immediately by combining Theorem 3.3.1 (1) and Proposition 4.1.3.

If we apply Theorem 3.3.1 (1) to the locus $Y_{n,k}$ and use Proposition 4.1.2, we obtain a bicyclic sieving result with polynomial

$$\begin{bmatrix} k \\ n \end{bmatrix} \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}(T)} f^{\mathrm{shape}(T)}(t) = \begin{bmatrix} k \\ n \end{bmatrix} \sum_{\lambda \vdash n} f^{\lambda}(t) f^{\lambda}(t) = Y_{n,k}(q,t);$$

this proves item (2) of this theorem.

Finally, if we apply Theorem 3.3.1 (1) to the locus $X_{n,k}$ and use Proposition 4.1.1, we obtain a bicyclic sieving result with polynomial

$$\sum_{\substack{\mu \text{ a partition} \\ \ell(\mu) \leq n,\ \mu_1 \leq k}} \sum_{\lambda \vdash n} q^{|\mu|} K_{\lambda,\mu} f^{\lambda}(t) \tag{4.2.1}$$

where we applied *Young's Rule*: for $\mu \vdash n$ we have $h_\mu = \sum_{\lambda \vdash n} K_{\lambda,\mu} s_\lambda$. For fixed μ, we claim that

$$\sum_{\lambda \vdash n} K_{\lambda,\mu} f^{\lambda}(t) = \begin{bmatrix} n \\ n - \ell(\mu), m_1(\mu), m_2(\mu), \ldots \end{bmatrix}_t. \tag{4.2.2}$$

Indeed, the left-hand side of Equation 4.2.2 is the generating function for the major index statistic over the set of words $w = w_1 \ldots w_n$ with $m_i(\mu)$ copies of i. The *Schensted correspondence* bijects such words w with ordered pairs (P, Q) of tableaux of the same shape with n boxes such that

- P is semistandard and Q is standard, and
- if $w \mapsto (P, Q)$ then $\mathrm{maj}(w) = \mathrm{maj}(Q)$.

Since $f^{\lambda}(t)$ is the generating function for major index on standard tableaux of shape λ, Equation (4.2.2) follows. Applying Equation (4.2.2), we see that the expression (4.2.1) equals the formula for $X_{n,k}(q, t)$ in the statement of the theorem. $\square$

4.3 The subgroup $G = \mathfrak{S}_n$

For our first application of Theorem 3.3.1 (2), we consider the case where $W = \mathfrak{S}_n$ and the subgroup $G = \mathfrak{S}_n$ is the full symmetric group. We phrase our sieving results in terms of compositions.

Let $\mathrm{WComp}_{n,k}$ be the family of weak compositions of n of length k and $\mathrm{Comp}_{n,k}$ be the set of compositions of n of length k. We have natural identifications of orbit sets

$$X_{n,k}/\mathfrak{S}_n = \mathrm{WComp}_{n,k}, \qquad Y_{n,k}/\mathfrak{S}_n = \binom{[k]}{n}, \qquad Z_{n,k}/\mathfrak{S}_n = \mathrm{Comp}_{n,k}$$

where $\binom{[k]}{n}$ is the family of n-element subsets of $[k]$. The relevant sieving result reads as follows.

Theorem 4.3.1. *Let n and k be positive integers. The following triples exhibit the cyclic sieving phenomenon.*

(1) The triple $(\mathrm{WComp}_{n,k}, \mathbb{Z}_k, \left[{n+k-1 \atop n}\right]_q)$ *where* $\mathbb{Z}_k$ *acts by rotation* $(\alpha_1, \alpha_2, \ldots, \alpha_k) \mapsto (\alpha_2, \ldots, \alpha_k, \alpha_1)$.

(2) The triple $\left(\binom{[k]}{n}, \mathbb{Z}_k, {k \brack n}_q\right)$ *where* $\mathbb{Z}_k$ *acts on* $\binom{[k]}{n}$ *by increasing entries by* 1 *modulo* k*.*

(3) The triple $(\mathrm{Comp}_{n,k}, \mathbb{Z}_k, {n-1 \brack k-1}_q)$ *where* $\mathbb{Z}_k$ *acts by rotation.*

Theorem 4.3.1 (2) is the 'first example' of the CSP of Reiner, Stanton, and White [RSW04]. Theorem 4.3.1 (1) and (3) seem to have not been explicitly stated in the literature so far.

Proof. For any graded $\mathfrak{S}_n$-module V, the Hilbert series $\mathrm{Hilb}(V^{\mathfrak{S}_n}; q)$ of the $\mathfrak{S}_n$-invariant subspace of V is the coefficient of $s_{(n)}$ in $\mathrm{grFrob}(V; q)$. Since the coefficient of $s_{(n)}$ in h_μ is 1 for any partition $\mu \vdash n$, Proposition 4.1.1 yields

$$\mathrm{Hilb}(\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X_{n,k})^{\mathfrak{S}_n}; q) = \sum_{\substack{\lambda \text{ a partition} \\ \lambda_1 < k,\ \ell(\lambda) \leq n}} q^{|\lambda|} = {n+k-1 \brack n}_q$$

and Theorem 3.3.1 (2) finishes the proof of item (1).

For item (2), we extract the coefficient of $s_{(n)}$ in Proposition 4.1.2 to get

$$\mathrm{Hilb}(\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(Y_{n,k})^{\mathfrak{S}_n}; q) = {k \brack n}_q$$

and apply Theorem 3.3.1 (2).

For item (3), extracting the coefficient of $s_{(n)}$ in Theorem 4.1.3 yields

$$\mathrm{Hilb}(\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(Z_{n,k})^{\mathfrak{S}_n}; q) = {n-1 \brack k-1}_q$$

and Theorem 3.3.1 (2) finishes the proof. $\square$

4.4 The subgroup $G = C_n$

In this section we consider the subgroup C_n of $\mathfrak{S}_n$ generated by the long cycle $(1, 2, \ldots, n)$. Recall that a *necklace* of length n over the bead set $[k]$ is an equivalence class of length n sequences $(b_1, b_2, \ldots, b_n)$ of beads in $[k]$ where two such sequences are considered equivalent if they differ by a cyclic shift. We have the following three families of necklaces over the bead set $[k]$.

$$\begin{cases} N^X_{n,k} := \{\text{all length } n \text{ necklaces with bead set } [k]\} \\ N^Y_{n,k} := \{\text{all length } n \text{ necklaces with bead set } [k] \text{ and no repeated beads}\} \\ N^Z_{n,k} := \{\text{all length } n \text{ necklaces with bead set } [k] \text{ in which every bead appears}\} \end{cases}$$

Each of the sets $N^X_{n,k}, N^Y_{n,k}, N^Z_{n,k}$ carries an action of $\mathbb{Z}_k$ by bead color rotation:

$$(b_1, b_2, \ldots, b_n) \mapsto (b_1{+}1, b_2{+}1, \ldots, b_n{+}1) \qquad \text{(letters interpreted modulo } k\text{)}$$

and we have the orbit set identifications

$$X_{n,k}/C_n = N^X_{n,k}, \qquad Y_{n,k}/C_n = N^Y_{n,k}, \qquad Z_{n,k}/C_n = N^Z_{n,k}.$$

Theorem 4.4.1. *Let n and k be positive integers. The following triples exhibit the cyclic sieving phenomenon.*

(1) The triple $(N^X_{n,k}, \mathbb{Z}_k, N^X_{n,k}(q))$ where

$$N^X_{n,k}(q) = \sum_{\substack{\mu \text{ a partition} \\ \ell(\mu) \leq n,\ \mu_1 < k}} q^{|\mu|} b_\mu$$

where b_μ is the number of length n words $w = w_1 \dots w_n$ satisfying $n \mid \mathrm{maj}(w)$ containing $n - \ell(\mu)$ copies of 0 and $m_i(\mu)$ copies of i for each $i > 0$.

(2) *The triple $(N^Y_{n,k}, \mathbb{Z}_k, N^Y_{n,k}(q))$ where*

$$N^Y_{n,k}(q) = \begin{bmatrix} k \\ n \end{bmatrix}_q \sum_{\lambda \vdash n} a_{\lambda,n} f^\lambda(q)$$

where $a_{\lambda,n}$ is the number of standard Young tableaux T of shape λ with $n \mid \mathrm{maj}(T)$.

(3) *The triple $(N^Z_{n,k}, \mathbb{Z}_k, N^Z_{n,k}(q))$ where*

$$N^Z_{n,k}(q) = \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}(T)} \begin{bmatrix} n - \mathrm{des}(T) - 1 \\ n - k \end{bmatrix}_q a_{\mathrm{shape}(T),n}.$$

Proof. For $0 \leq j \leq n-1$, let V_j be the linear representation of C_n on which $(1, 2, \dots, n-1)$ acts by $\exp(2\pi i j/n)$. Kraśkiewicz and Weyman proved [KW01] that the induction of V_j from C_n to $\mathfrak{S}_n$ may be written

$$V_j \uparrow^{\mathfrak{S}_n} \cong \bigoplus_{\mathrm{maj}(T) \equiv j \pmod{n}} S^{\mathrm{shape}(T)}$$

where the direct sum is over all standard tableaux T with n boxes such that $\mathrm{maj}(T) \equiv j$ modulo n. Specializing at $j = 0$, we see that the dimension of the C_n-fixed space of the $\mathfrak{S}_n$-irreducible S^λ is

$$\langle \mathbf{1}, S^\lambda \downarrow_{C_n} \rangle_{C_n} = \langle \mathbf{1} \uparrow^{\mathfrak{S}_n}, S^\lambda \rangle_{\mathfrak{S}_n} = a_{\lambda,n}$$

which proves (2) and (3). For (1), apply the RSK bijection to see that

$$\langle \mathbf{1}, M^{m(\mu)} \downarrow_{C_n} \rangle_{C_n} = \sum_{\lambda \vdash n} K_{\lambda, m(\mu)} \langle \mathbf{1}, S^{\lambda} \downarrow_{C_n} \rangle_{C_n} = \sum_{\lambda \vdash n} K_{\lambda, m(\mu)} \cdot a_{\lambda, n} = b_{m(\mu), n}.$$

□

4.5 The subgroup $G = H_r$

In this section we assume $n = 2r$ is even and let $H_r \subseteq \mathfrak{S}_n$ be the subgroup generated by the permutations

$$(1,2), (3,4), \ldots, (n-1, n), (1,3)(2,4), (3,5)(4,6), \ldots, (n-3, n-1)(n-2, n).$$

The group H_r is isomorphic to the hyperoctohedral group of signed permutations of an r-element set.

We consider undirected graphs on the vertex set $[k]$ in which multiple edges and loops are allowed. An *isolated vertex* in such a graph is an element $i \in [k]$ which is not incident to any edge (so that a vertex which has a loop is not isolated). We have the following families of graphs on the vertex set $[k]$.

$$\begin{cases} Gr^X_{n,k} := \{\text{all } r\text{-edge graphs on } [k]\} \\ Gr^Y_{n,k} := \{\text{all } r\text{-edge loopless graphs on } [k] \text{ in which each vertex has degree } \leq 1\} \\ Gr^Z_{n,k} := \{\text{all } r\text{-edge graphs on } [k] \text{ with no isolated vertices}\} \end{cases}$$

Observe that when $k = n$, the set $Gr^Y_{n,n}$ may be identified with the family of perfect matchings on $[n]$.

Each of the three sets $Gr^X_{n,k}, Gr^Y_{n,k}$, and $Gr^Z_{n,k}$ carries an action of $\mathbb{Z}_k$ by the vertex-rotating cycle $(1, 2, \ldots, k)$. This $\mathbb{Z}_k$-action is compatible with the orbit-set identifications

$$X_{n,k}/H_r = Gr^X_{n,k}, \qquad Y_{n,k}/H_r = Gr^Y_{n,k}, \qquad Z_{n,k}/H_r = Gr^Z_{n,k}.$$

In the following theorem, a partition λ is called *even* if each of its parts $\lambda_1, \lambda_2, \ldots$ is even.

Theorem 4.5.1. *We have the following cyclic sieving triples.*

(1) The triple $(Gr^X_{n,k}, \mathbb{Z}_k, Gr^X_{n,k}(q))$ exhibits the cyclic sieving phenomenon, where

$$Gr^X_{n,k}(q) = \sum_{\substack{\text{partitions } \mu \\ \mu \subseteq (k^n)}} \sum_{\lambda \vdash n \text{ even}} K_{\lambda, m(\mu)} \cdot q^{|\mu|} \cdot f^\lambda(q).$$

Here $m(\mu) \vdash n$ is the partition obtained by writing the part multiplicities in $(\mu_1, \mu_2, \ldots, \mu_n)$ in weakly decreasing order and $K_{\lambda, m(\mu)}$ is the Kotska number.

(2) The triple $(Gr^Y_{n,k}, \mathbb{Z}_k, Gr^Y_{n,k}(q))$ exhibits the cyclic sieving phenomenon, where

$$Gr^Y_{n,k}(q) = \begin{bmatrix} k \\ n \end{bmatrix}_q \cdot \sum_{\substack{\lambda \vdash n \\ \lambda \text{ even}}} f^\lambda(q).$$

(3) The triple $(Gr^Z_{n,k}, \mathbb{Z}_k, Gr^Z_{n,k}(q))$ exhibits the cyclic sieving phenomenon, where

$$Gr^Z_{n,k}(q) = \sum_{\substack{T \in \mathrm{SYT}(n) \\ \mathrm{shape}(T) \text{ even}}} q^{\mathrm{maj}(T)} \cdot \begin{bmatrix} n - \mathrm{des}(T) - 1 \\ n - k \end{bmatrix}_q.$$

This result should be compared with the result of Berget, Eu, and

Reiner. In [BER11, Theorem 5 (i)] they prove a result equivalent to Theorem 4.5.1 (1). Their proof, like ours, uses the symmetric function operation of *plethysm*.

Proof. For any value of d, the wreath product $\mathfrak{S}_d \wr \mathfrak{S}_r$ embeds naturally inside the larger symmetric group $\mathfrak{S}_{dr}$. The Frobenius image of the corresponding coset representation may be expressed using plethysm as

$$\mathrm{Frob}(\mathbf{1}\uparrow_{\mathfrak{S}_d \wr \mathfrak{S}_r}^{\mathfrak{S}_{dr}}) = \mathrm{Frob}(\mathbb{C}[\mathfrak{S}_{dr}/\mathfrak{S}_d \wr \mathfrak{S}_r]) = h_d[h_r].$$

Finding the Schur expansion of $h_d[h_r]$ is difficult in general, and is closely related to *Thrall's Problem.* In the special case $d = 2$, we may identify $H_r \cong \mathfrak{S}_2 \wr \mathfrak{S}_r$ and we have

$$h_2[h_r] = \sum_{\substack{\lambda \vdash n \\ \lambda \text{ even}}} s_\lambda$$

where $n = 2r$. Applying Frobenius Reciprocity, for any $\lambda \vdash n$ the dimension of the H_r-fixed subspace of the $\mathfrak{S}_n$-irreducible S^λ is given by the character inner product:

$$\dim(S^\lambda)^{H_r} = \langle \mathbf{1}, \chi^\lambda \downarrow_{H_r}^{\mathfrak{S}_n} \rangle_{H_r} = \langle \mathbf{1}\uparrow_{H_r}^{\mathfrak{S}_n}, \chi^\lambda \rangle_{\mathfrak{S}_n} = \begin{cases} 1 & \lambda \text{ is even} \\ 0 & \lambda \text{ is not even} \end{cases}$$

where we use $\mathbf{1}$ for the trivial character of H_r. Correspondingly, if V is any graded $\mathfrak{S}_n$-module with graded Frobenius image

$$\mathrm{grFrob}(V; q) = \sum_{\lambda \vdash n} c_\lambda(q) s_\lambda,$$

the Hilbert series of the H_r-fixed subspace will be

$$\mathrm{Hilb}(V^{H_r}; q) = \sum_{\substack{\lambda \vdash n \\ \lambda \text{ even}}} c_\lambda(q).$$

The polynomials $Gr^X_{n,k}(q), Gr^Y_{n,k}(q), Gr^Z_{n,k}(q)$ are obtained in this way from Propositions 4.1.1, 4.1.2, and 4.1.3. $\square$

Chapter 5

Other Combinatorial Loci

5.1 The Tanisaki Locus

Throughout this section, we fix a weak composition $\mu = (\mu_1, \ldots, \mu_k)$ of n into k parts which satisfies $\mu_i = \mu_{i+a}$ for all i, where indices are interpreted modulo k. If $\omega := \exp(2\pi i/k)$, define the *Tanisaki locus* $X_\mu \subseteq \mathbb{C}^n$ by

$$X_\mu := \{(\alpha_1, \ldots, \alpha_n) \,:\, \alpha_j = \omega^i \text{ for precisely } \mu_i \text{ values of } j\}.$$

As discussed in Section 3.2, Garsia-Procesi [GP92] proved that $\mathbf{T}(X_\mu) = I_\mu$ (the Tanisaki ideal) and $\mathrm{grFrob}(\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X_\mu); q) = \widetilde{Q}_\mu(\mathbf{x}; q)$ (the Hall-Littlewood symmetric function). We have the following bicyclic sieving result.

Theorem 5.1.1. *Let W_μ be the set of length n words $w_1 \ldots w_n$ which contain μ_i copies of the letter i for each $i = 1, 2, \ldots, k$. The set W_μ carries an action of $\mathbb{Z}_n \times \mathbb{Z}_{k/a}$, where $\mathbb{Z}_n$ acts by word rotation $w_1 w_2 \ldots w_n \mapsto w_2 \ldots w_n w_1$ and $\mathbb{Z}_{k/a}$ acts by $w_1 \ldots w_n \mapsto (w_1 + a) \ldots (w_n + a)$ where letter* *values are interpreted modulo k. The triple* $(W_\mu, \mathbb{Z}_n \times \mathbb{Z}_{k/a}, X(q,t))$ *exhibits the bicyclic sieving phenomenon, where*

$$X_\mu(q,t) = \sum_{\lambda \vdash n} \widetilde{K}_{\lambda, \mathrm{sort}(\mu)}(q) f^\lambda(t)$$

and $\mathrm{sort}(\mu)$ *is the partition obtained by sorting the parts of* μ *into weakly decreasing order.*

Proof. Apply Theorem 3.3.1 (1) to the point locus X_μ and use the Schur expansion

$$\widetilde{H}_\mu(\mathbf{x}; q) = \sum_{\lambda \vdash n} \widetilde{K}_{\lambda,\mu}(q) s_\lambda(\mathbf{x})$$

of the Hall-Littlewood polynomials. □

Theorem 5.1.1 was proven in the unpublished work of Reiner and White. A proof of Theorem 5.1.1 using Theorem 3.2.1 may be found in [Rho10].

Given a subgroup $G \subseteq \mathfrak{S}_n$, what happens when we apply Theorem 3.3.1 (2) to Y_μ? By reasoning analogous to that in Section 4 we obtain the following CSPs.

Example 5.1.1. When $G = \mathfrak{S}_n$, the locus X_μ is a single G-orbit. The $\mathfrak{S}_n$-invariant part of R_μ is simply the ground field in degree 0, so we get the trivial CSP $(\{*\}, \mathbb{Z}_{k/a}, 1)$ for the action of $\mathbb{Z}_k$ on a one-point set.

Example 5.1.2. When $G = C_n$, we may identify X_μ/G with the family of n-bead necklaces $(b_1, \ldots, b_n)$ with μ_i copies of the bead of color i. The cyclic group $\mathbb{Z}_{k/a}$ acts on these necklaces by a-fold color rotation $b_i \mapsto b_i + a$

and we get a CSP $(X_\mu/G, \mathbb{Z}_{k/a}, X(q))$ where

$$X(q) = \sum_{\lambda \vdash n} \widetilde{K}_{\lambda,\mathrm{sort}(\mu)}(q) a_{\lambda,n}$$

and $a_{\lambda,n}$ is the number of standard tableaux T of shape λ with $n \mid \mathrm{maj}(T)$.

Example 5.1.3. When $n = 2r$ is even and $G = H_r$, we may identify X_μ/G with the family of graphs (loops and multiple edges permitted) on the vertex set $[k]$ where the vertex i has degree μ_i (here a loop contributes two to the degree of its vertex). The orbit set X_μ/G is acted upon by $\mathbb{Z}_{k/a}$ by a-fold vertex rotation, and the triple $(X_\mu/G, \mathbb{Z}_{k/a}, X(q))$ exhibits the CSP where

$$X(q) = \sum_{\substack{\lambda \vdash n \\ \lambda \text{ even}}} \widetilde{K}_{\lambda,\mathrm{sort}(\mu)}(q).$$

5.2 The Springer Locus

In this section we return to the setting of an arbitrary complex reflection group $W \subseteq \mathrm{GL}_n(\mathbb{C})$ acting on $V := \mathbb{C}^n$. We fix a regular element $c \in W$ with regular eigenvector $v \in V$ and corresponding regular eigenvalue $\omega \in \mathbb{C}$, so that $c \cdot v = \omega v$. We also let $C := \langle c \rangle$ be the subgroup of W generated by c.

Recall that the Springer locus is the W-orbit $W \cdot v = \{w \cdot v : w \in W\} \subseteq V$. Section 3.1 shows that the Springer locus is closed under the action of the group $W \times C$, where W acts by its natural action on V and C acts by the rule $c : v' \mapsto \omega v'$ for all $v' \in V$. (Note that this is different from the $\circ$-action $c \circ v' := \omega^{-1} v'$ of C considered in Section 3.1.)

Theorem 5.2.1. *Let $c, c' \in W$ be regular elements and let $C = \langle c \rangle, C' = \langle c' \rangle$ be the cyclic subgroups which they generate. The product of cyclic* *groups $C \times C'$ acts on W by the rule $(c, c') \cdot w := c'wc$. The triple $(W, C \times C', W(q,t))$ exhibits the bicyclic sieving phenomenon where*

$$W(q,t) := \sum_{U} f^U(q) f^{U^*}(t)$$

and the sum is over all (isomorphism classes of) irreducible W-modules U.

Proof. The discussion in Section 3.1 shows that the homogeneous quotient $\mathbb{C}[V]/\mathbf{T}(W \cdot v)$ attached to the Springer locus $W \cdot v \subseteq V$ is given by the coinvariant ring

$$R_W = \mathbb{C}[V]/\mathbf{T}(W \cdot v).$$

The map $W \to W \cdot v$ given by $w \mapsto w \cdot v$ is a $W \times C$-equivariant bijection. Indeed, the generator c of the group C acts on $w \cdot v \in W \cdot v$ by

$$c : w \cdot v \mapsto \omega(w \cdot v) = w \cdot (\omega v) = w \cdot (c \cdot v) = (wc) \cdot v$$

which agrees with the action of C on W. By definition, the fake degree polynomial $f^U(q)$ is the graded multiplicity of U in the W-module R_W. Now apply Theorem 3.3.1 (1). $\square$

Theorem 5.2.1 is a result of Barcelo, Reiner, and Stanton [BRS08, Thm. 1.4]. In [BRS08] the polynomial $W(q,t)$ is referred to as a *bimahonian distribution*. More generally, Barcelo, Reiner, and Stanton consider 'Galois twisted' actions of $C \times C'$ on W as follows. Let d be the order of the regular element $c' \in W$ and let s be an integer coprime to d. The group $C \times C'$ acts on W by the rule

$$(c, c') := (c')^s \cdot w \cdot c. \tag{5.2.1}$$

If we let $\zeta := \exp(2\pi i/d)$, there is a unique Galois automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}[\zeta]/\mathbb{Q})$ satisfying $\sigma(\zeta) = \zeta^s$. Furthermore, if U is any W-module, there is a W-module $\sigma(U)$ obtained by applying σ entrywise to the matrices representing group elements in the action of W on U. The operation $U \mapsto \sigma(U)$ preserves the irreducibility of W-modules.

In [BRS08, Thm. 1.4] Barcelo, Reiner, and Stanton prove that

$$(W, C \times C', W^\sigma(q,t))$$

exhibits the biCSP where $C \times C'$ acts by the Galois-twisted action of (5.2.1) and $W^\sigma(q,t)$ is the *σ-bimahonian distribution* given by

$$W^\sigma(q,t) := \sum_U f^U(q) f^{\sigma^{-1}(U)}(t)$$

where U runs over all isomorphism classes of irreducible W modules. This more general biCSP can also be proven using orbit harmonics; one observes that $(c')^s \in C'$ is a regular element in W with regular eigenvalue ζ^s and applies Theorem 5.2.1 with $c' \mapsto (c')^s$.

5.3 Loci of colored words

The complex reflection group $G(r,1,n) \leq \mathrm{GL}_n(\mathbb{C})$ is a group of $n \times n$ monomial matrices whose nonzero entries are ζ^i for some i, where $\zeta := e^{\frac{2\pi i}{r}} \in \mathbb{C}$. Irreducible representations of $G(r,1,n)$ are in one-to-one correspondence with r-tuple $\lambda^\bullet = (\lambda^{(0)}, \ldots, \lambda^{(r-1)})$ of partitions with total size $|\lambda_0| + \cdots + |\lambda_{r-1}| = n$. We denote the irreducible representation corresponding to $\lambda^\bullet$ by $S^{\lambda^\bullet}$. Let $\Lambda^{G(r,1,n)} := \Lambda^{\otimes r}$ be the r^{th} tensor of the symmetric function ring Λ (with variables $x^{(i)}$'s). For any finite-dimensional

$G(r,1,n)$-module V, there extists unique multiplicity $c_{\lambda^\bullet}$ for each $\lambda^\bullet$ so that $V \cong \bigoplus_{\lambda^\bullet \vdash n} c_{\lambda^\bullet} S^{\lambda^\bullet}$. The *Frobenius image* of V is defined by

$$\mathrm{Frob}^{G(r,1,n)}(V) := \sum_{\lambda^\bullet \vdash n} c_{\lambda^\bullet} s_{\lambda^\bullet}(\mathbf{x}) \in \Lambda^{G(r,1,n)},$$

where $s_{\lambda^\bullet}(\mathbf{x}) = s_{\lambda^{(0)}}(\mathbf{x}^{(0)}) \cdots s_{\lambda^{(r-1)}}(\mathbf{x}^{(r-1)})$. We define $\mathrm{grFrob}^{G(r,1,n)}$ in a usual way.

Suppose X is invariant under the action of $\mathfrak{S}_n \times C_k$, where $\mathfrak{S}_n$ acts on X by permuting coordinates and a generator $c \in C_k$ acts on X by k^{th} root of unity scaling. Then an *r-colored version* of X,

$$\mathrm{Col}_r(X) := \{(\zeta^{c_1} x_1^{\frac{1}{r}}, \ldots, \zeta^{c_n} x_n^{\frac{1}{r}}) : (x_1, \ldots, x_n) \in X, c_1, \ldots, c_n \in \{0, 1, \ldots, r-1\}\}$$

is invariant under action of $G(r,1,n) \times C_{kr}$, where $G(r,1,n)$ acts by left multiplication and C_{kr} acts by scaling a kr^{th} root of unity. Then we have the following equivalence as ungraded $G(r,1,n) \times C_{kr}$-modules

$$\mathbb{C}[\mathrm{Col}_r(X)] \cong \mathbb{C}[x_1, \ldots, x_n] / \mathrm{Col}_r(T(X)),$$

where $\mathrm{Col}_r(T(X))$ is the image of $T(X)$ under r^{th} power ring homomorphism given by $x_k \mapsto x_k^r$. We apply Theorem 3.3.1 to $\mathrm{Col}_r(X)$ for a loci X to get a sieving result.

For example, following Section 5.1 let $\mu = (\mu_1, \ldots, \mu_k)$ be a weak composition of n into k parts with which satisfies $\mu_i = \mu_{i+a}$ for all i, where indices are interpreted modulo k and X_μ be the Tanisaki locus. Then $\mathrm{Col}_r(X_\mu)$ can be regarded as the set of words of length n where there are μ_i (colored) i's. To calculate graded Frobenius image, we use the following proposition.

Proposition 5.3.1. *[OS19, Proposition 9] For a locus $X \subseteq \mathbb{C}^n$, we have the following identity:*

$$\begin{aligned}\operatorname{grFrob}^{G(r,1,n)}(\mathbb{C}[\mathbf{x}_n]/\operatorname{Col}_r(T(X)); q) = \\ \operatorname{grFrob}(\mathbb{C}[\mathbf{x}_n]/T(X); q^r)[\mathbf{x}^{(0)} + q\mathbf{x}^{(1)} + \cdots + q^{r-1}\mathbf{x}^{(r-1)}],\end{aligned}$$

where grFrob *is the usual graded Frobenius image of $\mathfrak{S}_n$ module and $[\cdot]$ denotes the plethystic substitution.*

Before we calculate the graded Frobenius image of $\operatorname{Col}_r(X_\mu)$, let us define a statistic. Let $\lambda^\bullet = (\lambda^{(0)}, \ldots, \lambda^{(r-1)}) \vdash n$ be an r-tuple of partitions of n. We draw a Young diagram of $\lambda^\bullet$ having n cells total, consisting of r-tuples of Young diagrams arranged in the plane so that $\lambda^{(k-1)}$ is northwest of $\lambda^{(k)}$. A semistandard tableau of shape $\lambda^\bullet$ if its entries are weakly increasing from left to right in each row and strictly increasing from bottom to top in each column. A weight of a semistandard tableau T of shape $\lambda^\bullet$ is $(T_1, T_2, \ldots)$, where T_i denotes the number of i's in T. Then define the *flag-cocharge* of a tableau T of shape $\lambda^\bullet$ by

$$\operatorname{fcc}(T) := r \cdot \operatorname{cc}(T) + \sum_{i=0}^{r-1} i \cdot |\lambda^i|,$$

where cc is the usual cocharge statistic.

Since the graded Frobenius image of $\operatorname{Col}_r(X_\mu)$ is the Hall–Littlewood polynomial, by applying the above proposition to $\operatorname{Col}_r(X_\mu)$, we have

$$
\begin{aligned}
\operatorname{Col}_r(X_\mu)(\mathbf{x};q) &:= \operatorname{grFrob}^{G(r,1,n)}(\mathbb{C}[\mathbf{x}_n]/\operatorname{Col}_r(T(X_\mu));q^r)\\
&= \operatorname{grFrob}(\mathbb{C}[\mathbf{x}_n]/T(X);q^r)[\mathbf{x}^{(0)}+q\mathbf{x}^{(1)}+\cdots+q^{r-1}\mathbf{x}^{(r-1)}]\\
&= \widetilde{Q}_\mu(x;q^r)[\mathbf{x}^{(0)}+q\mathbf{x}^{(1)}+\cdots+q^{r-1}\mathbf{x}^{(r-1)}]\\
&= \sum_{\lambda\vdash n}\widetilde{K}_{\lambda,\mu}(q^r)s_\lambda[\mathbf{x}^{(0)}+q\mathbf{x}^{(1)}+\cdots+q^{r-1}\mathbf{x}^{(r-1)}]\\
&= \sum_{\lambda\vdash n}\sum_{\nu^\bullet\vdash n}\widetilde{K}_{\lambda,\mu}(q^r)q^{\sum_{i=0}^{r-1}i\nu_i}c^\lambda_{\nu^\bullet}s_{\nu^{(0)}}(\mathbf{x}^{(0)})\cdots s_{\nu^{(r-1)}}(\mathbf{x}^{(r-1)})\\
&= \sum_{\nu^\bullet\vdash n}\widetilde{K}^{flag}_{\nu^\bullet,\mu}(q)s_{\nu^\bullet}(\mathbf{x}).
\end{aligned}
$$

where $\widetilde{K}^{flag}_{\nu^\bullet,\mu}(q)$ is the generating function of flag-cocharge over all semistandard tabelau of shape $\nu^\bullet$ and content μ. To be precise, the last equality comes from a usual fact of plethysm and the last equality comes from the fact that the *jeu-de-taquin* provides a cocharge-preserving bijection between semistandard tableaux of shape λ and pairs of a Yamanouchi tableau and a semistandard tableau of shape $\nu^\bullet$.

Taking the Hilbert series to associated $G(r,1,n)$ module of $\operatorname{Col}_r(X_\mu)$, we have

$$
\operatorname{Col}_r(X_\mu)(q,t) := \sum_{\mu\vdash n}\sum_{\nu^\bullet\vdash n}\widetilde{K}^{flag}_{\lambda,\mu}(q)f^{\nu^{\bullet *}}(t),
$$

where $\nu^{\bullet *}(t)$ is the dual fake degree polynomial attached to the dual of the irreducible $G(r,1,n)$ module indexed by $\nu^\bullet$. By Theorem 3.3.1 we obtain a bicyclic sieving result concerning colored words with the action of 'twisted rotation' $(w_1^{c_1}\ldots w_n^{c_n})\mapsto(w_n^{c_n+1}w_1^{c_1}\ldots w_{n-1}^{c_{n-1}})$ and 'color rotation'

$(w_1^{c_1} \dots w_n^{c_n}) \mapsto (w_1^{c_1+1} \dots w_n^{c_n+1})$. When $\mu = (1^n)$,

$$\mathrm{Col}_r(X_{(1^n)})(q,t) := \sum_{\nu^\bullet \vdash n} f^{\nu^{\bullet *}}(t) f^{\nu^\bullet}(t)$$

this reduces to a bicyclic sieving theorem of [BRS08].

When $\mu = (1)$ and $r = 2$,

$$\mathrm{Col}_2(X_{(1)})(q,t) = \sum_{k=0}^{n} q^k t^k \begin{bmatrix} n \\ k \end{bmatrix}_{t^2}.$$

This provides a biCSP for binary words (words of length n with alphabets in $\{-1,1\}$) where one cyclic group acts by twisted rotation $(a_1,\dots,a_n) \mapsto (-a_n, a_1, \dots, a_{n-1})$ and the other cyclic group $\mathbb{Z}_2$ acts by $(a_1,\dots,a_n) \mapsto (-a_1,\dots,-a_n)$. This gives a desired representation theoretic proof of the sieving result for twisted rotation on binary words in [AU19].

Chapter 6

Macdonald polynomials and cyclic sieving

6.1 Main theorems

Since Macdonald [Mac88] defined Macdonald polynomials $\widetilde{H}_\mu(\mathbf{x}; q, t)$ and conjectured the Schur positivity of them, the Macdonald polynomial has been one of the central objects in algebraic combinatorics. Even though a combinatorial formula for Macdonald polynomials is given [HHL05] and the Schur positivity of Macdonald polynomials is proved [Hai01], not much is known about an explicit combinatorial formula for the (q, t)-Kostka polynomials, which are the Schur coefficients of the Macdonald polynomial. In this chapter, we discuss some enumerative results involving the (q, t)-Kostka polynomials in the words of cyclic sieving phenomena. This will provide a series of identities between the number of matrices with certain cyclic symmetries and evaluation of (q, t)-Kostka polynomials at a root of unity, uncovering a part of the mystery of the (q, t)-Kostka polynomials.

To begin, we generalize the cyclic seiving phenomena. Let X be a set with action of a direct product of k cyclic groups $C_1 \times C_2 \times \cdots \times C_k$. For each $i = 1, 2, \ldots, k$, fix a generator c_i for C_i. Let $X(q_1, q_2, \ldots, q_k) \in \mathbb{Z}[q_1, q_2, \ldots, q_k]$ be a polynomial in k variables. Following [BRS08], we say the triple $(X, C_1 \times C_2 \times \cdots \times C_k, X(q_1, q_2, \ldots, q_k))$ exhibits the *k-ary-cyclic sieving phenomenon (k-ari-CSP)* if for any integers $r_1, r_2, \ldots, r_k$ the number of fixed points of $(c_1^{r_1}, c_2^{r_2}, \ldots, c_k^{r_k})$ in X is equal to the evaluation of $X(q_1, q_2, \ldots, q_k)$ at $(q_1, q_2, \ldots, q_k) = (\zeta_1^{r_1}, \zeta_2^{r_2}, \ldots, \zeta_k^{r_k})$, i.e.

$$|X^{(c_1^{r_1}, c_2^{r_2}, \ldots, c_k^{r_k})}| = |\{x \in X \,:\, (c_1^{r_1}, c_2^{r_2}, \ldots, c_k^{r_k}) \cdot x = x\}| = X(\zeta_1^{r_1}, \zeta_2^{r_2}, \ldots, \zeta_k^{r_k}),$$

where ζ_i is a root of unity having the same multiplicative order as c_i. In this chapter, we provide instances of tricyclic sieving phenomena, i.e. k-ary-CSP for $k = 3$.

In this chapter, we adopt orbit harmonics to the diagonal orbit harmonics to obtain a 'generating theorem' (Theorem 6.3.1) for sieving results of the combinatorial locus $X \subseteq \mathbb{C}^{2n}$ with diagonal action of $\mathfrak{S}_n$ on X. A precise explanation of diagonal orbit harmonics is given in Section 6.3.

The main theorems of this chapter are Theorem 6.1.1 and Theorem 6.1.2 involving $(q, t)-$Kostka polynomials and enumeration of matrices under certain symmetries [Oh21].

Theorem 6.1.1. *Let $X_{(a^b)}$ be the set of $b \times a$ matrices of content (1^{ab}). The product of cyclic groups $\mathbb{Z}_b \times \mathbb{Z}_a \times \mathbb{Z}_{ab}$ acts on $X_{(a^b)}$ by row rotation, column rotation and adding 1 modulo ab to each entry. In addition for a composition $\nu \models ab$, let $X_{(a^b),\nu}$ be the set of $b \times a$ matrices of content ν where the product of cyclic groups $\mathbb{Z}_b \times \mathbb{Z}_a$ acts on $X_{(a^b),\nu}$ by row and column rotation. Then we have the following.*

- $(X_{(a^b)}, \mathbb{Z}_b \times \mathbb{Z}_a \times \mathbb{Z}_{ab}, X_{(a^b)}(q,t,z))$ *exhibits triCSP, where*

$$X_{(a^b)}(q,t,z) = \sum_{\lambda \vdash ab} \widetilde{K}_{\lambda,(a^b)}(q,t) f^{\lambda}(z).$$

- $(X_{(a^b),\nu}, \mathbb{Z}_b \times \mathbb{Z}_a, X_{(a^b),\nu}(q,t))$ *exhibits biCSP, where*

$$X_{(a^b),\nu}(q,t) = \sum_{\lambda \vdash ab} \widetilde{K}_{\lambda,(a^b)}(q,t) K_{\lambda,\nu}.$$

Here, $\widetilde{K}_{\lambda,\mu}(q,t)$ ($K_{\lambda,\mu}$, respectively) denotes the modified (q,t)-Kostka polynomial (Kostka number, respectively) and $f^{\lambda}(z)$ is the fake degree polynomial.

We say a composition ν has cyclic symmetry of order m if $\nu_i = \nu_{i+m}$ for all i, where the subscripts are interpreted modulo the length $l(\nu)$ of ν. In the second bullet point of the above theorem, if ν has a cyclic symmetry of order a dividing $l(\nu)$, the set $X_{(a^b),\nu}$ possesses an additional action of a cyclic group $\mathbb{Z}_{l(\nu)/m}$ by adding m modulo $l(\nu)$ to each entry. Then one might ask if there is a natural z-analogue of $X_{(a^b),\nu}(q,t)$ to give triCSP for $X_{(a^b),\nu}$. We give an answer of this question in the following theorem.

Theorem 6.1.2. *Let $\nu \models ab$ be a composition with cyclic symmetry of order m dividing $l(\nu)$. Let $X_{(a^b),\nu}$ be the set of $b \times a$ matrices of content ν. The product of cyclic groups $\mathbb{Z}_b \times \mathbb{Z}_a \times \mathbb{Z}_{l(\nu)/m}$ acts on $X_{(a^b),\nu}$ by row rotation, column rotation and adding a modulo $l(\nu)$ to each entry. Then the triple $(X_{(a^b),\nu}, \mathbb{Z}_b \times \mathbb{Z}_a \times \mathbb{Z}_{l(\nu)/m}, X_{(a^b),\nu}(q,t,z))$ exhibits the triCSP, where*

$$X_{(a^b),\nu}(q,t,z) = \sum_{\lambda \vdash ab} \widetilde{K}_{\lambda,(a^b)}(q,t) \widetilde{K}_{\lambda,\nu}(z).$$

Here, $\widetilde{K}_{\lambda,\mu}(q,t)$ ($\widetilde{K}_{\lambda,\mu}(z)$, respectively) denotes the modified (q,t)-Kostka polynomial (z-Kostka polynomial, respectively).

It should be remarked that there have been similar results discovered which relate root of unity specializations of q-Kostka polynomials and fixed point enumerations of matrices or fillings of tableaux (see [Rho10, AU19] for example). It should be mentioned that there is more resemblance between Theorem 6.1.2 and the results of Rhoades [Rho10] in which, using Hall–Littlewood polynomial, he showed that $\mathbb{N}$-matrices with fixed column content μ and row content ν exhibits biCSP [Rho10]. We modify the argument in [Rho10] to prove Theorem 6.1.2 in Section 6.5.2.

6.2 Modules of Garsia and Haiman

To each partition μ of n, one associates a matrix $(x_i^a y_i^b)_{1\le i\le n,(a,b)\in\mu}$. The $\mathfrak{S}_n$-module $\mathbf{H}_\mu$ is the smallest vector space over $\mathbb{C}$ that contains the determinant $\Delta_\mu := \det(x_i^a y_i^b)_{1\le i\le n,(a,b)\in\mu}$ and its partial derivatives with respect to any of the variables x_i's and y_i's for $1 \le i \le n$. For example, for a partition $\mu = (3,2)$, the corresponding matrix is given by

$$\Delta_\mu := \begin{bmatrix} 1 & x_1 & x_1^2 & y_1 & x_1y_1 \\ 1 & x_2 & x_2^2 & y_2 & x_3y_3 \\ 1 & x_3 & x_3^2 & y_3 & x_3y_3 \\ 1 & x_4 & x_4^2 & y_4 & x_4y_4 \\ 1 & x_5 & x_5^2 & y_5 & x_5y_5 \end{bmatrix}.$$

Then the module $\mathbf{H}_\mu$ is given by the $\mathbb{C}$-span

$$\mathbb{C}\{\partial_{\mathbf{x}_I}\partial_{\mathbf{y}_J}\Delta_\mu\}_{I,J} = \mathbb{C}\{\Delta_\mu, \partial_{x_1}\Delta_\mu, \partial_{x_2}\Delta_\mu, \partial_{y_1}\Delta_\mu, \partial_{y_2}\Delta_\mu, 1\},$$

where I, J are over all multisets with entries in $\{1, 2, \ldots, n\}$ and $\partial_{\mathbf{x}_I} := \partial_{x_{i_1}} \ldots \partial_{x_{i_k}}$ for a multiset $I = \{i_1, \ldots, i_k\}$ and $\partial_{\mathbf{y}_J}$ is defined similarly. The symmetric group $\mathfrak{S}_n$ acts on $\mathbf{H}_\mu$ diagonally i.e. permuting x and y coordinates in the same way. This bigraded $\mathfrak{S}_n$-module $\mathbf{H}_\mu$ is called the *Garsia–Haiman module*. Haiman [Hai01] proved the $n!$ *conjecture* which asserts that this module is of dimension $n!$ regardless of μ, and moreover, the graded Frobenius image of $\mathbf{H}_\mu$ is the modified Macdonald polynomial of μ:

$$\mathrm{grFrob}(\mathbf{H}_\mu; q, t) = \widetilde{H}_\mu(\mathbf{x}; q, t) = \sum_\lambda \widetilde{K}_{\lambda,\mu}(q, t) s_\lambda(\mathbf{x}). \tag{6.2.1}$$

6.3 Diagonal orbit harmonics and cyclic sieving

In this section, we introduce a systematic way to generate sieving results using orbit harmonics. The author and Rhoades provided a 'generating theorem' for sieving results (Theorem 3.4. in [OR20]) by exploiting orbit harmonics applied to a locus $X \subseteq \mathbb{C}^n$ with $\mathfrak{S}_n$ acting on X by permuting coordinates. To modify this idea for our purpose, we first explain the *diagonal orbit harmonics* (see [GH96] for more details). Consider $X \subseteq \mathbb{C}^{2n}$ which is closed under $\mathfrak{S}_n \times C_1 \times C_2$-action where

- a symmetric group $\mathfrak{S}_n$ acts on $\mathbb{C}^{2n}$ by permuting coordinates diagonally, i.e.

$$\sigma(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_{\sigma(1)}, \ldots, y_{\sigma(n)}),$$

- a finite cyclic group C_1 acts on $\mathbb{C}^n$ by scaling the x-coordinates by a root of unity, and

- a finite cyclic group C_2 acts on $\mathbb{C}^n$ by scaling the y-coordinates by a root of unity.

Then the method of orbit harmonics gives us an isomorphism of $\mathfrak{S}_n \times C_1 \times C_2$-modules:

$$\mathbb{C}[X] \cong \mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]/\mathbf{I}(X). \tag{6.3.1}$$

We further define a homogeneous ideal

$$\mathbf{T}(X) := \langle \tau_x \circ \tau_y(f) : f \in \mathbf{I}(X) \setminus \{0\} \rangle \subseteq \mathbb{C}[\mathbf{x}_n],$$

where τ_x and τ_y is the map taking top degree homogeneous part of $\mathbf{x}_n$ and $\mathbf{y}_n$, respectively. Then the isomorphism (6.3.1) extends to an isomorphism

$$\mathbb{C}[X] \cong \mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]/\mathbf{I}(X) \cong \mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]/\mathbf{T}(X),$$

where the last item $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]/\mathbf{T}(X)$ has an additional structure of graded $\mathfrak{S}_n \times C_1 \times C_2$-module on which C_1 and C_2 act by scaling in each fixed (bi)degree. Thanks to this isomorphism, we can provide a generating theorem for sieving results in diagonal orbit harmonics whose proof is analogous to the proof of Theorem 3.4 in [OR20].

Theorem 6.3.1. *Let C be the subgroup of $\mathfrak{S}_n$ generated by a long cycle $c = (1, 2, \ldots, n)$. Fix positive integers k_1 and k_2. For $j = 1, 2$, let $\zeta_j := \exp(2\pi i/k_j) \in \mathbb{C}^\times$ and $C_j = \langle c_j \rangle \cong \mathbb{Z}_{k_j}$ be a cyclic group of order k_j. Consider the action of $\mathfrak{S}_n \times C_1 \times C_2$ on $\mathbb{C}^{2n}$ where c_1 scales x-coordinates by ζ_1, c_2 scales y-coordinates by ζ_2 and $\mathfrak{S}_n$ acts by permuting coordinates diagonally. Let $X \subseteq \mathbb{C}^{2n}$ be a finite point set which is closed under the action of $\mathfrak{S}_n \times C_1 \times C_2$.*

1. *Suppose that for $d, e \geq 0$, the isomorphism type of the degree (d,e)-piece of $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]/\mathbf{T}(X)$ is given by*

$$(\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]/\mathbf{T}(X))_{d,e} \cong \bigoplus_{\lambda \vdash n} c_{\lambda,d,e} S^{\lambda}.$$

   The triple $(X, C_1 \times C_2 \times C, X(q,t,z))$ exhibits the tricyclic sieving phenomenon where

$$X(q,t,z) = \sum_{\lambda \vdash n} c_{\lambda}(q,t) f^{\lambda}(z).$$

   where $c_{\lambda}(q,t) := \sum_{d,e \geq 0} c_{\lambda,d,e} q^d t^e$.

2. *Let $G \subseteq \mathfrak{S}_n$ be a subgroup. The set X/G of G-orbits in X carries a natural $C_1 \times C_2$-action and the triple $(X/G, C_1 \times C_2, X/G(q,t))$ exhibits the cyclic sieving phenomenon where*

$$X/G(q,t) = \mathrm{Hilb}((\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]/\mathbf{T}(X))^G; q, t).$$

Proof. Applying orbit harmonics to the action of $\mathfrak{S}_n \times C_1 \times C_2$ on X yields an isomorphism of ungraded $\mathfrak{S}_n \times C_1 \times C_2$-modules

$$\mathbb{C}[X] \cong \mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]/\mathbf{T}(X). \tag{6.3.2}$$

Let $\zeta := \exp(2\pi i/n)$. To prove (1), apply Theorem 3.1.1 to obtain that for any integers r, s, k, the trace of $(c_1^r, c_2^s, c^k) \in C_1 \times C_2 \times C'$ acting on $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]/\mathbf{T}(X)$ is given by

$$\sum_{\lambda \vdash n} c_{\lambda}(\zeta_1^r, \zeta_2^s) f^{\lambda}(\zeta^k) = X(\zeta^r, \zeta_2^s, \zeta^k).$$

By the isomorphism (6.3.2), this coincides with the trace of (c_1^r, c_2^s, c^k) which is the number of fixed points of (c_1^r, c_2^s, c^k) acting on X, completing the proof of (1).

For (2), we take G-invariants of both sides of (6.3.2) to get an isomorphism of $C_1 \times C_2$-modules

$$\mathbb{C}[X/G] \cong (\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]/\mathbf{T}(X))^G. \tag{6.3.3}$$

Since $C_1 \times C_2$ acts on the graded vector space $(\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]/\mathbf{T}(X))^G$ by a root of unity scaling for each x and y variables, the trace of (c_1^r, c_2^s) on the right hand side of isomorphism (6.3.3) is given by

$$[\mathrm{Hilb}((\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]/\mathbf{T}(X))^G; q, t)]_{q=\zeta_1^r, t=\zeta_2^s} = X(\zeta_1^r, \zeta_2^s).$$

The trace of (c_1^r, c_2^s) on the left hand side of (6.3.3) coincides with the number of orbits in X/G fixed by (c_1^r, c_2^s). $\square$

Remark 6.3.1. In order to obtain a sieving result involving a combinatorial set X with a cyclic group action using Theorem 6.3.1, we must

- realize X (or its quotient X/G) and the relevant action on it as a point locus in $\mathbb{C}^{2n}$ and the compatible action,
- calculate the graded Frobenius image of $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]/\mathbf{T}(X)$ or the Hilbert series of the quotient $\mathrm{Hilb}((\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]/\mathbf{T}(X))^G; q, t)$.

6.4 Orbit harmonics and Garsia–Haiman module

There is a way to understand the Garsia–Haiman module $\mathbf{H}_\mu$ via orbit harmonics. Let μ be a partition of n with $l(\mu) = l$ and $l(\mu') = l'$. Let $\{\alpha_0, \ldots, \alpha_{l-1}\}$ and $\{\beta_0, \ldots, \beta_{l'-1}\}$ be two sets of distinct complex numbers. Recall that an *injective tableau* T of shape $\mu \vdash n$ is a filling of cells of μ by integers $1, 2, \ldots, n$ without repetition. The collection of such tableaux will be denoted by $\mathbf{IT}(\mu)$. For each $T \in \mathbf{IT}(\mu)$, we assign a point $p_T \in \mathbb{C}^{2n}$ by letting the i-th and the $(n+i)$-th coordinates of p_T record the position of i in T:

$$p_T = (\alpha_{y_T(1)}, \ldots, \alpha_{y_T(n)}, \beta_{x_T(1)}, \ldots, \beta_{x_T(n)}),$$

where $x_T(i)$ and $y_T(i)$ are the x and y coordinates of i in T. For example, for a partition $\mu = (2, 1)$ and an injective tableau T of shape μ in the figure below, the point assigned for T is $p_T = (\alpha_0, \alpha_1, \alpha_0, \beta_1, \beta_0, \beta_0)$. Let us

$$\begin{array}{|c|c}\hline 2 & \\ \hline 3 & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

denote the collection of points associated to the injective tableaux by

$$X_\mu = \{p_T \in \mathbb{C}^{2n} : T \in \mathbf{IT}(\mu)\}.$$

Note there are exactly $n!$ points in X_μ. The point locus X_μ possesses a natural diagonal action of $\mathfrak{S}_n$: For $\sigma \in \mathfrak{S}_n$,

$$\sigma(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_{\sigma(1)}, \ldots, y_{\sigma(n)}).$$

Using orbit harmonics, one can promote this ungraded $\mathfrak{S}_n$-module X_μ to the bigraded $\mathfrak{S}_n$-module. As usual, let $\mathbf{I}(X_\mu)$ be the ideal of polynomials in $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]$ which vanish on X and define a homogeneous ideal

$$\mathbf{T}(X_\mu) := \langle \tau_x \circ \tau_y(f) : f \in \mathbf{I}(X_\mu) \setminus \{0\} \rangle \subseteq \mathbb{C}[\mathbf{x}_n, \mathbf{y}_n].$$

Then the module $R_\mu := \mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]/\mathbf{T}(X_\mu)$ has an additional structure of (bi)graded $\mathfrak{S}_n$-module.

Garsia and Haiman [GH96] proved that the Garsia–Haiman module $\mathbf{H}_\mu$ embedds into this graded module R_μ. Thanks to the $n!$-conjecture, we can conclude the following isomorphism between R_μ and $\mathbf{H}_\mu$.

Theorem 6.4.1. *We have an isomorphism as bigraded $\mathfrak{S}_n$-modules:*

$$R_\mu \cong \mathbb{C}[X_\mu] \cong \mathbf{H}_\mu.$$

6.5 Proofs of main theorems

6.5.1 A proof of Theorem 6.1.1

Let $ab = n$. Consider a point locus $X_{(a^b)}$ associated to a rectangular partition $\mu = (a^b)$. Following Section 6.4, to consider $X_{(a^b)}$ as a point locus in $\mathbb{C}^{2n}$, we choose two sets of distinct complex numbers $\{\alpha_0, \ldots, \alpha_{b-1}\}$ and $\{\beta_0, \ldots, \beta_{a-1}\}$. For our purpose, let $\zeta_1 = \exp(\frac{2\pi i}{b})$ and $\zeta_2 = \exp(\frac{2\pi i}{a})$. Then set $\alpha_j = \zeta_1^j$ for $0 \leq j \leq b-1$ and $\beta_k = \zeta_2^k$ for $0 \leq k \leq a-1$. Then the corresponding locus $X_(a^b)$ possesses

- diagonal action of $\mathfrak{S}_n$,

- action of a cyclic group C_1 of order b acting by scaling a root of unity ζ_1 to each x-coordinates, and
- action of a cyclic group C_2 of order a acting by scaling a root of unity ζ_2 to each y-coordinates.

Now we can present a proof of Theorem 6.1.1. By the construction above, $\mathbb{C}[X_{(a^b)}]$ has $\mathfrak{S}_n \times C_1 \times C_2$ action which corresponds to permutation of letters, row rotation and column rotation on $\mathbf{IT}(\mu)$, respectively. Combining an isomorphism between R_μ and $\mathbf{H}_\mu$ (Theorem 6.4.1), Equation 6.2.1 and the sieving generating theorem (Theorem 6.3.1), the first bullet point of Theorem 6.1.1 immediately follows.

To proceed to the second bullet point, consider a composition ν of n. For the Young subgroup $G = \mathfrak{S}_\nu = \mathfrak{S}_{\nu_1} \times \mathfrak{S}_{\nu_2} \cdots$ of ν, the G-orbits of $X_{(a^b)}$ are in one-to-one correspondence with the set of $b \times a$ matrices with content ν.

To obtain a sieving result for $X_{(a^b)}/G$, we must calculate the Hilbert series of G-fixed subspace of R_μ. Let $\mathbf{1}$ be the trivial representation of $\mathfrak{S}_\nu$. It is a standard fact that the induction of $\mathbf{1}$ from $\mathfrak{S}_\nu$ to $\mathfrak{S}_n$ can be written as

$$\mathbf{1} \uparrow_{\mathfrak{S}_\nu}^{\mathfrak{S}_n} \cong \bigoplus_\lambda K_{\lambda,\nu} S^\lambda,$$

where $K_{\lambda,\nu}$ denotes the Kostka number. Applying Frobenius reciprocity, it follows that the dimension of the $\mathfrak{S}_\nu$-fixed subspace of the $\mathfrak{S}_n$-irreducible S^λ is given by the character inner product:

$$\dim(S^\lambda)^{\mathfrak{S}_\nu} = \langle \mathbf{1}, S^\lambda \downarrow_{\mathfrak{S}_\nu}^{\mathfrak{S}_n} \rangle_{\mathfrak{S}_\nu} = \langle \mathbf{1} \uparrow_{\mathfrak{S}_\nu}^{\mathfrak{S}_n}, S^\lambda \rangle_{\mathfrak{S}_n} = K_{\lambda,\nu}.$$

Correspondingly, if V is any bigraded $\mathfrak{S}_n$-module with Frobenius image

$$\operatorname{grFrob}(V;q,t)=\sum_{\lambda\vdash n}c_\lambda(q,t)S^\lambda,$$

the Hilbert series of $\mathfrak{S}_\nu$ fixed subspace will be

$$\operatorname{Hilb}(V^{\mathfrak{S}_\nu};q,t)=\sum_{\lambda\vdash n}c_\lambda(q,t)K_{\lambda,\nu}.$$

By (2) of Theorem 6.3.1, and Theorem 6.4.1 this concludes the second bullet point.

6.5.2 A proof of Theorem 6.1.2

In previous section, we proved that for a composition ν of n, the triple $(X_{(a^b),\nu},\mathbb{Z}_b\times\mathbb{Z}_a,\sum_\lambda\widetilde{K}_{\lambda,\mu}(q,t)K_{\lambda,\nu})$ exhibits biCSP. Suppose, furthermore, ν has a cyclic symmetry of order m. Then the set $X_{\mu,\nu}$ possesses another cyclic group action by adding m modulo $l(\nu)$ to each entry. Therefore, it is natural to seek for a sieving result that reflects this additional cyclic action.

Before we begin, we recall the Tanisaki locus. For a composition $\nu\models d$, let W_ν be the set of length d words $w=(w_1,\dots,w_d)$ of content ν. Let $\zeta=\exp(\frac{2\pi i}{l(\nu)})$. We assign a point p_w in $\mathbb{C}^d$ so that we can realize W_ν as a point locus Y_ν (called the Tanisaki locus) in $\mathbb{C}^d$ as follows:

$$p_w=(\zeta^{w_1},\dots,\zeta^{w_d}).$$

Garsia and Procesi [GP92] proved that the T-ideal corresponding to the Tanisaki locus is given by the ideal generated by elementary symmetric polynomials with extra conditions (for precise definition of this Tanisaki ideal, we refer [GP92]). By orbit harmonics, there is an isomorphism

$$\mathbb{C}[Y_\nu] \cong L_\nu := \mathbb{C}[\mathbf{x}_d]/\mathbf{T}(Y_\nu). \tag{6.5.1}$$

Moreover, they showed that the graded Frobenius image coincides with the Hall–Littlewood symmetric function,

$$\operatorname{grFrob}(L_\nu; q) = \widetilde{Q}_\mu(\mathbf{x}; q).$$

Furthermore, if a composition ν has a cyclic symmetry of order a, the set W_ν has additional cyclic group action given by adding a modulo $l(\nu)$ to each letter. This action corresponds with the action of scaling a root of unity ζ^a in each coordinates in Y_ν. In this setting, the isomorphism 6.5.1 extends to an isomorphism as graded $\mathfrak{S}_d \times C$-modules, where C is a cyclic group of order $l(\nu)/a$.

Now let $\mu = (a^b)$ a rectangular partition and ν be a composition of n with cyclic symmetry of order a. Then the product $X_\mu \times Y_\nu$ carries an $\mathfrak{S}_n \times \mathbb{Z}_b \times \mathbb{Z}_a \times \mathbb{Z}_{l(\nu)/m}$-action, where $\mathfrak{S}_n$ acts diagonally on X_μ and Y_ν and the cyclic groups $\mathbb{Z}_b$, $\mathbb{Z}_a$ and $\mathbb{Z}_{l(\nu)/m}$ acts by row rotation on X_μ, column rotation on X_μ and translation on the entries on Y_ν, respectively. By Theorem 6.4.1 and the isomorphism (6.5.1), we have an isomorphism

$$\mathbb{C}[X_\mu \times Y_\nu] \cong R_\mu \otimes_{\mathbb{C}} L_\nu \tag{6.5.2}$$

as $\mathfrak{S}_n \times \mathbb{Z}_b \times \mathbb{Z}_a \times \mathbb{Z}_{l(\nu)/m}$-modules. Since the graded Frobenius image of the module R_μ is given by the Macdonald polynomial and the Frobenius image of the Garsia–Procesi module L_ν is given by the Hall–Littlewood polynomial, the Frobenius image is given by

$$\mathrm{grFrob}(\mathbb{C}[X_\mu \times Y_\nu]; q, t, z) = \sum_{\rho,\lambda,\lambda' \vdash n} \widetilde{K}_{\lambda,\mu}(q,t)\widetilde{K}_{\lambda',\nu}(z) g^{\rho}_{\lambda,\lambda'} s_\rho, \tag{6.5.3}$$

where $g^{\rho}_{\lambda,\lambda'}$ denotes the Kronecker coefficient. By taking isotypic components for a trivial module of $\mathfrak{S}_n$ on both sides of equation (6.5.2), we have

$$\mathbb{C}[X_\mu \times Y_\nu]^{\mathfrak{S}_n} \cong [R_\mu \otimes_{\mathbb{C}} L_\nu]^{\mathfrak{S}_n}. \tag{6.5.4}$$

There is a natural basis of $\mathbb{C}[X_\mu \times Y_\nu]^{\mathfrak{S}_n}$ indexed by $\mathfrak{S}_n$-orbits of $X_\mu \times Y_\nu$, given by the sum of elements in each orbit. Note that each of these orbits corresponds to a $b \times a$ matrix with content of it eqauls to ν. It is clear that the cyclic groups $\mathbb{Z}_b$, $\mathbb{Z}_a$ and $\mathbb{Z}_{l(\nu)/m}$ act on these matrices by row rotation, column rotation and translation of the entries. For an element $g \in \mathbb{Z}_b \times \mathbb{Z}_a \times \mathbb{Z}_{l(\nu)/m}$, we can count the number of fixed points of g in $(X_\mu \times Y_\nu)/\mathfrak{S}_n$ is given by the trace of g acting on the left hand side of the isomorphism 6.5.4. On the other hand, this can be calculated by trigraded Hilbert series

$$\mathrm{Hilb}([R_\mu \otimes_{\mathbb{C}} L_\nu]^{\mathfrak{S}_n}; q, t, z) = \sum_{\lambda,\lambda' \vdash n} \widetilde{K}_{\lambda,(a^b)}(q,t)\widetilde{K}_{\lambda',\nu}(z) g^{(n)}_{\lambda,\lambda'}, \tag{6.5.5}$$

of $[R_\mu \otimes_{\mathbb{C}} L_\nu]^{\mathfrak{S}_n}$ at a root of unity. By Proposition 2.3.1, taking the coefficient of trivial Schur function $s_{(n)}$ in Equation (6.5.5), we have the following polynomial

$$\mathrm{Hilb}([R_\mu \otimes_{\mathbb{C}} L_\nu]^{\mathfrak{S}_n}; q, t, z) = X_{\mu,\nu}(q,t,z) := \sum_{\lambda \vdash n} \widetilde{K}_{\lambda,(a^b)}(q,t)\widetilde{K}_{\lambda,\nu}(z)$$

for sieving result. Thus we have proven Theorem 6.1.2. We end this section with an example of Theorem 6.1.2.

Example 6.5.1. Take $\mu = (2, 2)$ and $\nu = (2, 2)$. We have six 2×2 matrices with content ν listed in the following.

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 2 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & 1 \\ \hline \end{array}$$

Fixed points of $(0, 1, 1) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ correspond to the following four matrices, and there is no fixed point of $(0, 0, 1) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 2 & 1 \\ \hline \end{array}$$

The polynomial $X(q, t, z)$ is given by

$$\begin{aligned} X(q,t,z) &= \sum_{\lambda\vdash 4} \widetilde{K}_{\lambda,(2,2)}(q,t)\widetilde{K}_{\lambda,(2,2)}(z) \\ &= \widetilde{K}_{(4),(2,2)}(q,t)\widetilde{K}_{(4),(2,2)}(z) + \widetilde{K}_{(3,1),(2,2)}(q,t)\widetilde{K}_{(3,1),(2,2)}(z) \\ &+ \widetilde{K}_{(2,2),(2,2)}(q,t)\widetilde{K}_{(2,2),(2,2)}(z) \\ &\equiv 3 + qz + tz + qtz \qquad \bmod \quad (q^2-1, t^2-1, z^2-1). \end{aligned}$$

Note that $X(-1, 1, -1) = 4$ and $X(1, 1, -1) = 0$, which verifies the assertion of Theorem 6.1.2 for this example.

Chapter 7

Concluding remarks

7.1 A conjecture for Macdonald polynomials for rectangles

Since we have a specialization $\widetilde{K}_{\lambda,\mu}(1,1) = f^{\lambda}$, the (q,t)-Kostka polynomial can be considered as a '(q,t)-analogue of f^{λ}'. However, writing (q,t)-Kostka polynomial as a generating function with respect to certain statistics of standard Young tableaux is one of the major open problems in algebraic combinatorics. One can try to reduce one's attention to a small family of μ's such as hooks or rectangles. Theorem 6.1.1 suggests that Macdonald polynomials (or (q,t)-Kostka polynomials) for rectangles might have more structure than general partitions. We conjecture the following.

Conjecture 7.1.1. *Let $n = ab$. Then $\tilde{K}_{\lambda,(a^b)}(q, q^a) = \tilde{K}_{\lambda,(ab)}(q,t) = f^{\lambda}(q)$.*

The author has checked this conjecture for $n \leq 15$ by Sage. For $n = 4$, the polynomials $\tilde{K}_{\lambda,\mu}(q,t)$ for some rectangles μ is given in the table below and it is straightforward to check the conjecture by hand.

$\mu \backslash\lambda$	[4]	[3,1]	[2,2]	[2,1,1]	[1,1,1,1,1]
[4]	1	$q+q^2+q^3$	q^2+q^4	$q^3+q^4+q^5$	q^6
[2,2]	1	$t+qt+q$	t^2+q^2	qt^2+qt+q^2t	q^2t^2
[1,1,1,1]	1	$t+t^2+t^3$	t^2+t^4	$t^3+t^4+t^5$	t^6

Hopefully, we may define two statistics stat_1 and stat_2 such that

$$\tilde{K}_{\lambda,(a^b)}(q,t) = \sum_{T\in \mathrm{SYT}(\lambda)} (q^{\mathrm{stat}_1(T)}t^{\mathrm{stat}_2(T)}).$$

If Conjecture 7.1.1 is true, then for a standard tableaux T of shape λ, the two statistics may satisfy

$$\mathrm{stat}_1(T) + a \cdot \mathrm{stat}_2(T) = \mathrm{maj}(T).$$

Hopefully, this identity above may help to uncover a part of the mystery of the (q,t)-Kostka polynomials.

7.2 Other combinatorial loci

Recall that we obtained Theorem 6.1.2 by taking the tensor product of modules of Garsia–Haiman and Garsia–Procesi, then by taking $\mathfrak{S}_{mn}$-invariant part. We could replace one of those modules to obtain various sieving results. One way to do this is replacing Garsia–Procesi modules with the module $R_{n,k}$ defined in [HRS18]. They defined this module to construct a graded $\mathfrak{S}_n$-module for the Delta conjecture. The module $R_{n,k}$ can also be obtained by applying orbit harmonics to the locus corresponding to the set of surjective functions from $[k]$ to $[n]$. For $mn \leq k$ by taking $\mathfrak{S}_{mn}$ invariant part of $R_{(m^n)} \otimes R_{mn,k}$, we could obtain a triCSP for n times m matrices (or fillings of a rectangular partition) with entries given by nonempty set partitions of $[k]$ into $[mn]$ parts.

This process can be applied to a broad class of modules obtained via orbit harmonics. One of the interesting modules which we did not consider in this thesis is the module $R_{\mu,k}$ defined by Griffin [Gri21]. This module is a common generalization of the Garsia–Procesi module and the module $R_{n,k}$ of Haglund–Rhoades–Shimozono and it is possible to obtain this module via orbit harmonics.

7.3 CSP for bracelets

Inducing represenions of C_n to $\mathfrak{S}_n$ was studied by Kraśkiewicz and Weyman [KW01]. This was the key ingredient in Section 4.4 that allows us to deduce sieving results for necklaces as C_n orbits of words could be understood as necklaces. To obtain an analogous result for 'bracelets', a knowledge for inducing the trivial D_n representation to $\mathfrak{S}_n$ is needed. During the work of [OR20], the author conjectured that $b_{\lambda,n} := \langle \mathbf{1} \uparrow_{D_n}^{\mathfrak{S}_n}, S^\lambda \rangle$ has the following description, where $\langle\ ,\ \rangle$ stands for the Hall inner product.

Conjecture 7.3.1. *For a partition λ of a positive integer n, let $b_{\lambda,n}$ be given as above and $\mathrm{SYT}(\lambda, 0)$ be the set of standard tableaux of shape λ with major index equal to 0 modulo n. Then we have*

$$2b_{\lambda,n} = |\,\mathrm{SYT}(\lambda,0)| + (-1)^{e(\lambda)}|\,\mathrm{SYT}(\lambda,0)^{\mathrm{ev}}|,$$

where ev *is the operation on standard tableaux called the evacuation and $e(\lambda)$ determines a sign associated to a partition λ.*

Stembridge [Ste95] showed that the evacuation action on SYT coincides with the action of a longest element on a cellular basis. Using the results in [Ste95], the author managed to prove this conjecture for odd n. Although the same argument cannot resolve the conjecture for even n, it seems that there is a good chance to prove it using arguments in [Ste95].

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국문초록

궤도 조화 이론은 유한집합 X에 대한 선형군 G가 작용할 때, X를 복소공간 $\mathbb{C}^n$안의 G-안정한 점 자취로 이해함으로써 X에 대한 G의 작용을 등급환 구조를 갖는 다항식환으로 이해할 수 있게 해주는 조합적 표현론 분야의 도구이다.

순환체현상은 유한 순환 군 C가 유한집합 X에 작용할 때 고정점에 대한 정보를 특정 다항식의 단위근에서의 값을 통해 알아낼 수 있는 현상에 관한 세는 조합론 분야의 주제이다.

본 학위논문에서는 궤도 조화 이론을 적용하여 다양한 순환체 현상을 증명하는 방법을 다룬다. 특히, 조합적인 대상인 단어, 그래프, 행렬의 개수와 관련한 , 다른 한 편으로는 홀–리틀우드 다항식이나 맥도날드 다항식과 같은 대칭함수 이론의 중요한 예시들과 관련한 순환체현상을 제시한다.

주요어휘: 순환 체, 궤도 조화 이론, 점 자취, 변형, 복소 반사군

학번: 2015-20268

감사의 글

정말 많은 분의 관심과 지도, 격려 덕분에 무사히 학위 논문을 마치고 대학원을 졸업할 수 있게 된 것 같습니다. 도움을 주신 많은 분께 감사의 말을 올립니다. 먼저 본 학위 기간 동안 연구자의 길로 인도해주시고 아낌없는 지원을 해주신 국웅 교수님께 감사의 말씀을 전합니다. 그리고 바쁘신 와중에도 본 학위 논문 심사를 흔쾌히 맡아주신 권재훈 교수님, 김장수 교수님, 오영탁 교수님, 이승진 교수님께 감사드립니다. 특히 대학원 생활을 진행하며 공동연구와 지도를 아끼지 않으신 권재훈 교수님, 김장수 교수님, 김혁 교수님께 다시 한번 감사드립니다. 또한 이 논문 주제에 대해 함께 연구했던 Brendon Rhoades 교수님께 깊은 감사를 전합니다. 여러 교수님의 가르침 덕분에 오랜 기간 관심 있었던 주제에 대해서 좋은 결과를 낼 수 있었습니다. 그 가르침을 바탕으로 앞으로도 연구에 정진하여 앞서가신 훌륭한 수학자의 길을 따라가겠습니다.

6년 반이라는 대학원 생활 동안 힘들어도 항상 즐겁게 생활할 수 있던 것은 주변의 많은 동료가 있었기 때문입니다. 학부 때부터 함께하고 선배로서 끊임없이 조언을 아끼지 않은 규형, 대준, 석범, 태훈에게도 감사를 전합니다. 학부에서 연을 맺어 대학원 동기로 입학한 동욱, 두형, 성수, 재훈, 탁원과 함께 지내며 많은 힘이 되었습니다. 434호에서 많은 추억을 쌓은 정우와 현세에게도 감사의 인사를 전합니다.

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끝없는 사랑으로 항상 격려해주시고 멀리서 응원해주시는 부모님, 누나와 동생들에게 고맙습니다. 부족한 저를 지지하고 기다려주시는 장인어른, 장모님께도 감사를 전합니다. 학위 과정 동안 스트레스도 많이 받았는데 항상 힘을 준 아내 민혜에게 항상 미안하고 고맙다는 말과 석사 과정 끝까지 힘내라는 말을 전하고 싶습니다. 마지막으로 학교에서 돌아왔을 때 아빠가 제일 좋다는 말로 정말 큰 힘이 되어주는 딸 윤아에게 감사의 말을 전합니다.