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The Shapley value in cost allocation problems with network structure

네트워크 구조가 있는 비용 배분 문제에서의
샤플리 밸류에 관한 연구

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이 논문을 경제학박사 학위논문으로 제출함

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Abstract

This study consists of three chapters. Each chapter addresses independent issues. However they are connected in that they are analyzing economic phenomena using a network structure and they investigate the distribution of benefits or costs arising from cooperation using cooperative game theory. The first chapter investigate positional queueing problem which is a generalized problem of the classical queueing problem. In this chapter, we obtain generalized versions of the minimal transfer rule and of the maximal transfer rule. We also investigate properties of each rules and axiomatically characterized them. The second chapter investigate the minimum cost spanning tree problems with multiple sources. We investigate the properties and axiomatic characterization of the Kar rule for the minimum cost spanning tree problems with multiple sources. The final chapter investigate the profit allocation in the Korean automotive industry using the buyer-supplier network among the vehicle manufacturers and its first-tier vendors from the perspective of cooperative game theory. Some models are constructed and the Shapley values of each models are calculated. We compare them with real profit allocation of the Korean automotive industry.

Keywords: Shapley value, network, cooperative game theory, queueing problems, minimum cost spanning tree problems, Korean automotive industry.

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Introduction

Social networks are of great interest in economics. The network can be interpreted and utilized in various ways through its properties, that is, the presence or absence of a connection, the strength of the connection, and the direction of the connection. From an economic perspective, various meanings such as the possibility of cooperation, the strength of closeness, and structural relationship between economic agents can be modeled through the network, and the characteristics of the network contribute a lot to the analysis and understanding of economic phenomena.

Cooperative game theory is interested in maximizing the benefits of cooperation among participants and allocating the benefits. As networks may model cooperation among participants, it is naturally linked to cooperative game theory.

This study consists of three chapters. Each chapter addresses independent issues. However they are connected in that they are analyzing economic phenomena using a network structure and they investigate the distribution of benefits or costs arising from cooperation using cooperative game theory. The first chapter investigate the allocation problem of cost incurred when agents are forming a line to be served from a facility. The second chapter investigate the allocation problem of cost incurred when agents are forming a tree structure to be served from some facilities. In these two chapters, the Shapley values were investigated using an axiomatic approach. The third chapter investigate the profit allocation in the Korean automotive industry using the buyer-supplier network among the vehicle manufacturers and its first-tier vendors.

The first chapter investigate positional queueing problem which is a generalized problem of the classical queueing problem. A group of agents are waiting to be served in a facility. The facility can serve only one agent at a time and agents differ in their cost types. We are interested in finding the order in which to serve agents and the corresponding monetary transfers for the agents. In this chapter, we relax the assumption of classical queueing problem that the waiting cost of an agent is linear function of her waiting order. Instead, we assume that the cost function is non-decreasing and super-modular. We show that our generalized problem, the positional queueing problem, can be regarded as a generalization of the classical queueing problem, queueing problem with multiple parallel servers as well as the position allocation problem. By applying the Shapley value to the problem, we obtain generalized versions of the minimal transfer rule and of the maximal transfer rule. We also investigate properties of each rules and axiomatically characterized them.

The second chapter investigate the minimum cost spanning tree problems with multiple sources. There are a number of sources that provide services. A group of agents want to connect to these sources. An agent does not care if her connection to the source is direct or indirect, however she needs to connect to all of the sources. A connection entails a cost. We are interested in finding the efficient network and how to allocate the construction cost to each agents. The classical minimum cost spanning tree problem deals with problems with one source, and the Kar rule (the Shapley value) as well as the folk rule is investigated. As for the minimum cost spanning tree problems with multiple sources, the Kar rule is not yet addressed. Objective of this chapter is to investigate the properties and provide an axiomatic characterization of the Kar rule for the minimum cost spanning tree problems with multiple sources.

The last chapter investigate the Korean automotive industry from the perspective of cooperative game theory. An industry is a sector that produces a closely related goods or services within an economy. Individual firms produce their own product such as materials, intermediates and final products, and sell them to other firms (mostly) within the same industry or consumers. Firms in an specific industry are connected with the buyer-supplier relationships. In this point of view, an industry can be seen as a system consist of many firms working together to produce the final goods and distribute the value of the products among them. Therefore, we have possibility to investigate an industry with game theoretic approach. The Korean automotive industry is one of the biggest industries in Korea. Many researches have conducted on the Korean automotive industry, in particular, on the power imbalance in the industry, however it is not easy to find research conducted using game theoretic approach. We build models with network which reflect the relationships among vehicle manufacturers and first-tier vendors and we estimate the Shapley value of the game. We compare them with real profit allocation of the Korean automotive industry.

Chapter 1

The Shapley Value in Positional Queueing Problems and axiomatic characterizations

1.1 Introduction

A group of agents are waiting to be served in a facility. Only one agent can be served at a time, and a queue has to be organized to serve all agents. Agents differ in their cost types and her waiting cost is determined by her position and her cost type. Each agent's utility is assumed to be equal to the amount of her monetary transfer minus her waiting costs.

We are interested in finding the order of the queue and the monetary transfer they should receive. An *allocation* consist of each agents' position in the queue and the monetary transfer to her. An allocation rule, simply a *rule*, associates with each queueing problem and an allocation.

In the standard formulation(Maniquet, 2003; Chun, 2006; Chun and Hokari, 2007; van den Brink and Chun, 2012), an agent's waiting cost is a linear function of the order. The linear cost assumption of the classical queueing problem can be relaxed. For instance, Chun and Heo(2008) extended the model by assuming the facility has two or more parallel servers. They assumed that some agents are served at the same time.

In this paper, we relax the assumption further to the "positional queueing problems" by allowing the cost of each agent to depend on the cost type and the position assigned. The standard queueing problem is a special case of the positional queueing problem. In addition, the positional queueing problem can be used as a tool for analyzing the queueing problem with multiple parallel servers (Chun and Heo 2008) and position allocation problem studied in Essen and Wooders (2018).

This chapter is organized as follows. Section 2 presents the positional queueing problem and two rules for the problem; the minimal and the maximal transfer rules. Section 3 shows that the minimal transfer rule is the Shapley value of the optimistic positional queueing problem. Section 4 show that the maximal transfer rule is the Shapley value of the pessimistic positional queueing problem. Section 5 provides axiomatic characterizations of the two rules. Concluding remarks follow in Section 6.

1.2 The Positional Queueing Problem

Let $N = \{1, \dots, n\}$ be a set of agents, who want to be served in a facility which serves one agent at a time. Each agent $i \in N$ is characterized by her (cost) type, $\theta_i \in \mathbb{R}_+$. Let $\theta = (\theta_i)_{i \in N} \in \mathbb{R}^N_+$ be the profile of types. For each $S \subseteq N$, let $\Sigma(S)$ be the set of all possible orderings of S.

Given a set of agents N, each agent $i \in N$ is assigned to a position $\sigma_i \in \{1, \dots, n\}$ in a queue and a positive or negative monetary transfer t_i. If the monetary transfer of agent i, t_i, is positive, she receives a compensation from others. If negative, she pays the amount of money as a compensation to others. We are interested in finding the order in which to serve agents and the corresponding monetary transfers for the agents.

A *(positional) cost function* $C : \mathbb{R}_+ \times \mathbb{N} \to \mathbb{R}$ is a function that represent the cost incurred by cost type and position. If an agent i of type θ_i is positioned at σ_i, then the positional cost of agent i is $C(\theta_i, \sigma_i)$.[1]

For each θ'_i, $\theta_i \in \mathbb{R}_+$ and each σ'_i, $\sigma_i \in \mathbb{N}$, a cost function is non-decreasing in θ if $\theta'_i > \theta_i$ implies $C(\theta'_i, \sigma_i) \geq C(\theta_i, \sigma_i)$ and non-decreasing in σ if $\sigma'_i > \sigma_i$ implies $C(\theta_i, \sigma'_i) \geq C(\theta_i, \sigma_i)$. For each θ_i, $\theta_j \in \mathbb{R}_+$ and each σ_i, $\sigma_j \in \mathbb{N}$, a cost function C is *supermodular* with respect to the type and the position if $\theta_i \geq \theta_j$ and $\sigma_i \leq \sigma_j$ imply that $C(\theta_i, \sigma_i) + C(\theta_j, \sigma_j) \leq C(\theta_i, \sigma_j) + C(\theta_j, \sigma_i)$ and *strictly supermodular* with respect to the type and the position if $\theta_i > \theta_j$ and $\sigma_i < \sigma_j$ imply that $C(\theta_i, \sigma_i) + C(\theta_j, \sigma_j) < C(\theta_i, \sigma_j) + C(\theta_j, \sigma_i)$.

Let C be a cost function which is supermodular with respect to the type and the position and non-decreasing with respect to each argument. Let $\mathcal{C}$ be such family of all cost functions. All agents have an identical cost function, however they are differ in types. The utility of each agent is equal to the negative of her positional cost plus her monetary transfer, that is, $u_i(\sigma_i, t_i) = -C(\theta_i, \sigma_i) + t_i$.

A *positional queueing problem,* or a *problem,* is a tuple (θ, C), where $\theta \in \mathbb{R}^N_+$ is the profile of types and $C \in \mathcal{C}$ is the cost function. Let $\mathcal{Q}^N$ be the class of all problems for N. An *allocation* is a pair (σ, t) where for each $i \in N$, σ_i denotes agent i's position in the queue and t_i the monetary transfer assigned to her. An allocation is *feasible* if no two agents are assigned the same position and the sum of all transfers is not positive. Let $Z(\theta, C)$ be the set of all feasible allocations of (θ, C).

An allocation is *queue-efficient* if it minimizes sum of the waiting costs of all agents among all feasible allocations. With a supermodular cost function, if agents are served in the non-increasing order of their types, then it is queue-efficient. If the cost function is strictly supermodular and all agents are of different types, then the efficient queue is unique. However, the efficient queue may not be unique in other circumstances. For each problem (θ, C), let $E(\theta, C)$ be the set of all efficient queues for the problem and $\tilde{E}(\theta, C) \subseteq E(\theta, C)$ be the subset of efficient queues which serves agents in the non-increasing order of their types. Similarly, for each $S \subseteq N$, let $E(\theta_S, C)$ be the set of efficient queues for S and $\tilde{E}(\theta_S, C) \subseteq E(\theta_S, C)$ be the subset of efficient queues for S which serves agents in the non-increasing order of their types. An allocation is *budget-balanced* if $\sum_{i \in N} t_i = 0$.

[1] The positional cost function of the standard queueing problem is given by: for each $i \in N$, each θ_i, and each σ_i, $C(\theta_i, \sigma_i) = (\sigma_i - 1)\theta_i$.

An allocation rule, or a *rule,* is a mapping φ which associates to each problem a non-empty set of feasible allocations. An allocation rule is *efficient* if it is queue-efficient and budget-balanced.

For each $\sigma \in \Sigma(N)$ and each $i \in N$, let $P_i(\sigma)$ be the set of agents preceding agent i, that is, $P_i(\sigma) = \{j \in N|\ \sigma_j < \sigma_i\}$ and $F_i(\sigma)$ be the set of agents following agent i, that is, $F_i(\sigma) = \{j \in N|\ \sigma_j > \sigma_i\}$.

For each $\theta_0 \in \mathbb{R}_+$, let $\hat{P}(\theta_0) = \{j \in N|\theta_j > \theta_0\}$ be the set of agents with types larger than θ_0, $\hat{E}(\theta_0) = \{j \in N|\theta_j = \theta_0\}$ the set of agents with the type θ_0, and $\hat{F}(\theta_0) = \{j \in N|\theta_j < \theta_0\}$ the set of agents with types smaller than θ_0. Let $\hat{p}(\theta_0)$, $\hat{e}(\theta_0)$, and $\hat{f}(\theta_0)$ be the cardinality of each set. Let $\Theta(\hat{P}, i)$ be the set of distinct (cost) types in $\hat{P}(\theta_i)$ and $\Theta(\hat{F}, i)$ be the set of distinct types of agents in $\hat{F}(\theta_i)$ with a generic element denoted by θ_0. When there is no danger of confusion, we use for $i = 0, \ldots, n$, $\hat{p}_i$, $\hat{e}_i$, and $\hat{f}_i$ instead of $\hat{p}(\theta_i)$, $\hat{e}(\theta_i)$, $\hat{f}(\theta_i)$, respectively.

For the minimal transfer rule, after choosing an efficient queue in $E(\theta, C)$, the transfer to agent i is calculated as if all agents are positioned in the non-increasing order of their waiting costs so that agents with type θ_0 are assigned with positions from $\hat{p}_0+1$ to $\hat{p}_0+\hat{e}_0$; from each agent with the type larger than or equal to θ_i, agent i receives the sum of the cost difference between i's position and each of these agent's position divided by their total number (this number is equal to $\hat{p}_i + \hat{e}_i$) with the cost being evaluated by θ_i; for each agent with type $\theta_0 \in \Theta(\hat{F}, i)$ assigned with position k, we calculate the sum of cost difference between the assigned position and each position starting from 1 to the assigned position divided by the number of these positions (this number is equal to k) with the cost being evaluated by θ_0 and then, each precedent is equally responsible for this sum so that the sum is divided equally among the precedents (this number is equal to $k - 1$); finally, the transfer is determined by adding the amount paid to agent i and the amount agent i is paying to each agent with a smaller type. Note that for the minimal transfer rule, the cost of a smaller type is used to evaluate the cost difference. Moreover, if the efficient queue belongs to $\tilde{E}(\theta, C)$, then the transfer to agent i can be explained in terms of her precedents and followers similar to the standard queueing problem.

Minimal transfer rule, φ^M**:** For each $(\theta, C) \in \mathcal{Q}^N$,

$$\varphi^M(\theta, C) = \big\{(\sigma^M, t^M) \in Z(\theta, C)|\ \ \sigma^M \in E(\theta, C) \quad \text{and}$$

$$\forall i \in N,\ \ t_i^M = \sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i, \sigma_i^M) - C(\theta_i, k)}{\hat{p}_i + \hat{e}_i} - \sum_{\theta_0 \in \Theta(\hat{F}, i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_0, k) - C(\theta_0, \ell)}{k} \big\}.$$

Moreover, if $\sigma^M \in \tilde{E}(\theta, C)$, then the transfer is simplified as: for each $i \in N$,

$$t_i^M = \sum_{p \in P_i(\sigma^M)} \frac{C(\theta_i, \sigma_i^M) - C(\theta_i, \sigma_p^M)}{\sigma_i^M} - \sum_{f \in F_i(\sigma^M)} \frac{1}{\sigma_f^M - 1} \sum_{\ell \in P_f(\sigma^M)} \frac{C(\theta_f, \sigma_f^M) - C(\theta_f, \sigma_\ell^M)}{\sigma_f^M}.$$

For the maximal transfer rule, after choosing an efficient queue in $E(\theta, C)$, once again, the transfer to agent i is calculated as if all agents are positioned in the non-increasing order of their waiting costs so that agents with type θ_0 are assigned with positions from

$\hat{p}_0 + 1$ to $\hat{p}_0 + \hat{e}_0$; for each agent with type $\theta_0 \in \Theta(\hat{P}, i)$ assigned with position k, we calculate the sum of the cost difference between each position starting from the assigned position to the last position and the assigned position divided by the total number of these positions (this number is equal to $n - k + 1$) with the cost being evaluated by θ_0, and then, agent i receives this amount shared equally among all agents position from $k + 1$ to n (this number is equal to $n - k$); to each agent with the type smaller than or equal to θ_i, agent i pays the sum of the cost difference between each of these agent's position and i's position divided by their total number including herself (this number is equal to $\hat{e}_i + \hat{f}_i$) with the cost being evaluated by θ_i; finally, the transfer is determined by adding the amount paid to agent i by each agent with a larger type and the amount agent i is paying to each agent with a smaller or an equal type. Note that for the maximal transfer rule, differently from the minimal transfer rule, the cost of a larger type is used to evaluate the cost difference. Moreover, if the efficient queue belongs to $\tilde{E}(\theta, C)$, then the transfer to agent i can be explained in terms of her precedents and followers similar to the standard queueing problem.

Maximal transfer rule, φ^X**:** For each $(\theta, C) \in \mathcal{Q}^N$,

$$\varphi^X(\theta, C) = \Big\{(\sigma^X, t^X) \in Z(\theta, C) \big|\ \sigma^X \in E(\theta, C) \quad \text{and}$$

$$\forall i \in N,\ \ t_i^X = \sum_{\theta_0 \in \Theta(\hat{P}, i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{n-k} \sum_{\ell=k}^{n} \frac{C(\theta_0, \ell) - C(\theta_0, k)}{n-k+1} - \sum_{k=\hat{p}_i+1}^{n} \frac{C(\theta_i, k) - C(\theta_i, \sigma_i^X)}{\hat{e}_i + \hat{f}_i} \Big\}.$$

Moreover, if $\sigma^X \in \tilde{E}(\theta, C)$, then the transfer is simplified as: for each $i \in N$,

$$t_i^X = \sum_{p \in P_i(\sigma^X)} \frac{1}{n - \sigma_p^X} \sum_{\ell \in F_p(\sigma^X)} \frac{C(\theta_p, \sigma_\ell^X) - C(\theta_p, \sigma_p^X)}{n - \sigma_p^X + 1} - \sum_{f \in F_i(\sigma^X)} \frac{C(\theta_i, \sigma_f^X) - C(\theta_i, \sigma_i^X)}{n - \sigma_i^X + 1}.$$

To simplify notation, when there is no danger of confusion, we do not attach the superscripts M and X to σ and t.

Now we describe *cooperative games with transferable utility*, or simply *games*. Let $N = \{1, \ldots, n\}$ be the set of agents and $S \subseteq N$ be a coalition. A game is a real-valued function v defined on $S \subseteq N$ where $v(S)$ is the worth of coalition S. Let Γ^N be the set of all games with the set of players N. A *value* is a function ϕ which associates with each $v \in \Gamma^N$ a vector $\phi(v) = (\phi_i(v))_{i \in N}$, which represents the payoff to each player in game v.

The Shapley value (Shapley 1953) is the best-known value for games. It assigns to each member her own marginal contributions regarding all possible orderings which is calculated as below:

Shapley value: For each $v \in \Gamma^N$ and each $i \in N$,

$$\phi_i^{SV}(v) = \sum_{S \subseteq N, i \in S} \frac{(|S|-1)!|N \setminus S|!}{|N|!} \Big\{v(S) - v(S \setminus \{i\})\Big\}. \tag{1.1}$$

1.3 An optimistic approach and the minimal transfer rule

We solve the positional queueing problem using the Shapley value. We need to calculate the worth of each coalitions corresponding to a positional queueing problems. We first introduce the optimistic positional queueing game v_o introduced by Maniquet (2003) for the standard queueing problem. In this approach, the worth of each coalition is calculated under the optimistic assumption that the coalitional members are served before the non-coalitional members. The worth of a coalition S is defined as the negative of the minimum cost incurred by its members when the coalitional members are assigned to the first $|S|$ positions. For each $S \subseteq N$ and each $\tilde{\sigma}(S) \in E(\theta_S, C)$,

$$v_o(S) = \max_{\sigma \in \Sigma(S)} \big[-\sum_{i \in S} C(\theta_i, \sigma_i) \big] = -\sum_{i \in S} C(\theta_i, \tilde{\sigma}_i(S)).$$

Our first theorem shows that the utility of each agents assigned by the Shapley value under optimistic assumption can be reached using the minimal transfer rule.

Theorem 1.1. (Minimal transfer rule) Let $(\theta, C) \in \mathcal{Q}^N$, and (σ^M, t^M) be a feasible allocation such that agents' utilities are equal to the payoff vector obtained by applying the Shapley value to the optimistic positional queueing game v_o. Then, $\sigma^M \in E(\theta, C)$ and for each $i \in N$,

$$t_i^M = \sum_{k=1}^{\hat{p}_i + \hat{e}_i} \frac{C(\theta_i, \sigma_i^M) - C(\theta_i, k)}{\hat{p}_i + \hat{e}_i} - \sum_{\theta_0 \in \Theta(\hat{F}, i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_0, k) - C(\theta_0, \ell)}{k}. \quad (1.2)$$

Proof. The proof is given in Appendix A. □

If an efficient queue belongs to $\tilde{E}(\theta, C)$, then the transfer in Theorem 1.1 can be simplified similarly to the transfer of the minimal transfer rule in the standard queueing problem. Note that if C is strictly supermodular, then $E(\theta, C) = \tilde{E}(\theta, C)$.

Corollary 1.1. Let $(\theta, C) \in \mathcal{Q}^N$ and (σ^M, t^M) be a feasible allocation such that agents' utilities are equal to the payoff vector obtained by applying the Shapley value to v_o. If $\sigma^M \in \tilde{E}(\theta, C)$, then for each $i \in N$,

$$t_i^M = \sum_{p \in P_i(\sigma^M)} \frac{C(\theta_i, \sigma_i^M) - C(\theta_i, \sigma_p^M)}{\sigma_i^M} - \sum_{f \in F_i(\sigma^M)} \frac{1}{\sigma_f^M - 1} \sum_{\ell \in P_f(\sigma^M)} \frac{C(\theta_f, \sigma_f^M) - C(\theta_f, \sigma_\ell^M)}{\sigma_f^M}. \quad (1.3)$$

Proof. The proof is given in Appendix A. □

Example 1.1. (The minimal transfer rule applied to a problem with 3 agents) Let $N = \{1, 2, 3\}$ and θ be such that $\theta_1 > \theta_2 > \theta_3$. Then, the minimal transfer rule is given by setting for each $i \in N$, $\sigma_i^M = i$ and

$$t_1^M = -\frac{1}{1 \times 2}[C(\theta_2, 2) - C(\theta_2, 1)] - \frac{1}{2 \times 3}[C(\theta_3, 3) - C(\theta_3, 1)] - \frac{1}{2 \times 3}[C(\theta_3, 3) - C(\theta_3, 2)],$$
$$t_2^M = \frac{1}{2}[C(\theta_2, 2) - C(\theta_2, 1)] - \frac{1}{2 \times 3}[C(\theta_3, 3) - C(\theta_3, 1)] - \frac{1}{2 \times 3}[C(\theta_3, 3) - C(\theta_3, 2)],$$
$$t_3^M = \frac{1}{3}[C(\theta_3, 3) - C(\theta_3, 1)] + \frac{1}{3}[C(\theta_3, 3) - C(\theta_3, 2)].$$

Our theorem generalizes the results in Maniquet (2003) for the standard queueing problem, Chun and Heo (2008) for the queueing problem with ℓ-servers, and Essen and Wooders (2018) for the position allocation problem.

1.3.1 The minimal transfer rule for the standard queueing problem

In the standard queueing problem (Maniquet 2003), if agent i is served at the σ_ith position, then her waiting cost is $(\sigma_i - 1)\theta_i$. By substituting the cost function $C(\theta_i, \sigma_i) = (\sigma_i - 1)\theta_i$ into equation (1.3), we have:

$$\begin{aligned} t_i^M &= \sum_{p \in P_i(\sigma)} \frac{C(\theta_i, \sigma_i) - C(\theta_i, \sigma_p)}{\sigma_i} - \sum_{f \in F_i(\sigma)} \frac{1}{\sigma_f - 1} \sum_{k \in P_f(\sigma)} \frac{C(\theta_f, \sigma_f) - C(\theta_f, \sigma_k)}{\sigma_f} \\ &= \sum_{p \in P_i(\sigma)} \frac{(\sigma_i - \sigma_p)\theta_i}{\sigma_i} - \sum_{f \in F_i(\sigma)} \frac{1}{\sigma_f - 1} \sum_{k \in P_f(\sigma)} \frac{(\sigma_f - \sigma_k)\theta_f}{\sigma_f} \\ &= \{(\sigma_i - 1) - \frac{\sigma_i(\sigma_i - 1)}{2\sigma_i}\}\theta_i - \sum_{f \in F_i(\sigma)} \frac{1}{\sigma_f - 1}\{(\sigma_f - 1) - \frac{\sigma_f(\sigma_f - 1)}{2\sigma_f}\}\theta_f \\ &= (\sigma_i - 1)\frac{\theta_i}{2} - \sum_{f \in F_i(\sigma)} \frac{\theta_f}{2}, \end{aligned}$$

which is the transfer for the minimal transfer rule of the standard queueing problem.

1.3.2 The minimal transfer rule for queueing problems with multiple parallel servers

Chun and Heo (2008) introduced the queueing problem with multiple parallel servers. Now we explain that how the queueing problem with multiple parallel servers can be solved by our result. Assume that the facility has ℓ parallel servers, so that ℓ agents can simultaneously be served. Let σ be a queue for the standard queueing problem (with 1-server), that is, $\sigma \in \Sigma(N)$. For each $i \in N$, let $g_i = \lceil \frac{\sigma_i}{\ell} \rceil$ be the smallest integer larger than or equal to $\frac{\sigma_i}{\ell}$. If agent i is served at the g_ith position, then her utility is equal to $u(g_i, t_i; \theta_i) = -(g_i - 1)\theta_i + t_i$.

For each $S \subseteq N$, its worth under the optimistic assumption is defined as:

$$v_o(S) = -\sum_{i \in S} (g_i^S - 1)\theta_i,$$

where g_i^S is the service position of agent i in S, that is, for $\sigma(S) \in \tilde{E}(\theta_S, C)$, $g_i^S = \lceil \frac{\sigma_i(S)}{\ell} \rceil$.

The minimal transfer rule chooses an efficient queue and then, determines transfers as follows: for each $i \in N$,

$$t_i^M = \frac{\sum_{g_j^M < g_i^M} g_j^M \cdot \ell}{\sigma_i} \cdot \theta_i - \sum_{k \in F_i(\sigma)} \big(\frac{1}{\sigma_k - 1} \cdot \frac{\sum_{g_j^M < g_k^M} g_j^M \cdot \ell}{\sigma_k} \cdot \theta_k\big),$$

and the Shapey value assigns to agent i the following utility:

$$\phi_i^{SV}(v_o) = -(g_i^M - 1)\theta_i + \frac{\sum_{g_j^M < g_i^M} g_j^M \cdot \ell}{\sigma_i} \cdot \theta_i - \sum_{k \in F_i(\sigma)} \Big(\frac{1}{\sigma_k - 1} \cdot \frac{\sum_{g_j^M < g_k^M} g_j^M \cdot \ell}{\sigma_k} \cdot \theta_k \Big).$$

This ℓ-server queueing problem can be obtained from the positional queueing problem by setting $C(\theta_i, \sigma_i) = (g_i - 1)\theta_i$. By substituting this cost function into equation (1.3), we have[2]:

$$\begin{aligned} t_i^M &= \sum_{p \in P_i(\sigma)} \frac{C(\theta_i, \sigma_i) - C(\theta_i, \sigma_p)}{\sigma_i} - \sum_{f \in F_i(\sigma)} \frac{1}{\sigma_f - 1} \sum_{k \in P_f(\sigma)} \frac{C(\theta_f, \sigma_f) - C(\theta_f, \sigma_k)}{\sigma_f} \\ &= \sum_{p \in P_i(\sigma)} \frac{(g_i - g_p)\theta_i}{\sigma_i} - \sum_{f \in F_i(\sigma)} \frac{1}{\sigma_f - 1} \sum_{k \in P_f(\sigma)} \frac{(g_f - g_k)\theta_f}{\sigma_f} \end{aligned}$$

Since $\sum_{p \in P_i(\sigma)} (g_i - g_p) = (1 + \cdots + (g_i - 1)) \cdot \ell = \sum_{g_j < g_i} g_j \cdot \ell$ and $\sum_{k \in P_i(\sigma_f)} (g_f - g_k) = (1 + \cdots + (g_f - 1)) \cdot \ell = \sum_{g_j < g_f} g_j \cdot \ell$, t_i^M can be expressed as:

$$\begin{aligned} t_i^M &= \sum_{p \in P_i(\sigma)} \frac{(g_i - g_p)\theta_i}{\sigma_i} - \sum_{f \in F_i(\sigma)} \frac{1}{\sigma_f - 1} \sum_{k \in P_f(\sigma)} \frac{(g_f - g_k)\theta_f}{\sigma_f} \\ &= \frac{\sum_{g_j < g_i} g_j \cdot \ell}{\sigma_i} \cdot \theta_i - \sum_{f \in F_i(\sigma)} \frac{1}{\sigma_f - 1} \frac{\sum_{g_j < g_f} g_j \cdot \ell}{\sigma_f} \cdot \theta_f, \end{aligned}$$

which is the transfer for the minimal transfer rule of the ℓ-server queueing problem (as conjectured in Chun and Heo (2008)).

1.3.3 The minimal transfer rule for position allocation problems

Position allocation problem (Essen and Wooders 2018) is interested in allocating positions to each agent. Let $N = \{1, \ldots, n\}$ be the set agents and there are $|N|$ positions. Each position i, $1 \leq i \leq |N|$, has an inherent value α_i which is known to agents. Without loss of generality, let the positions are ordered so that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. Each position is assigned to each agent so that if agent i with type θ_i receives position σ_i, then her payoff is equal to $\alpha_{\sigma_i} \cdot \theta_i$.[3] Let v_o^{PA} be an optimistic position allocation game obtained from the position allocation problem under the optimistic assumption that the coalitional members are served before the non-coalitional members. Assuming that $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$, Essen

[2]Strictly speaking, since queueing problems with multiple parallel servers are supermodular, but not strictly supermodular, we have to apply Theorem 1.1 instead of Corollary 1.1. However, it is sufficient to consider efficient queues in $\tilde{E}(\theta, C)$ from the following reason. Let two efficient queues $\sigma_1, \sigma_2 \in E(\theta, C)$ are g-identical if for each $i \in N$, $g_i(\sigma_1) = g_i(\sigma_2)$. First, note that if $\sigma_1, \sigma_2 \in E(\theta, C)$ are g-identical, then the corresponding transfer is the same. In addition, for each $\sigma \in E(\theta, C)$, there is a g-identical queue $\tilde{\sigma}$ in $\tilde{E}(\theta.C)$. Therefore, by calculating the transfer for an efficient queue $\tilde{\sigma} \in \tilde{E}(\theta.C)$, we can identify the transfer for all efficient queues $\sigma \in E(\theta, C)$, which are g-identical to $\tilde{\sigma}$.

[3]Essen and Wooders(2018) uses $x_i \in \mathbb{R}$ for the player value, where as we use $\theta_i \in \mathbb{R}_+$ for the cost type. We can also extend our problem by allowing a negative θ_i, which would not affect our results.

and Wooders (2018, Proposition 1) shows that the assignment to agent i by the Shapley value is equal to:

$$\phi_i^{SV}(v_o^{PA}) = \frac{1}{i}(\sum_{m=1}^{i} \alpha_m)\theta_i - \sum_{m=1}^{n-i} \frac{1}{i+m-1} \left[\sum_{r=1}^{i+m-1} \frac{r}{i+m}(\alpha_r - \alpha_{r+1})\theta_{i+m} \right].$$

Let σ be an efficient queue. Note that

$$\begin{aligned}
\phi_i^{SV}(v_o^{PA}) &= \frac{1}{i}(\sum_{m=1}^{i} \alpha_m)\theta_i - \sum_{m=1}^{n-i} \frac{1}{i+m-1} \Big[\sum_{r=1}^{i+m-1} \frac{r}{i+m}(\alpha_r - \alpha_{r+1})\theta_{i+m} \Big] \\
&= \sum_{p=1}^{i} \frac{\alpha_p \theta_i}{i} - \sum_{m=1}^{n-i} \frac{1}{i+m-1} \Big[\sum_{r=1}^{i+m-1} \frac{1}{i+m}(\alpha_r - \alpha_{i+m})\theta_{i+m} \Big] \\
&= \sum_{p=1}^{i} \frac{\alpha_p \theta_i}{i} - \sum_{f=i+1}^{n} \frac{1}{f-1} \Big[\sum_{r=1}^{f-1} \frac{1}{f}(\alpha_r - \alpha_f)\theta_f \Big] \\
&= \sum_{p \in P_i(\sigma) \cup \{i\}} \frac{\alpha_p \theta_i}{i} - \sum_{f \in F_i(\sigma)} \frac{1}{f-1} \sum_{k \in P_f(\sigma)} \frac{(\alpha_k - \alpha_f)\theta_f}{f}.
\end{aligned}$$

Now from the definition of t_i,

$$\begin{aligned}
t_i &= \varphi_i^{SV}(v_o^{PA}) + C(\theta_i, \sigma_i) \\
&= \sum_{p \in P_i(\sigma) \cup \{i\}} \frac{\alpha_p \theta_i}{i} - \sum_{f \in F_i(\sigma)} \frac{1}{f-1} \sum_{k \in P_f(\sigma)} \frac{(\alpha_k - \alpha_f)\theta_f}{f} - \alpha_i \theta_i \\
&= \sum_{p \in P_i(\sigma)} \frac{(\alpha_{\sigma_p} - \alpha_{\sigma_i})\theta_i}{\sigma_i} - \sum_{f \in F_i(\sigma)} \frac{1}{\sigma_f - 1} \sum_{k \in P_f(\sigma)} \frac{(\alpha_{\sigma_k} - \alpha_{\sigma_f})\theta_f}{\sigma_f}.
\end{aligned}$$

Now let the positional cost function be $C(\theta_i, \sigma_i) = -\alpha_{\sigma_i} \cdot \theta_i$ for all $i \in N$ as the position allocation problem. Then we have:

$$\begin{aligned}
t_i^M &= \sum_{p \in P_i(\sigma)} \frac{C(\theta_i, \sigma_i) - C(\theta_i, \sigma_p)}{\sigma_i} - \sum_{f \in F_i(\sigma)} \frac{1}{\sigma_f - 1} \sum_{k \in P_f(\sigma)} \frac{C(\theta_f, \sigma_f) - C(\theta_f, \sigma_k)}{\sigma_f} \\
&= \sum_{p \in P_i(\sigma)} \frac{(\alpha_{\sigma_p} - \alpha_{\sigma_i})\theta_i}{\sigma_i} - \sum_{f \in F_i(\sigma)} \frac{1}{\sigma_f - 1} \sum_{k \in P_f(\sigma)} \frac{(\alpha_{\sigma_k} - \alpha_{\sigma_f})\theta_f}{\sigma_f},
\end{aligned}$$

which is the transfer for the minimal transfer rule of the positional allocation problem.

1.4 A pessimistic approach and the maximal transfer rule

Now we assume that all coalitional members are served after non-coalitional members (Chun 2006). The worth of each coalition $S \subseteq N$ is defined as the negative of the minimum cost when the coalitional members are served in the last $|S|$ positions. Formally, for each $S \subseteq N$ and each $\tilde{\sigma}(S) \in \tilde{E}(\theta_S, C)$, its worth $v_p(S)$ is defined as:

$$v_p(S) = \max_{\sigma \in \Sigma(S)} \big[-\sum_{i \in S} C(\theta_i, n - s + \sigma_i) \big] = -\sum_{i \in S} C(\theta_i, n - s + \tilde{\sigma}_i(S)).$$

Our second theorem shows that the utility of each agents assigned by the Shapley value under pessimistic assumption can be reached using the maximal transfer rule.

Theorem 1.2. (Maximal transfer rule) Let $(\theta, C) \in \mathcal{Q}^N$ and (σ, t) be a feasible allocation such that agents' utilities are equal to the payoff vector obtained by applying the Shapley value to the pessimistic positional queueing game v_p. Then, $\sigma^X \in E(\theta, C)$ and for each $i \in N$,

$$t_i^X = \sum_{\theta_0 \in \Theta(\hat{P}, i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{n-k} \sum_{\ell=k}^{n} \frac{C(\theta_0, \ell) - C(\theta_0, k)}{n-k+1} - \sum_{k=\hat{p}_i+1}^{n} \frac{C(\theta_i, k) - C(\theta_i, \sigma_i^X)}{\hat{e}_i + \hat{f}_i}. \tag{1.4}$$

Proof. The proof is given in Appendix B. □

If an efficient queue belongs to $\tilde{E}(\theta, C)$, then the transfer in Theorem 1.2 can be simplified similarly to the transfer of the maximal transfer rule in the standard queueing problem.

Corollary 1.2. Let $(\theta, C) \in \mathcal{Q}^N$ and (σ^X, t^X) be a feasible allocation such that agents' utilities are equal to the payoff vector obtained by applying the Shapley value to v_p. If $\sigma^X \in \tilde{E}(\theta, C)$, then for each $i \in N$,

$$t_i^X = \sum_{p \in P_i(\sigma^X)} \frac{1}{n - \sigma_p^X} \sum_{k \in F_p(\sigma^X)} \frac{C(\theta_p, \sigma_k^X) - C(\theta_p, \sigma_p^X)}{n - \sigma_p^X + 1} - \sum_{f \in F_i(\sigma^X)} \frac{C(\theta_i, \sigma_f^X) - C(\theta_i, \sigma_i^X)}{n - \sigma_i^X + 1}. \tag{1.5}$$

Proof. The proof is given in Appendix B. □

Example 1.2. (The maximal transfer rule applied to a problem with 3 agents) Let $N = \{1, 2, 3\}$ and θ be such that $\theta_1 > \theta_2 > \theta_3$. Then, the maximal transfer rule is given by setting for each $i \in N$, $\sigma_i^X = i$ and

$$t_1^X = -\frac{1}{3}[C(\theta_1, 2) - C(\theta_1, 1)] - \frac{1}{3}[C(\theta_1, 3) - C(\theta_1, 1)],$$
$$t_2^X = \frac{1}{2 \times 3}[C(\theta_1, 2) - C(\theta_1, 1)] + \frac{1}{2 \times 3}[C(\theta_1, 3) - C(\theta_1, 1)] - \frac{1}{2}[C(\theta_2, 3) - C(\theta_2, 2)],$$
$$t_3^X = \frac{1}{2 \times 3}[C(\theta_1, 2) - C(\theta_1, 1)] + \frac{1}{2 \times 3}[C(\theta_1, 3) - C(\theta_1, 1)] + \frac{1}{1 \times 2}[C(\theta_2, 3) - C(\theta_2, 2)].$$

Next, we show that how Theorem 1.2 is related to the results in Chun (2006) for the standard queueing problem and Chun and Heo (2008) for the queueing problem with ℓ-servers, We also discuss how the maximal transfer rule can be defined for the position allocation problem in Essen and Wooders (2018).

1.4.1 The maximal transfer rule for the standard queueing problem

As in Subsection 1.3.1, the transfer for the maximal transfer rule of the standard queueing problem (Chun 2006) can be obtained by setting for each $i \in N$, $C(\theta_i, \sigma_i) = (\sigma_i - 1)\theta_i$. By substituting the cost function into equation (1.5), we have:

$$
\begin{aligned}
t_i^X &= \sum_{p \in P_i(\sigma)} \frac{1}{n-\sigma_p} \sum_{k \in F_p(\sigma)} \frac{C(\theta_p,\sigma_k) - C(\theta_p,\sigma_p)}{n-\sigma_p+1} - \sum_{f \in F_i(\sigma)} \frac{C(\theta_i,\sigma_f) - C(\theta_i,\sigma_i)}{n-\sigma_i+1} \\
&= \sum_{p \in P_i(\sigma)} \frac{1}{n-\sigma_p} \sum_{k \in F_p(\sigma)} \frac{(\sigma_k - \sigma_p)}{n-\sigma_p+1}\theta_p - \sum_{f \in F_i(\sigma)} \frac{(\sigma_f - \sigma_i)}{n-\sigma_i+1}\theta_i \\
&= \sum_{p \in P_i(\sigma)} \frac{1}{n-\sigma_p} \frac{(n-\sigma_p)(n-\sigma_p+1)}{2} \frac{1}{n-\sigma_p+1}\theta_p - \frac{(n-\sigma_i)(n-\sigma_i+1)}{2} \frac{1}{n-\sigma_i+1}\theta_i \\
&= \sum_{p \in P_i(\sigma)} \frac{\theta_p}{2} - (n-\sigma_i)\frac{\theta_i}{2},
\end{aligned}
$$

the desired expression.

1.4.2 The maximal transfer rule for the queueing problem with multiple parallel servers

As in Subsection 1.3.2, the transfer for the maximal transfer rule of the ℓ-server queueing problem can be obtained by setting for each $i \in N$, $C(\theta_i, \sigma_i) = (g_i - 1)\theta_i$ where $g_i = \lceil \frac{\sigma_i}{\ell} \rceil$ and $\sigma \in \tilde{E}(\theta, C)$. By substituting the cost function into equation (1.5), we have:

$$
\begin{aligned}
t_i^X &= \sum_{p \in P_i(\sigma)} \frac{1}{n-\sigma_p} \sum_{k \in F_p(\sigma)} \frac{C(\theta_p,\sigma_k) - C(\theta_p,\sigma_p)}{n-\sigma_p+1} - \sum_{f \in F_i(\sigma)} \frac{C(\theta_i,\sigma_f) - C(\theta_i,\sigma_i)}{n-\sigma_i+1} \\
&= \sum_{p \in P_i(\sigma)} \frac{1}{n-\sigma_p} \sum_{k \in F_p(\sigma)} \frac{(g_k - g_p)\theta_p}{n-\sigma_p+1} - \sum_{f \in F_i(\sigma)} \frac{g_f - g_i}{n-\sigma_i+1}\theta_i.
\end{aligned}
$$

For each $i \in N$, let m_i be defined as:

$$
\begin{aligned}
m_i &= \sum_{f \in F_i(\sigma)} \frac{(g_f - g_i)}{n-\sigma_i+1}\theta_i \\
&= \frac{1}{n-\sigma_i+1}\Big[\sum_{f \in F_i(\sigma), g_f = g_i} (g_f - g_i) + \sum_{f \in F_i(\sigma), g_i < g_f < g_n} (g_f - g_i) + \sum_{f \in F_i(\sigma), g_f = g_n} (g_f - g_i) \Big]\theta_i \\
&= \frac{1}{n-\sigma_i+1}\Big[\sum_{g_i < g_f < g_n} (g_f - g_i)\ell + (g_n - g_i)\big(n - (g_n - 1)\ell\big) \Big]\theta_i \\
&= \frac{\sum_{g_i < g_f < g_n} (g_f - g_i)\ell}{n-\sigma_i+1}\theta_i + \frac{(g_n - g_i)[n - (g_n-1)\ell]}{n-\sigma_i+1}\theta_i.
\end{aligned}
$$

Therefore, t_i^X can be simplified by using m_i as:

$$
t_i^X = \sum_{p \in P_i(\sigma)} \frac{m_p}{(n-\sigma_p)} - m_i,
$$

and the corresponding utility $\phi_i^{SV}(v_p)$ is

$$\phi_i^{SV}(v_p) = -(g_i - 1)\theta_i + \sum_{p \in P_i(\sigma)} \frac{m_p}{(n - \sigma_p)} - m_i.$$

Note that if $\ell = 2$, then our formula is the same as the definition of the maximal transfer rule given in Chun and Heo (2008).

1.4.3 The maximal transfer rule for position allocation problems

As in Subsection 1.3.3, the transfer for the maximal transfer rule of the position allocation problem can be obtained by setting for each $i \in N$, $C(\theta_i, \sigma_i) = -\alpha_{\sigma_i} \cdot \theta_i$. By substituting the cost function into equation (1.5), we have:

$$\begin{aligned} t_i^X &= \sum_{p \in P_i(\sigma)} \frac{1}{n - \sigma_p} \sum_{k \in F_p(\sigma)} \frac{C(\theta_p, \sigma_k) - C(\theta_p, \sigma_p)}{n - \sigma_p + 1} - \sum_{f \in F_i(\sigma)} \frac{C(\theta_i, \sigma_f) - C(\theta_i, \sigma_i)}{n - \sigma_i + 1} \\ &= \sum_{p \in P_i(\sigma)} \frac{1}{n - \sigma_p} \sum_{k \in F_p(\sigma)} \frac{(\alpha_{\sigma_p} - \alpha_{\sigma_k})}{n - \sigma_p + 1}\theta_p - \sum_{f \in F_i(\sigma)} \frac{(\alpha_{\sigma_i} - \alpha_{\sigma_f})}{n - \sigma_i + 1}\theta_i. \end{aligned}$$

Therefore, the Shapley value assigns the following utility to agent i:

$$\begin{aligned} \phi_i^{SV}(v_p^{PA}) &= t_i^X - C(\theta_i, \sigma_i) \\ &= \sum_{p \in P_i(\sigma)} \frac{1}{n - \sigma_p} \sum_{k \in F_p(\sigma)} \frac{(\alpha_{\sigma_p} - \alpha_{\sigma_k})}{n - \sigma_p + 1}\theta_p - \sum_{f \in F_i(\sigma)} \frac{(\alpha_{\sigma_i} - \alpha_{\sigma_f})}{n - \sigma_i + 1}\theta_i - \alpha_{\sigma_i}\theta_i \\ &= \sum_{p \in P_i(\sigma)} \frac{1}{n - \sigma_p} \sum_{k \in F_p(\sigma)} \frac{(\alpha_{\sigma_p} - \alpha_{\sigma_k})}{n - \sigma_p + 1}\theta_p + \sum_{f \in F_i(\sigma) \cup \{i\}} \frac{\alpha_{\sigma_f}}{n - \sigma_i + 1}\theta_i. \end{aligned}$$

1.5 Axioms and characterizations

1.5.1 Axioms

As for the classical queueing problems, Maniquet(2003), Chun(2006), and van den Brink and Chun(2012) give us some axiomatic characterizations for the minimal transfer rule and the maximal transfer rule.

Maniquet(2003) shows that the minimal transfer rule is characterized by *efficiency*, *anonymity*, *equal treatment of equals*, *independence of preceding agents' impatience* together and *Pareto indifference*, *identical preferences lower bound*, *impatience monotonicity*, *equal responsibility* together.

Chun(2006) shows that the maximal transfer rule is characterized by *efficiency*, *Pareto indifference*, *equal treatment of equals*, *independence of following costs* together and *Pareto indifference*, *identical preferences lower bound*, *positive cost monotonicity*, *first-agent equal responsibility* together. In his paper, some axioms of Maniquet(2003) are renamed in order to compare the minimal transfer rule and the maximal transfer rule; *independence of preceding costs* instead of *independence of preceding agents' impatience* and *negative cost monotonicity* instead of *impatience monotonicity*. And he imposes *Pareto indifference* instead of *anonymity* since the same result can be obtained.

van den Brink and Chun(2012) show that the minimal transfer rule is characterized by *efficiency*, *Pareto indifference*, *balanced consistency* together and *efficiency*, *Pareto indifference*, *balanced cost reduction* together.

Now we state axioms for the positional queueing problems. Some of them have the same underlying idea as the axioms of classical queueing problems. In these cases, we name them same as the classical queueing problem and just modify the expressions to suitable for the positional queueing problem. However, some of them are similar but different from the original ones. Basically, the differences occur because the rules of positional queueing problems should consider the order of the cost types as well as the position in an efficient queue. As for the classical queueing problem, an efficient queue selects the ordering σ such that $\sigma_i < \sigma_j$ if $\theta_i > \theta_j$, therefore the order of the cost types and the order of the position in an efficient queue matches. However they do not have to match in the positional queueing problems.

Efficiency requires that a rule should choose an allocation which is *queue-efficient* and *budget balanced*. *Pareto indifference* requires that if a rule chooses an allocation, then it also chooses all other feasible allocations which assign the same utilities as the chosen one. *Equal treatment of equals* requires that a rule should assign the same utilities to the agents with the same cost types. *Identical preferences lower bound* requires that all agent are weakly better than the identical economy where all agents have the same cost type as her. It is easy to check that the minimal transfer rule and the maximal transfer rule satisfy all these axioms.

Axiom 1.1 (Efficiency)**.** *For all* $(\theta, C) \in \mathcal{Q}^N$ *and all* $(\sigma, t) \in \varphi(\theta, C)$, $\sigma \in E(\theta, C)$ *and* $\sum_{i\in N} t_i = 0$.

Axiom 1.2 (Pareto indifference)**.** *For all* $(\theta, C) \in \mathcal{Q}^N$, *all* $(\sigma, t) \in \varphi(\theta, C)$, *and* $(\sigma', t') \in Z(\theta, C)$, *if for all* $i \in N$, $u_i(\sigma', t'; \theta, C) = u_i(\sigma, t; \theta, C)$, *then* $(\sigma', t') \in \varphi(\theta, C)$.

Axiom 1.3 (Equal treatment of equals)**.** *For all* $(\theta, C) \in \mathcal{Q}^N$, *all* $(\sigma, t) \in \varphi(\theta, C)$, *for all* $i,\ j \in N$ *such that* $\theta_i = \theta_j$, $u_i(\sigma, t; \theta, C) = u_j(\sigma, t; \theta, C)$.

Axiom 1.4 (Identical preferences lower bound)**.** *For all* $(\theta, C) \in \mathcal{Q}^N$, *all* $(\sigma, t) \in \varphi(\theta, C)$, *all* $i \in N$,

$$u_i(\sigma, t; \theta, C) \geq -\sum_{k=1}^{n} \frac{C(\theta_i, k)}{n}.$$

Before we investigate further, we introduce Lemma 2 which is proved in Appendix A and its modification.

Lemma 2. For each $j \in N$ and any two positive integers p and q, we have

$$\sum_{k=1}^{p} \frac{C(\theta_j, k)}{p} - \sum_{k=1}^{p+q} \frac{C(\theta_j, k)}{p+q} = -\sum_{k=p+1}^{p+q} \sum_{\ell=1}^{k} \frac{C(\theta_j, k) - C(\theta_j, \ell)}{(k-1)k}.$$

Lemma 2 calculate the difference between the average from the first p elements and the average from the first $p + q$ elements. Here the word *first* means we count from 1 to

n. However if we count from n to 1 and use $r = p + q$, then the equation can be written as below:

Lemma 2 (modified). For each $j \in N$ and any two positive integers p and r such that $r > p$, we have

$$\sum_{k=n-p+1}^{n} \frac{C(\theta_j, k)}{p} - \sum_{k=n-r+1}^{n} \frac{C(\theta_j, k)}{r} = - \sum_{k=n-r+1}^{n-p} \sum_{\ell=k}^{n} \frac{C(\theta_j, k) - C(\theta_j, \ell)}{(n-k)(n-k+1)}.$$

Proposition 1.1. *Both of the minimal transfer rule and the maximal transfer rule satisfy efficiency, Pareto indifference, equal treatment of equals, and identical preferences lower bound.*

Proof. First, we show that the rules satisfy efficiency. From the definition of the minimal transfer rule and the maximal transfer rule, $\sum_{i \in N} u_i(\sigma, t; \theta, C) = - \sum_{i \in N} C(\theta_i, \sigma_i)$ if an allocation (σ, t) is chosen by one of these rules.

$$\begin{aligned}
\sum_{i \in N} t_i &= \sum_{i \in N} \big(u_i(\sigma, t; \theta, C) + C(\theta_i, \sigma_i) \big) \\
&= \sum_{i \in N} u_i(\sigma, t; \theta, C) + \sum_{i \in N} C(\theta_i, \sigma_i) \\
&= - \sum_{i \in N} C(\theta_i, \sigma_i) + \sum_{i \in N} C(\theta_i, \sigma_i) \\
&= 0.
\end{aligned}$$

Therefore, the rules satisfy *efficiency*.

The rules satisfy *Pareto indifference* by definition.

Next, we show that the rules satisfy equal treatment of equals. For all $i \in N$, the utility of agent i under the minimal transfer rule is calculated as

$$u_i^M(\sigma, t; \theta, C) = - \sum_{k=1}^{\hat{p}_i + \hat{e}_i} \frac{C(\theta_i, k)}{\hat{p}_i + \hat{e}_i} - \sum_{\theta_0 \in \Theta(\hat{F}, i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0 + \hat{e}_0} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_0, k) - C(\theta_0, \ell)}{k}, \tag{1.6}$$

and the utility of agent i under the maximal transfer rule is calculated as

$$u_i^X(\sigma, t; \theta, C) = \sum_{\theta_0 \in \Theta(\hat{P}, i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0 + \hat{e}_0} \frac{1}{n-k} \sum_{\ell=k}^{n} \frac{C(\theta_0, \ell) - C(\theta_0, k)}{n-k+1} - \sum_{k=\hat{p}_i+1}^{n} \frac{C(\theta_i, k)}{\hat{e}_i + \hat{f}_i}. \tag{1.7}$$

Since they are functions of θ's only, they ensures the same utilities for two agents with the same cost types. Therefore the rules satisfy *equal treatment of equals*.

Finally, we show that the rules satisfy identical preferences lower bound. For all $i \in N$, the utility of agent i under the minimal transfer rule would be

$$
\begin{aligned}
u_i^M(\sigma, t; \theta, C) &= -\sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i, k)}{\hat{p}_i + \hat{e}_i} - \sum_{\theta_0 \in \Theta(\hat{F}, i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \sum_{\ell=1}^{k} \frac{C(\theta_0, k) - C(\theta_0, \ell)}{(k-1)k} \\
&= -\sum_{k=1}^{n} \frac{C(\theta_i, k)}{n} + \sum_{k=\hat{p}_i+\hat{e}_i+1}^{n} \sum_{\ell=1}^{k} \frac{C(\theta_i, k) - C(\theta_i, \ell)}{(k-1)k} \\
&\quad - \sum_{\theta_0 \in \Theta(\hat{F}, i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \sum_{\ell=1}^{k} \frac{C(\theta_0, k) - C(\theta_0, \ell)}{(k-1)k} \\
&= -\sum_{k=1}^{n} \frac{C(\theta_i, k)}{n} + \sum_{\theta_0 \in \Theta(\hat{F}, i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \sum_{\ell=1}^{k} \frac{C(\theta_i, k) - C(\theta_i, \ell) - C(\theta_0, k) + C(\theta_0, \ell)}{(k-1)k} \\
&\geq -\sum_{k=1}^{n} \frac{C(\theta_i, k)}{n}.
\end{aligned}
$$

The second equality holds by Lemma 2. The last inequality holds since $\theta_i > \theta_0$, $k \geq \ell$ and C is supermodular. Therefore, the minimal transfer rule satisfies identical preferences lower bound. As for the maximal transfer rule, we have

$$
\begin{aligned}
u_i^X(\sigma, t; \theta, C) &= \sum_{\theta_0 \in \Theta(\hat{P}, i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{n-k} \sum_{\ell=k}^{n} \frac{C(\theta_0, \ell) - C(\theta_0, k)}{n-k+1} - \sum_{k=\hat{p}_i+1}^{n} \frac{C(\theta_i, k)}{\hat{e}_i + \hat{f}_i} \\
&= \sum_{\theta_0 \in \Theta(\hat{P}, i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{n-k} \sum_{\ell=k}^{n} \frac{C(\theta_0, \ell) - C(\theta_0, k)}{n-k+1} + \sum_{k=1}^{\hat{p}_i} \sum_{\ell=k}^{n} \frac{C(\theta_i, k) - C(\theta_i, \ell)}{(n-k)(n-k+1)} - \sum_{k=1}^{n} \frac{C(\theta_i, k)}{n} \\
&= \sum_{\theta_0 \in \Theta(\hat{P}, i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{n-k} \sum_{\ell=k}^{n} \frac{C(\theta_0, \ell) - C(\theta_0, k) + C(\theta_i, k) - C(\theta_i, \ell)}{n-k+1} - \sum_{k=1}^{n} \frac{C(\theta_i, k)}{n} \\
&\geq -\sum_{k=1}^{n} \frac{C(\theta_i, k)}{n}.
\end{aligned}
$$

The second equality hods by Lemma 2 (modified) with $p = n - \hat{p}_i$ and $r = n$. The last inequality holds since $\theta_i < \theta_0$, $k \leq \ell$ and C is supermodular. Therefore, the maximal transfer rule satisfies identical preferences lower bound.

□

We now introduce axioms concerning changes in the cost types. *Negative cost monotonicity* requires that an increase in a cost type of an agent makes all other agents weakly lose. On the other hand, *positive cost monotonicity* requires that an increase in a cost type of an agent makes all other agents weakly better. *Independence of larger costs* requires that an increase in an agent's cost type does not affect the utilities of agents with the same or smaller cost types than her original cost type. On the other hand, *independence of smaller costs* requires that an decrease in an agent's cost type does not affect the utilities of agents with the same or larger cost types than her original cost type.

Axiom 1.5 (Negative cost monotonicity)**.** *For all $(\theta, C) \in \mathcal{Q}^N$, all $(\theta', C) \in \mathcal{Q}^N$, all $(\sigma, t) \in \varphi(\theta, C)$, all $(\sigma', t') \in \varphi(\theta', C)$, and all $j \in N$, if for all $i \in N \setminus \{k\}$, $\theta_i = \theta'_i$ and $\theta_j < \theta'_j$, then for all $i \in N \setminus \{j\}$, $u_i(\sigma, t; \theta, C) \geq u_i(\sigma', t'; \theta', C)$.*

Axiom 1.6 (Positive cost monotonicity)**.** *For all* $(\theta, C) \in \mathcal{Q}^N$*, all* $(\theta', C) \in \mathcal{Q}^N$*, all* $(\sigma, t) \in \varphi(\theta, C)$*, all* $(\sigma', t') \in \varphi(\theta', C)$*, and all* $j \in N$*, if for all* $i \in N \setminus \{k\}$*,* $\theta_i = \theta'_i$ *and* $\theta_j < \theta'_j$*, then for all* $i \in N \setminus \{j\}$*,* $u_i(\sigma, t; \theta, C) \leq u_i(\sigma', t'; \theta', C)$*.*

Axiom 1.7 (Independence of larger costs)**.** *For all* $(\theta, C) \in \mathcal{Q}^N$*, all* $(\theta', C) \in \mathcal{Q}^N$*, all* $(\sigma, t) \in \varphi(\theta, C)$*, all* $(\sigma', t') \in \varphi(\theta', C)$*, and all* $k \in N$*, if for all* $j \in N \setminus \{k\}$*,* $\theta_j = \theta'_j$ *and* $\theta_k < \theta'_k$*, then for all* $i \in N$ *such that* $\theta_i \leq \theta_k$*,* $u_i(\sigma, t; \theta, C) = u_i(\sigma', t'; \theta', C)$*.*

Axiom 1.8 (Independence of smaller costs)**.** *For all* $(\theta, C) \in \mathcal{Q}^N$*, all* $(\theta', C) \in \mathcal{Q}^N$*, all* $(\sigma, t) \in \varphi(\theta, C)$*, all* $(\sigma', t') \in \varphi(\theta', C)$*, and all* $k \in N$*, if for all* $j \in N \setminus \{k\}$*,* $\theta_j = \theta'_j$ *and* $\theta_k > \theta'_k$*, then for all* $i \in N$ *such that* $\theta_i \geq \theta_k$*,* $u_i(\sigma, t; \theta, C) = u_i(\sigma', t'; \theta', C)$*.*

Proposition 1.2. *The minimal transfer rule satisfies negative cost monotonicity.*

Proof. Let θ' be such that for any $j \in N$, $\theta_i = \theta'_i$ for all $i \in N \setminus \{j\}$, and $\theta_j < \theta'_j$. We have 3 cases regarding relationships among θ_i, θ_j, and θ'_j; Case 1: $\theta_i \leq \theta_j < \theta'_j$, Case 2: $\theta_i \geq \theta'_j > \theta_j$, Case 3: $\theta_j < \theta_i \leq \theta'_j$. We show that for any cases, the utility of agent i is weakly worse off. For simplicity, we use u_i and u'_i instead of $u_i(\sigma, t; \theta, C)$ and $u_i(\sigma', t'; \theta', C)$, respectively.

Case 1: $\theta_i \leq \theta_j < \theta'_j$
$u_i = u'_i$ by independence of larger costs.

Case 2: $\theta_i \geq \theta'_j > \theta_j$
Let $\Theta(\hat{F}, i) = \{\hat{\theta}_1, \ldots, \hat{\theta}_m\}$ such that $\hat{\theta}_1 > \ldots > \hat{\theta}_m$. Let $\theta_j = \hat{\theta}_q, q \leq m$ and $\theta_i = \hat{\theta}_0$. Check that the utility of agent i is as below before the cost type of agent j changes;

$$u_i = -\sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i, k)}{\hat{p}_i + \hat{e}_i} - \sum_{\theta_0 \in \Theta(\hat{F}, i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_0, k) - C(\theta_0, \ell)}{k}.$$

We first claim that if $\theta'_j \leq \hat{\theta}_{q-1}$ then $u'_i \leq u_i$. If the cost type of agent j changes between $\theta_j (= \hat{\theta}_q)$ and $\hat{\theta}_{q-1}$, we have to consider the changes of cost type of θ_j and θ'_j only since $\hat{p}'_0 = \hat{p}_0$, $\hat{e}'_0 = \hat{e}_0$ for all $\theta_0 \in \Theta \setminus \{\theta_j, \theta'_j\}$. We have 2 cases; (1) $\theta'_j \leq \hat{\theta}_{q-1} < \theta_i$, and (3) $\theta'_j = \hat{\theta}_{q-1} = \theta_i$.

Case 2-1: $\theta'_j \leq \hat{\theta}_{q-1} < \theta_i$

$$\begin{aligned} u'_i - u_i = &- \sum_{k=\hat{p}_j+1}^{\hat{p}_j+1} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta'_j, k) - C(\theta'_j, \ell)}{k} + \sum_{k=\hat{p}_j+1}^{\hat{p}_j+1} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_j, k) - C(\theta_j, \ell)}{k} \\ =&- \sum_{k=\hat{p}_j+1}^{\hat{p}_j+1} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta'_j, k) - C(\theta'_j, \ell) - C(\theta_j, k) + C(\theta_j, \ell)}{k} \\ \leq& 0, \end{aligned}$$

where the last inequality holds since $\theta'_j > \theta_j$, $k \geq \ell$, and C is supermodular.

Case 2-2: $\theta'_j \leq \hat{\theta}_{q-1} = \theta_i$

$$\begin{aligned}
u'_i - u_i =& -\sum_{k=1}^{\hat{p}_i+\hat{e}_i+1} \frac{C(\theta_i,k)}{\hat{p}_i+\hat{e}_i+1} + \sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i,k)}{\hat{p}_i+\hat{e}_i} + \sum_{k=\hat{p}_j+1}^{\hat{p}_j+1} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_j,k)-C(\theta_j,\ell)}{k} \\
=& -\sum_{k=\hat{p}_i+\hat{e}_i+1}^{\hat{p}_i+\hat{e}_i+1} \sum_{\ell=1}^{k} \frac{C(\theta_i,k)-C(\theta_i,\ell)}{(k-1)k} + \sum_{k=\hat{p}_j+1}^{\hat{p}_j+1} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_j,k)-C(\theta_j,\ell)}{k} \\
=& -\sum_{k=\hat{p}_j+1}^{\hat{p}_j+1} \sum_{\ell=1}^{k} \frac{C(\theta_i,k)-C(\theta_i,\ell)}{(k-1)k} + \sum_{k=\hat{p}_j+1}^{\hat{p}_j+1} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_j,k)-C(\theta_j,\ell)}{k} \\
=& -\sum_{k=\hat{p}_j+1}^{\hat{p}_j+1} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_i,k)-C(\theta_i,\ell)-C(\theta_j,k)+C(\theta_j,\ell)}{k} \\
\leq& 0,
\end{aligned}$$

where the second equality holds by Lemma 2 (Han and Chun 2020) and the last inequality holds since $\theta_i > \theta_j$, $k \geq \ell$, and C is supermodular.

Altogether, if $\theta'_j \leq \hat{\theta}_{q-1}$ then $u'_i \leq u_i$. Now applying this result recursively. Let $\theta_j = \hat{\theta}_q$ and $\hat{\theta}_r \leq \theta'_j < \hat{\theta}_{r-1}$ such that $q \leq r$. Raise the cost type of agent j from $\theta_j = \hat{\theta}_q$ to $\hat{\theta}_{q-1}$, from $\hat{\theta}_{q-1}$ to $\hat{\theta}_{q-2}$, $\cdots$, from $\hat{\theta}_r$ to θ'_j. The utility of agent i is getting worse during the sequence as long as the new cost type of agent j is less than or equal to θ_i.

Case 3: $\theta_j < \theta_i \leq \theta'_j$

Combining the result of Case 1 and Case 2, $u'_i \leq u_i$.

Therefore if $\theta'_j > \theta_j$ then for all $i \in N \setminus \{j\}$, $u'_i \leq u_i$ as desired.

□

Proposition 1.3. *The minimal transfer rule satisfies independence of larger costs.*

Proof. Let θ' be such that for an agent $k \in N$, for all $j \in N \setminus \{k\}$, $\theta_j = \theta'_j$ and $\theta_k < \theta'_k$.

For all $i \in N$, the utility of agent i of games (θ, C) and (θ', C) under the minimal transfer rule are calculated as:

$$u_i(\sigma,t;\theta,C) = -\sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i,k)}{\hat{p}_i+\hat{e}_i} - \sum_{\theta_0 \in \Theta(\hat{F},i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_0,k)-C(\theta_0,\ell)}{k},$$

$$u_i(\sigma',t';\theta',C) = -\sum_{k=1}^{\hat{p}'_i+\hat{e}'_i} \frac{C(\theta_i,k)}{\hat{p}'_i+\hat{e}'_i} - \sum_{\theta_0 \in \Theta'(\hat{F},i)} \sum_{k=\hat{p}'_0+1}^{\hat{p}'_0+\hat{e}'_0} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_0,k)-C(\theta_0,\ell)}{k}.$$

Let $\theta_i \leq \theta_k$. If $\theta_i = \theta_k$ then $\hat{P}'_i = \hat{P}_i \cup \{k\}$, $\hat{E}'_i \setminus \{k\} = \hat{E}_i$, and $\hat{F}'_i = \hat{F}_i$. If $\theta_i < \theta_k$ then $\hat{P}'_i = \hat{P}_i$, $\hat{E}'_i = \hat{E}_i$, and $\hat{F}'_i = \hat{F}_i$. For any case, $\hat{p}'_i + \hat{e}'_i = \hat{p}_i + \hat{e}_i$. Therefore the first term of u_i and u'_i are same. The second term of u_i and u'_i are same since $\Theta(\hat{F},i) = \Theta'(\hat{F},i)$ and for all $\theta_0 \in \Theta(\hat{F},i)$, $\hat{p}'_0 = \hat{p}_0$, $\hat{e}'_0 = \hat{e}_0$, $\hat{f}'_0 = \hat{f}_0$. Altogether if $\theta_i \leq \theta_k$, then $u_i = u'_i$. Therefore, the minimal transfer rule satisfies independence of larger costs.

□

Remark 1.1. *It can be shown that the maximal transfer rule satisfies* positive cost monotonicity *and* independence of smaller costs *using similar proofs of proposition 1.2 and proposition 1.3.*

We now introduce axioms concerns with changes in the population. *Smallest agent equal responsibility* requires that if an agent who has the smallest cost type leaves, then all the transfer of remaining agents change by the same amount. On the other hand, *largest agent equal responsibility* requires that if an agent who has the largest cost type leaves, then all the transfer of remaining agents change by the same amount.

The last axioms we introduce are *balanced consistency* and *balanced consistency with external priority*. Choose agent $i, j \in N$. *Balanced consistency* requires that the change of agent j's utility incurred by the departure of agent i from the queue should be the same as the change of agent i's utility incurred by the departure of agent j from the queue. In the axiom, it is assumed that the leaving agent does not receive any services. On the other hand, we may think of a situation that the leaving agent receives services earlier than any other remaining agents. As she left from the queue, we do not consider her position or her transfer any more. However, as she receives services, the remaining agents cannot use the first position. *Balanced consistency with external priority* requires that the change of agent j's utility incurred by the departure of agent i should be the same as the change of agent i's utility incurred by the departure of agent j when the leaving agent receives services earlier than the remaining agents.

Axiom 1.9 (Smallest cost agent equal responsibility)**.** *For all $(\theta, C) \in \mathcal{Q}^N$, and all $(\sigma, t) \in \varphi(\theta, C)$, if agent m is such that $\theta_m \leq \theta_i$ for all $i \in N$, then for any $(\sigma', t') \in \varphi(\theta_{N\setminus\{m\}}, C)$, $u_i(\sigma, t; \theta, C) - u_i(\sigma', t'; \theta_{N\setminus\{m\}}, C) = u_j(\sigma, t; \theta, C) - u_j(\sigma', t'; \theta_{N\setminus\{m\}}, C)$ for all $i, j \in N \setminus \{m\}$.*

Axiom 1.10 (Largest cost agent equal responsibility)**.** *For all $(\theta, C) \in \mathcal{Q}^N$, and all $(\sigma, t) \in \varphi(\theta, C)$, if agent m is such that $\theta_m \geq \theta_i$ for all $i \in N$, then for any $(\sigma', t') \in \varphi(\theta_{N\setminus\{m\}}, C)$, $u_i(\sigma, t; \theta, C) - u_i(\sigma', t'; \theta_{N\setminus\{m\}}, C) = u_j(\sigma, t; \theta, C) - u_j(\sigma', t'; \theta_{N\setminus\{m\}}, C)$ for all $i, j \in N \setminus \{m\}$.*

Axiom 1.11 (Balanced consistency)**.** *For all $(\theta, C) \in \mathcal{Q}^N$, all $(\sigma, t) \in \varphi(\theta, C)$, all $i, j \in N$, all $(\sigma^{-i}, t^{-i}) \in \varphi(\theta_{N\setminus\{i\}}, C)$, all $(\sigma^{-j}, t^{-j}) \in \varphi(\theta_{N\setminus\{j\}}, C)$,*

$$u_i(\sigma, t; \theta, C) - u_i(\sigma^{-j}, t^{-j}; \theta_{N\setminus\{j\}}, C) = u_j(\sigma, t; \theta, C) - u_j(\sigma^{-i}, t^{-i}; \theta_{N\setminus\{i\}}, C).$$

Let $\tilde{\mathcal{Q}}^{N\setminus\{1\}}$ be a modified positional queueing problem such that the available positions are from 2 to $|N|$ with $|N| - 1$ players.

Axiom 1.12 (Balanced consistency with external priority)**.** *For all $(\theta, C) \in \mathcal{Q}^N$, all $(\sigma, t) \in \varphi(\theta, C)$, all $i, j \in N$, all $(\tilde{\theta}_{N\setminus\{i\}}, C)$, $(\tilde{\theta}_{N\setminus\{j\}}, C) \in \tilde{\mathcal{Q}}^{N\setminus\{1\}}$, all $(\sigma^{-i}, t^{-i}) \in \varphi(\tilde{\theta}_{N\setminus\{i\}}, C)$, and all $(\sigma^{-j}, t^{-j}) \in \varphi(\tilde{\theta}_{N\setminus\{j\}}, C)$,*

$$u_i(\sigma, t; \theta, C) - u_i(\sigma^{-j}, t^{-j}; \tilde{\theta}_{N\setminus\{j\}}, C) = u_j(\sigma, t; \theta, C) - u_j(\sigma^{-i}, t^{-i}; \tilde{\theta}_{N\setminus\{i\}}, C).$$

Proposition 1.4. *The minimal transfer rule satisfies smallest cost agent equal responsibility.*

Proof. Let $\theta_m \leq \theta_i$ for all $i \in N$. We compare the utilities of agent i ($i \in N \setminus \{m\}$) with and without agent m in the economy. For simplicity, we use u_i and u'_i instead of $u_i(\sigma, t; \theta, C)$ and $u_i(\sigma', t'; \theta_{N\setminus\{m\}}, C)$ respectively.

Case 1: $\theta_i = \theta_m$

$$\begin{aligned} u'_i - u_i &= -\sum_{k=1}^{\hat{p}_m+\hat{e}_m-1} \frac{C(\theta_m, k)}{\hat{p}_m + \hat{e}_m - 1} + \sum_{k=1}^{\hat{p}_m+\hat{e}_m} \frac{C(\theta_m, k)}{\hat{p}_m + \hat{e}_m} \\ &= -\sum_{k=1}^{n-1} \frac{C(\theta_m, k)}{n-1} + \sum_{k=1}^{n} \frac{C(\theta_m, k)}{n} \\ &= \sum_{\ell=1}^{n} \frac{C(\theta_m, k) - C(\theta_m, \ell)}{(k-1)k}, \end{aligned} \tag{1.8}$$

where the last equality holds by Lemma 2.

Case 2: $\theta_i > \theta_m$

$$\begin{aligned} u_i &= -\sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i, k)}{\hat{p}_i + \hat{e}_i} - \sum_{\theta_0 \in \Theta(\hat{F}, i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_0, k) - C(\theta_0, \ell)}{k} \\ &= -\sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i, k)}{\hat{p}_i + \hat{e}_i} - \sum_{\theta_0 \in \Theta(\hat{F}, i)\setminus\{\theta_m\}} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \sum_{\ell=1}^{k} \frac{C(\theta_0, k) - C(\theta_0, \ell)}{(k-1)k} - \sum_{k=\hat{p}_m+1}^{\hat{p}_m+\hat{e}_m} \sum_{\ell=1}^{k} \frac{C(\theta_m, k) - C(\theta_m, \ell)}{(k-1)k}. \end{aligned}$$

$$\begin{aligned} u'_i &= -\sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i, k)}{\hat{p}_i + \hat{e}_i} - \sum_{\theta_0 \in \Theta(\hat{F}, i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_0, k) - C(\theta_0, \ell)}{k} \\ &= -\sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i, k)}{\hat{p}_i + \hat{e}_i} - \sum_{\theta_0 \in \Theta(\hat{F}, i)\setminus\{\theta_m\}} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \sum_{\ell=1}^{k} \frac{C(\theta_0, k) - C(\theta_0, \ell)}{(k-1)k} - \sum_{k=\hat{p}_m+1}^{\hat{p}_m+\hat{e}_m-1} \sum_{\ell=1}^{k} \frac{C(\theta_m, k) - C(\theta_m, \ell)}{(k-1)k}. \end{aligned}$$

$$\begin{aligned} u'_i - u_i &= \sum_{\ell=1}^{\hat{p}_m+\hat{e}_m} \frac{C(\theta_m, k) - C(\theta_m, \ell)}{(k-1)k} \\ &= \sum_{\ell=1}^{n} \frac{C(\theta_m, k) - C(\theta_m, \ell)}{(k-1)k}. \end{aligned} \tag{1.9}$$

Therefore for all $i, j \in N \setminus \{m\}$, $u'_i - u_i = u'_j - u_j$.

□

Proposition 1.5. *The minimal transfer rule satisfies balanced consistency.*

Proof. Let $\theta_i = \theta_j$. $u_i(\sigma^{-j}, t^{-j}; \theta_{N\setminus\{j\}}, C) = u_j(\sigma^{-i}, t^{-i}; \theta_{N\setminus\{i\}}, C)$ by equal treatment of equals, so balanced consistency is satisfied.

Without loss of generality, let $\theta_i > \theta_j$.

$$u_i(\sigma, t; \theta, C) = -\sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i, k)}{\hat{p}_i + \hat{e}_i} - \sum_{\theta_0 \in \Theta(\hat{F}, i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_0, k) - C(\theta_0, \ell)}{k}. \tag{1.10}$$

$$\begin{aligned} u_i(\sigma^{-j}, t^{-j}; \theta_{N\setminus\{j\}}, C) = &-\sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i, k)}{\hat{p}_i + \hat{e}_i} - \sum_{\theta_0 \in \Theta(\hat{F}, i), \theta_0 > \theta_j} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \sum_{\ell=1}^{k} \frac{C(\theta_0, k) - C(\theta_0, \ell)}{(k-1)k} \\ &- \sum_{\theta_0 \in \Theta(\hat{F}, i), \theta_0 = \theta_j} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0-1} \sum_{\ell=1}^{k} \frac{C(\theta_0, k) - C(\theta_0, \ell)}{(k-1)k} \\ &- \sum_{\theta_0 \in \Theta(\hat{F}, i), \theta_0 < \theta_j} \sum_{k=\hat{p}_0}^{\hat{p}_0+\hat{e}_0-1} \sum_{\ell=1}^{k} \frac{C(\theta_0, k) - C(\theta_0, \ell)}{(k-1)k}. \end{aligned} \tag{1.11}$$

$$u_j(\sigma, t; \theta, C) = -\sum_{k=1}^{\hat{p}_j+\hat{e}_j} \frac{C(\theta_j, k)}{\hat{p}_j + \hat{e}_j} - \sum_{\theta_0 \in \Theta(\hat{F}, j)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_0, k) - C(\theta_0, \ell)}{k}. \tag{1.12}$$

$$u_j(\sigma^{-i}, t^{-i}; \theta_{N\setminus\{i\}}, C) = -\sum_{k=1}^{\hat{p}_j+\hat{e}_j-1} \frac{C(\theta_j, k)}{\hat{p}_j + \hat{e}_j - 1} - \sum_{\theta_0 \in \Theta(\hat{F}, j)} \sum_{k=\hat{p}_0}^{\hat{p}_0+\hat{e}_0-1} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_0, k) - C(\theta_0, \ell)}{k}. \tag{1.13}$$

$$\begin{aligned} &u_i(\sigma, t; \theta, C) - u_j(\sigma, t; \theta, C) \\ &= -\sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i, k)}{\hat{p}_i + \hat{e}_i} - \sum_{\theta_0 \in \Theta(\hat{F}, i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_0, k) - C(\theta_0, \ell)}{k} \\ &\quad + \sum_{k=1}^{\hat{p}_j+\hat{e}_j} \frac{C(\theta_j, k)}{\hat{p}_j + \hat{e}_j} + \sum_{\theta_0 \in \Theta(\hat{F}, j)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_0, k) - C(\theta_0, \ell)}{k} \\ &= -\sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i, k)}{\hat{p}_i + \hat{e}_i} + \sum_{k=1}^{\hat{p}_j+\hat{e}_j} \frac{C(\theta_j, k)}{\hat{p}_j + \hat{e}_j} - \sum_{\theta_0 \in \Theta(\hat{F}, i), \theta_0 \geq \theta_j} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \sum_{\ell=1}^{k} \frac{C(\theta_0, k) - C(\theta_0, \ell)}{(k-1)k}. \end{aligned} \tag{1.14}$$

$$
\begin{aligned}
&u_i(\sigma^{-j}, t^{-j}; \theta_{N\setminus\{j\}}, C) - u_j(\sigma^{-i}, t^{-i}; \theta_{N\setminus\{i\}}, C) \\
=& -\sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i, k)}{\hat{p}_i+\hat{e}_i} - \sum_{\theta_0\in\Theta(\hat{F},i),\theta_0>\theta_j} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \sum_{\ell=1}^{k} \frac{C(\theta_0,k)-C(\theta_0,\ell)}{(k-1)k} \\
& - \sum_{\theta_0\in\Theta(\hat{F},i),\theta_0=\theta_j} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0-1} \sum_{\ell=1}^{k} \frac{C(\theta_0,k)-C(\theta_0,\ell)}{(k-1)k} - \sum_{\theta_0\in\Theta(\hat{F},i),\theta_0<\theta_j} \sum_{k=\hat{p}_0}^{\hat{p}_0+\hat{e}_0-1} \sum_{\ell=1}^{k} \frac{C(\theta_0,k)-C(\theta_0,\ell)}{(k-1)k} \\
& + \sum_{k=1}^{\hat{p}_j+\hat{e}_j-1} \frac{C(\theta_j, k)}{\hat{p}_j+\hat{e}_j-1} + \sum_{\theta_0\in\Theta(\hat{F},j)} \sum_{k=\hat{p}_0}^{\hat{p}_0+\hat{e}_0-1} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_0,k)-C(\theta_0,\ell)}{k} \\
=& -\sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i, k)}{\hat{p}_i+\hat{e}_i} - \sum_{\theta_0\in\Theta(\hat{F},i),\theta_0>\theta_j} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \sum_{\ell=1}^{k} \frac{C(\theta_0,k)-C(\theta_0,\ell)}{(k-1)k} \\
& - \sum_{\theta_0\in\Theta(\hat{F},i),\theta_0=\theta_j} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0-1} \sum_{\ell=1}^{k} \frac{C(\theta_0,k)-C(\theta_0,\ell)}{(k-1)k} + \sum_{k=1}^{\hat{p}_j+\hat{e}_j-1} \frac{C(\theta_j, k)}{\hat{p}_j+\hat{e}_j-1} \\
=& -\sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i, k)}{\hat{p}_i+\hat{e}_i} + \sum_{k=1}^{\hat{p}_j+\hat{e}_j-1} \frac{C(\theta_j, k)}{\hat{p}_j+\hat{e}_j-1} \\
& - \sum_{\theta_0\in\Theta(\hat{F},i),\theta_0>\theta_j} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \sum_{\ell=1}^{k} \frac{C(\theta_0,k)-C(\theta_0,\ell)}{(k-1)k} - \sum_{\theta_0\in\Theta(\hat{F},i),\theta_0=\theta_j} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0-1} \sum_{\ell=1}^{k} \frac{C(\theta_0,k)-C(\theta_0,\ell)}{(k-1)k}.
\end{aligned} \tag{1.15}
$$

Using equations (1.14), (1.15),

$$
\begin{aligned}
&[u_i(\sigma, t; \theta, C) - u_j(\sigma, t; \theta, C)] - [u_i(\sigma^{-j}, t^{-j}; \theta_{N\setminus\{j\}}, C) - u_j(\sigma^{-i}, t^{-i}; \theta_{N\setminus\{i\}}, C)] \\
=& \sum_{k=1}^{\hat{p}_j+\hat{e}_j} \frac{C(\theta_j, k)}{\hat{p}_j+\hat{e}_j} - \sum_{k=1}^{\hat{p}_j+\hat{e}_j-1} \frac{C(\theta_j, k)}{\hat{p}_j+\hat{e}_j-1} - \sum_{\theta_0=\theta_j} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \sum_{\ell=1}^{k} \frac{C(\theta_0,k)-C(\theta_0,\ell)}{(k-1)k} \\
& + \sum_{\theta_0=\theta_j} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0-1} \sum_{\ell=1}^{k} \frac{C(\theta_0,k)-C(\theta_0,\ell)}{(k-1)k} \qquad (1.16) \\
=& \sum_{k=1}^{\hat{p}_j+\hat{e}_j} \frac{C(\theta_j, k)}{\hat{p}_j+\hat{e}_j} - \sum_{k=1}^{\hat{p}_j+\hat{e}_j-1} \frac{C(\theta_j, k)}{\hat{p}_j+\hat{e}_j-1} - \sum_{\ell=1}^{\hat{p}_j+\hat{e}_j} \frac{C(\theta_j,k)-C(\theta_j,\ell)}{(\hat{p}_j+\hat{e}_j-1)(\hat{p}_j+\hat{e}_j)} \\
=& 0,
\end{aligned}
$$

where the last equality holds by Lemma 2.

□

Remark 1.2. *It can be shown that the maximal transfer rule satisfies* largest cost agent equal responsibility *and* balanced consistency with external priority *using similar proofs of proposition 1.4 and proposition 1.5.*

1.5.2 Axiomatic characterizations

Now we show three axiomatic characterizations of the minimal transfer rule.

Theorem 1.3. *The minimal transfer rule is the only rule satisfying efficiency, Pareto indifference, equal treatment of equals, and independence of larger costs together.*

Proof. We show that there is only one solution satisfying all four axioms together. Let φ be a solution satisfies the four axioms. Let $(\theta, C) \in \mathcal{Q}^N$, and $(\sigma, t) \in \varphi(\theta, C)$. Let $\hat{\Theta}$ be the set of distinct cost types. Without loss of generality, let $\hat{\Theta} = \{\hat{\theta}_1, \ldots, \hat{\theta}_k\}$ and $\hat{\theta}_1 > \cdots > \hat{\theta}_k$.

For $q = 1, \cdots, k$, let (θ^q, C) be a problem that $\theta_i^q = \hat{\theta}_q$ if $\theta_i \geq \hat{\theta}_q$, and $\theta_i^q = \theta_i$ otherwise. Let $(\sigma^q, t^q) \in \varphi(\theta^q, C)$. By efficiency and Pareto indifference, $\sum_{i\in N} u_i(\sigma^q, t^q; \theta^q, C) = -\sum_{i\in N} C(\theta_i^q, \tilde{\sigma}_i)$ where $\tilde{\sigma} \in \tilde{E}(\theta, C)$. By independence of larger costs, $u_i(\sigma^q, t^q; \theta^q, C) = u_i(\sigma, t; \theta, C)$ for all i such that $\theta_i \leq \hat{\theta}_q$. By equal treatment of equals, $u_i(\sigma^q, t^q; \theta^q, C) = u_j(\sigma^q, t^q; \theta^q, C)$ for all $i, j \in \hat{P}_q \cup \hat{E}_q$. Altogether, the equation below holds for agent i such that $\theta_i = \hat{\theta}_q$.

$$
\begin{aligned}
\sum_{\ell\in N} u_\ell(\sigma^q, t^q; \theta^q, C) &= -\sum_{\ell\in N} C(\theta_\ell^q, \tilde{\sigma}_\ell)\\
\sum_{\ell\in \hat{P}_q\cup\hat{E}_q} u_\ell(\sigma^q, t^q; \theta^q, C) + \sum_{\ell\in\hat{F}_q} u_\ell(\sigma^q, t^q; \theta^q, C) &= -\sum_{\ell\in \hat{P}_q\cup\hat{E}_q} C(\theta_\ell^q, \tilde{\sigma}_\ell) - \sum_{\ell\in\hat{F}_q} C(\theta_\ell^q, \tilde{\sigma}_\ell)\\
(\hat{p}_i + \hat{e}_i) u_i(\sigma^q, t^q; \theta^q, C) + \sum_{\ell\in\hat{F}_i} u_\ell(\sigma^q, t^q; \theta^q, C) &= -\sum_{1\leq m\leq \hat{p}_i+\hat{e}_i} C(\theta_i, m) - \sum_{\ell\in\hat{F}_i} C(\theta_\ell, \tilde{\sigma}_\ell)\\
(\hat{p}_i + \hat{e}_i) u_i(\sigma, t; \theta, C) + \sum_{\ell\in\hat{F}_i} u_\ell(\sigma, t; \theta, C) &= -\sum_{1\leq m\leq \hat{p}_i+\hat{e}_i} C(\theta_i, m) - \sum_{\ell\in\hat{F}_i} C(\theta_\ell, \tilde{\sigma}_\ell)\\
(\hat{p}_i + \hat{e}_i) u_i(\sigma, t; \theta, C) &= -\sum_{1\leq m\leq \hat{p}_i+\hat{e}_i} C(\theta_i, m) - \sum_{\ell\in\hat{F}_i} C(\theta_\ell, \tilde{\sigma}_\ell) - \sum_{\ell\in\hat{F}_i} u_\ell(\sigma, t; \theta, C)\\
(\hat{p}_i + \hat{e}_i) u_i(\sigma, t; \theta, C) &= -\sum_{1\leq m\leq \hat{p}_i+\hat{e}_i} C(\theta_i, m) - \sum_{\ell\in\hat{F}_i} \Big[C(\theta_\ell, \tilde{\sigma}_\ell) + u_\ell(\sigma, t; \theta, C)\Big].
\end{aligned}
$$

Applying this equation to $q = k$. For all i such that $\theta_i = \hat{\theta}_k$,

$$
\begin{aligned}
(\hat{p}_i + \hat{e}_i) u_i(\sigma, t; \theta, C) &= -\sum_{1\leq m\leq \hat{p}_i+\hat{e}_i} C(\theta_i, m)\\
n \cdot u_i(\sigma, t; \theta, C) &= -\sum_{1\leq m\leq n} C(\theta_i, m)\\
\therefore u_i(\sigma, t; \theta, C) &= -\frac{\sum_{1\leq m\leq n} C(\theta_i, m)}{n}.
\end{aligned}
$$

Now we compare the utilities of agent i and j such that $\theta_i = \hat{\theta}_q$ and $\theta_j = \hat{\theta}_{q+1}$ for all $q = 1, \cdots, k-1$. By definition, $\hat{P}_i \cup \hat{E}_i = \hat{P}_j$ and $\hat{F}_i = \hat{E}_j \cup \hat{F}_j$.

$$
\begin{aligned}
(\hat{p}_i + \hat{e}_i)u_i(\sigma, t; \theta, C) =& - \sum_{1 \leq m \leq \hat{p}_i + \hat{e}_i} C(\theta_i, m) - \sum_{\ell \in \hat{F}_i} \Big[C(\theta_\ell, \tilde{\sigma}_\ell) + u_\ell(\sigma, t; \theta, C) \Big] \\
=& - \sum_{1 \leq m \leq \hat{p}_i + \hat{e}_i} C(\theta_i, m) - \sum_{\ell \in \hat{E}_j} \Big[C(\theta_\ell, \tilde{\sigma}_\ell) + u_\ell(\sigma, t; \theta, C) \Big] \\
& - \sum_{\ell \in \hat{F}_j} \Big[C(\theta_\ell, \tilde{\sigma}_\ell) + u_\ell(\sigma, t; \theta, C) \Big]
\end{aligned} \tag{1.17}
$$

$$
\begin{aligned}
(\hat{p}_j + \hat{e}_j)u_j(\sigma, t; \theta, C) =& - \sum_{1 \leq m \leq \hat{p}_j + \hat{e}_j} C(\theta_j, m) - \sum_{\ell \in \hat{F}_j} \Big[C(\theta_\ell, \tilde{\sigma}_\ell) + u_\ell(\sigma, t; \theta, C) \Big] \\
(\hat{p}_i + \hat{e}_i + \hat{e}_j)u_j(\sigma, t; \theta, C) =& - \sum_{1 \leq m \leq \hat{p}_i + \hat{e}_i + \hat{e}_j} C(\theta_j, m) - \sum_{\ell \in \hat{F}_j} \Big[C(\theta_\ell, \tilde{\sigma}_\ell) + u_\ell(\sigma, t; \theta, C) \Big] \\
(\hat{p}_i + \hat{e}_i)u_j(\sigma, t; \theta, C) =& - \sum_{1 \leq m \leq \hat{p}_i + \hat{e}_i + \hat{e}_j} C(\theta_j, m) - \sum_{\ell \in \hat{F}_j} \Big[C(\theta_\ell, \tilde{\sigma}_\ell) + u_\ell(\sigma, t; \theta, C) \Big] - \hat{e}_j \cdot u_j(\sigma, t; \theta, C)
\end{aligned} \tag{1.18}
$$

$$
\begin{aligned}
& (1.17) - (1.18) \\
=& (\hat{p}_i + \hat{e}_i) \Big(u_i(\sigma, t; \theta, C) - u_j(\sigma, t; \theta, C) \Big) \\
=& - \sum_{1 \leq m \leq \hat{p}_i + \hat{e}_i} C(\theta_i, m) + \sum_{1 \leq m \leq \hat{p}_i + \hat{e}_i + \hat{e}_j} C(\theta_j, m) \\
& - \sum_{\ell \in \hat{E}_j} \Big[C(\theta_\ell, \tilde{\sigma}_\ell) + u_\ell(\sigma, t; \theta, C) \Big] + \hat{e}_j \cdot u_j(\sigma, t; \theta, C) \\
=& - \sum_{1 \leq m \leq \hat{p}_i + \hat{e}_i} C(\theta_i, m) + \sum_{1 \leq m \leq \hat{p}_i + \hat{e}_i} C(\theta_j, m) + \sum_{\hat{p}_i + \hat{e}_i + 1 \leq m \leq \hat{p}_i + \hat{e}_i + \hat{e}_j} C(\theta_j, m) \\
& - \sum_{\ell \in \hat{E}_j} \Big[C(\theta_\ell, \tilde{\sigma}_\ell) + u_\ell(\sigma, t; \theta, C) \Big] + \hat{e}_j \cdot u_j(\sigma, t; \theta, C) \\
=& - \sum_{1 \leq m \leq \hat{p}_i + \hat{e}_i} C(\theta_i, m) + \sum_{1 \leq m \leq \hat{p}_i + \hat{e}_i} C(\theta_j, m) \\
=& - \sum_{1 \leq m \leq \hat{p}_i + \hat{e}_i} C(\theta_i, m) + \sum_{1 \leq m \leq \hat{p}_j} C(\theta_j, m).
\end{aligned}
$$

Therefore, for all $i, j \in N$ such that $\theta_i = \hat{\theta}_q$ and $\theta_j = \hat{\theta}_{q+1}$, $q = 1, \cdots, k-1$,

$$
\begin{aligned}
u_i(\sigma, t; \theta, C) =& - \sum_{1 \leq m \leq \hat{p}_i + \hat{e}_i} \frac{C(\theta_i, m)}{\hat{p}_i + \hat{e}_i} + \sum_{1 \leq m \leq \hat{p}_j} \frac{C(\theta_j, m)}{\hat{p}_j} + u_j(\sigma, t; \theta, C) \\
=& - \sum_{1 \leq m \leq \hat{p}_q + \hat{e}_q} \frac{C(\hat{\theta}_q, m)}{\hat{p}_q + \hat{e}_q} + \sum_{1 \leq m \leq \hat{p}_{q+1}} \frac{C(\hat{\theta}_{q+1}, m)}{\hat{p}_{q+1}} + u_j(\sigma, t; \theta, C).
\end{aligned}
$$

Let $\theta_i = \hat{\theta}_\ell$. Applying this equation from $q = \ell$ to k, we have

$$\begin{aligned}
u_i(\sigma, t; \theta, C) = &- \sum_{1 \le m \le \hat{p}_i + \hat{e}_i} \frac{C(\theta_i, m)}{\hat{p}_i + \hat{e}_i} + \sum_{1 \le m \le \hat{p}_{\ell+1})} \frac{C(\hat{\theta}_{\ell+1}, m)}{\hat{p}_{\ell+1}} \\
&- \sum_{1 \le m \le \hat{p}_{\ell+1} + \hat{e}_{\ell+1}} \frac{C(\hat{\theta}_{\ell+1}, m)}{\hat{p}_{\ell+1} + \hat{e}_{\ell+1}} + \sum_{1 \le m \le \hat{p}_{\ell+2}} \frac{C(\hat{\theta}_{\ell+2}, m)}{\hat{p}_{\ell+2}} \\
&\vdots \\
&- \sum_{1 \le m \le \hat{p}_k + \hat{e}_k} \frac{C(\hat{\theta}_k, m)}{\hat{p}_k + \hat{e}_k} \\
= &- \sum_{1 \le m \le \hat{p}_i + \hat{e}_i} \frac{C(\theta_i, m)}{\hat{p}_i + \hat{e}_i} - \sum_{\theta_0 \in \Theta(\hat{F}, i)} \Big[\sum_{m=1}^{\hat{p}_0 + \hat{e}_0} \frac{C(\theta_0, k)}{\hat{p}_0 + \hat{e}_0} - \sum_{m=1}^{\hat{p}_0} \frac{C(\theta_0, k)}{\hat{p}_0} \Big],
\end{aligned}$$

as desired. □

Theorem 1.4. *The minimal transfer rule is the only rule satisfying efficiency, Pareto indifference, identical preferences lower bound, negative cost monotonicity, and the smallest cost agent equal responsibility together.*

Proof. We show that there exist only one solution satisfying all the axioms together.

Step 1: Let $\theta_j \le \theta_i$ for all $i \in N$. We first claim that

$$u_j(\sigma, t; \theta, C) = - \sum_{k=1}^{n} \frac{C(\theta_j, k)}{n}.$$

By identical preferences lower bounds, $u_\ell(\sigma, t; \theta, C) \ge - \sum_{k=1}^{n} \frac{C(\theta_j, k)}{n}$. By way of contradiction, Suppose that $u_j(\sigma, t; \theta, C) > - \sum_{k=1}^{n} \frac{C(\theta_j, k)}{n}$. Let $\theta' \in \mathbb{R}_+^N$ be the cost type profile such that $\theta'_i = \theta_j$ for all $i \in N$. Let $(\sigma', t') \in \varphi(\theta', C)$. Applying negative cost monotonicity repeatedly, we get

$$u_j(\sigma', t'; \theta', C) \ge u_j(\sigma, t; \theta, C) > - \sum_{k=1}^{n} \frac{C(\theta_j, k)}{n}.$$

By identical preferences lower bounds, for all $i \in N \setminus \{j\}$,

$$u_i(\sigma', t'; \theta', C) \ge - \sum_{k=1}^{n} \frac{C(\theta_j, k)}{n}.$$

Altogether,

$$\begin{aligned}
\sum_{i \in N} u_i(\sigma', t'; \theta', C) &> - \sum_{k=1}^{n} C(\theta_j, k) \\
\sum_{i \in N} u_i(\sigma', t'; \theta', C) + \sum_{k=1}^{n} C(\theta_j, k) &> 0.
\end{aligned} \tag{1.19}$$

On the other hand, efficiency requires that

$$\sum_{i\in N} t'_i = \sum_{i\in N} u_i(\sigma', t'; \theta', C) + \sum_{k=1}^{n} C(\theta_\ell, k) = 0, \tag{1.20}$$

a contradiction. Therefore if $\theta_j \leq \theta_i$,

$$u_j(\sigma, t; \theta, C) = -\sum_{k=1}^{n} \frac{C(\theta_j, k)}{n}.$$

Step 2: We show that all agents should end up with the utilities assigned by the minimal transfer rule.

Step 2-1: Let agent $\mathbb{n}$ be such that $\theta_{\mathbb{n}} \leq \theta_i$ for all $i \in N$. By Step 1,

$$u_{\mathbb{n}}(\sigma, t; \theta, C) = -\sum_{\ell=1}^{n} \frac{C(\theta_{\mathbb{n}}, \ell)}{n}. \tag{1.21}$$

Step 2-2: Let $\theta^{-1} = \theta_{N\setminus\{\mathbb{n}\}}$ and $(\sigma^{-1}, t^{-1}) \in \varphi(\theta^{-1}, C)$. Let agent $\mathbb{n} - \mathbb{1}$ be such that $\theta_{\mathbb{n}-\mathbb{1}} \leq \theta_i$ for all $i \in N \setminus \{\mathbb{n}\}$. By Step 1,

$$u_{\mathbb{n}-\mathbb{1}}(\sigma^{-1}, t^{-1}; \theta^{-1}, C) = -\sum_{\ell=1}^{n-1} \frac{C(\theta_{\mathbb{n}-\mathbb{1}}, \ell)}{n-1}. \tag{1.22}$$

By efficiency and equation (1.21),

$$\begin{aligned} \sum_{i\in N} u_i(\sigma, t; \theta, C) &= -\sum_{i\in N} C(\theta_i, \sigma_i) \\ \sum_{i\in N\setminus\{\mathbb{n}\}} u_i(\sigma, t; \theta, C) &= -\sum_{i\in N} C(\theta_i, \sigma_i) + \sum_{\ell=1}^{n} \frac{C(\theta_{\mathbb{n}}, \ell)}{n}. \end{aligned} \tag{1.23}$$

By efficiency,

$$\sum_{i\in N\setminus\{\mathbb{n}\}} u_i(\sigma^{-1}, t^{-1}; \theta^{-1}, C) = -\sum_{i\in N\setminus\{\mathbb{n}\}} C(\theta_i, \sigma_i^{-1}). \tag{1.24}$$

By smallest cost agent equal responsibility, there exist $\Delta_1 \in \mathbb{R}$ such that for all $i \in N \setminus \{\mathbb{n}\}$,

$$u_i(\sigma, t; \theta, C) - u_i(\sigma^{-1}, t^{-1}; \theta^{-1}, C) = \Delta_1. \tag{1.25}$$

Using equations (1.23), (1.24) and (1.25),

$$\begin{aligned}\sum_{i\in N\setminus\{\mathbb{n}\}}\big(u_i(\sigma,t;\theta,C)-u_i(\sigma^{-1},t^{-1};\theta^{-1},C)\big) &= -\sum_{i\in N}C(\theta_i,\sigma_i)+\sum_{\ell=1}^{n}\frac{C(\theta_{\mathbb{n}},\ell)}{n}+\sum_{i\in N\setminus\{\mathbb{n}\}}C(\theta_i,\sigma_i^{-1})\\ (n-1)\Delta_1 &= -\sum_{i\in N\setminus\{\mathbb{n}\}}C(\theta_i,\sigma_i^{-1})-C(\theta_{\mathbb{n}},n)+\sum_{\ell=1}^{n}\frac{C(\theta_{\mathbb{n}},\ell)}{n}\\ &\quad+\sum_{i\in N\setminus\{\mathbb{n}\}}C(\theta_i,\sigma_i^{-1})\\ (n-1)\Delta_1 &= -C(\theta_{\mathbb{n}},n)+\sum_{\ell=1}^{n}\frac{C(\theta_{\mathbb{n}},\ell)}{n}\\ \Delta_1 &= -\frac{C(\theta_{\mathbb{n}},n)}{n-1}+\sum_{\ell=1}^{n}\frac{C(\theta_{\mathbb{n}},\ell)}{(n-1)n}\\ \Delta_1 &= -\sum_{\ell=1}^{n}\frac{C(\theta_{\mathbb{n}},n)-C(\theta_{\mathbb{n}},\ell)}{(n-1)n}.\end{aligned} \tag{1.26}$$

By equations (1.22), (1.25), and (1.26),

$$\begin{aligned}u_{\mathbb{n}-\mathbb{1}}(\sigma,t;\theta,C) &= u_{\mathbb{n}-\mathbb{1}}(\sigma^{-1},t^{-1};\theta^{-1},C)+\Delta_1\\ \therefore u_{\mathbb{n}-\mathbb{1}}(\sigma,t;\theta,C) &= -\sum_{\ell=1}^{n-1}\frac{C(\theta_{\mathbb{n}-\mathbb{1}},\ell)}{n-1}-\sum_{\ell=1}^{n}\frac{C(\theta_{\mathbb{n}},n)-C(\theta_{\mathbb{n}},\ell)}{(n-1)n}.\end{aligned} \tag{1.27}$$

Step 2-3: Let $\theta^{-2}=\theta_{N\setminus\{\mathbb{n},\mathbb{n}-\mathbb{1}\}}$ and $(\sigma^{-2},t^{-2})\in\varphi(\theta^{-2},C)$. Let agent $\mathbb{n}-\mathbb{2}$ be such that $\theta_{\mathbb{n}-\mathbb{2}}\le\theta_i$ for all $i\in N\setminus\{\mathbb{n},\mathbb{n}-\mathbb{1}\}$. By Step 1,

$$u_{\mathbb{n}-\mathbb{2}}(\sigma^{-2},t^{-2};\theta^{-2},C)=-\sum_{k=1}^{n-1}\frac{C(\theta_{\mathbb{n}-\mathbb{2}},k)}{n-2}. \tag{1.28}$$

By efficiency and equation (1.22),

$$\begin{aligned}\sum_{i\in N\setminus\{\mathbb{n}\}}u_i(\sigma^{-1},t^{-1};\theta^{-1},C) &= -\sum_{i\in N\setminus\{\mathbb{n}\}}C(\theta_i,\sigma_i^{-1})\\ \sum_{i\in N\setminus\{\mathbb{n},\mathbb{n}-\mathbb{1}\}}u_i(\sigma^{-1},t^{-1};\theta^{-1},C) &= -\sum_{i\in N\setminus\{\mathbb{n}\}}C(\theta_i,\sigma_i^{-1})+\sum_{\ell=1}^{n-1}\frac{C(\theta_{\mathbb{n}-\mathbb{1}},\ell)}{n-1}.\end{aligned} \tag{1.29}$$

By efficiency,

$$\sum_{i\in N\setminus\{\mathbb{n},\mathbb{n}-\mathbb{1}\}}u_i(\sigma^{-2},t^{-2};\theta^{-2},C)=-\sum_{i\in N\setminus\{\mathbb{n},\mathbb{n}-\mathbb{1}\}}C(\theta_i,\sigma_i^{-2}). \tag{1.30}$$

By smallest cost agent equal responsibility, there exist $\Delta_2\in\mathbb{R}$ such that for all $i\in N\setminus\{\mathbb{n},\mathbb{n}-\mathbb{1}\}$,

$$u_i(\sigma^{-1},t^{-1};\theta^{-1},C)-u_i(\sigma^{-2},t^{-2};\theta^{-2},C)=\Delta_2. \tag{1.31}$$

Using equations (1.28), (1.29), and (1.30),

$$\sum_{i\in N\setminus\{\mathbb{n},\mathbb{n}-\mathbb{1}\}}\big(u_i(\sigma^{-1},t^{-1};\theta^{-1},C)-u_i(\sigma^{-2},t^{-2};\theta^{-2},C)\big)=-\sum_{i\in N\setminus\{\mathbb{n}\}}C(\theta_i,\sigma_i^{-1})+\sum_{\ell=1}^{n-1}\frac{C(\theta_{\mathbb{n}-\mathbb{1}},\ell)}{n-1}+\sum_{i\in N\setminus\{\mathbb{n},\mathbb{n}-\mathbb{1}\}}C(\theta_i,\sigma_i^{-2})$$

$$\begin{aligned}(n-2)\Delta_2&=-\sum_{i\in N\setminus\{\mathbb{n},\mathbb{n}-\mathbb{1}\}}C(\theta_i,\sigma_i^{-2})-C(\theta_{\mathbb{n}-\mathbb{1}},n-1)+\sum_{\ell=1}^{n-1}\frac{C(\theta_{\mathbb{n}-\mathbb{1}},\ell)}{n-1}+\sum_{i\in N\setminus\{\mathbb{n},\mathbb{n}-\mathbb{1}\}}C(\theta_i,\sigma_i^{-2})\\(n-2)\Delta_2&=-C(\theta_{\mathbb{n}-\mathbb{1}},n-1)+\sum_{\ell=1}^{n-1}\frac{C(\theta_{\mathbb{n}-\mathbb{1}},\ell)}{n-1}\\\Delta_2&=-\frac{C(\theta_{\mathbb{n}-\mathbb{1}},n-1)}{n-2}+\sum_{\ell=1}^{n-1}\frac{C(\theta_{\mathbb{n}-\mathbb{1}},\ell)}{(n-2)(n-1)}\\\Delta_2&=-\sum_{\ell=1}^{n-1}\frac{C(\theta_{\mathbb{n}-\mathbb{1}},n-1)-C(\theta_{\mathbb{n}-\mathbb{1}},\ell)}{(n-2)(n-1)}.\end{aligned}\tag{1.32}$$

By equations (1.25), (1.26), (1.28), (1.31), and (1.32),

$$\begin{aligned}u_{\mathbb{n}-2}(\sigma,t;\theta,C)&=u_{\mathbb{n}-2}(\sigma^{-1},t^{-1};\theta^{-1},C)+\Delta_1\\&=u_{\mathbb{n}-2}(\sigma^{-2},t^{-2};\theta^{-2},C)+\Delta_2+\Delta_1\\\therefore u_{\mathbb{n}-2}(\sigma,t;\theta,C)&=-\sum_{\ell=1}^{n-2}\frac{C(\theta_{\mathbb{n}-2},\ell)}{n-2}-\sum_{\ell=1}^{n-1}\frac{C(\theta_{\mathbb{n}-\mathbb{1}},n-1)-C(\theta_{\mathbb{n}-\mathbb{1}},\ell)}{(n-2)(n-1)}-\sum_{\ell=1}^{n}\frac{C(\theta_{\mathbb{n}},n)-C(\theta_{\mathbb{n}},\ell)}{(n-1)n}.\end{aligned}\tag{1.33}$$

Step 2-m$(1\le m\le n)$**:** Let agent $\mathbb{n}-\mathbb{m}+\mathbb{1}$ be such that $\theta_{\mathbb{n}-\mathbb{m}+\mathbb{1}}\le\theta_i$ for all $i\in N\setminus\{\mathbb{n},\cdots,\mathbb{n}-\mathbb{m}+2\}$. The utility of agent $\mathbb{n}-\mathbb{m}+\mathbb{1}$ is calculated as below:

$$u_{\mathbb{n}-\mathbb{m}+\mathbb{1}}(\sigma,t;\theta,C)=-\sum_{\ell=1}^{n-m+1}\frac{C(\theta_{\mathbb{n}-\mathbb{m}+\mathbb{1}},\ell)}{n-m+1}-\sum_{k=n-m+2}^{n}\sum_{\ell=1}^{k}\frac{C(\theta_{\mathbb{k}},k)-C(\theta_{\mathbb{k}},\ell)}{(k-1)k}.\tag{1.34}$$

Step 3: Let σ be an ordering such that $\sigma_{\mathbb{n}-\mathbb{m}+\mathbb{1}}=n-m+1(1\le m\le n)$, then $\sigma\in\tilde{E}(\theta,C)$. The equation (1.34) is same as the utility calculated by Corollary 1. By Pareto indifference, we obtain the desired result.

□

Theorem 1.5. *The minimal transfer rule is the only rule satisfying efficiency, Pareto indifference, and balanced consistency together.*

Proof. We show that there exist only one rule that satisfies efficiency, Pareto indifference, and balanced consistency together. Let $(\theta,C)\in\mathcal{Q}^N$ and $(\sigma,t)\in\varphi(\theta,C)$ where φ is a rule satisfies efficiency, Pareto indifference, and balanced consistency.

Case 1: If $N=\{i\}$, then efficiency implies $\sigma_i=1$ and $t_i=0$.

Case 2: Let $N = \{i, j\}$. Without loss generality let $\theta_i \geq \theta_j$. Let $(\sigma, t) \in \varphi(\theta, C)$, $(\sigma^{-i}, t^{-i}) \in \varphi(\theta_{N\setminus\{i\}}, C)$, and $(\sigma^{-j}, t^{-j}) \in \varphi(\theta_{N\setminus\{j\}}, C)$. By efficiency, $u_i(\sigma^{-j}, t^{-j}; \theta_{N\setminus\{j\}}, C) = -C(\theta_i, 1)$ and $u_j(\sigma^{-i}, t^{-i}; \theta_{N\setminus\{i\}}, C) = -C(\theta_j, 1)$.

By balanced consistency,

$$u_i(\sigma, t; \theta, C) - u_j(\sigma, t; \theta, C) = u_i(\sigma^{-j}, t^{-j}; \theta_{N\setminus\{j\}}, C) - u_j(\sigma^{-i}, t^{-i}; \theta_{N\setminus\{i\}}, C) = -C(\theta_i, 1) + C(\theta_j, 1).$$

By efficiency,

$$u_i(\sigma, t; \theta, C) + u_j(\sigma, t; \theta, C) = -C(\theta_i, 1) - C(\theta_j, 2).$$

Altogether,

$$\begin{aligned} u_i(\sigma, t; \theta, C) &= - C(\theta_i, 1) - \frac{C(\theta_j, 2) - C(\theta_j, 1)}{2}, \\ u_j(\sigma, t; \theta, C) &= - \frac{C(\theta_j, 2) + C(\theta_j, 1)}{2}, \end{aligned}$$

and by Pareto indifference we may assume that $\sigma_i = 1, \sigma_j = 2$.

Case 3: We prove the case of $n \geq 3$ by induction hypothesis. Suppose that the claim holds for all games with n' agents, $1 \leq n' \leq n-1$. We want to show that it also holds for a game with n agents.

By balanced consistency, $u_i(\sigma, t; \theta, C) - u_i(\sigma^{-j}, t^{-j}; \theta_{N\setminus\{j\}}, C) = u_j(\sigma, t; \theta, C) - u_j(\sigma^{-i}, t^{-i}; \theta_{N\setminus\{i\}}, C)$ for all $i, j \in N$. Fix agent i and change j for all $j \in N \setminus \{i\}$. If we add up all equations,

$$\begin{aligned} (n-1)u_i(\sigma, t; \theta, C) - \sum_{j \in N\setminus\{i\}} u_i(\sigma^{-j}, t^{-j}; \theta_{N\setminus\{j\}}, C) &= \sum_{j \in N\setminus\{i\}} \Big(u_j(\sigma, t; \theta, C) - u_j(\sigma^{-i}, t^{-i}; \theta_{N\setminus\{i\}}, C) \Big) \\ n \cdot u_i(\sigma, t; \theta, C) = \sum_{j \in N} u_j(\sigma, t; \theta, C) - \sum_{j \in N\setminus\{i\}} &u_j(\sigma^{-i}, t^{-i}; \theta_{N\setminus\{i\}}, C) + \sum_{j \in N\setminus\{i\}} u_i(\sigma^{-j}, t^{-j}; \theta_{N\setminus\{j\}}, C). \end{aligned} \tag{1.35}$$

Let $\tilde{\sigma} \in \tilde{E}(\theta, C)$, $\tilde{\sigma}^{-i} \in \tilde{E}(\theta_{N\setminus\{i\}}, C)$ be such that $\tilde{\sigma}^{-i}_k < \tilde{\sigma}^{-i}_\ell$ implies $\tilde{\sigma}_k < \tilde{\sigma}_\ell$ for all $i, k, \ell \in N$. Let $\tilde{\theta}_i$ be the cost type of agent j such that $\tilde{\sigma}_j = i$, that is, the cost type of i-th agent in $\tilde{\sigma}$. Let $\tilde{\sigma}_i = i$.

By efficiency, $\sum_{j \in N} u_j(\sigma, t; \theta, C) = -\sum_{j \in N} C(\theta_j, \tilde{\sigma}_j) = -\sum_{j=1}^n C(\tilde{\theta}_j, j)$. Similarly by efficiency, $\sum_{j \in N\setminus\{i\}} u_j(\sigma^{-i}, t^{-i}; \theta_{N\setminus\{i\}}, C) = -\sum_{j \in N\setminus\{i\}} C(\theta_j, \tilde{\sigma}^{-i}_j) = -\sum_{j=1}^{\tilde{\sigma}_i - 1} C(\tilde{\theta}_j, j) - \sum_{j=\tilde{\sigma}_i+1}^n C(\tilde{\theta}_j, j-1)$. Therefore,

$$\begin{aligned} &\sum_{j \in N} u_j(\sigma, t; \theta, C) - \sum_{j \in N\setminus\{i\}} u_j(\sigma^{-i}, t^{-i}; \theta_{N\setminus\{i\}}, C) \\ &= - \sum_{j=1}^n C(\tilde{\theta}_j, j) + \sum_{j=1}^{\tilde{\sigma}_i - 1} C(\tilde{\theta}_j, j) + \sum_{j=\tilde{\sigma}_i+1}^n C(\tilde{\theta}_j, j-1) \\ &= - \sum_{j=\tilde{\sigma}_i}^n C(\tilde{\theta}_j, j) + \sum_{j=\tilde{\sigma}_i+1}^n C(\tilde{\theta}_j, j-1) \end{aligned} \tag{1.36}$$

By induction hypothesis, $u_i(\sigma^{-j}, t^{-j}; \theta_{N\setminus\{j\}}, C)$ is the utility calculated by the minimal transfer rule as below:

$$u_i(\sigma^{-j}, t^{-j}; \theta_{N\setminus\{j\}}, C) = -\sum_{p\in P_i(\tilde\sigma^{-j})\cup\{i\}} \frac{C(\theta_i,\tilde\sigma_p^{-j})}{\tilde\sigma_i^{-j}} - \sum_{f\in F_i(\tilde\sigma^{-j})}\sum_{\ell\in P_f(\tilde\sigma^{-j})} \frac{C(\theta_f,\tilde\sigma_f^{-j}) - C(\theta_f,\tilde\sigma_\ell^{-j})}{(\tilde\sigma_f^{-j}-1)\tilde\sigma_f^{-j}} \tag{1.37}$$

$$\begin{aligned}
&\sum_{j\in N\setminus\{i\}} u_i(\sigma^{-j}, t^{-j}; \theta_{N\setminus\{j\}}, C)\\
&= -\sum_{j\in N\setminus\{i\}}\sum_{p\in P_i(\tilde\sigma^{-j})\cup\{i\}} \frac{C(\theta_i,\tilde\sigma_p^{-j})}{\tilde\sigma_i^{-j}} - \sum_{j\in N\setminus\{i\}}\sum_{f\in F_i(\tilde\sigma^{-j})}\sum_{\ell\in P_f(\tilde\sigma^{-j})} \frac{C(\theta_f,\tilde\sigma_f^{-j}) - C(\theta_f,\tilde\sigma_\ell^{-j})}{(\tilde\sigma_f^{-j}-1)\tilde\sigma_f^{-j}}\\
&= -\sum_{j\in P_i(\tilde\sigma)}\sum_{p\in P_i(\tilde\sigma^{-j})\cup\{i\}} \frac{C(\theta_i,\tilde\sigma_p^{-j})}{\tilde\sigma_i^{-j}} - \sum_{j\in F_i(\tilde\sigma)}\sum_{p\in P_i(\tilde\sigma^{-j})\cup\{i\}} \frac{C(\theta_i,\tilde\sigma_p^{-j})}{\tilde\sigma_i^{-j}}\\
&\quad - \sum_{j\in P_i(\tilde\sigma)}\sum_{f\in F_i(\tilde\sigma^{-j})}\sum_{\ell\in P_f(\tilde\sigma^{-j})} \frac{C(\theta_f,\tilde\sigma_f^{-j}) - C(\theta_f,\tilde\sigma_\ell^{-j})}{(\tilde\sigma_f^{-j}-1)\tilde\sigma_f^{-j}} - \sum_{j\in F_i(\tilde\sigma)}\sum_{f\in F_i(\tilde\sigma^{-j})}\sum_{\ell\in P_f(\tilde\sigma^{-j})} \frac{C(\theta_f,\tilde\sigma_f^{-j}) - C(\theta_f,\tilde\sigma_\ell^{-j})}{(\tilde\sigma_f^{-j}-1)\tilde\sigma_f^{-j}}\\
&= -\sum_{j=1}^{\tilde\sigma_i-1}\sum_{p=1}^{\tilde\sigma_i-1}\frac{C(\theta_i,p)}{\tilde\sigma_i-1} - \sum_{j=\tilde\sigma_i+1}^{n}\sum_{p=1}^{\tilde\sigma_i}\frac{C(\theta_i,p)}{\tilde\sigma_i} - \sum_{j\in P_i(\tilde\sigma)}\sum_{f=\tilde\sigma_i}^{n-1}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_{f+1},f)-C(\tilde\theta_{f+1},\ell)}{(f-1)f}\\
&\quad - \sum_{j\in F_i(\tilde\sigma)}\Big[\sum_{f=\tilde\sigma_i+1}^{\tilde\sigma_j-1}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f)-C(\tilde\theta_f,\ell)}{(f-1)f} + \sum_{f=\tilde\sigma_j}^{n-1}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_{f+1},f)-C(\tilde\theta_{f+1},\ell)}{(f-1)f}\Big]\\
&= -\sum_{j=1}^{\tilde\sigma_i-1}\sum_{p=1}^{\tilde\sigma_i-1}\frac{C(\theta_i,p)}{\tilde\sigma_i-1} - \sum_{j=\tilde\sigma_i+1}^{n}\sum_{p=1}^{\tilde\sigma_i}\frac{C(\theta_i,p)}{\tilde\sigma_i} - \sum_{j=1}^{\tilde\sigma_i-1}\sum_{f=\tilde\sigma_i}^{n-1}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_{f+1},f)-C(\tilde\theta_{f+1},\ell)}{(f-1)f}\\
&\quad - \sum_{j=\tilde\sigma_i+1}^{n}\Big[\sum_{f=\tilde\sigma_i+1}^{\tilde\sigma_j-1}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f)-C(\tilde\theta_f,\ell)}{(f-1)f} + \sum_{f=\tilde\sigma_j}^{n-1}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_{f+1},f)-C(\tilde\theta_{f+1},\ell)}{(f-1)f}\Big]\\
&= -(\tilde\sigma_i-1)\sum_{p=1}^{\tilde\sigma_i-1}\frac{C(\theta_i,p)}{\tilde\sigma_i-1} - (n-\tilde\sigma_i)\sum_{p=1}^{\tilde\sigma_i}\frac{C(\theta_i,p)}{\tilde\sigma_i} - (\tilde\sigma_i-1)\sum_{f=\tilde\sigma_i}^{n-1}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_{f+1},f)-C(\tilde\theta_{f+1},\ell)}{(f-1)f}\\
&\quad - \sum_{j=\tilde\sigma_i+1}^{n}\Big[\sum_{f=\tilde\sigma_i+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f)-C(\tilde\theta_f,\ell)}{(f-1)f} - \sum_{f=\tilde\sigma_j}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f)-C(\tilde\theta_f,\ell)}{(f-1)f} + \sum_{f=\tilde\sigma_j}^{n-1}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_{f+1},f)-C(\tilde\theta_{f+1},\ell)}{(f-1)f}\Big]\\
&= -\sum_{p=1}^{\tilde\sigma_i-1}C(\theta_i,p) - n\sum_{p=1}^{\tilde\sigma_i}\frac{C(\theta_i,p)}{\tilde\sigma_i} + \sum_{p=1}^{\tilde\sigma_i}C(\theta_i,p) - \sum_{j=\tilde\sigma_i+1}^{n}\sum_{f=\tilde\sigma_i+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f)-C(\tilde\theta_f,\ell)}{(f-1)f}\\
&\quad - (\tilde\sigma_i-1)\sum_{f=\tilde\sigma_i}^{n-1}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_{f+1},f)-C(\tilde\theta_{f+1},\ell)}{(f-1)f} - \sum_{j=\tilde\sigma_i+1}^{n}\Big[-\sum_{f=\tilde\sigma_j}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f)-C(\tilde\theta_f,\ell)}{(f-1)f} + \sum_{f=\tilde\sigma_j}^{n-1}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_{f+1},f)-C(\tilde\theta_{f+1},\ell)}{(f-1)f}\Big]\\
&= -n\sum_{p=1}^{\tilde\sigma_i}\frac{C(\theta_i,p)}{\tilde\sigma_i} - \sum_{p=1}^{\tilde\sigma_i-1}C(\theta_i,p) + \sum_{p=1}^{\tilde\sigma_i}C(\theta_i,p) - (n-\tilde\sigma_i)\sum_{f=\tilde\sigma_i+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f)-C(\tilde\theta_f,\ell)}{(f-1)f}\\
&\quad - (\tilde\sigma_i-1)\sum_{f=\tilde\sigma_i}^{n-1}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_{f+1},f)-C(\tilde\theta_{f+1},\ell)}{(f-1)f} - \sum_{j=\tilde\sigma_i+1}^{n}\Big[-\sum_{f=\tilde\sigma_j}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f)-C(\tilde\theta_f,\ell)}{(f-1)f} + \sum_{f=\tilde\sigma_j}^{n-1}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_{f+1},f)-C(\tilde\theta_{f+1},\ell)}{(f-1)f}\Big]\\
&= -n\sum_{p=1}^{\tilde\sigma_i}\frac{C(\theta_i,p)}{\tilde\sigma_i} - n\sum_{f=\tilde\sigma_i+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f)-C(\tilde\theta_f,\ell)}{(f-1)f} + C(\theta_i,\tilde\sigma_i) + \tilde\sigma_i\sum_{f=\tilde\sigma_i+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f)-C(\tilde\theta_f,\ell)}{(f-1)f}\\
&\quad - (\tilde\sigma_i-1)\sum_{f=\tilde\sigma_i}^{n-1}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_{f+1},f)-C(\tilde\theta_{f+1},\ell)}{(f-1)f} - \sum_{j=\tilde\sigma_i+1}^{n}\Big[-\sum_{f=\tilde\sigma_j}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f)-C(\tilde\theta_f,\ell)}{(f-1)f} + \sum_{f=\tilde\sigma_j}^{n-1}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_{f+1},f)-C(\tilde\theta_{f+1},\ell)}{(f-1)f}\Big].
\end{aligned} \tag{1.38}$$

$$
\begin{aligned}
&(1.36)+(1.38)\\
=&-\sum_{j=\tilde\sigma_i}^{n} C(\tilde\theta_j,j)+\sum_{j=\tilde\sigma_i+1}^{n} C(\tilde\theta_j,j-1)\\
&-n\sum_{p=1}^{\tilde\sigma_i}\frac{C(\theta_i,p)}{\tilde\sigma_i}-n\sum_{f=\tilde\sigma_i+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f)-C(\tilde\theta_f,\ell)}{(f-1)f}+C(\theta_i,\tilde\sigma_i)+\tilde\sigma_i\sum_{f=\tilde\sigma_i+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f)-C(\tilde\theta_f,\ell)}{(f-1)f}\\
&-(\tilde\sigma_i-1)\sum_{f=\tilde\sigma_i}^{n-1}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_{f+1},f)-C(\tilde\theta_{f+1},\ell)}{(f-1)f}-\sum_{j=\tilde\sigma_i+1}^{n}\Big[-\sum_{f=\tilde\sigma_j}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f)-C(\tilde\theta_f,\ell)}{(f-1)f}+\sum_{f=\tilde\sigma_j}^{n-1}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_{f+1},f)-C(\tilde\theta_{f+1},\ell)}{(f-1)f}\Big]\\
=&-n\sum_{p=1}^{\tilde\sigma_i}\frac{C(\theta_i,p)}{\tilde\sigma_i}-n\sum_{f=\tilde\sigma_i+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f)-C(\tilde\theta_f,\ell)}{(f-1)f}-\sum_{j=\tilde\sigma_i+1}^{n} C(\tilde\theta_j,j)+\sum_{j=\tilde\sigma_i+1}^{n} C(\tilde\theta_j,j-1)+\tilde\sigma_i\sum_{f=\tilde\sigma_i+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f)-C(\tilde\theta_f,\ell)}{(f-1)f}\\
&-(\tilde\sigma_i-1)\sum_{f=\tilde\sigma_i}^{n-1}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_{f+1},f)-C(\tilde\theta_{f+1},\ell)}{(f-1)f}-\sum_{j=\tilde\sigma_i+1}^{n}\Big[-\sum_{f=\tilde\sigma_j}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f)-C(\tilde\theta_f,\ell)}{(f-1)f}+\sum_{f=\tilde\sigma_j}^{n-1}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_{f+1},f)-C(\tilde\theta_{f+1},\ell)}{(f-1)f}\Big].
\end{aligned}
\tag{1.39}
$$

Next we show that the equation from the third term is zero.

$$
\begin{aligned}
&-\sum_{j=\tilde\sigma_i+1}^{n} C(\tilde\theta_j,j)+\sum_{j=\tilde\sigma_i+1}^{n} C(\tilde\theta_j,j-1)+\tilde\sigma_i\sum_{f=\tilde\sigma_i+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f)-C(\tilde\theta_f,\ell)}{(f-1)f}\\
&-(\tilde\sigma_i-1)\sum_{f=\tilde\sigma_i}^{n-1}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_{f+1},f)-C(\tilde\theta_{f+1},\ell)}{(f-1)f}-\sum_{j=\tilde\sigma_i+1}^{n}\Big[-\sum_{f=\tilde\sigma_j}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f)-C(\tilde\theta_f,\ell)}{(f-1)f}+\sum_{f=\tilde\sigma_j}^{n-1}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_{f+1},f)-C(\tilde\theta_{f+1},\ell)}{(f-1)f}\Big]\\
=&-\sum_{j=\tilde\sigma_i+1}^{n} C(\tilde\theta_j,j)+\sum_{j=\tilde\sigma_i+1}^{n} C(\tilde\theta_j,j-1)+\tilde\sigma_i\sum_{f=\tilde\sigma_i+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f)-C(\tilde\theta_f,\ell)}{(f-1)f}\\
&-(\tilde\sigma_i-1)\sum_{f=\tilde\sigma_i+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f-1)-C(\tilde\theta_f,\ell)}{(f-2)(f-1)}+\sum_{j=\tilde\sigma_i+1}^{n}\sum_{f=\tilde\sigma_j}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f)-C(\tilde\theta_f,\ell)}{(f-1)f}-\sum_{j=\tilde\sigma_i+1}^{n}\sum_{f=\tilde\sigma_j+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f-1)-C(\tilde\theta_f,\ell)}{(f-2)(f-1)}\\
=&-\sum_{j=\tilde\sigma_i+1}^{n} C(\tilde\theta_j,j)+\sum_{j=\tilde\sigma_i+1}^{n} C(\tilde\theta_j,j-1)+\tilde\sigma_i\sum_{f=\tilde\sigma_i+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f)}{(f-1)f}-\tilde\sigma_i\sum_{f=\tilde\sigma_i+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,\ell)}{(f-1)f}\\
&-(\tilde\sigma_i-1)\sum_{f=\tilde\sigma_i+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f-1)}{(f-2)(f-1)}+(\tilde\sigma_i-1)\sum_{f=\tilde\sigma_i+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,\ell)}{(f-2)(f-1)}+\sum_{j=\tilde\sigma_i+1}^{n}\sum_{f=\tilde\sigma_j}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f)}{(f-1)f}\\
&-\sum_{j=\tilde\sigma_i+1}^{n}\sum_{f=\tilde\sigma_j}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,\ell)}{(f-1)f}-\sum_{j=\tilde\sigma_i+1}^{n}\sum_{f=\tilde\sigma_j+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,f-1)}{(f-2)(f-1)}+\sum_{j=\tilde\sigma_i+1}^{n}\sum_{f=\tilde\sigma_j+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,\ell)}{(f-2)(f-1)}\\
=&-\sum_{j=\tilde\sigma_i+1}^{n} C(\tilde\theta_j,j)+\sum_{j=\tilde\sigma_i+1}^{n} C(\tilde\theta_j,j-1)+\tilde\sigma_i\sum_{f=\tilde\sigma_i+1}^{n}\frac{C(\tilde\theta_f,f)}{f}-\tilde\sigma_i\sum_{f=\tilde\sigma_i+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,\ell)}{(f-1)f}\\
&-(\tilde\sigma_i-1)\sum_{f=\tilde\sigma_i+1}^{n}\frac{C(\tilde\theta_f,f-1)}{f-2}+(\tilde\sigma_i-1)\sum_{f=\tilde\sigma_i+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,\ell)}{(f-2)(f-1)}+\sum_{j=\tilde\sigma_i+1}^{n}\sum_{f=\tilde\sigma_j}^{n}\frac{C(\tilde\theta_f,f)}{f}\\
&-\sum_{j=\tilde\sigma_i+1}^{n}\sum_{f=\tilde\sigma_j}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,\ell)}{(f-1)f}-\sum_{j=\tilde\sigma_i+1}^{n}\sum_{f=\tilde\sigma_j+1}^{n}\frac{C(\tilde\theta_f,f-1)}{f-2}+\sum_{j=\tilde\sigma_i+1}^{n}\sum_{f=\tilde\sigma_j+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,\ell)}{(f-2)(f-1)}\\
=&-\sum_{j=\tilde\sigma_i+1}^{n} C(\tilde\theta_j,j)+\sum_{j=\tilde\sigma_i+1}^{n} C(\tilde\theta_j,j-1)\\
&+\tilde\sigma_i\sum_{f=\tilde\sigma_i+1}^{n}\frac{C(\tilde\theta_f,f)}{f}+\sum_{j=\tilde\sigma_i+1}^{n}\sum_{f=\tilde\sigma_j}^{n}\frac{C(\tilde\theta_f,f)}{f}-(\tilde\sigma_i-1)\sum_{f=\tilde\sigma_i+1}^{n}\frac{C(\tilde\theta_f,f-1)}{f-2}-\sum_{j=\tilde\sigma_i+1}^{n}\sum_{f=\tilde\sigma_j+1}^{n}\frac{C(\tilde\theta_f,f-1)}{f-2}\\
&+(\tilde\sigma_i-1)\sum_{f=\tilde\sigma_i+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,\ell)}{(f-2)(f-1)}+\sum_{j=\tilde\sigma_i+1}^{n}\sum_{f=\tilde\sigma_j+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,\ell)}{(f-2)(f-1)}-\tilde\sigma_i\sum_{f=\tilde\sigma_i+1}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,\ell)}{(f-1)f}-\sum_{j=\tilde\sigma_i+1}^{n}\sum_{f=\tilde\sigma_j}^{n}\sum_{\ell=1}^{f-1}\frac{C(\tilde\theta_f,\ell)}{(f-1)f}
\end{aligned}
$$

$$=-\sum_{j=\tilde{\sigma}_i+1}^{n} C(\tilde{\theta}_j, j) + \sum_{j=\tilde{\sigma}_i+1}^{n} C(\tilde{\theta}_j, j-1) + \sum_{f=\tilde{\sigma}_i+1}^{n} C(\tilde{\theta}_f, f) - \sum_{f=\tilde{\sigma}_i+1}^{n} C(\tilde{\theta}_f, f-1) + \sum_{f=\tilde{\sigma}_i+1}^{n} \sum_{\ell=1}^{f-1} \frac{C(\tilde{\theta}_f, \ell)}{f-1} - \sum_{f=\tilde{\sigma}_i+1}^{n} \sum_{\ell=1}^{f-1} \frac{C(\tilde{\theta}_f, \ell)}{f-1}$$
$$=0. \tag{1.40}$$

The second last equation holds since

$$\begin{aligned}
&\tilde{\sigma}_i \sum_{f=\tilde{\sigma}_i+1}^{n} \frac{C(\tilde{\theta}_f, f)}{f} + \sum_{j=\tilde{\sigma}_i+1}^{n} \sum_{f=\tilde{\sigma}_j}^{n} \frac{C(\tilde{\theta}_f, f)}{f} \\
=&\tilde{\sigma}_i \frac{C(\tilde{\theta}_{\tilde{\sigma}_i+1}, \tilde{\sigma}_i+1)}{\tilde{\sigma}_i+1} + \tilde{\sigma}_i \frac{C(\tilde{\theta}_{\tilde{\sigma}_i+2}, \tilde{\sigma}_i+2)}{\tilde{\sigma}_i+2} + \cdots + \tilde{\sigma}_i \frac{C(\tilde{\theta}_n, n)}{n} \\
&+ \frac{C(\tilde{\theta}_{\tilde{\sigma}_i+1}, \tilde{\sigma}_i+1)}{\tilde{\sigma}_i+1} + 2\frac{C(\tilde{\theta}_{\tilde{\sigma}_i+2}, \tilde{\sigma}_i+2)}{\tilde{\sigma}_i+2} + \cdots + (n-\tilde{\sigma}_i)\frac{C(\tilde{\theta}_n, n)}{n} \\
=& \sum_{f=\tilde{\sigma}_i+1}^{n} f \frac{C(\tilde{\theta}_f, f)}{f} \\
=& \sum_{f=\tilde{\sigma}_i+1}^{n} C(\tilde{\theta}_f, f),
\end{aligned} \tag{1.41}$$

and

$$\begin{aligned}
&(\tilde{\sigma}_i-1) \sum_{f=\tilde{\sigma}_i+1}^{n} \frac{C(\tilde{\theta}_f, f-1)}{f-2} + \sum_{j=\tilde{\sigma}_i+1}^{n} \sum_{f=\tilde{\sigma}_j+1}^{n} \frac{C(\tilde{\theta}_f, f-1)}{f-2} \\
=&(\tilde{\sigma}_i-1)\frac{C(\tilde{\theta}_{\tilde{\sigma}_i+1}, \tilde{\sigma}_i)}{\tilde{\sigma}_i-1} + (\tilde{\sigma}_i-1)\frac{C(\tilde{\theta}_{\tilde{\sigma}_i+2}, \tilde{\sigma}_i+1)}{\tilde{\sigma}_i} + \cdots + (\tilde{\sigma}_i-1)\frac{C(\tilde{\theta}_n, n-1)}{n-2} \\
&\qquad + \frac{C(\tilde{\theta}_{\tilde{\sigma}_i+2}, \tilde{\sigma}_i+1)}{\tilde{\sigma}_i} + \cdots + (n-\tilde{\sigma}_i-1)\frac{C(\tilde{\theta}_n, n-1)}{n-2} \\
=& \sum_{f=\tilde{\sigma}_i+1}^{n} (f-2)\frac{C(\tilde{\theta}_f, f-1)}{f-2} \\
=& \sum_{f=\tilde{\sigma}_i+1}^{n} C(\tilde{\theta}_f, f-1).
\end{aligned} \tag{1.42}$$

Therefore, we have the desired result.

$$\begin{aligned}
n \cdot u_i(\sigma, t; \theta, C) &= -n\sum_{p=1}^{\tilde{\sigma}_i} \frac{C(\theta_i, p)}{\tilde{\sigma}_i} - n \sum_{f=\tilde{\sigma}_i+1}^{n} \sum_{\ell=1}^{f-1} \frac{C(\tilde{\theta}_f, f) - C(\tilde{\theta}_f, \ell)}{(f-1)f} \\
\therefore u_i(\sigma, t; \theta, C) &= -\sum_{p=1}^{\tilde{\sigma}_i} \frac{C(\theta_i, p)}{\tilde{\sigma}_i} - \sum_{f=\tilde{\sigma}_i+1}^{n} \sum_{\ell=1}^{f-1} \frac{C(\tilde{\theta}_f, f) - C(\tilde{\theta}_f, \ell)}{(f-1)f},
\end{aligned} \tag{1.43}$$

By Pareto indifference, the result holds for all $\sigma \in E(\theta, C)$.

□

Remark 1.3. *Similar to Theorems 1.3, 1.4 and 1.5, it can be shown that the maximal transfer rule is the only rule satisfying efficiency, Pareto indifference, equal treatment of equals, and independence of smaller costs together; the only rule satisfying efficiency, Pareto indifference, identical preferences lower bound, positive cost monotonicity, and the largest cost agent equal responsibility together; and the only rule satisfying efficiency, Pareto indifference, and balanced consistency with external priority together.*

1.6 Concluding remarks

We conclude the paper with a discussion on some properties.

As in the standard queueing game, the optimistic positional queueing game is concave, and thus, the minimal transfer rule belongs to the anti-core of the game.

However, differently from the standard queueing game, the pessimistic positional queueing game is balanced, but not convex. Let σ be such that $\theta_{\sigma_1} \geq \cdots \geq \theta_{\sigma_n}$ and for each $i \in N$, $\varphi_i^1(\theta, C) = -C(\theta_i, \sigma_i)$. Since $\varphi^1(\theta, C)$ belongs to the core, the pessimistic positional queueing game is balanced.

Let $N = \{1, 2, 3\}$ and θ be such that $\theta_1 > \theta_2 > \theta_3$. Let $C(\theta_1, 3) = 1$ and 0 otherwise. It is easy to check that this cost function is supermodular with respect to the type and the position and non-decreasing with respect to each argument. The worth of each coalition is $v_p(\{1\}) = -1$ and $v_p(S) = 0$ for all $S \subseteq N$, $S \neq \{1\}$. Now choose $S = \{1, 2, 3\}$ and $T = \{1, 2\}$. Then, $v_p(S) - v_p(S \backslash \{2\}) = 0$ and $v_p(T) - v_p(T \backslash \{2\}) = 1$, which proves that this game is not convex. Moreover, the maximal transfer rule assigns $\varphi_1^X(\theta, C) = -\frac{1}{3}$ and $\varphi_2^X(\theta, C) = \varphi_3^X(\theta, C) = \frac{1}{6}$ which is not in the core since $v_p(\{1, 2\}) = 0 > \varphi_1^X(\theta, C) + \varphi_2^X(\theta, C) = -\frac{1}{6}$.

For the standard queueing problem, Chun and Hokari (2007) and Kar, Mitra and Mutuswami (2009) show the coincidence of the Shapley value and the prenucleolus (Schmeidler 1969) for the optimistic game and the coincidence of the Shapley value and the nucleolus (Schmeidler 1969) for the pessimistic game. However, the coincidence of the two solutions is not guaranteed for positional queueing problems. Note that these results are based on the 2-additivity of the standard queueing game, that is, the worth of any coalition with more than two agents can be expressed as a sum of the worths of all two-person coalitions.

It is easy to check that 2-additivity is not satisfied even for three agent problem. Let $N = \{1, 2, 3\}$ and θ be such that $\theta_1 > \theta_2 > \theta_3$. Since $v_p(N) = -C(\theta_1, 1) - C(\theta_2, 2) - C(\theta_3, 3)$ is not necessarily equal to $v_p(\{1, 2\}) + v_p(\{1, 3\}) + v_p(\{2, 3\}) = -\{C(\theta_1, 1) + C(\theta_2, 2)\} - \{C(\theta_1, 1) + C(\theta_3, 2)\} - \{C(\theta_2, 1) + C(\theta_3, 2)\}$, the positional queueing game does not satisfy 2-additivity.

Appendix A: The minimal transfer rule

To prove the theorems, we use the following three lemmas.

Lemma 1. For any two positive integers i and k such that $k \leq i$, we have

$$\sum_{s=k}^{i}(-1)^{s+k+1}\frac{1}{s}\binom{i-1}{s-1}\binom{s-1}{k-1} = -\frac{1}{i}.$$

Proof. Let i and k be two positive integers such that $k \leq i$. We first show that the equality holds for $k = i$. If $k = i$, then

$$\sum_{s=k}^{i}(-1)^{s+k+1}\frac{1}{s}\binom{i-1}{s-1}\binom{s-1}{k-1} = (-1)^{2i+1}\frac{1}{i}\binom{i-1}{i-1}\binom{i-1}{i-1} = -\frac{1}{i},$$

as desired.

Next, as induction hypothesis, suppose that the equality holds for all k' such that $k \leq k' \leq i$. We show that it also holds for $k-1$. From the induction hypothesis,

$$\sum_{s=k}^{i}(-1)^{s+k+1}\frac{1}{s}\binom{i-1}{s-1}\binom{s-1}{k-1} = -\frac{1}{i}.$$

Now we consider the equation for $k-1$, where the first equality is obtained from Pascal's formula.

$$\begin{aligned}
& \sum_{s=k-1}^{i}(-1)^{s+k}\frac{1}{s}\binom{i-1}{s-1}\binom{s-1}{k-2} \\
= & \sum_{s=k-1}^{i}(-1)^{s+k}\frac{1}{s}\binom{i-1}{s-1}\left[\binom{s}{k-1}-\binom{s-1}{k-1}\right] \\
= & \sum_{s=k-1}^{i}(-1)^{s+k}\frac{1}{s}\binom{i-1}{s-1}\binom{s}{k-1} + \sum_{s=k-1}^{i}(-1)^{s+k}\frac{1}{s}\binom{i-1}{s-1}(-1)\binom{s-1}{k-1} \\
= & \sum_{s=k-1}^{i}(-1)^{s+k}\frac{1}{s}\binom{i-1}{s-1}\binom{s}{k-1} + \sum_{s=k}^{i}(-1)^{s+k+1}\frac{1}{s}\binom{i-1}{s-1}\binom{s-1}{k-1}.
\end{aligned}$$

From the induction hypothesis, the second term is equal to $-\frac{1}{i}$. By using the binomial theorem, the first term can be simplified to:

$$
\begin{aligned}
& \sum_{s=k-1}^{i} (-1)^{s+k} \frac{1}{s} \binom{i-1}{s-1} \binom{s}{k-1} \\
= \quad & \sum_{s=k-1}^{i} (-1)^{s+k} \frac{(i-1)!}{(i-s)!(s-k+1)!(k-1)!} \\
= \quad & \sum_{s=k-1}^{i} (-1)^{s+k} \frac{(i-k+1)!}{(i-s)!(s-k+1)!} \frac{(i-1) \times \cdots \times (i-k+2)}{(k-1)!} \\
= \quad & \frac{(i-1) \times \cdots \times (i-k+2)}{(k-1)!} \sum_{s=k-1}^{i} (-1)^{s+k} \binom{i-k+1}{s-k+1} \\
= \quad & \frac{(i-1) \times \cdots \times (i-k+2)}{(k-1)!} \sum_{m=0}^{i-k+1} (-1)^{m+2k-1} \binom{i-k+1}{m} \\
= \quad & \frac{(i-1) \times \cdots \times (i-k+2)}{(k-1)!} (-1)^{2k-1} (1-1)^{i-k+1} \\
= \quad & 0.
\end{aligned}
$$

Altogether, we obtain the desired conclusion. □

Lemma 2. *For each $j \in N$ and any two positive integers p and q, we have*

$$
\sum_{k=1}^{p} \frac{C(\theta_j, k)}{p} - \sum_{k=1}^{p+q} \frac{C(\theta_j, k)}{p+q} = - \sum_{k=p+1}^{p+q} \sum_{\ell=1}^{k} \frac{C(\theta_j, k) - C(\theta_j, \ell)}{(k-1)k}.
$$

Proof. Let $j \in N$ and p, q be two positive integers. We prove by induction. First, we show that the equality holds when $q = 1$. If $q = 1$, then

$$
\begin{aligned}
& \sum_{k=1}^{p} \frac{C(\theta_j, k)}{p} - \sum_{k=1}^{p+1} \frac{C(\theta_j, k)}{p+1} + \sum_{\ell=1}^{p+1} \frac{C(\theta_j, p+1) - C(\theta_j, \ell)}{p(p+1)} \\
= \quad & \frac{1}{p(p+1)} \Big[\sum_{k=1}^{p} C(\theta_j, k)(p+1) - \sum_{k=1}^{p+1} C(\theta_j, k)p + \sum_{\ell=1}^{p+1} C(\theta_j, p+1) - \sum_{\ell=1}^{p+1} C(\theta_j, \ell) \Big] \\
= \quad & \frac{1}{p(p+1)} \Big[\sum_{k=1}^{p} C(\theta_j, k)(p+1) - \sum_{k=1}^{p+1} C(\theta_j, k)p + \sum_{\ell=1}^{p+1} C(\theta_j, p+1) - \sum_{\ell=1}^{p} C(\theta_j, \ell) - C(\theta_j, p+1) \Big] \\
= \quad & \frac{1}{p(p+1)} \Big[\sum_{k=1}^{p} C(\theta_j, k)(p+1) - \sum_{\ell=1}^{p} C(\theta_j, \ell) - \sum_{k=1}^{p+1} C(\theta_j, k)p + \sum_{\ell=1}^{p+1} C(\theta_j, p+1) - C(\theta_j, p+1) \Big] \\
= \quad & \frac{1}{p(p+1)} \Big[\sum_{k=1}^{p} C(\theta_j, k)p - \sum_{k=1}^{p+1} C(\theta_j, k)p + \sum_{\ell=1}^{p} C(\theta_j, p+1) \Big] \\
= \quad & \frac{1}{p(p+1)} \Big[-C(\theta_j, p+1)p + pC(\theta_j, p+1) \Big] \\
= \quad & 0.
\end{aligned}
$$

Next, as induction hypothesis, suppose the equation holds for all $\tilde{q}$ such that $1 \leq \tilde{q} \leq q$. Then, from the induction hypothesis,

$$\sum_{k=1}^{p} \frac{C(\theta_j, k)}{p} - \sum_{k=1}^{p+q} \frac{C(\theta_j, k)}{p+q} + \sum_{k=p+1}^{p+q} \sum_{\ell=1}^{k} \frac{C(\theta_j, k) - C(\theta_j, \ell)}{(k-1)k} = 0. \tag{1.44}$$

It is enough to show that the equation also holds for $q+1$, that is,

$$\sum_{k=1}^{p} \frac{C(\theta_j, k)}{p} - \sum_{k=1}^{p+q+1} \frac{C(\theta_j, k)}{p+q+1} + \sum_{k=p+1}^{p+q+1} \sum_{\ell=1}^{k} \frac{C(\theta_j, k) - C(\theta_j, \ell)}{(k-1)k} = 0. \tag{1.45}$$

Now we consider the following equation:

$$\begin{aligned}
& [(1.44) - (1.45)] \times (p+q)(p+q+1) \\
= \; & -(p+q+1) \sum_{k=1}^{p+q} C(\theta_j, k) + (p+q) \sum_{k=1}^{p+q+1} C(\theta_j, k) - \sum_{k=p+q+1}^{p+q+1} \sum_{\ell=1}^{k} \frac{C(\theta_j, k) - C(\theta_j, \ell)}{(k-1)k}(p+q)(p+q+1) \\
= \; & -(p+q+1) \sum_{k=1}^{p+q} C(\theta_j, k) + (p+q) \sum_{k=1}^{p+q+1} C(\theta_j, k) - \sum_{\ell=1}^{p+q+1} \big(C(\theta_j, p+q+1) - C(\theta_j, \ell)\big) \\
= \; & -(p+q+1) \sum_{k=1}^{p+q} C(\theta_j, k) + (p+q) \sum_{k=1}^{p+q+1} C(\theta_j, k) - \sum_{\ell=1}^{p+q+1} C(\theta_j, p+q+1) + \sum_{\ell=1}^{p+q+1} C(\theta_j, \ell) \\
= \; & -(p+q+1) \sum_{k=1}^{p+q} C(\theta_j, k) + (p+q+1) \sum_{k=1}^{p+q+1} C(\theta_j, k) - \sum_{\ell=1}^{p+q+1} C(\theta_j, p+q+1) \\
= \; & (p+q+1)C(\theta_j, p+q+1) - (p+q+1)C(\theta_j, p+q+1) \\
= \; & 0,
\end{aligned}$$

the desired conclusion. □

Lemma 3. *For each $j \in N$ and any two positive integers p and q such that $p+q \leq n$, we have*

$$\sum_{k=1}^{p} \frac{C(\theta_j, n-k+1)}{p} - \sum_{k=1}^{p+q} \frac{C(\theta_j, n-k+1)}{p+q} = - \sum_{k=n-p-q+1}^{n-p} \sum_{\ell=k}^{n} \frac{C(\theta_j, k) - C(\theta_j, \ell)}{(n-k)(n-k+1)}.$$

Proof. Let $j \in N$ and p and q be two positive integers such that $p+q \leq n$. For each k, let $\tilde{C}(\theta_j, k) = C(\theta_j, n-k+1)$. By Lemma 2,

$$\begin{aligned}
& \sum_{k=1}^{p} \frac{C(\theta_j, n-k+1)}{p} - \sum_{k=1}^{p+q} \frac{C(\theta_j, n-k+1)}{p+q} \\
= \; & \sum_{k=1}^{p} \frac{\tilde{C}(\theta_j, k)}{p} - \sum_{k=1}^{p+q} \frac{\tilde{C}(\theta_j, k)}{p+q} \\
= \; & - \sum_{k=p+1}^{p+q} \sum_{\ell=1}^{k} \frac{\tilde{C}(\theta_j, k) - \tilde{C}(\theta_j, \ell)}{(k-1)k} \\
= \; & - \sum_{k=p+1}^{p+q} \sum_{\ell=1}^{k} \frac{C(\theta_j, n-k+1) - C(\theta_j, n-\ell+1)}{(k-1)k}.
\end{aligned}$$

Let $m = n - k + 1$. Then,

$$
\begin{aligned}
& -\sum_{k=p+1}^{p+q}\sum_{\ell=1}^{k}\frac{C(\theta_j, n-k+1) - C(\theta_j, n-\ell+1)}{(k-1)k} \\
= & -\sum_{m=n-p-q+1}^{n-p}\sum_{\ell=1}^{k}\frac{C(\theta_j, m) - C(\theta_j, n-\ell+1)}{(n-m)(n-m+1)} \\
= & -\sum_{m=n-p-q+1}^{n-p}\sum_{\ell=m}^{n}\frac{C(\theta_j, m) - C(\theta_j, \ell)}{(n-m)(n-m+1)},
\end{aligned}
$$

as desired. □

Next, we calculate the unanimity coefficient of the optimistic positional queueing game.

Proposition 1.6. For the optimistic positional queueing game v_o, the unanimity coefficient λ_{v_o} is given by: for each $S \subseteq N$ such that $|S| = s$,

$$\lambda_{v_o}(S) = \sum_{k=1}^{s}(-1)^{s+k+1}\binom{s-1}{k-1}C(\underline{\theta}_S, k),$$

where $\underline{\theta}_S$ is the smallest type of agents in S.

Proof. For each game v_o and each $S \subseteq N$, the unanimity coefficient can be calculated as follows:

$$\lambda_{v_o}(S) = \sum_{T\subseteq S}(-1)^{s-t}v_o(T),$$

where $s = |S|$ and $t = |T|$.

For each $S \subseteq N$, let $i \in S$ be such that $\theta_i = \underline{\theta}_S$. Then, for each $T \subseteq S$ with $T \ni i$, $v_o(T) = v_o(T \setminus \{i\}) - C(\underline{\theta}_S, t)$. Therefore,

$$
\begin{aligned}
\lambda_{v_o}(S) &= \sum_{T\subseteq S}(-1)^{s-t}v_o(T) \\
&= \sum_{T\subseteq S, i\notin T}(-1)^{s-t}v_o(T) + \sum_{T\subseteq S, i\in T}(-1)^{s-t}v_o(T) \\
&= \sum_{T\subseteq S, i\notin T}(-1)^{s-t}v_o(T) + \sum_{T\subseteq S, i\in T}(-1)^{s-t}[v_o(T\setminus\{i\}) - C(\underline{\theta}_S, t)] \\
&= \sum_{T\subseteq S, i\notin T}(-1)^{s-t}v_o(T) + \sum_{T\subseteq S, i\notin T}(-1)^{s-t-1}v_o(T) - \sum_{T\subseteq S, i\in T}(-1)^{s-t}C(\underline{\theta}_S, t) \\
&= \sum_{T\subseteq S, i\in T}(-1)^{s-t+1}C(\underline{\theta}_S, t) \\
&= \sum_{t=1}^{s}\binom{s-1}{t-1}(-1)^{s-t+1}C(\underline{\theta}_S, t) \\
&= \sum_{t=1}^{s}\binom{s-1}{t-1}(-1)^{s+t+1}C(\underline{\theta}_S, t),
\end{aligned}
$$

as desired. □

Proof of Theorem 1:

Proof. We show that for each $i \in N$,

$$\phi_i^{SV}(v_o) = -\sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i, k)}{\hat{p}_i + \hat{e}_i} - \sum_{\theta_0 \in \Theta(\hat{F}, i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_0, k) - C(\theta_0, \ell)}{k}. \quad (1.46)$$

It is well-known that the assignment of the Shapley value to agent i can be calculated by using the unanimity coefficients as follows:

$$\phi_i^{SV}(v_o) = \sum_{S \subseteq N,\, S \ni i} \frac{\lambda_{v_o}(S)}{|S|}.$$

From Proposition 1.6,

$$\sum_{S \subseteq N,\, S \ni i} \frac{\lambda_{v_o}(S)}{|S|} = \sum_{S \subseteq N,\, S \ni i} \sum_{k=1}^{s} (-1)^{s+k+1} \frac{1}{s} \binom{s-1}{k-1} C(\underline{\theta}_S, k),$$

We partition the set of coalitions including agent i, $\mathcal{S}_i$, into two subsets: $\mathcal{S}_i^1 = \{S \in \mathcal{S}_i | \underline{\theta}_S = \theta_i\}$ and $\mathcal{S}_i^2 = \{S \in \mathcal{S}_i | \underline{\theta}_S < \theta_i\}$. Also, for each $\theta_0 \in \Theta(\hat{F}, i)$, $\mathcal{S}_0^1 = \{S \subseteq N | \underline{\theta}_S = \theta_0\}$.

Case 1.1: $S \in \mathcal{S}_i^1$.

$$\begin{aligned}
& \sum_{S \in \mathcal{S}_i^1} \sum_{k=1}^{s} (-1)^{s+k+1} \frac{1}{s} \binom{s-1}{k-1} C(\underline{\theta}_S, k) \\
= & \sum_{s=1}^{\hat{p}_i+\hat{e}_i} \sum_{S \in \mathcal{S}_i^1, |S|=s} \sum_{k=1}^{s} (-1)^{s+k+1} \frac{1}{s} \binom{s-1}{k-1} C(\theta_i, k) \\
= & \sum_{s=1}^{\hat{p}_i+\hat{e}_i} \binom{\hat{p}_i+\hat{e}_i-1}{s-1} \sum_{k=1}^{s} (-1)^{s+k+1} \frac{1}{s} \binom{s-1}{k-1} C(\theta_i, k) \\
= & \sum_{s=1}^{\hat{p}_i+\hat{e}_i} \sum_{k=1}^{s} (-1)^{s+k+1} \frac{1}{s} \binom{\hat{p}_i+\hat{e}_i-1}{s-1} \binom{s-1}{k-1} C(\theta_i, k) \\
= & \sum_{k=1}^{\hat{p}_i+\hat{e}_i} \sum_{s=k}^{\hat{p}_i+\hat{e}_i} (-1)^{s+k+1} \frac{1}{s} \binom{\hat{p}_i+\hat{e}_i-1}{s-1} \binom{s-1}{k-1} C(\theta_i, k) \\
= & \sum_{k=1}^{\hat{p}_i+\hat{e}_i} C(\theta_i, k) \sum_{s=k}^{\hat{p}_i+\hat{e}_i} (-1)^{s+k+1} \frac{1}{s} \binom{\hat{p}_i+\hat{e}_i-1}{s-1} \binom{s-1}{k-1} \\
= & -\sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i, k)}{\hat{p}_i+\hat{e}_i},
\end{aligned}$$

where the last equality comes from Lemma 1.

Case 1.2: $S \in \mathcal{S}_i^2$.

$$
\begin{aligned}
& \sum_{S\in\mathcal{S}_i^2}\sum_{k=1}^{s}(-1)^{s+k+1}\frac{1}{s}\binom{s-1}{k-1}C(\underline{\theta}_S,k) \\
= & \sum_{\theta_0\in\Theta(\hat{F},i)}\sum_{S\in\mathcal{S}_0^1,S\ni i}\sum_{k=1}^{s}(-1)^{s+k+1}\frac{1}{s}\binom{s-1}{k-1}C(\theta_0,k) \\
= & \sum_{\theta_0\in\Theta(\hat{F},i)}\sum_{s=2}^{\hat{p}_0+\hat{e}_0}\sum_{S\in\mathcal{S}_0^1,S\ni i,|S|=s}\sum_{k=1}^{s}(-1)^{s+k+1}\frac{1}{s}\binom{s-1}{k-1}C(\theta_0,k) \\
= & \sum_{\theta_0\in\Theta(\hat{F},i)}\sum_{s=2}^{\hat{p}_0+\hat{e}_0}\left[\binom{\hat{p}_0+\hat{e}_0-1}{s-1}-\binom{\hat{p}_0-1}{s-1}\right]\sum_{k=1}^{s}(-1)^{s+k+1}\frac{1}{s}\binom{s-1}{k-1}C(\theta_0,k) \\
= & \sum_{\theta_0\in\Theta(\hat{F},i)}\sum_{s=2}^{\hat{p}_0+\hat{e}_0}\sum_{k=1}^{s}(-1)^{s+k+1}\frac{1}{s}\left[\binom{\hat{p}_0+\hat{e}_0-1}{s-1}-\binom{\hat{p}_0-1}{s-1}\right]\binom{s-1}{k-1}C(\theta_0,k) \\
= & \sum_{\theta_0\in\Theta(\hat{F},i)}\sum_{k=1}^{\hat{p}_0+\hat{e}_0}\sum_{s=k}^{\hat{p}_0+\hat{e}_0}(-1)^{s+k+1}\frac{1}{s}\left[\binom{\hat{p}_0+\hat{e}_0-1}{s-1}-\binom{\hat{p}_0-1}{s-1}\right]\binom{s-1}{k-1}C(\theta_0,k) \\
= & \sum_{\theta_0\in\Theta(\hat{F},i)}\sum_{k=1}^{\hat{p}_0+\hat{e}_0}\sum_{s=k}^{\hat{p}_0+\hat{e}_0}(-1)^{s+k+1}\frac{1}{s}\binom{\hat{p}_0+\hat{e}_0-1}{s-1}\binom{s-1}{k-1}C(\theta_0,k) \\
& -\sum_{\theta_0\in\Theta(\hat{F},i)}\sum_{k=1}^{\hat{p}_0}\sum_{s=k}^{\hat{p}_0}(-1)^{s+k+1}\frac{1}{s}\binom{\hat{p}_0-1}{s-1}\binom{s-1}{k-1}C(\theta_0,k) \\
= & \sum_{\theta_0\in\Theta(\hat{F},i)}\sum_{k=1}^{\hat{p}_0+\hat{e}_0}C(\theta_0,k)\sum_{s=k}^{\hat{p}_0+\hat{e}_0}(-1)^{s+k+1}\frac{1}{s}\binom{\hat{p}_0+\hat{e}_0-1}{s-1}\binom{s-1}{k-1} \\
& -\sum_{\theta_0\in\Theta(\hat{F},i)}\sum_{k=1}^{\hat{p}_0}C(\theta_0,k)\sum_{s=k}^{\hat{p}_0}(-1)^{s+k+1}\frac{1}{s}\binom{\hat{p}_0-1}{s-1}\binom{s-1}{k-1} \\
= & \sum_{\theta_0\in\Theta(\hat{F},i)}\sum_{k=1}^{\hat{p}_0+\hat{e}_0}C(\theta_0,k)(-\frac{1}{\hat{p}_0+\hat{e}_0})-\sum_{\theta_0\in\Theta(\hat{F},i)}\sum_{k=1}^{\hat{p}_0}C(\theta_0,k)(-\frac{1}{\hat{p}_0}) \quad (*) \\
= & \sum_{\theta_0\in\Theta(\hat{F},i)}\left[\sum_{k=1}^{\hat{p}_0}\frac{C(\theta_0,k)}{\hat{p}_0}-\sum_{k=1}^{\hat{p}_0+\hat{e}_0}\frac{C(\theta_0,k)}{\hat{p}_0+\hat{e}_0}\right],
\end{aligned}
$$

where the 8th equality (with $(*)$) is obtained from Lemma 1.

Altogether,

$$\begin{aligned}
\phi_i^{SV}(v_o) &= \sum_{i\in S\subseteq N} \frac{\lambda_S(v)}{|S|} \\
&= -\sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i,k)}{\hat{p}_i+\hat{e}_i} - \sum_{\theta_0\in\Theta(\hat{F},i)} \left[\sum_{k=1}^{\hat{p}_0+\hat{e}_0} \frac{C(\theta_0,k)}{\hat{p}_0+\hat{e}_0} - \sum_{k=1}^{\hat{p}_0} \frac{C(\theta_0,k)}{\hat{p}_0}\right] \\
&= -\sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i,k)}{\hat{p}_i+\hat{e}_i} - \sum_{\theta_0\in\Theta(\hat{F},i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_0,k)-C(\theta_0,\ell)}{k},
\end{aligned}$$

where the last equality holds by Lemma 2.

Therefore, the transfer to agent i in v_o can be expressed as:

$$\begin{aligned}
t_i^M &= C(\theta_i,\sigma_i^M) + \phi_i^{SV}(v_o) \\
&= C(\theta_i,\sigma_i^M) - \sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i,k)}{\hat{p}_i+\hat{e}_i} - \sum_{\theta_0\in\Theta(\hat{F},i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_0,k)-C(\theta_0,\ell)}{k} \\
&= \sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i,\sigma_i^M)-C(\theta_i,k)}{\hat{p}_i+\hat{e}_i} - \sum_{\theta_0\in\Theta(\hat{F},i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_0,k)-C(\theta_0,\ell)}{k},
\end{aligned}$$

the desired expression. □

Proof of Corollary 1:

Proof. Let $\sigma^M \in \tilde{E}(\theta, C)$. By Theorem 1.1,

$$t_i^M = \sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i,\sigma_i^M)-C(\theta_i,k)}{\hat{p}_i+\hat{e}_i} - \sum_{\theta_0\in\Theta(\hat{F},i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_0,k)-C(\theta_0,\ell)}{k}. \tag{1.47}$$

For each $i \in N$, by definition of $\hat{p}_i$ and $\hat{e}_i$, $\hat{p}_i < \sigma_i^M \leq \hat{p}_i + \hat{e}_i$. Let e be a positive integer such that $\sigma_i^M = \hat{p}_i + e$. Then, $1 \leq e \leq \hat{e}_i$.

Suppose that $e < \hat{e}_i$. By Lemma 2,

$$\begin{aligned}
\sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i,k)}{\hat{p}_i+\hat{e}_i} &= \sum_{k=1}^{\hat{p}_i+e} \frac{C(\theta_i,k)}{\hat{p}_i+e} - \sum_{k=1}^{\hat{p}_i+e} \frac{C(\theta_i,k)}{\hat{p}_i+e} + \sum_{k=1}^{\hat{p}_i+\hat{e}_i} \frac{C(\theta_i,k)}{\hat{p}_i+\hat{e}_i} \\
&= \sum_{k=1}^{\hat{p}_i+e} \frac{C(\theta_i,k)}{\hat{p}_i+e} + \sum_{k=\hat{p}_i+e+1}^{\hat{p}_i+\hat{e}_i} \sum_{\ell=1}^{k} \frac{C(\theta_i,k)-C(\theta_i,\ell)}{(k-1)k}. \qquad (1.48)
\end{aligned}$$

If $e = \hat{e}_i$, then we use equation (1.47) without any modification.

By substituting equation (1.48) into equation (1.47), we have

$$\begin{aligned} t_i^M &= C(\theta_i,\sigma_i^M) - \sum_{k=1}^{\widehat{p_i}+\widehat{e_i}} \frac{C(\theta_i,k)}{\hat{p}_i+\hat{e}_i} - \sum_{\theta_0\in\Theta(\hat{F},i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_0,k)-C(\theta_0,\ell)}{k} \\ &= C(\theta_i,\sigma_i^M) - \sum_{k=1}^{\hat{p}_i+e} \frac{C(\theta_i,k)}{\hat{p}_i+e} - \sum_{k=\hat{p}_i+e+1}^{\hat{p}_i+\hat{e}_i} \sum_{\ell=1}^{k} \frac{C(\theta_i,k)-C(\theta_i,\ell)}{(k-1)k} \\ &\quad - \sum_{\theta_0\in\Theta(\hat{F},i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{k-1} \sum_{\ell=1}^{k} \frac{C(\theta_0,k)-C(\theta_0,\ell)}{k}. \end{aligned} \tag{1.49}$$

Note that if $\sigma^M \in \tilde{E}(\theta,C)$, then each $j \in \hat{P}(\theta_i)$ belongs to $P_i(\sigma^M)$ and each $j \in \hat{F}(\theta_i)$ belongs to $F_i(\sigma^M)$. In addition, each $j \in \hat{E}(\theta_i)$ such that $\sigma_j^M = \hat{p}_i+1,\ldots,\hat{p}_i+(e-1)$ belongs to $P_i(\sigma^M)$ and each $j \in \hat{E}(\theta_i)$ such that $\sigma_j^M = \hat{p}_i+(e+1),\ldots,\hat{p}_i+\hat{e}_i$ belongs to $F_i(\sigma^M)$. Altogether, equation (1.49) can be rewritten as:

$$\begin{aligned} & t_i^M \\ = \;& C(\theta_i,\sigma_i^M) - \sum_{k=1}^{\hat{p}_i+e} \frac{C(\theta_i,k)}{\hat{p}_i+e} - \sum_{k=\hat{p}_i+e+1}^{\hat{p}_i+\hat{e}_i} \sum_{\ell=1}^{k} \frac{C(\theta_i,k)-C(\theta_i,\ell)}{(k-1)k} \\ & - \sum_{\theta_0\in\Theta(\hat{F},i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \sum_{\ell=1}^{k} \frac{C(\theta_0,k)-C(\theta_0,\ell)}{(k-1)k} \\ = \;& C(\theta_i,\sigma_i^M) - \sum_{p\in P_i(\sigma^M)\cup\{i\}} \frac{C(\theta_i,\sigma_p^M)}{\sigma_i^M} - \sum_{f\in F_i(\sigma^M)} \sum_{\ell=1}^{\sigma_f^M} \frac{C(\theta_f,\sigma_f^M)-C(\theta_f,\ell)}{(\sigma_f^M-1)\sigma_f^M} \\ = \;& \sum_{p\in P_i(\sigma^M)\cup\{i\}} \frac{C(\theta_i,\sigma_i^M)-C(\theta_i,\sigma_p^M)}{\sigma_i^M} - \sum_{f\in F_i(\sigma^M)} \sum_{\ell\in P_f(\sigma^M)\cup\{f\}} \frac{C(\theta_f,\sigma_f^M)-C(\theta_f,\sigma_\ell^M)}{(\sigma_f^M-1)\sigma_f^M} \\ = \;& \sum_{p\in P_i(\sigma^M)} \frac{C(\theta_i,\sigma_i^M)-C(\theta_i,\sigma_p^M)}{\sigma_i^M} - \sum_{f\in F_i(\sigma^M)} \frac{1}{\sigma_f^M-1} \sum_{\ell\in P_f(\sigma^M)} \frac{C(\theta_f,\sigma_f^M)-C(\theta_f,\sigma_\ell^M)}{\sigma_f^M}, \end{aligned}$$

the desired expression. □

Appendix B: The maximal transfer rule

Once again, we begin with calculating the unanimity coefficient of the pessimistic positional queueing game.

Proposition 1.7. For the pessimistic positional queueing game v_p, the unanimity coefficient is given by: for each $S \subseteq N$ such that $|S| = s$,

$$\lambda_{v_p}(S) = \sum_{k=1}^{s} (-1)^{s+k+1} \binom{s-1}{k-1} C(\overline{\theta}_S, n-k+1),$$

where $\overline{\theta}_S$ is the largest type of agents in S.

Proof. For each game v_p and each $S \subseteq N$, the unanimity coefficient can be calculated as follows:

$$\lambda_{v_p}(S) = \sum_{T \subseteq S} (-1)^{s-t} v_p(T,$$

where $s = |S|$ and $t = |T|$.

For each $S \subseteq N$, let $i \in S$ be such that $\theta_i = \overline{\theta}_S$. Then, for each $T \subseteq S$ with $T \ni i$, $v_p(T) = v_p(T \setminus \{i\}) - C(\overline{\theta}_S, n - t + 1)$. Therefore,

$$\begin{aligned}
\lambda_{v_p}(S) &= \sum_{T \subseteq S} (-1)^{s-t} v_p(T) \\
&= \sum_{T \subseteq S, i \notin T} (-1)^{s-t} v_p(T) + \sum_{T \subseteq S, i \in T} (-1)^{s-t} v_p(T) \\
&= \sum_{T \subseteq S, i \notin T} (-1)^{s-t} v_p(T) + \sum_{T \subseteq S, i \in T} (-1)^{s-t} [v_p(T \setminus \{i\}) - C(\overline{\theta}_S, n - t + 1)] \\
&= \sum_{T \subseteq S, i \notin T} (-1)^{s-t} v_p(T) + \sum_{T \subseteq S, i \notin T} (-1)^{s-t-1} v_p(T) - \sum_{T \subseteq S, i \in T} (-1)^{s-t} C(\overline{\theta}_S, n - t + 1) \\
&= \sum_{T \subseteq S, i \in T} (-1)^{s-t+1} C(\overline{\theta}_S, n - t + 1) \\
&= \sum_{t=1}^{s} \binom{s-1}{t-1} (-1)^{s-t+1} C(\overline{\theta}_S, n - t + 1) \\
&= \sum_{t=1}^{s} \binom{s-1}{t-1} (-1)^{s+t+1} C(\overline{\theta}_S, n - t + 1),
\end{aligned}$$

as desired. □

Proof of Theorem 2:

Proof. We show that for each $i \in N$,

$$\phi_i^{SV}(v_p) = \sum_{\theta_0 \in \Theta(\hat{P}, i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{n-k} \sum_{\ell=k}^{n} \frac{C(\theta_0, \ell) - C(\theta_0, k)}{n-k+1} - \sum_{k=\hat{p}_i+1}^{n} \frac{C(\theta_i, k)}{\hat{e}_i + \hat{f}_i}.$$

It is well-known that the assignment of the Shapley value to agent i can be calculated by using the unanimity coefficient as follows:

$$\phi_i^{SV}(v_p) = \sum_{S \subseteq N, S \ni i} \frac{\lambda_{v_p}(S)}{|S|}.$$

From Proposition 1.7,

$$\sum_{S \subseteq N, S \ni i} \frac{\lambda_{v_p}(S)}{|S|} = \sum_{S \subseteq N, S \ni i} \sum_{k=1}^{s} (-1)^{s+k+1} \frac{1}{s} \binom{s-1}{k-1} C(\overline{\theta}_S, n - k + 1),$$

We partition the set of coalitions including agent i, $\mathcal{S}_i$, into two subsets: $\mathcal{S}_i^3 = \{S \in \mathcal{S}_i | \overline{\theta}_S = \theta_i\}$ and $\mathcal{S}_i^4 = \{S \in \mathcal{S}_i | \overline{\theta}_S > \theta_i\}$. Also, for each $\theta_0 \in \Theta(\hat{P}, i)$, $\mathcal{S}_0^3 = \{S \subseteq N | \overline{\theta}_S = \theta_0\}$.

Case 2.1: $S \in \mathcal{S}_i^3$.

$$\begin{aligned}
&\sum_{S \in \mathcal{S}_i^3} \sum_{k=1}^{s} (-1)^{s+k+1} \frac{1}{s} \binom{s-1}{k-1} C(\overline{\theta}_S, n-k+1) \\
= &\sum_{s=1}^{\hat{e}_i+\hat{f}_i} \sum_{S \in \mathcal{S}_i^3, |S|=s} \sum_{k=1}^{s} (-1)^{s+k+1} \frac{1}{s} \binom{s-1}{k-1} C(\theta_i, n-k+1) \\
= &\sum_{s=1}^{\hat{e}_i+\hat{f}_i} \binom{\hat{e}_i+\hat{f}_i-1}{s-1} \sum_{k=1}^{s} (-1)^{s+k+1} \frac{1}{s} \binom{s-1}{k-1} C(\theta_i, n-k+1) \\
= &\sum_{s=1}^{\hat{e}_i+\hat{f}_i} \sum_{k=1}^{s} (-1)^{s+k+1} \frac{1}{s} \binom{\hat{e}_i+\hat{f}_i-1}{s-1} \binom{s-1}{k-1} C(\theta_i, n-k+1) \\
= &\sum_{k=1}^{\hat{e}_i+\hat{f}_i} \sum_{s=k}^{\hat{e}_i+\hat{f}_i} (-1)^{s+k+1} \frac{1}{s} \binom{\hat{e}_i+\hat{f}_i-1}{s-1} \binom{s-1}{k-1} C(\theta_i, n-k+1) \\
= &\sum_{k=1}^{\hat{e}_i+\hat{f}_i} C(\theta_i, n-k+1) \sum_{s=k}^{\hat{e}_i+\hat{f}_i} (-1)^{s+k+1} \frac{1}{s} \binom{\hat{e}_i+\hat{f}_i-1}{s-1} \binom{s-1}{k-1} \\
= &-\sum_{k=1}^{\hat{e}_i+\hat{f}_i} \frac{C(\theta_i, n-k+1)}{\hat{e}_i+\hat{f}_i} \quad (*) \\
= &-\sum_{k=\hat{p}_i+1}^{n} \frac{C(\theta_i, k)}{\hat{e}_i+\hat{f}_i},
\end{aligned}$$

where the 6th equality (with (*)) is obtained from Lemma 1.

Case 2.2: $S \in \mathcal{S}_i^4$.

$$
\begin{aligned}
& \sum_{S\in\mathcal{S}_i^4}\sum_{k=1}^{s}(-1)^{s+k+1}\frac{1}{s}\binom{s-1}{k-1}C(\overline{\theta}_S,n-k+1) \\
= & \sum_{\theta_0\in\Theta(\hat{P},i)}\sum_{S\in\mathcal{S}_0^3,S\ni i}\sum_{k=1}^{s}(-1)^{s+k+1}\frac{1}{s}\binom{s-1}{k-1}C(\theta_0,n-k+1) \\
= & \sum_{\theta_0\in\Theta(\hat{P},i)}\sum_{s=2}^{\hat{e}_0+\hat{f}_0}\sum_{S\in\mathcal{S}_0^3,S\ni i,|S|=s}\sum_{k=1}^{s}(-1)^{s+k+1}\frac{1}{s}\binom{s-1}{k-1}C(\theta_0,n-k+1) \\
= & \sum_{\theta_0\in\Theta(\hat{P},i)}\sum_{s=2}^{\hat{e}_0+\hat{f}_0}\left[\binom{\hat{e}_0+\hat{f}_0-1}{s-1}-\binom{\hat{f}_0-1}{s-1}\right]\sum_{k=1}^{s}(-1)^{s+k+1}\frac{1}{s}\binom{s-1}{k-1}C(\theta_0,n-k+1) \\
= & \sum_{\theta_0\in\Theta(\hat{P},i)}\sum_{s=2}^{\hat{e}_0+\hat{f}_0}\sum_{k=1}^{s}(-1)^{s+k+1}\frac{1}{s}\left[\binom{\hat{e}_0+\hat{f}_0-1}{s-1}-\binom{\hat{f}_0-1}{s-1}\right]\binom{s-1}{k-1}C(\theta_0,n-k+1) \\
= & \sum_{\theta_0\in\Theta(\hat{P},i)}\sum_{k=1}^{\hat{e}_0+\hat{f}_0}\sum_{s=k}^{\hat{e}_0+\hat{f}_0}(-1)^{s+k+1}\frac{1}{s}\left[\binom{\hat{e}_0+\hat{f}_0-1}{s-1}-\binom{\hat{f}_0-1}{s-1}\right]\binom{s-1}{k-1}C(\theta_0,n-k+1) \\
= & \sum_{\theta_0\in\Theta(\hat{P},i)}\sum_{k=1}^{\hat{e}_0+\hat{f}_0}\sum_{s=k}^{\hat{e}_0+\hat{f}_0}(-1)^{s+k+1}\frac{1}{s}\binom{\hat{e}_0+\hat{f}_0-1}{s-1}\binom{s-1}{k-1}C(\theta_0,n-k+1) \\
& -\sum_{\theta_0\in\Theta(\hat{P},i)}\sum_{k=1}^{\hat{f}_0}\sum_{s=k}^{\hat{f}_0}(-1)^{s+k+1}\frac{1}{s}\binom{\hat{f}_0-1}{s-1}\binom{s-1}{k-1}C(\theta_0,n-k+1) \\
= & \sum_{\theta_0\in\Theta(\hat{P},i)}\sum_{k=1}^{\hat{e}_0+\hat{f}_0}C(\theta_0,n-k+1)\sum_{s=k}^{\hat{e}_0+\hat{f}_0}(-1)^{s+k+1}\frac{1}{s}\binom{\hat{e}_0+\hat{f}_0-1}{s-1}\binom{s-1}{k-1} \\
& -\sum_{\theta_0\in\Theta(\hat{P},i)}\sum_{k=1}^{\hat{f}_0}C(\theta_0,n-k+1)\sum_{s=k}^{\hat{f}_0}(-1)^{s+k+1}\frac{1}{s}\binom{\hat{f}_0-1}{s-1}\binom{s-1}{k-1} \\
= & \sum_{\theta_0\in\Theta(\hat{P},i)}\sum_{k=1}^{\hat{e}_0+\hat{f}_0}C(\theta_0,n-k+1)(-\frac{1}{\hat{e}_0+\hat{f}_0})-\sum_{\theta_0\in\Theta(\hat{P},i)}\sum_{k=1}^{\hat{f}_0}C(\theta_0,n-k+1)(-\frac{1}{\hat{f}_0}) \quad (*) \\
= & \sum_{\theta_0\in\Theta(\hat{P},i)}\left[\sum_{k=1}^{\hat{f}_0}\frac{C(\theta_0,n-k+1)}{\hat{f}_0}-\sum_{k=1}^{\hat{e}_0+\hat{f}_0}\frac{C(\theta_0,n-k+1)}{\hat{e}_0+\hat{f}_0}\right],
\end{aligned}
$$

where the 9th equality (with $(*)$) is obtained from Lemma 1.

Altogether,

$$\begin{aligned}
\phi_i^{SV}(v_p) &= \sum_{S\subseteq N, S\ni i} \frac{\lambda_{v_p}(S)}{|S|} \\
&= \sum_{\theta_0\in\Theta(\hat{P},i)} \left[\sum_{k=1}^{\hat{f}_0} \frac{C(\theta_0, n-k+1)}{\hat{f}_0} - \sum_{k=1}^{\hat{e}_0+\hat{f}_0} \frac{C(\theta_0, n-k+1)}{\hat{e}_0+\hat{f}_0}\right] - \sum_{k=\hat{p}_i+1}^{n} \frac{C(\theta_i,k)}{\hat{e}_i+\hat{f}_i} \\
&= \sum_{\theta_0\in\Theta(\hat{P},i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{n-k} \sum_{\ell=k}^{n} \frac{C(\theta_0,\ell)-C(\theta_0,k)}{n-k+1} - \sum_{k=\hat{p}_i+1}^{n} \frac{C(\theta_i,k)}{\hat{e}_i+\hat{f}_i},
\end{aligned}$$

where the last equality holds by Lemma 3.

Therefore, the transfer to agent i in v_o can be expressed as:

$$\begin{aligned}
t_i^X &= C(\theta_i, \sigma_i^X) + \phi_i^{SV}(v_p) \\
&= C(\theta_i, \sigma_i^X) + \sum_{\theta_0\in\Theta(\hat{P},i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{n-k} \sum_{\ell=k}^{n} \frac{C(\theta_0,\ell)-C(\theta_0,k)}{n-k+1} - \sum_{k=\hat{p}_i+1}^{n} \frac{C(\theta_i,k)}{\hat{e}_i+\hat{f}_i} \\
&= \sum_{\theta_0\in\Theta(\hat{P},i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{n-k} \sum_{\ell=k}^{n} \frac{C(\theta_0,\ell)-C(\theta_0,k)}{n-k+1} - \sum_{k=\hat{p}_i+1}^{n} \frac{C(\theta_i,k)-C(\theta_i,\sigma_i^X)}{\hat{e}_i+\hat{f}_i},
\end{aligned}$$

the desired expression. □

Proof of Corollary 2:

Proof. Let $\sigma^X \in \tilde{E}(\theta, C)$. By Theorem 1.2,

$$t_i^X = C(\theta_i, \sigma_i^X) + \sum_{\theta_0\in\Theta(\hat{P},i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \frac{1}{n-k} \sum_{\ell=k}^{n} \frac{C(\theta_0,\ell)-C(\theta_0,k)}{n-k+1} - \sum_{k=\hat{p}_i+1}^{n} \frac{C(\theta_i,k)}{\hat{e}_i+\hat{f}_i}. \tag{1.50}$$

For each $i \in N$, by definition of $\hat{p}_i$ and $\hat{e}_i$, $\hat{p}_i < \sigma_i^X \leq \hat{p}_i + \hat{e}_i$. Let e be a positive integer such that $\sigma_i^X = \hat{p}_i + \hat{e}_i - e + 1$. Then, $1 \leq e \leq \hat{e}_i$.

Suppose that $e < \hat{e}_i$. By Lemma 3,

$$\begin{aligned}
&\sum_{k=\hat{p}_i+1}^{n} \frac{C(\theta_i,k)}{\hat{e}_i+\hat{f}_i} \\
&= \sum_{m=1}^{\hat{e}_i+\hat{f}_i} \frac{C(\theta_i,n-m+1)}{\hat{e}_i+\hat{f}_i} \\
&= \sum_{m=1}^{e+\hat{f}_i} \frac{C(\theta_i,n-m+1)}{e+\hat{f}_i} - \sum_{m=1}^{e+\hat{f}_i} \frac{C(\theta_i,n-m+1)}{e+\hat{f}_i} + \sum_{m=1}^{\hat{e}_i+\hat{f}_i} \frac{C(\theta_i,n-m+1)}{\hat{e}_i+\hat{f}_i} \\
&= \sum_{m=1}^{e+\hat{f}_i} \frac{C(\theta_i,n-m+1)}{e+\hat{f}_i} + \sum_{m=n-\hat{e}_i-\hat{f}_i+1}^{n-e-\hat{f}_i} \sum_{\ell=m}^{n} \frac{C(\theta_i,m)-C(\theta_i,\ell)}{(n-m)(n-m+1)} \\
&= \sum_{m=1}^{e+\hat{f}_i} \frac{C(\theta_i,n-m+1)}{e+\hat{f}_i} + \sum_{k=n-\hat{e}_i-\hat{f}_i+1}^{n-e-\hat{f}_i} \sum_{\ell=k}^{n} \frac{C(\theta_i,k)-C(\theta_i,\ell)}{(n-k)(n-k+1)} \\
&= \sum_{k=\hat{p}_i+\hat{e}_i-e+1}^{n} \frac{C(\theta_i,k)}{e+\hat{f}_i} + \sum_{k=\hat{p}_i+1}^{\hat{p}_i+\hat{e}_i-e} \sum_{\ell=k}^{n} \frac{C(\theta_i,k)-C(\theta_i,\ell)}{(n-k)(n-k+1)}. \qquad (1.51)
\end{aligned}$$

If $e = \hat{e}_i$, we use equation (1.50) without any modification.

By substituting equation (1.51) into equation (1.50), we have

$$\begin{aligned}
t_i^X &= C(\theta_i,\sigma_i^X) + \sum_{\theta_0\in\Theta(\hat{P},i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \sum_{\ell=k}^{n} \frac{C(\theta_0,\ell)-C(\theta_0,k)}{(n-k)(n-k+1)} - \sum_{k=\hat{p}_i+1}^{n} \frac{C(\theta_i,k)}{\hat{e}_i+\hat{f}_i} \\
&= C(\theta_i,\sigma_i^X) + \sum_{\theta_0\in\Theta(\hat{P},i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \sum_{\ell=k}^{n} \frac{C(\theta_0,\ell)-C(\theta_0,k)}{(n-k)(n-k+1)} - \sum_{k=\hat{p}_i+1}^{\hat{p}_i+\hat{e}_i-e} \sum_{\ell=k}^{n} \frac{C(\theta_i,k)-C(\theta_i,\ell)}{(n-k)(n-k+1)} \\
&\quad - \sum_{k=\hat{p}_i+\hat{e}_i-e+1}^{n} \frac{C(\theta_i,k)}{e+\hat{f}_i} \\
&= C(\theta_i,\sigma_i^X) + \sum_{\theta_0\in\Theta(\hat{P},i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \sum_{\ell=k}^{n} \frac{C(\theta_0,\ell)-C(\theta_0,k)}{(n-k)(n-k+1)} + \sum_{k=\hat{p}_i+1}^{\hat{p}_i+\hat{e}_i-e} \sum_{\ell=k}^{n} \frac{C(\theta_i,\ell)-C(\theta_i,k)}{(n-k)(n-k+1)} \\
&\quad - \sum_{k=\hat{p}_i+\hat{e}_i-e+1}^{n} \frac{C(\theta_i,k)}{e+\hat{f}_i} \qquad (1.52)
\end{aligned}$$

Note that if $\sigma^X \in \tilde{E}(\theta, C)$, then each $j \in \hat{P}(\theta_i)$ belongs to $P_i(\sigma^X)$ and each $j \in \hat{F}(\theta_i)$ belongs to $F_i(\sigma^X)$. In addition, each $j \in \hat{E}(\theta_i)$ such that $\sigma_j^X = \hat{p}_i + 1, \ldots, \hat{p}_i + (e-1)$ belongs to $P_i(\sigma^X)$ and each $j \in \hat{E}(\theta_i)$ such that $\sigma_j^X = \hat{p}_i + (e+1), \ldots, \hat{p}_i + \hat{e}_i$ belongs to $F_i(\sigma^X)$. Altogether, equation (1.52) can be rewritten as:

$$
\begin{aligned}
t_i^X &= C(\theta_i, \sigma_i^X) + \sum_{\theta_0 \in \Theta(\hat{P}, i)} \sum_{k=\hat{p}_0+1}^{\hat{p}_0+\hat{e}_0} \sum_{\ell=k}^{n} \frac{C(\theta_0, \ell) - C(\theta_0, k)}{(n-k)(n-k+1)} + \sum_{k=\hat{p}_i+1}^{\hat{p}_i+\hat{e}_i-e} \sum_{\ell=k}^{n} \frac{C(\theta_i, \ell) - C(\theta_i, k)}{(n-k)(n-k+1)} \\
&\quad - \sum_{k=\hat{p}_i+\hat{e}_i-e+1}^{n} \frac{C(\theta_i, k)}{e + \hat{f}_i} \\
&= C(\theta_i, \sigma_i^X) + \sum_{p \in P_i(\sigma^X)} \sum_{\ell=\sigma_p^X}^{n} \frac{C(\theta_p, \ell) - C(\theta_p, \sigma_p^X)}{(n-\sigma_p^X)(n-\sigma_p^X+1)} - \sum_{f \in F_i(\sigma^X) \cup \{i\}} \frac{C(\theta_i, \sigma_f^X)}{n - \hat{p}_i - \hat{e}_i + e} \\
&= C(\theta_i, \sigma_i^X) + \sum_{p \in P_i(\sigma^X)} \sum_{\ell \in F_p(\sigma^X)} \frac{C(\theta_p, \sigma_\ell^X) - C(\theta_p, \sigma_p^X)}{(n-\sigma_p^X)(n-\sigma_p^X+1)} - \sum_{f \in F_i(\sigma^X) \cup \{i\}} \frac{C(\theta_i, \sigma_f^X)}{n - \sigma_i^X + 1} \\
&= \sum_{p \in P_i(\sigma^X)} \sum_{\ell \in F_p(\sigma^X)} \frac{C(\theta_p, \sigma_\ell^X) - C(\theta_p, \sigma_p^X)}{(n-\sigma_p^X)(n-\sigma_p^X+1)} - \sum_{f \in F_i(\sigma^X) \cup \{i\}} \frac{C(\theta_i, \sigma_f^X) - C(\theta_i, \sigma_i^X)}{n - \sigma_i^X + 1} \\
&= \sum_{p \in P_i(\sigma^X)} \frac{1}{n-\sigma_p^X} \sum_{\ell \in F_p(\sigma^X)} \frac{C(\theta_p, \sigma_\ell^X) - C(\theta_p, \sigma_p^X)}{n-\sigma_p^X+1} - \sum_{f \in F_i(\sigma^X)} \frac{C(\theta_i, \sigma_f^X) - C(\theta_i, \sigma_i^X)}{n - \sigma_i^X + 1},
\end{aligned}
$$

the desired expression. □

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Chapter 2

The Kar Solution for multi-source minimum cost spanning tree problems

2.1 Introduction

We consider a minimum cost spanning tree problem with multiple sources (hereafter, *m-mcstp*). There are a number of sources that provide services. A group of agents want to connect to these sources. An agent does not care if her connection to the source is direct or indirect, however she needs to connect to all of the sources. A connection entails a cost.

There are two objectives in such a problem: (1) how to construct a spanning network over all agents and sources which entails a minimal cost and (2) how to allocate the cost of the minimal network to each agent. There are many algorithms for the first problem, for instance, the Kruskal and Prim algorithm (Kruskal (1956), Prim (1957)).

As for the classical (single source) minimum cost spanning tree problems, there are many literatures regarding the allocation problem. The classical minimum cost spanning tree problem was first introduced by Claus and Kleitman (1973). Bird (1976) found an allocation rule which is at the core of the associated minimum cost spanning tree game. Granot and Huberman (1984) investigated the nucleolus and Kar (2002) investigated the Shapley value of the minimum cost spanning tree problem. Dutta and Kar (2004) introduced the Dutta-Kar rule, which is cost monotonic. Chun and Lee (2009) introduced the sequential contributions solution. Trudeau (2012) introduced the cycle-complete solution. The most widely-known solution for classical minimum cost spanning tree problems is the *folk rule*, which is studied separately as the Shapley value of the irreducible game (Bergantinos and Vidal-Puga (2007)), as an obligation rule (Tijs et al. (2006)), as a partition rule (Bergantinos et al. (2010)), and as a cone-wise decomposition calculation (Branzei et al. (2004)).

Bergantinos et al. (2019) introduced *m-mcstp* and studied the folk rule. We investigate the Kar rule, the Shapley value of the stand alone cost game. The objective of this chapter is to investigate the properties and provide an axiomatic characterization of the Kar rule for the *m-mcstp*.

This chapter is organized as follows. Section 2 introduces the minimum cost spanning tree problem with multiple sources (*m-mcstp*) and introduces the Kar rule. Section 3 studies the properties of the Kar rule and characterizes it axiomatically. In Section 4, we conclude this study with some remarks.

2.2 Model

2.2.1 Minimum cost spanning tree problems with multiple sources

Let $N = \{1, \cdots, n\}$ be a set of agents and let $M = \{s_1, \cdots, s_m\}$ be a set of sources. We call each element of $N \cup M$ a node. Given a node set $N \cup M$, a *cost matrix* $C = (c_{ij})_{i,j \in N \cup M}$ represents the cost of the direct link between any pair of nodes, with $c_{ij} = c_{ji} \geq 0$ and $c_{ii} = 0$, for all $i, j \in N \cup M$. The set of all cost matrices over $N \cup M$ is denoted by $\mathcal{C}_{N \cup M}$.

A *minimum cost spanning tree problem with multiple sources (m-mcstp)* is a triple (N, M, C) where N is a set of agents, M is a set of sources and C is a cost matrix over $N \cup M$.

A *graph* g over $N \cup M$ is a subset of a complete graph $\{(ij) | \forall i, j \in N \cup M, i \neq j\}$, whose element is a *link*. A *path* from i to j in g is a sequence of different links $\{(i_{k-1}i_k)\}_{k=1}^K$ such that $(i_{k-1}i_k) \in g$ for all $k \in \{1, 2, \cdots, K\}$, $i_0 = i$ and $i_K = j$. A *cycle* is a sequence of different links $\{(i_{k-1}i_k)\}_{k=1}^K$ such that $(i_{k-1}i_k) \in g$ for all $k \in \{1, 2, \cdots, K\}$, $i_0 = i_K$. Two distinct nodes i, j are *connected in* g if there exists a path from i to j in g, and a graph g is *connected* if all pairs of nodes are connected in g. A *tree* is a connected graph with a unique path from any node to another node. For any tree t, let t_{ij} be the unique path from i to j in t.

For any graph g over $N \cup M$, the construction cost of g is $c(N, M, C, g) = \sum_{(i,j) \in g} c_{ij}$. If there is no ambiguity, we use $c(g)$ instead of $c(N, M, C, g)$.

For any coalition $S \subseteq N$, let Γ_S be the set of all connected graphs with node set $S \cup M$. A minimum cost spanning tree (a minimal tree) over $S \cup M$, denoted by t_S, has the property

$$t_S = \underset{g \in \Gamma_S}{\text{argmin}} \Big[\sum_{(ij) \in g} c_{ij} \Big],$$

and let $c(S, M, C) = c(S) = c(t_S)$. According to the definition of t_S, $t_\emptyset$ is the tree network among the sources. However, we let $t_\emptyset = \emptyset$ and $c(\emptyset) = 0$ conventionally. Instead, we let t_M be the minimal tree over the sources only.

A minimum cost spanning tree game with multiple sources is a cost game (N, c) associated with a problem (N, M, C).

2.2.2 Rule

A cost allocation rule φ is a map associated with each problem (N, M, C) a vector of a cost shares $\varphi(N, M, C) \in \mathbb{R}^N$, where $\mathbb{R}$ is the set of real numbers. In a classical problem, Kar (2002) defines the Kar rule as the Shapley value (Shapley (1953)) of a minimum cost spanning tree game. Similarly we define the *(modified) Kar rule* as the Shapley value of *m-mcstp*. Given *m-mcstp* (N, M, C), we define the Kar rule as below:

$$\phi_i(N, M, C) = \sum_{s=1}^{n} \frac{(s-1)!(n-s)!}{n!} \Big[\sum_{i \in S \subseteq N, |S| = s} (c(S, M, C) - c(S \backslash \{i\}, M, C)) \Big] \ \forall i \in N.$$

Example 2.1. *Let there are two agents* $\{1, 2\}$ *and two sources* $\{s_1, s_2\}$*. Since* $c(\{1\}) = 5$*,* $c(\{2\}) = 9$*,* $c(\{1, 2\}) = 6$*, the Kar rule assigns 1 to agent 1 and 5 to agent 2.*

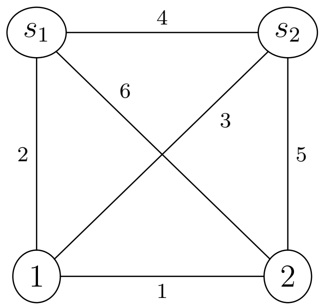

Figure 2.1: The Kar rule example

Here we introduce the definition of the folk rule to compare it with the Kar rule. There are many ways to define the folk rule (see Bergantinos et al. (2019)), here we use the definition *the Shapley value regarding the irreducible matrix*. Given an *m-mcstp*, let t_N be the minimal tree. The associated *irreducible matrix* of C is a cost matrix C^* with the cost $c^*_{ij} = \max_{(k,l) \in t_{N,ij}} c_{kl}$ where $t_{N,ij}$ denotes the unique path from i to j in t_N. The *m-mcstp* (N, M, C^*) is the irreducible problem of (N, M, C). Given an *m-mcstp* (N, M, C), the folk rule is defined as the Shapley value of irreducible minimum cost spanning tree problems with multiple sources. That is,

$$f_i(N, M, C) = \phi_i(N, M, C^*),$$

where C^* is the associated irreducible matrix of C.

Figure 2.2 shows the irreducible network of figure 2.1. The links with double lines comprise the unique minimal tree of the original game and each cost of the dotted lines are reduced according to the definition of the irreducible matrix. Since $c(\{1\}) = 5$, $c(\{2\}) = 5$, $c(\{1, 2\}) = 6$ in this case, the folk rule assigns 3 to agent 1 and 3 to agent 2.

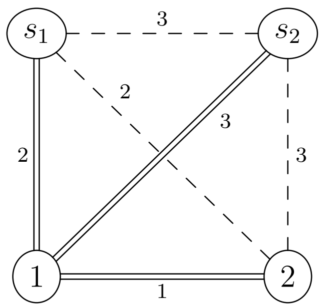

Figure 2.2: The folk rule example

2.3 An axiomatic characterization

2.3.1 Axioms

Kar (2002) used four axioms (*efficiency*, *absence of cross-subsidization*, *group independence*, and *equal treatment*) to characterize the Kar rule. We first look at those axioms and modify them if required. Next, we compare the Kar rule with the folk rule by investigating the axioms studied in Bergantinos et al. (2019).

Efficiency requires that the sum of the allocations for all agents is same as the construction cost of the network.

Axiom 2.1 (Efficiency). $\sum_{i \in N} \varphi_i(N, M, C) = c(N)$

Absence of cross-subsidization is similar to the definition of Kar (2002), however we modify the definition so it can be applied to *m-mcstp*. As for the classical problem, *absence of cross-subsidization* requires that if the minimum cost spanning tree is a *star-graph with a source in the center*, each agent pays her own connection cost to the source. In other words, if there is no positive synergy when constructing the network cooperatively, each agent pays her own cost. For example, let figure 2.3 be a minimal tree, then *absence of cross-subsidization* requires that $\varphi_i = c_{is}$ for all $i \in N$.

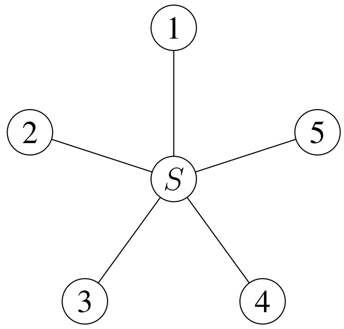

Figure 2.3: Star graph with a source in the center

If there are more than one source, a star-graph with a source in the center cannot happen. Therefore, we have to modify the concept of star-graph as well as the definition of *absence of cross-subsidization* axiom in order to apply it to *m-mcstp*.

Here we introduce the concept of *source cluster*. Suppose that given a minimum cost spanning tree, $t(N)$, the path from any source to another source, $t_{s_i s_j}$ $(s_i, s_j \in M,\ s_i \neq s_j)$, consists of links of a complete graph over M. It means that the sources are close enough so that the construction of the sources' network does not benefit from any help of agents. In this case, we name the efficient network of sources a *source cluster*. If there exists a *source cluster*, no positive synergy generated when constructing the minimal cost spanning tree. In this situation, total construction cost can be divided into two parts: (1) the cost of *source cluster* and (2) the cost of agents connecting to the *source cluster*.

A *star graph with a source cluster in the center* (hereafter, *modified star graph*) is a graph such that there exists a source cluster and all agents are directly linked to the source cluster (see Figure 2.4).

Absence of cross-subsidization requires that if the minimal tree t_N is a *modified star graph*, then each agent pays the construction cost of the *source cluster* equally and pays her own connection cost to the *source cluster*. Similar to the original definition of the absence of cross-subsidization, modified absence of cross-subsidization rules out cross-subsidization if there is no positive synergy to construct the spanning network cooperatively. Let $c(M)$ be the construction cost of *source cluster*, t_M, and let $c_{iM} = \min_{s \in M} c_{is}$.

Axiom 2.2 (Absence of cross-subsidization). *If a star graph with a source cluster in the center is a minimal tree, then* $\varphi_i(N, M, C) = \frac{c(M)}{n} + c_{iM}$ *for all* $i \in N$.

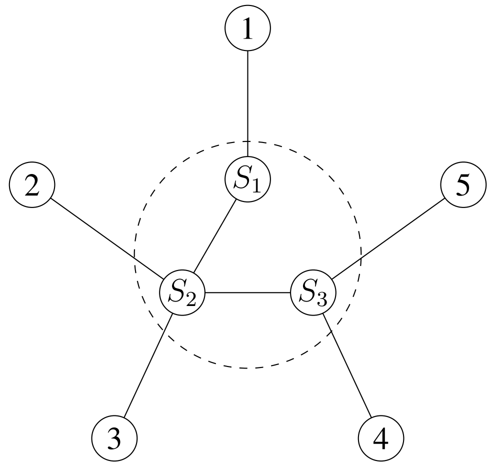

Figure 2.4: Star graph with a source cluster in the center

In order to introduce *group independence*, we define *irrelevant link*. For the classical minimum cost spanning tree problem, Kar (2002) defined the link between two agents i and j, (ij), is *irrelevant* if $c_{ij} > \max\{c_{is}, c_{js}\}$. It cannot be used when forming a minimal tree. We define a link (ij) is *irrelevant* if $c_{ij} > \max\{\min_{s\in M} c_{is}, \min_{s\in M} c_{js}, \max_{(pq)\in t_M} c_{pq}\}$. The first element gives agent i a better option of connecting directly to a source than connect to j, the second element gives agent j a better option of connecting to a source than connect to i, and the third element ensures the option of connecting to the source is better. A link which is not irrelevant is a *relevant* link.

A *relevant path* from i to j in g is a path where all links forming the path are *relevant* links. Suppose that the set of agents has partitions, $N = [N_1, \cdots, N_p]$, such that

(i) $i \in N_k,\ j \in N_k \Rightarrow$ there is a relevant path from i to j in N_k,

(ii) $i \in N_k,\ j \notin N_k \Rightarrow$ there is no relevant path from i to j in N.

Then an element of the partition is called a *group*.

Group independence requires that a change in the cost of a link within a group does not affect the cost allocation of an agent in another group.

Axiom 2.3 (Group independence)**.** *Given an m-mcstp (N, M, C), let $N = [N_1, \cdots, N_p]$ where each $N_k (1 \le k \le p)$ is a group. Let $i, j \in N_l$ and C' is the cost matrix where $c_{ij} \neq c'_{ij}$ and $c_{kl} = c'_{kl}$ for all $(kl) \neq (ij)$. Then,*

$$\varphi_k(N, M, C) = \varphi_k(N, M, C') \ \ \forall k \in N_t, \forall t \neq l.$$

Finally, *balanced contribution*, which was named as *equal treatment* in Kar (2002), requires that if a cost of a link between agent i and j changes, the cost allocations assigned to agent i and j change equally.

Axiom 2.4 (Balanced contribution)**.** *Given an m-mcstp (N, M, C), let $i, j \in N$. Let C' be a cost matrix such that $c_{ij} \neq c'_{ij}$ and $c_{kl} = c'_{kl}$ for all $(kl) \neq (ij)$. Then,*

$$\varphi_i(N, M, C) - \varphi_i(N, M, C') = \varphi_j(N, M, C) - \varphi_j(N, M, C').$$

Proposition 2.1. *The Kar rule satisfies efficiency, modified absence of cross-subsidization, modified group independence, and balanced contribution.*

Proof. First, it is trivial that the Kar rule satisfies *efficiency*.

Second, we show that the Kar rule satisfies *modified absence of cross-subsidization*. Let $\pi \in \Pi$ be an order of agents, where Π is the set of all permutations of N. Let $S_{\pi(i)} = \{k|\pi(k) < \pi(i)\}$. Suppose that given a problem (N, M, C), a modified star graph is a minimal tree corresponding to the problem. Then,

$$c(S_{\pi(i)} \cup i) - c(S_{\pi(i)}) = \begin{cases} c(M) + \min_{s \in M} c_{is}, & \text{if } S_{\pi(i)} = \emptyset, \\ \min_{s \in M} c_{is}, & \text{otherwise,} \end{cases}$$

where $c(M)$ is the construction cost of $t(\emptyset)$. Therefore, $\phi_i(N, M, C) = \frac{c(M)}{n} + \min_{s \in M} c_{is} \ \forall i \in N$.

Third, we show that the Kar rule satisfies *group independence*. Given an *m-mcstp* (N, M, C), let $N = [N_1, \cdots, N_p]$ be a partition. Let C' be a cost matrix $c_{ij} \neq c'_{ij}\ i, j \in N_l$ and the other link costs are the same as C. We show that $\phi_k(N, M, C) = \phi_k(N, M, C')\ \forall k \in N_t, t \neq l$.

We will show that $c(S) - c(S \setminus \{k\}) = c'(S) - c'(S \setminus \{k\})$ for all S such that $k \in S \subseteq N$. Without loss of generality, let $c'_{ij} = c_{ij} + \epsilon,\ \epsilon > 0$.

According to Spira and Pan (1975), when the cost of a link changes the minimal tree can be found as below:

(1) If the value of a tree link increases, consider all links running between the two sub-trees formed by deleting the link whose weight has increased. The new tree will be the union of the sub-trees and the connecting link of minimum weight.

(2) If the value of a non-tree link increases, the link of decreased weight appears in the new tree if and only if it is no longer the maximum-weight link in the cycle it forms when added to the old minimum spanning tree.

If i or j is not in S, $t(S, M, C') = t(S, M, C)$ since there exists no cost change (Case I). Consider a coalition S such that $i, j, k \in S \subseteq N$.

(1) If (ij) is a link of a minimum cost spanning tree for $S \cup M$, delete (ij) and find the alternative connecting link. If (ij) is still the connecting link of minimum weight, then $t(S, M, C') = t(S, M, C)$ (Case II). If there exists an alternative link with a weight less than $c_{ij} + \epsilon$, $t(S, M, C') \neq t(S, M, C)$ (Case III).

(2) If (ij) is not a link of a minimum cost spanning tree for $S \cup M$, $t(S, M, C') = t(S, M, C)$ (Case IV).

For Case II and Case IV, the minimum cost spanning tree is the same as (S, M, C) and (S, M, C'). For Case III, from the definition of irrelevant link, the alternative link cannot use the nodes in N_t.[1] Therefore, in any cases the tree structure in $N_t \cup M$ does not change between $t(S, M, C)$ and $t(S, M, C')$.

If a node in a minimum cost spanning tree deleted, the alternative minimum cost spanning tree can be constructed by connecting the sub-trees with minimum weighted

[1] The alternative link connects two nodes in N_l or one node in N_l and one source.

links which do not make cycle sequentially. For any coalition S, $k \in S \subseteq N$, consider the relationship between two trees $t(S, M, C)$ and $t(S \setminus \{k\}, M, C)$. $t(S \setminus \{k\}, M, C)$ can be found by deleting node k from $t(S, M, C)$ and connecting the sub-trees sequentially. When connecting the sub-trees to each other, we never use the inter-group connecting links from the definition of *irrelevant* link. Therefore deleting k makes changes only in the network of $N_t \cup M$. As we have already shown that $t(S, M, C)$ and $t(S, M, C')$ have the same tree structure within $N_t \cup M$ and $C_{N_t \cup M} = C'_{N_t \cup M}$ by definition, deleting k results in the same cost changes for both $t(S, M, C)$ and $t(S, M, C')$. Thus, $c(S) - c(S \setminus \{k\}) = c'(S) - c'(S \setminus \{k\})$ holds for any S, and therefore $\phi_k(N, M, C) = \phi_k(N, M, C') \;\; \forall k \in N_t, t \neq l$.

Finally, we show that the Kar rule satisfies *balanced contribution*. Given an *m-mcstp* (N, M, C), let C' be a cost matrix $c_{ij} \neq c'_{ij}$ $i, j \in N_l$ and the other link costs are the same as C. From the definition of the Kar rule,

$$\phi_i(N, M, C) = \sum_{s=1}^{n} \frac{(s-1)!(n-s)!}{n!} \Big[\sum_{i \in S \subseteq N, |S|=s} (c(S, M, C) - c(S \setminus \{i\}, M, C)) \Big],$$

$$\phi_i(N, M, C') = \sum_{s=1}^{n} \frac{(s-1)!(n-s)!}{n!} \Big[\sum_{i \in S \subseteq N, |S|=s} (c(S, M, C') - c(S \setminus \{i\}, M, C')) \Big].$$

And,

$$\phi_j(N, M, C) = \sum_{s=1}^{n} \frac{(s-1)!(n-s)!}{n!} \Big[\sum_{j \in S \subseteq N, |S|=s} (c(S, M, C) - c(S \setminus \{j\}, M, C)) \Big],$$

$$\phi_j(N, M, C') = \sum_{s=1}^{n} \frac{(s-1)!(n-s)!}{n!} \Big[\sum_{j \in S \subseteq N, |S|=s} (c(S, M, C') - c(S \setminus \{j\}, M, C')) \Big].$$

We want to show that for any s, $1 \leq s \leq n$,

$$\begin{aligned} &[\sum_{i \in S \subseteq N, |S|=s} (c(S, M, C) - c(S \setminus \{i\}, M, C))] - [\sum_{i \in S \subseteq N, |S|=s} (c(S, M, C') - c(S \setminus \{i\}, M, C'))] \\ =&[\sum_{j \in S \subseteq N, |S|=s} (c(S, M, C) - c(S \setminus \{j\}, M, C))] - [\sum_{j \in S \subseteq N, |S|=s} (c(S, M, C') - c(S \setminus \{j\}, M, C'))]. \end{aligned} \tag{2.1}$$

If i or j is not in S, $c(S, M, C') = c(S, M, C)$ since there exists no cost changes. Therefore, $c(S \setminus \{i\}, M, C') = c(S \setminus \{i\}, M, C)$ for all S, $i \in S \subseteq N$. Similarly, $c(S \setminus \{j\}, M, C') = c(S \setminus \{j\}, M, C)$ for all S, $j \in S \subseteq N$. Thus the equation (2.1) changes as below:

$$\sum_{i \in S \subseteq N, |S|=s} c(S, M, C) - \sum_{i \in S \subseteq N, |S|=s} c(S, M, C') = \sum_{j \in S \subseteq N, |S|=s} c(S, M, C) - \sum_{j \in S \subseteq N, |S|=s} c(S, M, C'). \tag{2.2}$$

Let $S_{i\wedge j}$ be coalition set such that $i \in S$ and $j \in S$. Similarly, let $S_{\neg i\wedge j}$ be coalition set such that $i \notin S$ and $j \in S$, let $S_{i\wedge\neg j}$ be coalition set such that $i \in S$ and $j \notin S$, and let $S_{\neg i\wedge\neg j}$ be coalition set such that $i \notin S$ and $j \notin S$. The equation (2.2) would be

$$\begin{aligned}
&\sum_{i\in S\subseteq N,|S|=s} c(S,M,C) - \sum_{i\in S\subseteq N,|S|=s} c(S,M,C') - \sum_{j\in S\subseteq N,|S|=s} c(S,M,C) + \sum_{j\in S\subseteq N,|S|=s} c(S,M,C') \\
=& \sum_{S\in S_{i\wedge j},|S|=s} c(S,M,C) + \sum_{S\in S_{i\wedge\neg j},|S|=s} c(S,M,C) - \sum_{S\in S_{i\wedge j},|S|=s} c(S,M,C') - \sum_{S\in S_{i\wedge\neg j},|S|=s} c(S,M,C') \\
-& \sum_{S\in S_{i\wedge j},|S|=s} c(S,M,C) - \sum_{S\in S_{\neg i\wedge j},|S|=s} c(S,M,C) + \sum_{S\in S_{i\wedge j},|S|=s} c(S,M,C') + \sum_{S\in S_{\neg i\wedge j},|S|=s} c(S,M,C') \\
=&0
\end{aligned}$$

The last equation holds since $\forall\, S \in S_{i\wedge\neg j} \cup S_{\neg i\wedge j},\ c(S,M,C) = c(S,M,C')$. Therefore, the Kar rule satisfies *equal treatment*.

□

Now we compare the Kar rule and the folk rule which is investigated in Bergantinos et al. (2019). They investigated some axioms the folk rule satisfies.

Core selection requires that no coalition of agents has incentive to deviate. *Population monotonicity* requires that if a new agent joins the problem, no agent in the original game should be worse off. *Independence of irrelevant trees* requires a cost allocation rule should depend only on the links which belong to a minimal tree.

Axiom 2.5 (Core selection)**.** *For each (N,M,C) and for all $S \subseteq N$, $\sum_{i\in S}\varphi_i(N,M,C) \leq c(S,M,C)$.*

Axiom 2.6 (Population monotonicity)**.** *For each (N,M,C), for all $i \in N$, and for all $i \in S \subseteq N$, $\varphi_j(S,M,C) \leq \varphi_j(S\setminus\{i\},M,C)$ for all $j \in S\setminus\{i\}$.*

Axiom 2.7 (Independence of irrelevant trees)**.** *For each (N,M,C) and (N,M,C'), if they have a common minimal tree t such that $c_{ij} = c'_{ij}$ for all $(ij) \in t$, then $\varphi_i(N,M,C) = \varphi_i(N,M,C')$ for all $i \in N$.*

Cost monotonicity is related to cost change of a link. It is defined by two kind of different concepts under the same name. The first concept requires that the cost of a link between two agent increases, the cost allocation for each of the two agent should not decrease. The first one can be found at Dutta and Kar (2004), Bergantinos and Vidal-Puga (2007), Trudeau (2013). We name the first type of cost monotonicity as *link cost monotonicity*. The second one requires that if some link costs increase then no agent should not be better off. The second one can be found at Bergantinos et al. (2019), Bergantinos and Navarro-Ramos (2019). We name the second type of cost monotonicity as *matrix cost monotonicity*. It is easy to check that matrix cost monotonicity implies link cost monotonicity.

Axiom 2.8 (Link cost monotonicity)**.** *For each (N,M,C) and (N,M,C') such that for any $i,j \in N$, $c'_{ij} < c_{ij}$ and $c_{kl} = c_{kl}$ for all $(kl) \neq (ij)$. Then, $\varphi_i(N,M,C') \leq \varphi_i(N,M,C)$ and $\varphi_j(N,M,C') \leq \varphi_j(N,M,C)$.*

Axiom 2.9 (Matrix cost monotonicity)**.** *For each* (N, M, C) *and* (N, M, C') *if* $C \leq C'$*, then* $\varphi_i(N, M, C) \leq \varphi_i(N, M, C')$*.*

Equal treatment of source costs requires that if a cost between two sources increases, then all agents should be affected by the same amount.

Axiom 2.10 (Equal treatment of source costs)**.** *For each* (N, M, C) *and* (N, M, C') *and each* $a, b \in M$*, if for each* $k, l \in M \cup N$ *such that* $\{k, l\} \neq \{a, b\}$*,* $c_{kl} = c'_{kl}$*, then for each* $i, j \in N$*,* $\varphi_i(N, M, C) - \varphi_i(N, M, C') = \varphi_j(N, M, C) - \varphi_j(N, M, C')$*.*

Cone-wise additivity requires that if two cost matrices have the same orders of costs, the cost allocation is an additive function of problems.

Axiom 2.11 (Cone-wise additivity)**.** *For each* (N, M, C) *and* (N, M, C') *and each order* $\sigma : \{\{ij\}\}_{i,j \in N \cup M, i<j} \to \{1, 2, \cdots, \frac{|N \cup M|(|N \cup M|+1)}{2}\}$*, if for each* $i, j, k, l \in N \cup M$ *such that* $\sigma\{i, j\} \leq \sigma\{k, l\}$*,* $c_{ij} \leq c_{kl}$ *and* $c'_{ij} \leq c'_{kl}$*, then* $\varphi_i(N, M, C + C') = \varphi_j(N, M, C) + \varphi_j(N, M, C')$*.*

Equal treatment of equals, which was named as *symmetry* in Bergantinos et al. (2019), requires that if two agents have the same link costs to all other nodes of $N \cup M$, then they must have the same cost allocation.

Axiom 2.12 (Equal treatment of equals)**.** *For each* (N, M, C) *and any* $i, j \in N$*, if* $c_{ik} = c_{jk}$ *for all* $k \in N \cup M \setminus \{i, j\}$*, then* $\varphi_i(N, M, C) = \varphi_j(N, M, C)$*.*

Separability requires that if two coalitions S and $N \setminus S$ have no incentive to cooperate, the allocation of each coalition can be made independently.

Axiom 2.13 (Separability)**.** *For each* (N, M, C) *and each* $S \in N$*, if* $c(N, M, C) = c(S, M, C) + c(N \setminus S, M, C)$*, then*

$$\varphi_i(N, M, C) = \begin{cases} \varphi_i(S, M, C) & \text{if } i \in S, \\ \varphi_i(N \setminus S, M, C) & \text{if } i \in N \setminus S. \end{cases}$$

Since the classical minimum cost spanning tree problem is a special case of the minimum cost spanning tree problem with multiple sources, if a rule does not satisfy an axiom for the classical minimum cost spanning tree problem then the rule cannot satisfy the axiom for *m-mcstp*. So we first check which axiom is not satisfied by a rule for the classical minimum cost spanning tree problem. Trudeau (2013) compared the Kar rule and the folk rule for the classical minimum cost spanning game, and we borrow his results only for the negative cases. We prove the rest results required to compare the two rules.

Proposition 2.2. *The folk rule satisfies absence of cross-subsidization.*

Proof. Let C^* be an irreducible matrix associated with C. From the definition of an irreducible matrix, the minimal tree $t(N, M, C)$ is also a minimal tree for the irreducible problem (N, M, C^*). If $t(N, M, C)$ is a star graph with a source cluster in the center, the irreducible game (N, M, C^*) has a star graph with a source in the center as the minimal tree which is $t(N, M, C)$. Since (N, M, C^*) is a *m-mcstp*, the allocation is $\varphi_i(N, M, C^*) = \frac{c(M)}{n} + c_{iM}$ for all $i \in N$ the same in the original game. □

Proposition 2.3. *The folk rule does not satisfy modified group independence.*

Proof. We show a counter example. Let there be two sources $\{S_1, S_2\}$, three agents $\{1, 2, 3\}$, and the cost matrix C and its irreducible matrix C^* as below. Check that there are two groups $\{1, 2\}$ and $\{3\}$.

$$C = \begin{array}{c} \\ S_1 \\ S_2 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} S_1 & S_2 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 288 & 270 & 132 & 390 \\ 288 & 0 & 72 & 306 & 114 \\ 270 & 72 & 0 & 156 & 372 \\ 132 & 306 & 156 & 0 & 348 \\ 390 & 114 & 372 & 348 & 0 \end{bmatrix} \end{array}, \quad C^* = \begin{array}{c} \\ S_1 \\ S_2 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} S_1 & S_2 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 156 & 156 & 132 & 156 \\ 156 & 0 & 72 & 156 & 114 \\ 156 & 72 & 0 & 156 & 114 \\ 132 & 156 & 156 & 0 & 156 \\ 156 & 114 & 114 & 156 & 0 \end{bmatrix} \end{array}$$

The characteristic function for (N, M, C^*) is $c(\{1\}) = 228, c(\{2\}) = 288, c(\{3\}) = 270, c(\{1, 2\}) = 360, c(\{1, 3\}) = 342, c(\{2, 3\}) = 402, c(\{1, 2, 3\}) = 474$. Therefore, the folk rule is $\{124, 184, 166\}$.

Now let the cost between agent 1 and agent 2 change from 156 to 216. The new cost matrix and its irreducible matrix can be seen as below. There are still two groups, $\{1, 2\}$ and $\{3\}$.

$$\hat{C} = \begin{array}{c} \\ S_1 \\ S_2 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} S_1 & S_2 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 288 & 270 & 132 & 390 \\ 288 & 0 & 72 & 306 & 114 \\ 270 & 72 & 0 & 216 & 372 \\ 132 & 306 & 216 & 0 & 348 \\ 390 & 114 & 372 & 348 & 0 \end{bmatrix} \end{array}, \quad \hat{C}^* = \begin{array}{c} \\ S_1 \\ S_2 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} S_1 & S_2 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 216 & 216 & 132 & 216 \\ 216 & 0 & 72 & 216 & 114 \\ 216 & 72 & 0 & 216 & 114 \\ 132 & 216 & 216 & 0 & 216 \\ 216 & 114 & 114 & 216 & 0 \end{bmatrix} \end{array}$$

The characteristic function for $(N, M, \hat{C}^*)$ is $c(\{1\}) = 288,\ c(\{2\}) = 348,\ c(\{3\}) = 330,\ c(\{1, 2\}) = 420,\ c(\{1, 3\}) = 402,\ c(\{2, 3\}) = 462,\ c(\{1, 2, 3\}) = 532$. Therefore, the folk rule is $\{144, 204, 186\}$. In this example, the allocation for agent 3 changes because of the cost change between agent 1 and agent 2 who belong to another group.

For comparison, the Kar rule assigns $\{93, 171, 210\}$ for the original game and $\{123, 201, 210\}$ after the cost changes therefore the allocation to agent 3 does not change.

□

Proposition 2.4. *The Kar rule satisfies cone-wise additivity, equal treatment of equals.*

Proof. First, given (N, M, C) and (N, M, C'), let σ be an order such that for each $i, j, k, l \in N \cup M$, $\sigma\{i, j\} \leq \sigma\{k, l\}$ implies $c_{ij} \leq c_{kl}$ and $c'_{ij} \leq c'_{kl}$. The Kruskal algorithm constructs a minimal tree by sequentially adding the cheapest link without forming a cycle. For any $S \subseteq N$, the Kruskal algorithm selects the same link at each step of C, C', and $C + C'$ since the order of the links of each cost matrix are the same. Therefore

$c(S, M, C+C') = c(S, M, C)+c(S, M, C')$ for all $S \subseteq N$, which implies $\varphi_i(N, M, C+C') = \varphi_j(N, M, C) + \varphi_j(N, M, C')$.

Second, let two agents $i, j \in N$ be such that $c_{ik} = c_{jk}$ for all $k \in N \cup M \setminus \{i, j\}$. Then for all $S \subset N \setminus \{i, j\}$, $c(S \cup \{i\}, M, C) = c(S \cup \{j\}, M, C)$ which implies $\phi_i(N, M, C) = \phi_j(N, M, C)$.

□

Proposition 2.5. *The Kar rule does not satisfy link cost monotonicity, matrix cost monotonicity and equal treatment of source costs.*

Proof. We show a counter examples. Let the cost be as figure 2.5. The left figure is the cost before cost changes and right figure is the cost after the cost changes. Before the cost changes, $c(\{1\}) = 4$, $c(\{1\}) = 6$, $c(\{1, 2\}) = 6$ therefore the Kar rule assigns $(2, 4)$. Now let the cost between agent 2 and the source increased. After the cost changes, the Kar rule assigns $(1, 5)$, therefore the Kar rule violates link cost monotonicity as well as matrix cost monotonicity.

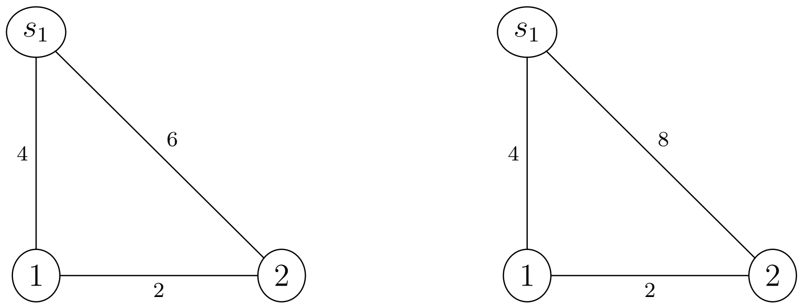

Figure 2.5: Example of the Kar rule violates cost monotonicity

Let the cost be as figure 2.6. The left figure is the cost before cost changes and right figure is the cost after the cost changes. Before the cost changes, the Kar rule assigns 1 to agent 1 and 5 to agent 2. However if the cost between the two sources changes from 4 to 10, $c(\{1\}) = 5$, $c(\{2\}) = 11$, $c(\{1, 2\}) = 6$, therefore the Kar rule assigns 0 to agent 1 and 6 to agent 2. Therefore the Kar rule violates equal treatment of source costs.

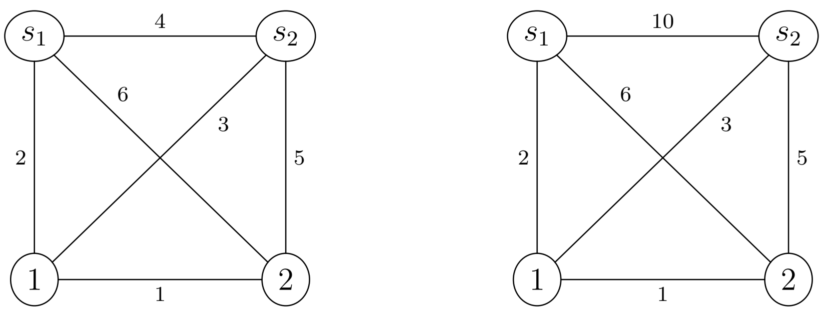

Figure 2.6: Example of the Kar rule violates equal treatment of source costs

□

We summarize the properties of the Kar rule and the folk rule for *m-mcstp* as below.

	Kar Rule	Folk Rule
Efficiency	Yes	Yes
Absence of cross-subsidization	Yes	Yes
Group independence	Yes	No
Balanced contribution	Yes	No[2]
Core selection	No[2]	Yes[1]
Population monotonicity	No[2]	Yes[1]
Independence of irrelevant trees	No[2]	Yes[1]
Link cost monotonicity	No	Yes[1]
Matrix cost monotonicity	No	Yes[1]
Equal treatment of source costs	No	Yes[1]
Cone-wise additivity	Yes	Yes[1]
Equal treatment of equals	Yes	Yes[1]
Separability	No[2]	Yes[1]

Source: 1: Bergantinos et al. (2019), 2: Trudeau (2013)

Table 2.1: Properties of the Kar and the folk rules for minimum cost spanning tree problems with multiple sources

Before we see the main result, we introduce the restricted domains of C which will be used in the proof of the main result. Let a link between an agent and a source (ij) be *inefficient* if $c_{ij} \geq \max\{\min_{s\in M} c_{is}, \max_{(pq)\in t_M} c_{pq}\}$ for any $i \in N$ and $j \in M$. Let a link between an agent and a source is *efficient* if it is not inefficient. Let $\mathbb{C}$ be the domain of a cost matrix C. Let $\mathbb{C}^1$ be a domain of cost matrix where each agent $i \in N$ has at most one efficient link, and let $\mathbb{C}^2 = \mathbb{C} \setminus \mathbb{C}^1$.

Proposition 2.6. *Given an m-mcstp (N, M, C) where $C \in \mathbb{C}^2$, a star graph with a source cluster in the center cannot be a minimal tree.*

Proof. Let agent i has two or more inefficient links, that is, at least two sources j, k such that $c_{ij} < \max\{\min_{s\in M} c_{is}, \max_{(pq)\in t_M} c_{pq}\}$ and $c_{ik} < \max\{\min_{s\in M} c_{is}, \max_{(pq)\in t_M} c_{pq}\}$

Case 1: $\max\{\min_{s\in M} c_{is}, \max_{(pq)\in t_M} c_{pq}\} = \min_{s\in M} c_{is}$

If this holds, $c_{ij} < \min_{s\in M} c_{is}$. This is impossible for both j and k.

Case 2: $\max\{\min_{s\in M} c_{is}, \max_{(pq)\in t_M} c_{pq}\} = \max_{(pq)\in t_M} c_{pq}$

In this case, $c_{ij} < \max_{(pq)\in t_M} c_{pq}$ and $c_{ik} < \max_{(pq)\in t_M} c_{pq}$. Suppose that the minimal tree is a star network with a source cluster in the center. It means at least one links of (ij) or (ik) is not in the minimal tree. Without loss of generality, suppose that (ij) is not in the minimal tree. Insert (ij) and check the cost of links forming the cycle made by inserting (ij). From the assumption of (ij), there exists at least one link between sources which is more expensive than (ij). We get a less expensive spanning tree by deleting the maximum cost link of the cycle. This is a contradiction. □

Lemma 4. *Given an m-mcstp (N, M, C) where $C \in \mathbb{C}^1$, if there exists no relevant link, a star network with a source cluster in the center is a minimum cost spanning tree of the problem.*

Proof. Choose a network consisting of t_M and (is) for all $i \in N$ where $(is) = \text{argmin}_{s\in M}\, c_{is}$. This network forms a spanning tree over $N \cup M$, and is a star network with a source cluster in the center.

Choose any link outside this tree. If we insert the link into the tree then a cycle is formed. If the link inserted is connecting an agent and a source, it is an inefficient link. The definition of the inefficient link ensures that the link inserted has the maximum cost among the links forming the cycle. If the link inserted is connecting two agents, it is a irrelevant link. The definition of the relevant link ensures that the link inserted has the maximum cost among the links forming the cycle. Therefore we have no way to find a tree with a lower cost than the original tree. □

Proposition 2.7. *On the domain $\mathbb{C}^1$, there exist only one allocation rule that satisfies efficiency, absence of cross-subsidization, group independence, and balanced contribution together.*

Proof. We prove this by induction on the number of relevant links. If there is no relevant link, the minimal tree is a *modified star graph* by lemma 4. The *absence of cross-subsidization* axiom ensures that the cost allocation is unique.

Let the proposition be true for all cost problems with at most $(k-1)$ relevant links. Then we show that the allocation is unique for the problem with k relevant links.

Let (N, M, C^k) be a *m-mcstp* with k relevant links. Suppose there exist two different cost allocations $\phi(N, M, C^k)$ and $\psi(N, M, C^k)$ those satisfy all four axioms.

Without loss of generality, let (mn) be a relevant link. Let $N = [N_1, \cdots, N_p]$, a partition where each element is a group and suppose that $m, n \in N_t,\ 1 \le t \le p$. Let C^{k-1} be a cost matrix such that $c^{k-1}_{mn} = \max\{\min_{s\in M} c_{is}, \min_{s\in M} c_{js}, \max_{(pq)\in t_M} c_{pq}\} + \epsilon,\ \epsilon > 0$, and the other costs are the same as C^k.

From the induction hypothesis, $\phi_i(N, M, C^{k-1}) = \psi_i(N, M, C^{k-1})\ \forall i \in N$.

Given the *group independence* axiom,

$$\phi_i(N, M, C^k) = \phi_i(N, M, C^{k-1}) = \psi_i(N, M, C^{k-1}) = \psi_i(N, M, C^k),\ \forall i \in N_l,\ \forall l \neq t. \tag{2.3}$$

Now,

$$\begin{aligned} &\phi_m(N, M, C^k) - \phi_n(N, M, C^k) \\ &= \phi_m(N, M, C^{k-1}) - \phi_n(N, M, C^{k-1}) \quad \text{(by \textit{balanced contribution})} \\ &= \psi_m(N, M, C^{k-1}) - \psi_n(N, M, C^{k-1}) \quad \text{(by the induction hypothesis)} \\ &= \psi_m(N, M, C^k) - \psi_n(N, M, C^k) \qquad \text{(by \textit{balanced contribution})} \\ &\Rightarrow \phi_m(N, M, C^k) - \psi_m(N, M, C^k) = \phi_n(N, M, C^k) - \phi_n(N, M, C^k). \end{aligned} \tag{2.4}$$

Choose any $i, j \in N_t$. There exists a relevant path from i to j by the definition of a group. We have

$$\phi_i(N, M, C^k) - \psi_i(N, M, C^k) = \phi_j(N, M, C^k) - \psi_j(N, M, C^k) \ \text{ for any } i,\ j \in N_t. \tag{2.5}$$

It means that for any node $i \in N_t$, $\phi_i(N, M, C^k) - \psi_i(N, M, C^k)$. Therefore,

$$\sum_{i \in N} \big[\phi_i(N, M, C^k) - \psi_i(N, M, C^k)\big] = 0 \quad \text{(using } \textit{efficiency}\text{)}$$
$$\Rightarrow \sum_{i \in N_t} \big[\phi_i(N, M, C^k) - \psi_i(N, M, C^k)\big] = 0 \quad \text{(using (2.3))}$$
$$\Rightarrow \phi_i(N, M, C^k) = \psi_i(N, M, C^k) \quad \forall i \in N_t \quad \text{(using (2.5))}$$
$$\Rightarrow \phi(N, M, C^k) = \psi(N, M, C^k) \quad \text{(using (2.3))}$$

This is a contradiction. □

We omit the independence of each axioms. If $|M| = 1$, then the four axioms are the same as the axioms of Kar (2002), and he showed the independence of each axioms.

Taking proposition 2.6 and proposition 2.7 together, there exist only one solution that satisfies efficiency, absence of cross-subsidization, group independence, and balanced contribution together. By proposition 2.1, the Kar rule is the only rule.

Theorem 2.1. *Given an m-mcstp (N, M, C), the Kar rule is the only cost allocation rule that satisfies efficiency, absence of cross-subsidization, group independence, and balanced contribution.*

2.4 Conclusion

In this chapter, we extend the definition of the Kar rule to suit the minimum cost spanning tree problem with multiple sources and study the properties of the rule. Like the result of the classical minimum cost spanning tree problem, the Kar rule is the only rule satisfying efficiency, absence of cross-subsidization, group independence, and balanced contribution together. The result of Kar (2002) is still valid for the minimum cost spanning tree problems with multiple sources.

We conclude this study with a remark. It is known that *separability* implies *group independence* for the classical minimum cost spanning tree problem. However as we can see in table 2.1, the folk rule satisfies *separability* however it does not satisfy *group independence*. Therefore, *separability* does not imply *group independence* for the minimum cost spanning tree problem with multiple sources.

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Chapter 3

A cooperative game theoretic approach on the profit allocation of the Korean automotive industry

3.1 Introduction

An industry is a sector that produces a closely related goods or services within an economy. Individual firms produce their own product such as materials, intermediates and final products, and sell them to other firms (mostly) within the same industry or consumers. Firms in an specific industry are connected with the buyer-supplier relationships. In this point of view, an industry can be seen as a system consist of many firms working together to produce the final goods and distribute the value of the products among them. Therefore, we have possibility to investigate an industry with game theoretic approach.

Many researches have been conducted on economic distributions of a project or an industry from the perspective of cooperative game theory. For example, Sung (2008) investigated the cost allocation of a quay construction, Bergantinos and Moreno-Ternero (2018) investigated the revenue allocation from the broadcasting sport events, Her et al. (2018) investigated the cost savings allocation of a power grid system, Teng et al. (2019) investigated the profit allocation of IPD project. However, in many cases, due to data limitations, research was mainly conducted using simulation method rather than using a real data. Moreover, due to the difficulty of calculation, it is hard to find a study regarding large number of firms.

In this chapter, we investigate the Korean automotive industry, one of the biggest industries of Korea. We study it empirically from the perspective of cooperative game theory. We collect various datum such as buyer-supplier relationships, financial statements of each firms, and patents each firms has. We build economic models with network which reflect the relationships and value of each coalitions. We estimate the Shapley value and evaluate the profit allocation of the Korean automotive industry.

This chapter is organized as follows. Section 2 introduces models regarding relationships among firms. Section 3 provides a detailed description of the datum and estimation method. In section 4, we show the computational results. Section 5 conclude this chapter.

3.2 Model

Owen (1975) introduced a linear production game model. In the linear production game model, each agents has their own resources which can be used to produce goods. Goods are sold in market prices and each goods needs pre-defined resources bundle to be produced. Agents in the economy has their own resources and work together to maximize the value of the goods.

In this chapter, we use the linear production game model. Let $N = \{1, \cdots, n\}$ be a set of agents, $P = \{1, \cdots, p\}$ be a set of types of products, $R = \{1, \cdots, r\}$ be a set of types of resources. We assume that when we produce one unit of product $j \in P$, a resources bundle $(a_{1j}, \cdots, a_{rj})$ is required. Each agent i has a vector of resources $b^i = (b^i_1, \cdots, b^i_r)$ for all $i \in N$.

Product j can be sold at price c_j in the market. Let z_j be the production volume of product j. The value of a production vector $(z_j)_{j\in P}$ is $\sum_{j\in P} c_j z_j$.

For any coalition $S \subseteq N$, the amount of resources of the coalition is represented by

$$b_k(S) = \sum_{i\in S} b^i_k, \ k = 1, \cdots, r. \tag{3.1}$$

The value of a coalition is represented by a maximized value of the production which can be calculated by solving the following linear programming:

$$\begin{aligned} v(S) = \max & \sum_{j\in P} c_j z_j \\ s.t. & \sum_{j\in P} a_{kj} z_j \le b_k(S), \quad \forall k \in R, \\ & z_j \ge 0, \quad \forall j \in P. \end{aligned} \tag{3.2}$$

The linear production game (N, v) is defined by the set of agents N and the value function $v(S)$ for all coalition $S \subseteq N$, where $v(S)$ be the value of equation (3.2).

Modularized and specialized industry assumption

In this chapter, we investigate the Korean automotive industry. To simplify the model and reflect the characteristics of the industry, we assume that it is modularized and specialized. In our model, *modularized* means that the role of firms in the industry can be divided into two types; final goods maker and module producer. A module producer makes modules those are parts of the final goods. A final goods maker buys modules from module producers and assembles them. Actually, in real world, a module produced need some materials or intermediate goods to produce a module. However, due to the limitation of data, we do not consider the second or lower tier vendors and only consider the final goods maker and its first-tier vendors. *Specialized* means that a module producer makes only one type of module and there is only one type of final goods in this industry. As for the Korean automotive industry, there are five major vehicle manufacturers, Hyundai motors, Kia motors, Renault-Samsung motors, General-Motors Korea, Ssangyong motors, who make vehicles and a lot of first-tier vendors who produce modules and supply to some of the vehicle manufacturers.

If we adopt the *modularized* and *specialized* industry assumptions, the value function of a linear production model becomes a Leontief production function. Assume that there is one type of final goods, and some resources required to make one unit of the final goods including assembly capacity that a final goods maker owns. Let $\{N_1, \cdots, N_r\}$ be a partition of N such that N_1 is the set of final goods makers, N_k, $2 \leq k \leq r$, be the set of module k producers. Given a coalition $S \subseteq N$, let $N_k(S) = S \cap N_k$. A linear production model can be simplified as below:

$$\begin{aligned} v(S) &= c_1 z_1 \\ &s.t.\ a_{k1} z_1 \leq b_k(S), \quad \forall k \in R, \\ &z_1 \geq 0, \\ \Rightarrow v(S) &= \min\{\frac{b_1(S)}{a_1}, \cdots, \frac{b_r(S)}{a_r}\} = \min\{\frac{b_1(N_1(S))}{a_1}, \cdots, \frac{b_k(N_r(S))}{a_r}\}. \end{aligned} \tag{3.3}$$

We omit the price of the product c_1 in the last line of the equation. a_k is the unit conversion factor of resource k which means the amount of resource k needed to make one unit of the final goods. Let $b_k^*(S) = b_k(S)/a_k$. From now on we use $b_k^*(S)$ instead of b_k so that each resource is evaluated by the unit of final goods.

Buyer-supplier relationships

We observe buyer-supplier relationships among firms within an industry. In particular, there is a special transaction relationship in the Korean automotive industry called *exclusive transaction*. We need to consider how to incorporate this particular relationship into our model, that is, we need to consider what is the meaning of the buyer-supplier relationship in the Korean automotive industry. See following example 3.1.

Example 3.1. *There are one type of final goods and two types of modules. Let there are 8 agents. Agent 1 and 2 are final goods maker, agent 3, 4, 5 produce module 1, and agent 6, 7, 8 produce module 2. Suppose that we observe buyer-seller relationship in this industry as figure 3.1.*

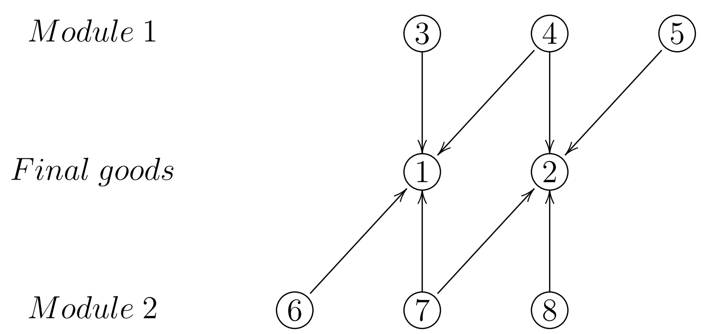


Figure 3.1: Buyer-supplier relationships

Here we define some notations which reflect a permission structure investigated by Gilles et al. (1992), van den Brink (1997), and Gilles and Owen (1999). If an agent i has a power to control agent j, agent i is denoted as a *superior* of agent j while agent j is denoted as a *subordinate* of agent i. A *directed network* on N is a collection of permission relation represented by a mapping $H : N \to 2^N$ such that $i \notin H(i)$ for all $i \in N$. An agent $j \in H(i)$ is a *subordinate* of agent i. Conversely, an agent $j \in H^{-1}$, where $H^{-1} = \{j \in N | i \in H(j)\}$, is called a *superior* of agent i. For any coalition $S \subseteq N$, let $H_i(S) = H(i) \cap S$.

The final goods makers usually have more power over his suppliers in the Korean automotive industry. Since we investigate the allocation among final goods maker and the first-tier vendors in this paper, we assume that a supplier in a network is a subordinate of his buyer. For the example above, we assume that $H(1) = \{3, 4, 6, 7\}$ and $H(2) = \{4, 5, 7, 8\}$. We introduce four approaches to incorporate the buyer-supplier relationships into our model.

Complete cooperation(Model 1)

In the first approach, we assume that a superior does not have a power to his subordinate. We assume that every firm can cooperate with each other without any restriction. In this approach, the buyer-supplier relationships in the market is merely an ex-post observed relationships among firms. In this approach, to calculate the value of a coalition, all we have to check is the amount of resources in the coalition. The agent 3 of the example 3.1 may sell his resources to agent 2 without any restriction. The characteristic function $v(S)$ can be calculated by solving the linear program equation (3.3), that is, $v^{NVP}(S) = \min_{k \in R} b_k(S)$.

Conjunctive approach(Model 2) and Disjunctive approach(Model 3)

The second approach is *cooperation under hierarchical authority (permission structure)* investigated by Gilles et al. (1992), van den Brink (1997), and Gilles and Owen (1999). In this approach, we want to reflect the power imbalance in the Korean automotive industry. We assume that a final goods maker has power to control her subordinates. In other words, we assume that a module producer cannot cooperate (sell modules) if he does not gain permission from his superiors. In the example above, agent 3 may cooperate if agent 1 exists in the coalition. Conversely, agent 3 cannot cooperate without permission of agent 1.

How about agent 4? Think of a coalition $\{1, 4\}$. Can we assume that agent 4 may cooperate with agent 1? There are two approaches in the hierarchical authority model; *conjunctive approach* and *disjunctive approach*. The *conjunctive approach* is developed in Gilles et al. (1992). A firm needs to gain permissions from *all* of his superiors to cooperate under conjunctive approach. Whereas, the *disjunctive approach* developed in Gilles and Owen (1999) assume that a firm needs to gain permission *at least one* of his superiors to cooperate. In our example of coalition $\{1, 4\}$, agent 4 may cooperate with agent 1 under disjunctive approach, whereas he cannot cooperate under conjunctive approach since he cannot gain permission from agent 2. In other words, a superior has full veto power to her subordinate in conjunctive approach, whereas she has limited veto power to her subordinate in disjunctive approach.

The characteristic function can be calculated by solving the linear program equation (3.3) with an adjusted coalition. Given a coalition $S \subseteq N$, let $S^{DH} = N_1(S) \cup (\cup_{i \in N_1(S)} H_i(S))$ and $S^{CH} = [N_1(S) \cup (\cup_{i \in N_1(S)} H_i(S))] \setminus (\cup_{i \in N_1 \setminus S} H_i(S))$. The characteristic function under disjunctive approach is $v^{DH}(S) = v^{CC}(S^{DH})$ and conjunctive approach is $v^{CH}(S) = v^{CC}(S^{CH})$.

Subcontracting (Model 4)

The last approach is *subcontracting*. We assume that if a buyer-supplier relationship is not observed, the relationship cannot happen because of the compatibility. Each module producer may sell his resources only to his observed direct superiors. In the example above, for any coalition $S \subseteq N$, agent 3 may sell his resource to agent 1 only, and he could not sell his resources if agent 1 is not in the coalition.

Subcontracting reflects a technological dependency between buyer and supplier. In reality, subcontracting is done by *on-demand production*. The buyer requests a customized product from the supplier. In this way, the product cannot be sold to other firms. Conversely, it is difficult for the buyer to purchase some product which is not ordered.

We have to modify the equation (3.3) to calculate the characteristic function of coalition S under subcontracting assumption. The value of a coalition S is the maximized value of final goods produced in the coalition under some constraints. The first line of the constraint is the supplier resource feasibility constraint which means a module producer sells to only his direct superiors in the coalition and the total amount sold cannot exceed his own resources. The second and the third constraint are the production feasibility constraints which means the production of a final goods production function is Leontief function under his own resources and the resources she buys from her direct subordinates in the coalition. The other constraints are non-negative constraints of final goods and resources.

$$\begin{aligned}
v^{sc}(S) = \max & \sum_{i \in N_1(S)} z_i \\
s.t. & \sum_{i \in H_j^{-1}(S)} z_{ji} \leq b^j, \quad \forall j \in S \setminus N_1(S), \\
& z_i \leq b^i, \quad \forall i \in N_1(S), \\
& z_i \leq \sum_{j \in N_k(S) \cap H_i(S)} z_{ji}, \quad \forall k \in R \setminus \{1\}, \forall i \in N_1(S), \\
& z_i \geq 0, \quad \forall i \in N_1(S), \\
& z_{ji} \geq 0, \quad \forall i \in N_1(S), \forall j \in H_i(S).
\end{aligned} \tag{3.4}$$

Each of the four models is extreme case and the reality exists may between them. We use the distribution status in each models as an evaluation criterion. We evaluate the real distribution by checking the closeness between the distribution states presented in each case and the reality.

The Shapley value

An allocation rule is a mapping which associates to each problem a non empty set of feasible allocations. Owen (1975) introduces a well-known solution for the linear production model. However it is not appropriate for our model. The Owen value only admits the contribution of the rarest resources in the industry under the modularized and specialized industry assumption.

The Shapley value(Shapley (1953)) is one of the most well-known allocation rule of cooperative game theory. The Shapley value assigns each member her own marginal contribution which is calculated as below:

Shapley value: For all $v \in \Gamma^N$, and all $i \in N$

$$\phi_i = \sum_{s=1}^{n} \frac{(s-1)!(n-s)!}{n!} \Big[\sum_{i \in S} (v(S) - v(S \setminus \{i\}) \Big], \tag{3.5}$$

where Γ^N is a class of games with player set N, n is the number of agents of N, and s is the number of agents of $S \subseteq N$. We use the Shapley value as our allocation rule. Example 3.2 shows results for each models.

Example 3.2. *There are one type of final goods and two types of modules. Agent 1, 2 are final goods maker, agent 3, 4 produce module 1, and agent 5 produces module 2. The resources of each agent from agent 1 to agent 5 are 5, 10, 8, 12, 20, respectively. The observed flow is as figure 3.2. The numbers in the parentheses means the resources each agent owns.*

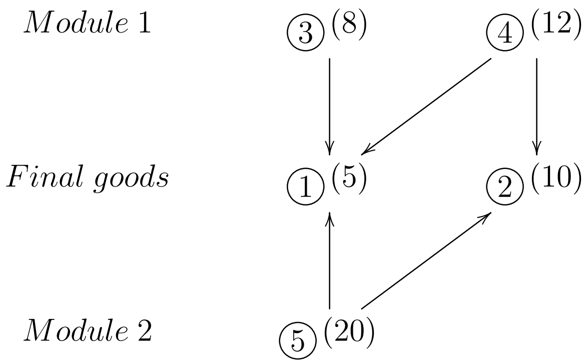


Figure 3.2: Flow network of example 3.2

We can see the characteristic functions under each restricted cooperation model at table 3.1 and the Shapley value at table 3.2.

	complete cooperation	conjunctive hierarchy	disjunctive hierarchy	subcontracting
{1, 3, 5}	5	0	5	5
{1, 4, 5}	5	0	5	5
{1, 3, 4, 5}	5	0	5	5
{2, 3, 5}	8	0	0	0
{2, 4, 5}	10	0	10	10
{2, 3, 4, 5}	10	0	10	10
{1, 2, 3, 5}	8	8	8	5
{1, 2, 4, 5}	12	12	12	12
{1, 2, 3, 4, 5}	15	15	15	15
others	0			

Table 3.1: Characteristic functions of example 3.2

	complete cooperation	conjunctive hierarchy	disjunctive hierarchy	subcontracting
Agent 1	101 /60	240 /60	125 /60	116 /60
Agent 2	216 /60	240 /60	200 /60	191 /60
Agent 3	86 /60	60 /60	70 /60	61 /60
Agent 4	156 /60	120 /60	180 /60	216 /60
Agent 5	341 /60	240 /60	325 /60	316 /60

Table 3.2: The Shapley values of example 3.2

3.3 Analysis method

3.3.1 Data and variable

In our model, we need resource type of each firm, resource amount of each firm, buyer-supplier relationships among firms, and profit allocations. We use datum of (1) relationships, (2) financial statements, (3) patents, and (4) number of employees. The financial statements is used to determine profit allocation, patents to resource type, and number of employees to resource amount. The source of the relationships data is KAICA(Korea auto industries coop. association) and we get the rest of the datum from KED(Korea enterprise data).

Relationships

We use a relationship data among vehicle manufacturers and first-tier vendors. Table 3.3 shows the example of the relationship data. It contains (1) information of firms such as name, ID, address, numbers of employees, main items, and (2) information of buyer-seller relationships of 421 first-tier vendors and 5 vehicle manufacturers.

Table 3.4 shows the distribution of the number of relationships. 148 firms supplied to only one vehicle manufacturers, 131 firms supplied to two vehicle manufacturers.

Table 3.5 shows the correlation coefficient of relationship data columns from Hyundai to Ssangyong. As we can see, there are many overlaps between vendors of Hyundai motors and vendors of Kia motors. Among the 257 vendors of Hyundai Motors or Kia Motors, 220 firms overlapped, 22 firms supplied only to Hyundai Motors, and 15 firms supplied only to Kia Motors.

We visualize the relationship of all firms in figure 3.3.

Firm	employee	main item	Hyundai	Kia	GM-	Renault-	Ssangyong
(Firm 1)	4,328	Brake system	O	O	O	O	O
(Firm 2)	3,213	Transmission	O	O	O		O
(Firm 3)	2,355	Wire Harness	O	O			
(Firm 4)	2,200	Head lamp	O	O	O		
(Firm 5)	2,079	Air conditioner	O	O			
(Firm 6)	1,678	Head lamp	O	O	O		
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Total (N=421)			242	235	137	203	128

Table 3.3: Example of the relationship data

No. of relationships	1	2	3	4	5	mean
No. of firms	148	131	63	49	30	2.24

Table 3.4: Distribution of the number of relationships

	Hyundai	Kia	GM-	Renault-	Ssangyong
Hyundai	1	0.82	-0.16	-0.07	0.01
Kia	0.82	1	-0.15	-0.08	0.02
GM-	-0.16	-0.15	1	0.17	0.16
Renault-	-0.07	-0.08	0.17	1	0.06
Ssangyong	0.01	0.02	0.16	0.06	1

Table 3.5: Correlation matrix of relationships

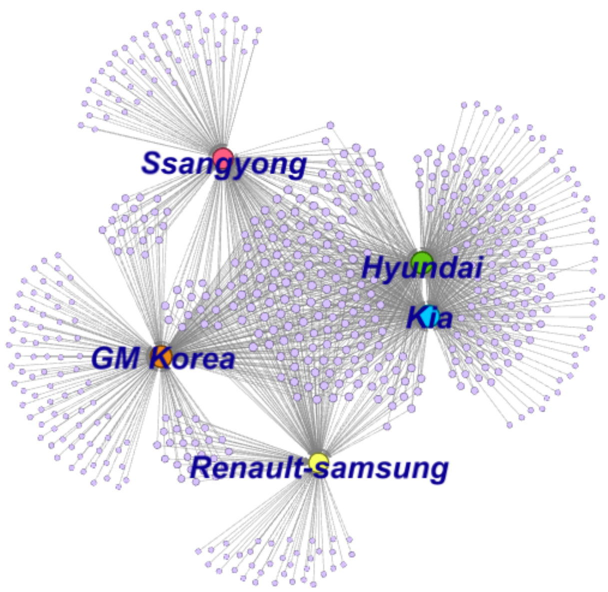


Figure 3.3: Relationship network of the Korean automotive industry

Resource type

We need to classify what resources each firm produces. We use the patents data to determine the type of each firms' resources. We use the IPC(international patent classification) when we classify each firms' patents. KIPO(Korean Intellectual Property Office) provides a linkage table between the industrial classification and the patent classification. Automotive industry is related to B60 and some other codes.[1] We count each firms' patents number those are related to the automotive vehicle industry. We compare it with firms' main item written in the relationship data. If the most patent classification codes and the main item of a firm match, we classify the firm according to the category of its patent. However if they do not match, we classify by referring to the business report and homepage. We classify first-tier vendors into 15 types. We assume that the same type of firms produce the same type of resources. A firm which produces non-core parts such as key, audio, mirror, wiper, lamp is classified as others of the resources category.

Type	No. of firms
Frame	35
Bumper	12
Door	29
Chair	17
Interior parts	34
Piston	26
Vaporizer	33
Exhaust system	17
Chilling system	17
Transmission	29
Brake system	11
Steering system	26
Parts control	16
others	108
Total	421

Table 3.6: Resource category of first-tier vendors

Resource amount

Once the classification of resource is done, we have to determine the amount of resources of each firm owns. We assume the resources (capacities to assemble) of vehicle manufactures are estimated to be the number of domestic vehicles sold in 2018.[2]

[1]Automotive industry is related to IPC code B60B(wheels), B60C(tyres), B60D(connections; components of brake system), B60F(vehicle for use both on rail and on road), B60G(suspension arrangements), B60H(heating, cooling, ventilating, or the air-treating devices), B60J(windows), B60K(arrangement of propulsion units), B60L(propulsion of electrically-propelled vehicles), B60N(seats), B60P(vehicles adapted to comprise special loads or objects), B60R(not otherwise provided for), B60S(servicing), B60T(brake control system), B60W(conjoint control of vehicle sub-units of different function), B62D(motor vehicles), E05F(keys), F02M(supplying combustion engines), F02N(combustion engines), F02P(ignition), F16J(pistons), G05G(control devices).

[2]In 2018, Hyundai motors' domestic sales of vehicle were 721,078. In 2018, Kia motors sold 531,700 vehicles, GM-Korea sold 109,140 vehicles, Renault-samsung sold 90,369 vehicles, and Ssangyong motors sold 93,317 vehicles.

As for the first-tier vendors, we use the number of employees building size of a first-tier vendor as a proxy variable of the resources amount of that firm. We assume that the resource each firm owns is proportional to the number of employees among the same type of firms. The unit conversion factor of resource k is calculated as the sum of effective number of employees of firms of resource type k divided by the sum of the vehicles sold in 2018. That is, we assume that the effective resources exactly support the vehicle production. Here we use the word *effective* since we have to consider the flow of resources of real world.

Example 3.3. *There are one type of final goods and one type of module. Agent 1, 2 are final goods maker, agent 3, 4 produce module. We observe that each final goods maker produces one unit of final goods. Resources of agent 3 is 1 and agent 4 is 3. The relationship is as figure 3.4. The numbers in the parentheses means the resources each agent owns.*

If we do not consider the flow, the conversion factor a *is* $\frac{4}{2} = 2$*, so agent 3 has* $\frac{1}{2}$ *resources and agent 4 has* $\frac{3}{2}$ *resources. However if we consider the flow, the conversion factor is 1 since agent 1 buys module from agent 3 only and the ratio is 1. So the* effective *resources of agent 3 and agent 4 are 1 and 1 respectively, and agent 4 has 2* redundant *resources.*

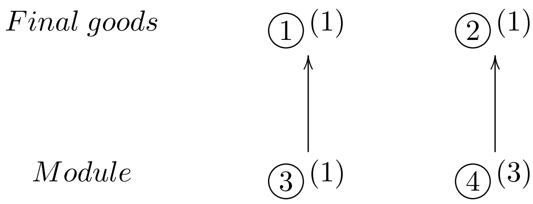


Figure 3.4: Flow network of example 3.3

The unit conversion factor of resource k regarding the flow network can be calculated using the linear programming below. The first constraints ensure that the adjusted amount of resources sold by a module maker j do not exceed his own adjusted resources. The second constraints ensure that each final goods maker i gets the amount of resources she needs. Unit conversion factor of resource k, a_k, is calculated by $a_k = \frac{1}{f_k^*}$ where f_k^* is the solution of following linear programming.

$$\begin{aligned} & \min f_k \\ s.t. \quad & \sum_{i \in H_j^{-1}(N)} z_{ji} \leq b^j f_k, \quad \forall j \in N_k(N), \\ & \sum_{j \in H_i(N)} z_{ji} \geq b^i, \quad \forall i \in N_1(N). \end{aligned} \tag{3.6}$$

Type	No. of firms	Total	Mean
Frame	35	2,625,857	75,024
Bumper	12	1,949,209	162,434
Door	29	1,585,202	54,662
Chair	17	2,172,956	127,821
Interior parts	34	1,545,604	45,459
Piston	26	1,545,604	59,446
Vaporizer	33	1,545,604	46,836
Exhaust system	17	2,058,688	121,099
Chilling system	17	1,545,792	90,929
Transmission	29	1,545,604	53,297
Wheel	11	1,545,604	140,509
Brake system	11	1,727,408	157,037
Steering system	26	1,789,200	68,815
Parts control	16	1,545,604	96,600
others	108	1,758,905	16,286

Table 3.7: Amount of resources (first-tier vendors)

Profit allocation

Here we define allocations using financial statements. The allocation we focus on is the distribution of the value from the sales of final goods. Therefore, we focus on *value added(V.A.)*. The value added is the difference between the value of a firm's output and the cost of the input materials. The market value of all final goods and services is same as sum of all V.A. of every stage of production. So the V.A. means the distribution of a total output to the firms in the production stage of the final goods. We use the latest financial statements of each firm to calculate V.A. Since we fail to obtain the financial statements of some firms, we cannot use some of the first-tier vendors.

There are many ways to define V.A. We use the definition of Bank of Korea (2019), which is calculated by *labour cost* + *operating surplus* + *tax processing* + *depreciation cost* + *interest cost*.[3] The descriptive statistics of V.A. are shown at table 3.8.

	No. of firms	mean (bil. KRW)	std (bil. KRW)	median (bil. KRW)
All firms	416	39.8	294.5	8.6
First-tier vendors	411	18.2	28.5	8.4

Five firms were excluded due to the availability of financial statements

Table 3.8: Descriptive statistics of value added

3.3.2 Estimation method of the Shapley value

Shapley value is one of the most important solution concepts in cooperative game theory. However it is challenging to compute the value if there are many agents since we have to

[3]Operating surplus = operating income + depreciation expenses - interest expense
Labour cost = (salary + retirement benefits)$_{income\ statement}$ + (labour cost + welfare benefits)$_{manufacturing\ cost\ statement}$

consider 2^n coalitions, except some special games such as airport games(Littlechild and Owen (1973)), sequencing games(Curiel, Pederzoli, Tijs (1989)).

Castro, Gomez, Tejada (2009) introduced a method to estimate the Shapley value based on sampling theory and many literatures tried to find more efficient method to calculate the Shapley value(Maleki et al. (2013), Castro et al. (2017), Campen et al. (2018), Le, Nguyen and Bektas (2020)).

The basic idea of the sampling method is as below. For each agent $i \in N$,

$$\begin{aligned} \phi_i &= \sum_{s=1}^{n} \frac{(s-1)!(n-s)!}{n!} \Big[\sum_{i \in S} (v(S) - v(S \setminus \{i\}) \Big], \\ &= \sum_{\sigma \in \Pi(N)} \frac{1}{n!} \big[v(Pre^i(\sigma) \cup \{i\}) - v(Pre^i(\sigma)) \big], \end{aligned} \tag{3.7}$$

where $\Pi(N)$ is the set of all possible orderings with agent set N, $Pre^i(\sigma)$ is the set of predecessors of agent i in order σ. Therefore, the Shapley value is the expectation of each agents' marginal contribution. The sampling methods replace the expectation with the sample average. Each literatures tried to find efficient ways to obtain better samples.

Here we use the *structured random sampling* method of Campen et al. (2018). The procedure of the structured random sampling is as below:

Procedure of the structured random sampling(Campen et al. (2018))
Input: n-person cooperative game (N, v). (Hence, n is fixed and determines the number of groups.)

1. Select a subset Π_r orderings from all $n!$ possible orderings, i.e., $\Pi_r \subset \Pi$, with $r = t \cdot n$ where n is a natural number. (Hence, the subset must be a multiple of n).

2. Divide the subset Π_r in n groups of size t. (This ensures that each player can attain each position in the ordering the same number of times.)

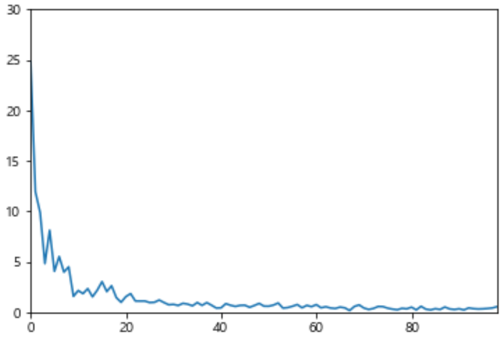
3. For each player i:

   (a) Swap player i with the player at position j for each t orderings in group j, where $j \in \{1, \cdots, n\}$, resulting is a set Π'_r of r new orderings. (This ensures that each player will attain each position in the ordering the same number of times.)

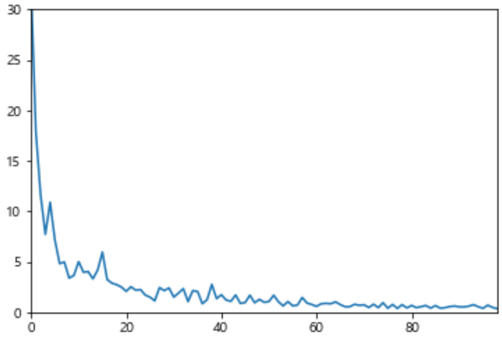
   (b) Compute the marginal contributions $m_v^\sigma(i)$ of player i for all new orderings $\sigma \in \Pi'_r$.

   (c) Approximate the Shapley value of player i by averaging the marginal contributions obtained at step $3b$, i.e., $\hat{\phi}_i = \frac{1}{r} \sum_{\sigma \in \Pi'_r} m_v^\sigma(i)$.

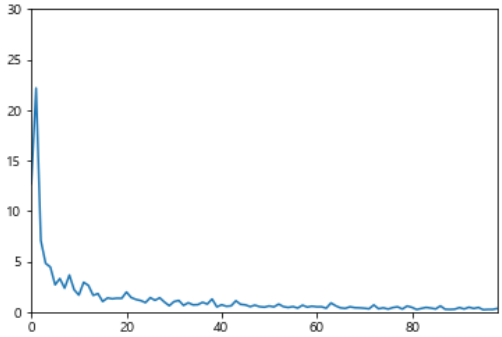
We first check the performance of this method for our 4 models with example 3.2. We use *average percentage error(APE)* as the error measurement. Given a true Shapley value $\phi(N, v)$ and its estimate $\hat{\phi}$, we define APE as below:



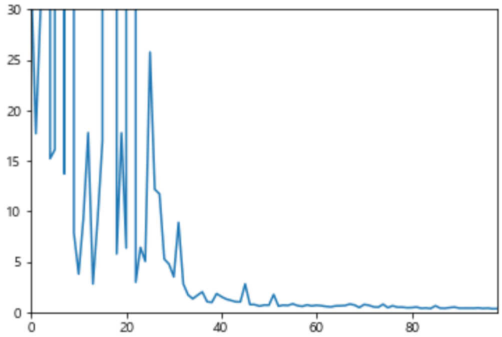
(a) Complete cooperation



(b) Conjunctive hierarchy



(c) Disjunctive hierarchy



(d) Subcontracting

Figure 3.5: Convergence of estimated Shapley value

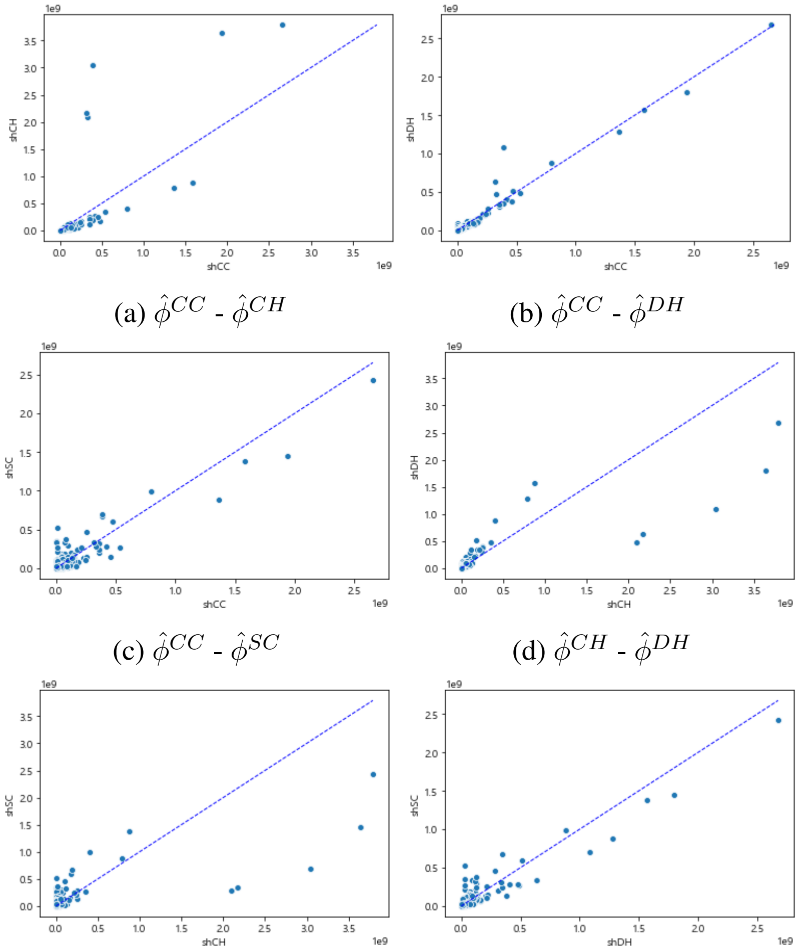
$$APE = \frac{1}{|N|} \sum_{i \in N} \frac{|\phi_i(N, v) - \hat{\phi}_i|}{|\phi_i(N, v)|}.$$

As mentioned in the procedure of the structured random sampling above, the sample size is calculated as t times n. We change t from 1 to 100, and repeat this procedure for 10 times to calculate the mean and standard deviation of APE. APE tends to lower and the width of confidence intervals tends to narrower as t grows. The average APE at $t = 100$ is around 5% for all cases.

3.3.3 Comparison among the estimated Shapley values

We estimate the Shapley value using the structured random sampling method with $t = 100$. We first check the convergence of each estimates. We calculate APE between two consecutive sequences of estimated Shapley value from $t = 1$ to 100. As we can see, the differences between two consecutive sequences is small around $t = 100$.

We compare each estimated Shapley values. See (a) of figure 3.6. We compare estimated Shapley values under complete cooperation assumption($\hat{\phi}^{CC}$) and conjunctive hierarchy assumption($\hat{\phi}^{CH}$). The dotted line is the 45 degree line from the origin. A dot means a firm. A dot below the dotted line means the allocation for a firm under $\hat{\phi}^{CC}$ is larger than that of $\hat{\phi}^{CH}$. Five dots are farther up the line, and the others are around or below the line. The five dots means five vehicle manufacturers. Since we assume a strong



(a) $\hat{\phi}^{CC}$ - $\hat{\phi}^{CH}$ (b) $\hat{\phi}^{CC}$ - $\hat{\phi}^{DH}$

(c) $\hat{\phi}^{CC}$ - $\hat{\phi}^{SC}$ (d) $\hat{\phi}^{CH}$ - $\hat{\phi}^{DH}$

(e) $\hat{\phi}^{CH}$ - $\hat{\phi}^{SC}$ (f) $\hat{\phi}^{DH}$ - $\hat{\phi}^{SC}$

Figure 3.6: Comparison of estimated Shapley values

veto power of a vehicle manufacturer under conjunctive hierarchy assumption, the $\hat{\phi}^{CH}$ allocates large amount to them compared to $\hat{\phi}^{CC}$.

The Shapley value under complete cooperation assumption($\hat{\phi}^{CC}$) and under disjunctive assumption($\hat{\phi}^{DH}$) are similar(See fig. (b)). Most of the dots did not deviate from the 45 degree line, and only a few were found deviate from the line. The weak veto does not significantly affect the allocation.

The Shapley value under subcontracting assumption($\hat{\phi}^{SC}$) allocates to the vehicle manufacturers more than $\hat{\phi}^{CC}$, $\hat{\phi}^{DH}$ and less than $\hat{\phi}^{CH}$.

3.4 Analysis result

Before we analyse, we remove outliers. We detect and remove them in the following way. We calculate mean and standard deviation of value added for each resource categories. We determine an observation as outlier if it lies outside of 3σ boundaries of value added for each resource categories. We determine 10 observations as outliers and remove them.

Table 3.9 shows the basic descriptive statistics of value added and the estimated Shap-

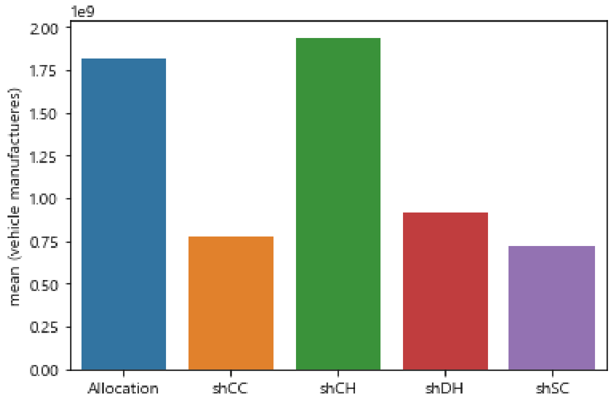


Figure 3.7: Average of V.A. of 5 vehicle manufacturers

	V.A.	$\hat{\phi}^{CC}$	$\hat{\phi}^{CH}$	$\hat{\phi}^{DH}$	$\hat{\phi}^{SC}$
Count	401	401	401	401	401
mean (bil. KRW)	15.9	28.9	14.4	27.1	29.6
std (bil. KRW)	24.5	71.1	37.8	67.9	70.3
median (bil. KRW)	8.2	6.9	2.9	8.3	6.8

Table 3.9: Descriptive statistics of V.A. and the estimated Shapley values (first-tier vendors)

ley values for the first vendors. As for the first vendors, the average of the allocation is 15.9 billion KRW, and that of $\hat{\phi}^{CC}$, $\hat{\phi}^{CH}$, $\hat{\phi}^{DH}$, $\hat{\phi}^{SC}$ are 28.9 billion KRW, 14.4 billion KRW, 27.1 billion KRW, 29.6 billion KRW, respectively. As we can see, the sum of the allocation of the vehicle manufacturers are larger than $\hat{\phi}^{CC}$, $\hat{\phi}^{DH}$, $\hat{\phi}^{SC}$, but smaller than $\hat{\phi}^{CH}$. The average of the $\hat{\phi}^{CH}$ is closer than any other estimates.

We analyse how close each estimated Shapley value to the real allocations. Here we use two proximity measures; *Average absolute error (AAE)*[4] and *average percentage error (APE)*[5]. AAE calculate the average of absolute distance of each allocations whereas APE calculate the average of relative distance of each allocations. Given a true allocation ϕ and its estimate $\hat{\phi}$, AAE and APE are defined as below:

$$AAE = \frac{1}{|N|}\sum_{i\in N}|\phi_i - \hat{\phi}_i|,$$

$$APE = \frac{1}{|N|}\sum_{i\in N}\frac{|\phi_i - \hat{\phi}_i|}{|\phi_i|}.$$

We show the proximity indices at table 3.10. Among the estimated Shapley values, $\hat{\phi}^{CH}$ is relatively closer to the real distribution.

We check resource-type based allocation distribution, that is, we compare the sum of V.A. of a same type firms and the sum of the Shapley value of the firms. As we can

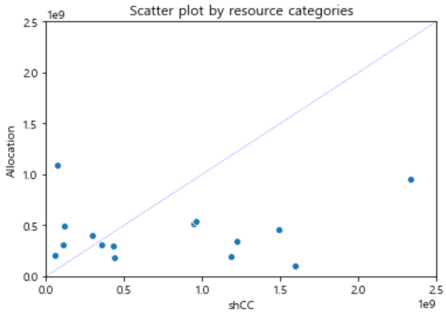
[4]AAE proximity measure is used in Maleki et al. (2013), Campen et al. (2018), Benati, Lopez-Blazquez and Puerto (2019).

[5]AAE proximity measure is used in Fatima, Wooldridge and Jennings (2008), Campen et al. (2018), Le, Nguyen and Bektas (2020).

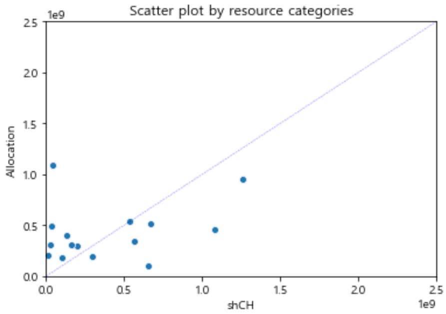
	$\hat{\phi}^{CC}$	$\hat{\phi}^{CH}$	$\hat{\phi}^{DH}$	$\hat{\phi}^{SC}$
AAE ($\times 10^6$)	23.81	**23.25**	23.35	25.07
APE	393	**230**	367	803

Table 3.10: Proximity index between true allocation and estimated Shapley values

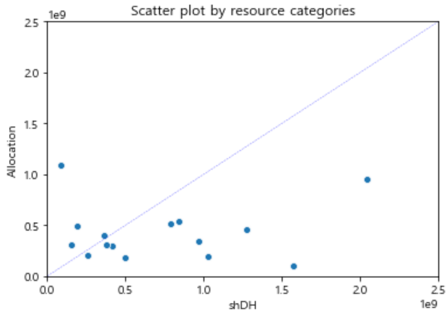
see at figure 3.8, $\hat{\phi}^{CH}$ is relatively closer to the real distribution as firm-based allocation distribution.



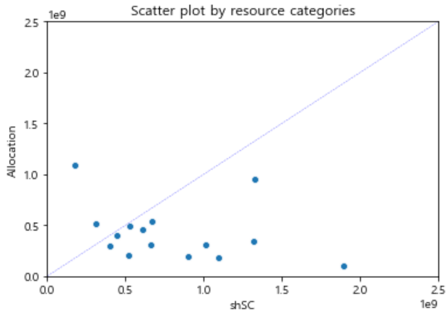
(a) $\hat{\phi}^{CC}$ - V.A.



(b) $\hat{\phi}^{CH}$ - V.A.



(c) $\hat{\phi}^{DH}$ - V.A.



(d) $\hat{\phi}^{SC}$ - V.A.

Figure 3.8: Scatter plot of V.A. and estimated Shapley values (first-tier vendors, resource category based)

3.5 Conclusion

We study the profit distribution in the Korean vehicle industry based on the cooperative game theory. Among the four models, distribution under conjunctive approach model which allows full veto power to the vehicle manufacturers is closest to the real profit distribution. From this result, we carefully talk about the power imbalance in the Korean automotive industry.

However, we have many limitations, and much of them stem from the limitation of data. First, measuring the type and the amount of each firms' resources is the biggest limitation of this study. Second, we only consider vehicle manufacturers and first-tier vendor in the Korean vehicle industry cause of the data limitations. We need to consider more firms. Third, we cannot separate domestic and overseas business of each firm. It is known that a large portion of first-tier vendor exports are sold to the Korean vehicle manufacturers, so there is possibility that the result does not change drastically, however we have to consider it explicitly.

It is obvious that we need more data and further study to fill the lack of limitations. We hope to address it in our future research.

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국문초록

네트워크 구조가 있는 비용 배분 문제에서의 샤플리 밸류에 관한 연구

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본 연구는 3개의 장으로 구성되어 있다. 각 장은 독립적인 문제를 다루고 있지만, 경제학적 현상을 네트워크 구조를 활용하여 분석하고 있다는 것과 협력에서 발생하는 이익 또는 비용의 배분 문제를 협조적 게임이론을 활용하여 분석하고 있다는 점에서 각 장은 상호 연결성을 갖는다. 첫 번째 장에서는 고전적인 대기열게임을 일반화한 문제(positional queueing problem)에서의 최소이전규칙(minimal transfer rule)과 최대이전규칙(maximal transfer rule)의 특성을 밝힌다. 두 번째 장에서는 소스가 여러 개인 최소신장가지문제(minimum cost spanning tree problem with multiple sources)에서의 카규칙(Kar rule)의 특성을 밝힌다. 마지막 장에서는 한국의 자동차 산업에서의 완성차 기업과 1차 벤더 사이의 이윤분배 문제를 협조적 게임이론적 접근법을 통해서 분석한다. 4가지 모형을 구축하고 각 모형에서 계산된 이윤분배와 현실의 이윤분배를 비교할 때, 완성차 기업의 영향력을 가장 크게 가정한 모형의 이윤분배 결과가 현실의 이윤분배와 가장 근접한 것을 확인하였다.

주요어: 샤플리 밸류, 네트워크, 협조적 게임이론, 대기열게임, 최소신장가지게임, 자동차산업

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