이학 박사 학위논문

Non-linear operators on fractal domains and homogenization for fully non-linear parabolic equations

(프랙탈 영역 위에서의 비선형 작용소 및 비선형 포물형 편미분 방정식의 균질화)

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Non-linear operators on fractal domains and homogenization for fully non-linear parabolic equations

A dissertation
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Abstract

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The analysis of fractals has been studied extensively in both analysis and probability approaches. In this thesis, we construct the non-linear elliptic equation involving second order terms on fractal spaces, and our main object is to exhibit the regularity of their solutions by using an analytic argument. Since a calculus on fractals is not available, our approach is based on the graph approximation argument to construct Dirichlet forms. The central concept is in finding suitable cut-off functions and weighted inequalities, which can be obtained by using the special geometric properties of the fractal domain.

Another topic in this thesis is the homogenization theory for fully nonlinear parabolic equations. In particular, we treat the case with different scales of the oscillating variables. The interesting point is that the homogenization occurs separately for time and space due to a mismatch in the scale of time and space fast variables. In addition, this phenomenon causes different order of convergence rates.

Key words: fractals, Sierpinski gasket, Harnack inequality, homogenization, convergence rate

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Chapter 1

Introduction

1.1 Part I : Non-linear operators on the fractal domains

Fractals have a very interesting structure called "self-similarity", which is a geometrically generated pattern that is reproducible at any magnification or reduction. During the last decade, various types of fractal spaces, such as Sierpinski gaskets, Sierpinski carpets, and more generally certain manifolds, graphs, and metric spaces have been studied extensively as an aspect of partial differential equations (see [3, 30, 31, 32, 36, 56]).

Fractal domains are of significant interest in both probability theory and analysis. The two fields are closely related and share the same goal. The main approach in probability theory is to construct diffusion processes on fractals, analogous to Brownian motion, and heat kernel estimates for these processes. This was first worked by Goldestein [22], Kusuoka [36, 37], and Lindtrom [44]. They proved independently the existence of Brownian motion as the scaling limit of a sequence of random walks on certain fractals. The advantage of this approach is that it is a very suitable method for finding Brownian motion and heat kernel estimates and hence makes it possible to extend to other fractal domains. Barlow–Bass [4, 5] followed the construction of Brownian motion on Sierpinski carpets, and Barlow–Perkins [3] use a similar approach to Sierpinski gaskets, which was proved by a probabilistic argument.

Another approach, based on analysis, is due to Kigami [30, 31, 32] by introducing Dirichlet forms and Laplacian on fractals. That is, he constructs analytic structures (Dirichlet forms in particular) on fractals and finds a harmonic function by minimizing the Dirichlet forms among all functions which have required boundary values. The Dirichlet forms can be obtained as the limit of a sequence of discrete energy on graphs approximating the fractal. For example, The Sierpinnski gasket, which is one of the simplest fractal set, energy forms on the Sierpinski gasket can be written as the limit of energy forms on a sequence of discrete graphs. This is essentially because we cannot define both gradient and integral in our energy. Nevertheless, this approach allows us to capture the structures of harmonic functions, Green's function, and solutions of Laplacian operators.

There are many results for linear cases such as Laplacian and Brownian motion, for example, we can refer to the following papers [6, 7, 18, 23, 24, 27, 54], but relatively few results for non-linear cases. [25, 55] and [15, 16] shows that the existence of p-harmonic functions and proved the Harnack inequality for non-negative p-harmonic functions on the Sierpinski gasket, respectively. On the other hand, a certain semi-linear parabolic equation on the Sierpinski gasket was studied independently in [26, 49].

As an analytic point of view, it is natural to wonder if general regularity theories such as the Hölder's continuity, Harnack inequality, and their applications hold on fractals. [5] gave the proof of the Harnack inequality in the case of harmonic functions on pre-Sierpinski carpets, and [6] proved the same result for linear operators in divergence form. See also [7, 18] and [23] for a similar statement for certain graphs and manifolds.

In this thesis, we would like to propose new non-linear operators on one particular class of fractals, domains in $\mathbb{R}^2$ which is the Sierpinski gasket. As mentioned above, the main tool in analytic approach is energy. We construct generalized energy functional on the Sierpinski gasket that covers the existing energy of Laplacian operators. We provide abstract formulations of these functional and show existence and uniqueness results for their minimizers. Our main interest is to obtain the Harnack inequality for non-negative minimizers. We develop an analytic approach in which we used very strongly the energy measure, the symmetry of the space, and the comparability of the non-linear operators.

The key step to achieving the Harnack ineuqality is to find suitable cut-off functions. In fractal domains, there is no analogue of the following Newton-Leibniz formula

$$u(x) - u(y) = \int_0^1 \langle \dot{\gamma}(s), \nabla u(\gamma(s)) \rangle ds$$

for every curve $\gamma : [0,1] \to \mathbb{R}^n$ connecting x and y. Due to this limitation, we cannot use the Sobolev inequality as in Euclidean space. Instead, by finding an appropriate cut-off function, we can combine it with the Hausdorff measure to create a new measure λ. Then the measure λ allows us to prove a weighted Sobolev inequality linking the $L^{2+\varepsilon}$ norm of λ to the energy measure. The Harnack inequality is achieved by involving the Caccioppoli type inequality and weighted Sobolev inequality which gives the local boundedness and the weak Harnack inequality of solutions with respect to the measure λ.

1.2 Part II : Homogenization for fully non-linear parabolic equations

In various fields of physics and engineering one need to solve partial differential equations in a composite media. In many cases the pattern of composite media is the periodic structure, in which case the heterogeneous media repeats for each cell. Generally, the size of the period is very small compared to the size of the entire media. In this case, we are mainly interested in the overall or macroscopic properties of a composite media and not so much about the properties in microscopic parts. From this point of view, the study of the asymptotic behavior when the size of period goes to zero and finding an averaged formulation is called homogenization. That is, homogenization is the process of seeking a macroscopic or effective aspect starting from a microscopic description of a problem. The most important goal of the homogenization is to find homogeneous effective parameters from heterogeneous media, or to justify of averaging process rigorously.

We investigate a physical problem of conductivity in a periodic domain $\Omega \subset \mathbb{R}^n$, since it is a natural example to see what homogenization is. The periodicity is $\varepsilon > 0$ and the rescaled unit periodic cell $Y = (0,1)^n$. The conductivity in Ω is a matrix $A(y)$, where $y = x/\varepsilon \in Y$ is the fast variable, while $x \in \Omega$ is the slow variable. Since the conductivity varies in Ω, the matrix A can be any second order tensor. Moreover, the matrix $A(y)$ is a periodic function of y with period Y. That is, the matrix A is a coefficient with rapidly oscillating structure in ε-scale, which is why we named y the fast variable. Then a homogenization problem can be formulated as follows.

$$-\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u^{\varepsilon}\right) = f \quad \text{in } \Omega. \tag{1.2.1}$$

The mathematical theory of homogenization can be interpreted as follows. Rather than solving a single problem (1.2.1), we look at the equation (1.2.1) as a sequence of such problems indexed by ε, which gets smaller and going to zero. The aim is to find the limit of this sequence of problems. More precisely, we want to find a function u which is the limit of u^{ε} in the appropriate sense, and limit problem which u solves. The first question is to determine an adequate topology where u^{ε} converges to u. We call u the effective limit. If we define the proper space for which $u^{\varepsilon} \to u$, the next thing we need to consider is finding the equation that u satisfies. That is, one can determine a coefficient $\overline{A}$ which satisfies the following equation:

$$-\operatorname{div}\left(\overline{A}\nabla u\right) = f \quad \text{in } \Omega.$$

The operator $\overline{A}$ is called the effective conductivity. Finally, the approximation can be rigorously justified by quantifying the resulting error. In other words, we need to quantitatively compare the difference between u^ε and the effective limit u.

The classical, but powerful approach is to use the well known two-scale asymptotic expansion method: the main idea is to assume that the solution u^ε of (1.2.1) can be represented by the following power series in ε, called ansatz,

$$u^\varepsilon(x) = \sum_{i=0}^{\infty} \varepsilon^i u^i(x, y)$$

where each term $u^i(x, y)$ is periodic in y. Then inserting this expansion in (1.2.1) and matching the order of ε gives equations that satisfies each u^i. In particular, the first term u^0 of this expansion will be identified with the effective limit u, and we can compute the exact form of the effective conductivity $\overline{A}$.

Classical results in the theory of homogenization can be seen in the books [1, 2, 10, 29]. The homogenization theory is started by Lions [48] about the first order evolutionary problems, and extended to second-order equations in [45, 46]. Evans [19, 20] introduced "perturbed test function method" to establish a periodic homogenization problem for certain fully non-linear, first and second order equations. Regarding the results about rates of convergence in periodic homogenization, for linear equation, it is well known that the $O(\varepsilon)$ rate proved to be optimal in [10]. For the case of fully non-linear equations, [14] proved a $O(\varepsilon^\alpha)$ rate and [34] improved this result to the higher order: $O(\varepsilon^{\left[\frac{m}{2}\right]})$ rate when the order of asymptotic expansion is m. On the other hand, a study of the stochastic homogenization for uniformly elliptic equations was introduced by Caffarelli, Souganidis, and Wang [12, 13]. Their approach extended to fully non-linear uniformly parabolic equations covered in [43]. For the homogenization theory in a perforated domain with oblique boundary condition, [38] obtained the effective operator by introducing the compatibility condition.

In this thesis, we cover the homogenization problem of non-divergence type elliptic and parabolic PDE, especially obtain the convergence rate in homogenization of non-linear PDEs. The first result concerns the parabolic fully non-linear equation when the space-time scaling factor k is different. Naturally, the space-time scaling factor k is 2 to match the order of ε. However, there are cases where the space-time scaling factor is not 2, such as a fractal. In this work, a mismatch will inevitably occur in a asymptotic expansion, and it causes a phenomenon in which the homogenization of time and space is separated. We overcome the difficulty by constructing appropriate k-multiple order effective limits and correctors. In fact, this approach is very natural, because the k-th order corrector will serves to connect the homogenization separated by time and space. One of the key features in this work is to recover the convergence rate up to ε, by considering effective limits whose order of ε is less than 1.

The second result studies the higher order convergence rate of the homogenization of non-divergence semi-linear equation with the oblique condition over a periodically perforated domain. The oblique condition is a generalization of the boundary condition in the well-known Skorokhod problem. In this case, The homogenization can be established when the diffusion term and drift term satisfy the compatibility condition. The compatibility condition will give the balance between the diffusion equation and the drift effect by the oblique condition, and then it gives the existence of global solution as it does in the standard divergence-type equation. In order to find the rate of convergence, we consider the higher order correctors. At each step of finding the higher order corrector, we need a compatibility condition which uniquely determines the corrector.

Chapter 2

Preliminaries

Let us summarize some notions, well known results and ways of notation that are frequently used throughout this work.

2.1 Part I : Non-linear operators on the fractal domains

2.1.1 Sierpinski gasket

Let $V_0 = \{p_0, p_1, p_2\} \subset \mathbb{R}^2$ with $p_0 = (0,0)$, $p_1 = (1,0)$, $p_2 = (1/2, \sqrt{3}/2)$ and consider a set of three mappings $F_i : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 0, 1, 2$ defined by

$$F_i(x) = 2^{-1}(x + p_i).$$

The m-lattice V_m, $m \in \mathbb{N}$ are the sets defined inductively by

$$V_m = \bigcup_{i=0,1,2} F_i V_{m-1}, \quad m \in \mathbb{N}_+.$$

We will regard the sets V_m as the vertices of a graph Γ_m, with edges written $x \sim_m y$ provided $|x - y| = 2^{-m}$. Then if we put $V_* = \bigcup_{m=0}^{\infty} V_m$, the Sierpinski gasket K is defined to be the closure $K = \mathrm{cl}(V_*)$. Let

$$W_* = \{w = w_1w_2w_3\cdots : w_i \in \{0,1,2\}, i \in \mathbb{N}_+\}$$

be the family of infinite sequences $w = w_1w_2w_3\cdots$ of symbols in $\{0,1,2\}$. For each $w \in W_*$, denote by $[w]_m = w_1w_2\cdots w_m$, the truncation of w of length m, we call $[w]_m$ a word of length m. Write $F_{[w]_m} = F_{w_1} \circ F_{w_2} \circ \cdots F_{w_m}$ for $[w]_m = w_1w_2\cdots w_m$, each $w_i \in \{0,1,2\}$. We call $F_{[w]_m}K$ m-cell or cell of length m. Then K satisfies the self-similar identity

$$K = \bigcup_{w\in W_*} F_{[w]_m}K.$$

This will be our decomposition of K into cells of length m. Note that distinct cells of length m are either disjoint or intersect at a single point. We will call such intersect points junction points. For any finite union of cells D, we write ∂D for the boundary of D defined by a set of points in D that are not junction point in D. We also define D^o for the interior of D such that $D^o = D\setminus\partial D$. In particular, $\partial K = V_0 = \{p_0, p_1, p_2\}$, $K^o = K\setminus\partial K = K\setminus V_0$, and for any m-cell $F_{[w]_m}K$, $\partial F_{[w]_m}K = \{F_{[w]_m}(p_0), F_{[w]_m}(p_1), F_{[w]_m}(p_2)\}$. Note that all points in a set $V_* \cap D^o$ are junction point.

Definition 2.1.1. The Hausdorff measure μ on K, normalized so that $\mu(K) = 1$, is the unique Borel measure on K such that $\mu(F_{[w]_m}K) = 3^{-m}$ for all $m \in \mathbb{N}$, $w \in W_*$.

Throughout this paper we define fractal dimension(or Hausdorff dimension), a dimension of the walk, spectral dimension of K, and Hölder's exponent by

$$\begin{aligned} d_f &= \log 3/\log 2,\\ d_w &= \log 5/\log 2,\\ d_s &= 2d_f/d_w = 2\log 3/\log 5,\\ \beta &= (d_w - d_f)/2 = \log(5/3)/2\log 2, \end{aligned}$$

respectively. We will require a certain amount of notation to proceed with our proof. We say that $A \asymp B$ if there are some constants c_1, $c_2 > 0$ such that $c_1 A \leq B \leq c_2 A$. For any connected finite union of cells D, let us denote

$$R_D := \operatorname{diam}(D).$$

If $I = F_{[w]_m} K$ for some $w \in W_*$ and $m \in \mathbb{N}$, then we call I is a single m_I-cell with length $m_I = m$. If $I \subset K$ is a single cell such that $I \cap V_0 = \emptyset$, then there are three cells of equal size as I that meet the boundaries of I. We define I^* as the union of I and these three cells. It is clear that $R_I \asymp R_{I^*} \asymp 2^{-m_I}$ and $\mu(I) \asymp \mu(I^*) \asymp 3^{-m_I} = 2^{-m_I d_f}$. Thus, we have

$$R_I^{d_w} \asymp R_I^{2\beta} \mu(I) \asymp R_{I^*}^{2\beta} \mu(I^*) \asymp 2^{-m_I(2\beta + d_f)} = 2^{-m_I d_w}$$

when I is a single m_I-cell. For any finite union of cells D and single cell I, write

$$N(I; D) := \mu(D)/\mu(I).$$

If $I \subset D$, then $N(I; D)$ means the number of I-sized cells contained in D.

2.1.2 Dirichlet forms and harmonic functions

Dirichlet forms on K can be defined as the limit of the sequence of energies. For any function $u : V_m \to \mathbb{R}$ and any finite union of cells $D \subset K$, define

$$\mathcal{E}_D^{(m)}(u) = \frac{r^{-m}}{2} \sum_{\substack{x \sim_m y \\ x,y \in V_m \cap D}} (u(x) - u(y))^2.$$

The scaling factor, where $r = \dfrac{3}{5}$, is chosen so that the sequence $\{\mathcal{E}_D^{(m)}\}$ of forms is consistent. That is, for any function u on V_m,

$$\mathcal{E}_D^{(m)}(u) = \min\{\mathcal{E}_D^{(m+1)}(v) : v \text{ is a function on } V_{m+1} \text{ and } v|_{V_m} = u\}.$$

Hence, the sequence of energies $\{\mathcal{E}_D^{(m)}\}$ is increasing(non-decreasing) for any function u defined on V_*, i.e.

$$\mathcal{E}_D^{(0)}(u) \leq \mathcal{E}_D^{(1)}(u) \leq \mathcal{E}_D^{(2)}(u) \leq \cdots . \tag{2.1.1}$$

In view of the monotonicity, it makes sense to define

$$\mathcal{E}_D(u) = \lim_{m\to\infty} \mathcal{E}_D^{(m)}(u)$$

allowing the value $+\infty$. Let

$$\mathcal{F}(D) = \{u : u \text{ is a function on } V_* \cap D \text{ and } \mathcal{E}_D(u) < \infty\},$$

and

$$\mathcal{F}_0(D) = \{u \in \mathcal{F}(D) : u|_{\partial D} = 0\}.$$

For simplicity, if $D = K$, we denote by $\mathcal{E}_K(u) = \mathcal{E}(u)$. Then for any m-cell $F_{[w]_m}K$ and any function $u \in \mathcal{F}(F_{[w]_m}K)$, the following scaling property holds

$$\mathcal{E}_{F_{[w]_m}K}(u) = r^{-m}\mathcal{E}(u \circ F_{[w]_m}).$$

Moreover, the following self-similar property holds: for subdivisions $K = \cup_{[w]_m\in\mathcal{P}} F_{[w]_m}K$, for any partition $\mathcal{P}$,

$$\mathcal{E}(u) = \sum_{[w]_m\in\mathcal{P}} r^{-m}\mathcal{E}(u \circ F_{[w]_m}). \tag{2.1.2}$$

It is well known that every function $u \in \mathcal{F}(D)$ is uniformly continuous on $V_* \cap D$, hence it has a unique continuous extension to D. In other words, we have $\mathcal{F}(D) \subset C(D)$. The form $(\mathcal{E}, \mathcal{F}(K))$, called as the standard Dirichlet form on K, is a local Dirichlet form on $L^2(K, \mu)$. Moreover, the following Hölder's inequality and Poincaré inequality hold, where the latter is a generalization to the anomalous diffusion case of the standard Poincaré inequality.

Lemma 2.1.2 ([56], Hölder inequality)**.** *Let D be a connected finite union* *of cells. Then for all* $u \in \mathcal{F}(K)$,

$$\sup_{x,y \in V_* \cap D} \frac{|u(x) - u(y)|^2}{|x-y|^{2\beta}} \leq c_1 \mathcal{E}_D(u)$$

where $\beta = \log(5/3)/2\log 2$. *In particular, if* $u(q_i) = 0$ *on for some* $q_i \in V_0$, $i = 0, 1, 2$ *then*

$$|u(x)|^2 \leq c_1 \mathcal{E}(u) \quad \text{for all } x \in K.$$

Lemma 2.1.3 (Poincaré inequality)**.** *Let* D *be a connected finite union of cells in* K*. Then for all* $u \in \mathcal{F}(K)$*, writing* $u_D = \mu(D)^{-1} \int_D u d\mu$,

$$\int_D (u - u_D)^2 d\mu \leq c_1 R_D^{2\beta} \mu(D) \mathcal{E}_D(u).$$

In particular, if D *is a single cell with length* m_D*, then for* $I = D$ *or* $I = D^*$,

$$\int_I (u - u_I)^2 d\mu \leq c_2 R_I^{d_w} \mathcal{E}_I(u).$$

Proof. We know that u has a unique continuous extension to K from the comment above this Lemma. Then by the density of V_* in D and Lemma 2.1.2, we have

$$|u(x) - u(y)|^2 \leq c_3 |x-y|^{2\beta} \mathcal{E}_D(u) \leq c_3 R_D^{2\beta} \mathcal{E}_D(u)$$

for all x, $y \in D$. On the other hand, since u is continuous on D and D is path connected, there exists $z \in D$ such that $u(z) = u_D$. Therefore, from the estimate above, we have

$$|u(x) - u_D|^2 \leq c_3 R_D^{2\beta} \mathcal{E}_D(u) \quad \text{for all } x \in D$$

and hence

$$\int_D (u - u_D)^2 d\mu \leq c_3 R_D^{2\beta} \mu(D) \mathcal{E}_D(u).$$

□

We can also look at the renormalized bilinear form: for any u, $v \in \mathcal{F}(D)$, set

$$\mathcal{E}_D^{(m)}(u,v) = \frac{r^{-m}}{2} \sum_{\substack{x \sim_m y \\ x,y \in V_m \cap D}} (u(x)-u(y))(v(x)-v(y)),$$

then Cauchy's inequality implies that

$$\mathcal{E}_D(u,v) = \lim_{m\to\infty} \mathcal{E}_D^{(m)}(u,v)$$

exists and it is finite. Hence, $\mathcal{F}(D)$ forms a Hilbert space with inner product $\int_D uvd\mu + \mathcal{E}_D(u,v)$. In particular, $\mathcal{F}(D)/constants$ forms a Hilbert space with inner product $\mathcal{E}_D(u,v)$ by Lemma 2.1.2.

Remark 2.1.4. Combining Lemma 2.1.2 and the Arzelá-Ascoli theorem, we can deduce that $\mathcal{F}_0(K)$ is compactly embedded in $C_0(K)$.

We can define a dual space in the same way we did on Euclidean spaces.

Definition 2.1.5. For any $v \in L^2(K,\mu)$, define

$$\|v\|_{\mathcal{F}^{-1}(K)} = \sup\left\{\int_K uvd\mu : u \in \mathcal{F}(K), \|u\|_{\mathcal{F}(K)} \le 1\right\}.$$

The space $\mathcal{F}^{-1}(K)$ is defined to be the $\|\cdot\|_{\mathcal{F}^{-1}}$-completion of $L^2(K,\mu)$. We will write $\langle\cdot,\cdot\rangle_\mu$ to denote the pairing between $\mathcal{F}^{-1}(K)$ and $\mathcal{F}_0(K)$.

It is noteworthy to observe that we have three Hilbert spaces $\mathcal{F}^{-1}(K)$, $L^2(K,\mu)$, $\mathcal{F}_0(K)$ and the embeddings

$$\mathcal{F}_0(K) \subset L^2(K,\mu) \subset \mathcal{F}^{-1}(K).$$

By considering bilinear energy $\mathcal{E}(\cdot,\cdot)$ and Hausdorff measure μ, we are in position to define a Laplacian Δ_μ on K via the weak formulation.

Definition 2.1.6. Let $u \in \mathcal{F}_0(K)$ and $f \in \mathcal{F}^{-1}(K)$. Then write $\Delta_\mu u = f$ if

$$\mathcal{E}(u,v) = -\langle f, v\rangle_\mu \quad \text{for all } v \in \mathcal{F}_0(K).$$

This relation defines uniquely the isomorphism

$$\Delta_\mu : \mathcal{F}_0(K) \simeq \mathcal{F}^{-1}(K),$$

and we call this operator the Laplacian.

The weak formulation is defined by the assumption that v vanishes on the boundary, and so it is just a special case of the Gauss-Green formula. It is well known that both the Gauss-Green formula and a definition of normal derivatives $\partial_n u$ at boundary points are well defined on SG.

Lemma 2.1.7 ([56], Gauss-Green formula)**.** *Suppose that $\Delta_\mu u = f$ for some $f \in L^2(K,\mu)$. Then $\partial_n u(x)$ exists for all $x \in V_0$, where ∂_n is defined by*

$$\partial_n u(x) = \lim_{m\to\infty} \frac{r^{-m}}{2} \sum_{\substack{x\sim_m y \\ y\in V_m}} (u(x) - u(y)),$$

and the Gauss-Green formula

$$\mathcal{E}(u,v) = -\int_K (\Delta_\mu u) v d\mu + \sum_{x\in V_0} v\partial_n u(x)$$

holds for all $v \in \mathcal{F}(K)$.

For any given function u on V_0, there exists a unique $h \in \mathcal{F}(K)$ such that $h|_{V_0} = u$ which has the minimum energy. In other words,

$$\mathcal{E}(h) = \min\{\mathcal{E}(v) : v \in \mathcal{F}(K) \text{ and } v|_{V_0} = u\}.$$

The function $h \in \mathcal{F}(K)$ is called the harmonic function in K with boundary value u, and satisfies

$$\mathcal{E}(h) = \mathcal{E}^{(m)}(h) = \mathcal{E}^{(0)}(h) \quad \text{for all } m \in \mathbb{N}.$$

The additivity in (2.1.2) suggests that we could think the energy as a measure. We point out that the energy may be regarded as the integral of a certain energy measure. For a function $u \in \mathcal{F}(K)$ we define $\nu_{\langle u\rangle}(I)$ for any cell I by the same definition as $\mathcal{E}(u)$ on I. This defines a regular measure on K using additivity. By the self-similarity of energy, we have $\nu_{\langle u\rangle}(F_w K) = r_w^{-1}\mathcal{E}(u \circ F_w)$.

Definition 2.1.8. For any $u \in \mathcal{F}(K)$, the energy measure $\nu_{\langle u\rangle}$ of u is a unique Borel measure on K such that for any finite union of cells $D \subset K$,

$$\int_D \phi d\nu_{\langle u\rangle} = \mathcal{E}_D(\phi u) - \frac{1}{2}\mathcal{E}_D(\phi, u^2) \quad \text{for all } \phi \in \mathcal{F}(K). \tag{2.1.3}$$

For any u, $v \in \mathcal{F}(K)$, the mutual energy measure $\nu_{\langle u,v\rangle}$ is defined by the polarisation $\nu_{\langle u,v\rangle} = \frac{1}{4}(\nu_{\langle u+v\rangle} - \nu_{\langle u-v\rangle})$.

Remark 2.1.9. The energy measure of u on D, $\nu_{\langle u\rangle}(D)$, may be identified with the quantity $\frac{1}{2}\int_D |\nabla u|^2 d\mu$ on $\mathbb{R}^n$. That is, $d\nu_{\langle u\rangle} = |\nabla u|^2 d\mu$ on Euclidean spaces. Using the identity $|\nabla u|^2 = \frac{1}{2}\Delta(u^2) - u\Delta u$ and applying integration by parts, we obtain

$$\int_D \phi|\nabla u|^2 d\mu = \int_D \nabla(\phi u) \cdot \nabla u d\mu - \frac{1}{2}\int_D \nabla\phi \cdot \nabla(u^2) d\mu,$$

which is exactly the same form as the (2.1.3).
However, note carefully that the analogy $d\nu_{\langle u\rangle} = |\nabla u|^2 d\mu$ breaks on K. In fact, the identity $d\nu_{\langle u\rangle} = |\nabla u|^2 d\mu$ on Euclidean spaces means that $|\nabla u|^2 \in L^1$ is the Radon-Nikodym derivative of the energy measure $\nu_{\langle u\rangle}$ of u with respect to the Hausdorff measure μ. But in the Sierpinski gasket, it is well known that the energy measure ν and the Hausdorff measure μ are mutually singular (see [8, 9]). Roughly speaking, this is because that the mass is concentrated too much near junction points.

It is clear by definition that $\nu_{\langle u\rangle}(K) = \mathcal{E}(u)$, and $\nu_{\langle u,v\rangle}(K) = \mathcal{E}(u, v)$ is a symmetric bilinear function of u and v with $\nu_{\langle u\rangle} = \nu_{\langle u,u\rangle}$. There is another formula for these measures, namely carré du champs,

$$\int_K \phi d\nu_{\langle u,v\rangle} = \frac{1}{2}\mathcal{E}(\phi u, v) + \frac{1}{2}\mathcal{E}(u, \phi v) - \frac{1}{2}\mathcal{E}(\phi, uv) \tag{2.1.4}$$

for any $\phi \in \mathcal{F}(K)$. Moreover, by simple computation we also have the following representation

$$\begin{aligned}\int_K \phi d\nu_{\langle u,v\rangle} &= \lim_{m\to\infty} \frac{r^{-m}}{2} \sum_{\substack{x\sim_m y\\ y\in V_m}} \left(\frac{\phi(x)+\phi(y)}{2}\right)(u(x)-u(y))(v(x)-v(y))\\ &= \lim_{m\to\infty} \frac{r^{-m}}{2} \sum_{x\in V_m} \phi(x) \sum_{\substack{x\sim_m y\\ y\in V_m}} (u(x)-u(y))(v(x)-v(y)).\end{aligned}$$

From this fact, we can easily derive the following chain rule which is frequently used in this paper.

Lemma 2.1.10 (Chain rule)**.** *Let $D \subset K$ be any finite union of cells and suppose f and $g \in C^1_{loc}(\mathbb{R})$. Then $f(u)$, $g(v) \in \mathcal{F}(K)$ and there holds*

$$\int_D \phi d\nu_{\langle f(u),g(v)\rangle} = \int_D \phi f'(u)g'(v)d\nu_{\langle u,v\rangle}$$

for all ϕ, u, and $v \in \mathcal{F}(K)$.

2.2 Part II : Homogenization for fully nonlinear parabolic equations

2.2.1 Cell problem

We summarize the main properties of the homogenization for second order equations, which frequently used in the thesis. Set $\mathcal{S}^n$ be the space of all real symmetric $n\times n$ matrices, endowed with (L^2, L^2)-norm. That is, $\|P\| = \left(\sum_{i,j=1}^n p_{ij}^2\right)^{1/2}$ for any $P \in \mathcal{S}^n$. To investigate the basic techniques, let us consider the model problem

$$\begin{cases} u^\varepsilon_t - F(D^2_x u^\varepsilon, x, t, x/\varepsilon, t/\varepsilon^2) = 0 & \text{in } S_T,\\ u^\varepsilon = \varphi(x,t) & \text{on } \partial_p S_T \end{cases} \tag{2.2.1}$$

where the state variable $(x, t, x/\varepsilon, t/\varepsilon^2)$ splits into the slow variable $(x, t) \in \overline{S_T}$ and in the fast variable $(x/\varepsilon, t/\varepsilon^2) = (y, s) \in \mathbb{R}^n \times [0, \infty)$. By Ω denotes a bounded smooth open domain of $\mathbb{R}^n$, $S_T = \Omega \times (0, T)$, the parabolic boundary $\partial_p S_T = (\partial\Omega \times [0, T)) \cup (\overline{\Omega} \times \{0\})$ and $F : \mathcal{S}^n \times \overline{S_T} \times \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is given smooth function. The important assumption is that $F(M, x, t, \cdot, \cdot)$ is (y, s)-periodic for all $(M, x, t) \in \mathcal{S}^n \times \overline{S_T}$. We make the additional uniform ellipticity assumption on F, that is, there are $0 < \lambda \leq \Lambda$ such that $\lambda\|N\| \leq F(M + N, x, t, y, s) - F(M, x, t, y, s) \leq \Lambda\|N\|$ for any $\|N\| \geq 0$, for all $(M, x, t, y, s) \in \mathcal{S}^n \times \overline{S_T} \times \mathbb{R}^n \times [0, \infty)$. We finally assume that F is convex in M-variable, $\varphi \in C^{0,1}(\overline{S_T})$, and F is Lipschitz on $\mathcal{S}^n \times \overline{S_T} \times \mathbb{R}^n \times [0, \infty)$ such that for each $L > 0$ with $B_L \subset \mathcal{S}^n$

$$\|F\|_{C^{0,1}(\overline{B_L} \times \overline{S_T} \times \mathbb{R}^n \times [0,\infty))} \leq \sigma(1 + \|M\|).$$

Let $Q_r(x_0, t_0) = \{(x, t) : |x - x_0| < r, 0 \leq t_0 - t < r^2\}$ and by Q_r we denote $Q_r(0, 0)$. We define the parabolic distance between (x_1, t_1) and (x_2, t_2) in $\mathbb{R}^n \times \mathbb{R}$ by

$$d((x_1, t_1), (x_2, t_2)) = \left(|x_1 - x_2|^2 + |t_1 - t_2|\right)^{1/2}.$$

For $\gamma \in (0, 1)$, $u \in C^\gamma(S_T)$ if

$$\|u\|_{C^\gamma(S_T)} = \|u\|_{L^\infty(S_T)} + \sup_{(x_1,t_1),(x_2,t_2) \in S_T} \frac{|u(x_1, t_1) - u(x_2, t_2)|}{d((x_1, t_1), (x_2, t_2))^\gamma}.$$

Moreover, $u \in C^l(\overline{S_T})$ if for all α, β such that $|\alpha| + 2\beta \leq l$, $D_x^\alpha D_t^\beta u$ is continuous on $\overline{S_T}$. By $C^{l,\gamma}(\overline{S_T})$ we denote the usual Hölder space on $\overline{S_T}$.

We first consider the following cell problem with respect to (2.2.1) : For every $(M, x, t) \in \mathcal{S}^n \times \overline{S_T}$, find a constant $\overline{F} = \overline{F}(M, x, t)$ such that there exists a (y, s)-periodic solution $w = w(y, s; M, x, t)$ to

$$w_s - F(M + D_y^2 w, x, t, y, s) = -\overline{F}(M, x, t) \quad \text{in } \mathbb{R}^n \times [0, \infty).$$

We begin by using the standard perturbed test-function argument. Although the proof can be found in [19, 20], we present the proof for completeness.

Lemma 2.2.1. *For each* $(M, x, t) \in \mathcal{S}^n \times \overline{S_T}$ *there exist a unique* (y, s)*-periodic solution* $w^\delta(y, s; M, x, t)$ *of following penalized problem,*

$$\delta w^\delta + w^\delta_s - F(M + D^2_y w^\delta, x, t, y, s) = 0 \quad \text{in } \mathbb{R}^n \times [0, \infty) \tag{2.2.2}$$

for each $\delta \in (0, 1)$*. Moreover,* $w^\delta(\cdot, \cdot; M, x, t)$ *lies in* $C^{2,\gamma}(\mathbb{R}^n \times [0, \infty))$ *with the uniform estimate*

$$\|\delta w^\delta\|_{C^{2,\gamma}(\mathbb{R}^n\times[0,\infty))} + osc_{\mathbb{R}^n\times[0,\infty)} w^\delta \leq C\,(1 + \|M\|)\,. \tag{2.2.3}$$

Proof. For brevity, we omit the dependency of M, x and t variables in the functions since these variables are fixed in this lemma. In view of [17], (2.2.2) has a comparison principle that the function $w^\delta_+ := \delta^{-1}(\sigma(1 + \|M\|)$ and $w^\delta_- = -\delta^{-1}(\sigma(1 + \|M\|)$ are super- and sub-solution of (2.2.2), respectively. Thus, there is a unique (y, s)-periodic viscosity solution w^δ to (2.2.2) such that $w^\delta_- \leq w^\delta \leq w^\delta_+$ in $\mathbb{R}^n \times [0, \infty)$ and

$$\|\delta w^\delta\|_{L^\infty(\mathbb{R}^n\times[0,\infty))} \leq \sigma(1 + \|M\|)$$

for all $\delta \in (0, 1)$. To show that $w^\delta \in C^{2,\gamma}(\mathbb{R}^n \times [0, \infty))$ we make use of classical regularity results. Since w^δ is a solution to (2.2.2) in $\mathbb{R}^n \times [0, \infty)$, if we restrict ourselves to the cylinder Q_3, the regularity results for parabolic equations([57]) ensures that $w^\delta \in C^{\tilde{\gamma}}(Q_2)$ and $\|w^\delta\|_{C^{\tilde{\gamma}}(Q_2)} \leq C\delta^{-1}(\sigma(1 + \|M\|)$. Since Q_2 contains a periodic cube of w^δ, we obtain a uniform Hölder estimate on δw^δ over $\mathbb{R}^n \times [0, \infty)$. On the other hand, we know that F is convex with respect to M and from hypothesis that for any (y, s), $(y_0, s_0) \in \mathbb{R}^n \times [0, \infty)$

$$\begin{aligned}\theta(y, s, y_0, s_0) &:= \sup_{N\in\mathcal{S}^n} \frac{|F(M + N, y, s) - F(M + N, y_0, s_0)|}{1 + \|N\|} \\ &\leq \sigma(1 + \|M\|)(|y - y_0|^\gamma + |s - s_0|^{\gamma/2})\end{aligned}$$

for some $0 < \gamma < 1$. Now the regularity results for parabolic equations([58]) and the periodicity of domain apply to w^δ so that we get a constant $C > 0$ for which

$$\|\delta w^\delta\|_{C^{2,\gamma}(\mathbb{R}^n \times [0,\infty))} \leq C\,(1 + \|M\|)\,. \tag{2.2.4}$$

Define $\widehat{w}^\delta(y,s) := w^\delta(y,s) - \min_{\mathbb{R}^n\times[0,\infty)} w^\delta \geq 0$ in $\mathbb{R}^n \times [0,\infty)$. Then $\widehat{w}^\delta$ solves the equation

$$\delta \widehat{w}^\delta + \widehat{w}^\delta_s - F(M + D_y^2 \widehat{w}^\delta, y, s) = -\delta \min_{\mathbb{R}^n\times[0,\infty)} w^\delta \quad \text{in } \mathbb{R}^n \times [0,\infty). \tag{2.2.5}$$

Let us restrict our domain to $Q_3(y_0, s_0)$ where (y_0, s_0) is an arbitrary point in $\mathbb{R}^n \times [0,\infty)$. Since Q_2 contains a periodic cube of w^δ, we have $\sup_{Q_2} \widehat{w}_\delta = \sup_{\mathbb{R}^n\times[0,\infty)} \widehat{w}_\delta$ and $\inf_{Q_2} \widehat{w}_\delta = \inf_{\mathbb{R}^n\times[0,\infty)} \widehat{w}_\delta = 0$. We apply the Harnack ineqaulity over Q_3 to (2.2.5) then

$$\sup_{Q_2} \widehat{w}^\delta \leq C\,(1 + \|M\|)\,.$$

Since the above bound is independent of $\delta \in (0,1)$, and since (y_0, s_0) is an arbitrary point, we have

$$\sup_{0<\delta<1} osc_{\mathbb{R}^n\times[0,\infty)} w^\delta = \sup_{0<\delta<1} \sup_{\mathbb{R}^n\times[0,\infty)} \widehat{w}^\delta \leq C\,(1 + \|M\|)\,.$$

□

Now we deal with a parabolic cell problem.

Lemma 2.2.2. *For each $(M, x, t) \in \mathcal{S}^n \times \overline{S_T}$ there exists a (y,s)-periodic function $w(y, s; M, x, t)$ such that $w(\cdot, \cdot; M, x, t) \in C^{2,\gamma}(\mathbb{R}^n \times [0,\infty))$, and a constant $\overline{F}(M, x, t) \in \mathbb{R}$ such that*

$$\begin{aligned}&\|\delta w^\delta(\cdot,\cdot; M, x, t) - \overline{F}(M, x, t)\|_{L^\infty(\mathbb{R}^n\times[0,\infty))} \\ &\quad + \|\widetilde{w}^\delta(\cdot,\cdot; M, x, t) - w(\cdot,\cdot; M, x, t)\|_{C^2(\mathbb{R}^n\times[0,\infty))} \to 0 \quad as \quad \delta \to 0,\end{aligned}$$

where $\widetilde{w}^\delta(y, s; M, x, t) = w^\delta(y, s; M, x, t) - w^\delta(0, 0; M, x, t)$. Moreover, $\overline{F}$ is *a unique constant where the equation has a unique solution w up to constant addition. It then immediately followed from Lemma 2.2.1 that $\overline{F}$, and w satisfy*

$$|\overline{F}(M,x,t)| + \|w(\cdot,\cdot;M,x,t\|_{C^{2,\gamma}(\mathbb{R}^n\times[0,\infty))} \leq C\,(1+\|M\|)$$

and solve the following cell problem:

$$w_s - F(M + D_y^2 w, x, t, y, s) = -\overline{F}(M,x,t) \quad \textit{in } \mathbb{R}^n \times [0,\infty). \tag{2.2.6}$$

Proof. Set $\widetilde{w}^\delta(y,s) := w^\delta(y,s) - w^\delta(0,0)$ and we will show that the family $\{\widetilde{w}^\delta\}_{\delta\in(0,1)}$ is uniformly bounded in $C^{2,\gamma}$. From Lemma 2.2.1, we have $\|\widetilde{w}^\delta(\cdot,\cdot;M,x,t)\|_{L^\infty(\mathbb{R}^n\times[0,\infty))} \leq C(1+\|M\|)$. Moreover, $\widetilde{w}^\delta \in C^{2,\gamma}(\mathbb{R}^n \times [0,\infty))$ and satisfies

$$\delta\widetilde{w}^\delta + (\widetilde{w}^\delta)_s - F(M + D_y^2\widetilde{w}^\delta, y, s) = -\delta w^\delta(0,0) \quad \text{in } \mathbb{R}^n\times[0,\infty).$$

Using the similar argument when proving (2.2.4), we obtain

$$\sup_{0<\delta<1} \|\widetilde{w}^\delta\|_{C^{2,\gamma}(\mathbb{R}^n\times[0,\infty))} \leq C(1+\|M\|). \tag{2.2.7}$$

In view of (2.2.3), we can take a subsequence $\{\delta_k w^{\delta_k}\}_{k=1}^\infty$ of $\{\delta w^\delta\}_{0<\delta<1}$ and a number $\overline{F}(M,x,t) \in \mathbb{R}$ such that $\delta_k w^{\delta_k}(\cdot,\cdot;M,x,t) \to \overline{F}(M,x,t)$ uniformly in $\mathbb{R}^n \times [0,\infty)$ as $k\to\infty$. On the other hand, by (2.2.7) and the compact embedding argument yield that there is a (y,s)-periodic function w and a further subsequence of $\{\delta_k\}_{k=1}^\infty$, which we denote again by $\{\delta_k\}_{k=1}^\infty$ for convenience, such that

$$\|\delta_k w^{\delta_k} - \gamma\|_{L^\infty(\mathbb{R}^n\times[0,\infty))} + \|\widetilde{w}^{\delta_k} - w\|_{C^2(\mathbb{R}^n\times[0,\infty))} \to 0 \;\text{ as }\; k\to\infty, \tag{2.2.8}$$

for some (y,s)-periodic $w \in C^{2,\alpha}(\mathbb{R}^n\times[0,\infty))$. Then by the stability of viscosity solutions, the function w solves following equation

$$w_s - F(M + D_y^2 w, x, t, y, s) = -\overline{F}(M, x, t) \quad \text{in } \mathbb{R}^n \times [0, \infty).$$

Now we show that the constant $\overline{F}$ is unique. We assume to the contrary that that there is another a subsequence of $\{\delta w^\delta\}_{0<\delta<1}$ converges to $\widetilde{F} \in \mathbb{R}$ uniformly in $\mathbb{R}^n \times [0, \infty)$, where $\widetilde{F} \neq \overline{F}$. Also, let w' be the solution of (2.2.6) corresponding limit of a subsequence of $\{\widetilde{w}^\delta\}_{0<\delta<1}$. Without loss of generality, suppose that $\overline{F} > \widetilde{F}$. Since w and w' are bounded, add a constant h_0 to w such that $w'(y_0, s_0) + h_0 < w(y_0, s_0)$ at a point $(y_0, s_0) \in \mathbb{R}^n \times [0, \infty)$. Let

$$h_1 := \inf\{h : w'(y, s) + h \geq w(y, s)\}.$$

Then $w' + h_1$ touches w by above at a point (y_1, s_1). Therefore, we deduce that

$$\begin{aligned}
-\widetilde{F}(M, x, t) &= (w' + h_1)_s(y_1, s_1) - F(M + D_y^2(w' + h_1)(y_1, s_1), x, t, y_1, s_1) \\
&\leq w_s(y_1, s_1) - F(M + D_y^2 w(y_1, s_1), x, t, y_1, s_1) \\
&= -\overline{F}(M, x, t),
\end{aligned}$$

which is a contradiction. This shows that the constant $\overline{F}$ must be unique. Finally, by the maximum principle we can also observe that the uniform convergence (2.2.8) could be made along the full sequence. Consequently, the limit function w is also unique (up to constant). □

2.2.2 Effective operators and effective limits

The functional $\overline{F} : \mathcal{S}^n \times \overline{S_T} \to \mathbb{R}$ in Lemma 2.2.2 is called the effective operator. It is natural to predict that the effective operator $\overline{F}$ has similar properties to F.

Lemma 2.2.3. *(i) $\overline{F}$ is uniformly elliptic with the same ellipticity constants of F and convex with respect to M-variable.*

(ii) For each $L > 0$, $\overline{F} \in C^{0,1}(\overline{B_L} \times \overline{S_T})$.

Proof. (i) In this proof, let us get rid of the dependence of (x,t)-variable for convenience. It is enough to show that

$$\overline{F}(M+N,x,t) - \overline{F}(M,x,t) \geq \lambda\|N\| \quad \text{if } N \geq 0.$$

For fixed $(x,t) \in \overline{S_T}$, let $w^{M+N}(y,s) := w(y,s;M+N,x,t)$ and $w^M(y,s) := w(y,s;M,x,t)$. Adding a constant to w^{M+N} if necessary, we may assume that $w^{M+N} < w^M$. Assume for a contradiction that

$$\overline{F}(M+N,x,t) - \overline{F}(M,x,t) < \lambda\|N\|.$$

Then by the uniform ellipticity of F we obtain

$$\begin{aligned} w_s^M - F(M+N+D_y^2 w^M, y, s) &\leq w_s^M - F(M + D_y^2 w^M, y, s) - \lambda\|N\| \\ &= -\overline{F}(M) - \lambda\|N\| \\ &< -\overline{F}(M+N,x,t) \\ &= w_s^{M+N} - F(M+N+D_y^2 w^{M+N}, y, s) \end{aligned}$$

in $\mathbb{R}^n \times [0,\infty)$. Hence by the comparison principle, we have $w^{M+N} \geq w^M$, which is the desired contradiction.
Now we will prove the convexity of $\overline{F}$. Let M, $N \in \mathcal{S}^n$ and $(x,t) \in \overline{S_T}$ be fixed. We write w^M as before. Suppose toward a contradiction that there is some $\theta \in (0,1)$ and M, $N \in \mathcal{S}^n$ such that

$$\overline{F}(\theta M + (1-\theta)N, x, t) > \theta\overline{F}(M,x,t) + (1-\theta)\overline{F}(N,x,t).$$

Put $X := \theta M + (1-\theta)N$. We may assume that $w^X > \theta w^M + (1-\theta) w^N$

in $\mathbb{R}^n \times \mathbb{R}$. Then we obtain from the convexity of F that

$$\begin{aligned}
&\left(\theta w^M + (1-\theta)w^N\right)_s - F\left(X + D_y^2\left(\theta w^M + (1-\theta)w^N\right), y, s\right) \\
&\quad \geq \theta\left[w_s^M - F\left(M + D_y^2 w^M, y, s\right)\right] + (1-\theta)\left[w_s^N - F\left(N + D_y^2 w^N, y, s\right)\right] \\
&\quad = -\theta\overline{F}(M, x, t) - (1-\theta)\overline{F}(N, x, t) \\
&\quad > -\overline{F}(\theta M + (1-\theta)N, x, t) \\
&\quad = w_s^X - F(X + D_y^2 w^X, y, s)
\end{aligned}$$

in $\mathbb{R}^n \times [0, \infty)$. Hence the comparison principle implies that $w^X \leq \theta w^M + (1-\theta)w^N$ in $\mathbb{R}^n \times [0, \infty)$, which is a contradiction.

(ii) We drop the dependence of (y, s)-variable for convenience. Fix (M_1, x_1, t_1), $(M_2, x_2, t_2) \in \overline{B_L} \times \overline{S_T}$. We denote v_1^δ, v_2^δ the functions $w^\delta(y, s; M_1, x_1, t_1)$, $w^\delta(y, s; M_2, x_2, t_2)$ respectively for simplicity of notation, where w^δ is in Lemma 2.2.2. By Lipschitz continuity of F, we have

$$\begin{aligned}
&(v_1^\delta)_s - F(M_2 + D_y^2 v_1^\delta, x_2, t_2) \\
&\quad \leq (v_1^\delta)_s - F(M_1 + D_y^2 v_1^\delta, x_1, t_1) \\
&\qquad + \sigma(1+L)\left(\|M_1 - M_2\| + |x_1 - x_2| + |t_1 - t_2|^{1/2}\right) \\
&\quad = -\delta v_1^\delta + \sigma(1+L)\left(\|M_1 - M_2\| + |x_1 - x_2| + |t_1 - t_2|^{1/2}\right)
\end{aligned}$$

uniformly $(y, s) \in \mathbb{R}^n \times [0, \infty)$, which means that

$$v_1^\delta - \delta^{-1}\sigma(1+L)\left(\|M_1 - M_2\| + |x_1 - x_2| + |t_1 - t_2|^{1/2}\right)$$

is a sub-solution of (2.2.2). Therefore, by comparison we obtain

$$\delta v_2^\delta - \delta v_1^\delta \leq \sigma(1+L)\left(\|M_1 - M_2\| + |x_1 - x_2| + |t_1 - t_2|^{1/2}\right)$$

in $\mathbb{R}^n \times [0, \infty)$. By a similar argument for v_2^δ, we deduce that

$$|\delta v_2^\delta - \delta v_1^\delta| \leq \sigma(1+L)\left(\|M_1 - M_2\| + |x_1 - x_2| + |t_1 - t_2|^{1/2}\right).$$

Then the conclusion comes by taking limits on both sides.

□

Now we can find the effective limit u which solves the following homogenized equation.

Lemma 2.2.4. *Let* $\{u^\varepsilon\}_{\varepsilon>0} \subset C(\overline{S_T})$ *be the family of viscosity solutions to* (2.2.1)*. Then there exists a unique function* u *such that* $u^\varepsilon \to u$ *uniformly in* $\overline{S_T}$*, and* u *solves the following homogenized equation:*

$$\begin{cases} u_t - \overline{F}(D_x^2 u, x, t) = 0 & \text{in } S_T, \\ u = \varphi(x,t) & \text{on } \partial_p S_T. \end{cases} \tag{2.2.9}$$

Proof. Owing to estimates [57], there exists $\widetilde{\gamma} > 0$ for which

$$\sup_{0<\varepsilon<1} \|u^\varepsilon\|_{C^{\widetilde{\gamma}}(\overline{S_T})} < \infty.$$

Thus, we may extract a subsequence $\{u^{\varepsilon_l}\}_{l=1}^{\infty}$ of $\{u^\varepsilon\}_{\varepsilon>0}$ and a function $u \in C^{\widetilde{\gamma}}(\overline{S_T})$ with $u^{\varepsilon_l} \to u$ uniformly on $\overline{S_T}$. Moreover, since $u^\varepsilon = \varphi$ on $\partial_p S_T$ for all $\varepsilon > 0$, we have $u = \varphi$ on $\partial_p S_T$. For convenience, we will not use subsequencial notation. Let P be a paraboloid with $M_0 = D^2 P$ which touches u by above at (x_0, t_0) in a neighborhood. Without loss of generality, we may assume that P touches u strictly by above. Assume, to the contrary, that

$$P_t - \overline{F}(M_0, x_0, t_0) > 3\eta > 0$$

for some $\eta > 0$. Put $\widehat{w}(y,s) := w(y,s; M_0, x_0, t_0)$. Then by Lemma 2.2.2 we can observe that $\widehat{w}$ satisfies

$$\widehat{w}_s - F(M_0 + D_y^2 \widehat{w}, x_0, t_0, y, s) = -\overline{F}(M_0, x_0, t_0) \quad \text{in } \mathbb{R}^n \times [0, \infty). \tag{2.2.10}$$

By the continuity of F and $\overline{F}$(Lemma 4.2.2), we can choose $\rho > 0$ in such way that $Q_\rho(x_0,t_0)\subset S_T$,

$$\begin{aligned}&P_t-\overline{F}(M_0,x,t)>3\eta, \quad \text{and}\\ &|F(M_0+D_y^2\widehat{w},x,t,y,s)-F(M_0+D_y^2\widehat{w},x_0,t_0,y,s)|\\ &\quad+|\overline{F}(M_0,x,t)-\overline{F}(M_0,x_0,t_0)|<\eta\end{aligned} \tag{2.2.11}$$

for any $(x,t)\in Q_\rho(x_0,t_0)$, uniformly $(y,s)\in\mathbb{R}^n\times[0,\infty)$. Moreover, $u(x,t)-P(x,t)\le-\mu$ on ∂Q_ρ, for some $\mu>0$. Define

$$P^\varepsilon(x,t):=P(x,t)+\varepsilon^2\widehat{w}\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right). \tag{2.2.12}$$

For a while, let us drop the dependency of $(x/\varepsilon,t/\varepsilon^2)$. Then in view of (2.2.10), (2.2.11), and (2.2.12), we have

$$\begin{aligned}P_t^\varepsilon-F\left(D_x^2P^\varepsilon,x,t\right)&=P_t+\widehat{w}_s-F\left(M_0+D_y^2\widehat{w},x,t\right)\\ &\ge P_t+\widehat{w}_s-F\left(M_0+D_y^2\widehat{w},x_0,t_0\right)-\eta\\ &=P_t-\overline{F}(M_0,x_0,t_0)-\eta\\ &\ge P_t-\overline{F}(M_0,x,t)-2\eta\\ &>0\end{aligned}$$

in $Q_\rho(x_0,t_0)$. As $u^\varepsilon\to u$ and $P^\varepsilon\to P$ uniformly in $Q_\rho(x_0,t_0)$, we can easily check that for some $\varepsilon_0\in(0,1)$ there holds

$$u^\varepsilon(x,t)-P^\varepsilon(x,t)<-\mu/2 \quad \text{on } \partial Q_\rho(x_0,t_0), \quad \varepsilon<\varepsilon_0.$$

Hence $P^\varepsilon-\mu/4$ is a super-solution to the following initial-boundary value problem:

$$\begin{cases}v_t-F(D_x^2v,x,t,x/\varepsilon,t/\varepsilon^2)=0 & \text{in } Q_\rho(x_0,t_0),\\ v=u^\varepsilon(x,t) & \text{on } \partial_pQ_\rho(x_0,t_0).\end{cases}$$

Therefore, the comparison principle implies $u^\varepsilon\le P^\varepsilon-\mu/4$ in $Q_\rho(x_0,t_0)$.

Letting $\varepsilon \to 0$ then $u(x_0, t_0) \leq P(x_0, t_0) - \mu/4$ which contradicts assumption that $u(x_0, t_0) = P(x_0, t_0)$. It shows that u is a viscosity sub-solution of (2.2.9). In a similar manner, we are able to prove that u is a viscosity super-solution of (2.2.9). Finally, the uniqueness of u is obtained by the comparison principle, and hence the convergence of $u^\varepsilon \to u$ does not need to extract a subsequence. This completes the proof. □

Chapter 3

Non-linear operators of divergence form on the Sierpinski gasket

3.1 Introduction

In this paper, we consider one particular class of fractals, domains in $\mathbb{R}^2$ which are Sierpinski gaskets. The Sierpinski gasket(SG), also called the Sierpinski triangle, is a kind of fractal sets with the overall shape of an equilateral triangle, subdivided recursively into smaller equilateral triangles(see [27, 56]). This is one of the basic examples of self-similar sets. There is a remarkable difference between analysis on Euclidean spaces and that on fractals: different measures are involved to measure the volume of sets and energy of functions, and these measures are singular to each other in general. We develop an analytic approach in which we used very strongly the energy measure, the symmetry of the space, and the comparability of the non-linear operators. We have chosen to work on SG since this makes the simplest context to employ our methods. However, we expect that our methods will apply with only minor changes to these other spaces of fractal type.

3.1.1 Main results

It is natural to expect whether the Harnack inequality can be extended to non-linear energy forms on SG. For example, the existence of p-harmonic functions on SG has been proved in [25, 55], and [15, 16] proved the Harnack inequality for non-negative p-harmonic functions on metric fractals, which contain SG. But the study of non-linear operators of divergence type on SG is new, to our best knowledge, hence we have to first define operators properly. Consider the divergence form operator

$$Af(x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial f}{\partial x_j}\right)(x)$$

taking on functions on $\mathbb{R}^n$, where $a = (a_{ij}(x))$ is bounded, measurable, and uniformly elliptic. Moser [53] states that an elliptic Harnack inequality holds for non-negative functions u that are harmonic with respect to the operator A. Classically, most of the proofs for the elliptic Harnack inequality use in an essential way the fact that the energy forms for the Laplacian and the divergence operator A, given by $E(f) = \int |\nabla f|^2$ and $E_A(f) = \int \nabla f \cdot a\nabla f$, respectively, are comparable each other. In this way, it is reasonable to define non-linear operator L so that the energy forms of the L are comparable to that of the existing Laplacian. Then we define a $\mathcal{L}$-harmonic function u to be one that minimizes the energy form of the operator L for the given boundary values. In the next section, we will discuss more the operators L and its energy forms.

The main result of this paper is the following elliptic Harnack inequality for $\mathcal{L}$-harmonic functions. We will use the symbol K to denote SG and let V_0 be a set of three boundary points of K.

Theorem 3.1.1. *If K' is a compact subset of K that is contained in a connected component of $K \setminus V_0$, then there exists a constant $c_1 > 0$ depending* *only on K', such that*

$$u(x) \leq c_1 u(y) \quad x, y \in K'$$

for any non-negative $\mathcal{L}$-harmonic function u on K.

3.1.2 Main strategies

We now summarize the main strategies of this paper and make some remarks on the key ingredients observed in achieving the result. In the following, we mainly use Moser's approaches [53] to prove the Harnack inequalities, but the standard techniques of Moser iteration encounter difficulties in the fractal case. Given a harmonic function $u \geq 0$, and for $f = u^p$, the standard Moser iteration argument uses the Caccioppoli inequality, Sobolev inequality, and cut-off functions η with the minimum energy that satisfy

$$\int_{B(x,R)} |\nabla \eta|^2 d\mu \approx R^{-2} \mu(B(x,R)), \quad R \leq 1$$

to bound

$$\int_{B(x,R/2)} |f|^{2+\varepsilon} d\mu \leq \int_{B(x,R/2)} |\nabla f|^2 d\mu \leq \int_{B(x,R)} |f|^2 d\mu.$$

Iterating and passing to the limit, one obtains local boundedness of harmonic functions. As hinted above, the key steps are to prove the Caccioppoli type inequality and weighted Sobolev inequality. The Caccioppoli type inequality on SG can be established by carrying out an interesting self-similarity property, which suggests that we can consider energy as a measure. This special characteristic of fractals allows us to link the energy of f to the L^2 norm of f with respect to the energy measure of cut-off functions. On the other hand, The difficulty in capturing the Sobolev inequality is that there is no suitable analogue of the following Newton-Leibniz formula

$$u(x) - u(y) = \int_0^1 \langle \dot{\gamma}(s), \nabla u(\gamma(s)) \rangle ds$$

for every curve $\gamma : [0,1] \to \mathbb{R}^n$ connecting x and y. Moreover, we notice that the order of cut-off function, R^{-2}, plays an important role in Moser's method since it cancels terms involving R^2 which arise from the Poincaré inequality. But on fractal domains in $\mathbb{R}^n$, for example, Sierpinski gasket, such functions do not exist (see [36]). Instead, we focus on the "anomalous" scaling in the Poincaré inequality

$$\inf_a \int_{B(x,R)} |f-a|^2 d\mu \leq CR^{d_w} \int_{B(x,R)} |\nabla f|^2 d\mu \leq CR^2 \int_{B(x,R)} |\nabla f|^2 d\mu, \tag{3.1.1}$$

where $d_w > 2$, called 'walk dimension', means the space-time scaling relation for the diffusion process on SG. Since $R < 1$, (3.1.1) means that we can establish a more appropriate estimate for the Poincaré inequality. Then this estimate allows us to use cut-off functions derived from the potentials associated with the Laplacian on SG. That is, a rescaled Poincaré inequality implies the existence of enough 'moderate energy' cut-off functions on the space. In fact, we can find a cut-off function with minimal energy of order $R^{-d_w} \gg R^{-2}$. The important point is to create a cut-off function by using Green functions and combine it with the Caccioppoli type inequality to prove that

$$\int_{B(x,R/2)} |\nabla f|^2 d\lambda \leq C \int_{B(x,R)} |f|^2 d\mu$$

for a new measure $\lambda = \mu + R^{-d_w}\nu_\eta$, where ν_η is a energy measure of the cut-off function defined as in Definition 2.1.8. The characteristic of the new measure λ is comparable to the existing Hausdorff measure μ, and serves to match the order R^{d_w} of the Poincaré inequality. Then we prove a weighted Sobolev inequality linking the $L^{2+\varepsilon}$ norm of f with respect to λ to the energy of f.

We point out that the process to prove theorem 3.1.1 is similar to those [6, 7], however in working with SG one faces several difficulties arising from special characteristics of the domain. On SG and related fractals, most operators such as Laplacian or Green function will be defined as limits of discrete operations on a sequence of graphs whose vertices approximate the fractal. This approach occurs essentially because there is no gradient terminology. To overcome this difficulty, we will use the concept of 'cell' to describe various subsets of the domain. SG is a union of three smaller copies of cells(self-similarity), and these copies intersect each other at a finite set of points. This property allows us to define the energy measure on each cell, and we can describe the local behavior of functions on the SG. In addition, by capturing symmetric property of the cell, we can overcome the consistent issue arising from [6].

3.1.3 Outline

This paper is organized as follows: In Section 3.2, we provide abstract formulations of generalized energy forms and show existence and uniqueness results for their minimizers. In Section 3.3, we formulate the construction of a cut-off function and weighted measure λ. We give the proof of the weighted Sobolev inequality involving measure λ in Subsection 3.3.2. In Subsection 3.4.1 we present the proof of the local boundedness and weak Harnack inequality, and finally prove main theorem in Subsection 3.4.2.

3.2 L-harmonic functions

In this section, we construct divergence structure non-linear operators and their solutions. We begin by considering a general notion of energy on the Sierpinski gasket. Suppose we are given functions $L : \mathbb{R} \times V_* \times V_* \to \mathbb{R}$ and $G : \mathbb{R} \times K \to \mathbb{R}$ which possess the following structure conditions,

(a) There are positive real numbers c_0, c_1 and c_2 such that

$$\begin{aligned} &1/c_0|p|^2 \le L(p,x,y) \le c_0|p|^2 && \text{for all } x, y \in V_*, \\ &G(z,x) \ge -c_1 && \text{for all } z \in \mathbb{R} \text{ and } x \in K, \\ &|D_z G(z,x)| \le c_2|z| && \text{for all } x \in K. \end{aligned} \tag{3.2.1}$$

(b) L is convex in p-variable.

Then we may consider the generalized energy on Γ_m: for any function $u : K \to \mathbb{R}$ and any finite union of cells $D \subset K$, define

$$(\mathcal{E}_D^{\mathcal{L}})^{(m)}(u) := \frac{r^{-m}}{2} \sum_{\substack{x \sim_m y \\ x,y \in V_m \cap D}} L(u(x)-u(y),x,y) + \int_D G(u(x),x)d\mu.$$

This is a natural extension if there is a weight on the Sierpinski gasket, but in the case of generalized energy, monotonicity (2.1.1) is not clear since (2.1.2) does not hold. So it makes sense to define generalized energy on $D \subset K$ as

$$\mathcal{E}_D^{\mathcal{L}}(u) := \limsup_{m\to\infty} (\mathcal{E}_D^{\mathcal{L}})^{(m)}(u).$$

We also simply write $\mathcal{E}_K^{\mathcal{L}}(u) = \mathcal{E}^{\mathcal{L}}(u)$. Then by the structure condition (3.2.1) of L, it is obvious that

$$\frac{1}{c_0}\mathcal{E}_D(u) \le \mathcal{E}_D^{\mathcal{L}}(u) \le c_0\mathcal{E}_D(u),$$

hence for any u on V_*, $\mathcal{E}_D^{\mathcal{L}}(u) < \infty$ if and only if $u \in \mathcal{F}(D)$. As a simplest example, we can consider that

$$L(p,x,y) = a_{xy}|p|^2 \quad \text{and} \quad G \equiv 0$$

where a_{xy} is a positive function defined on the $V_* \times V_*$. We call $a = (a_{xy})$, x, $y \in V_*$, a conductance matrix if $a_{xy} \ge 0$, $a_{xy} = a_{yx}$ for all x, $y \in V_*$ and $a_{xy} = 0$ if $\{x, y\}$ is not an edge in Γ_m for any $m \in \mathbb{N}$. If there exists $c_0 > 0$

such that for any $m \in \mathbb{N}$,

$$1/c_0 \leq a_{xy} \leq c_0 \quad \text{whenever \{x,y\} is an any edge in } \Gamma_m,$$

in a physical sense, we interpret a_{xy} as conductances and the reciprocals of resistances. Then we can think of the energy applied to the weight for each edge of the Sierpinski gasket. In this case, for any $u \in \mathcal{F}(K)$ the energy on $D \subset K$ are defined by

$$\mathcal{E}_D^{\mathcal{L}}(u) = \limsup_{m\to\infty} \frac{r^{-m}}{2} \sum_{\substack{x \sim_m y \\ x,y \in V_m \cap D}} a_{xy}(u(x)-u(y))^2.$$

In particular, if $a_{xy} = 1$ for all x, $y \in V_*$, then $\mathcal{E}_D^{\mathcal{L}}(u) = \mathcal{E}_D(u)$.

What we can naturally expect is that, like a harmonic function, there exists a function to be one that minimizes $\mathcal{E}^{\mathcal{L}}(\cdot)$ for the given boundary values on V_0. We can show that the answer is true, and we call a $\mathcal{L}$-harmonic function u to be one that minimizes $\mathcal{E}^{\mathcal{L}}(\cdot)$ for the given boundary values on V_0.

Lemma 3.2.1 (Existence of minimizer)**.** *Suppose that $u \in \mathcal{F}(K)$ and define*

$$\mathcal{B}(u) := \{v \in \mathcal{F}(K) : v = u \ \text{on } V_0\}.$$

Suppose that the mapping $p \mapsto L(p,x,y)$ is smooth and convex for each x, $y \in V_*$. Then there exists at least one function $\widetilde{u} \in \mathcal{B}(u)$ solving

$$\mathcal{E}^{\mathcal{L}}(\widetilde{u}) = \min_{v\in\mathcal{B}(u)} \mathcal{E}^{\mathcal{L}}(v).$$

Proof. Without loss of generality, assume $u(q_0) = 0$ and it is convenient to identify $\mathcal{F}(K)/constants$ with the space $\widetilde{\mathcal{F}}(K) := \{w \in \mathcal{F}(K) : w(q_0) = 0\}$. Note that $\widetilde{\mathcal{F}}(K)$ forms a Hilbert space with inner product $\mathcal{E}_D(\cdot,\cdot)$, which is endowed with the norm $\|u\|_D = \mathcal{E}_D(u)^{1/2}$. We introduce the notation

$$\mathcal{L}(v) := \limsup_{m\to\infty} \frac{r^{-m}}{2} \sum_{\substack{x\sim_m y \\ x,y\in V_m\cap K}} L(v(x)-v(y),x,y),$$

and let us first claim that a function $\mathcal{L}(\cdot)$ is lower semi-continuous on $\widetilde{\mathcal{F}}(K)$, i.e, $w_k \to w$ in $\widetilde{\mathcal{F}}(K)$ implies $\mathcal{L}(w) \le \liminf_{k\to\infty} \mathcal{L}(w_k)$. To see this, suppose $w_k \to w$ in $\widetilde{\mathcal{F}}(K)$ and set $a := \liminf_{k\to\infty} \mathcal{L}(w_k)$. Upon passing to a subsequence if necessary, we may as well also suppose $a = \lim_{k\to\infty} \mathcal{L}(w_k)$. Then we must show $\mathcal{L}(w) \le a$.

Since L is convex with respect to p-variable, we can observe that

$$\mathcal{L}(w_k) \ge \mathcal{L}(w) - \limsup_{m\to\infty} I_m^k,$$

where

$$I_m^k = \frac{r^{-m}}{2} \sum_{\substack{x\sim_m y \\ x,y\in V_m}} D_pL(w(x)-w(y),x,y)(w(x)-w(y)-(w_k(x)-w_k(y))).$$

Let us estimate $\limsup_{m\to\infty} I_m^k$. Note that

$$|D_pL(p,x,y)| \le c_1|p| \quad \text{for all } x,y\in V_*,$$

which following from the structure conditions (3.2.1) and convexity of L. So for each $\varepsilon > 0$, we have

$$|D_pL(p,x,y)||p-q| \le c_2|p||p-q| \le c_3\left(\varepsilon p^2 + \frac{1}{4\varepsilon}(p-q)^2\right)$$

for all p, $q \in \mathbb{R}$. Therefore, we obtain

$$\left|\limsup_{m\to\infty} I_m^k\right| \le c_3\left(\varepsilon\mathcal{E}(w) + \frac{1}{4\varepsilon}\mathcal{E}(w-w_k,w-w_k)\right).$$

Here, the second term of right-hand side goes to zero since $w_k \to w$ in $\widetilde{\mathcal{F}}(K)$.

Hence $\lim_{k\to\infty}\left|\limsup_{m\to\infty} I_m^k\right| \le c_3\varepsilon\mathcal{E}(w)$ and since this inequality holds for each $\varepsilon > 0$, we obtain that

$$\limsup_{m\to\infty} I_m^k \to 0 \quad \text{as} \quad k \to \infty. \tag{3.2.2}$$

Consequently, in view of (3.2.2) we deduce that

$$a = \lim_{k\to\infty} \mathcal{L}(w_k) \ge \mathcal{L}(w) - \lim_{k\to\infty}\left(\limsup_{m\to\infty} I_m^k\right) = \mathcal{L}(w).$$

Thus, $\mathcal{L}(\cdot)$ is lower semi-continuous.
Now we prove the existence of minimizer. Set

$$l := \inf_{v\in\mathcal{B}(u)} \mathcal{E}^{\mathcal{L}}(v).$$

Select a minimizing sequence $\{v_k\}_{k=1}^{\infty}$. Then

$$\mathcal{E}^{\mathcal{L}}(v_k) \to l.$$

Since $\mathcal{E}(v) \le c_0\left(\mathcal{E}^{\mathcal{L}}(v) + c_4\right)$ for any $v \in \widetilde{\mathcal{F}}(K)$ and l is finite, we have $\|v_k\| < c_5$ for any k. Consequently, by the weak compactness theorem, there exists a subsequence $\{v_{k_i}\}_{i=1}^{\infty}$ of $\{v_k\}_{k=1}^{\infty}$ which converges weakly to $\widetilde{u} \in \widetilde{\mathcal{F}}(K)$. i.e.

$$\mathcal{E}(v_{k_i}, \varphi) \to \mathcal{E}(\widetilde{u}, \varphi) \quad \text{for all } \varphi \in \widetilde{\mathcal{F}}(K).$$

On the other hand, lemma 2.1.2 allows us to use the Arzelá-Ascoli theorem, from which we deduce that there is a further subsequence of $\{v_{k_i}\}_{i=1}^{\infty}$, which we denote again by $\{v_{k_i}\}_{i=1}^{\infty}$ for convenience, such that $\{v_{k_i}\}_{i=1}^{\infty}$ converges uniformly to $\widetilde{u} \in C(K)$, and thus $\widetilde{u} \in \mathcal{B}(u)$. Upon passing to a subsequence if necessary, we may also suppose

$$l = \lim_{k\to\infty} \mathcal{E}^{\mathcal{L}}(v_k).$$

Now it remains to show that $\widetilde{u}$ is in fact the minimizer among functions in

$\mathcal{B}(u)$. So we have to show that

$$\mathcal{E}^{\mathcal{L}}(\widetilde{u}) \leq l.$$

To see this, for each $\varepsilon > 0$ let

$$K_\varepsilon = \left\{ w \in \mathcal{B}(u) : \mathcal{L}(w) + \int_K G(\widetilde{u}(x), x) d\mu \leq l + \varepsilon \right\}.$$

Then the convexity of L and lower semi-continuity of $\mathcal{L}(\cdot)$ imply that K_ε is convex and closed. Thus it is weakly closed according to Mazur's Theorem. Since $\{v_k\}_{k=1}^\infty$ converges uniformly to $\widetilde{u}$ we have

$$\int_K G(v_k(x), x) d\mu \to \int_K G(\widetilde{u}(x), x) d\mu,$$

and since $\{v_k\}_{k=1}^\infty$ converges weakly to $\widetilde{u}$, we conclude that all but finitely many of the points $\{v_k\}_{k=1}^\infty$ lie in K_ε, $\widetilde{u}$ lies in K_ε, and consequently

$$\mathcal{E}^{\mathcal{L}}(\widetilde{u}) = \mathcal{L}(\widetilde{u}) + \int_K G(\widetilde{u}(x), x) d\mu \leq l + \varepsilon.$$

This is true for each $\varepsilon > 0$ and thus $\mathcal{E}^{\mathcal{L}}(\widetilde{u}) \leq l$. Finally, since $\widetilde{u} \in \mathcal{B}(u)$, it follows that

$$\mathcal{E}^{\mathcal{L}}(\widetilde{u}) = l = \min_{v \in \mathcal{B}(u)} \mathcal{E}^{\mathcal{L}}(v).$$

□

We turn next to the problem of uniqueness. In general, there can be many minimizers, and so to ensure uniqueness we require further assumptions.

Lemma 3.2.2 (Uniqueness of minimizer)**.** *Suppose that the mapping* $p \mapsto L(p, x, y)$ *is smooth and uniformly convex for each* $x, y \in V_*$*, and the mapping* $z \mapsto G(z, x)$ *is smooth and convex for each* $x \in K$*. Then a minimizer* $\widetilde{u} \in \mathcal{B}(u)$ *of* $\mathcal{E}^{\mathcal{L}}(\cdot)$ *is unique.*

Proof. Assume $u_1, u_2 \in \mathcal{B}(u)$ are both minimizer of $\mathcal{E}^{\mathcal{L}}(\cdot)$ over $\mathcal{B}(u)$. Then

$w := \dfrac{u_1 + u_2}{2} \in \mathcal{B}(u)$. We claim

$$\mathcal{E}^{\mathcal{L}}(w) \leq \frac{\mathcal{E}^{\mathcal{L}}(u_1) + \mathcal{E}^{\mathcal{L}}(u_2)}{2},$$

with a strict inequality, unless $u_1 = u_2$.
Setting $\widetilde{w} = \dfrac{u_1 - u_2}{2}$. It is easy to observe that $\widetilde{w} \in \mathcal{F}(K)$. Put

$$\begin{aligned} \mathcal{L}(v) &= \frac{r^{-m}}{2} \sum_{\substack{x \sim_m y \\ x,y \in V_m}} L(v(x) - v(y), x, y), \\ I_m &= \frac{r^{-m}}{2} \sum_{\substack{x \sim_m y \\ x,y \in V_m}} D_p L(w(x) - w(y), x, y)(\widetilde{w}(x) - \widetilde{w}(y)) \end{aligned}$$

Note from the uniform convexity assumption that for all p, $q \in \mathbb{R}$ and x, $y \in V_*$, there exists $c_1 > 0$ such that

$$L(p, x, y) \geq L(q, x, y) + D_p L(q, x, y)(p - q) + \frac{c_1}{2}|p - q|^2.$$

Thus, the definition of $\mathcal{E}^{\mathcal{L}}$ and the convexity of G imply that for each $\varepsilon > 0$ there exists an index N that for all $m \geq N$ we have

$$\begin{aligned} \mathcal{E}^{\mathcal{L}}(u_1) + \varepsilon &\geq (\mathcal{E}^{\mathcal{L}})^{(m)}(u_1) \\ &= \mathcal{L}(u_1) + \int_D G(u_1) d\mu \\ &\geq \mathcal{L}(w) + I_m + \frac{c_1}{2}\mathcal{E}^{(m)}(\widetilde{w}, \widetilde{w}) + \int_D G(w) d\mu + D_z G(w) \widetilde{w} d\mu \\ &= (\mathcal{E}^{\mathcal{L}})^{(m)}(w) + I_m + \frac{c_1}{2}\mathcal{E}^{(m)}(\widetilde{w}, \widetilde{w}) + \int_D D_z G(w) \widetilde{w} d\mu, \end{aligned}$$

and

$$\begin{aligned}\mathcal{E}^{\mathcal{L}}(u_2)+\varepsilon &\geq (\mathcal{E}^{\mathcal{L}})^{(m)}(u_2)\\ &= \mathcal{L}(u_2)+\int_D G(u_2)d\mu\\ &\geq \mathcal{L}(w)-I_m+\frac{c_1}{2}\mathcal{E}^{(m)}(\widetilde{w},\widetilde{w})+\int_D G(w)d\mu - D_zG(w)\widetilde{w}d\mu\\ &= (\mathcal{E}^{\mathcal{L}})^{(m)}(w)-I_m+\frac{c_1}{2}\mathcal{E}^{(m)}(\widetilde{w},\widetilde{w})-\int_D D_zG(w)\widetilde{w}d\mu.\end{aligned}$$

Here, each second inequality we used the uniform convexity of L. Add and divide by 2, to deduce

$$\frac{\mathcal{E}^{\mathcal{L}}(u_1)+\mathcal{E}^{\mathcal{L}}(u_2)}{2}+\varepsilon \geq (\mathcal{E}^{\mathcal{L}})^{(m)}(w)+\frac{c_1}{2}\mathcal{E}^{(m)}(\widetilde{w},\widetilde{w})$$

and this inequality holds for all $m \geq N$ so we have

$$\frac{\mathcal{E}^{\mathcal{L}}(u_1)+\mathcal{E}^{\mathcal{L}}(u_2)}{2}+\varepsilon \geq \mathcal{E}^{\mathcal{L}}(w)+\frac{c_1}{2}\mathcal{E}(\widetilde{w},\widetilde{w}) \geq \mathcal{E}^{\mathcal{L}}(w).$$

We now let ε tend to zero we have

$$\frac{\mathcal{E}^{\mathcal{L}}(u_1)+\mathcal{E}^{\mathcal{L}}(u_2)}{2} \geq \mathcal{E}^{\mathcal{L}}(w).$$

As $\mathcal{E}^{\mathcal{L}}(u_1)=\mathcal{E}^{\mathcal{L}}(u_2)=\min_{v\in\mathcal{B}(u)}\mathcal{E}^{\mathcal{L}}(v) \leq \mathcal{E}^{\mathcal{L}}(w)$, we deduce that

$$\frac{c_1}{2}\mathcal{E}(\widetilde{w},\widetilde{w})=0.$$

Since $\widetilde{w}=0$ on V_0, by Lemma 2.1.2, $|u_1(x)-u_2(x)|^2=4|\widetilde{w}(x)|^2 \leq c_2\mathcal{E}(\widetilde{w},\widetilde{w})=0$ for all $x \in K$ so we deduce that $u_1 \equiv u_2$. □

3.3 Weighted inequalities

In this section, we find a suitable cut-off function to obtain weighted Sobolev and Poincaré inequalities. We can then use these inequalities to drive the

Moser iteration so that we can estimate the supremum and infimum of $\mathcal{L}$-harmonic functions.

3.3.1 Barriers

We begin to construct cut-off functions for which one has good enough control of its energy. For given connected finite union of cells D, write $G_D(x, y)$ for the Green function on D. Then G_D is symmetric and continuous, and for $v \in \mathcal{F}(K)$ with support in D we have

$$\mathcal{E}(G_D(x, \cdot), v) = -\int_D \Delta_\mu G(x, \cdot) v d\mu(y) = v(x).$$

Since $G_D(x, y) = 0$ if $y \in \partial D$, so we can extend G_D to $K \times K$ by taking it to be zero off $D \times D$. A more general details of Green functions on SG is contained in [33].

Now let $A \subset D$ be connected finite union of cells. Define

$$U(x, A, D) = \int_A G_D(x, y) d\mu(y) \quad x \in K.$$

Then we notice that $U = 0$ on ∂D, U is strictly positive on L, and

$$\Delta_\mu U = -1_A.$$

In other words, for any $v \in \mathcal{F}(K)$ with support in D we have

$$\mathcal{E}(U(\cdot, A, D), v) = \int_A v d\mu.$$

Note also that U is monotone in L and D: if A' and D' be finite union of cells which is connected respectively satisfying $A \subset A' \subset D \subset D'$, then

$$U(x, A, D) \leq U(x, A', D) \leq U(x, A', D').$$

We now investigate some estimates of U.

Lemma 3.3.1. *$\mathcal{E}(U)$ satisfies the bound*

$$\mu(A)\inf_A U \leq \mathcal{E}(U) \leq \mu(A)\sup_A U. \tag{3.3.1}$$

Proof. Note that $\Delta_\mu U = -1_A$. Then by Gauss-Green formula in the domain D,

$$\mathcal{E}_D(U) = -\int_D (\Delta_\mu U) U d\mu + \sum_{\partial D} U \partial_n U = \int_A U d\mu + \sum_{\partial D} U \partial_n U.$$

As $U = 0$ on ∂D we have

$$\mathcal{E}(U) = \mathcal{E}_D(U) = \int_A U d\mu.$$

Since $U \geq 0$, we obtain (3.3.1). □

For any single cell I, recall that $R_I \asymp 2^{-m_I}$ is the diameter of I.

Lemma 3.3.2. *Let I be a single m_I-cell. Then*

$$U(x, I, I) \leq c_1 R_I^{d_w} \qquad \textit{for } x \in K,$$
$$U(x, I, I^*) \geq c_2 R_I^{d_w} \qquad \textit{for } x \in I.$$

Proof. We prove the first inequality. By Lemma 3.3.1, we have

$$\mathcal{E}(U) \leq \mu(I) \sup_I U.$$

Thus, by Lemma 2.1.2 we have

$$\sup_I |U|^2 \leq c_3 R_I^{2\beta} \mathcal{E}(U) \leq c_3 R_I^{2\beta} \mu(I) \sup_I U \leq c_4 R_I^{d_w} \sup_I U$$

and the result is now immediate.

Next we prove the second inequality. Suppose $x \in I^o$. The function $G_{I^*}(x, y)$

is super-harmonic in I, and, by the minimum principle,

$$\inf_{y\in I} G_{I^*}(x,y) = \min_{y\in\partial I} G_{I^*}(x,y).$$

Moreover, the function $G_{I^*}(x,\cdot)$ is harmonic in $I^* \setminus I^o$, so by the behavior near a boundary point of harmonic functions ([27] Lemma 2.7.1) we have

$$G_{I^*}(x,y) \geq c_5 r^{m_I} = c_5 2^{-2m_I\beta} \geq c_6 R_I^{2\beta} \quad \text{for } y \in I^* \setminus I^o.$$

Thus, if $x \in I^o$,

$$\begin{aligned} U(x,I,I^*) &= \int_I G_{I^*}(x,y)d\mu(y) \geq \int_I \inf_{y\in I} G_{I^*}(x,y)d\mu(y) \\ &= \mu(I)\min_{y\in\partial I} G_{I^*}(x,y) \geq c_6 R_I^{2\beta}\mu(I) \geq c_7 R_I^{d_w}. \end{aligned}$$

Finally, by continuity of U on I, we obtain $U(x,I,I^*) \geq c_7 R_I^{d_w}$ on I. □

3.3.2 Weighted inequalities

In this subsection, we will prove the weighted Sobolev and the Poincaré inequalities by defining a new measure, called λ, that involves the barrier constructed in previous subsection. Let us fix two connected finite union of cells $D_1 \subset D_2$, and set

$$w(x) = U(x, D_1, D_2), \quad x \in K. \tag{3.3.2}$$

Then by a similar argument as the proof of Lemma 3.3.2 we have

$$\begin{aligned} \sup_{D_2} |w| &\leq c_1 R_{D_2}^{2\beta}\mu(D_2), \text{ and} \\ \mathcal{E}(w) &\leq \mu(D_2)\sup_{D_2}|w| \leq c_1 R_{D_2}^{2\beta}\mu(D_2)^2. \end{aligned} \tag{3.3.3}$$

For the remainder of this subsection, we assume that for any single m_I-cell I, D_2 contains at least one cell of the same size as I. In other words, we can make I to be included in D_2 by translation.

We will use the following elementary result.

Lemma 3.3.3. *Let x, y, $z \geq 0$. If $x \leq c_1(x^{1/2}z^{1/2} + y)$, then*

$$x \leq 2c_1 y + 4c_1^2 z.$$

We begin by proving a weighted Poincaré inequality.

Lemma 3.3.4 (Weighted Poincaré inequality)**.** *Let I be a single m_I-cell and suppose $f \in \mathcal{F}(K)$. Then we have*

$$\int_I f^2 d\nu_{\langle w\rangle} \leq c_1 (R_{D_2} R_I)^{2\beta} \mu(D_2)^2 \left(\mathcal{E}_{I^*}(f) + R_I^{-d_w} \int_{I^*} f^2 d\mu \right). \tag{3.3.4}$$

Proof. For brevity, put

$$P = \int_I f^2 d\nu_{\langle w\rangle}.$$

Let $\varphi = U(\cdot, I, I^*)$, and write $\Phi_0 = \inf_I \varphi$, $\Phi_1 = \sup_{I^*} \varphi$. Then by Lemma 3.3.2 we have

$$c_2 R_I^{d_w} \leq \Phi_0 \leq \Phi_1 \leq c_3 R_I^{d_w}.$$

Set

$$\begin{aligned} A &= \int_{I^*} f^2 \varphi^2 d\nu_{\langle w\rangle}, \\ B &= \int_{I^*} \varphi^2 d\nu_{\langle f\rangle}, \\ C &= \int_{I^*} f^2 d\mu, \\ D &= \int_{I^*} f^2 d\nu_{\langle \varphi\rangle}, \\ E &= \frac{1}{2}\mathcal{E}_{I^*}(f). \end{aligned}$$

Then

$$P \leq (\inf_I \varphi)^{-2} \int_I f^2 \varphi^2 d\nu_{\langle w\rangle} \leq \Phi_0^{-2} A.$$

If I^* and D_2 are either disjoint or intersect at a single point, then $P = 0$ and

$A = 0$ since $w = 0$ on $(D_2^o)^c$. So we assume that $\mu(I^* \cap D_2) > 0$.
We begin by bounding L. Choose $x_0 \in I$ and set $\widetilde{w}$ defined by

$$\begin{cases} \widetilde{w} = w & \text{if } I^* \nsubseteq D_2, \\ \widetilde{w} = w - w(x_0) & \text{if } I^* \subseteq D_2. \end{cases}$$

In either case we see that there exists a point in I^* at which $\widetilde{w}$ is zero. Set

$$S = \sup_{I^*} \widetilde{w}.$$

Then by Lemma 2.1.2,

$$S^2 \leq c_4 \mathcal{E}(\widetilde{w}, \widetilde{w}) R_{I^*}^{2\beta} = c_4 \mathcal{E}(w) R_{I^*}^{2\beta} \leq c_5 (R_{D_2} R_I)^{2\beta} \mu(D_2)^2.$$

Now the definition of energy measure implies that

$$A = \mathcal{E}_{D_2}(f^2\varphi^2\widetilde{w}, \widetilde{w}) - \frac{1}{2}\mathcal{E}_{D_2}(f^2\varphi^2, \widetilde{w}^2). \tag{3.3.5}$$

We first consider the first term on the right-hand side of (3.3.5). If $I^* \nsubseteq D_2$, then $\widetilde{w} = w = 0$ on ∂D_2 and if $I^* \subseteq D_2$, then $f^2\varphi^2 = 0$ on ∂D_2. So by Gauss-Green formula,

$$\mathcal{E}_{D_2}(f^2\varphi^2\widetilde{w}, \widetilde{w}) = -\int_{D_2} (\Delta_\mu \widetilde{w}) f^2 \varphi^2 \widetilde{w}\ d\mu.$$

Hence

$$\mathcal{E}_{D_2}(f^2\varphi^2\widetilde{w}, \widetilde{w}) \leq \int_{D_1} f^2\varphi^2\widetilde{w} d\mu \leq S \int_{I^*} f^2\varphi^2 d\mu \leq S\Phi_1^2 \int_{I^*} f^2 d\mu = S\Phi_1^2 C.$$

Now consider the second term on the right-hand side of (3.3.5). Set $F = f\varphi$.

Then by Hölder's inequality,

$$\begin{aligned}
&\frac{1}{4}\sum_{x\sim_m y}(F(x)^2-F(y)^2)(\widetilde{w}(x)^2-\widetilde{w}(y)^2) \\
&\quad\le \frac{1}{4}\sum |F(x)+F(y)||F(x)-F(y)||\widetilde{w}(x)^2-\widetilde{w}(y)^2| \\
&\quad\le \left(\frac{1}{4}\sum \left|F(x)^2+F(y)^2\right| |\widetilde{w}(x)^2-\widetilde{w}(y)^2|^2\right)^{\frac{1}{2}} \left(\frac{1}{2}\sum |F(x)-F(y)|^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.3.6}$$

By simple computation, the second term on the last line of (3.3.6) can be bounded

$$\begin{aligned}
\left(\frac{1}{2}\sum_{x\sim_m y}|F(x)-F(y)|^2\right)^{\frac{1}{2}} &\le \left(\frac{1}{4}\sum_{x\sim_m y}(\varphi(x)^2+\varphi(y)^2)(f(x)-f(y))^2\right)^{\frac{1}{2}} \\
&\quad+\left(\frac{1}{4}\sum_{x\sim_m y}(f(x)^2+f(y)^2)(\varphi(x)-\varphi(y))^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Thus, using the representation (2.1.4) we have

$$\begin{aligned}
&-\frac{1}{2}\mathcal{E}_{D_2}(f^2\varphi^2,\widetilde{w}^2) \\
&\le \left(\int_{D_2} f^2\varphi^2 d\nu_{\langle \widetilde{w}^2\rangle}\right)^{1/2} \mathcal{E}_{D_2}(f\varphi,f\varphi)^{1/2} \\
&\le \left(\int_{D_2} f^2\varphi^2 d\nu_{\langle \widetilde{w}^2\rangle}\right)^{1/2} \left[\left(\int_{D_2}\varphi^2 d\nu_{\langle f\rangle}\right)^{1/2}+\left(\int_{D_2} f^2 d\nu_{\langle \varphi\rangle}\right)^{1/2}\right] \\
&\le (2S^2A)^{1/2}(B^{1/2}+D^{1/2}).
\end{aligned}$$

Consequently, we can obtain the bounds of L,

$$\begin{aligned}
A &= -\frac{1}{2}\mathcal{E}_{D_2}(f^2\varphi^2,\widetilde{w}^2)+\mathcal{E}_{D_2}(f^2\varphi^2\widetilde{w},\widetilde{w}) \\
&\le (2S^2A)^{1/2}(B^{1/2}+D^{1/2})+S\Phi_1^2C \\
&\le c_6(S^2A(B+D))^{1/2}+S\Phi_1^2C.
\end{aligned}$$

Then by Lemma 3.3.3,

$$A \le c_7 S^2(B + D) + c_7 S\Phi_1^2 C.$$

We next bound D. We also obtain that D is of the form

$$D = \mathcal{E}_{I^*}(f^2\varphi, \varphi) - \frac{1}{2}\mathcal{E}_{I^*}(f^2, \varphi^2).$$

We can bound each term on the right-hand side by using similar argument as L. First, since $f^2\varphi = 0$ on ∂I^*, by Gauss Green formula

$$\mathcal{E}_{I^*}(f^2\varphi, \varphi) = -\int_{I^*}(\Delta_\mu \varphi) f^2 \varphi d\mu = \int_I f^2\varphi d\mu \le \Phi_1^2 \int_I f^2 d\mu \le \Phi_1^2 C.$$

Secondly, we calculate

$$\begin{aligned}
&\frac{1}{4}\sum_{x\sim_m y}(f(x)^2 - f(y)^2)(\varphi(x)^2 - \varphi(y)^2) \\
&\quad \le 2\left(\frac{1}{4}\sum_{x\sim_m y}(f(x)^2 + f(y)^2)(\varphi(x) - \varphi(y))^2\right)^{1/2} \\
&\quad\quad \times \left(\frac{1}{4}\sum_{x\sim_m y}(\varphi(x)^2 + \varphi(y)^2)(f(x) - f(y))^2\right)^{1/2}
\end{aligned}$$

and hence again using representation (2.1.4) we have

$$\mathcal{E}_{I^*}(f^2, \varphi^2) \le 2\left(\int_{I^*} f^2 d\nu_{\langle\varphi\rangle}\right)^{1/2}\left(\int_{I^*} \varphi^2 d\nu_{\langle f\rangle}\right)^{1/2} = 2B^{1/2}D^{1/2}.$$

So we obtain

$$D \le c_8(B^{1/2}D^{1/2} + \Phi_1 C).$$

Using Lemma 3.3.3 again we conclude that

$$D \le c_9(B + \Phi_1 C).$$

Finally, as $B \leq \Phi_1^2 E$, we deduce that

$$\begin{aligned} A &\leq c_7 S^2(B+D) + c_7 S\Phi_1^2 C \\ &\leq c_7 S^2(\Phi_1^2 E + c_9 B + c_9 \Phi_1 C) + c_7 S \Phi_1^2 C \\ &\leq c_{10} S^2 \Phi_1^2 E + c_{10}(S^2\Phi_1 + S\Phi_1^2)C. \end{aligned}$$

Thus,

$$\begin{aligned} P \leq \Phi_0^{-2} A &\leq c_{10} S^2 \left(\frac{\Phi_1}{\Phi_0}\right)^2 E + c_{10}\left(S^2\left(\frac{\Phi_1}{\Phi_0}\right)^2 \Phi_1^{-1} + S\left(\frac{\Phi_1}{\Phi_0}\right)^2\right) C \\ &\leq c_{11} S^2 E + c_{11}(S^2\Phi_1^{-1} + S)C \\ &\leq c_{11} S^2 E + c_{11}\Phi_1^{-1}(S^2 + S\Phi_1)C. \end{aligned}$$

Since $R_I^\beta \mu(I) \leq R_{D_2}^\beta \mu(D_2)$ from the assumption above this lemma, we have

$$\Phi_1 \leq c_3 R_I^{d_w} \leq c_{12} R_I^{2\beta}\mu(I) \leq c_{12}(R_{D_2} R_I)^\beta \mu(D_2).$$

Therefore, we conclude that

$$\begin{aligned} P &\leq c_{11} S^2 E + c_{11}\Phi_1^{-1}(S^2 + S\Phi_1)C \\ &\leq c_{13}(R_{D_2} R_I)^{2\beta}\mu(D_2)^2\left(E + \Phi_1^{-1} C\right), \end{aligned}$$

which verifies (3.3.4). □

Now we define a measure λ as

$$\lambda_{\langle w\rangle} = \mu + R_{D_2}^{-2\beta}\mu(D_2)^{-1}\nu_{\langle w\rangle}$$

where the coefficient appears in the above definition is needed to cancel terms involving $R_{D_2}^{2\beta}\mu(D_2)$ which arise from the weighted Poincaré inequality. It is clear that $\mu \ll \lambda_{\langle w\rangle}$. Recall that $N(I; D_2) = \mu(D_2)/\mu(I)$, and we denote $N := N(I; D_2) \geq 1$ for convenience.

Lemma 3.3.5. *The measure $\lambda_{\langle w\rangle}$ satisfies bounds*

$$\lambda_{\langle w\rangle}(D_2) \asymp \mu(D_2) \quad \text{and} \quad \mu(I) \le \lambda_{\langle w\rangle}(I) \le c_2(1+N)\mu(I). \tag{3.3.7}$$

Proof. By definition of $\lambda_{\langle w\rangle}$ and (3.3.3),

$$\begin{aligned}\lambda_{\langle w\rangle}(D_2) &= \mu(D_2) + R_{D_2}^{-2\beta}\mu(D_2)^{-1}\nu_{\langle w\rangle}(D_2)\\ &\le \mu(D_2) + c_1 R_{D_2}^{-2\beta}\mu(D_2)^{-1}R_{D_2}^{2\beta}\mu(D_2)^2\\ &\le c_3\mu(D_2)\end{aligned}$$

which proves the first inequality of (3.3.7). If we apply Lemma 3.3.4 with $f \equiv 1$ to deduce

$$\begin{aligned}\lambda_{\langle w\rangle}(I) &= \mu(I) + R_{D_2}^{-2\beta}\mu(D_2)^{-1}\nu_{\langle w\rangle}(I)\\ &\le \mu(I) + c_4(R_{D_2}R_I)^{2\beta}\mu(D_2)^2 R_{D_2}^{-2\beta}\mu(D_2)^{-1}R_I^{-d_w}\int_{I^*} 1 d\mu\\ &= \mu(I) + c_4(R_{D_2}R_I)^{2\beta}\mu(D_2)^2 R_{D_2}^{-2\beta}\mu(D_2)^{-1}R_I^{-2\beta}\mu(I)^{-1}\mu(I^*)\\ &\le c_5(\mu(I)+\mu(D_2))\\ &= c_5(1+N)\mu(I).\end{aligned}$$

Then the second inequality of (3.3.7) follows. □

Corollary 3.3.6. *Let f, I, and I^* be as in Lemma 3.3.4. Then write $f_{I^*} = \mu(I^*)^{-1}\int_{I^*} f d\mu = ⨍_{I^*} f d\mu$, we have*

$$\int_I (f - f_{I^*})^2 d\lambda_{\langle w\rangle} \le c_1 N R_I^{d_w}\mathcal{E}_{I^*}(f) \tag{3.3.8}$$

and

$$\int_I f^2 d\lambda_{\langle w\rangle} \le c_2 N R_I^{d_w}\mathcal{E}_{I^*}(f) + \lambda_{\langle w\rangle}(I)^{-1}\left(\int_I |f| d\lambda_{\langle w\rangle}\right)^2. \tag{3.3.9}$$

Proof. By the Poincaré inequality Lemma 2.1.3 we have

$$\int_{I^*}(f - f_{I^*})^2 d\mu \leq c_3 R_{I^*}^{d_w} \mathcal{E}_{I^*}(f) \leq c_4 R_I^{d_w} \mathcal{E}_{I^*}(f).$$

Thus, applying Lemma 3.3.4 to $f - f_{I^*}$ we deduce

$$\int_I (f - f_{I^*})^2 d\nu_{\langle w\rangle} \leq c_5 (R_{D_2} R_I)^{2\beta} \mu(D_2)^2 \mathcal{E}_{I^*}(f).$$

Consequently, by definition of $\lambda_{\langle w\rangle}$ and assumption $N \geq 1$ we obtain

$$\begin{aligned}
\int_I (f - f_{I^*})^2 d\lambda_{\langle w\rangle} &= \int_I (f - f_{I^*})^2 d\mu + R_{D_2}^{-2\beta} \mu(D_2)^{-1} \int_I (f - f_{I^*})^2 d\nu_{\langle w\rangle} \\
&\leq c_6 \left(R_I^{d_w} + R_I^{2\beta} \mu(D_2) \right) \mathcal{E}_{I^*}(f) \\
&\leq c_7 \left(R_I^{d_w} + N R_I^{d_w} \right) \mathcal{E}_{I^*}(f) \\
&\leq c_8 N R_I^{d_w} \mathcal{E}_{I^*}(f)
\end{aligned}$$

which shows (3.3.8).

Let $b = \lambda_{\langle w\rangle}(I)^{-1} \int_I f d\lambda_{\langle w\rangle} = ⨍_I f d\lambda_{\langle w\rangle}$. Then using (3.3.8) we have

$$\begin{aligned}
\int_I f^2 d\lambda_{\langle w\rangle} &= \int_I (f - b)^2 d\lambda_{\langle w\rangle} + \int_I b^2 d\lambda_{\langle w\rangle} \\
&\leq \int_I (f - f_{I^*})^2 d\lambda_{\langle w\rangle} + b^2 \lambda_{\langle w\rangle}(I) \\
&= \int_I (f - f_{I^*})^2 d\lambda_{\langle w\rangle} + \lambda_{\langle w\rangle}(I)^{-1} \left(\int_I f d\lambda_{\langle w\rangle} \right)^2 \\
&\leq c_8 N R_I^{d_w} \mathcal{E}_{I^*}(f) + \lambda_{\langle w\rangle}(I)^{-1} \left(\int_I f d\lambda_{\langle w\rangle} \right)^2.
\end{aligned}$$

This complete the proof of (3.3.9). □

Using the fact that I is a single cell, we can obtain a sharper result. The proof is based on the rotationally symmetric property of the cell.

Corollary 3.3.7. *Let I be a single m_I-cell and suppose $f \in \mathcal{F}(K)$. Then*

$$\int_I f^2 d\nu_{\langle w \rangle} \le c_1 (R_{D_2} R_I)^{2\beta} \mu(D_2)^2 \left(\mathcal{E}_I(f) + R_I^{-d_w} \int_I f^2 d\mu \right) \tag{3.3.10}$$

and

$$\int_I f^2 d\lambda_{\langle w \rangle} \le c_2 N R_I^{d_w} \mathcal{E}_I(f) + \lambda_{\langle w \rangle}(I)^{-1} \left(\int_I |f| d\lambda_{\langle w \rangle} \right)^2. \tag{3.3.11}$$

Proof. Note that the left-hand sides of (3.3.10) and (3.3.11) do not depend on the values of f outside I. Recall that I^* is the union of the 4 single cells of length m_I, so extend $f|_I$ to a function $\widetilde{f}$ on K by rotation. Then

$$\int_{I^*} \widetilde{f}^2 d\mu = 4 \int_I f^2 d\mu, \qquad \mathcal{E}_{I^*}(\widetilde{f}, \widetilde{f}) \le 4\mathcal{E}_I(f)$$

and (3.3.10) and (3.3.11) now follow from Lemma 3.3.4 and Corollary 3.3.6 for $\widetilde{f}$. □

Next we proceed to a Nash inequality for a single cell.

Lemma 3.3.8. *Let I, f be as in Lemma 3.3.4 and suppose that $\int_I f^2 d\lambda_{\langle w \rangle} < \infty$. Then write $\lambda_{\langle w \rangle}(I)^{-1} \int_I f d\lambda_{\langle w \rangle} = ⨍_I f d\lambda_{\langle w \rangle}$, we have*

$$⨍_I f^2 d\lambda_{\langle w \rangle} \le c_1 A^{d_f/d_w} B^{2\beta/d_w}$$

where

$$A = N^{d_w/d_f} R_I^{2\beta} \mathcal{E}_I(f) + N^{2\beta/d_f} ⨍_I f^2 d\lambda_{\langle w \rangle} \quad \text{and} \quad B = \left(⨍_I |f| d\lambda_{\langle w \rangle} \right)^2.$$

Proof. The result is trivial if $A = 0$, so we may assume $A > 0$. Let $t \in (0, R_I)$. We can find a covering of I by cells I_i such that $t/2 \le R_{I_i} \le t$, $I = \bigcup I_i$, and the I_i^o are disjoint. Note that $\mu(I_i) \asymp R_{I_i}^{d_f} \asymp t^{d_f}$. As $I_i \subset I$, I_i and I are both single cells, write $N_i = N(I_i; D_2)$ we have

$$\begin{aligned} N_i R_{I_i}^{d_w} &\asymp R_{I_i}^{2\beta}\mu(D_2) = R_{I_i}^{2\beta}\mu(D_2)\left(\frac{\mu(I_i)}{\mu(D_2)}\right)^{2\beta/d_f}\left(\frac{\mu(D_2)}{\mu(I_i)}\right)^{2\beta/d_f} \\ &\asymp \mu(D_2)^{d_w/d_f}\left(\frac{\mu(I_i)}{\mu(D_2)}\right)^{2\beta/d_f} \asymp \mu(D_2)^{d_w/d_f}\left(\frac{t^{d_f}}{\mu(D_2)}\right)^{2\beta/d_f}. \end{aligned}$$

and by Lemma 3.3.5,

$$\frac{\lambda_{\langle w\rangle}(I)}{\lambda_{\langle w\rangle}(I_i)} \le \frac{c_2 N\mu(I)}{\mu(I_i)} = c_2\frac{\mu(D_2)}{\mu(I_i)} \asymp c_3\frac{\mu(D_2)}{t^{d_f}}.$$

We apply Corollary 3.3.7 and sum. Then write $d(t) = t^{d_f}/\mu(D_2)$ we obtain

$$\begin{aligned} &\int_I f^2 d\lambda_{\langle w\rangle} \\ &= \sum_i \int_{I_i} f^2 d\lambda_{\langle w\rangle} \\ &\le c_4 \sum_i \left[N_i R_{I_i}^{d_w}\mathcal{E}_{I_i}(f) + \frac{\lambda_{\langle w\rangle}(I)}{\lambda_{\langle w\rangle}(I_i)}\lambda_{\langle w\rangle}(I)^{-1}\left(\int_{I_i}|f|d\lambda_{\langle w\rangle}\right)^2\right] \\ &\le c_5 \sum_i \left[\mu(D_2)^{\frac{d_w}{d_f}} d(t)^{\frac{2\beta}{d_f}}\mathcal{E}_{I_i}(f) + d(t)^{-1}\lambda_{\langle w\rangle}(I)^{-1}\left(\int_{I_i}|f|d\lambda_{\langle w\rangle}\right)^2\right] \\ &\le c_5 \left[\mu(D_2)^{\frac{d_w}{d_f}} d(t)^{\frac{2\beta}{d_f}}\mathcal{E}_{I}(f) + d(t)^{-1}\lambda_{\langle w\rangle}(I)^{-1}\left(\int_{I}|f|d\lambda_{\langle w\rangle}\right)^2\right]. \end{aligned} \tag{3.3.12}$$

Since $\lambda_{\langle w\rangle}(I) \geq \mu(I)$, we deduce

$$\begin{aligned}
&⨍_I f^2 d\lambda_{\langle w\rangle} \\
&\leq c_5 \left[\left(\frac{\mu(D_2)^{\frac{d_w}{d_f}}}{\mu(I)} \right) d(t)^{\frac{2\beta}{d_f}} \mathcal{E}_I(f) + d(t)^{-1} \lambda_{\langle w\rangle}(I)^{-2} \left(\int_I |f| d\lambda_{\langle w\rangle} \right)^2 \right] \\
&= c_6 \left[N^{\frac{d_w}{d_f}} \mu(I)^{\frac{2\beta}{d_f}} d(t)^{\frac{2\beta}{d_f}} \mathcal{E}_I(f) + d(t)^{-1} \lambda_{\langle w\rangle}(I)^{-2} \left(\int_I |f| d\lambda_{\langle w\rangle} \right)^2 \right] \\
&\leq c_6 \left(d(t)^{\frac{2\beta}{d_f}} A + d(t)^{-1} B \right).
\end{aligned}$$

If $t \geq R_I$, then

$$d(t)^{\frac{2\beta}{d_f}} N^{\frac{2\beta}{d_f}} \geq \left(\frac{R_I^{d_f}}{\mu(D_2)} \right)^{\frac{2\beta}{d_f}} \left(\frac{\mu(D_2)}{\mu(I)} \right)^{\frac{2\beta}{d_f}} \geq 1.$$

so the above inequality is trivial. If we choose t_0 so that $d(t_0)^{2\beta/d_f} A = d(t_0)^{-1} B$, then we have $d(t_0) = (B/A)^{d_f/d_w}$. Now let $t = t_0$ and substitute in (3.3.12) to conclude the proof. □

Next, we use this to derive a weighted Sobolev inequality linking the $L^{2+\varepsilon}$ norm of f with respect to $\lambda_{\langle w\rangle}$ to the energy of f.

Lemma 3.3.9 (Weighted Sobolev inequality)**.** *Let I be a single m_I-cell and f be as above. Then for any $q \in (2, 2 + 4\beta/d_w)$ there exists $c_1(q) < \infty$ such that*

$$\left(⨍_I |f|^q d\lambda_{\langle w\rangle} \right)^{2/q} \leq c_1(q) \left(N^{1+\alpha} R_I^{2\beta} \mathcal{E}_I(f) + N^{\alpha} ⨍_I f^2 d\lambda_{\langle w\rangle} \right).$$

where $\alpha = 2\beta/(d_w + 2\beta)$.

Proof. Since $\mathcal{E}_I(f^+, f^+) \leq \mathcal{E}_I(f)$ and $|f| \leq f^+ + f^-$, it suffices to consider non-negative f. Write

$$A_0(f) = N^{d_w/d_f} R_I^{2\beta} \mathcal{E}_I(f) + N^{2\beta/d_f} \fint_I f^2 d\lambda_{\langle w\rangle},$$
$$B_0(f) = \left(\fint_I |f| d\lambda_{\langle w\rangle} \right)^2.$$

Let us first assume that $A_0(f) = 1$. This assumption will be removed at the end of the proof.

Fix $0 < \varepsilon < \min\left(N^{-2\beta/d_f}, \dfrac{1}{2}\right)$. Set $I_i \subset I$ be a finite union of cells with the same size, containing $\{f \geq 2^i\} \cap I$ and satisfying $f(x) + \varepsilon/2^i \geq 2^i$ on I_i. This is possible since f is continuous and self-similarity structure of K (see [52], Theorem 2.1). Without loss of generality, we may assume that $I_{i+1} \subset I_i$. Define

$$p_n = \frac{\lambda_{\langle w\rangle}(I_n)}{\lambda_{\langle w\rangle}(I)} \quad \text{and} \quad f_n = ((f + \varepsilon/2^n) \wedge 2^{n+1}) - ((f + \varepsilon/2^n) \wedge 2^n).$$

Note that $f_n \leq 2^n$ on I, $f_n = 2^n$ on I_{n+1}, and $f_n \leq \varepsilon/2^n$ on $(I_n^o)^c$. Therefore,

$$\begin{aligned} \fint_I f_n d\lambda_{\langle w\rangle} &= \lambda_{\langle w\rangle}(I)^{-1} \left(\int_{I_n} f_n d\lambda_{\langle w\rangle} + \int_{(I_n^o)^c} f_n d\lambda_{\langle w\rangle} \right) \\ &\leq \lambda_{\langle w\rangle}(I)^{-1} \left(2^n \lambda_{\langle w\rangle}(I_n) + \varepsilon/2^n \lambda_{\langle w\rangle}((I_n^o)^c) \right) \\ &\leq \lambda_{\langle w\rangle}(I)^{-1} \left(2^n \lambda_{\langle w\rangle}(I_n) + \varepsilon/2^n \lambda_{\langle w\rangle}(I) \right) \\ &\leq 2^n p_n + \varepsilon/2^n, \end{aligned} \tag{3.3.13}$$

while

$$\begin{aligned} \fint_I f_n^2 d\lambda_{\langle w\rangle} &= \lambda_{\langle w\rangle}(I)^{-1} \left(\int_{I_n} f_n^2 d\lambda_{\langle w\rangle} + \int_{(I_n^o)^c} f_n^2 d\lambda_{\langle w\rangle} \right) \\ &\geq \lambda_{\langle w\rangle}(I)^{-1} \int_{I_{n+1}} f_n^2 d\lambda_{\langle w\rangle} = 2^{2n} p_{n+1}. \end{aligned} \tag{3.3.14}$$

Since $f_n \le f + \varepsilon/2^n$ on I and $\mathcal{E}_I(f_n, f_n) \le \mathcal{E}_I(f)$, we have

$$\begin{aligned} A_0(f_n) &= N^{d_w/d_f} R_I^{2\beta} \mathcal{E}_I(f_n, f_n) + N^{2\beta/d_f} \fint_I f_n^2 d\lambda_{\langle w\rangle} \\ &\le N^{d_w/d_f} R_I^{2\beta} \mathcal{E}_I(f) + 2N^{2\beta/d_f} \fint_I \left[f^2 + (\varepsilon/2^n)^2 \right] d\lambda_{\langle w\rangle} \\ &\le 2A_0(f) + 2N^{2\beta/d_f} (\varepsilon/2^n)^2 \\ &\le c_2. \end{aligned}$$

So, from above (3.3.14) we deduce

$$p_n \le 2^{-2(n-1)} \fint_I f_{n-1}^2 d\lambda_{\langle w\rangle} \le 2^{-2(n-1)} N^{-2\beta/d_f} A_0(f_{n-1}) \le c_3 N^{-2\beta/d_f} 2^{-2n}. \tag{3.3.15}$$

Applying Lemma 3.3.8 to f_n,

$$\fint_I f_n^2 d\lambda_{\langle w\rangle} \le c_4 A_0(f_n)^{d_f/d_w} B_0(f_n)^{2\beta/d_w} \le c_5 B_0(f_n)^{2\beta/d_w}.$$

Using this, (3.3.13), (3.3.14) and (3.3.15) we obtain

$$\begin{aligned} 2^{2n} p_{n+1} &\le \fint_I f_n^2 d\lambda_{\langle w\rangle} \le c_5 B_0(f_n)^{2\beta/d_w} \le c_5 \left(2^n p_n + \varepsilon/2^n\right)^{4\beta/d_w} \\ &\le c_5 (c_3 N^{-2\beta/d_f} 2^{-n} + N^{-2\beta/d_f}/2^n)^{4\beta/d_w} \le c_6 N^{-8\beta^2/d_f d_w} 2^{-4n\beta/d_w}. \end{aligned}$$

So we have

$$p_n \le c_7 N^{-8\beta^2/d_f d_w} 2^{-n(2+4\beta/d_w)} \tag{3.3.16}$$

where $c_7 = c_7(\beta, d_w, d_f)$. Consequently,

$$\begin{aligned}
⨍_I |f|^q d\lambda_{\langle w\rangle} &\le \lambda_{\langle w\rangle}(I)^{-1} \sum_{n=0}^{\infty} \int_{I_{n-1}\setminus I_n^o} |f|^q d\lambda_{\langle w\rangle} \\
&\le \lambda_{\langle w\rangle}(I)^{-1} \sum_{n=0}^{\infty} \int_{I_{n-1}\setminus I_n^o} 2^{nq} d\lambda_{\langle w\rangle} \\
&\le \lambda_{\langle w\rangle}(I)^{-1} \sum_{n=0}^{\infty} 2^{nq} \lambda_{\langle w\rangle}(I_{n-1}) \\
&= \sum_{n=0}^{\infty} 2^{nq} p_{n-1}
\end{aligned}$$

where $I_{-1} = I$. Hence (3.3.16) shows that

$$⨍_I |f|^q d\lambda_{\langle w\rangle} \le c_8 N^{-8\beta^2/d_f d_w} \tag{3.3.17}$$

if $q \in (2, 2+4\beta/d_w)$.

In the general case, let $g := A_0(f)^{-1/2} f$. Then $A_0(g) = 1$ so g satisfies (3.3.17). Since $q \in (2, 2+4\beta/d_w)$ we have

$$\begin{aligned}
0 < -\frac{8\beta^2}{d_f d_w} \cdot \frac{2}{q} + \frac{d_w}{d_f} &\le -\frac{8\beta^2}{d_f d_w} \cdot \frac{d_w}{d_w + 2\beta} + \frac{d_w}{d_f} = \frac{-8\beta^2 + d_w^2 + 2\beta d_w}{d_f(d_w + 2\beta)} \\
&= \frac{(d_w + 4\beta)(d_w - 2\beta)}{d_f(d_w + 2\beta)} = \frac{d_w + 4\beta}{d_w + 2\beta} = 1 + \alpha
\end{aligned}$$

and

$$0 < -\frac{8\beta^2}{d_f d_w} \cdot \frac{2}{q} + \frac{2\beta}{d_f} = -\frac{8\beta^2}{d_f d_w} \cdot \frac{2}{q} + \left(\frac{d_w}{d_f} - 1\right) \le \alpha.$$

Therefore, since $N \ge 1$, we conclude that

$$\begin{aligned}
\left(⨍_I |f|^q d\lambda_{\langle w\rangle}\right)^{2/q} &= A_0(f) \left(⨍_I |g|^q d\lambda_{\langle w\rangle}\right)^{2/q} \le c_9 A_0(f) N^{\frac{-8\beta^2}{d_f d_w} \cdot \frac{2}{q}} \\
&= c_9 N^{\frac{-8\beta^2}{d_f d_w} \cdot \frac{2}{q}} \left(N^{d_w/d_f} R_I^{2\beta} \mathcal{E}_I(f) + N^{2\beta/d_f} ⨍_I f^2 d\lambda_{\langle w\rangle}\right) \\
&\le c_9 \left(N^{1+\alpha} R_I^{2\beta} \mathcal{E}_I(f) + N^{\alpha} ⨍_I f^2 d\lambda_{\langle w\rangle}\right),
\end{aligned}$$

which completes the proof. $\square$

We can also observe the following estimate.

Corollary 3.3.10. *Let f and I be as in Lemma 3.3.9. Then for any $q \in (2, 2+4\beta/d_w)$ there exists $c_1(q) < \infty$ such that*

$$\left(\fint_I |f|^q d\lambda_{\langle w \rangle} \right)^{2/q} \leq c_1 N^{1+\alpha} \left(R_I^{2\beta} \mathcal{E}_I(f) + \fint_I f^2 d\mu \right).$$

Proof. Using Corollary 3.3.7 and the fact that $R_I^{2\beta}\mu(D_2) \asymp N R_I^{d_w}$ we obtain

$$\begin{aligned}
\int_I f^2 d\lambda_{\langle w \rangle} &= \int_I f^2 d\mu + R_{D_2}^{-2\beta} \mu(D_2)^{-1} \int_I f^2 d\nu_{\langle w \rangle} \\
&\leq \int_I f^2 d\mu + c_2 R_I^{2\beta} \mu(D_2) \left(\mathcal{E}_I(f) + R_I^{-d_w} \int_I f^2 d\mu \right) \\
&\leq \int_I f^2 d\mu + c_3 N R_I^{d_w} \left(\mathcal{E}_I(f) + R_I^{-d_w} \int_I f^2 d\mu \right) \\
&\leq c_3 N \left(R_I^{d_w} \mathcal{E}_I(f) + \int_I f^2 d\mu \right).
\end{aligned}$$

Now applying Lemma 3.3.9 and from the fact that $\lambda_{\langle w \rangle}(I) \geq \mu(I)$ we have

$$\begin{aligned}
&\left(\fint_I |f|^q d\lambda_{\langle w \rangle} \right)^{2/q} \\
&\leq c_4 \left(N^{1+\alpha} R_I^{2\beta} \mathcal{E}_I(f) + N^\alpha \fint_I f^2 d\lambda_{\langle w \rangle} \right) \\
&\leq c_4 \left[N^{1+\alpha} R_I^{2\beta} \mathcal{E}_I(f) + N^{1+\alpha} \lambda_{\langle w \rangle}(I)^{-1} \left(R_I^{d_w} \mathcal{E}_I(f) + \int_I f^2 d\mu \right) \right] \\
&\leq c_5 N^{1+\alpha} \left(R_I^{2\beta} \mathcal{E}_I(f) + \fint_I f^2 d\mu \right).
\end{aligned}$$

$\square$

We now show the weighted Sobolev inequality on D_2. We can find a covering of D_2 by cells I_i, where each I_i has the same size, such that $D_2 = \cup_i I_i$ and the I_i^o are disjoint. Then clearly, $\mu(D_2)/\mu(I_i) = \mu(D_2)/\mu(I_j)$ and $R_{I_i} = R_{I_j}$ for all i and j.

Corollary 3.3.11. *Suppose that $f \in \mathcal{F}(K)$. If D_2 is covered by cells I_i, where each I_i has the same size, such that $D_2 = \cup_i I_i$ and the I_i^o are disjoint, then*

$$\left(\fint_{D_2} |f|^q d\lambda_{\langle w \rangle} \right)^{2/q} \leq c_1 N^{1+\alpha} \left(R_I^{2\beta} \mathcal{E}_{D_2}(f) + N \fint_{D_2} f^2 d\mu \right).$$

where $N := \mu(D_2)/\mu(I_i)$ and $R_I := R_{I_i}$ for any i.

Proof. Since $I_i \subset D_2$ for all i, applying Corollary 3.3.10 to each of the I_i, we obtain

$$\begin{aligned}
\fint_{D_2} |f|^q d\lambda_{\langle w \rangle} &= \sum_i \frac{\lambda_{\langle w \rangle}(I_i)}{\lambda_{\langle w \rangle}(D_2)} \left(\fint_{I_i} |f|^q d\lambda_{\langle w \rangle} \right) \\
&\leq c_2 \sum_i \left[N^{1+\alpha} \left(R_I^{2\beta} \mathcal{E}_{I_i}(f) + \fint_{I_i} f^2 d\mu \right) \right]^{q/2} \\
&\leq c_3 N^{q(1+\alpha)/2} \sum_i \left(R_I^{2\beta} \mathcal{E}_{I_i}(f) + \mu(I_i)^{-1} \int_{I_i} f^2 d\mu \right)^{q/2}.
\end{aligned}$$

Since $q/2 > 1$,

$$\begin{aligned}
\fint_{D_2} |f|^q d\lambda_{\langle w \rangle} &\leq c_4 N^{q(1+\alpha)/2} \left[\sum_i \left(R_I^{2\beta} \mathcal{E}_{I_i}(f) + \mu(I_i)^{-1} \int_{I_i} f^2 d\mu \right) \right]^{q/2} \\
&= c_4 N^{q(1+\alpha)/2} \left(R_I^{2\beta} \mathcal{E}_{D_2}(f) + \mu(I_i)^{-1} \int_{D_2} f^2 d\mu \right)^{q/2} \\
&\leq c_5 N^{q(1+\alpha)/2} \left(R_I^{2\beta} \mathcal{E}_{D_2}(f) + N \fint_{D_2} f^2 d\mu \right)^{q/2}.
\end{aligned}$$

□

3.4 Harnack inequality

In this section, we prove the Harnack inequality for non-negative $\mathcal{L}$-harmonic functions. Our basic approach is the ideas of Moser [53] from the general metric measure space case [11]. We use the Moser method by using weighted

Sobolev inequality to estimate the supremum of $\mathcal{L}$-harmonic functions by their averaged L^q-norms. Next by careful choice of appropriate cells and iteration, we will obtain the result.

3.4.1 Caccioppoli type inequality and local boundedness

The aim of this section is to prove two components of the Harnack inequality, namely, Caccioppoli type inequality and local boundedness. We first need a logarithmic Caccioppoli type inequality which is a special case of the following Caccioppoli inequality for minimizers.

Lemma 3.4.1 (Caccioppoli inequality)**.** *Let D be a connected finite union of cells in K, and suppose $u > 0$ is a minimizer of $\mathcal{E}^{\mathcal{L}}(\cdot)$. Let $\gamma \neq 0$, and suppose $\eta \in \mathcal{F}(K)$ with support in D. Then*

$$\int_D u^{-1-\gamma}\eta^2 d\nu_{\langle u\rangle} \leq c_1 \left(\frac{1}{\gamma^2}\int_D u^{1-\gamma} d\nu_{\langle \eta\rangle} + \frac{|\gamma|+1}{\gamma^2}\int_D \eta^2 u^{1-\gamma} d\mu\right). \quad (3.4.1)$$

Proof. Using the homogeneity of (3.4.1) we can replace η by $a\eta$ and so we can assume that $0 \leq \eta \leq 1$ in D. We first assume that $\gamma > 0$. We note that u is continuous, and so $u \geq \delta$ for some $\delta = \delta(u) > 0$ due to compactness of domain. Thus using the homogeneity again, we may assume that $u \geq |\gamma|^{1/(|\gamma|+1)}$.

Let $w = u + \eta^2 u^{-\gamma}$ then by Lemma 2.1.10 $w \in \mathcal{F}(K)$ and so $\mathcal{E}^{\mathcal{L}}(w) < \infty$. Let

$$p_{xy} = \begin{cases} -\dfrac{u(x)^{-\gamma} - u(y)^{-\gamma}}{u(x) - u(y)} & \text{if } u(x) \neq u(y), \\ 0 & \text{if } u(x) = u(y). \end{cases}$$

Then the assumption $u \geq |\gamma|^{1/(|\gamma|+1)}$ implies that

$$u(x) + u(x)^{-\gamma} \geq u(y) + u(y)^{-\gamma} \quad \text{if } u(x) > u(y).$$

Thus, $0 \leq p_{xy} \leq 1$. For brevity, let us denote by $\overline{v}_{xy} = 1/2(v(x)+v(y))$. Then

$(\overline{v}_{xy})^2 = 1/4(v(x)+v(y))^2 \leq 1/2(v(x)^2+v(y)^2) = \overline{v^2}_{xy}$ in our notation. By elementary computation,

$$\begin{aligned} w(x)-w(y) &= u(x)-u(y)+\eta(x)^2u(x)^{-\gamma}-\eta(y)^2u(y)^{-\gamma} \\ &= u(x)-u(y)+\overline{\eta^2}_{xy}(u(x)^{-\gamma}-u(y)^{-\gamma}) \\ &\quad +\overline{u^{-\gamma}}_{xy}(\eta(x)^2-\eta(y)^2) \\ &= (1-\overline{\eta^2}_{xy}p_{xy})(u(x)-u(y))+\overline{u^{-\gamma}}_{xy}(\eta(x)^2-\eta(y)^2). \end{aligned}$$

Since $1-\overline{\eta^2}_{xy}p_{xy} \geq 0$, if $u(x) \neq u(y)$ and $\max(\eta(x),\eta(y)) > 0$ we then have by the convexity of the function $p \mapsto L(p,x,y)$ and structure conditions (3.2.1) of L that

$$\begin{aligned} &L(w(x)-w(y)) \\ &\leq (1-\overline{\eta^2}_{xy}p_{xy})L(u(x)-u(y))+\overline{\eta^2}_{xy}p_{xy}L\left(\frac{\overline{u^{-\gamma}}_{xy}(\eta(x)^2-\eta(y)^2)}{\overline{\eta^2}_{xy}p_{xy}}\right) \\ &\leq (1-\overline{\eta^2}_{xy}p_{xy})L(u(x)-u(y))+\frac{c_0(\overline{u^{-\gamma}}_{xy})^2}{\overline{\eta^2}_{xy}p_{xy}}(\eta(x)^2-\eta(y)^2)^2 \\ &\leq (1-\overline{\eta^2}_{xy}p_{xy})L(u(x)-u(y))+\frac{4c_0\overline{u^{-2\gamma}}_{xy}}{\overline{\eta^2}_{xy}p_{xy}}\overline{\eta^2}_{xy}(\eta(x)-\eta(y))^2 \\ &\leq L(u(x)-u(y))-\frac{\overline{\eta^2}_{xy}p_{xy}}{c_0}(u(x)-u(y))^2+\frac{4c_0\overline{u^{-2\gamma}}_{xy}}{p_{xy}}(\eta(x)-\eta(y))^2. \end{aligned} \tag{3.4.2}$$

Due to the uniform continuity of u, given arbitrary $\varepsilon > 0$ we have

$$|p_{xy}-\gamma u(x)^{-\gamma-1}| \leq \varepsilon$$

for any x, $y \in V_m$ with $x \sim_m y$ if m is large enough. Hence the second term and the third term on the last line of (3.4.2) can be bounded

$$\begin{aligned}
&-\frac{r^{-m}}{2}\sum_{\substack{x\sim_m y\\ y\in D}}\overline{\eta^2}_{xy}p_{xy}(u(x)-u(y))^2\\
&\quad=-\frac{r^{-m}}{2}\sum_{x\in V_m\cap D}\eta(x)^2\sum_{\substack{x\sim_m y\\ y\in D}}p_{xy}(u(x)-u(y))^2\\
&\quad\le\frac{r^{-m}}{2}\sum_{x\in V_m\cap D}\eta(x)^2\left(-\gamma u(x)^{-\gamma-1}+\varepsilon\right)\sum_{\substack{x\sim_m y\\ y\in D}}(u(x)-u(y))^2
\end{aligned}$$

and

$$\begin{aligned}
&\frac{r^{-m}}{2}\sum_{\substack{x\sim_m y\\ y\in D}}\frac{\overline{u^{-2\gamma}}_{xy}}{p_{xy}}(\eta(x)-\eta(y))^2\\
&\quad=\frac{r^{-m}}{2}\sum_{x\in V_m\cap D}u(x)^{-2\gamma}\sum_{\substack{x\sim_m y\\ y\in D}}\frac{(\eta(x)-\eta(y))^2}{p_{xy}}\\
&\quad\le\frac{r^{-m}}{2}\sum_{x\in V_m\cap D}u(x)^{-2\gamma}(\gamma^{-1}u(x)^{\gamma+1}+\varepsilon)\sum_{\substack{x\sim_m y\\ y\in D}}(\eta(x)-\eta(y))^2,
\end{aligned}$$

respectively. We notice that all of the following terms exist and are finite.

$$\int_D\eta^2u^{-\gamma-1}d\nu_{\langle u\rangle},\ \int_D\eta^2d\nu_{\langle u\rangle},\ \int_D u^{1-\gamma}d\nu_{\langle\eta\rangle},\ \int_D u^{-2\gamma}d\nu_{\langle\eta\rangle}<\infty.$$

On the other hand, since $u^{1-\gamma}\ge|\gamma|u^{-2\gamma}$, and by the growth condition (3.2.1) of G we obtain

$$\begin{aligned}
G(w)&=G(u)+G(u+\eta^2u^{-\gamma})-G(u)\\
&\le G(u)+c_2\eta^2u^{-\gamma}(u+\eta^2u^{-\gamma})\\
&\le G(u)+c_2\left(1+\frac{1}{|\gamma|}\right)\eta^2u^{1-\gamma}.
\end{aligned}$$

Therefore, since u is a minimizer, $u = w$ on ∂K, and $\mathcal{E}^{\mathcal{L}}(w) < \infty$, we have

$$\begin{aligned}\mathcal{E}^{\mathcal{L}}(u) &\le \mathcal{E}^{\mathcal{L}}(w)\\ &= \limsup_{m\to\infty} \frac{r^{-m}}{2} \sum_{\substack{x\sim_m y\\ x,y\in V_m}} L(w(x)-w(y)) + \int_K G(w)d\mu\\ &\le \mathcal{E}^{\mathcal{L}}(u) + \frac{1}{c_0}\int_D \eta^2\left(-\gamma u^{-\gamma-1}+\varepsilon\right) d\nu_{\langle u\rangle}\\ &\quad + 4c_0\int_D u^{-2\gamma}\left(\gamma^{-1}u^{\gamma+1}+\varepsilon\right) d\nu_{\langle \eta\rangle} + \int_D c_2\left(1+\frac{1}{|\gamma|}\right)\eta^2 u^{1-\gamma}d\mu.\end{aligned}$$

So after subtracting $\mathcal{E}^{\mathcal{L}}(u) < \infty$ from both sides we obtain

$$\begin{aligned}\frac{\gamma}{c_0}\int_D \eta^2 u^{-\gamma-1}d\nu_{\langle u\rangle} &\le \frac{4c_0}{\gamma}\int_D u^{1-\gamma}d\nu_{\langle \eta\rangle} + \int_D c_2\left(1+\frac{1}{|\gamma|}\right)\eta^2u^{1-\gamma}d\mu\\ &\quad + \varepsilon\left(\frac{1}{c_0}\int_D \eta^2 d\nu_{\langle u\rangle} + 4c_0\int_D u^{-2\gamma}d\nu_{\langle\eta\rangle}\right)\end{aligned}$$

Letting ε tend to zero, the result follows. The argument for the $\gamma < 0$ is similar so we omit it to avoid the redundancy. This finishes the proof. $\square$

Lemma 3.4.2 (Logarithmic Caccioppoli inequality)**.** *Assume that $u > 0$ is a minimizer of $\mathcal{E}^{\mathcal{L}}(\cdot)$ and let $w = \log u$. Then for any single m_I-cell I with $I^* \subset K$, there exists c_1, not depending on u, such that*

$$\mathcal{E}_I(w) \le c_1 R_I^{-d_w}\mu(I).$$

Proof. Let $\phi = U(\cdot, I, I^*)$. Then by Lemma 3.3.2 $\phi \ge c_2R_I^{d_w}$ on I and $\phi \le c_3R_I^{d_w}$ on I^*. So by applying Lemma 2.1.10 with $f(x) = \log x$ and Lemma 3.4.1 with $\gamma = 1$ we have

$$\begin{aligned}\mathcal{E}_I(w) = \int_I d\nu_{\langle w\rangle} &\le c_2R_I^{-2d_w}\int_{I^*}\phi^2 d\nu_{\langle w\rangle} = c_2R_I^{-2d_w}\int_{I^*}\phi^2u^{-2}d\nu_{\langle u\rangle}\\ &\le c_4R_I^{-2d_w}\left(\int_{I^*}d\nu_{\langle\phi\rangle} + \int_{I^*}\phi^2d\mu\right)\\ &\le c_5R_I^{-2d_w}\left(\mathcal{E}_{I^*}(\phi,\phi) + R_I^{2d_w}\mu(I)\right).\end{aligned}$$

From Lemma 3.3.1 and 3.3.2 we see that $\mathcal{E}_{I^*}(\phi,\phi) \leq c_4 R_I^{d_w}\mu(I)$. Therefore,

$$\mathcal{E}_I(w) \leq c_5 R_I^{-2d_w}\left(R_I^{d_w}\mu(I) + R_I^{2d_w}\mu(I)\right) \leq c_6 R_I^{-d_w}\mu(I).$$

□

Let u be $\mathcal{L}$-harmonic and non-negative in K. By looking at $u+\varepsilon$ for $\varepsilon > 0$ and letting $\varepsilon \to 0$ we may without loss of generality suppose u is strictly positive.

Now we are ready to prove the following local boundedness for $\mathcal{L}$-harmonic functions. To do this, we construct a family of shrinking cells and using the weighted Sobolev inequality to find a recursive relation for averaged L^q-norms of solutions. We first discuss the construction of a family of shrinking cells in more detail. Let I be a single m_I-cell with $I^* \subset K$ and let $0 \leq k \leq \infty$. Since I has three boundary points, so there are three cells of length $m_I + k$ that meet the boundaries of I. We define Q_k as the union of I and these three cells. Then obviously $Q_0 = I^*$, $Q_\infty = I$ and $Q_{k+1} \subset Q_k$. We can assume that Q_k is covered by cells I_i^k, where each I_i^k has the same size with length $m_I + k$, such that $Q_k = \cup_i I_i^k$ and the $(I_i^k)^o$ are disjoint. Then $N_k := N(I_i^k; Q_k) = 2^{kd_f} + 3 \leq 2^{(k+1)d_f} = c_1(d_f)2^{kd_f}$.

The following is the local boundedness of Sierpinski gasket version for $\mathcal{L}$-harmonic functions when the domain is a single cell.

Lemma 3.4.3. *(Local boundedness) Let I be a single m_I-cell with $I^* \subset K$ and let v be either u or u^{-1}. If $0 < q < 2$, then there exists $c_1 > 0$ such that*

$$\sup_I v^{2q} \leq c_1\left(R_I^{2\beta}\mathcal{E}_{I^*}(v^q, v^q) + ⨍_{I^*} v^{2q}\,d\mu\right).$$

Proof. For brevity, let us denote $\mu(Q_k)$ and R_{Q_k} by μ_k and R_k respectively.

Note that $R_k = (2+2^k)2^{-m_I-k}$ and $\mu_k = (3+3^k)3^{-m_I-k}$. For $0 \le k < \infty$ let

$$w_k = U(\cdot, Q_{k+1}, Q_k), \quad \text{and} \quad \lambda_{\langle w_k \rangle} = \mu + R_k^{-2\beta}\mu_k^{-1}\nu_{\langle w_k \rangle}. \tag{3.4.3}$$

Then $w_k \le U(\cdot, I^*, I^*) \le c_2 R_I^{d_w}$ on K by Lemma 3.3.2. Let $f = u^p$, where $p \in \mathbb{R}$, $p \ne 1/2$. We notice that Corollary 3.3.11 can be applied for f by replacing D_2 to Q_k and w to w_k. So, applying Corollary 3.3.11 with $t \in (2, 2+4\beta/d_w)$, we have

$$\begin{aligned} \left(\fint_{Q_k} |f|^t d\lambda_{\langle w \rangle} \right)^{2/t} &\le c_3 N_k^{1+\alpha} \left(R_{I_i^k}^{2\beta} \mathcal{E}_{Q_k}(f) + N_k \fint_{Q_k} f^2 d\mu \right) \\ &\le c_4 2^{k(1+\alpha)d_f} \left(R_{I_i^k}^{2\beta} \mathcal{E}_{Q_k}(f) + 2^{kd_f} \fint_{Q_k} f^2 d\mu \right). \end{aligned} \tag{3.4.4}$$

If $x \in Q_{k+1}$, then there exists a single cell J with length $m_J = m_I + (k+1)$ such that $x \in J \subset Q_{k+1}$. Then $J^* \subset Q_k$, so by Lemma 3.3.2 again we have

$$w_k = U(\cdot, Q_{k+1}, Q_k) \ge U(\cdot, J, J^*) \ge c_5 R_J^{d_w} \quad \text{on } Q_{k+1}.$$

Thus, using (3.4.3), Lemma 2.1.10 and 3.4.1 with $\gamma = 1-2p$ we deduce that

$$\begin{aligned} \mathcal{E}_{Q_{k+1}}(f) &= \int_{Q_{k+1}} d\nu_{\langle f \rangle} \le c_6 R_J^{-2d_w} \int_{Q_{k+1}} w_k^2 d\nu_{\langle f \rangle} \\ &\le c_6 R_J^{-2d_w} \int_{Q_k} w_k^2 d\nu_{\langle f \rangle} = c_4 R_J^{-2d_w} \int_{Q_k} p^2 w_k^2 u^{2p-2} d\nu_{\langle u \rangle} \\ &\le c_7(p) R_J^{-2d_w} \left(\int_{Q_k} u^{2p} d\nu_{\langle w_k \rangle} + \int_{Q_k} w_k^2 u^{2p} d\mu \right) \\ &\le c_8 R_J^{-2d_w} \left(\int_{Q_k} f^2 d\nu_{\langle w_k \rangle} + R_I^{2d_w} \int_{Q_k} f^2 d\mu \right) \\ &= c_8 R_J^{-2d_w} \left[R_k^{2\beta}\mu_k \int_{Q_k} f^2 d\lambda_{\langle w_k \rangle} + (R_I^{2d_w} - R_k^{2\beta}\mu_k) \int_{Q_k} f^2 d\mu \right] \\ &\le c_8 \frac{R_k^{2\beta}\mu_k}{R_J^{2d_w}} \int_{Q_k} f^2 d\lambda_{\langle w_k \rangle}. \end{aligned}$$

Here, we use the relation $R_k^{2\beta}\mu_k \ge R_I^{2\beta}\mu(I) \ge R_I^{2d_w}$ on the last inequality.

From the definition of R_k and μ_k, we can easily check that $R_{I_i^{k+1}}/R_J = 1$, $R_{k+1}/R_J \le R_k/R_J \le 2(2+2^k) \le c_9 2^k$, $\mu_k/\mu(J) \le 3(3+3^k) \le c_{10}2^{kd_f}$, and $\lambda_{\langle w_k \rangle}(Q_k) \le c_{11}\mu_k$ (Lemma 3.3.5). Therefore,

$$\begin{aligned} R_{I_i^{k+1}}^{2\beta}\mathcal{E}_{Q_{k+1}}(f) &\le c_8 \frac{(R_{I_i^{k+1}}R_k)^{2\beta}\mu_k}{R_J^{2d_w}} \int_{Q_k} f^2 d\lambda_{\langle w_k \rangle} \\ &\le c_{12}\left(\frac{R_{I_i^{k+1}}R_k}{R_J^2}\right)^{2\beta}\left(\frac{\mu_k}{\mu(J)}\right)^2 ⨍_{Q_k} f^2 d\lambda_{\langle w_k \rangle} \\ &\le c_{13}2^{2k\beta+2kd_f} ⨍_{Q_k} f^2 d\lambda_{\langle w_k \rangle}. \end{aligned} \tag{3.4.5}$$

Moreover, since $\mu_k/\mu_{k+1} \le 4$, we have

$$⨍_{Q_{k+1}} f^2 d\mu \le c_{14} ⨍_{Q_k} f^2 d\lambda_{\langle w_k \rangle}, \tag{3.4.6}$$

so applying (3.4.5) and (3.4.6) to (3.4.4) we obtain

$$\begin{aligned} \left(⨍_{Q_{k+1}} |f|^t d\lambda_{\langle w_{k+1} \rangle}\right)^{2/t} &\le c_{14}2^{k[(1+\alpha)d_f+2\beta+2d_f]} ⨍_{Q_k} f^2 d\lambda_{\langle w_k \rangle} \\ &\le c_{14}2^{6k} ⨍_{Q_k} f^2 d\lambda_{\langle w_k \rangle}. \end{aligned} \tag{3.4.7}$$

Choose $q' > 0$ such that $\inf_{l\in\mathbb{Z}} |q'(t/2)^l - 1/2| \ge c_{15} > 0$. First set $q_0 = q'(t/2)^{-i}$ for some i. Let $p_k = 2q_0(t/2)^k$ for $k \ge 0$ and write

$$\Psi_k = \left(⨍_{Q_k} v^{p_k} d\lambda_{\langle w_k \rangle}\right)^{1/p_k}$$

Note that $p_{k+1}/t = p_k/2 \ne 1/2$. Applying (3.4.7) to $f = v^{p_{k+1}/t} = v^{p_k/2}$ we have

$$\begin{aligned}\Psi_{k+1}^{2p_{k+1}/t} &= \left(\fint_{Q_{k+1}} v^{p_{k+1}} d\lambda_{\langle w_{k+1}\rangle}\right)^{2/t} = \left(\fint_{Q_{k+1}} |f|^t d\lambda_{\langle w_{k+1}\rangle}\right)^{2/t} \\ &\le c_{14} 2^{6k} \fint_{Q_k} f^2 d\lambda_{\langle w_k\rangle} = c_{14} 2^{6k} \fint_{Q_k} v^{p_k} d\lambda_{\langle w_k\rangle} \\ &= c_{14} 2^{6k} \Psi_k^{p_k}.\end{aligned}$$

Thus,

$$\Psi_{k+1} \le \left(c_{14} 2^{6k}\right)^{t/2p_{k+1}} \Psi_k^{p_k t/2p_{k+1}} = \left(c_{14} 2^{6k}\right)^{t/2p_{k+1}} \Psi_k$$

or for every l

$$\log_2 \Psi_l \le \log_2 \Psi_0 + \sum_{k=0}^{l-1} \frac{c_{16}(t) + 3tk}{p_k}. \tag{3.4.8}$$

Since

$$\left(\int_{Q_k} v^{p_k} d\mu\right)^{1/p_k} \le \left(\int_{Q_k} v^{p_k} d\lambda_{\langle w_k\rangle}\right)^{1/p_k}$$

we have $\sup_I v = \sup_{Q_\infty} v \le \limsup_{k\to\infty} \Psi_k$. Therefore, as (3.4.8) converges, we obtain

$$\sup_I v \le c_{17}\Psi_0 = c_{17}\left(\fint_{I^*} v^{2q_0} d\lambda_{\langle w_0\rangle}\right)^{1/2q_0}.$$

Now let $q \in (0,2)$. Take $q_0 = q' S^{-i} < q$. Then by Hölder's inequality we have

$$\fint_{I^*} v^{2q_0} d\lambda_{\langle w_0\rangle} \le c_{18}(q)\left(\fint_{I^*} v^{2q} d\lambda_{\langle w_0\rangle}\right)^{q_0/q}.$$

Hence we obtain

$$\sup_I v^{2q} \le c_{19} \fint_{I^*} v^{2q} d\lambda_{\langle w_0\rangle}.$$

Consequently, by Corollary 3.3.7 we conclude that

$$\begin{aligned}
\sup_I v^{2q} &\le c_{19} \fint_{I^*} v^{2q} d\lambda_{\langle w_0\rangle} \\
&\le c_{20}\mu(I^*)^{-1}\left(\int_{I^*} v^{2q}d\mu + R_I^{-d_w}\int_{I^*} v^{2q}d\nu_{\langle w_0\rangle}\right) \\
&\le c_{21}\mu(I^*)^{-1}\left[\int_{I^*} v^{2q}d\mu + \frac{R_I^{2d_w}}{R_I^{d_w}}\left(\mathcal{E}_{I^*}(v^q,v^q) + R_I^{-d_w}\int_{I^*} v^{2q}d\mu\right)\right] \\
&= c_{21}\mu(I^*)^{-1}\left(R_I^{d_w}\mathcal{E}_{I^*}(v^q,v^q) + \int_{I^*} v^{2q}d\mu\right) \\
&\le c_{22}\left(R_I^{2\beta}\mathcal{E}_{I^*}(v^q,v^q) + \fint_{I^*} v^{2q}d\mu\right).
\end{aligned}$$

□

3.4.2 Harnack inequality

We now follow the ideas of Moser [53]. To use the Moser's iteration method, we need to construct suitable choice of 'balls' $\{B_k\}_{0\le k<\infty}$ growing inductively. For a given single m_I-cell I with $I^* \subset K$, construct inductively B_k, $0 \le k < \infty$ the finite union of cells as follows: First, let $B_0 = I$. As I has three boundary points, so there are three cells of length $m_I + 2$ that meet the boundaries of I. We define B_1 as the union of I and these three cells. In general, for given $0 < i < \infty$, if $2^i - 1 \le k \le 2^{i+1} - 3$ we consider all cells of length $m_I + 2i$ that meet at the boundaries of B_k. Then we define B_{k+1} be the union of B_k and these cells. If $k = 2^{i+1} - 2$, we consider all cells of length $m_I + 2(i+1)$ that meet at the boundaries of B_k. Then similarly B_{k+1} be defined as the union of B_k and these cells inductively. Then we can easily check that $B_k \subset B_{k+1} \subset I^*$ for any k.

In this subsection, for simplicity we denote $\mu(B_k)$ and R_{B_k} by μ_k and R_k respectively. We first obtain a more general result for local boundedness to link the L^∞ norms of u.

Lemma 3.4.4. *For each* $0 < i < \infty$ *and* $2^i - 1 \le k \le 2^{i+1} - 2$*, let*

$$w_k = U(\cdot, B_k, B_{k+1}), \qquad \text{and} \qquad \lambda_{\langle w_k \rangle} = \mu + R_{k+1}^{-2\beta} \mu_{k+1}^{-1} \nu_{\langle w_k \rangle}. \tag{3.4.9}$$

Then if $0 < q < 1/3$*,*

$$\sup_{B_k} v^{2q} \le c_1 2^{4(\beta + d_f)i} \fint_{B_{k+1}} v^{2q} d\lambda_{\langle w_k \rangle}.$$

Proof. Note that $w_k \le U(\cdot, I^*, I^*) \le c_2 R_I^{d_w}$ on K by Lemma 3.3.2. Consider all cells of length $m_I + 2i + 3 = m_I + 2(i+1) + 1$ that meet at the boundaries of B_k. Let B'_k be the union of B_k and these cells. Then since $m_I + 2(i+1) < m_I + 2i + 3$, $B_k \subset B'_k \subset B_{k+1}$. Hence for each single cell $J \subset B_k$ with length $m_I + 2i + 3$, we have $J^* \subset B'_k$. Thus, by Lemma 3.4.3,

$$\sup_J v^{2q} \le c_3 \left(R_J^{2\beta} \mathcal{E}_{J^*}(v^q, v^q) + \fint_{J^*} v^{2q} d\mu \right).$$

We notice that $\mu(B'_k)/\mu(J^*) \le \mu(I^*)/\mu(J^*) = 2^{-m_I d_f} / 2^{-(m_I + 2i + 3) d_f} \le c_4 2^{2 d_f i}$. Therefore, an easy covering argument gives us

$$\sup_{B_k} v^{2q} \le c_5 \left(R_J^{2\beta} \mathcal{E}_{B'_k}(v^q, v^q) + 2^{2 d_f i} \fint_{B'_k} v^{2q} d\mu \right). \tag{3.4.10}$$

If $x \in B'_k$, then there exist a single $(m_I + 2(i+1))$-cell $J_0 \subset B_k$, and $J_0^* \subset B_{k+1}$ such that $x \in J_0^*$. So by Lemma 3.3.2 we have $w_k \ge c_6 R_{J_0}^{d_w}$ on B'_k. Recall that $v = u$ or $v = u^{-1}$, and $w_k = 0$ on ∂B_{k+1}. So if $v = u$ then we have by (3.4.9), Lemma 2.1.10 with $f(x) = x^q$ and Lemma 3.4.1 with $\gamma = 1 - 2q$ (We can achieve the same result when $v = u^{-1}$ by applying Lemma 2.1.10 with

$f(x) = x^{-q}$ and Lemma 3.4.1 with $\gamma = 1 + 2q$),

$$\begin{aligned}
\mathcal{E}_{B'_k}(v^q, v^q) &= \int_{B'_k} d\nu_{\langle v^q \rangle} \leq c_6 R_{J_0}^{-2d_w} \int_{B'_k} w_k^2 d\nu_{\langle v^q \rangle} \\
&\leq c_6 R_{J_0}^{-2d_w} \int_{B_{k+1}} w_k^2 d\nu_{\langle v^q \rangle} = c_6 q^2 R_{J_0}^{-2d_w} \int_{B_{k+1}} w_k^2 u^{2q-2} d\nu_{\langle u \rangle} \\
&\leq c_7(q) R_{J_0}^{-2d_w} \left(\int_{B_{k+1}} u^{2q} d\nu_{\langle w_k \rangle} + \int_{B_{k+1}} w_k^2 u^{2q} d\mu \right) \\
&\leq c_8 R_{J_0}^{-2d_w} \left(\int_{B_{k+1}} v^{2q} d\nu_{\langle w_k \rangle} + R_I^{2d_w} \int_{B_{k+1}} v^{2q} d\mu \right) \\
&\leq c_8 \frac{R_{k+1}^{2\beta} \mu_{k+1}}{R_{J_0}^{2d_w}} \int_{B_{k+1}} v^{2q} d\lambda_{\langle w_k \rangle}.
\end{aligned}$$

Here, we used the definition of λ, (3.4.9), and the relation $R_{k+1}^{2\beta}\mu_{k+1} \geq R_I^{2\beta}\mu(I) \geq R_I^{2d_w}$ on the last inequality. Since $\lambda_{\langle w_k \rangle}(B_{k+1}) \leq c_9 \mu(B_{k+1})$, $R_J/R_{J_0} \leq c_{10} 2^{-(m_I+2i+3)}/2^{-(m_I+2i+2)} = 1/2$,

$$\frac{R_{k+1}}{R_{J_0}} \leq \frac{R_{I^*}}{R_{J_0}} = \frac{3 \cdot 2^{-m_I}}{2^{-(m_I+2i+2)}} \leq c_{11} 2^{2i},$$

and

$$\frac{\mu_{k+1}}{\mu(J_0)} \leq \frac{\mu(I^*)}{\mu(J_0)} = \frac{4 \cdot 2^{-m_I d_f}}{2^{-(m_I+2i+2)d_f}} \leq c_{12} 2^{2i d_f},$$

so from these facts we obtain

$$\begin{aligned}
R_J^{2\beta} \mathcal{E}_{B'_k}(v^q, v^q) &\leq c_8 \frac{(R_J R_{k+1})^{2\beta} \mu_{k+1}}{R_{J_0}^{2d_w}} \int_{B_{k+1}} v^{2q} d\lambda_{\langle w_k \rangle} \\
&\leq c_{13} \left(\frac{R_J R_{k+1}}{R_{J_0}^2} \right)^{2\beta} \left(\frac{\mu_{k+1}}{\mu(J_0)} \right)^2 ⨍_{B_{k+1}} v^{2q} d\lambda_{\langle w_k \rangle} \qquad (3.4.11) \\
&\leq c_{14} 2^{4(\beta+d_f)i} ⨍_{B_{k+1}} v^{2q} d\lambda_{\langle w_k \rangle}.
\end{aligned}$$

Moreover,

$$\fint_{B'_k} v^{2q} d\mu \le \frac{\mu(B_{k+1})}{\mu(B'_k)} \fint_{B_{k+1}} v^{2q} d\mu \le \frac{\mu(I^*)}{\mu(I)} \fint_{B_{k+1}} v^{2q} d\mu \le 4 \fint_{B_{k+1}} v^{2q} d\lambda_{\langle w_k \rangle}, \tag{3.4.12}$$

so applying (3.4.11) and (3.4.12) to (3.4.10) we have

$$\begin{aligned}\sup_{B_k} v^{2q} &\le c_{15} \left(2^{4(\beta+d_f)i} \fint_{B_{k+1}} v^{2q} d\lambda_{\langle w_k \rangle} + 2^{2d_f i} \fint_{B_{k+1}} v^{2q} d\lambda_{\langle w_k \rangle} \right) \\ &\le c_{15} 2^{4(\beta+d_f)i} \fint_{B_{k+1}} v^{2q} d\lambda_{\langle w_k \rangle}.\end{aligned}$$

□

Set $I_1 = I$ and $I_2 = I^*$. Note that I_2 is the union of I_1 and all cells of the same size as I that meet at the boundaries of I_1. Likewise, we can define I_{k+1} as the union of I_k and all cells of the same size as I that meet at the boundaries of I_k. Recall that $v = u$ or $v = u^{-1}$. Let $\alpha(k) = 1/\mu(I_k) \int_{I_k} \log v d\mu$.

Lemma 3.4.5. *Let $w = \log v$ and $I = I_1$ be a single m_I-cell with $I_4 \subset K$. For given $0 < \varepsilon_0 < M$, let $B_k^M \subset B_k$ be finite union of cells such that $\{|w - \alpha(3)| \ge M\} \cap B_k \subset B_k^M$ and $|w - \alpha(3)| \ge M - \varepsilon_0$ on B_k^M. Then*

$$\lambda_{\langle w_k \rangle}(B_k^M) \le \frac{c_1 \mu(I)}{(M - \varepsilon_0)^2}. \tag{3.4.13}$$

Proof. Since $|w - \alpha(3)| \ge M - \varepsilon_0$ on B_k^M and $B_k^M \subset B_k \subset I_2$,

$$\begin{aligned}\lambda_{\langle w_k \rangle}(B_k^M) &= \int_{B_k^M} d\lambda_{\langle w_k \rangle} \le \int_{B_k^M} \left| \frac{w - \alpha(3)}{M - \varepsilon_0} \right|^2 d\lambda_{\langle w_k \rangle} \\ &\le \int_{I_2} \left| \frac{w - \alpha(3)}{M - \varepsilon_0} \right|^2 d\lambda_{\langle w_k \rangle}.\end{aligned}$$

Let $J_i \subset I_2$, $1 \le i \le 4$ be a single cell of the same size as I such that $\cup_{i=1}^4 J_i = I_2$. We note that for any $0 \le k < \infty$, $R_{J_i} = R_I \le R_k \le 3R_I$ and $\mu(J_i) = \mu(I) \le \mu_k \le 4\mu(I)$. Since $I \subset B_k$, we can make J_i, $1 \le i \le 4$, to be included in B_k by translation. Hence, we apply Lemma 3.3.4 with $D_1 = B_k$,$D_2 = B_{k+1}$ to $w - \alpha(3)$ for each J_i, then we have

$$\begin{aligned}
&\int_{J_i} (w - \alpha(3))^2 d\nu_{\langle w_k \rangle} \\
&\quad \le c_2 (R_{k+1} R_{J_i})^{2\beta} \mu_{k+1}^2 \left(\mathcal{E}_{J_i^*}(w) + R_{J_i}^{-d_w} \int_{J_i^*} (w - \alpha(3))^2 d\mu \right) \\
&\quad \le c_3 R_I^{2d_w} \left(\mathcal{E}_{J_i^*}(w) + R_I^{-d_w} \int_{J_i^*} (w - \alpha(3))^2 d\mu \right).
\end{aligned}$$

By the Poincaré inequality Lemma 2.1.3 we have

$$\int_{J_i^*} (w - \alpha(3))^2 d\mu \le \int_{I_3} (w - \alpha(3))^2 d\mu \le c_4 R_I^{d_w} \mathcal{E}_{I_3}(w)$$

and hence

$$\int_{I_2} (w - \alpha(3))^2 d\nu_{\langle w_k \rangle} = \sum_{i=1}^{4} \int_{J_i} (w - \alpha(3))^2 d\nu_{\langle w_k \rangle} \le c_5 R_I^{2d_w} \mathcal{E}_{I_3}(w).$$

Therefore, we deduce that

$$\begin{aligned}
\int_{I_2} (w - \alpha(3))^2 d\lambda_{\langle w_k \rangle} &= \int_{I_2} (w - \alpha(3))^2 d\mu + R_{k+1}^{-2\beta} \mu_{k+1}^{-1} \int_{I_2} (w - \alpha(3))^2 d\nu_{\langle w_k \rangle} \\
&\le c_6 \left(R_I^{d_w} \mathcal{E}_{I_3}(w) + R_{k+1}^{-2\beta} \mu_{k+1}^{-1} R_I^{2d_w} \mathcal{E}_{I_3}(w) \right) \\
&\le c_7 R_I^{d_w} \mathcal{E}_{I_3}(w).
\end{aligned}$$

and by Lemma 3.4.2, $\mathcal{E}_{I_3}(w) \le c_8 R_I^{-d_w} \mu(I)$, so we obtain (3.4.13). □

For fixed $a > 0$, put $\widetilde{u} := a^{-1} u$, $\widetilde{L}(p, x, y) := \frac{1}{a^2} L(ap, x, y)$, and $\widetilde{G}(z, x) := \frac{1}{a^2} G(az, x)$. Then $\widetilde{L}$ is convex with respect to p-variable, and we can also easily check that $\widetilde{L}$ and $\widetilde{G}$ satisfy the structure conditions (3.2.1). By definition of

$\widetilde{L}$ and $\widetilde{G}$, we can define energy of $\widetilde{L}$ such that

$$\begin{aligned}\mathcal{E}^{\widetilde{\mathcal{L}}}(\widetilde{u}) &:= \limsup_{m\to\infty} \frac{r^{-m}}{2} \sum_{\substack{x\sim_m y\\ x,y\in V_m\cap K}} \widetilde{L}(\widetilde{u}(x)-\widetilde{u}(y),x,y) + \int_K \widetilde{G}(\widetilde{u},x)d\mu \\ &= \frac{1}{a^2}\left(\limsup_{m\to\infty} \frac{r^{-m}}{2} \sum_{\substack{x\sim_m y\\ x,y\in V_m\cap K}} L(u(x)-u(y),x,y) + \int_K G(u,x)d\mu\right) \\ &= \frac{1}{a^2}\mathcal{E}^{\mathcal{L}}(u).\end{aligned}$$

Hence we conclude that u is a minimizer of $\mathcal{E}^{\mathcal{L}}(\cdot)$ if and only if $\widetilde{u}$ is a minimizer of $\mathcal{E}^{\widetilde{\mathcal{L}}}(\cdot)$. So without loss of generality, for $v = u$ or $v = u^{-1}$ we can assume that $\alpha(3) = 1/\mu(I_3)\int_{I_3} \log v d\mu = 0$ by multiplying u for some constant $a > 0$.

Define

$$\varphi_k = \sup_{B_k} \log v.$$

Lemma 3.4.6. *For each $0 \le i < \infty$ and $2^i - 1 \le k \le 2^{i+1} - 2$, there holds*

$$\varphi_k \le \frac{3}{4}\varphi_{k+1} + c_1 2^{4(\beta+d_f)i}. \tag{3.4.14}$$

Proof. Choose $0 < \varepsilon_0 < \min(\varphi_{k+1}, e)$, and let $c_2 > 4e$ satisfy $6\log(c_2 - \varepsilon_0) = c_2$. Since $\varepsilon_0 < e$, c_2 exists. If $\varphi_{k+1} \le c_2$ then

$$\varphi_k \le \varphi_{k+1} \le \frac{3}{4}\varphi_{k+1} + \frac{1}{4}c_2,$$

so that (3.4.14) holds provided $c_1 \ge c_2/4$.
Now suppose $\varphi_{k+1} > c_2$. Let $B_{k+1}^{\varphi_{k+1}} \subset B_{k+1}$ be finite union of cells such that $\{|\log v| \ge \varphi_{k+1}/2\} \cap B_{k+1} \subset B_{k+1}^{\varphi_{k+1}}$ and $|\log v| \ge (\varphi_{k+1} - \varepsilon_0)/2$ on $B_{k+1}^{\varphi_{k+1}}$. Note that $|\log v| < \varphi_{k+1}/2$ on $B_{k+1} \setminus (B_{k+1}^{\varphi_{k+1}})^o$. Then from the facts that

$v^q \leq e^{q\varphi_{k+1}}$ on B_{k+1} and $v^{2q} \leq e^{q\varphi_{k+1}}$ on $B_{k+1} \setminus (B_{k+1}^{\varphi_{k+1}})^o$,

$$\begin{aligned}\int_{B_{k+1}} v^{2q} d\lambda_{\langle w_k \rangle} &\leq \int_{B_{k+1}^{\varphi_{k+1}}} v^{2q} d\lambda_{\langle w_k \rangle} + \int_{B_{k+1} \setminus (B_{k+1}^{\varphi_{k+1}})^o} v^{2q} d\lambda_{\langle w_k \rangle} \\ &\leq e^{2q\varphi_{k+1}} \lambda_{\langle w_k \rangle}(B_{k+1}^{\varphi_{k+1}}) + e^{q\varphi_{k+1}} \lambda_{\langle w_k \rangle}(B_{k+1}).\end{aligned}$$

We note that $\lambda_{\langle w_k \rangle}(B_{k+1}) \leq c_3 \mu(B_{k+1}) \leq 4c_3 \mu(I)$ for any k, by Lemma 3.3.5. Hence Lemma 3.4.5 implies that,

$$\int_{B_{k+1}} v^{2q} d\lambda_{\langle w_k \rangle} \leq c_4 \left(\frac{e^{2q\varphi_{k+1}}}{(\varphi_{k+1} - \varepsilon_0)^2} + e^{q\varphi_{k+1}} \right) \mu(I).$$

Let $q = \dfrac{2\log(\varphi_{k+1} - \varepsilon_0)}{\varphi_{k+1}}$, so that $e^{q\varphi_{k+1}} = (\varphi_{k+1} - \varepsilon_0)^2$. As $\varphi_{k+1} > c_2$ we have $q < \dfrac{2\log(c_2 - \varepsilon_0)}{c_2} = 1/3$. Then

$$⨍_{B_{k+1}} v^{2q} d\lambda_{\langle w_k \rangle} \leq \mu(I)^{-1} \int_{B_{k+1}} v^{2q} d\lambda_{\langle w_k \rangle} \leq c_4 e^{q\varphi_{k+1}}.$$

Hence by Corollary 3.4.4 we have

$$\begin{aligned}\varphi_k &= \frac{1}{2q} \log(\sup_{B_k} v^{2q}) \leq \frac{1}{2q} \log \left(c_5 2^{4(\beta + d_f)i} ⨍_{B_{k+1}} v^{2q} d\nu_{\langle w_k \rangle} \right) \\ &\leq \frac{1}{2q} \log \left(c_6 2^{4(\beta+d_f)i} e^{q\varphi_{k+1}} \right) = \frac{\varphi_{k+1}}{2} \left(1 + \frac{\log c_6 2^{4(\beta+d_f)i}}{2\log(\varphi_{k+1} - \varepsilon_0)} \right).\end{aligned}$$

We may assume that $c_6 > c_2$. If $\varphi_{k+1} - \varepsilon_0 \geq c_6 2^{4(\beta+d_f)i}$, then

$$\varphi_k \leq \frac{\varphi_{k+1}}{2}(1 + \frac{1}{2}) = \frac{3}{4}\varphi_{k+1}.$$

If $\varphi_{k+1} - \varepsilon_0 \leq c_6 2^{4(\beta+d_f)i}$, then since $\varepsilon_0 < e$, we have $\varphi_k \leq \varphi_{k+1} \leq c_7 2^{4(\beta+d_f)i}$ and also (3.4.14) holds. □

We now prove the Harnack inequality.

Theorem 3.4.7. *Let I be a single m_I-cell with $I_4 \subset K$, and $u > 0$ is a* *minimizer of* $\mathcal{E}^{\mathcal{L}}(\cdot)$*. Then there exists* c_1*, not depending on* u*, such that*

$$\frac{\sup_I u}{\inf_I u} \leq c_1.$$

Proof. Writing $\theta = 4(\beta + d_f)$ and multiplying u by a constant we may assume $\int_{I_3} \log u d\mu = 0$. First let $v = u$. Then for each $0 \leq i < \infty$ and $2^i - 1 \leq k \leq 2^{i+1} - 2$,

$$\begin{aligned}
\varphi_0 &\leq \frac{3}{4}\varphi_1 + c_2 2^{2\theta \cdot 0} \\
&\leq \left(\frac{3}{4}\right)^2 \varphi_2 + c_2 2^{2\theta \cdot 0} + \frac{3}{4} c_2 2^{2\theta \cdot 1} \\
&\leq \left(\frac{3}{4}\right)^3 \varphi_3 + c_2 2^{2\theta \cdot 0} + \frac{3}{4} c_2 2^{2\theta \cdot 1} + \left(\frac{3}{4}\right)^2 c_2 2^{2\theta \cdot 1} \\
&\leq \left(\frac{3}{4}\right)^4 \varphi_4 + c_2 2^{2\theta \cdot 0} + \frac{3}{4} c_2 2^{2\theta \cdot 1} + \left(\frac{3}{4}\right)^2 c_2 2^{2\theta \cdot 1} + \left(\frac{3}{4}\right)^3 c_2 2^{2\theta \cdot 2} \\
&\leq \cdots \\
&\leq \left(\frac{3}{4}\right)^n \varphi_n + c_2 \frac{16}{3} \sum_{k=0}^{n} \left[\left(\frac{3}{4}\right)^{2^k} - \left(\frac{3}{4}\right)^{2^{k+1}} \right] \cdot 2^{2\theta k}.
\end{aligned}$$

Since $\varphi_n \leq \sup_{I^*} \log v < \infty$, and

$$\sum_{k=0}^{\infty} \left[\left(\frac{3}{4}\right)^{2^k} - \left(\frac{3}{4}\right)^{2^{k+1}} \right] \cdot 2^{2\theta k} \leq c_3 < \infty,$$

so we obtain

$$\sup_I \log v \leq c_4.$$

If $v = u^{-1}$, then $\log v = -\log u$ so we still have $\int_{I_3} \log u d\mu = 0$. The same argument as above implies

$$\sup_I \log v \leq c_4 \quad \text{or} \quad \inf_I \log u \geq -c_4.$$

Combining we deduce

$$e^{-c_4} \leq \inf_I u \leq \sup_I u \leq e^{c_4}$$

hence we have desired results. □

Theorem 3.1.1 follows from Theorem 3.4.7 by covering argument.

Chapter 4

Homogenization of fully non-linear parabolic equations with different oscillations in space and time

4.1 Introduction

In this paper, we consider a periodic homogenization of fully non-linear parabolic equations of non-divergence form with different scales in space and time. Let $\Omega \subset \mathbb{R}^n$ be an open and connected domain with smooth boundary. We denote $S_T = \Omega \times (0, T)$, and the parabolic boundary $\partial_p S_T = (\partial\Omega \times [0, T)) \cup (\overline{\Omega} \times \{0\})$. Let u^ε be the viscosity solution of

$$\begin{cases} u^\varepsilon_t - F(D^2_x u^\varepsilon, x, t, x/\varepsilon, t/\varepsilon^k) = 0 & \text{in } S_T, \\ u^\varepsilon = \varphi(x, t) & \text{on } \partial_p S_T \end{cases} \tag{4.1.1}$$

Here, the parameter k, which we call the space-time scaling factor, can be any positive real number which affects the different oscillation in space and time. It is well known that the case when $k = 2$ is a classical homogenization problem for fully non-linear uniformly parabolic equations. In this case, various results have already been well established for the homogenization problem (see [20, 28, 35, 43, 50]). They proved that under the standard assumptions on F and φ, the solution u^ε converges uniformly to the solution u of the following homogenized equation:

$$\begin{cases} u_t - \overline{F}(D_x^2 u, x, t) = 0 & \text{in } S_T, \\ u = \varphi(x,t) & \text{on } \partial_p S_T. \end{cases}$$

We call $\overline{F}$ and u the effective operator and the effective limit respectively, which are uniquely defined by the cell problem. That is, with the slow spatial and temporal variable $(x,t) \in \overline{S_T}$ and fast spatial and temporal variable $(y,s) = (x/\varepsilon, t/\varepsilon^2) \in \mathbb{R}^n \times [0,\infty)$, we can find a unique (y,s)-periodic solution (up to constant) w, which is said to be a corrector, and a unique value $\overline{F}(M,x,t)$ satisfying the following equation:

$$w_s - F(M + D_y^2 w, x, t, y, s) = -\overline{F}(M,x,t) \quad \text{in } \mathbb{R}^n \times [0,\infty). \tag{4.1.2}$$

Also, it is well known that the error between u^ε and u is of order ε (see [35, 43]). In other words, we can observe a rate of convergence in a such way that

$$\|u^\varepsilon - u\|_{L^\infty(S_T)} \leq C\varepsilon.$$

The aim of this paper is to study the limiting behavior of solutions $u^\varepsilon = u^\varepsilon(x,t,x/\varepsilon,t/\varepsilon^k)$ as the space-time scaling factor k varies. Roughly speaking, when $k=2$, which is the natural space-time scaling factor, the homogenization process occurs simultaneously for time and space as we can see above. But if $k \neq 2$, we have to consider the homogenization process for space and time separately. This is fundamentally because of the mismatch between the highly oscillating spatial and temporal variables: When $k=2$, the scaling invariant property remains as ε goes to zero. But in case of $k \in (0,2)$, the spatial variable oscillates faster than the temporal variable, whereas when

$k \in (2,\infty)$, the opposite occurs. As a result, we can expect that the homogenization process does not occur simultaneously when $k \neq 2$. In fact, by looking at the asymptotic expansion, we can observe that the homogenization occurs in the order of space followed by time when $k \in (0,2)$, whereas in the case of $k \in (2,\infty)$, homogenization occurs in the reverse order.

4.1.1 Main results

Let $\mathcal{S}^n$ be the all real symmetric matrices of order n, endowed with (L^2, L^2)-norm. That is, $\|P\| = \left(\sum_{i,j=1}^n p_{ij}^2\right)^{1/2}$ for any $P = (p_{ij}) \in \mathcal{S}^n$. Let F be a smooth functional on $\mathcal{S}^n$. We denote by $F_{p_{ij}}(P)$ the derivative of F in direction E_{ij} at P, where $\{E^{ij} : 1 \leq i,j \leq n\}$ be the set of standard basis matrices. Let $Q_r(x_0,t_0) = \{(x,t) : |x - x_0| < r, 0 \leq t_0 - t < r^2\}$ and $S_T = \Omega \times (0,T)$. By Q_r we denote $Q_r(0,0)$. We define the parabolic distance between (x_1,t_1) and (x_2,t_2) in $\mathbb{R}^n \times \mathbb{R}$ by

$$d((x_1,t_1),(x_2,t_2)) = (|x_1 - x_2|^2 + |t_1 - t_2|)^{1/2}.$$

For $\gamma \in (0,1)$, $u \in C^\gamma(\overline{S_T})$ if

$$\|u\|_{C^\gamma(S_T)} = \|u\|_{L^\infty(\overline{S_T})} + \sup_{(x_1,t_1),(x_2,t_2)\in\overline{S_T}} \frac{|u(x_1,t_1) - u(x_2,t_2)|}{d((x_1,t_1),(x_2,t_2))^\gamma}.$$

Moreover, $u \in C^l(\overline{S_T})$ if for all α, β such that $|\alpha| + 2\beta \leq l$, $D_x^\alpha D_t^\beta u$ is continuous on $\overline{S_T}$. By $C^{l,\gamma}(\overline{S_T})$ we denote the usual Hölder space on $\overline{S_T}$.

We assume that $F : \mathcal{S}^n \times \overline{S_T} \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ and $\varphi : \overline{S_T} \to \mathbb{R}$ satisfy the following structure conditions.

(a) (Uniformly ellipticity) F is uniformly elliptic on $\mathcal{S}^n$:

$$\lambda\|N\| \leq F(M+N,x,t,y,s) - F(M,x,t,y,s) \leq \Lambda\|N\|$$

for any M, $N \in \mathcal{S}^n$, $N \geq 0$.

(b) (Periodicity) $F(M,x,t,y,s)$ is periodic in the (y,s)-variable: for every $(l,m) \in \mathbb{Z}^n \times \mathbb{Z}_{\geq 0}$, we have

$$F(M,x,t,y+l,s+m) = F(M,x,t,y,s).$$

(c) (Regularity) For each $L > 0$, $F \in C^\infty(\overline{B_L} \times \overline{S_T} \times \mathbb{R}^n \times [0,\infty))$ and $\varphi \in C^\infty(\overline{S_T})$. Moreover, there is a constant $C_r > 0$ such that

$$\|F\|_{C^r(\overline{B_L}\times\overline{S_T}\times\mathbb{R}^n\times\mathbb{R})} \leq C_r(1+L) \quad \text{and} \quad \|\varphi\|_{C^r(\overline{S_T})} \leq C_r$$

for each $r \geq 0$.

(d) (Convexity) F is convex in M-variable.

In this section, we would like to propose the qualitative and quantitative behavior of u^ε as k value changes. Interestingly, there are something remarkable points in each behavior. In terms of effective operators, there are only two type of homogenized equations depending on whether k is greater than or less than 2. This is essentially because the cell problem which create the effective operator does not depend on the value k. On the other hand, the asymptotic expansion depends on k, which results in the convergence rate being dependent on k.

Our first results concerning the homogenized equation are stated as follows.

Theorem 4.1.1 (Homogenization when $k \in (0,2)$)**.** *Let $\{u^\varepsilon\}_{\varepsilon>0} \subset C(\overline{S_T})$ be the family of viscosity solutions to* (4.1.1) *when $k \in (0,2)$. Then there exist a effective operator $\overline{F_1} : \mathcal{S}^n \times \overline{S_T} \to \mathbb{R}$ which is independent of k, such that the u^ε converges to a function u_1 uniformly, where u_1 is the solution of the following homogenized equation:*

$$\begin{cases} (u_1)_t - \overline{F_1}(D_x^2 u_1, x, t) = 0 & \textit{in } S_T, \\ u_1 = \varphi(x,t) & \textit{on } \partial_p S_T. \end{cases} \tag{4.1.3}$$

Theorem 4.1.2 (Homogenization when $k \in (2,\infty)$). *Let $\{u^\varepsilon\}_{\varepsilon>0} \subset C(\overline{S_T})$ be the family of viscosity solutions to (4.1.1) when $k \in (2,\infty)$. Then there exist a effective operator $\overline{F_3} : \mathcal{S}^n \times \overline{S_T} \to \mathbb{R}$ which is independent of k, such that the u^ε converges to a function u_3 uniformly, where u_3 is the solution of the following homogenized equation:*

$$\begin{cases} (u_3)_t - \overline{F_3}(D_x^2 u_3, x, t) = 0 & \text{in } S_T, \\ u_3 = \varphi(x,t) & \text{on } \partial_p S_T. \end{cases} \tag{4.1.4}$$

The results for the rate of convergence are stated below.

Theorem 4.1.3 (Convergence rate for $k \in (0,2)$). *Assume that $F : \mathcal{S}^n \times \overline{S_T} \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ and $\varphi : \overline{S_T} \to \mathbb{R}$ satisfy the structure conditions. Let $\{u^\varepsilon\}_{\varepsilon>0} \subset C(\overline{S_T})$ be the family of viscosity solutions to (4.1.1), and u_1 be the solution of homogenized equation (4.1.3). Then for any $\varepsilon_0 > 0$, $\varepsilon < \varepsilon_0$, the followings hold.*

(i) The case $k \in (0,1]$: Let $m \geq 1$ be an integer, and for k satisfying $k \in \left(\frac{1}{m+1}, \frac{1}{m}\right]$, there exists a sequence of the lk-th order effective limits $\{v^l\}_{l=1}^m$ on $\overline{S_T}$, such that

$$\left\| u^\varepsilon - u_1 - \sum_{l=1}^{m} \varepsilon^{kl} v^l \right\|_{L^\infty(S_T)} \leq C\varepsilon, \tag{4.1.5}$$

where C depends only on n, k, ε_0, λ, Λ, F, φ, and S_T. In particular, we have

$$\|u^\varepsilon - u_1\|_{L^\infty(S_T)} \leq C\varepsilon^k.$$

(ii) The case $k \in (1,2)$: Let $m \geq 1$ be an integer, and for k satisfying $k \in \left(\frac{2m-1}{m}, \frac{2m+1}{m+1}\right]$, there exists a sequence of the $l(2-k)$-th order effective limits $\{v^l\}_{l=1}^m$ on $\overline{S_T}$, such that

$$\left\| u^\varepsilon - u_1 - \sum_{l=1}^{m} \varepsilon^{(2-k)l} v^l \right\|_{L^\infty(S_T)} \leq C\varepsilon, \tag{4.1.6}$$

where C depends only on n, k, ε_0, λ, Λ, F, φ, and S_T. In particular, we have

$$\|u^\varepsilon - u_1\|_{L^\infty(S_T)} \leq C\varepsilon^{2-k}.$$

Theorem 4.1.4 (Convergence rate for $k \in (2, \infty)$)**.** *Assume that $F : \mathcal{S}^n \times \overline{S_T} \times \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ and $\varphi : \overline{S_T} \to \mathbb{R}$ satisfy the structure conditions. Let $\{u^\varepsilon\}_{\varepsilon>0} \subset C(\overline{S_T})$ be the family of viscosity solutions to (4.1.1), and u_3 be the solution of homogenized equation (4.1.4). Then for any $\varepsilon_0 > 0$, $\varepsilon < \varepsilon_0$, the followings hold.*

(i) The case $k \in (2, 3)$: Let $m \geq 1$ be an integer, and for k satisfying $k \in \left[\dfrac{2m+3}{m+1}, \dfrac{2m+1}{m}\right)$, there exists a sequence of the $l(k-2)$-th order effective limits $\{v^l\}_{l=1}^m$ on $\overline{S_T}$, such that

$$\left\| u^\varepsilon - u_3 - \sum_{l=1}^{m} \varepsilon^{(k-2)l} v^l \right\|_{L^\infty(S_T)} \leq C\varepsilon,$$

where C depends only on n, k, ε_0, λ, Λ, F, φ, and S_T. In particular, we have

$$\|u^\varepsilon - u_3\|_{L^\infty(S_T)} \leq C\varepsilon^k.$$

(ii) The case $k \in [3, \infty)$: If $k \in [3, \infty)$, then

$$\|u^\varepsilon - u_3\|_{L^\infty(S_T)} \leq C\varepsilon,$$

where C depends only on n, k, ε_0, λ, Λ, F, φ, and S_T.

4.1.2 Heuristics discussion and main strategies

Before we make our argument rigorous, we want to provide the heuristic calculation to understand the key idea. We first investigate the two interesting cases, where $k = 1$ or 3, and next look at the general case. Let us consider the following classical asymptotic expansion

$$u^\varepsilon(x,t) = u^0(x,t,x/\varepsilon,t/\varepsilon^k) + \varepsilon u^1(x,t,x/\varepsilon,t/\varepsilon^k) + \varepsilon^2 u^2(x,t,x/\varepsilon,t/\varepsilon^k) + \cdots$$

which occur inside and outside of the operator F. Then a simple computation gives following

$$\begin{aligned} u^\varepsilon_t - F(D^2_x u^\varepsilon) &\simeq (u^0 + \varepsilon u^1 + \varepsilon^2 u^2)_t - F\left[D^2_x(u^0 + \varepsilon u^1 + \varepsilon^2 u^2)\right] \\ &= \varepsilon^{-k} u^0_s + u^0_t + \varepsilon^{1-k} u^1_s + \varepsilon u^1_t + \varepsilon^{2-k} u^2_s + \varepsilon^2 u^2_t \\ &\quad - F\left[\varepsilon^{-2} D^2_y u^0 + \varepsilon^{-1}(D_x D_y u^0 + D_y D_x u^0) + D^2_x u^0 \right. \\ &\qquad + \varepsilon^{-1} D^2_y u^1 + (D_x D_y u^1 + D_y D_x u^1) + \varepsilon D^2_x u^1 \\ &\qquad \left. + D^2_y u^2 + \varepsilon(D_x D_y u^2 + D_y D_x u^2) + \varepsilon^2 D^2_x u^2\right]. \end{aligned} \tag{4.1.7}$$

Here we have dropped the dependency on $(x,t,x/\varepsilon,x/\varepsilon^k)$. By comparing the ε-power in (4.1.7), we will roughly look at how the effective operator varies according to the values of k. Assume for a while that F is linear with respect to the Hessian.

The case when $k = 1$.

If we compare each of ε-powers then first we can get the equation for u^0,

$$-F(D^2_y u^0) = 0 \quad \text{in } \mathbb{R}^n \times [0,\infty),$$

which implies that u^0 is y-independent. Moreover, from the equation for u^1,

$$u^0_s - F(D^2_y u^1) = 0 \quad \text{in } \mathbb{R}^n \times [0,\infty),$$

To solve the above equation, we shall use the following elementary result.

Lemma 4.1.5 ([10]). *The following equation*

$$-F(D^2_y w, y) = h \quad \textit{in } \mathbb{R}^n$$

admits y-periodic solution only when $h = 0$.

Thanks to the above lemma, we obtain that u^0 is s-independent and u^1 is y-independent. Finally, u^2 satisfies the following equation:

$$u_t^0 + u_s^1 - F(D_x^2 u^0 + D_y^2 u^2) = 0 \quad \text{in } \mathbb{R}^n \times [0, \infty). \tag{4.1.8}$$

Note that the above equation becomes an elliptic equation for u^2 if we regard the forcing term as $-u_t^0 - u_s^1$. Therefore, by considering the cell problem for an elliptic equation, we can obtain the s-periodic constant $\overline{E}_1(M, x, t, s)$ for each $s \in [0, \infty)$ such that

$$F(M + D_y^2 w, x, t, y, s) = \overline{E}_1(M, x, t, s) \quad \text{in } \mathbb{R}^n.$$

In this case, we can expect the effective operator $\overline{F_1} : \mathcal{S}^n \times \overline{S_T} \to \mathbb{R}$ to be of the form

$$\overline{F_1}(M, x, t) = \int_0^1 \overline{E}_1(M, x, t, s) ds,$$

and the effective limit u^0 satisfies following homogenized equation

$$\begin{cases} u_t^0 - \overline{F_1}(D_x^2 u^0, x, t) = 0 & \text{in } S_T, \\ u^0 = \varphi(x, t) & \text{on } \partial_p S_T. \end{cases}$$

The case when $k \in (1, 2)$.

Let's look at the case of $k \in (1, 3/2)$ first and then the general case. In this case, the previously applied expansion is inappropriate since there is no term to match the power ε^{i-k} or ε^{k-i} which would emergent. As a consequence, it is natural to expect u^ε to be of the form

$$\begin{aligned} u^\varepsilon(x, t) = {} & u^0(x, t, x/\varepsilon, t/\varepsilon^k) + \varepsilon u^1(x, t, x/\varepsilon, t/\varepsilon^k) + \varepsilon^2 u^2(x, t, x/\varepsilon, t/\varepsilon^k) + \cdots \\ & + \varepsilon^{k-1} \widetilde{v}(x, t, x/\varepsilon, t/\varepsilon^k) \\ & + \varepsilon^{2-k} v^1(x, t, x/\varepsilon, t/\varepsilon^k) + \varepsilon^{3-k} v^2(x, t, x/\varepsilon, t/\varepsilon^k) + \cdots \\ & + \varepsilon^k \xi^0(x, t, x/\varepsilon, t/\varepsilon^k) + \varepsilon^{k+1} \xi^1(x, t, x/\varepsilon, t/\varepsilon^k) + \cdots . \end{aligned}$$

As before, identifying the coefficients of ε^{-2}, ε^{k-3}, we obtain $-F(D_y^2 u^0) = -F(D_y^2 \widetilde{v}) = 0$ on $\mathbb{R}^n \times [0, \infty)$, which gives that u_0 and $\widetilde{v}$ are independent of y. Similarly, identifying the coefficients of ε^{-k} we obtain the equation

$$u_s^0 - F(D_y^2 v^1) = 0 \quad \text{in } \mathbb{R}^n \times [0, \infty),$$

then Lemma 4.1.5 implies that u^0 is s-independent and v^1 is y-independent. Next, from the ε^{-1} coefficients we obtain

$$\widetilde{v}_s - F(D_y^2 u^1) = 0 \quad \text{in } \mathbb{R}^n \times [0, \infty),$$

and since $\widetilde{v}$ is y-independent, Lemma 4.1.5 again we can observe that $\widetilde{v}$ is s-independent and u^1 is y-independent. Similarly, one can obtain that u^1 is s-independent and v^2 is y-independent. We note that the coefficient of ε^{k-1} is only related to $\widetilde{v}$, hence we have an equation which $\widetilde{v}$ satisfies:

$$\begin{cases} \widetilde{v}_t - F(D_x^2 \widetilde{v}) = 0 & \text{in } S_T, \\ \widetilde{v} = 0 & \text{on } \partial_p S_T. \end{cases}$$

Therefore, $\widetilde{v} \equiv 0$, so it can be considered that there is no ε^{k-1} term. In short, we have obtained

$$u^0 = u^0(x,t), \quad u^1 = u^1(x,t), \quad \widetilde{v} \equiv 0, \quad v^1 = v^1(x,t), \quad v^2 = v^2(x,t,s).$$

Now let's focus on the coefficients of ε^0 and ε^{2-k}, each satisfies the following equation:

$$u_t^0 + \xi_s^0 - F(D_x^2 u^0 + D_y^2 u^2) = 0 \quad \text{in } \mathbb{R}^n \times [0, \infty),$$
$$v_t^1 + u_s^2 - F(D_x^2 v^1 + D_y^2 v^3) = 0 \quad \text{in } \mathbb{R}^n \times [0, \infty).$$

As we can see, the two equations have the same form. That is, by repeating the previous process when $k = 1$, we can find two effective limits u^0 (of ε^0-order) and v^1 (of ε^{2-k}-order) corresponding effective operators. In particular, the effective operator corresponding to u^0 is exactly the same as $\overline{F_1}$, what this fact tells us that the effective operator and the effective limit does not depend on k.

On the other hand, the difference from the case of $k = 1$ is the existence of another effective limit v^1, which order is ε^{2-k}. From this observation, we can expect that the presence of v^1 affects the rate of convergence.

In general, when $k \in \left(\frac{2m-1}{m}, \frac{2m+1}{m+1}\right)$ for each $m \in \{1, 2, \cdots\}$, the ansatz for the asymptotic expansion of u^ε will be

$$\begin{aligned} u^\varepsilon(x,t) &= u^0(x,t,x/\varepsilon,t/\varepsilon^k) + \varepsilon u^1(x,t,x/\varepsilon,t/\varepsilon^k) + \varepsilon^2 u^2(x,t,x/\varepsilon,t/\varepsilon^k) + \cdots \\ &\quad + \sum_{l=1}^{m} \varepsilon^{l(2-k)} v^l(x,t,x/\varepsilon,t/\varepsilon^k) + \cdots + \varepsilon^k \xi^0(x,t,x/\varepsilon,t/\varepsilon^k) + \cdots . \end{aligned}$$

For each case, the effective operator $\overline{F_1}$ and the effective limit u^0 does not depend on k, but the structure of u^ε depends on k.

Now we investigate the one-dimensional simplest case to capture the asymptotic behavior of solutions u^ε when $k \in (0,1)$. Let $\Omega = (0,1)$ and consider the following initial boundary problem:

$$\begin{cases} u^\varepsilon_t - a(t/\varepsilon^k) D_{x_i x_j} u^\varepsilon = f(t/\varepsilon^k) & \text{in } S_T, \\ u^\varepsilon = -\frac{1}{2}x^2 & \text{on } \partial_p S_T, \end{cases}$$

where we normalize the diffusion coefficient a and the forcing term f by $\int_0^1 a(s)ds = \int_0^1 f(s)ds = 1$. From above heuristic computation, one can easily find the effective limit $u^0 = -x^2/2$ solving

$$\begin{cases} u^0_t - D_{x_i x_j} u^0 = 1 & \text{in } S_T, \\ u^0 = -\frac{1}{2}x^2 & \text{on } \partial_p S_T. \end{cases}$$

Put $\xi^0(s) = \int_0^s (f(\tau) - a(\tau))d\tau$. Then the asymptotic expansion of u^ε is

$u^0(x,t) + \varepsilon^k \xi^0(t/\varepsilon^k)$ since

$$u_t^\varepsilon - a(t/\varepsilon^k) D_{x_i x_j} u^\varepsilon = \xi_s^0 + a(t/\varepsilon^k) = f(t/\varepsilon^k) - a(t/\varepsilon^k) + a(t/\varepsilon^k) = f(t/\varepsilon^k)$$

in S_T, and $u^\varepsilon(x,0) = u^0(x,0) = -x^2/2$. To conclude, we get the convergence rate $\|u^\varepsilon - u^0\|_{L^\infty(S_T)} \le C\varepsilon^k$, in particular, u^ε cannot be faster than ε^k when $k \in (0,1)$. Hence, this is the optimal rate of convergence when $k \in (0,1)$.

The case when k=3

We will only look at the case of $k = 3$, since the general case proceeds along the line of $k \in (0,2)$ case. First, we can easily check that

$$\frac{1}{\varepsilon^3} u_s^0 = 0 \quad \text{in } \mathbb{R}^n \times [0,\infty),$$

by collecting highest order term, which means that u^0 is s-independent. Next, if we see the equation for u^1,

$$u_s^1 - F(D_y^2 u^0) = 0 \quad \text{in } \mathbb{R}^n \times [0,\infty),$$

then Lemma 4.1.5 implies that u^1 is s-independent and u^0 is y-independent. Similarly, from ε^{-1} order terms, we obtain

$$u_s^2 - F(D_y^2 u^1) = 0 \quad \text{in } \mathbb{R}^n \times [0,\infty),$$

so by Lemma 4.1.5 again we conclude that u^2 is s-independent and u^1 is y-independent. Finally, from ε^0 order terms, we obtain the corrector equation in such a way that

$$u_t^0 + u_s^3 - F(D_x^2 u^0 + D_y^2 u^2) = 0 \quad \text{in } \mathbb{R}^n \times [0,\infty). \tag{4.1.9}$$

Since u^0, u^2 are independent on s-variable, and u^3 is s-periodic, we can integrate the above equation with respect to s to obtain

$$u_t^0 - \int_0^1 F(D_x^2 u^0 + D_y^2 u^2) ds = 0 \quad \text{in } \mathbb{R}^n. \tag{4.1.10}$$

Let $\widehat{F}(M, x, t, y) = \int_0^1 F(M, x, t, y, s) ds$. Then we may expect that the effective operator $\overline{F_3}$ is given by the following cell problem

$$\widehat{F}(M + D_y^2 w) = \overline{F_3}(M) \quad \text{in } \mathbb{R}^n,$$

and the effective limit u^0 satisfies following homogenized equation

$$\begin{cases} u_t^0 - \overline{F_3}(D_x^2 u^0) = 0 & \text{in } S_T, \\ u^0 = \varphi & \text{on } \partial_p S_T. \end{cases}$$

4.1.3 Outline

This paper is organized as follows: we review the basic scheme of homogenization in Subsection 4.2. Section 4.3 and 4.4 are devoted to the cases when $k \in (0, 2)$ and $k \in (2, \infty)$ respectively. For each case we give the proof of the homogenization in Subsection 4.3.1 and 4.4.1, and present the rate of convergence in Subsection 4.3.2 and 4.4.2 respectively.

4.2 basic homogenization process

As we saw in the previous heuristic calculations, the cell problems change depending on whether k is greater or less than 2. So we need to look at a general homogenization scheme that can cover all of these cases. Fortunately, we point out that the cell problems are always elliptic regadless of the value k. So we present some general observations of the following homogenization results for elliptic equation which will be frequently used throughout the paper. We refer to [17, 19, 20] for proofs of Lemma 4.2.1 and Lemma 4.2.2.

Lemma 4.2.1. *Assume that F verifies the structure conditions. Then for fixed $(M, x, t, s) \in \mathcal{S}^n \times \overline{S_T} \times [0, \infty)$, there exists a (y, s)-periodic function $w(y; M, x, t, s)$ such that $w(\cdot; M, x, t, s) \in C^\infty(\mathbb{R}^n)$, and a constant $\overline{E}(M, x, t, s) \in \mathbb{R}$ which solve the following cell problem:*

$$F(M + D_y^2 w, x, t, y, s) = \overline{E}(M, x, t, s) \quad \text{in } \mathbb{R}^n. \tag{4.2.1}$$

Moreover, $\overline{E}$ is a unique constant where the equation has a unique solution w up to constant addition with the uniform estimate

$$|\overline{E}(M, x, t, s)| + \|w(\cdot; M, x, t, s)\|_{C^{\sigma+2,\gamma}(\mathbb{R}^n)} \leq C\,(1 + \|M\|)$$

for all $\sigma \geq 0$, $\gamma \in (0, 1)$, where C depends only on n, γ, σ, λ, and Λ.

We notice that the structure of (4.2.1) and the uniqueness of $\overline{E}$ imply that $\overline{E}$ is also a s-periodic function. Let us now observe additional properties of the functional $\overline{E}$.

Lemma 4.2.2. *(i) $\overline{E}$ is uniformly elliptic with the same ellipticity constants of F and convex with respect to M-variable.*

(ii) For fixed $L > 0$, suppose that $F \in C^{0,1}(\overline{B_L} \times \overline{S_T} \times \mathbb{R}^n \times [0, \infty))$. Then $\overline{E}$, $w(y; \cdot, \cdot, \cdot, \cdot) \in C^{0,1}(B_L \times \overline{S_T} \times [0, \infty))$ uniformly in $y \in \mathbb{R}^n$.

The next lemma summarizes the improved regularities of $\overline{E}$ and w. In fact, from the regularity point of view, we only use Lemma 4.2.2 to find a effective limit and the corresponding effective operator. More precisely, by freezing the slow variable we just need a decoupled regularity of the slow variable (x, t) and the fast variable (y, s) to accept standard arguments of perturbed test function method. But in seeking the convergence rate the mixed regularity occurs. By assuming that the operator F has a good enough regularity, we can present the appropriate regularities of $\overline{E}$ and w.

Lemma 4.2.3. *([34, 35]). For fixed $L > 0$, suppose that $F \in C^\infty(\overline{B_L} \times \overline{S_T} \times \mathbb{R}^n \times [0, \infty))$. Then $\overline{E}$, $w(y; \cdot, \cdot, \cdot, \cdot) \in C^\infty(B_L \times \overline{S_T} \times [0, \infty))$ uniformly in*

$y \in \mathbb{R}^n$ and for any $(M, x, t, s) \in \overline{B_L} \times \overline{S_T} \times [0, \infty)$ there holds

$$\sum_{l+\mu+2\nu+2\rho=r} \left(|D_p^l D_x^\mu \partial_t^\nu \partial_s^\rho \overline{E}| + \|D_p^l D_x^\mu \partial_t^\nu \partial_s^\rho w(\cdot; M, x, t, s)\|_{C^{\sigma+2,\gamma}(\mathbb{R}^n)}\right)$$
$$\leq C_{\gamma,\sigma,r}(1 + \|M\|)$$

for all r, $\sigma \geq 0$, and $\gamma \in (0, 1)$, where $C_{\gamma,\sigma,r}$ depends on n, λ, and Λ.

Remark 4.2.4. It is worth pointing out that the argument of $C^{2,\gamma}$-regularity of $w(\cdot; M, x, t, s)$ is valid by only assuming that $F(M, x, t, \cdot, s) \in C^\gamma(\mathbb{R}^n)$.

4.3 Homogenization when $k \in (0, 2)$

In this section, we consider the case of $k \in (0, 2)$. From the heuristic calculation (4.1.8) we can observe that the second corrector u^2 solves the elliptic equation. This fact implies that we first have to take a strategy of finding a homogenization for space, then for time.

4.3.1 The effective operator and the effective limit

We start with the cell problem.

Lemma 4.3.1. *Assume that F verifies the structure conditions. Then for each $(M, x, t, s) \in \mathcal{S}^n \times \overline{S_T} \times [0, \infty)$ there exists a (y, s)-periodic function $w(y; M, x, t, s)$ such that $w(\cdot; M, x, t, s) \in C^{2,\gamma}(\mathbb{R}^n)$, and a s-periodic constant $\overline{E}_1(s; M, x, t) \in \mathbb{R}$ which solve the following cell problem:*

$$F(M + D_y^2 w, x, t, y, s) = \overline{E}_1(s; M, x, t) \quad \text{in } \mathbb{R}^n \tag{4.3.1}$$

with the uniform estimate

$$|\overline{E}_1(M, x, t, s)| + \|w(\cdot; M, x, t, s)\|_{C^{2,\gamma}(\mathbb{R}^n)} \leq C\,(1 + \|M\|)\,,$$

where C depends only on n, γ, σ, λ, and Λ. Moreover, $\overline{E}_1$ is a unique constant where the equation has a unique solution w up to constant addition.

This is a re-statement of Lemma 4.2.1. The s-periodicity of w and $\overline{E}_1$ comes directly from the fact that F is also s-periodic, and the uniqueness of w. Define

$$\overline{F_1}(M,x,t) := \int_0^1 \overline{E}_1(s;M,x,t)ds.$$

We will call $\overline{F_1}$ the effective operator when $k \in (0,2)$. From Lemma 4.2.2, we can observe that $\overline{F_1}$ is uniformly elliptic with the same ellipticity constants of F and convex with respect to M-variable. Moreover, the regularity results in Lemma 4.2.3 also hold for the w and $\overline{F_1}$.

Recall that the heuristic calculation (4.1.8). If we consider of (4.1.8) as a PDE for u^0, we can observe that the forcing term of (4.1.8) is actually $-u^1_s$. This fact gives us that when constructing the solution of (4.1.1) using the asymptotic expansion, we should make it reflect the influence on the k-th order corrector u^1. So, let us consider the function $\xi : [0,\infty) \times \mathcal{S}^n \times \overline{S_T} \to \mathbb{R}$ defined by

$$\xi(s;M,x,t) := \int_0^s \overline{E}_1(\tau;M,x,t)d\tau - s\overline{F_1}(M,x,t). \tag{4.3.2}$$

Since $\overline{E}_1(s;M,x,t)$ is s-periodic, we can observe that

$$\begin{aligned}\xi(s+1,M,x,t) &= \int_0^{s+1} \overline{E}_1(\tau;M,x,t)d\tau - (s+1)\overline{F_1}(M,x,t)\\ &= \xi(s;M,x,t) + \int_s^{s+1} \overline{E}_1(\tau;M,x,t)d\tau - \overline{F_1}(M,x,t)\\ &= \xi(s;M,x,t),\end{aligned}$$

so $\xi(s;M,x,t)$ is also s-periodic. Moreover, $\xi_s(s;M,x,t) = \overline{E}_1(s;M,x,t) - \overline{F_1}(M,x,t)$. Thus, we can expect that ξ will serve as the k-th order corrector.

Let us now establish an homogenized equation.

proof of theorem 4.1.1. Owing to estimates [57], there exists $\widetilde{\gamma} > 0$ for which $\sup_{0<\varepsilon<1} \|u^\varepsilon\|_{C^{\widetilde{\gamma}}(\overline{S_T})} < \infty$. Thus, we may extract a subsequence $\{u^{\varepsilon_l}\}_{l=1}^\infty$ of

$\{u^\varepsilon\}_{\varepsilon>0}$ and a function $u_1 \in C^{\widetilde{\gamma}}(\overline{S_T})$ with $u^{\varepsilon_l} \to u_1$ uniformly on $\overline{S_T}$. Moreover, since $u^\varepsilon = \varphi$ on $\partial_p S_T$ for all $\varepsilon > 0$, we have $u_1 = \varphi$ on $\partial_p S_T$. For convenience, we will not use subsequencial notation. Let P be a paraboloid with $M_0 = D_x^2 P$ which touches u_1 by above at (x_0, t_0) in a neighborhood. Without loss of generality, we may assume that P touches u_1 strictly by above. Assume, to the contrary, that

$$P_t(x_0, t_0) - \overline{F_1}(M_0, x_0, t_0) > 3\eta > 0$$

for some $\eta > 0$. Put $\widehat{w}(y, s) := w(y; M_0, x_0, t_0, s)$. Then from Lemma 4.3.1 we can observe that $\widehat{w}$ satisfies

$$F(M_0 + D_y^2 \widehat{w}, x_0, t_0, y, s) = \overline{E}_1(s; M_0, x_0, t_0) \quad \text{in } \mathbb{R}^n. \tag{4.3.3}$$

By the continuity of F and $\overline{F_1}$, we can choose $\rho > 0$ in such way that $Q_\rho(x_0, t_0) \subset S_T$,

$$\begin{aligned} &P_t(x, t) - \overline{F_1}(M_0, x, t) > 3\eta, \quad \text{and} \\ &|F(M_0 + D_y^2 \widehat{w}, x, t, y, s) - F(M_0 + D_y^2 \widehat{w}, x_0, t_0, y, s)| \\ &\quad + |\overline{F_1}(M_0, x, t) - \overline{F_1}(M_0, x_0, t_0)| < \eta \end{aligned} \tag{4.3.4}$$

for any $(x, t) \in Q_\rho(x_0, t_0)$, uniformly $(y, s) \in \mathbb{R}^n \times [0, \infty)$. Moreover, $u_1(x, t) - P(x, t) \leq -\mu$ on ∂Q_ρ, for some $\mu > 0$.
Now define

$$\widehat{\xi}(s) := \xi(s, M_0, x_0, t_0), \tag{4.3.5}$$

where the definition of ξ is in (4.3.2), and set

$$P^\varepsilon(x, t) := P(x, t) + \varepsilon^k \widehat{\xi}\left(\frac{t}{\varepsilon^k}\right) + \varepsilon^2 \widehat{w}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k}\right).$$

Note from the comment above this lemma that $\widehat{\xi}$ is s-periodic, and $\widehat{\xi}_s(s) = \overline{E}_1(s; M_0, x_0, t_0) - \overline{F_1}(M_0, x_0, t_0)$. For a while, let us drop the dependency of

$(x/\varepsilon, t/\varepsilon^k)$. Since $2-k>0$, in view of (4.3.3), (4.3.4), and (4.3.5) we have

$$\begin{aligned}
P_t^\varepsilon - F\left(D_x^2 P^\varepsilon, x, t\right) &= P_t + \widehat{\xi}_s + \varepsilon^{2-k}\widehat{w}_s - F\left(M_0 + D_y^2\widehat{w}, x, t\right) \\
&\geq P_t + \overline{E}_1\left(\frac{t}{\varepsilon^k}; M_0, x_0, t_0\right) - \overline{F_1}(M_0, x_0, t_0) + \varepsilon^{2-k}\widehat{w}_s \\
&\quad - F\left(M_0 + D_y^2 w, x_0, t_0\right) - \eta \\
&= P_t + \overline{E}_1\left(\frac{t}{\varepsilon^k}; M_0, x_0, t_0\right) - \overline{F_1}(M_0, x_0, t_0) + \varepsilon^{2-k}\widehat{w}_s \\
&\quad - \overline{E}_1\left(\frac{t}{\varepsilon^k}; M_0, x_0, t_0\right) - \eta \\
&= P_t - \overline{F_1}(M_0, x_0, t_0) + \varepsilon^{2-k}\widehat{w}_s - \eta \\
&\geq P_t - \overline{F_1}(M_0, x, t) + \varepsilon^{2-k}\widehat{w}_s - 2\eta \\
&\geq P_t - \overline{F_1}(M_0, x, t) - \varepsilon^{2-k}\|\widehat{w}_s\|_{L^\infty(\mathbb{R}^n\times[0,\infty))} - 2\eta \\
&> 0.
\end{aligned}$$

if ε is small enough, in $Q_r(x_0,t_0)$. As $u^\varepsilon \to u_1$ and $P^\varepsilon \to P$ uniformly in $Q_r(x_0,t_0)$, we can easily check that for some $\varepsilon_0 \in (0,1)$ there holds

$$u^\varepsilon(x,t) - P^\varepsilon(x,t) < -\mu/2 \quad \text{on } \partial Q_r(x_0,t_0), \quad \varepsilon < \varepsilon_0.$$

Hence $P^\varepsilon - \mu/4$ is a super-solution to the following initial-boundary value problem:

$$\begin{cases} v_t - F(D_x^2 v, x, t, x/\varepsilon, t/\varepsilon^k) = 0 & \text{in } Q_r(x_0,t_0), \\ v = u^\varepsilon(x,t) & \text{on } \partial_p Q_r(x_0,t_0). \end{cases}$$

Hence, the comparison principle implies $u^\varepsilon \leq P^\varepsilon - \mu/4$ in $Q_r(x_0,t_0)$. Letting $\varepsilon \to 0$ then $u_1(x_0,t_0) \leq P(x_0,t_0) - \mu/4$ which contradicts assumption that $u_1(x_0,t_0) = P(x_0,t_0)$. It shows that u_1 is a viscosity sub-solution of (4.1.3). In a similar manner, we are able to prove that u_1 is a viscosity super-solution of (4.1.3). The uniqueness of u_1 is clear, by the comparison principle, and hence the convergence of $u^\varepsilon \to u$ does not need to extract a subsequence. Finally, since $\overline{F_1}(M,\cdot,\cdot) \in C^\infty(\overline{S_T})$, and $\overline{F_1}(\cdot,x,t)$ is convex from the comment above this proof, the fact that $u_1 \in C^\infty(\overline{S_T})$ follows from the standard regularity argument for fully non-linear parabolic equations (see [58]). This completes the proof. □

As a corollary, we obtain the regularity of effective limit u_1, which is important later when calculating the convergence rate.

Corollary 4.3.2. *Assume that F and φ verify the structure conditions. Then $u_1 \in C^\infty(\overline{S_T})$ and*

$$\|u_1\|_{C^{r+2,\gamma}(\overline{S_T})} \le C_r$$

for each $r \ge 0$, where C_r depends only on n, γ, λ, Λ, φ, and S_T.

Remark 4.3.3. One can observe that u_1 and $\overline{F_1}$ are independent of $k \in (0,2)$, since k has no effect on the cell problem Lemma 4.3.1, and φ. That is, for any $k \in (0,2)$, $u^\varepsilon \to u$ converge uniformly in $\overline{S_T}$, where u is the unique solution of (4.1.3).

4.3.2 Rate of convergence for the homogenization

We are now in a position to give the proof of the convergence rate when $k \in (0,2)$. We in particular suppose that $k \in (1,2)$, this is because the process of $k \in (0,1]$ case is similar to when $k \in (1,2)$. Before we start, let us discuss the difficulties which arise given the effect of k. First, we emphasize that the asymptotic expansion of u^ε depends on k. As we saw in the proof of 4.1.1, a additional term $\varepsilon^{2-k}w_s$ emerges due to the influence of the second corrector when we calculate (4.1.1) of u^ε, which induces an additional error differ from the $k=2$ case. This is essentially a problem that occurs because k is not an integer, and the ansatz itself depends on k. Therefore we need to find k-multiple effective limits and correctors to obtain ε convergence rate by offsetting $(2-k)$-th order term.

Now let us sketch our key idea. From now on, we write

$$\begin{aligned} v^0(x,t) &= u_1(x,t), \\ \xi^0(x,t,s) &= \xi(s; D_x^2 u_1, x, t), \\ w^0(x,t,y,s) &= w(y; D_x^2 u_1, x, t, s). \end{aligned}$$

It is noteworthy to see that, we combine (4.1.3), (4.3.1), and (4.3.2) to get

$$\begin{aligned} & v_t^0 + \xi_s^0 - F(D_x^2 v^0 + D_y^2 w^0) \\ & \quad = v_t^0 + \overline{E}_1(s; D_x^2 v^0, x, t) - \overline{F_1}(D_x^2 v^0, x, t) - F(D_x^2 v^0 + D_y^2 w^0) = 0. \end{aligned} \tag{4.3.6}$$

For fixed $k \in (1,2)$, choose $m \in \{1, 2, \cdots\}$ in a such way that

$$k \in \left(\frac{2m-1}{m}, \frac{2m+1}{m+1} \right], \tag{4.3.7}$$

and consider the following expansion:

$$v^\varepsilon(x,t) = \sum_{l=0}^{m} \varepsilon^{l(2-k)} \left[v^l(x,t) + \varepsilon^k \xi^l \left(x, t, \frac{t}{\varepsilon^k} \right) + \varepsilon^2 w^l \left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^k} \right) \right], \tag{4.3.8}$$

where the families $\{v^l : \overline{S_T} \to \mathbb{R}\}_{1 \le l \le m}$, $\{\xi^l : \overline{S_T} \times [0,\infty) \to \mathbb{R}\}_{1 \le l \le m}$, and $\{w^l : \overline{S_T} \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}\}_{1 \le l \le m}$ will be determined later. In fact, each of v^l, ξ^l and w^l plays the role of the l-th effective limit u_1, the correctors ξ, and w. For a while, let us assume that all the functions v^l, ξ^l, and w^l are regular enough. To simplify, let us drop the dependency of $(x, t, x/\varepsilon, t/\varepsilon^k)$. Put

$$\begin{aligned} X^l &= D_x^2 v^l(x,t) + D_y^2 w^l \left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^k} \right), \\ A^l &= \varepsilon^k D_x^2 \xi^l \left(x, t, \frac{t}{\varepsilon^k} \right) + \varepsilon D_{x,y} w^l \left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^k} \right) + \varepsilon^2 D_x^2 w^l \left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^k} \right), \\ Y^m &= X^1 + \cdots + \varepsilon^{(m-1)(2-k)} X^m = \sum_{l=1}^{m} \varepsilon^{(l-1)(2-k)} X^l. \end{aligned} \tag{4.3.9}$$

Here we have denoted by $D_{x,y}$ the operator $D_xD_y + D_yD_x$. Then by the assumption, it follows for fixed $\varepsilon_0 \in (0,1)$ that

$$\sup_{0<\varepsilon\leq\varepsilon_0} \sum_{l=0}^{m} \|A^l(\cdot,\cdot,\cdot/\varepsilon,\cdot/\varepsilon^k)\|_{L^\infty(\overline{S_T})} \leq C_0(\varepsilon_0,m)\varepsilon.$$

Under these settings, we compute (4.3.8) with respect to the operator F that we have

$$\begin{aligned} F(D_x^2 v^\varepsilon) &= F\left(\sum_{l=0}^{m} \varepsilon^{l(2-k)} X^l + \sum_{l=0}^{m} \varepsilon^{l(2-k)} A^l\right) \\ &= F\left(X_0 + \varepsilon^{2-k} Y^m\right) + O(\varepsilon). \end{aligned}$$

Moreover, a Taylor expansion of F with respect to the Hessian gives

$$\begin{aligned} &F(X^0 + \varepsilon^{2-k}Y^m) \\ &\quad= F(X^0) + \sum_{l=1}^{m} \frac{\varepsilon^{l(2-k)}}{l!} F_{p_{i_1j_1}\cdots p_{i_lj_l}}(X^0) Y^m_{i_1j_1}\cdots Y^m_{i_lj_l} + O(\varepsilon^{(m+1)(2-k)}) \\ &\quad= F(X^0) + \sum_{l=1}^{m} \varepsilon^{l(2-k)} \sum_{d=1}^{l} \frac{1}{d!} \sum_{n_1+\cdots+n_d=l} F_{p_{i_1j_1}\cdots p_{i_dj_d}}(X^0) X^{n_1}_{i_1j_1}\cdots X^{n_d}_{i_dj_d} \\ &\qquad+ \sum_{l=1}^{m} \sum_{m+1\leq n_1+\cdots+n_l\leq lm} \frac{\varepsilon^{n_1+\cdots+n_l}}{l!} F_{p_{i_1j_1}\cdots p_{i_lj_l}}(X^0) X^{n_1}_{i_1j_1}\cdots X^{n_l}_{i_lj_l} \\ &\qquad+ O(\varepsilon^{(m+1)(2-k)}). \end{aligned} \tag{4.3.10}$$

Since $(m+1)(2-k) \geq 1$, if we can control the m-th derivatives of F with respect to Hessian, then last two terms including the error term $O(\varepsilon^{(m+1)(2-k)})$ are dominated by $O(\varepsilon)$. It illustrates the reason why we restrict the range of k as in (4.3.7) and have to find correctors until $l \leq m$. As we have observed the heuristic calculation, the method of finding correctors is also to compare the order of ε. To do this, we rewrite each term in (4.3.10) as

$$\begin{aligned}
&\sum_{d=1}^{l}\frac{1}{d!}\sum_{n_1+\cdots+n_d=l}F_{p_{i_1j_1}\cdots p_{i_dj_d}}(X^0)X^{n_1}_{i_1j_1}\cdots X^{n_d}_{i_dj_d}\\
&\quad=F_{p_{ij}}(X^0)X^l_{ij}+\sum_{d=2}^{l}\frac{1}{d!}\sum_{n_1+\cdots+n_d=l}F_{p_{i_1j_1}\cdots p_{i_dj_d}}(X^0)X^{n_1}_{i_1j_1}\cdots X^{n_d}_{i_dj_d}\\
&\quad:=F_{p_{ij}}(X^0)\left(D_{x_ix_j}v^l+D_{y_iy_j}w^l\right)+\Phi_l(x,t,y,s).
\end{aligned}$$

where

$$\Phi^l(x,t,y,s)=\sum_{d=2}^{l}\frac{1}{d!}\sum_{n_1+\cdots+n_d=l}F_{p_{i_1j_1}\cdots p_{i_dj_d}}(X^0)X^{n_1}_{i_1j_1}\cdots X^{n_d}_{i_dj_d}.$$

As a first step, put $a_{ij}=F(X^0)$. Then similar to the way we found u_1, ξ, and w, we can choose the function $v^1:\overline{S_T}\to\mathbb{R}$, the s-periodic function $\xi^1:\overline{S_T}\times[0,\infty)\to\mathbb{R}$, and the (y,s)-periodic function $w^1:\overline{S_T}\times\mathbb{R}^n\times[0,\infty)\to\mathbb{R}$ satisfying the following linear elliptic equation:

$$v^1_t+\xi^1_s-a_{ij}\left(D_{x_ix_j}v^1+D_{y_iy_j}w^1\right)=-w^0_s\quad\text{in }\overline{S_T}\times\mathbb{R}^n\times[0,\infty).$$

Note that this equation belongs to the same class of (4.3.6). Secondly, we will choose the function $v^l:\overline{S_T}\to\mathbb{R}$, the s-periodic function $\xi^l:\overline{S_T}\times[0,\infty)\to\mathbb{R}$, and the (y,s)-periodic function $w^l:\overline{S_T}\times\mathbb{R}^n\times[0,\infty)\to\mathbb{R}$ to offset w^{l-1}_s in an inductive manner. That is, v^l, ξ^l, and w^l satisfy the following linear elliptic equation:

$$v^l_t+\xi^l_s-a_{ij}\left(D_{x_ix_j}v^l+D_{y_iy_j}w^l\right)=-w^{l-1}_s-\Phi^l\quad\text{in }\overline{S_T}\times\mathbb{R}^n\times[0,\infty).\tag{4.3.11}$$

Note that the term Φ^l does not contain the function w^l, so we can solve the (4.3.11) to obtain v^l, ξ^l, and w^l.

It is noteworthy to see that $m\to\infty$ as $k\to 2$. That is, we need a more correctors as $k\to 2$, and we have to control the supremum norm of all these correctors. This is the reason why we need a C^∞-regularity of F, f, and φ.

Now we make our observation rigorous.

Lemma 4.3.4. *Suppose that $k \in (1,2)$, and let $m \in \mathbb{Z}$ be chosen to satisfy $k \in \left(\dfrac{2m-1}{m}, \dfrac{2m+1}{m+1}\right]$. Assume that F and φ satisfy the structure conditions. Then there exist families of $\{v^l : \overline{S_T} \to \mathbb{R}\}_{0\le l\le m}$, s-periodic functions $\{\xi^l : \overline{S_T} \times [0,\infty) \to \mathbb{R}\}_{0\le l\le m}$, and (y,s)-periodic functions $\{w^l : \overline{S_T} \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}\}_{0\le l\le m}$, which satisfy following conditions:*

(i) ξ^l, $w^l(\cdot,\cdot,y,\cdot) \in C^\infty(\overline{S_T} \times [0,\infty))$, and $w^l(x,t,\cdot,s) \in C^{2,\gamma}(\mathbb{R}^n)$ uniformly for all $(x,t,s) \in \overline{S_T} \times [0,\infty)$ with

$$\sum_{\mu+2\nu+2\rho=r} \left(|D_x^\mu \partial_t^\nu \partial_s^\rho \xi^l| + \|D_x^\mu \partial_t^\nu \partial_s^\rho w^l(x,t,\cdot,s)\|_{C^{2,\gamma}(\mathbb{R}^n)}\right) \le C_{k,r}$$

for all $r \ge 0$, where $C_{k,r}$ depends only on n, γ, λ, Λ, φ, and S_T.

(ii) $v^l \in C^\infty(\overline{S_T})$ with

$$\|v^l\|_{C^{r+2,\gamma}(\overline{S_T})} \le C_{k,r}$$

for all $r \ge 0$, where $C_{k,r}$ depends only on n, γ, λ, Λ, φ, and S_T.

(iii) For each $1 \le l \le m$, v^l, ξ^l and w^l satisfy the following recursive relation:

$$v_t^l + \xi_s^l - a_{ij}\left(D_{x_i x_j} v^l + D_{y_i y_j} w^l\right) = -w_s^{l-1} + \Phi^l \quad \text{in } \overline{S_T} \times \mathbb{R}^n \times [0,\infty), \tag{4.3.12}$$

where

$$X^l(x,t,y,s) = D_x^2 v^l(x,t) + D_y^2 w^l(x,t,y,s),$$
$$a_{ij}(x,t,y,s) = F_{p_{ij}}(X^0,x,t,y,s),$$
$$\Phi^1(x,t,y,s) = 0,$$
$$\Phi^l(x,t,y,s) = \sum_{d=2}^{l} \frac{1}{d!} \sum_{n_1+\cdots+n_d=l} F_{p_{i_1 j_1}\cdots p_{i_d j_d}}(X^0) X_{i_1 j_1}^{n_1} \cdots X_{i_d j_d}^{n_d}, \quad (l \ge 2).$$

Proof. In this proof, we are going to use a modification of the technique introduced in [34]. As the first step, we linearize the equation (4.3.1). Pick $(x,t,s) \in \overline{S_T} \times [0,\infty)$, and from now on we omit the dependence on (x,t,s) for notational convenience. Let $\chi_h^{\alpha\beta}(y) = h^{-1}[w(y;D_x^2v^0+hE^{\alpha\beta})-w(y;D_x^2v^0)]$, where $\{E^{ij} : 1 \le i,j \le n\}$ be the set of standard basis matrices. Then we can observe from (4.3.1) that $\chi_h^{\alpha\beta}$ satisfies

$$a_{ij,h}D_{y_iy_j}\chi_h^{\alpha\beta} + a_{\alpha\beta,h} = \overline{a}_{\alpha\beta,h},$$

where

$$\begin{aligned}
a_{ij,h} &= \int_0^1 F_{p_{ij}}(N_{\theta,h})d\theta,\\
N_{\theta,h} &= \theta D_y^2 w(y, D_x^2v^0 + hE^{\alpha\beta}) + (1-\theta)D_y^2w(y,D_x^2v^0) + D_x^2v^0 + \theta hE^{\alpha\beta},\\
\overline{a}_{\alpha\beta,h} &= \frac{\overline{E}_1(D_x^2v^0 + hE^{\alpha\beta}) - \overline{E}_1(D_x^2v^0)}{h}.
\end{aligned}$$

We can observe that $a_{ij,h}$ is uniform elliptic with the same ellipticity constants of F uniformly in h. In addition, Lemma 4.2.3 and Theorem 4.1.1 imply that for all $r \ge 0$, for any h with $|h|$ small, $a_{ij,h}(x,t,\cdot,s) \in C^\gamma(\mathbb{R}^n)$ with

$$|\overline{a}_{\alpha\beta,h}| + \|a_{ij,h}(x,t,\cdot,s)\|_{C^\gamma(\mathbb{R}^n)} \le C,$$

and

$$a_{ij,h}(x,t,y,s) \to a_{ij}(x,t,y,s) = F_{p_{ij}}(D_x^2v^0 + D_y^2w, x,t,y,s)$$

uniformly in $\mathbb{R}^n$ as $h \to 0$. Consequently, by the same argument of the perturbed test function method as in Lemma 4.2.1 (see [34], Lemma 2.1.2), there exists a unique constant $\overline{a}_{\alpha\beta}(x,t,s) = (\overline{E}_1)_{p_{\alpha\beta}}(s;D_x^2v^0,x,t)$ and a bounded (y,s)-periodic function $\chi^{\alpha\beta}(x,t,y,s) = D_{p_{\alpha\beta}}w(y;D_x^2v^0,x,t,s)$ with $\chi^{\alpha\beta}(x,t,\cdot,s) \in C^{2,\gamma}(\mathbb{R}^n)$ such that

$$|\overline{a}_{\alpha\beta,h} - \overline{a}_{\alpha\beta}| + \|\chi_h^{\alpha\beta} - \chi^{\alpha\beta}\|_{C^2(\mathbb{R}^n)} \to 0$$

as $h \to 0$. Then $\chi^{\alpha\beta}$ satisfies

$$a_{ij} D_{y_i y_j} \chi^{\alpha\beta} + a_{\alpha\beta} = \overline{a}_{\alpha\beta}.$$

Since we also have $a_{ij}(x,t,\cdot,s) \in C^\gamma(\mathbb{R}^n)$ with $\|a_{ij}(x,t,\cdot,s)\|_{C^\gamma(\mathbb{R}^n)} \leq C$, then from Lemma 4.2.3 we can observe that $\overline{a}_{\alpha\beta} = \overline{a}_{\alpha\beta}(x,t,s) \in C^\infty(\overline{S_T} \times [0,\infty))$, and $\chi^{\alpha\beta} = \chi^{\alpha\beta}(\cdot,\cdot,y,\cdot) \in C^\infty(\overline{S_T} \times [0,\infty))$ with

$$\sum_{\mu+2\nu+2\rho=r} \left(|D_x^\mu \partial_t^\nu \partial_s^\rho \overline{a}_{\alpha\beta}| + \|D_x^\mu \partial_t^\nu \partial_s^\rho \chi^{\alpha\beta}(\cdot; x,t,s)\|_{C^{2,\gamma}(\mathbb{R}^n)} \right) \leq C_r \qquad (4.3.13)$$

for all $r \geq 0$. Now putting $A^{\alpha\beta}(x,t,s) = \displaystyle\int_0^s \overline{a}_{\alpha\beta}(x,t,\tau)d\tau$ and we define a additional function $\widehat{\chi}^{\alpha\beta} : \overline{S_T} \times [0,\infty) \to \mathbb{R}$ in a similar way to finding the corrector ξ such that

$$\widehat{\chi}^{\alpha\beta}(x,t,s) = A^{\alpha\beta}(x,t,s) - sA^{\alpha\beta}(x,t,1).$$

Then $\widehat{\chi}^{\alpha\beta}$ is s-periodic, and we can deduce that

$$\widehat{\chi}^{\alpha\beta}_s - a_{ij} D_{y_i y_j} \chi^{\alpha\beta} = \overline{a}_{\alpha\beta} - A^{\alpha\beta}(x,t,1) + a_{\alpha\beta} - \overline{a}_{\alpha\beta} = -A^{\alpha\beta}(x,t,1) + a_{\alpha\beta}. \qquad (4.3.14)$$

Now we construct the family of functions $\{v^l : \overline{S_T} \to \mathbb{R}\}_{0 \leq l \leq m}$, $\{\xi^l : \overline{S_T} \times [0,\infty) \to \mathbb{R}\}_{0 \leq l \leq m}$, and $\{w^l : \overline{S_T} \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}\}_{1 \leq l \leq m}$ by using an induction argument. As we wrote before, we define $v^0 = u_1$, $\xi^0(x,t,s) = \xi(s; D_x^2 u_1, x, t)$, and $w^0(x,t,y,s) = w(y,s; D_x^2 u_1, x, t)$. Then the assertions (i) and (ii) are then immediate from Lemma 4.2.3, Theorem 4.1.1, and Corollary 4.3.2. We choose $1 \leq l \leq m$ and suppose that we have already found the families $\{v^l : \overline{S_T} \to \mathbb{R}\}_{0 \leq l \leq m}$, $\{\xi^l : \overline{S_T} \times [0,\infty) \to \mathbb{R}\}_{0 \leq l \leq m}$, and $\{w^l : \overline{S_T} \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}\}_{0 \leq l \leq m}$, which satisfy (i), (ii), and (iii).

Consider the following problem: For each $(x,t,s) \in \overline{S_T} \times [0,\infty)$, there exists a (y,s)-periodic function $\phi^l : \overline{S_T} \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ such that $\phi^l(x,t,\cdot,s) \in C^{2,\gamma}(\mathbb{R}^n)$, and a constant $\overline{E}_1^l(x,t,s) \in \mathbb{R}$ which solve the following linear cell problem:

$$a_{ij}(x,t,y,s)D_{y_iy_j}\phi^l = \overline{E}_1^l(x,t,s) + w_s^{l-1}(x,t,y,s) - \Phi^l(x,t,y,s) \quad \text{in } \mathbb{R}^n.$$

Note that Φ^l does not contain w^l, and $\Phi^l(x,t,\cdot,s) \in C^\gamma(\mathbb{R}^n)$, $w_s^{l-1}(x,t,\cdot,s) \in C^\gamma(\mathbb{R}^n)$ by the induction hypothesis. Hence the existence of ϕ^l and $\overline{E}_1^l$ follows from the same argument as in Lemma 4.3.1 with the uniform estimate

$$\begin{aligned}
&|\overline{E}_1^l(x,t,s)| + \|\phi^l(x,t,\cdot,s)\|_{C^{2,\gamma}(\mathbb{R}^n)} \\
&\quad \le C\left(\|\Phi^l(x,t,\cdot,s)\|_{C^\gamma(\mathbb{R}^n)} + \|w_s^{l-1}(x,t,\cdot,s)\|_{C^\gamma(\mathbb{R}^n)}\right) \le C_k.
\end{aligned}$$

Moreover, the induction hypothesis again we get $\overline{E}_1^l \in C^\infty(\overline{S_T}\times[0,\infty))$, and $\phi^l(\cdot,\cdot,y,\cdot) \in C^\infty(\overline{S_T}\times[0,\infty))$ with

$$\sum_{\mu+2\nu+2\rho=r}\left(|D_x^\mu\partial_t^\nu\partial_s^\rho\overline{E}_1^l| + \|D_x^\mu\partial_t^\nu\partial_s^\rho\phi^l(x,t,\cdot,s)\|_{C^{2,\gamma}(\mathbb{R}^n)}\right) \le C_{k,r} \tag{4.3.15}$$

for all $r \ge 0$. If we put

$$\widehat{\phi}^l(x,t,s) = \int_0^s \overline{E}_1^l(x,t,\tau)d\tau - s\int_0^1 \overline{E}_1^l(x,t,\tau)d\tau$$

then we deduce that

$$\begin{aligned}
\widehat{\phi}_s^l - a_{ij}D_{y_iy_j}\phi^l &= \overline{E}_1^l(x,t,s) - \int_0^1 \overline{E}_1^l(x,t,\tau)d\tau - \overline{E}_1^l(x,t,s) - w_s^{l-1} + \Phi^l \\
&= -\int_0^1 \overline{E}_1^1(x,t,\tau)d\tau - w_s^{l-1} + \Phi^l.
\end{aligned} \tag{4.3.16}$$

To this end, we choose the function $v^l : \overline{S_T} \to \mathbb{R}$ by the solution of

$$\begin{cases} v_t^l - A^{ij}(x,t,1)D_{x_ix_j}v^l = \int_0^1 \overline{E}_1^l(x,t,\tau)d\tau & \text{in } S_T, \\ v^l = 0 & \text{on } \partial_p S_T. \end{cases} \tag{4.3.17}$$

Recall from Lemma 4.2.3, Corollary 4.3.2 and (4.3.13) that $A^{ij}(x,t,1)$ is uniform elliptic in S_T and $A(x,t,1) \in C^\infty(\overline{S_T})$ whose $C^{r,\gamma}$-norm is bounded above by C_r for all $r \geq 0$. Since we also have the same regularity for $\overline{E}_1^l$ (depends on k), we obtain that $v^l \in C^\infty(\overline{S_T})$ whose $C^{r+2,\gamma}$-norm is bounded above by $C_{k,r}$ for all $r \geq 0$, which verifies (ii). Now let

$$\begin{aligned} w^l(x,t,y,s) &= \phi^l(x,t,y,s) + \chi^{ij}(x,t,y,s) D_{x_i x_j} v^l(x,t), \\ \xi^l(x,t,s) &= \widehat{\phi}^l(x,t,s) + \widehat{\chi}^{ij}(x,t,s) D_{x_i x_j} v^l(x,t). \end{aligned}$$

Then (i) can be obtained by combining (ii), (4.3.13), and (4.3.15). Finally, in view of (4.3.14), (4.3.16) and (4.3.17) that we have

$$\begin{aligned} & v_t^l + \xi_s^l - a_{ij}\left(D_{x_i x_j} v^l + D_{y_i y_j} w^l\right) - \Phi^l \\ &= v_t^1 - a_{ij} D_{x_i x_j} v^l + \left(\widehat{\phi}_s^l - a_{ij} D_{y_i y_j} \phi^l\right) + \left(\widehat{\chi}_s^{ij} - a_{ij} D_{y_i y_j} \chi^{ij}\right) D_{x_i x_j} v^l - \Phi^l \\ &= v_t^l - a_{ij} D_{x_i x_j} v^l - \int_0^1 \overline{E}_1^1(x,t,\tau)d\tau - w_s^{l-1} + \Phi^l \\ &\quad - \left(A^{ij}(x,t,1) - a_{ij}\right) D_{x_i x_j} v^l - \Phi^l \\ &= \left(v_t^l - A^{ij}(x,t,1) D_{x_i x_j} v^l - \int_0^1 \overline{E}_1^l(x,t,\tau)d\tau\right) - w_s^{l-1} \\ &= -w_s^{l-1}, \end{aligned}$$

in $\overline{S_T} \times \mathbb{R}^n \times [0,\infty)$, which shows (iii). □

We call the solution v^l of (4.3.17) the $l(2-k)$-th order effective limit. In particular, the solution $u_1 = v^0$ of (4.1.3) is 0-th order effective limit. Now we are ready to prove Theorem 4.1.3.

proof of theorem 4.1.3. Here we only investigate when $k \in (1,2)$, since the proof for the $k \in (0,1]$ case is very similar, so we omit it. Let us choose $m \in \mathbb{Z}$ to satisfy $k \in \left(\frac{2m-1}{m}, \frac{2m+1}{m+1}\right]$. We choose the families $\{v^l : \overline{S_T} \to \mathbb{R}\}_{0 \leq l \leq m}$, s-periodic functions $\{\xi^l : \overline{S_T} \times [0,\infty) \to \mathbb{R}\}_{0 \leq l \leq m}$, and (y,s)-periodic functions $\{w^l : \overline{S_T} \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}\}_{0 \leq l \leq m}$ from Lemma 4.3.4. Next, we define the families $\{X^l\}_{0 \leq l \leq m}$, $\{A^l\}_{0 \leq l \leq m}$, and the function Y^m as in (4.3.9).

We firstly observe from (4.3.6) that

$$v_t^0 + \xi_s^0 - F(X^0) = 0. \tag{4.3.18}$$

Now fix $\varepsilon_0 \in (0,1)$. Then Lemma 4.3.4 provide us the following uniform bound

$$\begin{aligned} \|X^l(\cdot,\cdot,\cdot/\varepsilon,\cdot/\varepsilon^k)\|_{L^\infty(\overline{S_T})} &\le C_k, \\ \sup_{0<\varepsilon\le\varepsilon_0} \sum_{l=0}^m \|A^l(\cdot,\cdot,\cdot/\varepsilon,\cdot/\varepsilon^k)\|_{L^\infty(\overline{S_T})} &\le C_1\varepsilon, \end{aligned} \tag{4.3.19}$$

where C_1 depends on k, ε_0, n, γ, λ, Λ, φ, and S_T. Hence, we can easily check that

$$\sup_{0<\varepsilon\le\varepsilon_0} \|Y^m(\cdot,\cdot,\cdot/\varepsilon,\cdot/\varepsilon^k)\|_{L^\infty(\overline{S_T})} \le C_2, \tag{4.3.20}$$

where C_2 also depends on k, ε_0, n, γ, λ, Λ, φ, and S_T. From now on, we fix $\varepsilon \in (0,\varepsilon_0)$, and we omit the dependency on $(x,t,x/\varepsilon,t/\varepsilon^k)$ for simplicity. Choose $R>0$ in such a way that $\overline{S_T} \subset Q_R(0,0)$ and $K>0$. Let us define $\theta_m^{\varepsilon,\pm} : \overline{S_T} \to \mathbb{R}$ by

$$\begin{aligned} \theta_m^{\varepsilon,\pm}(x,t) = &\sum_{l=0}^m \varepsilon^{l(2-k)} \left[v^l(x,t) + \varepsilon^k \xi^l \left(x,t,\frac{t}{\varepsilon^k}\right) + \varepsilon^2 w^l \left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^k}\right)\right] \\ &\pm C_0\varepsilon \pm \varepsilon K(R^2 - |x|^2 + t) \\ =: &\ \eta_m^\varepsilon \pm C_0\varepsilon \pm \varepsilon K(R^2-|x|^2+t). \end{aligned}$$

where K will be determined later, and

$$\begin{aligned} \eta_m^\varepsilon &= \sum_{l=0}^m \varepsilon^{l(2-k)} \left(v^l + \varepsilon^k \xi^l + \varepsilon^2 w^l\right), \\ C_0 &= \sum_{l=0}^m \left(\|\xi^l\|_{L^\infty(\overline{S_T}\times[0,\infty))} + \|w^l\|_{L^\infty(\overline{S_T}\times\mathbb{R}^n\times[0,\infty))}\right). \end{aligned}$$

We claim that $\theta_m^{\varepsilon,+}$ is a (viscosity) super-solution to (4.1.1). To do this, we first take a look at the spatial Hessian of this function. We notice that each ξ^l does not depend on $y = x/\varepsilon$-variable. Hence, it then follows by a direct computation that

$$\begin{aligned} D_x^2\eta_m^\varepsilon &= \sum_{l=0}^{m} \varepsilon^{l(2-k)} \left(D_x^2 v^l + \varepsilon^k D_x^2 \xi^l + \varepsilon^2 D_x^2 w^l + \varepsilon D_{x,y} w^l + D_y^2 w^l\right) \\ &= \sum_{l=0}^{m} \varepsilon^{l(2-k)} (X^l + A^l). \end{aligned}$$

Thus, by the Lipschitz continuity of F we obtain

$$\begin{aligned} F(D_x^2\eta_m^\varepsilon) &= F\left(\sum_{l=0}^{m} \varepsilon^{l(2-k)}(X^l + A^l)\right) \\ &\le F\left(\sum_{l=0}^{m} \varepsilon^{l(2-k)} X^l\right) + C\sum_{l=0}^{m} \varepsilon^{l(2-k)} \|A^l\|_{L^\infty(\overline{S_T})} \\ &\le F\left(X^0 + \varepsilon^{2-k} Y^m\right) + C_3\varepsilon. \end{aligned} \tag{4.3.21}$$

As we have seen in (4.3.10), a Taylor expansion for the last term of (4.3.21) gives

$$F(X^0 + \varepsilon^{2-k} Y^m) = F(X^0) + \sum_{l=1}^{m} \varepsilon^{l(2-k)} W^l + R_m^\varepsilon + \widetilde{R}_m^\varepsilon, \tag{4.3.22}$$

where

$$\begin{aligned} W^l &= \sum_{d=1}^{l} \frac{1}{d!} \sum_{n_1+\cdots+n_d=l} F_{p_{i_1j_1}\cdots p_{i_dj_d}}(X^0) X_{i_1j_1}^{n_1} \cdots X_{i_dj_d}^{n_d}, \\ R_m^\varepsilon &= \frac{\varepsilon_*^{(m+1)(2-k)}}{(m+1)!} F_{p_{i_1j_1}\cdots p_{i_{m+1}j_{m+1}}}(X^0) Y_{i_1j_1}^m \cdots Y_{i_{m+1}j_{m+1}}^m, \\ \widetilde{R}_m^\varepsilon &= \sum_{l=1}^{m} \sum_{m+1\le n_1+\cdots+n_l\le lm} \frac{\varepsilon^{n_1+\cdots+n_l}}{l!} F_{p_{i_1j_1}\cdots p_{i_lj_l}}(X^0) X_{i_1j_1}^{n_1} \cdots X_{i_lj_l}^{n_l} \end{aligned}$$

for some $\varepsilon_* \in [0,\varepsilon]$. Due to (4.3.19), (4.3.20), the fact that $(m+1)(2-k) \ge 1$, and the regularity assumption of F, there hold

$$|R_m^\varepsilon| + |\widetilde{R}_m^\varepsilon| \le C_4\varepsilon, \tag{4.3.23}$$

where C_4 also depends on k, ε_0, n, γ, λ, Λ, φ, and S_T. Moreover, one can check that W^l can be rewritten by

$$\begin{aligned} W^l &= \sum_{d=1}^{l} \frac{1}{d!} \sum_{n_1+\cdots+n_d=l} F_{p_{i_1j_1}\cdots p_{i_dj_d}}(X^0) X^{n_1}_{i_1j_1} \cdots X^{n_d}_{i_dj_d} \\ &= F_{p_{ij}}(X^0)X^l_{ij} + \sum_{d=2}^{l} \frac{1}{d!} \sum_{n_1+\cdots+n_d=l} F_{p_{i_1j_1}\cdots p_{i_dj_d}}(X^0) X^{n_1}_{i_1j_1} \cdots X^{n_d}_{i_dj_d} \\ &= F_{p_{ij}}(X^0)X^l_{ij} + \Phi^l(x,t,y,s), \end{aligned} \tag{4.3.24}$$

where Φ^l is as defined in Lemma 4.3.4.
On the other hand, the time derivative of $\theta^{\varepsilon,+}_m$ follows directly from the definition of $\theta^{\varepsilon,+}_m$ that

$$\begin{aligned} (\theta^{\varepsilon,+}_m)_t &= (\eta^{\varepsilon}_m)_t + K\varepsilon \\ &= \sum_{l=0}^{m} \varepsilon^{l(2-k)} \left(v^l_t + \varepsilon^k \xi^l_t + \xi^l_s + \varepsilon^2 w^l_t + \varepsilon^{2-k} w^l_s \right) + K\varepsilon \\ &= v^0_t + \xi^0_s + \sum_{l=1}^{m} \varepsilon^{l(2-k)} \left(v^l_t + \xi^l_s + w^{l-1}_s \right) \\ &\quad + \sum_{l=0}^{m} \varepsilon^{l(2-k)} \left(\varepsilon^k \xi^l_t + \varepsilon^2 w^l_t \right) + \varepsilon^{(m+1)(2-k)} w^m_s + K\varepsilon \\ &=: v^0_t + \xi^0_s + \sum_{l=1}^{m} \varepsilon^{l(2-k)} \Xi^l + \widehat{R}^{\varepsilon}_m + K\varepsilon, \end{aligned} \tag{4.3.25}$$

where

$$\begin{aligned} \Xi^l &= v^l_t + \xi^l_s + w^{l-1}_s, \\ \widehat{R}^{\varepsilon}_m &= \sum_{l=0}^{m} \varepsilon^{l(2-k)} \left(\varepsilon^k \xi^l_t + \varepsilon^2 w^l_t \right) + \varepsilon^{(m+1)(2-k)} w^m_s. \end{aligned}$$

Since $(m+1)(2-k) \geq 1$, and $l(2-k)+2 \geq l(2-k)+k \geq 1$ for any $l \geq 0$, we can deduce from Lemma 4.3.4 that

$$|\widehat{R}^{\varepsilon}_m| \leq C_5 \varepsilon. \tag{4.3.26}$$

Moreover, in view of the definition of W^l, (4.3.24) and combining the resultant with the recursive equation (4.3.12) in Lemma 4.3.4, we arrive at

$$\begin{aligned}
&\sum_{l=1}^{m} \varepsilon^{l(2-k)} \Xi^l - \sum_{l=1}^{m} \varepsilon^{l(2-k)} W^l \\
&\quad = \sum_{l=1}^{m} \varepsilon^{l(2-k)} \left[\left(v_t^l + \xi_s^l + w_s^{l-1} \right) - F_{p_{ij}}(X^0) X_{ij}^l - \Phi^l \right] \\
&\quad = \sum_{l=1}^{m} \varepsilon^{l(2-k)} \left[\left(v_t^l + \xi_s^l + w_s^{l-1} \right) - a_{ij} \left(D_{x_i x_j} v^l + D_{y_i y_j} w^l \right) - \Phi^l \right] \\
&\quad = 0
\end{aligned} \tag{4.3.27}$$

in $\overline{S_T} \times \mathbb{R}^n \times [0, \infty)$.

We have obtained so far the spatial Hessian and time derivative of $\theta_m^{\varepsilon,+}$. We will mix these results. Choose $K \geq (2\lambda)^{-1}(C_3 + C_4 + C_5)$. Then from (4.3.21), (4.3.22), (4.3.23), and the uniform ellipticity of F that

$$\begin{aligned}
F(D_x^2 \theta_m^{\varepsilon,+}) &\leq F(D_x^2 \eta_m^{\varepsilon}) - 2\lambda K \varepsilon \\
&\leq F(X_0 + \varepsilon^{2-k} Y^m) + C_3 \varepsilon - 2\lambda K \varepsilon \\
&= F(X^0) + \sum_{l=1}^{m} \varepsilon^{l(2-k)} W^l + R_m^{\varepsilon} + \widetilde{R}_m^{\varepsilon} + C_3 \varepsilon - 2\lambda K \varepsilon \\
&\leq F(X^0) + \sum_{l=1}^{m} \varepsilon^{l(2-k)} W^l.
\end{aligned}$$

Consequently, we combine (4.3.18), (4.3.25), (4.3.26), (4.3.27), and above estimate that

$$\begin{aligned}
&(\theta_m^{\varepsilon,+})_t - F(D_x^2\theta_m^{\varepsilon,+}) \\
&\geq v_t^0 + \xi_s^0 + \sum_{l=1}^{m} \varepsilon^{l(2-k)}\Xi^l + \widehat{R}_m^\varepsilon + K\varepsilon - F(X^0) - \sum_{l=1}^{m} \varepsilon^{l(2-k)} W^l \\
&\geq v_t^0 + \xi_s^0 + \sum_{l=1}^{m} \varepsilon^{l(2-k)} \left(\Xi^l - W^l\right) - F(X^0) \\
&= v_t^0 + \xi_s^0 - F(X^0) \\
&= 0.
\end{aligned}$$

Now we investigate the boundary value of $\theta_m^{\varepsilon,+}$. First, we note that $v^0 = u_1 = \varphi$, and $v^l = 0$ on $\partial_p S_T$ for all $1 \leq l \leq m$. Hence, by definition of C_0 we have for $\varepsilon < \varepsilon_0$ that

$$\begin{aligned}
\theta_m^{\varepsilon,+} - \varphi &= \sum_{l=0}^{m} \varepsilon^{l(2-k)} \left(v^l + \varepsilon^k \xi^l + \varepsilon^2 w^l\right) + C_0\varepsilon + \varepsilon K(R^2 - |x|^2 + t) - \varphi \\
&\geq (v^0 - \varphi) + \varepsilon \left[C_0 + \sum_{l=0}^{m} \varepsilon^{l(2-k)} \left(\varepsilon^{k-1}\xi^l + \varepsilon w^l\right)\right] \\
&\geq 0.
\end{aligned}$$

Thus, $\theta_m^{\varepsilon,+}$ is a viscosity super-solution of (4.1.1). In a similar manner, one can verify that $\theta_m^{\varepsilon,-}$ is a viscosity sub-solution of (4.1.1). Thus, the comparison principle yields $\theta_m^{\varepsilon,-} \leq u^\varepsilon \leq \theta_m^{\varepsilon,+}$ in $\overline{S_T}$. It then follows that

$$\left\| u^\varepsilon - u_1 - \sum_{l=1}^{m} \varepsilon^{(2-k)l} v^l \right\|_{L^\infty(S_T)} \leq \widetilde{C}\varepsilon,$$

where $\widetilde{C}$ depends on k, ε_0, n, γ, λ, Λ, φ, and S_T. □

Remark 4.3.5. The proofs of Lemma 4.3.4 and Theorem 4.1.3 for the case $k \in (0,1]$ share the same idea presented above, when $k \in (1,2)$.There are several differences from the case $k \in (1,2)$, which are as follows. First, the interval to which k belongs is changed to $\left(\frac{1}{m+1}, \frac{1}{m}\right]$, for each integer $m \geq$

1. Second, the order of barriers constructed in the proof of Theorem 4.1.3 is ε^{lk}, not $\varepsilon^{l(2-k)}$. That is,

$$\eta_m^\varepsilon = \sum_{l=0}^{m} \varepsilon^{lk} \left[v^l(x,t) + \varepsilon^k \xi^l \left(x, t, \frac{t}{\varepsilon^k} \right) + \varepsilon^2 w^l \left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^k} \right) \right]$$

where η_m^ε is a function defined in the proof of Theorem 4.1.3. Therefore, the order of the convergence rate is also changed to (4.1.5) compared to (4.1.6) due to this effect.

4.4 Homogenization when $k \in (2, \infty)$

In this section, we consider the case of $k \in (2,\infty)$. Contrary to the case of $k \in (0,2)$, it is natural attempt to find a homogenization for time, then for space. To do this, we consider following time homogenized operators:

$$\widehat{F}(M,x,t,y) := \int_0^1 F(M,x,t,y,\tau) d\tau.$$

Then we can easily check that $\widehat{F}$ satisfies the structure conditions of F.

4.4.1 The effective operator and the effective limit

Now we first look at the cell problem, the modified version of Lemma 4.2.1. Before we start, we point out that the y-variable regularity of the corrector should be improved since it is used importantly in the process of finding the convergence rate.

Lemma 4.4.1. *For each $(M,x,t) \in \mathcal{S}^n \times \overline{S_T}$ there exists a y-periodic function $w(y;M,x,t)$ such that $w(\cdot;M,x,t) \in C^{\sigma+2,\gamma}(\mathbb{R}^n)$ for any $\sigma \geq 0$, and a constant $\overline{E}_3(M,x,t) \in \mathbb{R}$ which solve the following cell problem:*

$$\widehat{F}(M + D_y^2 w, x, t, y, s) = \overline{E}_3(M,x,t) \quad \text{in } \mathbb{R}^n \tag{4.4.1}$$

with the uniform estimate

$$|\overline{E}_3(M,x,t)| + \|w(\cdot;M,x,t)\|_{C^{\sigma+2,\gamma}(\mathbb{R}^n)} \le C_\sigma \left(1 + \|M\|\right).$$

Moreover, $\overline{E}_3$ is a unique constant where the equation has a unique solution w up to constant addition.

Proof. The proof of Lemma 4.4.1 is similar to that in Lemma 4.2.1, so the details are omitted. We make a remark here that the regularity and the uniform estimate of w only depend on the space fast variable y. □

Define

$$\overline{F_3}(M,x,t) := \overline{E}_3(M,x,t).$$

We will call $\overline{F_3}$ the effective operator when $k \in (2,\infty)$. By the same argument as in the case of $k \in (0,2)$, we can observe that the effective operator $\overline{F_3}$ is uniformly elliptic with the same ellipticity constants of F and convex with respect to M-variable. Moreover, the regularity results for the w and $\overline{E}_3$ stated in Lemma 4.2.3 also hold.

Recall that the heuristic calculation (4.1.9) and (4.1.10), and let's consider of these two equations as PDEs for u^0. If we know the functions u^0 and u^2, then the extra term u_s^3, which has the information of the third correctors, can be calculated explicitly by subtracting (4.1.10) from (4.1.9). So, let us consider the function $\xi : \mathbb{R}^n \times [0,\infty) \times \mathcal{S}^n \times \overline{S_T} \to \mathbb{R}$ defined by

$$\xi(y,s;M,x,t) = \int_0^s F(M + D_y^2 w, x, t, y, \tau)d\tau - s\overline{F_3}(M,x,t). \tag{4.4.2}$$

It is clear that ξ is y-periodic. Moreover, since F is s-periodic, and w is s-independent, ξ is also s-periodic.

Let us now establish an homogenized equation.

proof of theorem 4.1.2. We already proved the existence of u_3 and uniform convergence(up to subsequence) $u^\varepsilon \to u_3$ on $\overline{S_T}$ in Theorem 4.1.3. Let P

be a paraboloid with $M_0 = D_x^2 P$ which touches u_3 by above at (x_0, t_0) in a neighborhood. Without loss of generality, we may assume that P touches u_3 strictly by above. Assume, to the contrary, that

$$P_t(x_0, t_0) - \overline{F_3}(M_0, x_0, t_0) > 3\eta > 0$$

for some $\eta > 0$. Put $\widehat{w}(y) := w(y; M_0, x_0, t_0)$. Then from Lemma 4.4.1 we can observe that $\widehat{w}$ satisfies

$$\widehat{F}(M_0 + D_y^2 \widehat{w}, x_0, t_0, y) = \overline{F_3}(M_0, x_0, t_0) \quad \text{in } \mathbb{R}^n. \tag{4.4.3}$$

On the other hand, the continuity of F and $\overline{F_3}$ imply that we can choose $\rho > 0$ in such way that $Q_\rho(x_0, t_0) \subset S_T$,

$$\begin{aligned} &P_t - \overline{F_3}(M_0, x, t) > 3\eta, \quad \text{and} \\ &|F(M_0 + D_y^2 \widehat{w} + N, x, t, y, s) - F(M_0 + D_y^2 \widehat{w}, x_0, t_0, y, s)| \\ &\quad + |\overline{F_3}(M_0, x, t) - \overline{F_3}(M_0, x_0, t_0)| < \eta \end{aligned} \tag{4.4.4}$$

for any $(x, t) \in Q_\rho(x_0, t_0)$, and $N \in \mathcal{S}^n$ with $\|N\| < \rho$, uniformly $(y, s) \in \mathbb{R}^n \times [0, \infty)$. Moreover, $u_3(x, t) - P(x, t) \leq -\mu$ on ∂Q_ρ, for some $\mu > 0$.
Now define

$$\widehat{\xi}(y, s) = \xi(y, s; M_0, x_0, t_0), \tag{4.4.5}$$

where the definition of ξ is in (4.4.2), and set

$$P^\varepsilon(x, t) := P(x, t) + \varepsilon^2 \widehat{w}\left(\frac{x}{\varepsilon}\right) + \varepsilon^k \widehat{\xi}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k}\right).$$

For a while, let us drop the dependency of $(x/\varepsilon, t/\varepsilon^k)$. Since $k - 2 > 0$, in view of (4.4.3), (4.4.4), and (4.4.5) we have

$$\begin{aligned}
P_t^\varepsilon - F\left(D_x^2 P^\varepsilon, x, t\right) &= P_t + \widehat{\xi}_s - F\left(M_0 + D_y^2\widehat{w} + \varepsilon^{k-2}D_y^2\widehat{\xi}, x, t\right)\\
&= P_t + \left[F\left(M_0 + D_y^2\widehat{w}, x_0, t_0\right) - \overline{F_3}(M_0, x_0, t_0)\right]\\
&\quad - F\left(M_0 + D_y^2\widehat{w} + \varepsilon^{k-2}D_y^2 h, x, t\right)\\
&\geq P_t - \overline{F_3}(M_0, x_0, t_0) - \eta\\
&\geq P_t - \overline{F_3}(M_0, x, t) - 2\eta\\
&> 0
\end{aligned}$$

if ε is small enough, in $Q_r(x_0, t_0)$. As $u^\varepsilon \to u_3$ and $P^\varepsilon \to P$ uniformly in $Q_r(x_0, t_0)$, we can easily check that for some $\varepsilon_0 \in (0, 1)$ there holds

$$u^\varepsilon(x, t) - P^\varepsilon(x, t) < -\mu/2 \quad \text{on } \partial Q_r(x_0, t_0), \quad \varepsilon < \varepsilon_0.$$

Hence $P^\varepsilon - \mu/4$ is a super-solution to the following initial-boundary value problem:

$$\begin{cases} v_t - F(D_x^2 v, x, t, x/\varepsilon, t/\varepsilon^k) = 0 & \text{in } Q_r(x_0, t_0),\\ v = u^\varepsilon(x, t) & \text{on } \partial_p Q_r(x_0, t_0). \end{cases}$$

Hence, the comparison principle implies $u^\varepsilon \leq P^\varepsilon - \mu/4$ in $Q_r(x_0, t_0)$. Letting $\varepsilon \to 0$ then $u_3(x_0, t_0) \leq P(x_0, t_0) - \mu/4$ which contradicts assumption that $u_3(x_0, t_0) = P(x_0, t_0)$. It shows that u_3 is a viscosity sub-solution of (4.1.4). In a similar manner, we are able to prove that u_3 is a viscosity super-solution of (4.1.4). The rest of the process follows the same argument as in Theorem 4.1.1, so is omitted. This completes the proof. □

We also have the following regularity result.

Corollary 4.4.2. *Assume that F and φ verify the structure conditions. Then $u_3 \in C^\infty(\overline{S_T})$ and*

$$\|u_1\|_{C^{r+2,\gamma}(\overline{S_T})} \leq C_r$$

for each $r \geq 0$, where C_r depends only on n, γ, λ, Λ, φ, and S_T.

Remark 4.4.3. Like the case $k \in (0,2)$, we remark that u and $\overline{F_3}$ are independent of $k \in (2,\infty)$.

4.4.2 Rate of convergence for the homogenization

From now on, we write

$$\begin{aligned} v^0(x,t) &= u_3(x,t), \\ w^0(x,t,y) &= w(y; D_x^2 u, x, t), \\ \xi^0(x,t,y,s) &= \xi(y,s; D_x^2 u, x, t). \end{aligned}$$

Then in view of (4.4.1), (4.4.2), and (4.1.4) we have

$$\begin{aligned} &v_t^0 + \xi_s^0 - F(D_x^2 v^0 + D_y^2 w^0) \\ &\quad = v_t^0 + F(D_x^2 v^0 + D_y^2 w^0) - \overline{F_3}(D_x^2 v^0, x, t) - F(D_x^2 v^0 + D_y^2 w^0) = 0. \end{aligned} \tag{4.4.6}$$

We are now in a position to state and give the proof of the convergence rate when $k \in (2,\infty)$. The overall process is similar to the case of $k \in (0,2)$. The only difference is the order in which the homogenization occurs first in time or space, and this is reflected in the role of k-th order corrector ξ.

We first introduce the result similar to Lemma 4.3.4.

Lemma 4.4.4. *Suppose that $k \in (2,3)$, and let $m \in \mathbb{Z}$ be chosen to satisfy $k \in \left[\frac{2m+3}{m+1}, \frac{2m+1}{m}\right)$. Assume that F and φ satisfy the structure conditions. Then there exist families of $\{v^l : \overline{S_T} \to \mathbb{R}\}_{1\leq l\leq m}$, y-periodic functions $\{w^l : \overline{S_T} \times \mathbb{R}^n \to \mathbb{R}\}_{1\leq l\leq m}$, and (y,s)-periodic functions $\{\xi^l : \overline{S_T} \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}\}_{1\leq l\leq m}$, which satisfy following conditions:*

(i) $w^l(\cdot,\cdot,y) \in C^\infty(\overline{S_T})$, $\xi^l(\cdot,\cdot,y,\cdot) \in C^\infty(\overline{S_T} \times [0,\infty))$ uniformly for $y \in \mathbb{R}^n$, and $w^l(x,t,\cdot)$, $\xi^l(x,t,\cdot,s) \in C^\infty(\mathbb{R}^n)$ uniformly for all $(x,t,s) \in$

$\overline{S_T} \times [0,\infty)$ with

$$\sum_{\mu+2\nu+2\rho=r} \left(\|D_x^\mu \partial_t^\nu w^l(x,t,\cdot)\|_{C^{\sigma,\gamma}(\mathbb{R}^n)} + \|D_x^\mu \partial_t^\nu \partial_s^\rho \xi^l(x,t,\cdot,s)\|_{C^{\sigma,\gamma}(\mathbb{R}^n)}\right) \le C_{k,r,\sigma}$$

for all r, $\sigma \ge 0$, where $C_{k,r,\sigma}$ depends only on n, γ, λ, Λ, φ, and S_T.

(ii) $v^l \in C^\infty(\overline{S_T})$ with

$$\|v^l\|_{C^{r+2,\gamma}(\overline{S_T})} \le C_{k,r}$$

for all $r \ge 0$, where $C_{k,r}$ depends only on n, γ, λ, Λ, φ, and S_T.

(iii) For each $1 \le l \le m$, v^l, ξ^l and w^l satisfy the following recursive relation:

$$v_t^l + \xi_s^l - a_{ij}\left(D_{x_ix_j} v^l + D_{y_iy_j} w^l\right) = a_{ij} D_{y_iy_j} \xi^{l-1} + \Phi^l \quad \text{in } \overline{S_T} \times \mathbb{R}^n \times [0,\infty), \tag{4.4.7}$$

where

$$\begin{aligned}
X^0(x,t,y) &= D_x^2 v^0(x,t) + D_y^2 w^0(x,t,y), \\
X^l(x,t,y) &= D_x^2 v^l(x,t) + D_y^2 w^l(x,t,y) + D_y^2 \xi^{l-1}(x,t,y,s), \quad (l \ge 1), \\
a_{ij}(x,t,y,s) &= F_{p_{ij}}(X^0,x,t,y,s), \\
\Phi^1(x,t,y,s) &= 0, \\
\Phi^l(x,t,y,s) &= \sum_{d=2}^{l} \frac{1}{d!} \sum_{n_1+\cdots+n_d=l} F_{p_{i_1j_1}\cdots p_{i_dj_d}}(X^0) X_{i_1j_1}^{n_1} \cdots X_{i_dj_d}^{n_d}, \quad (l \ge 2).
\end{aligned}$$

Proof. As the first step, we linearize the equation (4.4.1). Pick $(x,t) \in \overline{S_T}$, and from now on we omit the dependence on (x,t) for notational convenience.

Let $\chi_h^{\alpha\beta}(y) = h^{-1}[w(y; D_x^2 v^0 + hE^{\alpha\beta}) - w(y; D_x^2 v^0)]$, and

$$\begin{aligned}
a_{ij,h} &= \int_0^1 F_{p_{ij}}(N_{\theta,h}) d\theta, \\
\widehat{a}_{ij,h} &= \int_0^1 \widehat{F}_{p_{ij}}(N_{\theta,h}) d\theta, \\
N_{\theta,h} &= \theta D_y^2 w(y, D_x^2 v^0 + hE^{\alpha\beta}) + (1-\theta) D_y^2 w(y, D_x^2 v^0) + D_x^2 v^0 + \theta h E^{\alpha\beta}, \\
\overline{\widehat{a}}_{\alpha\beta,h} &= \frac{\overline{E}_3(D_x^2 v^0 + hE^{\alpha\beta}) - \overline{E}_3(D_x^2 v^0)}{h},
\end{aligned}$$

where $\{E^{ij}\}_{1\le i,j\le n}$ be the set of standard basis matrices. Note that since u_3 and w are independent of s-variable, we get

$$\widehat{a}_{ij,h}(x,t,y) = \int_0^1 a_{ij,h}(x,t,y,\tau) d\tau.$$

We then deduce that $\chi_h^{\alpha\beta}$ satisfies

$$\widehat{a}_{ij,h} D_{y_i y_j} \chi_h^{\alpha\beta} + \widehat{a}_{\alpha\beta,h} = \overline{\widehat{a}}_{\alpha\beta,h}.$$

We can observe that $\widehat{a}_{ij,h}$ is uniform elliptic with the same ellipticity constants of $\widehat{F}$(and so F) uniformly in h. In addition, Lemma 4.2.3 and Theorem 4.1.2 imply that for any h with $|h|$ small, $a_{ij,h}(x,t,\cdot,s) \in C^\infty(\mathbb{R}^n)$, $\widehat{a}_{ij,h}(x,t,\cdot) \in C^\infty(\mathbb{R}^n)$ with

$$|\overline{\widehat{a}}_{\alpha\beta,h}| + \|a_{ij,h}(x,t,\cdot), s\|_{C^{\sigma,\gamma}(\mathbb{R}^n)} + \|\widehat{a}_{ij,h}(x,t,\cdot)\|_{C^{\sigma,\gamma}(\mathbb{R}^n)} \le C_\sigma,$$

for all $\sigma \ge 0$, and

$$\begin{aligned}
\widehat{a}_{ij,h}(x,t,y) &\to \widehat{a}_{ij}(x,t,y) = \widehat{F}_{p_{ij}}(D_x^2 u + D_y^2 w, x, t, y), \\
a_{ij,h}(x,t,y,s) &\to a_{ij}(x,t,y,s) = F_{p_{ij}}(D_x^2 u + D_y^2 w, x, t, y, s)
\end{aligned}$$

uniformly in $\mathbb{R}^n$ as $h \to 0$. Note that we can easily check that

$$\widehat{a}_{ij}(x,t,y) = \int_0^1 a_{ij}(x,t,y,\tau) d\tau.$$

Consequently, by the same argument of the perturbed test function method as in Lemma 4.2.1 (see [34], Lemma 2.1.2), there exists a unique constant $\overline{\overline{a}}_{\alpha\beta}(x,t) = (\overline{E}_3)_{p_{\alpha\beta}}(D_x^2 v^0, x, t)$ and a bounded y-periodic function $\chi^{\alpha\beta}(x,t,y) = D_{p_{\alpha\beta}} w(y; D_x^2 v^0, x, t)$ with $\chi^{\alpha\beta}(x,t,\cdot) \in C^\infty(\mathbb{R}^n)$ such that

$$|\overline{\overline{a}}_{\alpha\beta,h} - \overline{\overline{a}}_{\alpha\beta}| + \|\chi_h^{\alpha\beta} - \chi^{\alpha\beta}\|_{L^\infty(\mathbb{R}^n)} \to 0$$

as $h \to 0$. Then $\chi^{\alpha\beta}$ satisfies

$$\widehat{a}_{ij} D_{y_i y_j} \chi^{\alpha\beta} + \widehat{a}_{\alpha\beta} = \overline{\overline{a}}_{\alpha\beta}.$$

Since we also have $\widehat{a}_{ij}(x,t,\cdot) \in C^{\sigma,\gamma}(\mathbb{R}^n)$ with $\|\widehat{a}_{ij}(x,t,\cdot)\|_{C^{\sigma,\gamma}(\mathbb{R}^n)} \le C_\sigma$, then from Lemma 4.2.3 we can observe that $\overline{\overline{a}}_{\alpha\beta} = \overline{\overline{a}}_{\alpha\beta}(x,t) \in C^\infty(\overline{S_T})$, and $\chi^{\alpha\beta} = \chi^{\alpha\beta}(\cdot,\cdot,y) \in C^\infty(\overline{S_T})$ with

$$\sum_{\mu+2\nu=r} \left(|D_x^\mu \partial_t^\nu \overline{\overline{a}}_{\alpha\beta}| + \|D_x^\mu \partial_t^\nu \chi^{\alpha\beta}(x,t,\cdot)\|_{C^{\sigma+2,\gamma}(\mathbb{R}^n)} \right) \le C_{r,\sigma} \tag{4.4.8}$$

for all r, $\sigma \ge 0$. Now putting $A^{ij}(x,t,y,s) = \int_0^s a_{ij}(x,t,y,\tau)d\tau$ and we define a additional function $\widehat{\chi}^{\alpha\beta} : \overline{S_T} \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ in a similar way to finding the corrector ξ such that

$$\widehat{\chi}^{\alpha\beta}(x,t,y,s) = A^{ij}(x,t,y,s)\left(D_{y_i y_j}\chi^{\alpha\beta} + \delta_{\alpha\beta}\right) - s\overline{\overline{a}}_{\alpha\beta}(x,t),$$

where $\delta_{\alpha\beta}$ is the Kronecker delta function. Then $\widehat{\chi}^{\alpha\beta}$ is the (y,s)-periodic function, and we deduce that

$$\widehat{\chi}_s^{\alpha\beta} - a_{ij} D_{y_i y_j}\chi^{\alpha\beta} = a_{ij}\left(D_{y_i y_j}\chi^{\alpha\beta} + \delta_{\alpha\beta}\right) - \overline{\overline{a}}_{\alpha\beta} - a_{ij} D_{y_i y_j}\chi^{\alpha\beta} = a_{\alpha\beta} - \overline{\overline{a}}_{\alpha\beta}. \tag{4.4.9}$$

Now we construct the family of functions $\{v^l : \overline{S_T} \to \mathbb{R}\}_{0 \le l \le m}$, $\{w^l : \overline{S_T} \times \mathbb{R}^n \to \mathbb{R}\}_{1 \le l \le m}$, and $\{\xi^l : \overline{S_T} \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}\}_{0 \le l \le m}$ by using an induction argument. As we wrote before, we define $v^0 = u_3$, $w^0(x,t,y) = w(y; D_x^2 u_3, x, t)$, and $\xi^0(x,t,y,s) = \xi(y,s; D_x^2 u_3, x, t)$. Then the assertions

(i) and (ii) are then immediate from Lemma 4.2.3, Theorem 4.1.2, and Corollary 4.4.2. We choose $1 \le l \le m$ and suppose that we have already found the families $\{v^l : \overline{S_T} \to \mathbb{R}\}_{0\le l\le m}$, $\{w^l : \overline{S_T} \times \mathbb{R}^n \to \mathbb{R}\}_{0\le l\le m}$, and $\{\xi^l : \overline{S_T} \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}\}_{0\le l\le m}$, which satisfy (i), (ii), and (iii).
Consider the following problem: For each $(x,t) \in \overline{S_T}$, there exists a y-periodic function $\phi^l(x,t,y)$ such that $\phi^l(x,t,\cdot) \in C^{\sigma+2,\gamma}(\mathbb{R}^n)$, and a constant $\overline{E}^l_3(x,t) \in \mathbb{R}$ which solve the following linear cell problem:

$$\widehat{a}_{ij}(x,t,y)D_{y_iy_j}\phi^l = \int_0^1 a_{ij}(x,t,y,s)D_{y_iy_j}\xi^{l-1}ds + \overline{E}^l_3(x,t) - \widehat{\Phi}^l(x,t,y) \quad \text{in } \mathbb{R}^n,$$

where

$$\widehat{\Phi}^l(x,t,y) = \int_0^1 \Phi^l(x,t,y,\tau)d\tau.$$

Note that $\widehat{\Phi}^l$ does not contain w^l and ξ^l. In addition, we also know that $\widehat{\Phi}^l(x,t,\cdot) \in C^{\sigma,\gamma}(\mathbb{R}^n)$, $\xi^{l-1}(x,t,\cdot,s) \in C^{\sigma+2,\gamma}(\mathbb{R}^n)$ by the induction hypothesis. Hence, the existence of ϕ^l and $\overline{E}^l_3$ follows from the same argument as in Lemma 4.3.1 with the uniform estimate

$$\begin{aligned} &|\overline{E}^l_3(x,t)| + \|\phi^l(\cdot;x,t)\|_{C^{\sigma+2,\gamma}(\mathbb{R}^n)} \\ &\quad \le C\left(\|\widehat{\Phi}^l(x,t,\cdot)\|_{C^{\sigma,\gamma}(\mathbb{R}^n)} + \|D^2_y\xi^{l-1}(x,t,\cdot,s)\|_{C^{\sigma,\gamma}(\mathbb{R}^n)}\right) \le C_{k,\sigma}. \end{aligned}$$

Moreover, the induction hypothesis again we get $\overline{E}^l_3$, $\phi^l(\cdot,\cdot,y) \in C^\infty(\overline{S_T})$ with

$$\sum_{\mu+2\nu=r}\left(|D^\mu_x\partial^\nu_t\overline{E}^l_3| + \|D^\mu_x\partial^\nu_t\phi^l(x,t,\cdot)\|_{C^{\sigma+2,\gamma}(\mathbb{R}^n)}\right) \le C_{k,r,\sigma} \tag{4.4.10}$$

for all r, $\sigma \ge 0$. If we put

$$\begin{aligned} \widehat{\phi}^l(x,t,y,s) = A^{ij}D_{y_iy_j}\phi^l &- \int_0^s a_{ij}(x,t,y,\tau)D_{y_iy_j}\xi^{l-1}(x,t,y,\tau)d\tau \\ &+ \int_0^s \Phi^l(x,t,y,\tau)d\tau - s\overline{E}^l_3(x,t), \end{aligned}$$

then which shows

$$\begin{aligned}\widehat{\phi}^l_s - a_{ij}D_{y_iy_j}\phi^l &= a_{ij}D_{y_iy_j}(\phi^l - \xi^{l-1}) + \Phi^l - \overline{E}^l_3 - a_{ij}D_{y_iy_j}\phi^l \\ &= -a_{ij}D_{y_iy_j}\xi^{l-1} + \Phi^l - \overline{E}^l_3.\end{aligned} \tag{4.4.11}$$

To this end, we choose the function $v^l : \overline{S_T} \to \mathbb{R}$ by the solution of

$$\begin{cases} v^l_t - \overline{\widehat{a}}_{ij}D_{x_ix_j}v^l = \overline{E}^l_3(x,t) & \text{in } S_T, \\ v^l = 0 & \text{on } \partial_p S_T. \end{cases} \tag{4.4.12}$$

Recall from Lemma 4.2.3, Corollary 4.4.2 and (4.4.8) that $\overline{\widehat{a}}_{ij}$ is uniform elliptic in S_T and $\overline{\widehat{a}}_{ij} \in C^\infty(\overline{S_T})$ whose $C^{r,\gamma}$-norm is bounded above by C_r for all $r \geq 0$. Since we also have the same regularity for $\overline{E}^l_3$(we choose $\sigma = r$, depends on k), we obtain that $v^l \in C^\infty(\overline{S_T})$ whose $C^{r+2,\gamma}$-norm is bounded above by $C_{k,r}$ for all $r \geq 0$, which verifies (ii). Now let

$$\begin{aligned} w^l(x,t,y) &= \phi^l(x,t,y) + \chi^{ij}(x,t,y)D_{x_ix_j}v^l(x,t), \\ \xi^l(x,t,y,s) &= \widehat{\phi}^l(x,t,y,s) + \widehat{\chi}^{ij}(x,t,y,s)D_{x_ix_j}v^l(x,t). \end{aligned}$$

Then (i) can be obtained by combining (ii), (4.4.8), and (4.4.10). Finally, in view of (4.4.9), (4.4.11) and (4.4.12) that we have

$$\begin{aligned} & v^l_t + \xi^l_s - a_{ij}\left(D_{x_ix_j}v^l + D_{y_iy_j}w^l\right) - \Phi^l \\ &= v^1_t - a_{ij}D_{x_ix_j}v^l + \left(\widehat{\phi}^l_s - a_{ij}D_{y_iy_j}\phi^l\right) + \left(\widehat{\chi}^{ij}_s - a_{ij}D_{y_iy_j}\chi^{ij}\right)D_{x_ix_j}v^l - \Phi^l \\ &= v^l_t - a_{ij}D_{x_ix_j}v^l - a_{ij}D_{y_iy_j}\xi^{l-1} + \Phi^l - \overline{E}^l_3 + \left(a_{ij} - \overline{\widehat{a}}_{ij}\right)D_{x_ix_j}v^1 - \Phi^l \\ &= \left(v^l_t - \overline{\widehat{a}}_{ij}D_{x_ix_j}v^l - \overline{E}^l_3(x,t)\right) + a_{ij}D_{y_iy_j}\xi^{l-1} \\ &= a_{ij}D_{y_iy_j}\xi^{l-1}, \end{aligned}$$

in $\overline{S_T} \times \mathbb{R}^n \times [0,\infty)$, which shows (iii). □

We call the solution v^l of (4.4.12) the $l(k-2)$-th order effective limit. In particular, the solution $u_3 = v^0$ of (4.1.4) is 0-th order effective limit. Now we are ready to prove 4.1.4.

proof of theorem 4.1.4. Let's first assume that $k \in (2,3)$. Let us choose $m \in \mathbb{Z}$ to satisfy $k \in \left[\frac{2m+3}{m+1}, \frac{2m+1}{m}\right)$. We choose the families $\{v^l : \overline{S_T} \to \mathbb{R}\}_{0\le l\le m}$, y-periodic functions $\{w^l : \overline{S_T} \times \mathbb{R}^n \to \mathbb{R}\}_{0\le l\le m}$, and (y,s)-periodic functions $\{\xi^l : \overline{S_T} \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}\}_{0\le l\le m}$ from Lemma 4.4.4. Next, we define the families $X^0 : \overline{S_T} \times \mathbb{R}^n \to \mathbb{R}$, $\{X^l : \overline{S_T} \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}\}_{1\le l\le m}$, $\{A^l : \overline{S_T} \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}\}_{0\le l\le m}$, and the function $Y^m : \overline{S_T} \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ as

$$
\begin{aligned}
X^0 &= D_x^2 v^0(x,t) + D_y^2 w^0\left(x,t,\frac{x}{\varepsilon}\right),\\
X^l &= D_x^2 v^l(x,t) + D_y^2 w^l\left(x,t,\frac{x}{\varepsilon}\right) + D_y^2 \xi^{l-1}\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^k}\right), \quad (l \ge 1),\\
A^l &= \varepsilon D_{x,y} w^l\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^k}\right) + \varepsilon^2 D_x^2 w^l\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^k}\right)\\
&\quad + \varepsilon^{k-1} D_{x,y}\xi^l\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^k}\right) + \varepsilon^k D_x^2 \xi^l\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^k}\right), \quad (l \le m-1),\\
A^m &= \varepsilon D_{x,y} w^m\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^k}\right) + \varepsilon^2 D_x^2 w^m\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^k}\right)\\
&\quad + \varepsilon^{k-1} D_{x,y}\xi^m\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^k}\right) + \varepsilon^k D_x^2 \xi^m\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^k}\right)\\
&\quad + \varepsilon^{(m+1)(k-2)} D_y^2 \xi^m\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^k}\right),\\
Y^m &= X^1 + \cdots + \varepsilon^{(m-1)(k-2)} X^m = \sum_{l=1}^{m} \varepsilon^{(l-1)(k-2)} X^l.
\end{aligned}
$$

We firstly observe from (4.4.6) that

$$
v_t^0 + \xi_s^0 - F(X^0) = 0. \tag{4.4.13}
$$

Now fix $\varepsilon_0 \in (0,1)$. Then Lemma 4.4.4 provide us the following uniform bound

$$\begin{aligned} \|X^l(\cdot,\cdot,\cdot/\varepsilon,\cdot/\varepsilon^k)\|_{L^\infty(\overline{S_T})} &\le C_k, \\ \sup_{0<\varepsilon\le\varepsilon_0}\sum_{l=0}^m \|A^l(\cdot,\cdot,\cdot/\varepsilon,\cdot/\varepsilon^k)\|_{L^\infty(\overline{S_T})} &\le C_1\varepsilon, \end{aligned} \tag{4.4.14}$$

where C_1 depends on k, ε_0, n, γ, λ, Λ, φ, and S_T. Hence, we can easily check that

$$\sup_{0<\varepsilon\le\varepsilon_0} \|Y^m(\cdot,\cdot,\cdot/\varepsilon,\cdot/\varepsilon^k)\|_{L^\infty(\overline{S_T})} \le C_2, \tag{4.4.15}$$

where C_2 also depends on k, ε_0, n, γ, λ, Λ, φ, and S_T. From now on, we fix $\varepsilon \in (0,\varepsilon_0)$, and we omit the dependency on $(x,t,x/\varepsilon,t/\varepsilon^k)$ for simplicity. Choose $R>0$ in such a way that $\overline{S_T} \subset Q_R(0,0)$ and $K>0$. Let us define $\theta_m^{\varepsilon,\pm} : \overline{S_T} \to \mathbb{R}$ by

$$\begin{aligned} \theta_m^{\varepsilon,\pm}(x,t) &= \sum_{l=0}^m \varepsilon^{l(k-2)}\left[v^l(x,t) + \varepsilon^2 w^l\left(x,t,\frac{x}{\varepsilon}\right) + \varepsilon^k \xi^l\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^k}\right)\right] \\ &\quad \pm C_0\varepsilon \pm \varepsilon K(R^2 - |x|^2 + t) \\ &=: \eta_m^\varepsilon(x,t) \pm C_0\varepsilon \pm \varepsilon K(R^2 - |x|^2 + t). \end{aligned}$$

where K will be determined later, and

$$\begin{aligned} \eta_m^\varepsilon &= \sum_{l=0}^m \varepsilon^{l(k-2)}\left(v^l + \varepsilon^2 w^l + \varepsilon^k \xi^l\right) \\ C_0 &= \sum_{l=0}^m \left(\|w^l\|_{L^\infty(\overline{S_T}\times\mathbb{R}^n)} + \|\xi^l\|_{L^\infty(\overline{S_T}\times\mathbb{R}^n\times[0,\infty))}\right). \end{aligned}$$

We claim that $\theta_m^{\varepsilon,+}$ is a (viscosity) super-solution to (4.1.1). To do this, we first take a look at the spatial Hessian of this function. It then follows by a direct computation that

$$\begin{aligned} D_x^2\eta_m^\varepsilon &= \sum_{l=0}^{m} \varepsilon^{l(k-2)} \big(D_x^2 v^l + \varepsilon^2 D_x^2 w^l + \varepsilon D_{x,y} w^l + D_y^2 w^l \\ &\quad + \varepsilon^k D_x^2 \xi^l + \varepsilon^{k-1} D_{x,y}\xi^l + \varepsilon^{k-2} D_y^2 \xi^l\big) \\ &= \sum_{l=0}^{m} \varepsilon^{l(k-2)} (X^l + A^l). \end{aligned}$$

Thus, by the Lipschitz continuity of F we obtain

$$\begin{aligned} F(D_x^2\eta_m^\varepsilon) &= F\left(\sum_{l=0}^{m} \varepsilon^{l(k-2)}(X^l + A^l)\right) \\ &\le F\left(\sum_{l=0}^{m} \varepsilon^{l(k-2)} X^l\right) + C\sum_{l=0}^{m} \varepsilon^{l(k-2)} \|A^l\|_{L^\infty(\overline{S_T})} \\ &\le F\left(X^0 + \varepsilon^{k-2} Y^m\right) + C_3\varepsilon. \end{aligned} \tag{4.4.16}$$

As we have seen in (4.3.10), a Taylor expansion for the last term of (4.4.16) gives

$$F(X^0 + \varepsilon^{k-2}Y^m) = F(X^0) + \sum_{l=1}^{m} \varepsilon^{l(k-2)} W^l + R_m^\varepsilon + \widetilde{R}_m^\varepsilon, \tag{4.4.17}$$

where

$$\begin{aligned} W^l &= \sum_{d=1}^{l} \frac{1}{d!} \sum_{n_1+\cdots+n_d=l} F_{p_{i_1 j_1}\cdots p_{i_d j_d}}(X^0) X_{i_1 j_1}^{n_1} \cdots X_{i_d j_d}^{n_d}, \\ R_m^\varepsilon &= \frac{\varepsilon_*^{(m+1)(k-2)}}{(m+1)!} F_{p_{i_1 j_1}\cdots p_{i_{m+1} j_{m+1}}}(X^0) Y_{i_1 j_1}^m \cdots Y_{i_{m+1} j_{m+1}}^m, \\ \widetilde{R}_m^\varepsilon &= \sum_{l=1}^{m} \sum_{m+1\le n_1+\cdots+n_l\le lm} \frac{\varepsilon^{n_1+\cdots+n_l}}{l!} F_{p_{i_1 j_1}\cdots p_{i_l j_l}}(X^0) X_{i_1 j_1}^{n_1} \cdots X_{i_l j_l}^{n_l} \end{aligned}$$

for some $\varepsilon_* \in [0, \varepsilon]$. Due to (4.4.14), (4.4.15), the fact that $(m+1)(k-2) \ge 1$, and the regularity assumption of F, there hold

$$|R^{\varepsilon}_{m}| + |\widetilde{R}^{\varepsilon}_{m}| \leq C_4\varepsilon, \tag{4.4.18}$$

where C_4 also depends on k, ε_0, n, γ, λ, Λ, φ, and S_T. Moreover, one can check that W^l can be rewritten by

$$\begin{aligned} W^l &= \sum_{d=1}^{l} \frac{1}{d!} \sum_{n_1+\cdots+n_d=l} F_{p_{i_1j_1}\cdots p_{i_dj_d}}(X^0) X^{n_1}_{i_1j_1} \cdots X^{n_d}_{i_dj_d} \\ &= F_{p_{ij}}(X^0) X^l_{ij} + \sum_{d=2}^{l} \frac{1}{d!} \sum_{n_1+\cdots+n_d=l} F_{p_{i_1j_1}\cdots p_{i_dj_d}}(X^0) X^{n_1}_{i_1j_1} \cdots X^{n_d}_{i_dj_d} \\ &= F_{p_{ij}}(X^0)\left(D_{x_ix_j} v^l + D_{y_iy_j} w^l\right) + F_{p_{ij}}(X^0) D_{y_iy_j}\xi^{l-1} + \Phi_l(x,t,y,s), \end{aligned} \tag{4.4.19}$$

where Φ^l is as defined in Lemma 4.4.4.
On the other hand, we will look at the time derivative of $\theta^{\varepsilon,+}_m$. Note that each w^l does not depend on $s = x/\varepsilon^k$-variable. Hence, it follows directly from the definition of $\theta^{\varepsilon,+}_m$ that

$$\begin{aligned} (\theta^{\varepsilon,+}_m)_t &= (\eta^{\varepsilon}_m)_t + K\varepsilon \\ &= \sum_{l=0}^{m} \varepsilon^{l(k-2)} \left(v^l_t + \varepsilon^2 w^l_t + \varepsilon^k \xi^l_t + \xi^l_s\right) + K\varepsilon \\ &= v^0_t + \xi^0_s + \sum_{l=1}^{m} \varepsilon^{l(k-2)} \left(v^l_t + \xi^l_s\right) + \sum_{l=0}^{m} \varepsilon^{l(k-2)} \left(\varepsilon^2 w^l_t + \varepsilon^k \xi^l_t\right) + K\varepsilon \\ &=: v^0_t + \xi^0_s + \sum_{l=1}^{m} \varepsilon^{l(k-2)} \Xi^l + \widehat{R}^{\varepsilon}_m + K\varepsilon, \end{aligned} \tag{4.4.20}$$

where

$$\begin{aligned} \Xi^l &= v^l_t + \xi^l_s, \\ \widehat{R}^{\varepsilon}_m &= \sum_{l=0}^{m} \varepsilon^{l(k-2)} \left(\varepsilon^2 w^l_t + \varepsilon^k \xi^l_t\right). \end{aligned}$$

Since $k \in (2, \infty)$, we can deduce from Lemma 4.4.4 that

$$|\widehat{R}_m^\varepsilon| \le C_5 \varepsilon. \tag{4.4.21}$$

Moreover, in view of the definition of W^l, (4.4.19) and combining the resultant with the recursive equation (4.4.7) in Lemma 4.4.4, we arrive at

$$\begin{aligned}
&\sum_{l=1}^m \varepsilon^{l(k-2)} \Xi^l - \sum_{l=1}^m \varepsilon^{l(k-2)} W^l \\
&\quad = \sum_{l=1}^m \varepsilon^{l(k-2)} \left[\left(v_t^l + \xi_s^l\right) - a_{ij} \left(D_{x_i x_j} v^l + D_{y_i y_j} w^l\right) - a_{ij} D_{y_i y_j} \xi^{l-1} - \Phi_l \right] \\
&\quad = 0.
\end{aligned} \tag{4.4.22}$$

We have obtained so far the spatial Hessian and time derivative of $\theta_m^{\varepsilon,+}$. We will mix these results. Choose $K \ge (2\lambda)^{-1}(C_3 + C_4 + C_5)$. Then from (4.4.16), (4.4.17), (4.4.18), and the uniform ellipticity of F that

$$\begin{aligned}
F(D_x^2 \theta_m^{\varepsilon,+}) &\le F(D_x^2 \eta_m^\varepsilon) - 2\lambda K \varepsilon \\
&\le F(X_0 + \varepsilon^{k-2} Y^m) + C_3 \varepsilon - 2\lambda K \varepsilon \\
&= F(X^0) + \sum_{l=1}^m \varepsilon^{l(k-2)} W^l + R_m^\varepsilon + \widetilde{R}_m^\varepsilon + C_3 \varepsilon - 2\lambda K \varepsilon \\
&\le F(X^0) + \sum_{l=1}^m \varepsilon^{l(k-2)} W^l.
\end{aligned}$$

Consequently, we combine (4.4.13), (4.4.20), (4.4.21), (4.4.22), and above estimate that

$$\begin{aligned}
&(\theta_m^{\varepsilon,+})_t - F(D_x^2\theta_m^{\varepsilon,+}) \\
&\geq v_t^0 + \xi_s^0 + \sum_{l=1}^{m} \varepsilon^{l(k-2)}\Xi^l + \widehat{R}_m^\varepsilon + K\varepsilon - F(X^0) - \sum_{l=1}^{m} \varepsilon^{l(k-2)} W^l \\
&\geq v_t^0 + \xi_s^0 + \sum_{l=1}^{m} \varepsilon^{l(k-2)} \left(\Xi^l - W^l\right) - F(X^0) \\
&= v_t^0 + \xi_s^0 - F(X^0) \\
&= 0.
\end{aligned}$$

Now we investigate the boundary value of $\theta_m^{\varepsilon,+}$. First, we note that $v^0 = u_3 = \varphi$, and $v^l = 0$ on $\partial_p S_T$ for all $1 \leq l \leq m$. Hence, by definition of C_0 we have for $\varepsilon < \varepsilon_0$ that

$$\begin{aligned}
\theta_m^{\varepsilon,+} - \varphi &= \sum_{l=0}^{m} \varepsilon^{l(k-2)} \left(v^l + \varepsilon^2 w^l + \varepsilon^k \xi^l\right) + C_0\varepsilon + \varepsilon K(R^2 - |x|^2 + t) - \varphi \\
&\geq (v^0 - \varphi) + \varepsilon \left[C_0 + \sum_{l=0}^{m} \varepsilon^{l(k-2)} \left(\varepsilon w^l + \varepsilon^{k-1}\xi^l\right)\right] \\
&\geq 0.
\end{aligned}$$

Thus, $\theta_m^{\varepsilon,+}$ is a viscosity super-solution of (4.1.1). In a similar manner, one can verify that $\theta_m^{\varepsilon,-}$ is a viscosity sub-solution of (4.1.1). Thus, the comparison principle yields $\theta_m^{\varepsilon,-} \leq u^\varepsilon \leq \theta_m^{\varepsilon,+}$ in $\overline{S_T}$. It then follows that

$$\left\| u^\varepsilon - u_3 - \sum_{l=1}^{m} \varepsilon^{(k-2)l} v^l \right\|_{L^\infty(S_T)} \leq \widetilde{C}\varepsilon,$$

where $\widetilde{C}$ depends only on k, ε_0, n, γ, λ, Λ, φ, and S_T.
Finally, the proof for case $k \in [3, \infty)$ is similar to the above, but rather easier. In this case, we simply choose $m = 0$ and

$$\eta_m^\varepsilon(x,t) = v^0(x,t) + \varepsilon^2 w^0\left(x, t, \frac{x}{\varepsilon}\right) + \varepsilon^k \xi^0\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^k}\right).$$

Thus, we can prove directly without an expansion (4.4.17). The rest of the proof is exactly the same, so is omitted. □

Chapter 5

Higher order convergence rate for the homogenization of soft inclusions with non-divergence structure

5.1 Introduction

In this paper, we study a homogenization problem of non-divergence type elliptic partial differential equation defined in a perforated domain. On the boundary of perforations, we consider an oblique boundary condition instead of Neumann boundary condition. Let Ω be a domain whose boundary is $C^{2,\alpha}$, $0 < \alpha < 1$. We define a periodically perforated domain Ω_ε as follows: let $B_r(z)$, $z \in \mathbb{Z}^n$, be holes distributed periodically and $T(r) = \cup_{z\in\mathbb{Z}^n} B_r(z)$ with radius $r \in (0, 1/2)$. Let $\varepsilon > 0$ be a small parameter which eventually tends to zero. Then the periodically perforated domain Ω_ε is represented by

$$\Omega_\varepsilon := \Omega \setminus T_\varepsilon,\ T_\varepsilon = \varepsilon T(r).$$

We consider the following non-divergence type equation with oblique conditions on the boundary of the holes:

$$\begin{cases} a^\varepsilon_{ij}(x) D_{ij} u^\varepsilon(x) + c(u^\varepsilon, x, x/\varepsilon) = f^\varepsilon(x) & \text{in } \Omega_\varepsilon, \\ b^\varepsilon_i(x) D_i u^\varepsilon(x) = \varepsilon g^\varepsilon(x) & \text{on } \Omega \cap \partial T_\varepsilon, \\ u^\varepsilon(x) = \varphi(x) & \text{on } \partial\Omega \setminus T_\varepsilon. \end{cases} \tag{L_ε}$$

Here, $A^\varepsilon(x) := (a^\varepsilon_{ij}(x)) = (a_{ij}(x/\varepsilon))$ is a $n \times n$ matrix which is uniformly elliptic with elliptic constants λ, Λ, that is, $A(y) := (a_{ij}(y))$ satisfies

$$\lambda|\xi|^2 \leq a_{ij}(y)\xi_i\xi_j \leq \Lambda|\xi|^2 \tag{5.1.1}$$

for all $\xi \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. Also, $b^\varepsilon(x) := (b^\varepsilon_i(x)) = (b_i(x/\varepsilon))$ is a vector field defined on ∂T satisfying

$$|b| \leq \Lambda, \quad \text{and} \quad b \cdot \nu \geq \lambda \tag{5.1.2}$$

for the unit-outward normal vector field $\nu = (\nu_i)$ of $\mathbb{R}^n \setminus T$, and $c(z, x, x/\varepsilon)$, $f^\varepsilon(x) := f(x, x/\varepsilon)$, $g^\varepsilon(x) := g(x, x/\varepsilon)$, and $\varphi(x)$ are all continuous functions. Moreover, all of the coefficients and the functions are periodic in the fast variable $y = x/\varepsilon$.

Non-divergence type elliptic equations can be applied in many fields, such as optimal control, stochastic differential games, and geometry. For related applications, see the [17] and [21]. Especially, the oblique condition is a generalization of the boundary condition in the well-known Skorokhod problem. In probability theory, the Skorokhod problem is the problem of solving a stochastic differential equation with a reflecting boundary condition, and obliquely reflecting Brownian motions in the Skorokhod equations arise naturally in the diffusion approximation in stochastic theory. For more detailed explanation, see [47] and [53].

The authors of [38] suggest a sufficient condition, called a compatibility condition to have a homogenization process under the oblique condition. The compatibility condition will give the balance between the diffusion equation in $\mathbb{R}^n \setminus T$ and the drift effect by the oblique condition on ∂T, and then it gives the existence of global solution as it does in the standard divergence-type equation.

Definition 5.1.1 (Compatibility condition)**.** (A, b) satisfies a compatibility condition if the following equation

$$\begin{cases} a_{ij}(y) D_{y_i y_j} v = 0 & \text{in } \mathbb{R}^n \setminus T, \\ b_i(y) \left(\xi_i + D_{y_i} v\right) = 0 & \text{on } \partial T \end{cases} \tag{5.1.3}$$

admits a periodic solution v for any given $\xi \in \mathbb{R}^n$.

[38] showed that the homogenization takes place when the coefficients (A, b) hold a compatibility condition and the size of holes r is less than a constant $r_0 = r_0(n, \lambda, \Lambda)$. As a result, we obtain the existence of the effective(homogenized) equation given by

$$\begin{cases} \overline{L}(D^2\overline{u}, \overline{u}, x) = \overline{a}_{ij} D_{x_i x_j} \overline{u}(x) + \overline{c}(\overline{u}, x) - \overline{f}(x) = 0 & \text{in } \Omega, \\ \overline{u}(x) = \varphi(x) & \text{on } \partial\Omega, \end{cases} \tag{5.1.4}$$

where the solution $\overline{u}$, which is called an effective limit, is the uniform limit of u^ε.

In this paper, we are going to study an estimate of the rate of convergence between the solution u^ε and the effective limit $\overline{u}$. We will show a rigorous justification of the following asymptotic expansion of the solution u^ε:

$$u^\varepsilon(x) = \overline{u}(x) + \varepsilon w_1^\varepsilon(x) + \cdots + \varepsilon^m w_m^\varepsilon(x) + \theta_m^\varepsilon(x) + O(\varepsilon^{m-1}),$$

where $w_k^\varepsilon(x) = w_k(x, x/\varepsilon)$ and $\theta_k^\varepsilon(x)$ are the k-th order correctors which fix the error occurring in the interior and on the boundary respectively. For the homogenization theory in a perforated domain with oblique boundary condition, [38] obtained the effective operator by introducing the compatibility condition. However, the study of convergence rate including higher orders in homogenization theory is new, to our best knowledge, for second order uniformly elliptic equations in non-divergence form with oblique conditions on the boundary of the holes. As an important by-product we can provide an estimate of the rate of convergence (namely, of $\|u^\varepsilon - \overline{u}\|_{L^\infty}$) and establish that the solutions u^ε converges uniformly to $\overline{u}$.

5.1.1 Main results

In order to find the higher order correctors, as mentioned in the introduction, we have to use the basic method for the existence and the regularity of the correctors for each order in an inductive manner. For the purpose, we need regularity assumptions on the coefficients that play an essential role in our analysis. The following conditions are assumed in this paper.

(C1) $A(y)$, $b(y)$, $c(z,x,y)$, $f(x,y)$ and $g(x,y)$ are periodic in y-variable.

(C2) $c(0,x,y) = 0$ and $c(z,x,y)$ is non-increasing with z-variable.

(C3) $a_{ij} \in C^\alpha(\mathbb{R}^n \setminus T)$ and $b_i \in C^{1,\alpha}(\partial T)$ for some $\alpha \in (0,1)$.

For given any subsets X and Y of $\mathbb{R}^n$ and a continuous function $v(x,y)$ defined on $X \times Y$, we define the space $C^l(X; C^{k,\alpha}(Y))$ of all $v : X \times Y \to \mathbb{R}$ satisfying

$$\|v\|_{C^l(X;C^{k,\alpha}(Y))} := \sup_{0\le|\gamma|\le l} \sup_{x\in X} \|D_x^\gamma v(x,\cdot)\|_{C^{k,\alpha}(Y)} < \infty.$$

Also, we set

$$[v]_{C^{l,\beta}(X;C^{k,\alpha}(Y))} = \sup_{|\gamma|=l} \sup_{x_1,x_2\in X} \frac{\|D_x^\gamma v(x_1,\cdot) - D_x^\gamma v(x_2,\cdot)\|_{C^{k,\alpha}(Y)}}{|x_1-x_2|^\beta}$$

for some $\alpha, \beta \in (0,1]$ and $k, l \in \mathbb{Z}$. With this semi-norm, we can define the space $C^{l,\beta}(X; C^{k,\alpha}(Y))$ of all $v : X \times Y \to \mathbb{R}$ satisfying

$$\|v\|_{C^{l,\beta}(X;C^{k,\alpha}(Y))} := \|v\|_{C^l(X;C^{k,\alpha}(Y))} + [v]_{C^{l,\beta}(X;C^{k,\alpha}(Y))} < \infty.$$

For simplicity we write $C^{0,\beta}(X;C^{k,\alpha}(Y)) = C^{\beta}(X;C^{k,\alpha}(Y))$ when $\beta \in (0,1)$. Let us make few remarks on this space. By definition, we first note that if $v \in C^{l,\beta}(X;C^{k,\alpha}(Y))$, then $v(\cdot,y) \in C^{l,\beta}(X)$ uniformly for all $y \in Y$ and $v(x,\cdot) \in C^{k,\alpha}(Y)$ uniformly for all $x \in X$.

Suppose that $v \in C^{0,1}(X;C^{k,\alpha}(Y))$ and $v(\cdot,y) \in C^1(X)$ for any $y \in Y$. let $Y = \mathbb{R}^n \setminus T$ with $v(x,\cdot)$ is y-periodic. Hence, we may assume that $v(x,\cdot)$ can be considered to be defined on a compact metric space $(\mathbb{R}^n \setminus T)/\mathbb{Z}^n$. Now we put

$$D^h_{x_k} v(x,y) := \frac{v(x+he_k,y) - v(x,y)}{h}$$

for fixed $y \in Y$, and for some unit coordinate vector e_k. Then the definition of $C^{0,1}(X;C^{k,\alpha}(Y))$ space implies that

$$\|D^h_{x_k} v(x,\cdot)\|_{C^{k,\alpha}(Y)} \leq [v]_{C^{0,1}(X;C^{k,\alpha}(Y))} < \infty. \tag{5.1.5}$$

Therefore, the compact embedding result (or Arzelá-Ascoli theorem) ensures the existence of a limit function $w : X \times Y \to \mathbb{R}$ along a subsequence of h, which is y-periodic and belongs to $C^{k,\alpha}(\mathbb{R}^n \setminus T)$. But since $v(\cdot,y) \in C^1(X)$, we know that the limit of $D^h_{x_k} v(x,\cdot) \to w(x,\cdot)$ (in C^k-norm) takes place for the full sequence of h. Consequently, by definition, $w(x,y) = D_{x_k} v(x,y)$ and from (5.1.5) we have

$$\|D_x v(x,\cdot)\|_{C^{k,\alpha}(Y)} \leq \|v\|_{C^{0,1}(X;C^{k,\alpha}(Y))}.$$

In general, suppose that $v \in C^{l,1}(X;C^{k,\alpha}(Y))$ and $v(\cdot,y) \in C^{l+1}(X)$ for any $y \in Y$. Then we can deduce that

$$\|D^\gamma_x v(x,\cdot)\|_{C^{k,\alpha}(Y)} \leq \|v\|_{C^{l,1}(X;C^{k,\alpha}(Y))},$$

where $|\gamma| = l+1$.

Moreover, we also have

$$\|v\|_{C^{l,\beta}(X;C^{k,\alpha}(Y))} \leq \|v\|_{C^{l',\beta'}(X;C^{k,\alpha}(Y))}$$

if $l+\beta \le l'+\beta'$.

The followings are regularities on c, f and g:

(C4) c, f, and g satisfy

$$\|c\|_{C^{m,\alpha}(\mathbb{R}\times\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))} + \|f\|_{C^{m,\alpha}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))} + \|g\|_{C^{m,\alpha}(\overline{\Omega};C^{1,\alpha}(\partial T))} < \infty.$$

Now we can state our main results on the rate of convergence. Define $\eta_m^\varepsilon : \overline{\Omega}_\varepsilon \to \mathbb{R}$ by

$$\eta_m^\varepsilon(x) = \overline{u}(x) + \varepsilon w_1^\varepsilon(x) + \cdots + \varepsilon^m w_m^\varepsilon(x)$$

where w_k^ε is an k-th order interior corrector defined by (5.3.1). In addition, to correct the error occurring on the boundary, we need the boundary corrector θ_m^ε defined by the solution of (5.3.2). Then we can obtain the higher order convergence rate.

Theorem 5.1.2 (Main theorem). *Let $m \ge 2$ be an integer, $\{u^\varepsilon\}_{\varepsilon>0}$ be family of (viscosity) solutions of (L_ε), and $\overline{u}$ is the effective limit of $\{u^\varepsilon\}_{\varepsilon>0}$ which solves* (5.1.4). *Assume the conditions* (5.1.1), (5.1.2), *(C1)-(C4) hold and (A,b) satisfies a compatibility condition. Then there are interior corrector $\eta_m^\varepsilon(x)$ and boundary corrector $\theta_m^\varepsilon(x)$ such that for any $\varepsilon \in (0,1)$*

$$\|u^\varepsilon - \eta_m^\varepsilon - \theta_m^\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \le C\varepsilon^{m-1}$$

where C is a constant depending on n, m, α, λ, Λ, r, Ω, $\|a_{ij}\|_{C^{\alpha}(\mathbb{R}^n\setminus T)}$, $\|b_i\|_{C^{1,\alpha}(\partial T)}$, $\|c\|_{C^{m,\alpha}(\mathbb{R}\times\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))}$, $\|f\|_{C^{m,\alpha}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))}$, $\|g\|_{C^{m,\alpha}(\overline{\Omega};C^{1,\alpha}(\partial T))}$ and $\|\varphi\|_{C^{m+2,\alpha}(\overline{\Omega})}$. In particular, we have

$$\|u^\varepsilon - \overline{u}\|_{L^\infty(\Omega_\varepsilon)} \le C\varepsilon.$$

5.1.2 Heuristics discussion and main strategies

We now make a few remarks on the key features observed in achieving the rates. In order to find the next order corrector, we consider the effective operator which can be derived from the equation given by the previous correctors as source terms. In each step, all the effective operators are turn out to be still in the same format of the previous one. For this reason, we are able to employ the basic approach for the existence and the regularity of the correctors for each order in an inductive manner. We notice that the compatibility condition guarantees the solvability of a boundary value problem. That is, at each step of finding the k-th order interior corrector, we need a compatibility condition which uniquely determines the corrector.

The strategies to prove Theorem 5.1.2 are based on the barrier argument, however there are several difficulties which arise given the oblique condition. The classical proof presented by Evans ([19, 20]) can be established using a appropriate test function created by adding a second corrector to the perturbed term. But in our case this is not enough due to the effect of the oblique condition. The basic idea to overcome this hurdle is to reflect the influence of the first corrector in the test function. We point out that the first corrector w_1 depends on $D_x\overline{u}(x)$, in other words, the first corrector cancels the effect of the one-time derivative from the oblique condition. As a result, this allows one to create a barrier that satisfies the oblique condition.

Concerning the regularity of the correctors, we are faced with the coupling effect of the fast variable $y = x/\varepsilon$ and the slow variable x of the interior correctors $w_k(x, y)$. The interior correctors w_k can be represented in the form of (5.2.33), which is the summation of the functions whose (x, y)-variable is coupled. This phenomenon occurs due to the influence of the (x, y)-coupled effect of the low order terms. As a result, the function $y \to D_{x_i}w_k(x, y)$ turns out to have a lower regularity than that of $y \to w_k(x, y)$ (see [34]). In order to overcome this difficulty, we introduce the coupled regularity to the low order terms, and by combining this with the basic homogenization scheme, we can restore the regularity of the correctors.

Finally, it is worth mentioning the need for the boundary correctors. We have to correct the boundary oscillation occurred by the interior correctors by solving the corresponding boundary value problem (5.3.2). One may notice from the regularity results of the interior corrector in Section 5.2 that the existence of the boundary correctors is guaranteed in the viscosity sense (see [3, 17]).

5.1.3 Outline

This paper is organized as follows: Section 5.2 is devoted to the existence and the regularity of correctors. In Subsection 5.2.1, we investigate the existence and the regularity of the solutions for the general corrector equations and review the basic homogenization scheme via the viscosity method. And then we apply the asymptotic expansion method to define the first and second correctors and find an effective equation in Subsection 5.2.2. Subsection 5.2.3 we study the higher order interior correctors, especially find the explicit formulas of the higher order correctors, which play crucial roles in the proof of the main theorem. Finally, we present the proof of the main theorem in Section 5.3.

5.2 Homogenization and correctors

5.2.1 Basic homogenization process and regularity of solutions

In this subsection, we investigate the existence and the regularity of the solutions $w(x, y)$ and $w_\varepsilon(x, y)$ of the general corrector equations

$$\begin{cases} a_{ij}(y) D_{y_i y_j} w = f(x, y) & \text{in } \mathbb{R}^n \setminus T, \\ b_i(y) D_{y_i} w = g(x, y) & \text{on } \partial T \end{cases}$$

and

$$\begin{cases} a_{ij}(y)D_{y_iy_j}w_\varepsilon - \varepsilon^2 w_\varepsilon = f(x,y) & \text{in } \mathbb{R}^n \setminus T, \\ b_i(y)D_{y_i}w_\varepsilon + \varepsilon^2 w_\varepsilon = g(x,y) & \text{on } \partial T. \end{cases} \tag{5.2.1}$$

Here, $x \in \overline{\Omega}$ is a slow-variable, and all of the a_{ij}, b_i, $f(x,\cdot)$, and $g(x,\cdot)$ are periodic in the fast variable $y = x/\varepsilon$. We also assume that $(a_{ij}(y))$ is uniformly elliptic, $(b_i(y))$ is uniformly oblique, f and g are continuous.

We note that the equation (5.2.1) is obtained by subtracting and adding $\varepsilon^2 w_\varepsilon(x,y)$ to the interior and boundary equations respectively. Due to the auxiliary term $\varepsilon^2 w_\varepsilon$ in equation (5.2.1), we can use comparison principle and hence we obtain the existence. For more details, one may refer to [19, 20] and [38].

Lemma 5.2.1. *(Comparison) Suppose that $w^+(x,y)$ is a super-solution of* (5.2.1) *and $w^-(x,y)$ is a sub-solution of* (5.2.1)*. Then for fixed $x \in \overline{\Omega}$ we have*

$$w^+(x,y) \geq w^-(x,y) \quad \text{in } \mathbb{R}^n \setminus T.$$

Lemma 5.2.2. *(Existence) There exists a unique bounded y-periodic solution $w_\varepsilon(x,y)$ of the equation* (5.2.1) *satisfying*

$$\|\varepsilon^2 w_\varepsilon(x,\cdot)\|_{L^\infty(\mathbb{R}^n\setminus T)} \leq \|f(x,\cdot)\|_{L^\infty(\mathbb{R}^n\setminus T)} + \|g(x,\cdot)\|_{L^\infty(\partial T)}.$$

Lemma 5.2.3. *The solution w_ε of the equation* (5.2.1) *satisfies*

$$\begin{aligned} osc_{\mathbb{R}^n\setminus T} w_\varepsilon(x,\cdot) + \|\widetilde{w}_\varepsilon(x,\cdot)\|_{C^{1,\alpha}(\mathbb{R}^n\setminus T)} \\ \leq C(\|f(x,\cdot)\|_{L^\infty(\mathbb{R}^n\setminus T)} + \|g(x,\cdot)\|_{C^\alpha(\partial T)}). \end{aligned}$$

Moreover, we have

$$\|\widetilde{w}_\varepsilon(x,\cdot)\|_{C^{2,\alpha}(\mathbb{R}^n\setminus T)} \leq C(\|f(x,\cdot)\|_{C^\alpha(\mathbb{R}^n\setminus T)} + \|g(x,\cdot)\|_{C^{1,\alpha}(\partial T)})$$

where $\widetilde{w}_\varepsilon(x,y) := w_\varepsilon(x,y) - w_\varepsilon(x,0)$ and C depends on n, α, λ, Λ, r, $\|a_{ij}\|_{C^\alpha(\mathbb{R}^n\setminus T)}$, and $\|b_i\|_{C^{1,\alpha}(\partial T)}$.

Remark 5.2.4. By Lemma 5.2.2 and 5.2.3 we can also obtain

$$\|\varepsilon^2 w_\varepsilon(x,\cdot)\|_{C^{2,\alpha}(\mathbb{R}^n\setminus T)} \le C\left(\|f(x,\cdot)\|_{C^{\alpha}(\mathbb{R}^n\setminus T)} + \|g(x,\cdot)\|_{C^{1,\alpha}(\partial T)}\right).$$

where C depends on n, α, λ, Λ, r, $\|a_{ij}\|_{C^{\alpha}(\mathbb{R}^n\setminus T)}$, and $\|b_i\|_{C^{1,\alpha}(\partial T)}$.

Now we are ready to find the effective operator. We notice that $w_\varepsilon(x,\cdot)$ (or $\widetilde{w}_\varepsilon(x,\cdot)$) is y-periodic, hence, we may assume that $w_\varepsilon(x,\cdot)$ (or $\widetilde{w}_\varepsilon(x,\cdot)$) can be considered to be defined on a compact metric space $(\mathbb{R}^n\setminus T)/\mathbb{Z}^n$.

Lemma 5.2.5. *Assume that $a_{ij}\in C^{\alpha}(\mathbb{R}^n\setminus T)$ and $b_i\in C^{1,\alpha}(\partial T)$ for some $\alpha\in(0,1]$. Then there exist a y-periodic function $w(x,y)$, $w(x,\cdot)\in C^{2,\alpha}(\mathbb{R}^n\setminus T)$, and a constant $\gamma(x)\in\mathbb{R}$ such that*

$$\|\varepsilon^2 w_\varepsilon(x,\cdot)-\gamma(x)\|_{L^\infty(\mathbb{R}^n\setminus T)} + \|\widetilde{w}_\varepsilon(x,\cdot)-w(x,\cdot)\|_{C^{2}(\mathbb{R}^n\setminus T)} \to 0 \quad \text{as} \quad \varepsilon\to 0 \tag{5.2.2}$$

where $\widetilde{w}_\varepsilon(x,y):=w_\varepsilon(x,y)-w_\varepsilon(x,0)$. Moreover, γ is a unique constant where the equation (5.2.4) *has a unique solution up to constant addition.*

It then immediately followed from Lemma 5.2.5 that $\gamma(x)$, and $w(x,y)$ satisfy

$$|\gamma(x)| + \|w(x,\cdot)\|_{C^{2,\alpha}(\mathbb{R}^n\setminus T)} \le C(\|f(x,\cdot)\|_{C^{\alpha}(\mathbb{R}^n\setminus T)} + \|g(x,\cdot)\|_{C^{1,\alpha}(\partial T)}) \tag{5.2.3}$$

and solve the following cell problem:

$$\begin{cases} a_{ij}(y)D_{y_iy_j}w = \gamma(x)+f(x,y) & \text{in } \mathbb{R}^n\setminus T,\\ b_i(y)D_{y_i}w = -\gamma(x)+g(x,y) & \text{on } \partial T. \end{cases} \tag{5.2.4}$$

Proof. Fix $y_0\in\mathbb{R}^n\setminus T$. Then in view of Lemma 5.2.2, we can take a subsequence $\{\varepsilon_k^2 w_{\varepsilon_k}(y_0)\}_{k=1}^{\infty}$ of $\{\varepsilon^2 w_\varepsilon\}_{0<\varepsilon<1}$ and a number $\gamma\in\mathbb{R}$ such that $\varepsilon_k^2 w_{\varepsilon_k}(y_0)\to\gamma$ as $k\to 0$. Then Lemma 5.2.3 implies that $\varepsilon_k^2 w_{\varepsilon_k}\to\gamma$ uniformly in $\mathbb{R}^n\setminus T$ as $k\to 0$.

On the other hand, Lemma 5.2.3 allows us to use the compact embedding theorem, from which we deduce that there is $w(x,\cdot)\in C^{2,\alpha}(\mathbb{R}^n\setminus T)$

and a further subsequence of $\{\varepsilon_k\}_{k=1}^{\infty}$, which we denote again by $\{\varepsilon_k\}_{k=1}^{\infty}$ for convenience, such that $\widetilde{w}_{\varepsilon_k} \to w$ with respect to $C^2(\mathbb{R}^n \setminus T)$-norm; i.e.

$$\|\varepsilon_k^2 w_{\varepsilon_k} - \gamma\|_{L^\infty(\mathbb{R}^n \setminus T)} + \|\widetilde{w}_{\varepsilon_k} - w\|_{C^2(\mathbb{R}^n \setminus T)} \to 0 \quad \text{as} \quad k \to \infty.$$

Clearly, w satisfies (5.2.4) and by using proof of Lemma 5.2.3 similarly, we have $w(x, \cdot) \in C^{2,\alpha}(\mathbb{R}^n \setminus T)$.

Now we are going to show the uniqueness of γ and w. Let $w_1(x, y)$ and $w_2(x, y)$ be two solutions of the equation (5.2.4) with corresponding to constants $\gamma_1(x)$ and $\gamma_2(x)$ respectively. To obtain contradiction, assume that $\gamma_1 \neq \gamma_2$ and without loss of generality, assume $\gamma_1 < \gamma_2$. Since w_1 and w_2 are bounded, we can find a constant c such that $w_1 + c$ touches w_2 by above at $y_0 \in \mathbb{R}^n \setminus T$. Suppose that y_0 is a interior point, then $w_1 + c - w_2$ has a local minimum at y_0. But since $a_{ij}(y) D_{y_i y_j} \left((w_1 + c - w_2) \right)(y) = \gamma_1 - \gamma_2 < 0$, $w_1 + c - w_2$ cannot have its minimum at interior point by the strong maximum principle. Hence y_0 cannot be in the interior point. Now suppose that $y_0 \in \partial T$. But in this case, since $D_{y_i} \left(w_1 + c - w_2 \right)(y_0) = 0$, we have

$$\begin{aligned} g(x, y_0) &= b_i(y_0) D_{y_i} w_2(x, y_0) + \gamma_2(x) \\ &= b_i(y_0) D_{y_i} (w_1 + c)(x, y_0) + \gamma_2(x) \\ &= g(x, y_0) - \gamma_1(x) + \gamma_2(x) \\ &> g(x, y_0). \end{aligned}$$

So we get a contradiction and hence $\gamma_1 = \gamma_2$.

Let $v(x, y) = w_1(x, y) - w_2(x, y)$. Then, since $\gamma_1 = \gamma_2$, v solves

$$\begin{cases} a_{ij}(y) D_{y_i y_j} v = 0 & \text{in } \mathbb{R}^n \setminus T, \\ b_i(y) D_{y_i} v = 0 & \text{on } \partial T \end{cases}$$

Now, from the strong maximum principle and Hopf's boundary maximum principle in [21], we can conclude that v is a constant. Hence w is unique up to a constant addition.

Due to the normalization that $\widetilde{w}_\varepsilon(x,0)=0$, it is noteworthy to observe that $w(x,0)=0$. That is, the solution w of (5.2.4) is also unique, and hence the uniqueness of (γ,w) implies that every convergence subsequence $(\varepsilon_k^2 w_{\varepsilon_k}, \widetilde{w}_{\varepsilon_k})$ has the same limit (γ,w). Hence we can conclude that (5.2.2) holds.

□

As the next step, we investigate the regularity of w and γ, in particular in the x-variable. Roughly speaking, since the x-dependency of $w(\cdot,y)$ and $\gamma(\cdot)$ depends only on the $f(\cdot,y)$ and $g(\cdot,y)$, it is natural to ask whether a higher regularity f and g in x-variable gives a higher regularity for $w(\cdot,y)$ and $\gamma(\cdot)$, and we now prove that the answer is affirmative. This regularity result plays the key role in the rest of this paper, especially in seeking higher order interior correctors. To be precise, we observe the following.

Proposition 5.2.6. *Let $\gamma(x)$, and $w(x,y)$ be functions which solve a equation* (5.2.4)*. Assume that*

$$\|f\|_{C^{m,\beta}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))}+\|g\|_{C^{m,\beta}(\overline{\Omega};C^{1,\alpha}(\partial T))}<\infty.$$

Then for any integer $m\geq 0$, $\gamma, w(\cdot,y)\in C^{m,\beta}(\overline{\Omega})$, where the Hölder continuity of the latter is uniform in $y\in\mathbb{R}^n\setminus T$. Moreover, there holds

$$\begin{aligned}\|\gamma\|_{C^{m,\beta}(\overline{\Omega})}&+\|w\|_{C^{m,\beta}(\overline{\Omega};C^{2,\alpha}(\mathbb{R}^n\setminus T))}\\&\leq C(\|f\|_{C^{m,\beta}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))}+\|g\|_{C^{m,\beta}(\overline{\Omega};C^{1,\alpha}(\partial T))})\end{aligned}$$

where C depends only on n, m, α, λ, Λ, r, $\|a_{ij}\|_{C^{\alpha}(\mathbb{R}^n\setminus T)}$, and $\|b_i\|_{C^{1,\alpha}(\partial T)}$.

Before we begin the proof, give an heuristics explanation of our argument. First, we only assume that $f(\cdot,y)$ and $g(\cdot,y)$ are C^β in $\overline{\Omega}$ for each y and end up with the conclusion that $w(\cdot,y)$ and $\gamma(\cdot)$ are also C^β in $\overline{\Omega}$ for each y. We also observe that the equation, which involves the partial derivatives of w and γ in x-variable, satisfies same structure of the equation of w and γ. This implies that we can iterate the argument recursively to get $C^{m,\beta}$ regularity of w and γ.

Lemma 5.2.7. *Let $w_\varepsilon(x,y)$ be a solution of (5.2.1) and $\widetilde{w}_\varepsilon(x,y) := w_\varepsilon(x,y) - w_\varepsilon(x,0)$. Assume that*

$$\|f\|_{C^\beta(\overline{\Omega};C^\alpha(\mathbb{R}^n\setminus T))} + \|g\|_{C^\beta(\overline{\Omega};C^{1,\alpha}(\partial T))} < \infty. \tag{5.2.5}$$

Then for each $x_1, x_2 \in \overline{\Omega}$, there holds

$$\begin{aligned}\|\varepsilon^2 w_\varepsilon(x_1,\cdot) - \varepsilon^2 w_\varepsilon(x_2,\cdot)\|_{C^{2,\alpha}(\mathbb{R}^n\setminus T)} + \|\widetilde{w}_\varepsilon(x_1,\cdot) - \widetilde{w}_\varepsilon(x_2,\cdot)\|_{C^{2,\alpha}(\mathbb{R}^n\setminus T)}\\ \leq C|x_1-x_2|^\beta(\|f\|_{C^\beta(\overline{\Omega};C^\alpha(\mathbb{R}^n\setminus T))} + \|g\|_{C^\beta(\overline{\Omega};C^{1,\alpha}(\partial T))})\end{aligned}$$

where C depends on n, α, λ, Λ, r, $\|a_{ij}\|_{C^\alpha(\mathbb{R}^n\setminus T)}$, and $\|b_i\|_{C^{1,\alpha}(\partial T)}$.

Proof. Let $v_\varepsilon(\cdot) = w_\varepsilon(x_1,\cdot) - w_\varepsilon(x_2,\cdot)$. Then v_ε satisfies the following equation:

$$\begin{cases} a_{ij}(y)D_{y_iy_j}v_\varepsilon - \varepsilon^2 v_\varepsilon = f(x_1,y) - f(x_2,y) & \text{in } \mathbb{R}^n\setminus T, \\ b_i(y)D_{y_i}v_\varepsilon + \varepsilon^2 v_\varepsilon = g(x_1,y) - g(x_2,y) & \text{on } \partial T. \end{cases}$$

Note that this equation belongs to the same class of (5.2.1). So by applying Lemma 5.2.2 and 5.2.3 to v_ε, we can see that $v_\varepsilon \in C^{2,\alpha}(\mathbb{R}^n\setminus T)$. Moreover, condition (5.2.5) implies that we have

$$\begin{aligned}&\|\varepsilon^2 v_\varepsilon\|_{C^{2,\alpha}(\mathbb{R}^n\setminus T)} + \|\widetilde{v}_\varepsilon\|_{C^{2,\alpha}(\mathbb{R}^n\setminus T)}\\ &\quad\leq C\left(\|f(x_1,\cdot) - f(x_2,\cdot)\|_{C^\alpha(\mathbb{R}^n\setminus T)} + \|g(x_1,\cdot) - g(x_2,\cdot)\|_{C^{1,\alpha}(\partial T)}\right)\\ &\quad\leq C|x_1-x_2|^\beta\left(\|f\|_{C^\beta(\overline{\Omega};C^\alpha(\mathbb{R}^n\setminus T))} + \|g\|_{C^\beta(\overline{\Omega};C^{1,\alpha}(\partial T))}\right).\end{aligned}$$

where $\widetilde{v}_\varepsilon(y) := v_\varepsilon(y) - v_\varepsilon(0) = \widetilde{w}_\varepsilon(x_1,y) - \widetilde{w}_\varepsilon(x_2,y)$. □

Since γ and w solving equation (5.2.4) can be understood as limits of $\varepsilon^2 w_\varepsilon(x,y)$ and $\widetilde{w}_\varepsilon(x,y)$, we obtain the following lemma from Lemma 5.2.5 and 5.2.7:

Lemma 5.2.8. *Let $w(x,y)$ be a solution of* (5.2.4)*. Assume all the conditions in Lemma 5.2.7 hold. Then $\gamma, w(\cdot,y) \in C^{\beta}(\overline{\Omega})$, where the Hölder continuity of the latter is uniform in $y \in \mathbb{R}^n \setminus T$. Moreover, there holds*

$$\|\gamma\|_{C^{\beta}(\overline{\Omega})} + \|w\|_{C^{\beta}(\overline{\Omega};C^{2,\alpha}(\mathbb{R}^n\setminus T))} \leq C\Big(\|f\|_{C^{\beta}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))} + \|g\|_{C^{\beta}(\overline{\Omega};C^{1,\alpha}(\partial T))}\Big)$$

where C depends only on n, α, λ, Λ, r, $\|a_{ij}\|_{C^{\alpha}(\mathbb{R}^n\setminus T)}$, and $\|b_i\|_{C^{1,\alpha}(\partial T)}$.

Proof. From Lemma 5.2.5 and 5.2.7, it is easily check that for each $x_1, x_2 \in \overline{\Omega}$ there holds

$$\begin{aligned}|\gamma(x_1) - \gamma(x_2)| &+ \|w(x_1,\cdot) - w(x_2,\cdot)\|_{C^{2,\alpha}(\mathbb{R}^n\setminus T)} \\ &\leq C|x_1 - x_2|^{\beta}(\|f\|_{C^{\beta}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))} + \|g\|_{C^{\beta}(\overline{\Omega};C^{1,\alpha}(\partial T))}).\end{aligned}$$

Consequently, from Lemma 5.2.5 again and above estimate we obtain

$$\begin{aligned}\|w\|_{C^{\beta}(\overline{\Omega};C^{2,\alpha}(\mathbb{R}^n\setminus T))} &= \sup_{x\in\overline{\Omega}} \|w(x,\cdot)\|_{C^{2,\alpha}(\mathbb{R}^n\setminus T)} + [w]_{C^{\beta}(\overline{\Omega};C^{2,\alpha}(\mathbb{R}^n\setminus T))} \\ &\leq C(\|f\|_{C^{\beta}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))} + \|g\|_{C^{\beta}(\overline{\Omega};C^{1,\alpha}(\partial T))}).\end{aligned}$$

□

By Lemma 5.2.8, we have $w(\cdot,y)$ and corresponding effective operator $\gamma(\cdot)$ are Hölder continuous(uniform in y) on $\overline{\Omega}$. Now we are left with proving that the x-partial derivative of $w(\cdot,y)$ and $\gamma(\cdot)$ are Hölder continuous on $\overline{\Omega}$ for each $y \in \mathbb{R}^n \setminus T$. Let us make our argument precisely.

Lemma 5.2.9. *Let $w_\varepsilon(x,y)$ be a solution of* (5.2.1)*. Assume that f and g are differentiable with respect to x-variable and*

$$\|f\|_{C^{0,1}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))} + \|g\|_{C^{0,1}(\overline{\Omega};C^{1,\alpha}(\partial T))} < \infty.$$

Then $D_{x_k} w_\varepsilon$ $(1 \leq k \leq n)$ exist and satisfy

$$\begin{cases} a_{ij}(y) D_{y_i y_j} D_{x_k} w_\varepsilon - \varepsilon^2 D_{x_k} w_\varepsilon = D_{x_k} f(x,y) & \text{in } \mathbb{R}^n \setminus T, \\ b_i(y) D_{y_i} D_{x_k} w_\varepsilon + \varepsilon^2 D_{x_k} w_\varepsilon = D_{x_k} g(x,y) & \text{on } \partial T. \end{cases} \tag{5.2.6}$$

Proof. For some unit coordinate vector e_k, let

$$D^h_{x_k} w_\varepsilon(x,\cdot) = \frac{w_\varepsilon(x+he_k,\cdot) - w_\varepsilon(x,\cdot)}{h}.$$

Substituting x with $x + he_k$ in equation (5.2.1) and substracting original equation, we have following equation:

$$\begin{cases} a_{ij}(y) D_{y_i y_j} D^h_{x_k} w_\varepsilon - \varepsilon^2 D^h_{x_k} w_\varepsilon = D^h_{x_k} f(x,y) & \text{in } \mathbb{R}^n \setminus T, \\ b_i(y) D_{y_i} D^h_{x_k} w_\varepsilon + \varepsilon^2 D^h_{x_k} w_\varepsilon = D^h_{x_k} g(x,y) & \text{on } \partial T \end{cases}$$

and from Lemma 5.2.7($\beta = 1$ case), there holds

$$\|D^h_{x_k} w_\varepsilon(x,\cdot)\|_{C^{2,\alpha}(\mathbb{R}^n \setminus T)} \le \varepsilon^{-2} C \left(\|f\|_{C^{0,1}(\overline{\Omega}; C^\alpha(\mathbb{R}^n \setminus T))} + \|g\|_{C^{0,1}(\overline{\Omega}; C^{1,\alpha}(\partial T))} \right).$$

Then the Arzelá-Ascoli theorem ensures the existence of a limit function v_ε, which is bounded y-periodic and belongs to $C^{2,\alpha}(\mathbb{R}^n \setminus T)$ for each ε along a subsequence of h, satisfying following equation:

$$\begin{cases} a_{ij}(y) D_{y_i y_j} v_\varepsilon - \varepsilon^2 v_\varepsilon = D_{x_k} f(x,y) & \text{in } \mathbb{R}^n \setminus T, \\ b_i(y) D_{y_i} v_\varepsilon + \varepsilon^2 v_\varepsilon = D_{x_k} g(x,y) & \text{on } \partial T. \end{cases}$$

Note that above equation has the same form as (5.2.1). Therefore, due to the uniqueness of the solution of above (5.2.1), we know that the limit of $D^h_{x_k} w_\varepsilon(x,\cdot) \to v_\varepsilon(x,\cdot)$ (in C^2-norm) takes place for the full sequence of h. Consequently, by definition, $v_\varepsilon(x,y) = D_{x_k} w_\varepsilon(x,y)$ and hence $D_{x_k} w_\varepsilon$ satisfies (5.2.6). □

Lemma 5.2.10. *Let $w(x,y)$ and $\gamma(x)$ be solutions of* (5.2.4). *Assume that f and g are differentiable with respect to x-variable and*

$$\|f\|_{C^{0,1}(\overline{\Omega}; C^\alpha(\mathbb{R}^n \setminus T))} + \|g\|_{C^{0,1}(\overline{\Omega}; C^{1,\alpha}(\partial T))} < \infty.$$

Then $D_{x_k}\gamma$ and $D_{x_k}w$ $(1 \le k \le n)$ exist and satisfy

$$\begin{cases} a_{ij}(y)D_{y_iy_j}D_{x_k}w = D_{x_k}\gamma(x) + D_{x_k}f(x,y) & \text{in } \mathbb{R}^n \setminus T, \\ b_i(y)D_{y_i}D_{x_k}w = -D_{x_k}\gamma(x) + D_{x_k}g(x,y) & \text{on } \partial T. \end{cases} \tag{5.2.7}$$

Moreover, we have

$$\begin{aligned} &|D_{x_k}\gamma(x)| + ||D_{x_k}w(x,\cdot)||_{C^{2,\alpha}(\mathbb{R}^n\setminus T)} \\ &\qquad \le C\left(\|f\|_{C^{0,1}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))} + \|g\|_{C^{0,1}(\overline{\Omega};C^{1,\alpha}(\partial T))}\right) \end{aligned} \tag{5.2.8}$$

where C depends only on n, α, λ, Λ, r, $\|a_{ij}\|_{C^{\alpha}(\mathbb{R}^n\setminus T)}$, and $\|b_i\|_{C^{1,\alpha}(\partial T)}$.

Proof. Fix $x \in \overline{\Omega}$ and define

$$D^h_{x_k}w(x,\cdot) = \frac{w(\cdot;x+he_k) - w(x,\cdot)}{h}.$$

Then $D^h_{x_k}w$ satisfies

$$\begin{cases} a_{ij}(y)D_{y_iy_j}D^h_{x_k}w = D^h_{x_k}\gamma(x) + D^h_{x_k}f(x,y) & \text{in } \mathbb{R}^n \setminus T, \\ b_i(y)D_{y_i}D^h_{x_k}w = -D^h_{x_k}\gamma(x) + D^h_{x_k}g(x,y) & \text{on } \partial T \end{cases}$$

and from Lemma 5.2.8, we have

$$\begin{aligned} &|D^h_{x_k}\gamma(x)| + \|D^h_{x_k}w(x,\cdot)\|_{C^{2,\alpha}(\mathbb{R}^n\setminus T)} \\ &\qquad \le C\left(\|f\|_{C^{0,1}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))} + \|g\|_{C^{0,1}(\overline{\Omega};C^{1,\alpha}(\partial T))}\right). \end{aligned} \tag{5.2.9}$$

Hence we deduce from the proof of Lemma 5.2.5 that there exist a unique constant $\widehat{\gamma}_k(x)$ and a bounded y-periodic function $\widehat{w}_k(x,\cdot) \in C^{2,\alpha}(\mathbb{R}^n \setminus T)$ such that

$$\left|D^h_{x_k}\gamma(x) - \widehat{\gamma}_k(x)\right| + \|D^h_{x_k}w(x,\cdot) - \widehat{w}_k(x,\cdot)\|_{C^2(\mathbb{R}^n\setminus T)} \to 0$$

as $h \to 0$ and $\widehat{\gamma}_k$ and $\widehat{w}_k$ satisfy

$$\begin{cases} a_{ij}(y) D_{y_i y_j} \widehat{w}_k = \widehat{\gamma}_k(x) + D_{x_k} f(x,y) & \text{in } \mathbb{R}^n \setminus T, \\ b_i(y) D_{y_i} \widehat{w}_k = -\widehat{\gamma}_k(x) + D_{x_k} g(x,y) & \text{on } \partial T. \end{cases}$$

Due to the uniqueness of the solution of above (5.2.4), we know that the limit of $D^h_{x_k} w(x,\cdot) \to \widehat{w}(x,\cdot)$ (in C^2-norm) takes place for the full sequence of h. Consequently, by definition, $\widehat{\gamma}_k(x) = D_{x_k}\gamma(x)$ and $\widehat{w}_k(x,y) = D_{x_k} w(x,y)$. Moreover, estimate (5.2.9) implies that $D_{x_k}\gamma$ and $D_{x_k} w$ satisfy (5.2.8). □

We are now in position to show the proof of Proposition 5.2.6.

proof of Proposition 5.2.6. First, assume that

$$\|f\|_{C^{1,\beta}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))} + \|g\|_{C^{1,\beta}(\overline{\Omega};C^{1,\alpha}(\partial T))} < \infty.$$

Then from Lemma 5.2.10 the first order partial derivatives of $w(\cdot,y)$ with respect to x-variable satisfies the equations (5.2.7) which belong to the same class of (5.2.4), and admit the ε-approximating equation (5.2.6). More precisely, the uniqueness of the solution $w(x,y)$ implies that the limit of the normalized function $\widetilde{v}_\varepsilon(x,y) = v_\varepsilon(x,y) - v_\varepsilon(x,0)$, where v_ε is the solution of (5.2.6), solves (5.2.7). Consequently, we can apply Lemma 5.2.7 and 5.2.8 again to obtain

$$\begin{aligned} |D_{x_k}\gamma(x_1) - D_{x_k}\gamma(x_2)| + \|D_{x_k} w(x_1,\cdot) - D_{x_k} w(x_2,\cdot)\|_{C^{2,\alpha}(\mathbb{R}^n\setminus T)} \\ \leq C|x_1 - x_2|^{\beta}\left(\|f\|_{C^{1,\beta}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))} + \|g\|_{C^{1,\beta}(\overline{\Omega};C^{1,\alpha}(\partial T))}\right) \end{aligned} \tag{5.2.10}$$

for each $x_1, x_2 \in \overline{\Omega}$ and $k \in \{1,\cdots,n\}$. Then in view of (5.2.3), (5.2.8) and (5.2.10) we conclude that

$$\begin{aligned} \|\gamma\|_{C^{1,\beta}(\overline{\Omega})} + \|w\|_{C^{1,\beta}(\overline{\Omega};C^{2,\alpha}(\mathbb{R}^n\setminus T))} \\ \leq C\left(\|f\|_{C^{1,\beta}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))} + \|g\|_{C^{1,\beta}(\overline{\Omega};C^{1,\alpha}(\partial T))}\right) \end{aligned}$$

where C depends only on n, α, λ, Λ, r, $\|a_{ij}\|_{C^\alpha(\mathbb{R}^n\setminus T)}$, and $\|b_i\|_{C^{1,\alpha}(\partial T)}$. Thus, if the condition

$$\|f\|_{C^{m,\beta}(\overline{\Omega};C^\alpha(\mathbb{R}^n\setminus T))} + \|g\|_{C^{m,\beta}(\overline{\Omega};C^{1,\alpha}(\partial T))} < \infty$$

holds, then we can repeat the argument used through Lemma 5.2.8 and Lemma 5.2.10 again to get the Hölder continuity of the second order partial derivatives of $w(\cdot, y)$. Hence for any $m \in \{0, 1, 2, \cdots\}$, we iterate this process by m-times to reach the conclusion. □

Lemma 5.2.11. *Let $w(x, y)$ be a solution of the equation* (5.2.4)*. Assume that*

$$\|f\|_{C^{1,\beta}(\overline{\Omega};C^\alpha(\mathbb{R}^n\setminus T))} + \|g\|_{C^{1,\beta}(\overline{\Omega};C^{1,\alpha}(\partial T))} < \infty.$$

Then $D_{x_k}D_{y_l}w = D_{y_l}D_{x_k}w$ $(1 \leq k, l \leq n)$ and satisfies

$$\|D_{x_i}D_{y_j}w\|_{C^\beta(\overline{\Omega};C^{1,\alpha}(\mathbb{R}^n\setminus T))} \leq C\left(\|f\|_{C^{0,1}(\overline{\Omega};C^\alpha(\mathbb{R}^n\setminus T))} + \|g\|_{C^{0,1}(\overline{\Omega};C^{1,\alpha}(\partial T))}\right)$$

where C depends only on n, α, λ, Λ, r, $\|a_{ij}\|_{C^\alpha(\mathbb{R}^n\setminus T)}$, and $\|b_i\|_{C^{1,\alpha}(\partial T)}$.

Proof. From the proof of Lemma 5.2.10, it is clear that $D_{y_l}D_{x_k}w$ exists and satisfies

$$\|D_{y_l}D_{x_k}w\|_{C^\beta(\overline{\Omega};C^{1,\alpha}(\mathbb{R}^n\setminus T))} \leq C\left(\|f\|_{C^{0,1}(\overline{\Omega};C^\alpha(\mathbb{R}^n\setminus T))} + \|g\|_{C^{0,1}(\overline{\Omega};C^{1,\alpha}(\partial T))}\right).$$

To show that existence of $D_{x_k}D_{y_l}w$, fix $x \in \overline{\Omega}$ and consider the following difference quotient

$$\begin{aligned} D^h_{x_k}D_{y_l}w(x, \cdot) :=& \frac{D_{y_l}w(x + he_k, \cdot) - D_{y_l}w(x, \cdot)}{h} \\ =& D_{y_l}\left(\frac{w(x + he_k, \cdot) - w(x, \cdot)}{h}\right) = D_{y_l}D^h_{x_k}w(x, \cdot). \end{aligned}$$

We note that $D^h_{x_k} w(x,\cdot)$ solves following equation:

$$\begin{cases} a_{ij}(y) D_{y_i y_j} D^h_{x_k} w = D^h_{x_k}\gamma(x) + D^h_{x_k} f(x,y) & \text{in } \mathbb{R}^n \setminus T, \\ b_i(y) D_{y_i} D^h_{x_k} w = -D^h_{x_k}\gamma(x) + D^h_{x_k} g(x,y) & \text{on } \partial T. \end{cases}$$

From Lemma 5.2.8, we know that

$$\|D^h_{x_k} w(x,\cdot)\|_{C^{2,\alpha}(\mathbb{R}^n\setminus T)} \le C\left(\|f\|_{C^{0,1}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))} + \|g\|_{C^{0,1}(\overline{\Omega};C^{1,\alpha}(\partial T))}\right)$$

and hence

$$\begin{aligned} \|D^h_{x_k} D_{y_l} w(x,\cdot)\|_{C^{1,\alpha}(\mathbb{R}^n\setminus T)} &= \|D_{y_l} D^h_{x_k} w(x,\cdot)\|_{C^{1,\alpha}(\mathbb{R}^n\setminus T)} \\ &\le C\left(\|f\|_{C^{0,1}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))} + \|g\|_{C^{0,1}(\overline{\Omega};C^{1,\alpha}(\partial T))}\right). \end{aligned}$$

Consequently, the conditions for the Arzelá-Ascoli theorem met, which ensures the existence of a subsequence $\{D^{h_m}_{x_k} D_{y_l} w\}_{m=1}^{\infty}$ of $\{D^h_{x_k} D_{y_l} w\}_{h>0}$ which converges to $v(x,\cdot)$ in C^1-norm. But the uniqueness of $D_{y_l} D_{x_k} w$ (Lemma 5.2.10) implies that $D_{y_l} D^h_{x_k} w(x,\cdot) \to D_{y_l} D_{x_k} w(x,\cdot)$ in C^1-norm takes place for the full sequence of h. Hence we conclude that $v(x,y) = D_{x_k} D_{y_l} w(x,y) = D_{y_l} D_{x_k} w(x,y)$. □

5.2.2 Asymptotic expansions and correctors

In this subsection, we define corrector equations from the asymptotic expansion of u^ε. We will take a heuristic approach first, and then rigorously investigate the results. Assume that u^ε has the following asymptotic expansions:

$$\begin{aligned} u^\varepsilon(x) &\simeq u_0(x,x/\varepsilon) + \varepsilon u_1(x,x/\varepsilon) + \varepsilon^2 u_2(x,x/\varepsilon) + \cdots + \varepsilon^m u_m(x,x/\varepsilon) \\ &= u_0(x,x/\varepsilon) + \varepsilon q_m(x,x/\varepsilon) \end{aligned} \tag{5.2.11}$$

where $x \in \overline{\Omega}$, $y := x/\varepsilon \in \mathbb{R}^n \setminus T$, and

$$q_m(x, x/\varepsilon) = u_1(x, x/\varepsilon) + \varepsilon u_2(x, x/\varepsilon) + \cdots \varepsilon^{m-1} u_m(x, x/\varepsilon).$$

To simplify our notation, let us drop the dependency of $(x, x/\varepsilon)$. For a while, let us assume that all the functions c and $\{u_k\}_{0 \le k \le m}$ are regular enough. Then a Taylor expansion of $c(u^\varepsilon, x, x/\varepsilon)$ with respect to u^ε gives

$$\begin{aligned} c(u^\varepsilon) &= c(u_0) + \varepsilon \frac{\partial c}{\partial z}(u_0) q_m + \cdots + \frac{\varepsilon^{m-2}}{(m-2)!} \frac{\partial^{m-2} c}{\partial z^{m-2}}(u_0) q_m^{m-2} + O(\varepsilon^{m-1}) \\ &= c(u_0) + \varepsilon \frac{\partial c}{\partial z}(u_0) u_1 + \varepsilon^2 \left(\frac{\partial c}{\partial z}(u_0) u_2 + \frac{1}{2!} \frac{\partial^2 c}{\partial z^2}(u_0) u_1^2 \right) + \cdots \\ &\quad + \varepsilon^{m-2} \left(\sum_{i=1}^{m-2} \frac{1}{i!} \frac{\partial^i c}{\partial z^i}(u_0) \sum_{\substack{n_1 + \cdots + n_i = m-2 \\ n_1, \cdots, n_i \neq 0}} u_{n_1} u_{n_2} \cdots u_{n_i} \right) + O(\varepsilon^{m-1}) \\ &= c(u_0) + \varepsilon \Psi_1 + \cdots + \varepsilon^{m-2} \Psi_{m-2} + O(\varepsilon^{m-1}) \end{aligned} \tag{5.2.12}$$

where

$$\Psi_k(u_0, u_1, \cdots, u_k, x, x/\varepsilon) = \sum_{i=1}^{k} \frac{1}{i!} \frac{\partial^i c}{\partial z^i}(u_0, x, x/\varepsilon) \sum_{\substack{n_1 + \cdots + n_i = k \\ n_1, \cdots, n_i \neq 0}} u_{n_1} u_{n_2} \cdots u_{n_i}.$$

Then by putting (5.2.11) and (5.2.12) to our main equation (L_ε), we have

$$\begin{aligned} & a_{ij}^\varepsilon(x) D_{ij} u^\varepsilon + c(u^\varepsilon, x, x/\varepsilon) \\ &\quad = a_{ij}^\varepsilon(x) \Big(D_{x_i x_j} u_0 + \frac{1}{\varepsilon} D_{x_i y_j} u_0 + \frac{1}{\varepsilon} D_{y_i x_j} u_0 + \frac{1}{\varepsilon^2} D_{y_i y_j} u_0 + \varepsilon D_{x_i x_j} u_1 \\ &\qquad + D_{x_i y_j} u_1 + D_{y_i x_j} u_1 + \frac{1}{\varepsilon} D_{y_i y_j} u_1 + \varepsilon^2 D_{x_i x_j} u_2 + \varepsilon D_{x_i y_j} u_2 \\ &\qquad + \varepsilon D_{y_i x_j} u_2 + D_{y_i y_j} u_2 + \cdots \Big) + c(u_0, x, x/\varepsilon) \\ &\qquad + \varepsilon \Psi_1(u_0, u_1, x, x/\varepsilon) + \varepsilon^2 \Psi_2(u_0, u_1, u_2, x, x/\varepsilon) + \cdots \\ &\quad = f(x, x/\varepsilon) \end{aligned} \tag{5.2.13}$$

with oblique boundary condition

$$b_i^\varepsilon(x)\cdot D_i u^\varepsilon = b_i^\varepsilon \cdot \left(D_{x_i}u_0 + \frac{1}{\varepsilon}D_{y_i}u_0 + \varepsilon D_{x_i}u_1 + D_{y_i}u_1 + \cdots \right) = \varepsilon g(x, x/\varepsilon). \tag{5.2.14}$$

From above expansions, we can expect to see the appropriate correctors heuristically by comparing the order of ε. If we focus on $1/\varepsilon^2$ order terms, we can obtain the following cell problem

$$\begin{cases} a_{ij}(y)D_{y_iy_j}u_0 = 0 & \text{in } \mathbb{R}^n \setminus T, \\ b_i(y)D_{y_i}u_0 = 0 & \text{on } \partial T. \end{cases} \tag{5.2.15}$$

As mentioned in the introduction, (5.2.15) admits solutions if (A, b) satisfies the compatibility condition, hence we deduce that u_0 does not depend on y. That is, $u_0(x,y) = u_0(x)$. From this fact, and by comparing $1/\varepsilon$ order terms, $u_1(x,y)$ satisfies the following equation

$$\begin{cases} a_{ij}(y)D_{y_iy_j}u_1 = 0 & \text{in } \mathbb{R}^n \setminus T, \\ b_i(y)\left(D_{x_i}u_0(x,y) + D_{y_i}u_1\right) = 0 & \text{on } \partial T. \end{cases} \tag{5.2.16}$$

Let $\chi_1^k = \chi_1^k(y)$, $1 \le k \le n$ be a solution of the equation (5.2.16) when $u_0(x) = x_k$. i.e. χ_1^k solves

$$\begin{cases} a_{ij}(y)D_{y_iy_j}\chi_1^k = 0 & \text{in } \mathbb{R}^n \setminus T, \\ b_k(y) + b_i(y)D_{y_i}\chi_1^k = 0 & \text{on } \partial T. \end{cases} \tag{5.2.17}$$

We assume that $\chi_1^k(0) = 0$ for the uniqueness of solution. Then the general solution u_1 of (5.2.16) is represented by

$$u_1(x,y) = \sum_k \chi_1^k(y)D_{x_k}u_0(x) + \psi_1(x) \tag{5.2.18}$$

for some function ψ_1 defined in $\overline{\Omega}$.

If we focus on the ε^0 order terms, then we can obtain following cell problem for $u_2(x,y)$:

$$\begin{cases} a_{ij}(y)\left(D_{x_ix_j}u_0(x) + D_{x_iy_j}u_1(x,y) + D_{y_ix_j}u_1(x,y) + D_{y_iy_j}u_2\right) \\ \qquad\qquad\qquad +c(u_0,x,y) = f(x,y) & \text{in } \mathbb{R}^n \setminus T, \\ b_i(y)\left(D_{x_i}u_1(x,y) + D_{y_i}u_2\right) = g(x,y) & \text{on } \partial T. \end{cases} \tag{5.2.19}$$

Now we investigate the homogenization process rigorously. Let v_ε be a solution of the following corrector equation:

$$\begin{cases} a_{ij}(y)\left(M_{ij} + M_{ik}D_{y_j}\chi_1^k(y) + D_{y_i}\chi_1^k(y)M_{kj} + D_{y_iy_j}v_\varepsilon\right) \\ \qquad\qquad\qquad +c(z,x,y) - \varepsilon^2 v_\varepsilon = f(x,y) & \text{in } \mathbb{R}^n \setminus T, \\ b_i(y)\left(M_{ik}\chi_1^k(y) + D_{y_i}v_\varepsilon\right) + \varepsilon^2 v_\varepsilon = g(x,y) & \text{on } \partial T, \end{cases} \tag{5.2.20}$$

obtained by using (5.2.18) with an assumption that $\psi_1 \equiv 0$, by freezing $M = D^2u_0(x)$ and $z = u_0(x)$, and by subtracting and adding an auxiliary term $\varepsilon^2 v_\varepsilon$ to the equation (5.2.19). Then from Lemma 5.2.2 there is a unique bounded y-periodic solution v_ε, which we denote $v_\varepsilon(y;M,z,x)$ for given $n\times n$ matrix M, $x \in \overline{\Omega}$ and $z \in \mathbb{R}$. Additionally, Lemma 5.2.5 implies that there exist a unique y-periodic function $w(y;M,z,x)$ with $w(\cdot;M,x,z) \in C^{2,\alpha}(\mathbb{R}^n \setminus T)$, and a unique constant $\overline{L}(M,z,x) \in \mathbb{R}$ such that

$$\begin{aligned} &\|\varepsilon^2 v_\varepsilon(\cdot;M,z,x) - \overline{L}(M,z,x)\|_{L^\infty(\mathbb{R}^n\setminus T)} \\ &\qquad + \|\widetilde{v}_\varepsilon(\cdot;M,z,x) - w(\cdot;M,z,x)\|_{C^2(\mathbb{R}^n\setminus T)} \to 0 \ \text{ as } \ \varepsilon \to 0 \end{aligned}$$

where $\widetilde{v}_\varepsilon(y;M,z,x) := v_\varepsilon(y;M,z,x) - v_\varepsilon(0;M,z,x)$. Now we define an effective operator $\overline{L}(M,z,x)$ as

$$\overline{L}(M,z,x) = \lim_{\varepsilon\to 0} \varepsilon^2 v_\varepsilon(y;M,z,x). \tag{5.2.21}$$

In [38], the authors showed some properties which are related with the existence of solutions of the effective equation.

Lemma 5.2.12. *Let $\overline{L}(M,z,x)$ be an operator defined in* (5.2.21) *obtained* *from the coefficients in* (L_ε). *Assume that* $A(y)$, $b(y)$, $c(z,x,y)$, $f(x,y)$, *and* $g(x,y)$ *satisfy* (5.1.1), (5.1.2), (5.1.3), *(C1) and (C2). Then we have the followings:*

(i) For each $x \in \overline{\Omega}$ *and* $z \in \mathbb{R}$, $\overline{L}(\cdot, z, x)$ *is an affine function on the set of* $n \times n$ *matrices* $\mathcal{S}^n$.

(ii) $\overline{L}(M, z, x)$ *is non-increasing with* z *variable.*

(iii) (Uniform ellipticity) There is a positive real number r_0 *depending only on* n, λ *and* Λ *such that if the size of holes* r *is less than or equal to* r_0, *then* $\overline{L}(M, z, x)$ *is uniformly elliptic for each* $x \in \overline{\Omega}$ *and* $z \in \mathbb{R}$, *i.e., there is a positive constant* $\overline{\lambda} = \overline{\lambda}(r_0)$ *satisfying*

$$\overline{L}(M + N, z, x) \geq \overline{L}(M, z, x) + \overline{\lambda} \|N\|$$

for any $M \in \mathcal{S}^n$, *positive matrix* N, $x \in \overline{\Omega}$ *and* $z \in \mathbb{R}$.

Let $-\overline{f}(x) = \overline{L}(O, 0, x)$ and $\overline{c}(z, x) = \overline{L}(O, z, x) - \overline{L}(O, 0, x)$ where O is the $n \times n$ zero matrix. Then, due to its linear structure of $\overline{L}$, there exists a constant matrix $(\overline{a}_{ij})$ such that

$$\overline{L}(M, z, x) = \overline{a}_{ij} M_{ij} + \overline{c}(z, x) - \overline{f}(x)$$

and hence we can find a solution $\overline{u}$ that solves following Dirichlet problem:

$$\begin{cases} \overline{L}(D^2\overline{u}, \overline{u}, x) = \overline{a}_{ij} D_{x_i x_j} \overline{u}(x) + \overline{c}(\overline{u}, x) - \overline{f}(x) = 0 & \text{in } \Omega, \\ \overline{u}(x) = \varphi(x) & \text{on } \partial\Omega. \end{cases} \tag{5.2.22}$$

We call $\overline{L}(M, z, x)$ as the effective operator in the sense of the following theorem.

Theorem 5.2.13 (Effective operator)**.** *Let* u^ε *be a viscosity solution of* (L_ε) *and* $\overline{u}$ *is a solution of* (5.2.22). *Assume all the conditions in Lemma 5.2.12 hold. Then* $\overline{u}$ *is unique and* u^ε *converges uniformly to* $\overline{u}$.

Proof. See [38] for detailed proof. □

We finish this subsection by introducing the first and second interior correctors by investigating their existence and regularity. Let $\phi_2(x,y) = w(y;O,\overline{u},x)$ and $\chi_2^{kl}(y) = w(y;E^{kl},0,x) - w(y;O,0,x)$ where $\overline{u}$ is the solution of the effective operator (5.2.22) and $\{E^{kl}|k,l=1,\cdots,n\}$ is the standard basis of $\mathcal{S}^n$. Then ϕ_2 and χ_2^{kl} solve the following equations respectively,

$$\begin{cases} a_{ij}(y)D_{y_iy_j}\phi_2 + c(\overline{u},x,y) - f(x,y) = \overline{c}(\overline{u},x) - \overline{f}(x) & \text{in } \mathbb{R}^n \setminus T, \\ b_i(y)D_{y_i}\phi_2 - g(x,y) = \overline{f}(x) - \overline{c}(\overline{u},x) & \text{on } \partial T \end{cases} \tag{5.2.23}$$

and

$$\begin{cases} a_{kl}(y) + a_{kj}(y)D_{y_j}\chi_1^l + a_{il}(y)D_{y_i}\chi_1^k + a_{ij}(y)D_{y_iy_j}\chi_2^{kl} = \overline{a}_{kl} & \text{in } \mathbb{R}^n \setminus T, \\ b_k(y)\chi_1^l(y) + b_i(y)D_{y_i}\chi_2^{kl} = -\overline{a}_{kl} & \text{on } \partial T. \end{cases} \tag{5.2.24}$$

Now we are ready to define the first and second interior corrector. Define $w_1, w_2 : \overline{\Omega} \times \mathbb{R}^n \setminus T \to \mathbb{R}$ by

$$\begin{cases} w_1(x,y) = \chi_1^k(y)D_{x_k}\overline{u}(x) + \psi_1(x) \\ w_2(x,y) = \phi_2(x,y) + \chi_2^{kl}(y)D_{x_kx_l}\overline{u}(x) + \chi_1^k(y)D_{x_k}\psi_1(x) + \psi_2(x) \end{cases} \tag{5.2.25}$$

where $\overline{u}$ is the solution of (5.2.22), ψ_1 and ψ_2 will be determined later. Then we utilize (5.2.17), (5.2.22), (5.2.23) and (5.2.24) to obtain

$$\begin{cases} a_{ij}(y)D_{y_iy_j}w_1 = 0 & \text{in } \mathbb{R}^n \setminus T, \\ b_i(y)\left(D_{x_i}\overline{u}(x) + D_{y_i}w_1\right) = 0 & \text{on } \partial T \end{cases}$$

and

$$\begin{cases} a_{ij}(y)\left(D_{x_ix_j}\overline{u}(x) + D_{x_iy_j}w_1(x,y) + D_{y_ix_j}w_1(x,y) + D_{y_iy_j}w_2\right) & \\ \qquad\qquad\qquad\qquad +c(\overline{u},x,y) = f(x,y) & \text{in } \mathbb{R}^n \setminus T, \\ b_i(y)\left(D_{x_i}w_1(x,y) + D_{y_i}w_2\right) = g(x,y) & \text{on } \partial T. \end{cases}$$

We call w_1 and w_2 as the first and second order interior corrector respectively in the sense that w_1 and w_2 satisfy (5.2.16) and (5.2.19) respectively.

5.2.3 Higher order interior correctors

In this subsection, we are going to determine the k-th order interior correctors when $k \geq 3$. Through the heuristic calculation of (5.2.13) and (5.2.14), we obtain equations for u_k, $3 \leq k \leq m$:

$$\begin{cases} a_{ij}(y)\left(D_{x_ix_j}u_{k-2} + D_{x_iy_j}u_{k-1} + D_{y_ix_j}u_{k-1}\right. & \\ \qquad\qquad \left.+D_{y_iy_j}u_k\right) + \Psi_{k-2}(u_0,\cdots,u_{k-2},x,y) = 0 & \text{in } \Omega_\varepsilon \times (\mathbb{R}^n \setminus T), \\ b_i(y)\left(D_{x_i}u_{k-1} + D_{y_i}u_k\right) = 0 & \text{on } \partial T_\varepsilon \times \partial T. \end{cases} \tag{5.2.26}$$

We are going to construct a family of correctors $\{w_k\}_{3\leq k\leq m}$ satisfying equation (5.2.26). To see the structure of corrector equation, we assume that w_k has the following representation:

$$w_k(x,y) = \phi_k(x,y) + \chi_2^{i_1i_2}(y)D_{x_{i_1}x_{i_2}}\psi_{k-2}(x) + \chi_1^{i_1}(y)D_{x_{i_1}}\psi_{k-1}(x) + \psi_k(x). \tag{5.2.27}$$

for $3 \leq k \leq m$. If we set $\psi_{-1} \equiv 0$, $\psi_0 = \overline{u}$, $\phi_1 \equiv 0$ and ϕ_2 chosen as the solution of (5.2.23), then w_1 and w_2 defined in (5.2.25) also can be represented as in (5.2.27).

In order for $\{w_k\}_{3\leq k\leq m}$ to satisfy equation (5.2.26), We will define Ψ_k inductively using the k-th order correctors $\{w_k\}_{1\leq k\leq m}$, including the solution

$\overline{u}$ of (5.2.22). So we assume that Ψ_k is of the form

$$\Psi_k(\overline{u}, w_1, \cdots, w_k, x, y) = \sum_{i=1}^{k} \frac{1}{i!} \frac{\partial^i c}{\partial z^i}(\overline{u}, x, y) \sum_{\substack{n_1+\cdots+n_i=k \\ n_1,\cdots,n_i \neq 0}} w_{n_1} w_{n_2} \cdots w_{n_i}. \tag{5.2.28}$$

By putting (5.2.25) and (5.2.27) to equation (5.2.26), we obtain that

$$\begin{aligned}
\mathcal{L}_k :=& a_{ij}\left(D_{y_i y_j} w_k + D_{x_i y_j} w_{k-1} + D_{y_i x_j} w_{k-1} + D_{x_i x_j} w_{k-2}\right) + \Psi_{k-2} \\
=& a_{ij}\big(D_{y_i y_j}\phi_k + D_{y_i y_j}\chi_2^{i_1 i_2} D_{x_{i_1} x_{i_2}}\psi_{k-2} + D_{x_i y_j}\phi_{k-1} \\
& + D_{y_j}\chi_2^{i_1 i_2} D_{x_i x_{i_1} x_{i_2}}\psi_{k-3} + D_{y_j}\chi_1^{i_1} D_{x_i x_{i_1}}\psi_{k-2} + D_{y_i x_j}\phi_{k-1} \\
& + D_{y_i}\chi_2^{i_1 i_2} D_{x_j x_{i_1} x_{i_2}}\psi_{k-3} + D_{y_i}\chi_1^{i_1} D_{x_j x_{i_1}}\psi_{k-2} + D_{x_i x_j}\phi_{k-2} \\
& + \chi_2^{i_1 i_2} D_{x_i x_j x_{i_1} x_{i_2}}\psi_{k-4} + \chi_1^{i_1} D_{x_i x_j x_{i_1}}\psi_{k-3} + D_{x_i x_j}\psi_{k-2}\big) \\
& + \frac{\partial c}{\partial z}(\overline{u}, x, y)\psi_{k-2} + \widetilde{\Psi}_{k-2} \\
=& \left(a_{ij} D_{y_i y_j}\phi_k + \frac{\partial c}{\partial z}(\overline{u}, x, y)\psi_{k-2}\right) \\
& + \Big\{\left(a_{i_3 j} D_{y_j}\chi_2^{i_1 i_2} + a_{i i_3} D_{y_i}\chi_2^{i_1 i_2} + a_{i_2 i_3}\chi_1^{i_1}\right) D_{x_{i_1} x_{i_2} x_{i_3}}\psi_{k-3} \\
& + a_{ij}\big(\chi_2^{i_1 i_2} D_{x_i x_j x_{i_1} x_{i_2}}\psi_{k-4} + D_{x_i y_j}\phi_{k-1} + D_{y_i x_j}\phi_{k-1} \\
& + D_{x_i x_j}\phi_{k-2}\big) + \widetilde{\Psi}_{k-2}\Big\} \\
& + \left(a_{ij} D_{y_i y_j}\chi_2^{i_1 i_2} + a_{i_2 j} D_{y_j}\chi_1^{i_1} + a_{i i_2} D_{y_j}\chi_1^{i_1} + a_{i_1 i_2}\right) D_{x_{i_1} x_{i_2}}\psi_{k-2} \\
=& \left(a_{ij} D_{y_i y_j}\phi_k + \frac{\partial c}{\partial z}(\overline{u}, x, y)\psi_{k-2}\right) - f_k + \overline{a}_{i_1 i_2} D_{x_{i_1} x_{i_2}}\psi_{k-2}
\end{aligned} \tag{5.2.29}$$

and

$$\begin{aligned}
\mathcal{N}_k :=& b_i\left(D_{x_i}w_{k-1} + D_{y_i}w_k\right)\\
=& b_i\left(D_{x_i}\phi_{k-1} + \chi_2^{i_1i_2}D_{x_ix_{i_1}x_{i_2}}\psi_{k-3} + \chi_1^{i_1}D_{x_ix_{i_1}}\psi_{k-2} + D_{x_i}\psi_{k-1}\right)\\
&+ b_i\left(D_{y_i}\phi_k + D_{y_i}\chi_2^{i_1i_2}D_{x_{i_1}x_{i_2}}\psi_{k-2} + D_{y_i}\chi_1^{i_1}D_{x_{i_1}}\psi_{k-1}\right)\\
=& b_iD_{y_i}\phi_k + \left(b_{i_2}\chi_1^{i_1} + b_iD_{y_i}\chi_2^{i_1i_2}\right)D_{x_{i_1}x_{i_2}}\psi_{k-2}\\
&+ \left\{\left(b_{i_1} + b_iD_{y_i}\chi_1^{i_1}\right)D_{x_{i_1}}\psi_{k-1} + b_i\chi_2^{i_1i_2}D_{x_ix_{i_1}x_{i_2}}\psi_{k-3} + b_iD_{x_i}\phi_{k-1}\right\}\\
=& b_iD_{y_i}\phi_k - \overline{a}_{i_1i_2}D_{x_{i_1}x_{i_2}}\psi_{k-2} + b_i\left(\chi_2^{i_1i_2}D_{x_ix_{i_1}x_{i_2}}\psi_{k-3} + D_{x_i}\phi_{k-1}\right)\\
=& b_iD_{y_i}\phi_k - \overline{a}_{i_1i_2}D_{x_{i_1}x_{i_2}}\psi_{k-2} - g_k
\end{aligned} \tag{5.2.30}$$

where $\widetilde{\Psi}_{k-2}$, f_k and g_k are functions given by

$$\begin{aligned}
\widetilde{\Psi}_{k-2}(x,y) =& \Psi_{k-2}(\overline{u}, w_1, \cdots, w_{k-2}, x, y) - \frac{\partial c}{\partial z}(\overline{u}, x, y)\psi_{k-2}(x),\\
f_k(x,y) :=& - a_{ij}(y)\left(D_{x_iy_j}\phi_{k-1}(x,y) + D_{y_ix_j}\phi_{k-1}(x,y) + D_{x_ix_j}\phi_{k-2}(x,y)\right.\\
&\left. + \chi_2^{i_1i_2}(y)D_{x_ix_jx_{i_1}x_{i_2}}\psi_{k-4}(x)\right) - \left(a_{i_3j}(y)D_{y_j}\chi_2^{i_1i_2}(y)\right.\\
&\left. + a_{ji_3}(y)D_{y_i}\chi_2^{i_1i_2}(y) + a_{i_2i_3}(y)\chi_1^{i_1}(y)\right)D_{x_{i_1}x_{i_2}x_{i_3}}\psi_{k-3}(x)\\
&- \widetilde{\Psi}_{k-2}(x,y),\\
g_k(x,y) :=& - b_i(y)\left(\chi_2^{i_1i_2}(y)D_{x_ix_{i_1}x_{i_2}}\psi_{k-3}(x) + D_{x_i}\phi_{k-1}(x,y)\right).
\end{aligned} \tag{5.2.31}$$

Now we will see how to obtain ϕ_k and ψ_{k-2} for $3 \le k \le m$. We are going to use an induction argument, so suppose that we have already found the families $\{\psi_{l-2}\}_{1\le l\le k-1}$ and $\{\phi_l\}_{1\le l\le k-1}$. We then define f_k and g_k as in (5.2.31) and consider v_ε as a solution of

$$\begin{cases}
a_{ij}(y)\left(M_{ij} + M_{ii_1}D_{y_j}\chi_1^{i_1}(y) + D_{y_i}\chi_1^{i_1}(y)M_{i_1j} + D_{y_iy_j}v_\varepsilon\right) & \\
\qquad\qquad + c_0(x,y)z - \varepsilon^2 v_\varepsilon = f_k(x,y) & \text{in } \mathbb{R}^n \setminus T,\\
b_i(y)\left(M_{ii_1}\chi_1^{i_1} + D_{y_i}v_\varepsilon\right) + \varepsilon^2 v_\varepsilon = g_k(x,y) & \text{on } \partial T
\end{cases} \tag{5.2.32}$$

where $c_0(x,y) = \frac{\partial c}{\partial z}(\overline{u}, x, y)$. One may notice that f_k and g_k do not involve the functions ψ_{k-2} and ϕ_k.

As long as c_0, f_k and g_k are regular enough, (5.2.32) belongs to the same class of (5.2.20), only $c(z,x,y)$, $f(x,y)$ and $g(x,y)$ are replaced by $c_0(x,y)z$, $f_k(x,y)$ and $g_k(x,y)$. Consequently, from Lemma 5.2.1-5.2.5 there exists a unique bounded y-periodic solution $v_\varepsilon(y;M,z,x)$ of the equation (5.2.32) and there exists a y-periodic function $v(\cdot;M,z,x) \in C^{2,\alpha}(\mathbb{R}^n \setminus T)$, a unique number $\overline{L}_k(M,z,x) \in \mathbb{R}$ such that

$$\|\varepsilon^2 v_\varepsilon - \overline{L}_k\|_{L^\infty(\mathbb{R}^n\setminus T)} + \|\widetilde{v}_\varepsilon - v\|_{C^2(\mathbb{R}^n\setminus T)} \to 0 \quad \text{as} \quad \varepsilon \to \infty$$

where $\widetilde{v}_\varepsilon(x,y) := v_\varepsilon(x,y) - v_\varepsilon(x,0)$. Then $v(y;M,z,x)$ and $\overline{L}_k(M,z,x)$ solve following equation:

$$\begin{cases} a_{ij}(y)\left(M_{ij} + M_{ii_1}D_{y_j}\chi_1^{i_1}(y) + D_{y_i}\chi_1^{i_1}(y)M_{i_1j} + D_{y_iy_j}v\right) & \\ \qquad\qquad +c_0(x,y)z = \overline{L}_k(M,z,x) + f_k(x,y) & \text{in } \mathbb{R}^n \setminus T, \\ b_i(y)\left(M_{ii_1}\chi_1^{i_1}(y) + D_{y_i}v\right) = -\overline{L}_k(M,z,x) + g_k(x,y) & \text{on } \partial T. \end{cases} \tag{5.2.33}$$

Due to Lemma 5.2.12, the operator $\overline{L}_k(M,z,x)$ is uniformly elliptic when the size of holes are sufficiently small. Also, similar to the form of $\overline{L}$, $\overline{L}_k$ can be represented by

$$\overline{L}_k(M,z,x) = \overline{a}_{ij}M_{ij} + \overline{c}_k(x)z - \overline{f}_k(x)$$

where $\overline{a}_{ij}$ are in (5.2.22), $\overline{f}_k(x) = -\overline{L}_k(O,0,x)$ and $\overline{c}_k(x) = \overline{L}_k(O,1,x) - \overline{L}_k(O,0,x)$.

Now we define $\psi_{k-2}(x)$ and $\phi_k(x,y)$ as solutions of

$$\begin{cases} \overline{a}_{ij}D_{x_ix_j}\psi_{k-2} + \overline{c}_k(x)\psi_{k-2} = \overline{f}_k(x) & \text{in } \Omega, \\ \psi_{k-2}(x) = 0 & \text{on } \partial\Omega \end{cases} \tag{5.2.34}$$

and

$$\begin{cases} a_{ij}(y)D_{y_iy_j}\phi_k + c_0(x,y)\psi_{k-2}(x) = \overline{L}_k(O,\psi_{k-2},x) + f_k(x,y) & \text{in } \mathbb{R}^n \setminus T, \\ b_i(y)D_{y_i}\phi_k = -\overline{L}_k(O,\psi_{k-2},x) + g_k(x,y) & \text{on } \partial T \end{cases} \tag{5.2.35}$$

respectively for $3 \le k \le m$. We notice that ψ_{k-2} exists by the same argument as the case of showing the existence of $\overline{u}$. Finally, choose $\psi_{m-1} \in C^{3,\alpha}(\overline{\Omega})$ and $\psi_m \in C^{2,\alpha}(\overline{\Omega})$ arbitrary functions. For example, we can choose $\psi_{m-1} \equiv \psi_m \equiv 0$. We assume that ψ_{m-1} and ψ_m satisfy estimate (5.2.39) without any loss of generality.

Now we make our argument rigorous. we must first enhance the regularity of $\overline{u}$ since the regularity of $\overline{u}$ plays an essential role in proving the existence of the higher order correctors.

Lemma 5.2.14. *Let $m \ge 2$ with $\varphi \in C^{m+2,\alpha}(\overline{\Omega})$ and $\partial\Omega \in C^{m+2,\alpha}$. Let $\overline{u}$ be the solution of* (5.2.22) *and assume that condition (C4) holds. Then $\overline{u} \in C^{m+2,\alpha}(\overline{\Omega})$ and*

$$\begin{aligned} &\|\overline{u}\|_{C^{m+2,\alpha}(\overline{\Omega})} \\ &\quad \le C\left(\|f\|_{C^{m,\alpha}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))} + \|g\|_{C^{m,\alpha}(\overline{\Omega};C^{1,\alpha}(\partial T))} + \|\varphi\|_{C^{m+2,\alpha}(\overline{\Omega})}\right) \end{aligned} \tag{5.2.36}$$

where C depends on n, m, α, λ, Λ, r, Ω, $\|a_{ij}\|_{C^{\alpha}(\mathbb{R}^n\setminus T)}$, $\|b_i\|_{C^{1,\alpha}(\partial T)}$ and $\|c\|_{C^{m,\alpha}(\mathbb{R}\times\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))}$.

Proof. From regularity assumptions of c, f and g, and from Proposition 5.2.6, we obtain that $\overline{c} - \overline{f} \in C^{m,\alpha}(\mathbb{R}\times\overline{\Omega})$ and there holds

$$\|\overline{c}\|_{C^{m,\alpha}(\mathbb{R}\times\overline{\Omega})} + \|\overline{f}\|_{C^{m,\alpha}(\overline{\Omega})} \le C\left(\|f\|_{C^{m,\alpha}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))} + \|g\|_{C^{m,\alpha}(\overline{\Omega};C^{1,\alpha}(\partial T))}\right)$$

where C depends only on n, m, α, λ, Λ, r, $\|a_{ij}\|_{C^{\alpha}(\mathbb{R}^n\setminus T)}$, $\|b_i\|_{C^{1,\alpha}(\partial T)}$ and $\|c\|_{C^{m,\alpha}(\mathbb{R}\times\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))}$. On the other hand, since $\overline{u}$ satisfies equation (5.2.22), the regularity theory in [21] implies that $\overline{u} \in C^{m+2,\alpha}(\overline{\Omega})$ satisfying

$$\|\overline{u}\|_{C^{m+2,\alpha}(\overline{\Omega})} \le \widetilde{C}\left(\|\overline{f}\|_{C^{m,\alpha}(\overline{\Omega})} + \|\varphi\|_{C^{m+2,\alpha}(\overline{\Omega})}\right)$$

where $\widetilde{C}$ depends only on n, m, α, λ, Λ, Ω and $\|\bar{c}\|_{C^{m,\alpha}(\mathbb{R}\times\overline{\Omega}))}$. Consequently, combining above two estimates, we can obtain (5.2.36). □

Lemma 5.2.15. *Assume that $\varphi \in C^{m+2,\alpha}(\overline{\Omega})$, $\partial\Omega \in C^{m+2,\alpha}$ and condition (C4) holds. Then there exist families of $\{\psi_k : \overline{\Omega}\times\mathbb{R}^n\setminus T \to \mathbb{R}\}_{-1\le k\le m}$ defined by the solutions of* (5.2.34)*, y-periodic functions $\{\phi_k : \overline{\Omega}\times\mathbb{R}^n\setminus T \to \mathbb{R}\}_{1\le k\le m}$ defined by the solutions of* (5.2.35) *respectively, which verify the following conditions.*

(i) For each $1 \le k \le m$, $\phi_k(x,\cdot) \in C^{2,\alpha}(\mathbb{R}^n \setminus T)$ uniformly for all $x \in \overline{\Omega}$, $\phi_k(\cdot, y) \in C^{m-k+2,\alpha}(\overline{\Omega})$ uniformly for all $y \in \mathbb{R}^n \setminus T$ and

$$\begin{aligned}&\|\phi_k\|_{C^{m-k+2,\alpha}(\overline{\Omega};C^{2,\alpha}(\mathbb{R}^n\setminus T))}\\ &\qquad\le C\left(\|f\|_{C^{m,\alpha}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))} + \|g\|_{C^{m,\alpha}(\overline{\Omega};C^{1,\alpha}(\partial T))} + \|\varphi\|_{C^{m+2,\alpha}(\overline{\Omega})}\right)\end{aligned} \tag{5.2.37}$$

(ii) For each $0 \le k \le m$, $\psi_k \in C^{m-k+2,\alpha}(\overline{\Omega})$ and

$$\begin{aligned}&\|\psi_k\|_{C^{m-k+2,\alpha}(\overline{\Omega})}\\ &\qquad\le C\left(\|f\|_{C^{m,\alpha}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))} + \|g\|_{C^{m,\alpha}(\overline{\Omega};C^{1,\alpha}(\partial T))} + \|\varphi\|_{C^{m+2,\alpha}(\overline{\Omega})}\right)\end{aligned} \tag{5.2.38}$$

where C depends on n, m, α, λ, Λ, r, Ω, $\|a_{ij}\|_{C^{\alpha}(\mathbb{R}^n\setminus T)}$, $\|b_i\|_{C^{1,\alpha}(\partial T)}$ and $\|c\|_{C^{m,\alpha}(\mathbb{R}\times\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))}$.

Proof. We are going to use an induction argument. As we set $\psi_{-1} \equiv 0$, $\psi_0 = \overline{u}$, $\phi_1 \equiv 0$ and ϕ_2 chosen as the solution of (5.2.23), we already know ψ_{-1}, ψ_0, ϕ_1 and ϕ_2 satisfy the assertion (i) and (ii) respectively, which immediately follows from Lemma 5.2.5, 5.2.6 and 5.2.14. Thus, we consider $3 \le k \le m$ and in order to run the induction argument, suppose that the families $\{\psi_{l-2}\}_{0\le l\le k-1}$ and $\{\phi_l\}_{1\le l\le k-1}$ satisfy above conditions (i) and (ii) respectively. Define f_k and g_k as (5.2.31) and $c_0(x,y) = \frac{\partial c}{\partial z}(\overline{u}, x, y)$. Then by induction hypotheses, Proposition 5.2.6, Lemma 5.2.11 and 5.2.14, we can observe that $c_0(x,\cdot)z$, $f_k(x,\cdot) \in C^{\alpha}(\mathbb{R}^n \setminus T)$, $g_k(x,\cdot) \in C^{1,\alpha}(\partial T)$,

$c_0(\cdot,y)\cdot \in C^{m-1,\alpha}(\mathbb{R}\times\overline{\Omega})$, $f_k(\cdot,y)\in C^{m-k+2,\alpha}(\overline{\Omega})$,$g_k(\cdot,y)\in C^{m-k+2,\alpha}(\overline{\Omega})$ and

$$\begin{aligned}&\|c_0\|_{C^{m-1,\alpha}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))}+\|f_k\|_{C^{m-k+2,\alpha}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))}+\|g_k\|_{C^{m-k+2,\alpha}(\overline{\Omega};C^{\alpha}(\partial T))}\\&\quad\le C\left(\|f\|_{C^{m,\alpha}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))}+\|g\|_{C^{m,\alpha}(\overline{\Omega};C^{1,\alpha}(\partial T))}+\|\varphi\|_{C^{m+2,\alpha}(\overline{\Omega})}\right).\end{aligned}$$

From this observation and Lemma 5.2.1-5.2.5, we obtain that there exist a function $v_k(y;M,z,x)$, $v_k(\cdot;M,z,x)\in C^{2,\alpha}(\mathbb{R}^n\setminus T)$ and a constant $\overline{L}_k(M,z,x)\in\mathbb{R}$, which solve (5.2.33). Therefore, in the same way that we found $\overline{u}$, there exists $\psi_{k-2}:\overline{\Omega}\to\mathbb{R}$ which solves (5.2.34). Moreover, From Proposition 5.2.6, $\overline{L}_k(O,\cdot,\cdot)=\overline{c}_k(\cdot)\cdot-\overline{f}_k(\cdot)\in C^{m-k+2,\alpha}(\mathbb{R}\times\overline{\Omega})$ and there holds

$$\begin{aligned}&\|\overline{c}_k\|_{C^{m-k+2,\alpha}(\overline{\Omega})}+\|\overline{f}\|_{C^{m-k+2,\alpha}(\overline{\Omega})}\\&\quad\le C\Big(\|c_0\|_{C^{m-1,\alpha}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))}+\|f_k\|_{C^{m-k+2,\alpha}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))}\\&\qquad+\|g_k\|_{C^{m-k+2,\alpha}(\overline{\Omega};C^{\alpha}(\partial T))}\Big).\end{aligned}$$

Hence by similar argument as in lemma 5.2.14, we can also observe that $\psi_{k-2}\in C^{m-k+4,\alpha}(\overline{\Omega})$ and

$$\begin{aligned}&\|\psi_{k-2}\|_{C^{m-k+4,\alpha}(\overline{\Omega})}\\&\quad\le C\left(\|f\|_{C^{m,\alpha}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))}+\|g\|_{C^{m,\alpha}(\overline{\Omega};C^{1,\alpha}(\partial T))}+\|\varphi\|_{C^{m+2,\alpha}(\overline{\Omega})}\right).\end{aligned}\tag{5.2.39}$$

On the other hand, if we set $\phi_k(x,y)=v_k(y;D^2\psi_{k-2},\psi_{k-2},x)$ then ϕ_k solves (5.2.35). Hence we apply Lemma 5.2.5 and Proposition 5.2.6 to obtain that $\phi_k(x,\cdot)\in C^{2,\alpha}(\mathbb{R}^n\setminus T)$ uniformly for all $x\in\overline{\Omega}$, $\phi_k(\cdot,y)\in C^{m-k+2,\alpha}(\overline{\Omega})$ uniformly for all $y\in\mathbb{R}^n\setminus T$ and

$$\begin{aligned}&\|\phi_k\|_{C^{m-k+2,\alpha}(\overline{\Omega};C^{2,\alpha}(\mathbb{R}^n\setminus T))}\\&\quad\le C\Big(\|c_0\psi_{k-2}\|_{C^{m-k+4,\alpha}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))}+\|f_k\|_{C^{m-k+2,\alpha}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))}\\&\qquad+\|g_k\|_{C^{m-k+2,\alpha}(\overline{\Omega};C^{\alpha}(\partial T))}\Big).\end{aligned}$$

Consequently, ϕ_k satisfies (5.2.37). Finally, choose $\psi_{m-1}\in C^{3,\alpha}(\overline{\Omega})$ and $\psi_m\in$

$C^{2,\alpha}(\overline{\Omega})$ arbitrary functions that satisfy (5.2.38). Then the proof now finishes by the induction principle. □

We are now in position to present the proof of our main lemma of this subsection : The construction of the higher order correctors.

Lemma 5.2.16. *There exist a family of y-periodic functions* $\{w_k : \overline{\Omega} \times \mathbb{R}^n \setminus T \to \mathbb{R}\}_{1\le k\le m}$ *defined by* (5.2.27)*, which verify the following conditions.*

(i) $w_k(x,\cdot) \in C^{2,\alpha}(\mathbb{R}^n \setminus T)$ *uniformly for all* $x \in \overline{\Omega}$*,*
$w_k(\cdot, y) \in C^{m-k+2,\alpha}(\overline{\Omega})$ *uniformly for all* $y \in \mathbb{R}^n \setminus T$ *and*

$$\begin{aligned}&\|w_k\|_{C^{m-k+2,\alpha}(\overline{\Omega};C^{2,\alpha}(\mathbb{R}^n\setminus T))}\\ &\qquad\le C\left(\|f\|_{C^{m,\alpha}(\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))} + \|g\|_{C^{m,\alpha}(\overline{\Omega};C^{1,\alpha}(\partial T))} + \|\varphi\|_{C^{m+2,\alpha}(\overline{\Omega})}\right)\end{aligned}$$

where C *depends on* n, m, α, λ, Λ, r, Ω, $\|a_{ij}\|_{C^{\alpha}(\mathbb{R}^n\setminus T)}$, $\|b_i\|_{C^{1,\alpha}(\partial T)}$ *and* $\|c\|_{C^{m,\alpha}(\mathbb{R}\times\overline{\Omega};C^{\alpha}(\mathbb{R}^n\setminus T))}$.

(ii) For each $3 \le k \le m$, w_k *solves*

$$\begin{cases} a_{ij}\left(D_{y_iy_j}w_k + D_{x_iy_j}w_{k-1} + D_{y_ix_j}w_{k-1}\right.\\ \qquad\qquad\qquad\left.+D_{x_ix_j}w_{k-2}\right) + \Psi_{k-2} = 0 & \text{in } \mathbb{R}^n \setminus T \times \Omega_\varepsilon,\\ b_i\left(D_{x_i}w_{k-1} + D_{y_i}w_k\right) = 0 & \text{on } \partial T \times \partial T_\varepsilon \end{cases}$$

where Ψ_k *be defined as in* (5.2.28).

Proof. The assertion (i) immediately follows from the definition of w_k and Lemma 5.2.15. Now we prove the assertion (ii). In view of (5.2.27), (5.2.29), (5.2.30), (5.2.31), (5.2.34) and (5.2.35) we obtain that

$$\begin{aligned}\mathcal{L}_k &= a_{ij}\left(D_{y_iy_j}w_k + D_{x_iy_j}w_{k-1} + D_{y_ix_j}w_{k-1} + D_{x_ix_j}w_{k-2}\right) + \Psi_{k-2}\\ &= \left(a_{ij}D_{y_iy_j}\phi_k + \frac{\partial c}{\partial z}(\overline{u}, x, y)\psi_{k-2}\right) - f_k + \overline{a}_{i_1i_2}D_{x_{i_1}x_{i_2}}\psi_{k-2}\\ &= \left(\overline{L}_k(O, \psi_{k-2}, x) + f_k(x,y)\right) - f_k(x,y) + \overline{a}_{i_1i_2}D_{x_{i_1}x_{i_2}}\psi_{k-2}\\ &= 0\end{aligned}$$

in $\mathbb{R}^n \setminus T \times \Omega_\varepsilon$ and

$$\begin{aligned}\mathcal{N}_k &= b_i\left(D_{x_i}w_{k-1} + D_{y_i}w_k\right)\\ &= b_i D_{y_i}\phi_k - \overline{a}_{i_1 i_2} D_{x_{i_1}x_{i_2}}\psi_{k-2} - g_k\\ &= \left(-\overline{L}_k(O,\psi_{k-2},x) + g_k(x,y)\right) - \overline{a}_{i_1 i_2} D_{x_{i_1}x_{i_2}}\psi_{k-2} - g_k(x,y)\\ &= 0\end{aligned}$$

on $\partial T \times \partial T_\varepsilon$. Hence we have desired result. □

5.3 Higher order convergence rate

In this section, we are going to prove the main theorem 5.1.2. Define the k-th order interior corrector w_k^ε for each $1 \leq k \leq m$ by

$$w_k^\varepsilon(x) := w_k\left(x, \frac{x}{\varepsilon}\right) \quad (x \in \overline{\Omega}_\varepsilon) \tag{5.3.1}$$

and define $\eta_m^\varepsilon : \overline{\Omega}_\varepsilon \to \mathbb{R}$ by

$$\eta_m^\varepsilon(x) = \overline{u}(x) + \varepsilon w_1^\varepsilon\left(x, \frac{x}{\varepsilon}\right) + \cdots + \varepsilon^m w_m^\varepsilon\left(x, \frac{x}{\varepsilon}\right).$$

Now we are going to construct the boundary corrector. Define $\theta_m^\varepsilon : \overline{\Omega}_\varepsilon \to \mathbb{R}$ by the solution of the following PDE,

$$\begin{cases} a_{ij}\left(\frac{x}{\varepsilon}\right) D_{ij}\theta_m^\varepsilon + c\left(\eta_m^\varepsilon + \theta_m^\varepsilon, x, \frac{x}{\varepsilon}\right) = c\left(\eta_m^\varepsilon, x, \frac{x}{\varepsilon}\right) & \text{in } \Omega_\varepsilon,\\ b_i\left(\frac{x}{\varepsilon}\right) D_i\theta_m^\varepsilon = 0 & \text{on } \partial T_\varepsilon,\\ \theta_m^\varepsilon = -\eta_m^\varepsilon + \varphi & \text{on } \partial\Omega. \end{cases} \tag{5.3.2}$$

Note from Lemma 5.2.11 and 5.2.16 that $\eta_m^\varepsilon \in C^{2,\alpha}(\overline{\Omega}_\varepsilon)$, so this equation belongs to the same class of (L_ε). Thus, the Comparison principle and Perron's method ensure the unique existence of a viscosity solution $\theta_m^\varepsilon \in C(\overline{\Omega}_\varepsilon)$ of (5.3.2), see [17].

proof of theorem 5.1.2. Fix $\varepsilon > 0$. Define η_m^ε and θ_m^ε as the comment above this lemma. Recall (5.2.28), the definition of Ψ_k. i.e.

$$\Psi_k(x, x/\varepsilon) = \sum_{i=1}^{k} \frac{1}{i!} \frac{\partial^i c}{\partial z^i}(\overline{u}, x, x/\varepsilon) \sum_{\substack{n_1+\cdots+n_i=k \\ n_1,\cdots,n_i\neq 0}} w^\varepsilon_{n_1} w^\varepsilon_{n_2} \cdots w^\varepsilon_{n_i}.$$

We omit the dependency $(x, x/\varepsilon)$ for simplicity. We first observe from the heuristic calculation (5.2.12) that

$$c(\eta^\varepsilon_m, x, x/\varepsilon) = c(\overline{u}) + \varepsilon\Psi_1 + \cdots + \varepsilon^{m-2}\Psi_{m-2} + \varepsilon_*^{m-1}\Psi_{m-1}$$

for some $\varepsilon_* \in [0, \varepsilon]$. By Lemma 5.2.16, $\{\Psi_k\}_{0\leq k\leq m-1}$ have uniform bounds independent of ε, namely,

$$\begin{aligned} &\|\Psi_k\|_{L^\infty(\Omega_\varepsilon\times\mathbb{R}^n\setminus T)} \\ &\quad\leq C\left(\|f\|_{C^{m,\alpha}(\overline{\Omega};C^\alpha(\mathbb{R}^n\setminus T))} + \|g\|_{C^{m,\alpha}(\overline{\Omega};C^{1,\alpha}(\partial T))} + \|\varphi\|_{C^{m+2,\alpha}(\overline{\Omega})}\right) \\ &\quad\leq C_1(f, g, \varphi), \end{aligned} \tag{5.3.3}$$

where $C_1(f, g, \varphi)$ depends on n, m, α, λ, Λ, r, Ω, $\|a_{ij}\|_{C^\alpha(\mathbb{R}^n\setminus T)}$, $\|b_i\|_{C^{1,\alpha}(\partial T)}$, and $\|c\|_{C^{m,\alpha}(\mathbb{R}\times\overline{\Omega};C^\alpha(\mathbb{R}^n\setminus T))}$. From these observations, Lemma 5.2.16 (2), and (5.3.2), it follows that

$$\begin{aligned} &a_{ij}\left(\frac{x}{\varepsilon}\right) D_{ij}(\eta^\varepsilon_m + \theta^\varepsilon_m) + c\left(\eta^\varepsilon_m + \theta^\varepsilon_m\right) \\ &= a_{ij}\left(\frac{x}{\varepsilon}\right) D_{ij}\eta^\varepsilon_m + c\left(\eta^\varepsilon_m\right) \\ &= \varepsilon^{-1} a_{ij}\left(\frac{x}{\varepsilon}\right) D_{y_iy_j} w^\varepsilon_1 + a_{ij}\left(\frac{x}{\varepsilon}\right)\left(D_{x_ix_j}\overline{u} + 2D_{x_iy_j} w^\varepsilon_1 + D_{y_iy_j} w^\varepsilon_2\right) + c\left(\overline{u}\right) \\ &\quad + \varepsilon^{k-2}\left\{\sum_{k=3}^{m} a_{ij}\left(\frac{x}{\varepsilon}\right)\left(D_{x_ix_j} w^\varepsilon_{k-2} + 2D_{x_iy_j} w^\varepsilon_{k-1} + D_{y_iy_j} w^\varepsilon_k\right) + \Psi_{k-2}\right\} \\ &\quad + \varepsilon^{m-1} a_{ij}\left(\frac{x}{\varepsilon}\right)\left(D_{x_ix_j} w^\varepsilon_{m-1} + \varepsilon D_{x_ix_j} w^\varepsilon_m + 2D_{x_iy_j} w^\varepsilon_m\right) + \varepsilon_*^{m-1}\Psi_{m-1} \\ &= \overline{L}(D^2\overline{u}, \overline{u}, x) + f \\ &\quad + \varepsilon^{m-1} a_{ij}\left(\frac{x}{\varepsilon}\right)\left(D_{x_ix_j} w^\varepsilon_{m-1} + \varepsilon D_{x_ix_j} w^\varepsilon_m + 2D_{x_iy_j} w^\varepsilon_m\right) + \varepsilon_*^{m-1}\Psi_{m-1} \\ &= f + \varepsilon^{m-1}\Phi_m \end{aligned}$$

in Ω_ε, where

$$\Phi_m\left(x,\frac{x}{\varepsilon}\right)=a_{ij}\left(\frac{x}{\varepsilon}\right)\left(D_{x_ix_j}w_{m-1}^\varepsilon+\varepsilon D_{x_ix_j}w_m^\varepsilon+2D_{x_iy_j}w_m^\varepsilon\right)\\ +\left(\frac{\varepsilon_*}{\varepsilon}\right)^{m-1}\Psi_{m-1}.$$

From Lemma 5.2.16, and (5.3.3) we can observe that $\Phi_m(x,\cdot)\in C^\alpha(\mathbb{R}^n\setminus T)$ and

$$\|\Phi_m\|_{L^\infty(\Omega_\varepsilon\times\mathbb{R}^n\setminus T)}\le C_2, \tag{5.3.4}$$

where C_2 depends on ε_* and $C_1(f,g,\varphi)$. On the other hand,

$$\begin{aligned} b_i\left(\frac{x}{\varepsilon}\right)D_i\left(\eta_m^\varepsilon+\theta_m^\varepsilon\right)&=b_i\left(\frac{x}{\varepsilon}\right)D_i\left(\overline{u}+\varepsilon w_1^\varepsilon++\varepsilon^2w_2^\varepsilon+\cdots+\varepsilon^m w_m^\varepsilon\right)\\ &=b_i\left(\frac{x}{\varepsilon}\right)\sum_{k=1}^m\varepsilon^{k-1}\left(D_{x_i}w_{k-1}+D_{y_i}w_k\right)\\ &=\varepsilon g \end{aligned}$$

on ∂T_ε, here we understand $w_0(x)=\overline{u}(x)$. Thus, $\eta_m^\varepsilon+\theta_m^\varepsilon$ solves the following equation:

$$\begin{cases} a_{ij}\left(\frac{x}{\varepsilon}\right)D_{ij}(\eta_m^\varepsilon+\theta_m^\varepsilon)+c\left(\eta_m^\varepsilon+\theta_m^\varepsilon,x,\frac{x}{\varepsilon}\right)\\ \qquad\qquad =f\left(x,\frac{x}{\varepsilon}\right)+\varepsilon^{m-1}\Phi_m\left(x,\frac{x}{\varepsilon}\right) & \text{in } \Omega_\varepsilon,\\ b_i\left(\frac{x}{\varepsilon}\right)D_i(\eta_m^\varepsilon+\theta_m^\varepsilon)=\varepsilon g\left(x,\frac{x}{\varepsilon}\right) & \text{on } \partial T_\varepsilon,\\ \eta_m^\varepsilon+\theta_m^\varepsilon=\varphi(x) & \text{on } \partial\Omega. \end{cases} \tag{5.3.5}$$

Consider the following problem: For each $x\in\Omega$,

$$\begin{cases} a_{ij}(y)\left(M_{ij}+M_{ik}D_{y_j}\chi_1^k(y)+D_{y_i}\chi_1^k(y)M_{kj}+D_{y_iy_j}v_\varepsilon\right)\\ \qquad\qquad -\varepsilon^2 v_\varepsilon=\Phi_m(x,y) & \text{in } \mathbb{R}^n\setminus T,\\ b_i(y)\left(M_{ik}\chi_1^k(y)+D_{y_i}v_\varepsilon\right)+\varepsilon^2 v_\varepsilon=0 & \text{on } \partial T. \end{cases}$$

Then by the same argument as Lemma 5.2.1-5.2.5, there exist the $v(\cdot;M,x)\in C^{2,\alpha}(\mathbb{R}^n\setminus T)$ and unique constant $\widetilde{L}(M,x)\in\mathbb{R}$ satisfying following equation and estimate:

$$\begin{cases} a_{ij}(y)\left(M_{ij}+M_{ik}D_{y_j}\chi_1^k(y)+D_{y_i}\chi_1^k(y)M_{kj}+D_{y_iy_j}v\right) \\ \qquad\qquad\qquad\qquad = \widetilde{L}(M,x)+\Phi_m(x,y) & \text{in } \mathbb{R}^n\setminus T, \\ b_i(y)\left(M_{ik}\chi_1^k(y)+D_{y_i}v\right) = -\widetilde{L}(M,x) & \text{on } \partial T, \end{cases}$$

and

$$\begin{aligned} osc_{\mathbb{R}^n\setminus T} v_\varepsilon + |\widetilde{L}(M,x)| &\le C_3\left(\|M\| + \|\Phi_m(x,\cdot)\|_{L^\infty(\mathbb{R}^n\setminus T)}\right) \\ &\le C_3\left(\|M\| + \|\Phi_m\|_{L^\infty(\Omega\times\mathbb{R}^n\setminus T)}\right) \\ &\le C_3(\|M\|+C_2) \end{aligned} \tag{5.3.6}$$

where $C_3 = C_3(n,\lambda,\Lambda,r)$. Here, we notice that $\widetilde{L}$ is an effective operator of (L_ε) when $c(z,x,y)$, $g(x,y)\equiv 0$, and $f(x,y)$ is replaced by $\Phi_m(x,y)$. That is, from Lemma 5.2.12 we can observe that there is a positive real number r_0 depending only on n, λ and Λ such that if the size of holes r is less than or equal to r_0, then $\widetilde{L}(M,x)$ is uniformly elliptic for each $x\in\overline{\Omega}$. In other words, there is a positive constant $\widetilde{\lambda}=\widetilde{\lambda}(r_0)$ satisfying

$$\widetilde{L}(M+N,x)\ge \widetilde{L}(M,x)+\widetilde{\lambda}\|N\|$$

for any $M\in\mathcal{S}^n$, positive matrix N, and $x\in\overline{\Omega}$.

Now we will construct barriers. Fix $x_0\in\Omega$ and choose $d>\operatorname{diam}(\Omega)$. Define

$$P(x)=K\left(d^2-\frac{|x|^2}{2}\right)$$

and $\xi(x)=DP(x)=-Kx$, where $K>0$ will be determined later. Set

$$\begin{cases} \chi\left(x,\frac{x}{\varepsilon}\right)=\xi(x)\chi_1\left(\frac{x}{\varepsilon}\right) \\ v_\varepsilon\left(\frac{x}{\varepsilon}\right)=v_\varepsilon\left(\frac{x}{\varepsilon};-KI_n,x_0\right) \end{cases}$$

where $\chi_1(y) = (\chi_1^1(y), \chi_1^2(y), \cdots, \chi_1^n(y))$. Now consider the functions $Q_\varepsilon^\pm : \overline{\Omega}_\varepsilon \to \mathbb{R}$ defined by

$$Q_\varepsilon^\pm(x) = \pm\varepsilon^{m-1}\left(P(x) + \varepsilon\chi\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2\widetilde{v}_\varepsilon\left(\frac{x}{\varepsilon}\right) + \varepsilon\|\xi\|_{L^\infty(\overline{\Omega})}\|\chi_1\|_{L^\infty(\mathbb{R}^n\setminus T)}\right)$$

where $\widetilde{v}_\varepsilon\left(\frac{x}{\varepsilon}\right) = v_\varepsilon\left(\frac{x}{\varepsilon}\right) - \min_{y\in\mathbb{R}^n\setminus T} v_\varepsilon(y)$. Then we can easily check that $Q_\varepsilon^+(x) \geq 0$ and $Q_\varepsilon^-(x) \leq 0$.

We will show that $u^\varepsilon + Q_\varepsilon^+(x)$[resp. $u^\varepsilon + Q_\varepsilon^-(x)$] is a viscosity super-solution[resp. viscosity sub-solution] of equation (5.3.5) if we choose K properly. First, let us check at the interior,

$$\begin{aligned}
&a_{ij}\left(\frac{x}{\varepsilon}\right) D_{x_ix_j}(u^\varepsilon(x) + Q_\varepsilon^+(x)) + c\left(u^\varepsilon + Q_\varepsilon^+, x, \frac{x}{\varepsilon}\right) - f\left(x, \frac{x}{\varepsilon}\right) \\
&\leq a_{ij}\left(\frac{x}{\varepsilon}\right) D_{x_ix_j}u^\varepsilon(x) + c\left(u^\varepsilon, x, \frac{x}{\varepsilon}\right) + a_{ij}\left(\frac{x}{\varepsilon}\right) D_{x_ix_j}Q_\varepsilon^+(x) - f\left(x, \frac{x}{\varepsilon}\right) \\
&= \varepsilon^{m-1}a_{ij}\left(\frac{x}{\varepsilon}\right)\Big\{(-KI_n)_{ij} + D_{y_j}\chi_1^l\left(\frac{x}{\varepsilon}\right) D_{x_i}\xi_l(x) + \varepsilon(-KI_n)_{lj}D_{x_i}\chi_1^l\left(\frac{x}{\varepsilon}\right) \\
&\quad + D_{y_iy_j}v_\varepsilon\left(\frac{x}{\varepsilon}\right)\Big\} \\
&= \varepsilon^{m-1}a_{ij}\left(\frac{x}{\varepsilon}\right)\Big((-KI_n)_{ij} + D_{y_j}\chi_1^l\left(\frac{x}{\varepsilon}\right)(-KI_n)_{li} + (-KI_n)_{lj}D_{y_i}\chi_1^l\left(\frac{x}{\varepsilon}\right) \\
&\quad + D_{y_iy_j}v_\varepsilon\left(\frac{x}{\varepsilon}\right)\Big) \\
&= \varepsilon^{m-1}\left(\Phi_m\left(x_0, \frac{x}{\varepsilon}\right) + \varepsilon^2 v_\varepsilon\left(\frac{x}{\varepsilon}; -KI_n, x_0\right)\right).
\end{aligned}$$

Then by the uniform ellipticity of $\widetilde{L}$, and (5.3.6) we get

$$\varepsilon^2 v_\varepsilon\left(\frac{x}{\varepsilon}; -KI_n, x_0\right) \leq \widetilde{L}(-KI_n, x_0) + o(\varepsilon) \leq \widetilde{L}(O, x_0) - \frac{\widetilde{\lambda}K}{2} \leq C_2C_3 - \frac{\widetilde{\lambda}K}{2}$$

if ε is small enough, where C_2 and C_3 are constants as in (5.3.4) and (5.3.6) respectively. It then follows from this estimate that if we put

$$K = 2\widetilde{\lambda}^{-1}C_2(C_3 + 2)$$

then

$$\begin{aligned}
&a_{ij}\left(\frac{x}{\varepsilon}\right) D_{x_ix_j}(u^\varepsilon(x)+Q_\varepsilon^+(x)) + c\left(u^\varepsilon + Q_\varepsilon^+, x, \frac{x}{\varepsilon}\right) - f\left(x, \frac{x}{\varepsilon}\right)\\
&\le \varepsilon^{m-1}\left(\Phi_m\left(x_0, \frac{x}{\varepsilon}\right) + \varepsilon^2 v_\varepsilon\left(\frac{x}{\varepsilon}; -KI_n, x_0\right)\right)\\
&\le \varepsilon^{m-1}\left(\Phi_m\left(x_0, \frac{x}{\varepsilon}\right) - 2C_2\right)\\
&\le \varepsilon^{m-1}\Phi_m^\varepsilon\left(x, \frac{x}{\varepsilon}\right)
\end{aligned}$$

for every $x \in \Omega_\varepsilon$. Secondly, let us check the boundary condition,

$$\begin{aligned}
&b_i\left(\frac{x}{\varepsilon}\right) D_{x_i}(u^\varepsilon(x)+Q_\varepsilon^+(x)) - \varepsilon g\left(x, \frac{x}{\varepsilon}\right)\\
&= \varepsilon^{m-1} b_i\left(\frac{x}{\varepsilon}\right) D_{x_i}\left(P(x) + \varepsilon\chi\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2\widetilde{v}_\varepsilon\left(\frac{x}{\varepsilon}\right)\right)\\
&= \varepsilon^{m-1} b_i\left(\frac{x}{\varepsilon}\right)\left\{\xi_i(x) + D_{y_i}\chi_1^l\left(\frac{x}{\varepsilon}\right)\xi_l(x)\right.\\
&\quad \left. +\varepsilon\left((-KI_n)_{il}\chi_1^l\left(\frac{x}{\varepsilon}\right) + D_{y_i}v_\varepsilon\left(\frac{x}{\varepsilon}\right)\right)\right\}\\
&= -\varepsilon^m\left(\varepsilon^2 v_\varepsilon\left(\frac{x}{\varepsilon}; -KI_n, x_0\right)\right)\\
&\ge 2C_2\varepsilon^m\\
&\ge 0.
\end{aligned}$$

Consequently, $Q_\varepsilon^+(x)$ is the super-solution of (5.3.5). In the same manner, one can verify that $Q_\varepsilon^-(x)$ is the sub-solution of (5.3.5). Thus, the comparison principle yields $u^\varepsilon + Q_\varepsilon^- \le \eta_m^\varepsilon + \theta_m^\varepsilon \le u^\varepsilon + Q_\varepsilon^+$ in $\overline{\Omega}_\varepsilon$, in particular,

$$\begin{aligned}
&\|u^\varepsilon - \eta_m^\varepsilon - \theta_m^\varepsilon\|_{L^\infty(\Omega_\varepsilon)}\\
&\quad \le \|Q_\varepsilon^\pm\|_{L^\infty(\Omega_\varepsilon)}\\
&\quad \le \varepsilon^{m-1}\left(\|P\|_{L^\infty(\Omega_\varepsilon)} + 2C\varepsilon\|\xi\|_{L^\infty(\Omega_\varepsilon)} + \varepsilon^2 osc_{\mathbb{R}^n\setminus T} v_\varepsilon(\cdot; -KI_n, x_0)\right)\\
&\quad \le \varepsilon^{m-1}\left(CK + 2CK\varepsilon + \varepsilon^2 osc_{\mathbb{R}^n\setminus T} v_\varepsilon(\cdot; -KI_n, x_0)\right)\\
&\quad \le \varepsilon^{m-1}\left(CK + 2CK\varepsilon + \varepsilon^2 C_3(K + C_2)\right)\\
&\quad \le C\varepsilon^{m-1}.
\end{aligned}$$

□

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국문초록

프랙탈 영역 위에서의 해석학은 해석적 접근과 확률론적 접근을 통해 다양하게 연구되고 있다. 본 학위논문에서는 프랙탈 영역에서 2차항을 포함하는 비선형 타원 방정식를 구성하고, 해석적 논증을 이용하여 해의 정칙성을 구하고자 한다. 프랙탈 영역에서는 기존의 편미분 이론을 사용할 수 없기 때문에, 우리의 접근 방식은 그래프 근사 논증을 이용하여 디리클레 형식을 구성하는 것에 기반을 두고 있다. 가장 중점적인 개념은 프랙탈 영역의 특수한 기하학적 특성을 사용하여 적절한 차단 함수와 가중치 부등식을 찾는 것이다.

본 학위논문의 또 다른 주제는 완전 비선형 포물형 방정식에 대한 균질화 이론이다. 특히, 우리는 진동 변수들의 척도가 기존과 다른 경우에 대해서 다룬다. 흥미로운 점은 시공간 빠른 변수의 척도가 일치하지 않기 때문에 균질화가 시간과 공간에 대해 개별적으로 발생한다는 점이다. 또한 이 현상은 기존과 다른 수렴속도를 야기한다.

주요어휘: 프랙탈, 시어핀스키 가스킷, Harnack 부등식, 균질화, 수렴속도
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감사의 글

졸업을 앞두고 대학원 생활을 뒤돌아보니 고마운 분들이 정말 많습니다. 먼저 박사과정동안 저를 이끌어주신 이기암 교수님께 깊은 감사의 말씀을 드립니다. 항상 연구에 열정적이신 교수님의 모습을 본받아서 저 역시 연구에 매진할 수 있었고, 부족한 저를 위해 아낌없는 조언과 격려를 해주셔서 학위 과정을 무사히 마칠 수 있었습니다. 교수님을 본받아 성실하고 열정적인 연구자가 되도록 끊임없이 노력하겠습니다. 그리고 바쁘신 와중에 저의 학위 논문을 심사하러 와주신 변순식 교수님, 국웅 교수님, 김성훈 교수님, 김수정 교수님께 감사드립니다. 교수님들이 말씀해주신 조언 덕분에 앞으로의 연구 방향에 대해서도 깊이 생각해 볼 수 있었습니다.

박사과정동안 매주 초대해주셔서 함께 연구할 기회를 주신 국가수리과학 연구소의 유민하 박사님께 감사의 말씀을 전합니다. 연구실 선배로서 많은 조언을 해주시고 연구 방향을 잡도록 도와주셔서 저에게 많은 동기부여가 되었고 덕분에 지금까지 성장할 수 있었다고 생각합니다.

지금까지 함께한 동료와 선후배님들 역시 감사한 분들이 많습니다. 특히 연구실 동료인 효석이형, 상필이형, 형성이형, 진완이형, 성한, 민현, 태훈, 탁원, 종명, 세찬, 성은이 덕분에 대학원 생활을 즐겁게 보낼 수 있었습니다. 좋은 동료들과 함께 했기에 지루할수도 있는 연구실 생활이 항상 활기차고 즐거웠습니다. 박사과정을 함께 시작하고 공부하며 많은 도움을 준 동기들에게도 고마운 마음을 전합니다. 특히 종봉, 상훈, 그리고 자현이는 제가 힘들 때 옆에서 많은 힘이 되어주었습니다. 그 밖에도 많은 시간을 함께 보낸 경민, 규호, 동희, 민정누나, 병찬, 병학, 승민, 응범, 지승, 진제, 찬호에게 고맙다는 말을 전하고 싶습니다.

마지막으로 항상 저를 사랑하시는 부모님과 형에게 깊은 감사의 말을 전합니다. 가족들은 제가 학업을 이어갈 수 있는 가장 큰 원동력이자 동기부여였습니다. 가족이 함께 했기에 힘든 시기도 웃으면서 보낼 수 있었고, 또한 행복했고 소중했던 시간을 함께 해주었습니다. 진심으로 감사드리고 사랑합니다.

그리고 이 글에서 언급드리지 못했지만 저에게 도움을 주셨던 많은 분들께도 진심으로 감사의 인사를 전해드리며 이 글을 마칩니다. 다시 한번 감사드립니다.