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Kummer-type Congruences for Multi-poly-Bernoulli Numbers

by

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Abstract. The multi-poly-Bernoulli numbers are generalizations of the Bernoulli numbers. In this paper, we will prove Kummer-type congruences for multi-poly-Bernoulli numbers via p-adic distributions.

1. Introduction

For a non-negative integer n, the (n-th) Bernoulli number B_n is defined by the generating function

$$\frac{te^t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}$$

as formal power series over $\mathbb{Q}$. It is well known that the following congruence holds (cf. [2, Theorem 11.6]). For positive integers m, n, N and an odd prime p, if $m \equiv n \bmod (p-1)p^{N-1}$, then we have

$$(1-p^{m-1})\frac{B_m}{m}\equiv(1-p^{n-1})\frac{B_n}{n} \bmod p^N .$$

This congruence is called the Kummer congruence.

In [6] and [3], Arakawa and Kaneko introduced the poly-Bernoulli numbers $B_n^{(k)}$ and $C_n^{(k)}$, which are generalizations of the Bernoulli numbers, as follows. Let k be an integer and n be a non-negative integer. Poly-Bernoulli numbers $B_n^{(k)}$ and $C_n^{(k)}$ are defined by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}=\sum_{n=0}^{\infty}B_n^{(k)}\frac{t^n}{n!},$$

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}=\sum_{n=0}^{\infty}C_n^{(k)}\frac{t^n}{n!}$$

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respectively, as formal power series over $\mathbb{Q}$. Here,

$$\mathrm{Li}_k(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k}$$

is the k-th polylogarithm. Note that $\mathrm{Li}_1(t) = -\log(1-t)$ and $B_n^{(1)} = (-1)^n C_n^{(1)} = B_n$ for $n \geq 0$. Kitahara proved the following congruence for poly-Bernoulli numbers by using p-adic distributions.

THEOREM 1.1 ([7, Theorem 12]). *Let k be an integer, p be an odd prime, and m, n and N be positive integers with $m, n \geq N$ and $k < p-1$. If $m \equiv n \bmod (p-1)p^{N-1}$, then we have*

$$p^{2k'} B_m^{(k)} \equiv p^{2k'} B_n^{(k)} \bmod p^N ,$$

where $k' = \max\{k, 0\}$.

REMARK 1.2. Sakata gave an elementary proof of Theorem 1.1 in the case $k < 0$ ([9, Theorem 6.1]).

In this paper, we will consider a further generalization of Theorem 1.1.

DEFINITION 1.3 ([5, Section 1]). For $\mathbf{k} = (k_1, \cdots, k_r) \in \mathbb{Z}^r$, define the multiple polylogarithm to be

$$\mathrm{Li}_{\mathbf{k}}(t) = \sum_{0<m_1<\cdots<m_r} \frac{t^{m_r}}{m_1^{k_1} \cdots m_r^{k_r}} .$$

Multi-poly-Bernoulli numbers $B_n^{(\mathbf{k})}$ and $C_n^{(\mathbf{k})}$ are defined to be the rational numbers satisfying

$$\frac{\mathrm{Li}_{\mathbf{k}}(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(\mathbf{k})} \frac{t^n}{n!} ,$$

$$\frac{\mathrm{Li}_{\mathbf{k}}(1-e^{-t})}{e^t-1} = \sum_{n=0}^{\infty} C_n^{(\mathbf{k})} \frac{t^n}{n!}$$

respectively, as formal power series over $\mathbb{Q}$. (Note that the order of the summation indices of $\mathrm{Li}_{\mathbf{k}}(t)$ in [5] are reversed. Hence, $B_n^{(k_1,\cdots,k_r)}$ in this paper coincide with $B_n^{(k_r,\cdots,k_1)}$ in [5].)

REMARK 1.4. In [5], some relations between $B_n^{(\mathbf{k})}$ and $C_n^{(\mathbf{k})}$ were proved. For example, we have relations

$$B_n^{(\mathbf{k})} = \sum_{i=0}^{n} \binom{n}{i} C_i^{(\mathbf{k})} ,$$

$$C_n^{(\mathbf{k})} = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} B_i^{(\mathbf{k})} ,$$

$$B_n^{(\mathbf{k})} = C_n^{(\mathbf{k})} + C_{n-1}^{(k_1,k_2,\cdots,k_r-1)}$$

for any $r \geq 1$, $\mathbf{k} = (k_1, k_2, \cdots, k_r) \in \mathbb{Z}^r$ and $n \geq 1$ ([5, Section 2]).

REMARK 1.5. The multiple polylogarithm was introduced in [3]. It is expected to have relations with the multiple zeta values and the multiple zeta functions. It is also known that the multi-poly-Bernoulli numbers $C_n^{(\mathbf{k})}$ are described as the finite multiple zeta values ([5, Theorem 8]).

We call $\mathbf{k} = (k_1, \cdots, k_r) \in \mathbb{Z}^r$ an index. For an index $\mathbf{k}$, we define the weight of $\mathbf{k}$ to be $\mathrm{wt}(\mathbf{k}) = k_1 + \cdots + k_r$ and write $k_i' = \max\{k_i, 0\}$ and $\mathbf{k}^+ = (k_1', \cdots, k_r')$. We will prove the following result in Section 3.

THEOREM 1.6. *Let $\boldsymbol{k} \in \mathbb{Z}^r$ be an index, p be an odd prime and m, n and N be positive integers with $m, n \geq N$ and $\mathrm{wt}(\boldsymbol{k}^+) < p - 1$. If $m \equiv n \bmod (p-1)p^{N-1}$, then we have*

$$p^{2\,\mathrm{wt}(\boldsymbol{k}^+)} B_m^{(\boldsymbol{k})} \equiv p^{2\,\mathrm{wt}(\boldsymbol{k}^+)} B_n^{(\boldsymbol{k})} \bmod p^N ,$$

$$p^{2\,\mathrm{wt}(\boldsymbol{k}^+)} C_m^{(\boldsymbol{k})} \equiv p^{2\,\mathrm{wt}(\boldsymbol{k}^+)} C_n^{(\boldsymbol{k})} \bmod p^N .$$

Note that $\mathrm{wt}(\mathbf{k}^+) = 1$ for the case of ordinary Bernoulli numbers, and hence the assumption of Theorem 1.6 holds naturally for any odd prime p.

In Section 4, we will consider the multi-poly-Bernoulli-star numbers, which were introduced in [4], and find Kummer-type congruences for the multi-poly-Bernoulli-star numbers which are similar to Theorem 1.6.

NOTATION: In this paper, let p be a prime. For $x \in \mathbb{Q}_p$, we denote the p-adic valuation by $\mathrm{ord}_p(x)$. For a real number x, $\lfloor x \rfloor$ means the greatest integer less than or equal to x.

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2. Preliminaries

In this section, we will recall a theory of p-adic distributions.

DEFINITION 2.1. Let h be a non-negative integer. Define $\mathrm{LA}_h(\mathbb{Z}_p, \mathbb{Q}_p)$ to be the set of functions $f : \mathbb{Z}_p \to \mathbb{Q}_p$ which is locally analytic at each point with radius of convergence $\geq p^{-h}$. For $f \in \mathrm{LA}_h(\mathbb{Z}_p, \mathbb{Q}_p)$, the norm of f is given by

$$||f||_h = \sup_{n \geq 0,\, a \in \mathbb{Z}_p}\{|p^{nh} a_n|_p\}$$

for the expansion $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ on $a + p^h\mathbb{Z}_p$. The set $\mathrm{LA}_h(\mathbb{Z}_p, \mathbb{Q}_p)$ is a $\mathbb{Q}_p$-vector space equipped with the topology induced by the norm. Since there exist natural inclusions $\mathrm{LA}_h(\mathbb{Z}_p, \mathbb{Q}_p) \to \mathrm{LA}_{h+1}(\mathbb{Z}_p, \mathbb{Q}_p)$ for all $h \geq 0$, we may define $\mathrm{LA}(\mathbb{Z}_p, \mathbb{Q}_p) =$

$\cup_{h\geq 0}\mathrm{LA}_h(\mathbb{Z}_p,\mathbb{Q}_p)$ equipped with the inductive limit topology. A continuous $\mathbb{Q}_p$-linear map $\mu:\mathrm{LA}(\mathbb{Z}_p,\mathbb{Q}_p)\to\mathbb{Q}_p$ is called a p-adic distribution and we write

$$\int_{\mathbb{Z}_p} f(x)d\mu(x) := \mu(f)$$

for $f\in\mathrm{LA}(\mathbb{Z}_p,\mathbb{Q}_p)$. We denote by $D(\mathbb{Z}_p)$ the set of p-adic distributions.

It is known that the following theorems hold.

THEOREM 2.2 ([8, Lemma 1]). *Let $f:\mathbb{Z}_p\to\mathbb{Q}_p$. The function f is continuous if and only if there exist $a_n\in\mathbb{Q}_p$ such that*

$$f(x)=\sum_{n=0}^{\infty}a_n\binom{x}{n}$$

and $a_n\to 0$ as $n\to\infty$. Here, we define

$$\binom{x}{0}=1,\ \binom{x}{n}=\frac{x(x-1)\cdots(x-n+1)}{n!}\in\mathbb{Q}[x]$$

for $n\geq 1$.

THEOREM 2.3 ([1, Théorème 3]). *Let h be a non-negative integer. For $f:\mathbb{Z}_p\to\mathbb{Q}_p$, $f\in\mathrm{LA}_h(\mathbb{Z}_p,\mathbb{Q}_p)$ if and only if there exist $a_n\in\mathbb{Q}_p$ such that*

$$f(x)=\sum_{n=0}^{\infty}a_n\left\lfloor\frac{n}{p^h}\right\rfloor!\binom{x}{n}$$

and $a_n\to 0$ as $n\to\infty$. Moreover, $||f||_h\leq 1$ holds if and only if $a_n\in\mathbb{Z}_p$ for all $n\geq 0$.

THEOREM 2.4 ([10, Theorem 2.3]). *Let R be the set of formal power series $f(T)$ over $\mathbb{Q}_p$ which converges on the open unit disk. Then the map $D(\mathbb{Z}_p)\to R$ given by*

$$\mu\mapsto\int_{\mathbb{Z}_p}(1+T)^x d\mu(x) := \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\binom{x}{n}d\mu(x)T^n$$

is bijective. The inverse map sends $\sum_{n=0}^{\infty}c_nT^n\in R$ to the element of $D(\mathbb{Z}_p)$ given by

$$\mathrm{LA}(\mathbb{Z}_p,\mathbb{Q}_p)\to\mathbb{Q}_p;\ f(x)=\sum_{n=0}^{\infty}a_n\binom{x}{n}\mapsto\sum_{n=0}^{\infty}a_nc_n\,. \tag{1}$$

REMARK 2.5. Since $f\in\mathrm{LA}(\mathbb{Z}_p,\mathbb{Q}_p)$ is continuous on $\mathbb{Z}_p$, it follows from Theorem 2.2 that f has the expansion as (1) and the infinite sum in (1) is convergent.

Note that, if a formal power series $f(T)\in R$ corresponds to a p-adic distribution μ, we have

$$\left((1+T)\frac{d}{dT}\right)f(T)=\int_{\mathbb{Z}_p}x(1+T)^x d\mu(x)=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}x\binom{x}{n}d\mu(x)T^n$$

and

$$\left((1+T)\frac{d}{dT}\right)^n f(T)\Big|_{T=0} = \int_{\mathbb{Z}_p} x^n d\mu(x) \tag{2}$$

for $n \geq 0$. Indeed, we can check these by using the property

$$x\binom{x}{n} = (n+1)\binom{x}{n+1} + n\binom{x}{n}.$$

3. Proof of Theorem 1.6

In this section, we will prove Theorem 1.6. Our proof is inspired by the proof of [7, Theorem 12]. In the following, let p be an odd prime.

For positive integers m, n and N, by applying Theorem 2.3 to the case $h = 1$ and $p^{-N}(x^m - x^n) \in \mathrm{LA}_1(\mathbb{Z}_p, \mathbb{Q}_p)$, we obtain $a_j \in \mathbb{Q}_p$ satisfying

$$\frac{x^m - x^n}{p^N} = \sum_{j=0}^{\infty} a_j \left\lfloor \frac{j}{p} \right\rfloor ! \binom{x}{j}$$

and $|a_j|_p \to 0$ as $j \to \infty$.

LEMMA 3.1. *If $m, n \geq N$ and $m \equiv n \bmod (p-1)p^{N-1}$, then we have $a_j \in \mathbb{Z}_p$ for any $j \geq 0$.*

Proof. Put $P(x) = p^{-N}(x^m - x^n)$. According to Theorem 2.3, we must prove $||P(x)||_1 \leq 1$ and it suffices to show that $Q(y) := P(c + py) \in \mathbb{Z}_p[y]$ for any $c = 0, 1, \cdots, p-1$. If $c = 0$, it is clear.

Suppose that $c \neq 0$. We put $m - n = (p-1)p^{N-1}d$ with $d \in \mathbb{Z}_{>0}$ and

$$Q(y) = p^{-N}(c+py)^n\{(c+py)^{(p-1)p^{N-1}d} - 1\}.$$

We will check that $(c+py)^{(p-1)p^{N-1}d} \equiv 1 \bmod p^N\mathbb{Z}_p[y]$ by induction on N. When $N = 1$, we see that $(c+py)^{(p-1)d} \equiv c^{(p-1)d} \equiv 1 \bmod p\mathbb{Z}_p[y]$. Let $N > 0$ and suppose that the assertion holds for N. Then there exists a polynomial $R_N(y) \in \mathbb{Z}_p[y]$ such that $(c + py)^{(p-1)p^{N-1}d} = 1 + p^N R_N(y)$ and we have

$$\begin{aligned}(c+py)^{(p-1)p^N d} &= (1 + p^N R_N(y))^p \\ &= \sum_{i=0}^{p} \binom{p}{i} p^{Ni} R_N(y)^i \equiv 1 \bmod p^{N+1}\mathbb{Z}_p[y].\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.6. We omit the proof for $C_n^{(\mathbf{k})}$ because it can be checked by the same argument as the following proof for $B_n^{(\mathbf{k})}$. Put

$$f(x) = \frac{\mathrm{Li}_{\mathbf{k}}(1-e^x)}{1-e^x}$$

and $g(T) = f(\log(1+T))$. In other words, we set

$$f(x) = \sum_{0<m_1<\cdots<m_r} \frac{(1-e^x)^{m_r-1}}{m_1^{k_1}\cdots m_r^{k_r}} = \sum_{n=0}^{\infty}(-1)^n B_n^{(\mathbf{k})}\frac{x^n}{n!},$$

$$g(T) = \sum_{0<m_1<\cdots<m_r} \frac{(-1)^{m_r-1}}{m_1^{k_1}\cdots m_r^{k_r}} T^{m_r-1}.$$

We can check that $g(T)$ converges on the open unit disk. Indeed, since we have

$$\left| \sum_{0<m_1<\cdots<m_r} \frac{(-1)^{m_r-1}}{m_1^{k_1}\cdots m_r^{k_r}} \right|_p \leq m_r^{\mathrm{wt}(\mathbf{k}^+)},$$

it follows that

$$\limsup_{m_r\to\infty} \left| \sum_{0<m_1<\cdots<m_r} \frac{(-1)^{m_r-1}}{m_1^{k_1}\cdots m_r^{k_r}} \right|_p^{\frac{1}{m_r}} \leq 1.$$

Using Theorem 2.4, we get a p-adic distribution μ corresponding to g. The p-adic distribution $\mu : \mathrm{LA}(\mathbb{Z}_p, \mathbb{Q}_p) \to \mathbb{Q}_p$ is given by

$$\varphi \mapsto \sum_{j=r-1}^{\infty}(-1)^j a_j \sum_{0<m_1<\cdots<m_{r-1}<j+1} \frac{1}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}(j+1)^{k_r}},$$

where φ has the expansion $\varphi(x) = \sum_{j=0}^{\infty} a_j \binom{x}{j}$. According to (2), we obtain that

$$\begin{aligned}\int_{\mathbb{Z}_p} x^n d\mu(x) &= \left((1+T)\frac{d}{dT}\right)^n g(T)\Big|_{T=0} \\ &= \left(\frac{d}{dx}\right)^n f(x)\Big|_{x=0} = (-1)^n B_n^{(\mathbf{k})}\end{aligned}$$

for $n \geq 0$.

For positive integers m, n and N with $m \equiv n \bmod (p-1)p^{N-1}$, Theorem 2.3 implies that there exist $a_j \in \mathbb{Q}_p$ such that

$$\frac{x^m - x^n}{p^N} = \sum_{j=0}^{\infty} a_j \left\lfloor \frac{j}{p} \right\rfloor ! \binom{x}{j}$$

and $|a_j|_p \to 0$ as $j \to \infty$. Then we have $a_j \in \mathbb{Z}_p$ for any $j \geq 0$ by Lemma 3.1. We see that

$$\begin{aligned}\int_{\mathbb{Z}_p} \frac{x^m - x^n}{p^N} d\mu(x) &= \sum_{j=0}^{\infty} a_j \left\lfloor \frac{j}{p} \right\rfloor ! \int_{\mathbb{Z}_p} \binom{x}{j} d\mu(x) \\ &= \sum_{j=0}^{\infty} (-1)^j a_j \left\lfloor \frac{j}{p} \right\rfloor ! \sum_{0<m_1<\cdots<m_{r-1}<j+1} \frac{1}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}(j+1)^{k_r}}.\end{aligned}$$

Put

$$h(j) = \left\lfloor \frac{j}{p} \right\rfloor ! \sum_{0<m_1<\cdots<m_{r-1}<j+1} \frac{1}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} (j+1)^{k_r}} \tag{3}$$

for $j \geq r-1$. Note that the summation in the R.H.S. of (3) is empty for $0 \leq j \leq r-2$ and understood to be 0. We will prove the following lemma soon later.

LEMMA 3.2. *If* $\mathrm{wt}(\boldsymbol{k}^+) < p-1$*, then we have*

$$\min_{j \geq r-1} \{\mathrm{ord}_p(h(j))\} \geq -2\,\mathrm{wt}(\boldsymbol{k}^+) .$$

It follows from the above lemma that

$$p^{2\,\mathrm{wt}(\mathbf{k}^+)} \int_{\mathbb{Z}_p} \frac{x^m - x^n}{p^N} d\mu(x) = p^{2\,\mathrm{wt}(\mathbf{k}^+)-N} \left\{ (-1)^m B_m^{(\mathbf{k})} - (-1)^n B_n^{(\mathbf{k})} \right\} \in \mathbb{Z}_p .$$

It is equivalent to the congruence

$$p^{2\,\mathrm{wt}(\mathbf{k}^+)} B_m^{(\mathbf{k})} \equiv p^{2\,\mathrm{wt}(\mathbf{k}^+)} B_n^{(\mathbf{k})} \bmod p^N .$$

□

We will show Lemma 3.2.

Proof of Lemma 3.2. Let $\mathbf{k} = (k_1, \cdots, k_r)$. For $j \leq p-1$, we see that $\mathrm{ord}_p(h(j)) \geq -k_r$. Set $j = ap + i$ $(\geq p)$ with $a \geq 1$ and $0 \leq i \leq p-1$. Then we have

$$\begin{aligned}
&\min_{0 \leq i \leq p-1} \left\{ \mathrm{ord}_p(h(ap+i)) \right\} \\
&= \min_{0 \leq i \leq p-1} \left\{ \mathrm{ord}_p(a!) - k_r \,\mathrm{ord}_p(ap+i+1) + \mathrm{ord}_p \left(\sum_{0<m_1<\cdots<m_{r-1}<ap+i+1} \frac{1}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}}} \right) \right\} \\
&\geq \min_{0 \leq i \leq p-1} \left\{ \mathrm{ord}_p(a!) - k_r' \,\mathrm{ord}_p(ap+i+1) + \min_{0<m_1<\cdots<m_{r-1}<ap+i+1} \left\{ -\sum_{s=1}^{r-1} k_s' \,\mathrm{ord}_p(m_s) \right\} \right\} \\
&= \mathrm{ord}_p(a!) - k_r' \,\mathrm{ord}_p(a+1) - \max_{0<m_1<\cdots<m_{r-1}<(a+1)p} \left\{ \sum_{s=1}^{r-1} k_s' \,\mathrm{ord}_p(m_s) \right\} - k_r' \\
&\geq \mathrm{ord}_p(a!) - k_r' \,\mathrm{ord}_p(a+1) - \max_{0<b_1<\cdots<b_{r-1}\leq a} \left\{ \sum_{s=1}^{r-1} k_s' \,\mathrm{ord}_p(b_s) \right\} - \mathrm{wt}(\mathbf{k}^+) =: F(a) .
\end{aligned}$$

It is enough to prove that $\min_{a \geq 1}\{F(a)\} \geq -2\,\mathrm{wt}(\mathbf{k}^+)$. For $t \geq 0$ and $0 \leq u \leq p-1$, since we see that

$$\mathrm{ord}_p((tp+u)!) = \mathrm{ord}_p((tp+p-1)!)$$

and

$$\max_{0<b_1<\cdots<b_{r-1}\leq tp+u} \left\{ \sum_{s=1}^{r-1} k_s' \,\mathrm{ord}_p(b_s) \right\} \leq \max_{0<b_1<\cdots<b_{r-1}\leq tp+p-1} \left\{ \sum_{s=1}^{r-1} k_s' \,\mathrm{ord}_p(b_s) \right\} ,$$

it suffices to check the case $a \equiv p-1 \mod p$. Putting $a = qp^l - 1$ with $l \geq 1$, $q \geq 1$ and $p \nmid q$, we have

$$\begin{aligned}
&F(qp^l-1)\\
&=\mathrm{ord}_p\left(\frac{(qp^l)!}{qp^l}\right) - k'_r \,\mathrm{ord}_p(qp^l) - \max_{0<b_1<\cdots<b_{r-1}\leq qp^l-1}\left\{\sum_{s=1}^{r-1} k'_s \,\mathrm{ord}_p(b_s)\right\} - \mathrm{wt}(\mathbf{k}^+)\\
&=\mathrm{ord}_p((qp^l)!) - (k'_r+1)\,\mathrm{ord}_p(qp^l) - \max_{0<b_1<\cdots<b_{r-1}\leq qp^l-1}\left\{\sum_{s=1}^{r-1} k'_s \,\mathrm{ord}_p(b_s)\right\} - \mathrm{wt}(\mathbf{k}^+)\\
&=q\frac{p^l-1}{p-1} + \mathrm{ord}_p(q!) - (k'_r+1)l - \max_{0<b_1<\cdots<b_{r-1}\leq qp^l-1}\left\{\sum_{s=1}^{r-1} k'_s \,\mathrm{ord}_p(b_s)\right\} - \mathrm{wt}(\mathbf{k}^+)\,.
\end{aligned}$$

If $1 \leq q \leq p-1$, since $b_s \leq (p-1)p^l - 1 < p^{l+1}$ and $\mathrm{ord}_p(b_s) \leq l$ for $1 \leq s \leq r-1$, we find that

$$\begin{aligned}
F(qp^l-1) &= q\frac{p^l-1}{p-1} - (k'_r+1)l - \max_{0<b_1<\cdots<b_{r-1}\leq qp^l-1}\left\{\sum_{s=1}^{r-1} k'_s \,\mathrm{ord}_p(b_s)\right\} - \mathrm{wt}(\mathbf{k}^+)\\
&\geq q\frac{p^l-1}{p-1} - (k'_r+1)l - \left(\sum_{s=1}^{r-1} k'_s\right) l - \mathrm{wt}(\mathbf{k}^+)\\
&\geq \frac{p^l-1}{p-1} - (\mathrm{wt}(\mathbf{k}^+)+1)l - \mathrm{wt}(\mathbf{k}^+)\\
&\begin{cases} = -2\,\mathrm{wt}(\mathbf{k}^+) & \text{if } l=1\\ \geq p+1-2(\mathrm{wt}(\mathbf{k}^+)+1)-(p-2) & \text{if } l\geq 2\end{cases}\\
&\geq -2\,\mathrm{wt}(\mathbf{k}^+)\,.
\end{aligned}$$

Note that we used the assumption $\mathrm{wt}(\mathbf{k}^+) < p-1$ in the case $l \geq 2$.

If $q \geq p+1$, set $q = \sum_{i=0}^{d} c_i p^i$ with $0 \leq c_i \leq p-1$, $c_0 c_d \neq 0$ and $d \geq 1$. Then it follows that

$$\begin{aligned}
&F(qp^l-1)\\
&\geq\frac{p^l-1}{p-1}\sum_{i=0}^{d} c_i p^i + \frac{1}{p-1}\sum_{i=1}^{d} c_i(p^i-1) - (k'_r+1)l - \left(\sum_{s=1}^{r-1} k'_s\right)(d+l) - \mathrm{wt}(\mathbf{k}^+)\\
&\geq\frac{p^l-1}{p-1}(p^d+1) + \frac{p^d-1}{p-1} - (\mathrm{wt}(\mathbf{k}^+)+1)l - \left(\sum_{s=1}^{r-1} k'_s\right) d - \mathrm{wt}(\mathbf{k}^+)\\
&=\frac{p^{l+d}+p^l-2}{p-1} - (\mathrm{wt}(\mathbf{k}^+)+1)l - \left(\sum_{s=1}^{r-1} k'_s\right) d - \mathrm{wt}(\mathbf{k}^+)
\end{aligned}$$

$$\geq \frac{p^{d+1}+p-2}{p-1} - \left(\sum_{s=1}^{r-1} k'_s\right) d - 2\,\mathrm{wt}(\mathbf{k}^+) - 1$$
$$= \left(1+\frac{1}{p-1}\right) p^d - \left(\sum_{s=1}^{r-1} k'_s\right) d - 2\,\mathrm{wt}(\mathbf{k}^+) - \frac{1}{p-1}$$
$$\geq \left(1+\frac{1}{p-1}\right) p - \sum_{s=1}^{r-1} k'_s - 2\,\mathrm{wt}(\mathbf{k}^+) - \frac{1}{p-1}$$
$$= \left(p - \sum_{s=1}^{r-1} k'_s\right) + 1 - 2\,\mathrm{wt}(\mathbf{k}^+) > -2\,\mathrm{wt}(\mathbf{k}^+)\,.$$

This completes the proof. □

REMARK 3.3. We obtain the explicit formula of $B_n^{(\mathbf{k})}$ by using the p-adic distribution μ in the proof of Theorem 1.6 as follows. For $n \geq 0$, it is known that we have

$$x^n = \sum_{j=0}^{n} \left\{ {n \atop j} \right\} j! \binom{x}{j},$$

where, for any integers a and b, $\left\{ {a \atop b} \right\}$ are called the Stirling numbers of the second kind and defined by the recurrence formula

$$\left\{ {a+1 \atop b} \right\} = \left\{ {a \atop b-1} \right\} + b \left\{ {a \atop b} \right\}$$

with the conditions $\left\{ {0 \atop 0} \right\} = 1$ and $\left\{ {a \atop b} \right\} = 0$ for $a < b$ ([2, Definition 2.2, Proposition 2.6]). Then we find that

$$B_n^{(\mathbf{k})} = (-1)^n \int_{\mathbb{Z}_p} x^n d\mu(x) = (-1)^n \sum_{j=0}^{n} \left\{ {n \atop j} \right\} j! \int_{\mathbb{Z}_p} \binom{x}{j} d\mu(x)$$
$$= (-1)^n \sum_{j=0}^{n} \left\{ {n \atop j} \right\} j! \sum_{0<m_1<\cdots<m_{r-1}<j+1} \frac{(-1)^j}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}(j+1)^{k_r}}$$
$$= (-1)^n \sum_{0<m_1<\cdots<m_{r-1}<m_r\leq n+1} \frac{(-1)^{m_r-1}(m_r-1)!\left\{ {n \atop m_r-1} \right\}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}} m_r^{k_r}}\,.$$

By exactly the same way, we get

$$C_n^{(\mathbf{k})} = (-1)^n \sum_{0<m_1<\cdots<m_{r-1}<m_r\leq n+1} \frac{(-1)^{m_r-1}(m_r-1)!\left\{ {n+1 \atop m_r} \right\}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}} m_r^{k_r}}\,.$$

These formulas were proved in [5, Theorem 3] by using the generating functions.

EXAMPLE 3.4. We see that $B_4^{(-1,2)} = \frac{31}{60} \in 5^{-1}\mathbb{Z}_5$ and $B_{504}^{(-1,2)} = A/44375269362060 \in 5^{-1}\mathbb{Z}_5$, where A is a 757-digit integer, by computer calculation. Hence, it follows that $5^4 B_4^{(-1,2)} \equiv 5^4 B_{504}^{(-1,2)} \bmod 5^3$.

REMARK 3.5. It was claimed in [7, Theorem 13] that, given an odd prime p and positive integers m, n, k, N with $p \geq \max\{k+2, (N+k)/2\}$ and $m \equiv n \bmod (p-1)p^N$, one has $p^k B_m^{(k)} \equiv p^k B_n^{(k)} \bmod p^N$. However, there are counterexamples: $pB_1^{(1)} = p/2 \not\equiv 0 = pB_m^{(1)} \bmod p^N$ for $N \geq 2$ and $m = (p-1)p^N + 1$. (Its proof breaks down at [7, Proposition 11], for which $j = p^2 + p - 1$ yields a counterexample.)

4. Multi-poly-Bernoulli-star numbers

At the end of this paper, we will give Kummer-type congruences for other Bernoulli numbers.

DEFINITION 4.1 ([4, Section 1]). For $\mathbf{k} = (k_1, \cdots, k_r) \in \mathbb{Z}^r$, define the non-strict multiple polylogarithm to be

$$\mathrm{Li}^{\star}_{\mathbf{k}}(t) = \sum_{0<m_1\leq\cdots\leq m_r} \frac{t^{m_r}}{m_1^{k_1}\cdots m_r^{k_r}} .$$

The multi-poly-Bernoulli-star numbers $B_{n,\star}^{(\mathbf{k})}$ and $C_{n,\star}^{(\mathbf{k})}$ are defined to be the rational numbers satisfying

$$\frac{\mathrm{Li}^{\star}_{\mathbf{k}}(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_{n,\star}^{(\mathbf{k})} \frac{t^n}{n!} ,$$

$$\frac{\mathrm{Li}^{\star}_{\mathbf{k}}(1-e^{-t})}{e^{t}-1} = \sum_{n=0}^{\infty} C_{n,\star}^{(\mathbf{k})} \frac{t^n}{n!}$$

respectively, as formal power series over $\mathbb{Q}$.

REMARK 4.2. Similar relations to Remark 1.4 were proved in [4, Proposition 2.3, 2.4]. Furthermore, the multi-poly-Bernoulli-star numbers $B_{n,\star}^{(\mathbf{k})}$ and $C_{n,\star}^{(\mathbf{k})}$ satisfy a duality relation for $\mathbf{k} = (k_1, \cdots, k_r) \in \mathbb{Z}_{>0}^r$ ([4, Theorem 3.2]).

REMARK 4.3. It is known that the multi-poly-Bernoulli-star numbers $C_{n,\star}^{(\mathbf{k})}$ are described as finite multiple zeta-star values ([4, Section 4]).

The following theorem can be shown by exactly the same argument as Theorem 1.6 and hence is omitted.

THEOREM 4.4. *Let $\boldsymbol{k} \in \mathbb{Z}^r$ be an index, p be an odd prime and m, n and N be positive integers with $m, n \geq N$ and $\mathrm{wt}(\boldsymbol{k}^+) < p-1$. If $m \equiv n \bmod (p-1)p^{N-1}$, then we have*

$$p^{2\,\mathrm{wt}(\boldsymbol{k}^+)} B_{m,\star}^{(\boldsymbol{k})} \equiv p^{2\,\mathrm{wt}(\boldsymbol{k}^+)} B_{n,\star}^{(\boldsymbol{k})} \bmod p^N ,$$

$$p^{2\,\mathrm{wt}(\boldsymbol{k}^+)} C_{m,\star}^{(\boldsymbol{k})} \equiv p^{2\,\mathrm{wt}(\boldsymbol{k}^+)} C_{n,\star}^{(\boldsymbol{k})} \bmod p^N .$$

REMARK 4.5. We can check the following formulas

$$B_{n,\star}^{(\mathbf{k})} = (-1)^n \sum_{0<m_1\le\cdots\le m_{r-1}\le m_r\le n+1} \frac{(-1)^{m_r-1}(m_r-1)!\left\{ {n \atop m_r-1} \right\}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}} m_r^{k_r}},$$

$$C_{n,\star}^{(\mathbf{k})} = (-1)^n \sum_{0<m_1\le\cdots\le m_{r-1}\le m_r\le n+1} \frac{(-1)^{m_r-1}(m_r-1)!\left\{ {n+1 \atop m_r} \right\}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}} m_r^{k_r}}$$

by the same computation as Remark 3.3. These were obtained in [4, Proposition 2.2] by using the generating functions.

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