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Chapter 1

Questioning some paradigms of signal processing via concrete examples

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Abstract We are proposing an algebraic estimation theory of noisy signals. This approach which is bypassing probabilistic techniques does not necessitate any *a priori* sampling of the signals. Several examples are illustrating the efficiency of our techniques, which yield very fast computations.

Keywords: Linear systems, identifiability, parametric identification, module theory, differential algebra, operational calculus.

1.1 Introduction

Some of the underlying methods of *modern* signal processing (see, *e.g.*, [28]) may be summarised as follows:

- An extensive use of highly developed probabilistic tools has become quite universal.
- Shannon's information theory and digital computers have imposed an almost exclusive analysis of discrete-time signals.

Our algebraic standpoint (see also [13]), which is based on differential fields[1], ring theory, and operational calculus, leads to the following facts:

- No precise statistical knowledge of the noise is required[2].
- We are keeping the "true" physical nature of the continuous-time signals, which might be forgotten when sampling[3].
- There is no distinction between stationary and non-stationary signals.
- The computations of the estimates can be done on-line.

Mikusiński's approach to operational calculus[4] (see [23, 24] and [31]) permits a straightforward introduction of the field theoretic language, which is most

[1]Differential fields [3, 18] are already playing some rôle in non-linear control (see, *e.g.*, [4, 9]).

[2]*Unknown but Bounded* and *Interval Analysis* are other ways of a complete different nature for avoiding probability and statistics in estimation. See, *e.g.*, [16], [29] and the references therein.

[3]Note that the *differentially flat* systems [9], which are so useful in practice (see, *e.g.*, [30]), have also shed a new light on sampling in control.

[4]This setting ignores the Laplace transform (see, *e.g.*, [7, 26] for an introduction to operational calculus via the Laplace transform). What matters are the convolution product and the operational properties.

convenient. After a brief review of differential algebra, of non-commutative differential operators and of operational calculus, section 1.2 defines various types of identifiability, which are often encountered in practice (compare with [12]). We are able to handle perturbations, which are *structured*, *i.e.*, which, like a constant perturbation of unknown amplitude, are solutions of a given homogeneous linear differential equation. By annihilating them via a suitable differential operator, we are avoiding any bias due, for instance, to a random noise of unknown constant mean. Unstructured noises, which are understood as high frequency perturbations, are attenuated via low pass filters.

Section 1.6.1 is providing the determination of A, ω, ϕ in $A\sin(\omega t + \phi)$, which under various guises is crucial in signal analysis[5]. For the non-stationary piecewise polynomial signal in section 1.6.2, borrowed from [21], we are able to calculate the coefficients of the various polynomials in the presence of a non-classic random noise, and to determine quite precisely the discontinuities. These results seem to be new. Parameter estimation in non-stationary context is presented further through section 1.6.3 with a chirp signal and section 1.6.4 with a polynomial phase signal with time-varying amplitude. Computer simulations are indicating the efficiency of our approach.

The algebraic approach presented here has also proved to be successful in other estimation problems [12, 14]

Acknowledgement.

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1.2 Basic mathematical notions

1.2.1 Differential algebra[6]

Basic definitions

A *differential ring*, or, more precisely, an *ordinary differential ring*, R is a commutative ring which is equipped with a single derivation, written here $\frac{d}{ds}$, *i.e*, a map $R \to R$ such that, $\forall\ x, y \in R$,

[5] We are thus solving an engineering problem which is largely open when the random noise is not Gaussian and/or with poorly known statistics.

[6] See [3, 18] for more details.

- $\frac{d}{ds}(x+y) = \frac{dx}{ds} + \frac{dy}{ds}$,
- $\frac{d}{ds}(xy) = \frac{dx}{ds}y + x\frac{dy}{ds}$.

A *differential field*, or, more precisely, an *ordinary differential field*, is a differential ring which is a field[7]. A *constant* $c \in R$ is such that $\frac{dc}{ds} = 0$. The set of all constants of a given differential ring (resp. field) is a differential subring (resp. subfield), called the *subring* (resp. *subfield*) *of constants*. A *(differential) ring* (resp. *field*) *of constants* is a differential ring (resp. field) whose elements are constant.

Example 1.2.1. Let k be a differential field of constants. The field $k(s)$ of rational functions in the indeterminate s, with coefficients in k, possesses an obvious structure of differential field with respect to $\frac{d}{ds}$. Its subfield of constants is k.

A *differential morphism* $\phi : R_1 \to R_2$ between two differential rings is a ring morphism such that, $\forall x \in R_1$, $\frac{d}{ds}(\phi(x)) = \phi(\frac{dx}{ds})$. A *differential specialisation* $R \to K$ is a differential morphism where R is differential ring and K a differential field.

Differential field extension

A *differential field extension* L/K is given by two differential fields K, L such that

- $K \subseteq L$,
- the restriction to K of the derivation of L is the derivation of K.

An element $x \in L$ is said to be *differentially algebraic* over K if, and only if, x satisfies an algebraic differential equation over K, *i.e.*, $P(x, \frac{dx}{ds}, \ldots, \frac{d^n x}{ds^n}) = 0$, where P is a polynomial over K. The extension L/K is said to be *differentially algebraic* if, and only if, any element of L is differentially algebraic over K.

An element of L which is not differentially algebraic over K is said to be *differentially transcendental*. A *differentially transcendental* extension L/K is an extension which is not differentially algebraic.

Notation Let S be a subset of L. The differential overfield (resp. overring) of K generated by S is written $K < S >$ (resp. $K\{S\}$).

[7]All fields are assumed here to be of characteristic 0. See [1] for basic notions in commutative algebra.

1.2.2 Linear differential operators

Let K be a differential field. The ring $K[\frac{d}{ds}]$ of linear differential operators of the form $\sum_{\text{finite}} a_\alpha \frac{d^\alpha}{ds^\alpha}$, $a_\alpha \in K$, is commutative if, and only if, K is a field of constants. For seeing it, take an element $a \in K$. Then, $\frac{d}{ds}a = \frac{da}{ds} + a\frac{d}{ds}$. Even in the general non-commutative case, it is known (see, *e.g.*, [22]) that $K[\frac{d}{ds}]$ is a principal left and right ideal ring: any left or right ideal may be generated by a single element.

1.2.3 The differential field of Mikusiński's operators[8]

Mikusiński's field of operators

Endow the set $\mathcal{C}$ of continuous functions $[0, +\infty) \to \mathbb{C}$ with a structure of commutative ring with respect to the addition $(f+g)(t) = f(t) + g(t)$ and to the convolution (product) $(f \star g)(t) = (g \star f)(t) = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t g(\tau)f(t-\tau)d\tau$. According to a famous theorem due to Titchmarsh (see [23, 24, 31]), $\mathcal{C}$ does not possess zero divisors. Any element of the *Mikusiński field* $\mathcal{M}$, *i.e.*, the quotient field of $\mathcal{C}$, is called an *operator*. Any function $f : \mathbb{R} \to \mathbb{C}$, which belongs to $\mathcal{M}$, may also be written $\{f\}$. Note that in general the product of two elements $a, b \in \mathcal{M}$ will be written ab and not $a \star b$. Some examples are in order:

1. The neutral element $1 \in \mathcal{M}$ with respect to the convolution is the analogue of the Dirac measure at $t = 0$ in Schwartz's distribution theory.

2. Any locally Lebesgue-integrable function $\mathbb{R} \to \mathbb{C}$ with a left bounded support belongs to $\mathcal{M}$.

3. The inverse in $\mathcal{M}$ of the Heaviside function

$$\mathbf{1}(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ 1 & \text{if} \quad t \geq 0 \end{cases}$$

   is the derivation operator s. Let $f : \mathbb{R} \to \mathbb{C}$ be a C^1-function with a left bounded support. Then $s\{f\} = \{f'\}$. Let $g : \mathbb{R} \to \mathbb{C}$ be a locally Lebesgue-integrable function with a left bounded support. Then $\frac{\{g\}}{s} = \{\int_{-\infty}^t g(\sigma)d\sigma\}$ has also a left bounded support. The meaning of the subfield $\mathbb{C}(s) \subset \mathcal{M}$ of rational functions over $\mathbb{C}$ in the indeterminate s is the usual one in operational calculus (see, *e.g.*, [7, 23, 24, 26, 31]).

[8] See [23, 24] and [31] where the notion of *differential field* is of course absent.

4. The meaning of the *delay* operator e^{-Ls}, $L \in \mathbb{R}$, is the usual one in operational calculus (see, *e.g.*, [7, 23, 24, 26, 31]). It is the analogue of the Dirac measure at $t = L$ in the theory of distributions.

The algebraic derivative

For any $f \in \mathcal{C}$, it is known (see [23, 24, 31]) that the mapping $f \mapsto \frac{df}{ds} = \{-tf\}$ satisfies the properties of a derivation, *i.e.*,

$$\frac{d}{ds}(f+g) = \frac{df}{ds} + \frac{dg}{ds}$$

and

$$\frac{d}{ds}(f \star g) = \frac{df}{ds} \star g + f \star \frac{dg}{ds}$$

It can be trivially extended to a derivation, called the *algebraic derivative*, of $\mathcal{M}$ by setting, if $g \neq 0$,

$$\frac{d}{ds}(\{f\} \star \{g\}^{-1}) = \frac{\frac{df}{ds} \star g - f \star \frac{dg}{ds}}{\{g\}^2}$$

Endowed with the algebraic derivative, $\mathcal{M}$ becomes a differential field, whose subfield of constants is $\mathbb{C}$.

1.3 Identifiability

All fields are subfields of a differential field which is a *universal extension* [18] of the field $\mathbb{Q}$ of rational numbers.

1.3.1 The mathematical framework

Let k_0 be a given ground field, which is assumed to be a differential field of constants. Let k be a finite algebraic extension of $k_0(\boldsymbol{\Theta})$ where $\boldsymbol{\Theta} = (\theta_1, \ldots, \theta_r)$ is a finite set of *unknown parameters*. Thus the transcendence degree of the extension k/k_0 is $\leq r$. Moreover we give to k a canonical structure of a differential field of constants. Let $K/k(s)$ be a finitely generated differentially algebraic extension. A *signal* is an element of K. Take a finite set $\boldsymbol{x} = (x_1, \ldots, x_\kappa)$ of signals. The parameters $\boldsymbol{\Theta}$ are said to be

- *algebraically* (resp. *rationally*) *identifiable*[9] with respect to $\boldsymbol{x}$ if, and only if, $\theta_1, \ldots, \theta_r$ are algebraic over (belong to) $k_0 < s, \boldsymbol{x} >$;

[9] Those definitions are borrowed from [5, 6].

- *linearly identifiable*[10] with respect to $\boldsymbol{x}$ if, and only if,

$$P \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_r \end{pmatrix} = Q \tag{1.1}$$

where

  - P and Q are respectively $r \times r$ and $r \times 1$ matrices,
  - the entries of P and Q belong to $\mathrm{span}_{k_0(s)[\frac{d}{ds}]}(1, \boldsymbol{x})$,
  - $\det(P) \neq 0$.

- *projectively linearly identifiable* with respect to $\boldsymbol{x}$ if, and only if, there exists $\theta_\epsilon \neq 0$ such that $\frac{\theta_1}{\theta_\epsilon}, \ldots, \frac{\theta_{\epsilon-1}}{\theta_\epsilon}, \frac{\theta_{\epsilon+1}}{\theta_\epsilon}, \ldots, \frac{\theta_r}{\theta_\epsilon}$ are linearly identifiable.

- *weakly linearly identifiable* with respect to $\boldsymbol{x}$ if, and only if, there exists a finite set $\boldsymbol{\Theta}' = (\theta'_1, \ldots, \theta'_{q'})$ such that

  - the components of $\boldsymbol{\Theta}'$ (resp. $\boldsymbol{\Theta}$) are algebraic over $k_0(\boldsymbol{\Theta})$ (resp. $k_0(\boldsymbol{\Theta}')$),
  - $\boldsymbol{\Theta}'$ is linearly identifiable.

The following result is clear:

Proposition 1.3.1. *Linear (resp. rational) identifiability implies rational (resp. algebraic) identifiability. Linear (resp. weak linear) identifiability implies weak linear (resp. algebraic) identifiability.*

1.3.2 Rational signals

A *rational signal* is an element of $k(s)$.

Proposition 1.3.2. *Assume that the numerator and the denominator of*

$$x = \frac{b_0 + b_1 s + \cdots + b_{n-1} s^{n-1}}{a_0 + a_1 s + \cdots + a_{n-1} s^{n-1} + s^n} \tag{1.2}$$

are coprime. Then, the coefficients $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}$ *are linearly identifiable*[11] *with respect to* x.

---

[10] This definition as well as the two below are adapted from [12].

[11] $k_0 = \mathbb{Q}$, $k = k_0(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1})$.

Proof. Equation (1.2) yields the linear system of equations of type (1.1):

$$\frac{d^\nu}{ds^\nu}\left[b_0 + \cdots + b_{n-1}s^{n-1} - (a_0 + \cdots + a_{n-1}s^{n-1})x\right] = \frac{d^\nu(s^n x)}{ds^n} \qquad \nu = 0, 1, \ldots, 2n-1 \tag{1.3}$$

□

1.3.3 Introducing exponentials

Lemma 1.3.3. *The expression*

$$x = \sum_{\nu=0}^{M} c_\nu^{(1)} \left(\frac{1}{s^{\nu+1}} - \sum_{\mu=0}^{\nu} \frac{L^\mu}{\mu!} \frac{1}{s^{\nu-\mu+1}} \right) + e^{-Ls} \sum_{\nu=0}^{M} \frac{c_\nu^{(2)}}{s^{\nu+1}}$$

where

- $k_0 = \mathbb{Q}$, $k = k_0(c_\nu^{(1)}, c_\nu^{(2)}, L)$,
- e^{-Ls} *satisfies the differential equation* $(\frac{d}{ds} - L)e^{-Ls} = 0$,

is differentially algebraic over $k(s)$*.*

Proof. It follows from the fact that finite sums and products of differentially algebraic elements over $k(s)$ are again differentially algebraic over $k(s)$. □

Proposition 1.3.4. *The parameters* L, $c_\nu^{(\iota)}$, $\nu = 1, \ldots, N$, $\iota = 1, 2$, *are algebraically identifiable with respect to the signal* x*.*

Proof. Take as in the proof of theorem 1.3.2 sufficiently many derivatives of $s^M x$ with respect to s. The conclusion follows from the transcendence[12] of e^{-Ls} over $k(s)$. □

1.3.4 Differentially rational signals

A signal x is said to be *differentially rational* if, and only if, $L(x) = p$, where $L \in k(s)[\frac{d}{ds}]$, $p \in k(s)$. Set

$$\left(\sum_{\text{finite}} a_{\alpha\beta} s^\alpha \frac{d^\beta}{ds^\beta} \right) x = \sum_{\text{finite}} b_\gamma s^\gamma \tag{1.4}$$

$k_0 = \mathbb{Q}$, $k = k_0(a_{\alpha\beta}, b_\gamma)$. The next result, which is a direct generalisation of proposition 1.3.2, may be proved in the same way.

[12] See [11] for a direct proof without having recourse to analytic functions.

Proposition 1.3.5. *Assume that in Equation 1.4 the polynomials* $\sum_\alpha a_{\alpha\beta} s^\alpha$ *and* $\sum_\gamma b_\gamma s^\gamma$ *are coprime. Then, the coefficients* $a_{\alpha\beta}$, b_γ *are projectively linearly identifiable with respect to* x.

1.4 Structured noises and linear estimators

1.4.1 Noises

Let k_1/k_0 be a differential field extension such that

- k_1 is a differential field of constants,
- k and k_1 are linearly disjoint over k_0.

A *noise* ϖ is an element of a differential overfield N of $k_1(s)$ such that K and N are linearly disjoint over $k_0(s)$. It is said to be *structured*[13] if, and only if, it is annihilated by $\Pi \in k_0(s)[\frac{d}{ds}]$, $\Pi \neq 0$.

Example 1.4.1. Consider the noise $\frac{\gamma}{s^\nu}$, $\gamma \in k_1$. It is annihilated by $\nu s^{\nu-1} + s^\nu \frac{d}{ds} \in k_0(s)[\frac{d}{ds}]$, which does not depend on γ.

1.4.2 Noisy signals

A *signal with an additive noise* is a sum $x + \varpi$, where $x \in K$ is a signal and $\varpi \in N$ a noise. Let $\boldsymbol{y} = (y_1, \ldots, y_\kappa)$, where $y_\iota = x_\iota + \varpi_\iota$ be a finite set of such noisy signals. If the parameters $\boldsymbol{\Theta}$ are linearly identifiable, Equation (1.1) becomes

$$P \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_r \end{pmatrix} = Q + Q' \tag{1.5}$$

where

- the matrices P and Q are obtained from (1.1) by substituting $\boldsymbol{y}$ to $\boldsymbol{x}$,
- the entries of the $r \times 1$ matrix Q' belong to $\mathrm{span}_{k'(s)[\frac{d}{ds}]}(\boldsymbol{\varpi})$, where k' is the quotient field of $k \otimes_{k_0} k_1$, and $\boldsymbol{\varpi} = (\varpi_1, \ldots, \varpi_\kappa)$.

[13] This definition might also be expressed in terms of *Picard-Vessiot extensions*. See, *e.g.*, [17, 18].

1.4.3 Linear estimators

Assume that the components of $\boldsymbol{\varpi}$ are structured. The next fundamental theorem follows at once from the fact that $k_0(s)[\frac{d}{ds}]$ is a principal left ideal ring.

Theorem 1.4.2. *There exists $\Delta \in k_0(s)[\frac{d}{ds}]$, such that Equation (1.5) becomes*

$$\Delta P \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_r \end{pmatrix} = \Delta Q \tag{1.6}$$

Equation (1.6), which is independent of the noises, is called a *linear estimator* of the unknown parameters if, and only if, $\det(\Delta P) \neq 0$.

Let ℓ be a differential field of constants. A differential operator in $\ell(s)[\frac{d}{ds}]$ is said to be *proper* (resp. *strictly proper*) if, and only if, the coefficients of $\frac{d^\alpha}{ds^\alpha}$ are proper (resp. strictly proper) rational functions in $\ell(s)$. The estimator (1.6) is said to be *proper* (resp. *strictly proper*) if, and only if, the entries of ΔP and ΔQ are proper (resp. strictly proper) differential operators. Multiplying both sides of equation (1.6) by a suitable proper element of $k_0(s)$ yields the

Proposition 1.4.3. *Any linear estimator may be replaced by a proper (resp. strictly proper) one.*

1.4.4 Linear estimator of a noisy rational signal

Set

$$y = x + \frac{\gamma}{s}$$

where x is given by equation (1.2). The analogue of equation (1.5) is obtained by substituting $y - \frac{\kappa}{s}$ to x in equation (1.3). The next result is adapted from section 1.4.3:

Proposition 1.4.4. *The system of linear equations*

$$\frac{d^{n+\nu}}{ds^{n+\nu}} \left[s^{\nu+1} \left(\frac{d^\nu}{ds^\nu} \left(b_0 + \cdots + b_{n-1}s^{n-1} - (a_0 + \cdots + a_{n-1}s^{n-1})y \right) - \frac{d^\nu (s^n y)}{ds^n} \right) \right] = 0$$

$$\nu = 0, 1, \ldots, 2n-1$$

is a linear identifier if, and only if, the residue of x at $s = 0$ is 0.

Proof. If $c_{-1} \neq 0$ in the Laurent series expansion $x = \sum_{\nu \in \mathbb{Z}} c_\nu s^\nu$, $\frac{c_{-1}}{s}$ is annihilated in equation (1.7) with $\frac{\kappa}{s}$. □

1.5 Exploiting linear estimators

1.5.1 Obtaining numerical values

For obtaining "numerical" values of the unknown parameters $\boldsymbol{\Theta}$ define a differential specialisation $k_0\{\boldsymbol{y}\} \to \mathcal{M}$ over k_0.

1.5.2 Unstructured noises

A noise which is not structured is said to be *unstructured*. In practice we will assume that when specialised to $\mathcal{M}$ such a noise corresponds to a "high frequency" time-function which may be attenuated by a low pass filter.

Example 1.5.1. Set, for instance,

$$\varpi(t) = \begin{cases} 0 \quad \text{if} \quad t < 0 \\ \sin(\Omega t + \varphi) \quad \text{if} \quad t \geq 0 \end{cases}$$

Then

$$\frac{\varpi}{s} = \int_0^t \varpi(\tau) d\tau = \begin{cases} 0 \quad \text{if} \quad t < 0 \\ \frac{\cos(\Omega t + \varphi)}{\Omega} \quad \text{if} \quad t \geq 0 \end{cases}$$

It goes to 0 when $\Omega \to \infty$.

1.6 Examples

1.6.1 A corrupted sinusoid

Set, for $t \geq 0$,

$$y(t) = A \sin(\omega t + \phi) + \kappa + n(t), \tag{1.7}$$

where κ represents an unknown constant bias attached to an unstructured noise $n(t)$. Consider first the noise-free signal $\tilde{y}(t) = A \sin(\omega t + \phi) + \kappa$ which also reads

$$\tilde{y} = A \frac{\omega \cos \phi + s \sin \phi}{s^2 + \omega^2} + \frac{\kappa}{s}$$

The numerical simulations of figure 1.1 were obtained for

$$\tilde{y}(t) = 117.5 \sin(2\pi f t + 0.44) + \kappa$$

where $f = 60$ Hz, by a direct application of section 1.4.4. They depict the precision and speed of calculations.

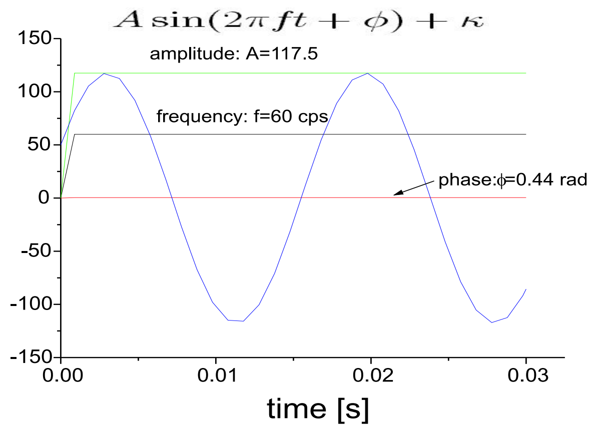

Figure 1.1: Estimation of the unknown parameters of a sinusoid corrupted by a constant perturbation of unknown amplitude.

The robustness of the proposed estimation was tested when the biased sinusoid signal is also corrupted by a high frequency perturbation signal $n(t)$.

$$y(t) = A\sin(2\pi ft + \phi) + \kappa + n(t)$$

We set for the computer simulations $n(t) = 0.1\sin(900t + 300\cos(1000t))$. Figure 1.2 shows the accuracy of the estimations in the simulation results. It can be seen that the mean values of the estimates coincide with the actual values of the parameters.

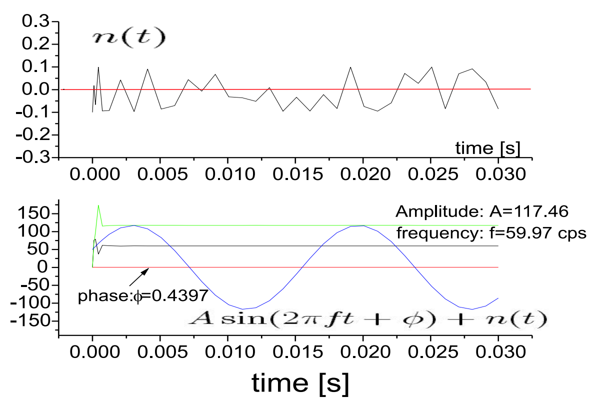

Figure 1.2: Estimation of the unknown parameters of a sinusoid in the presence of a high frequency noise.

Although this example is somehow of academic nature, the speed of calculations make it very relevant in many real-life applications. The demodulation of continuous phase modulated (CPM) signals [28] is such an application. Indeed, a full response (1REC) CPM signal has, in each symbol period interval, the form given by equation (1.7). An on-line symbol-by-symbol detection method exploiting the results of this example are presented in [10].

1.6.2 A piecewise polynomial signal

Set

$$\begin{aligned} p_0(t) &= -3(t-t_0)+3, \\ p_1(t) &= -4(t-t_1)^3/6+5(t-t_1)^2/2-2(t-t_1)+2, \\ p_2(t) &= (t-t_2)^2-2(t-t_2)+2 \end{aligned}$$

to be a sequence of unknown time polynomial signals measured by $y_i(t) = p_i(t) + \varpi(t)$ where $\varpi(t)$ is a zero mean value stochastic process constituted, at each time t, by a rectangularly distributed computer-generated random variable[14].

To carry out the simulations, the polynomial signals were generated as solutions of a perturbed linear differential equation of the form: $z^{(5)} = \nu(t), y = z + \varpi(t)$, with suitable (unknown) initial conditions and $\nu(t)$ being also a zero mean stochastic process.

Figure 1.3 shows the sequence of polynomials estimates, which are seen to converge quite fast to the ideal signal and the results of the constant parameter identification in the noisy environment. It should be pointed out that in the previous simulations, the instants t_i, at which the polynomial signal $p_i(t)$ changed into a new one $p_{i+1}(t)$, were known beforehand. It is not difficult to see that the proposed identification algorithm is also capable of depicting the instant at which the new polynomial signal arrives, when such discontinuity instants are randomly selected. Being unaware of the signal change, results in a noticeable drifting of the constant values of the parameters being currently identified. This allows for a simple and timely re-initialization of the estimation algorithm. Figure 1.4 depicts an example of the estimated parameters drift that occurs when a second order polynomial signal is suddenly changed to a different one.

[14]Proposition 1.4.4 is telling us that such a signal cannot be identified if it is corrupted by a constant perturbation.

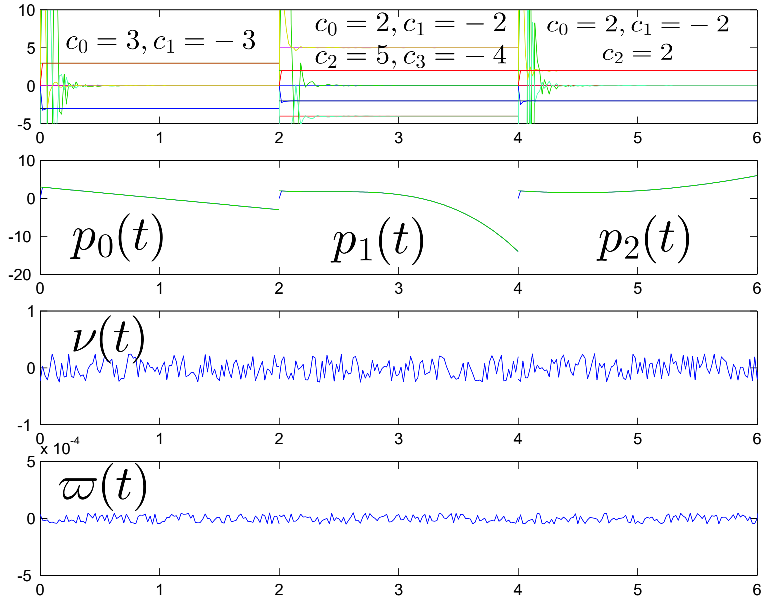

Figure 1.3: A sequence of noisy measured polynomial signals, generated by a noisy system, and their estimated parameter values.

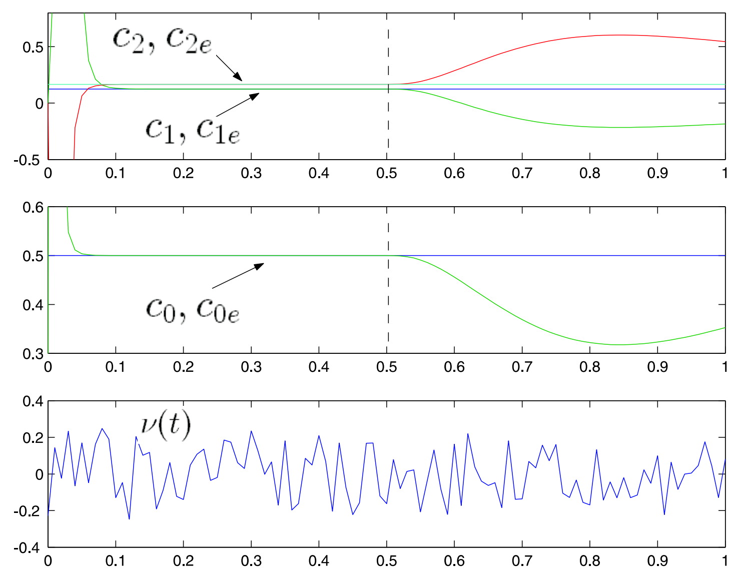

Figure 1.4: Identification of a discontinuity time in a perturbed time-polynomial signal parameter identification process.

1.6.3 A chirp signal

Chirp signals are of interest in many signal processing applications such as frequency modulation, sonar and radar systems [27], and Vibroseis system in seismology [2], [32]. Such signals can be represented as:

$$y(t) = A \sin \varphi(t) + n(t) \tag{1.8}$$

where $\varphi(t)$ is a second order polynomial, $\varphi(t) = \varphi_0 + \varphi_1 t + \varphi_2 t^2$, A is a constant amplitude and $n(t)$ represents a noise corruption. The estimation of the parameters $\mathbf{\Theta} = (A, \varphi_0, \varphi_1, \varphi_2)$ has long been investigated in the litterature essentially in a discrete-time setting [19], [20], [15], [25], [8].

Keeping the continuous-time nature of the signal, we readily observe that the noise-free signal $x(t) = y(t) - n(t)$ satisfies the following linear differential equation with time-varying coefficients

$$\dddot{x}(t) + \dot{\varphi}(t)^2 \dot{x}(t) + 3\dot{\varphi}(t)\ddot{\varphi}(t)x(t) = 0 \tag{1.9}$$

which also reads as

$$\begin{aligned} &\left\{(2\varphi_1\varphi_2 + \varphi_1^2 s + s^3) + 4\varphi_2(\varphi_2 + \varphi_1 s)\frac{d}{ds} + 4\varphi_2^2 s \frac{d^2}{ds^2}\right\} x \\ &\qquad = (\ddot{x}(0) + x(0)\varphi_1^2) + \dot{x}(0)s + x(0)s^2 \end{aligned} \tag{1.10}$$

The signal is thus differentially rational (compare with equation (1.4)). Setting

$$\begin{array}{lll} \theta_1' = \ddot{x}(0) + \varphi_1^2 x(0) = 2\varphi_2 A \cos\varphi_0 & \theta_3' = x(0) = A \sin\varphi_0 & \theta_5' = 2\varphi_1\varphi_2 \\ \theta_2' = \dot{x}(0) = \varphi_1 A \cos\varphi_0 & \theta_4' = -\varphi_1^2 & \theta_6' = -4\varphi_2^2 \end{array}$$

allows us to rewrite equation (1.10) in the more convenient form

$$\theta_1' + \theta_2' s + \theta_3' s^2 + \theta_4' s x + \theta_5' \left(2s\frac{d}{ds} - 1\right) x + \theta_6' \left(s\frac{d^2}{ds^2} - \frac{d}{ds}\right) x = s^3 x \tag{1.11}$$

Now, the parameters $\mathbf{\Theta}' = (\theta_1', \ldots, \theta_6')$ are clearly linearly identifiable from the system of linear equations

$$\begin{aligned} &s^{-\nu}\frac{d^m}{ds^m}\left\{\theta_1' + \theta_2' s + \theta_3' s^2 + \theta_4' s x + \theta_5' \left(2s\frac{d}{ds} - 1\right) x + \theta_6' \left(s\frac{d^2}{ds^2} - \frac{d}{ds}\right) x\right\} \\ &\qquad = s^{-\nu}\frac{d^m}{ds^m}\left\{s^3 x\right\}, \quad m = 0, 1, \ldots, 5 \end{aligned} \tag{1.12}$$

where $\nu > 0$ is large enough so that the coefficients of the system and the left hand sides only contain terms of the form cs^{-k} and $cs^{-k}\frac{d^i x}{ds^i}$, $k \geqslant 1$, $i \geqslant 0$. By the well know rules of the operational calculus, we obtain, in the time domain, expressions of the forms τ^k, where τ is the estimation time, and

$$\int_0^\tau \int_0^{\tau_{k-1}} \cdots \int_0^{\tau_1} \lambda^i x(\lambda) d\tau_k \cdots d\tau_1 d\lambda = \frac{1}{(k-1)!} \int_0^\tau (\tau - \lambda)^{k-1} \lambda^i x(\lambda) d\lambda$$

We now proceed with numerical simulations in which the chirp signal is corrupted by an additive Gaussian noise as in equation (1.8). We consider, for sake of simplicity, that $n(t)$ has zero-mean. Note that if a constant bias were present, then either it could be ruled out as described in section 1.4.4 or else it could be estimated as part of the parameters.

The noise-free chirp signal considered in the following simulations was set to

$$x(t) = 2.2911 \sin(1.524 + 0.876t - 1.892t^2)$$

The noisy signal, $y(t) = x(t) + n(t)$ is represented in figure 1.5 for a signal to noise ratio of $20dB$. The estimates are obtained from (1.12) for $\nu = 4$ and $\tau = 1.4$, by substituting the noisy measured signal y to the noise-free signal x.

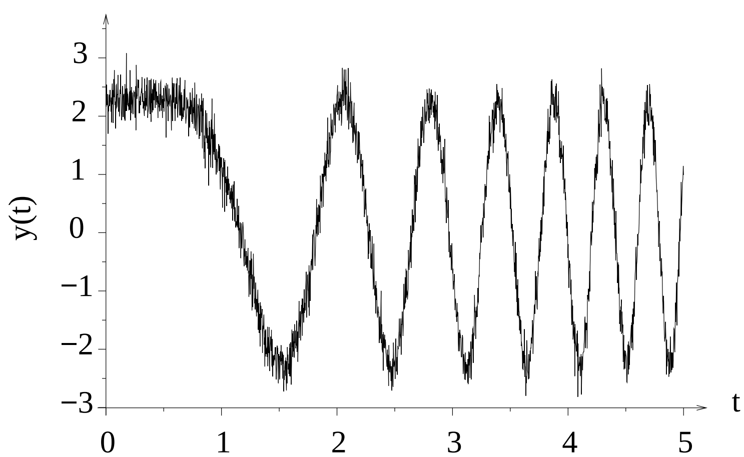

Figure 1.5: Noisy chirp signal: $SNR = 20dB$

Figure 1.6 displays the estimated signal (solid line) in comparison with the noise-free signal (dotted line). The two curves are undistinguishable, showing that the estimates are quite precise under a reasonable level of noise.

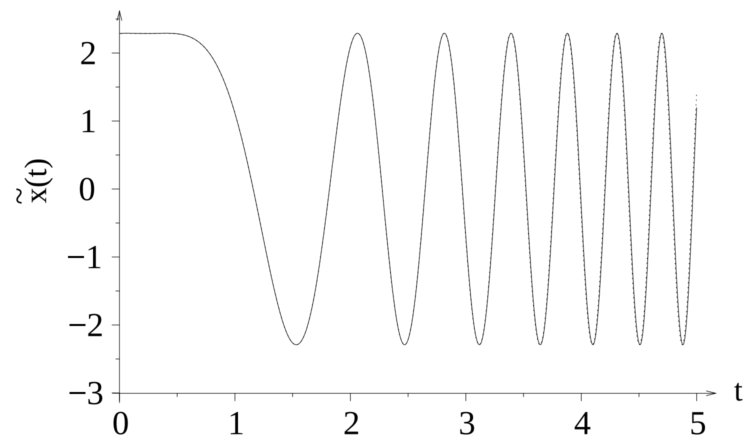

Figure 1.6: Estimation of a chirp signal corrupted by a zero-mean Gaussian noise: $SNR = 20dB$

The parameters obtained after different estimation times τ are also given in table 1.1 below, to illustrate the fastness of the proposed estimator.

τ	1.4	1.6	1.8	2	2.2	2.4
$\tilde{A}$	2.2919	2.2902	2.2911	2.2904	2.2922	2.2912
$\tilde{\varphi}_0$	1.5279	1.5100	1.5241	1.5137	1.5234	1.5238
$\tilde{\varphi}_1$	0.8678	0.8999	0.8762	0.8895	0.8758	0.8762
$\tilde{\varphi}_2$	-1.8914	-1.900	-1.8927	-1.8952	-1.8920	-1.8920

Table 1.1: Numerical values for different estimation times: $SNR = 20dB$

The same simulation is reproduced but with a high level of noise: the signal to noise ratio is now set to $10dB$. Again, figure 1.7 display the noisy signal while figure 1.8 shows the noise-free signal (dotted line) along with the estimated one (solid line) obtained for $\tau = 2$. Note that even in this case, which seems to correspond to the worst result among that given in table 1.2, the signal is correctly recovered up to an initial phase shift.

The obtained results are a nice illustration of the robustness and speed of the estimation.

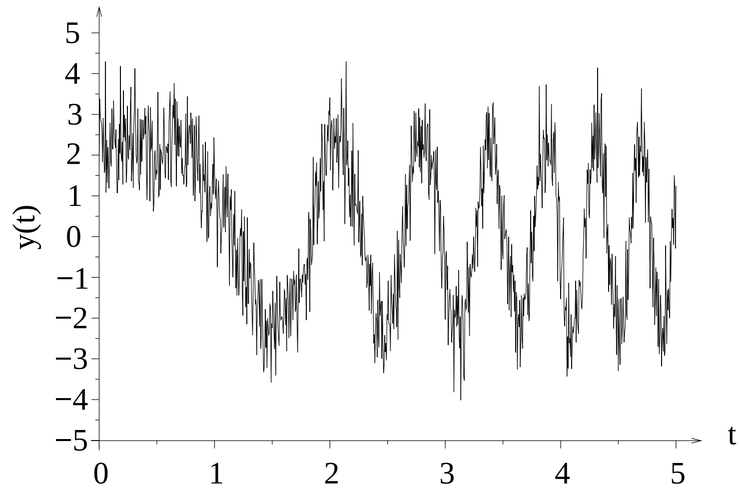

Figure 1.7: Noisy chirp signal: $SNR = 10dB$

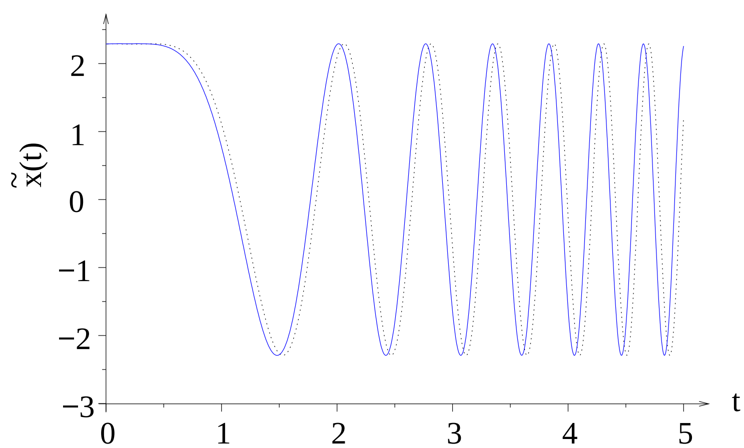

Figure 1.8: Estimation of a chirp signal corrupted by a zero-mean Gaussian noise: $SNR = 10dB$

τ	2	2.2	2.4	2.6
$\tilde{A}$	2.2919	2.2986	2.2958	2.2912
$\tilde{\varphi}_0$	1.5137	1.5024	1.5342	1.5219
$\tilde{\varphi}_1$	0.7018	0.8619	0.7257	0.8796
$\tilde{\varphi}_2$	-1.8953	-1.8961	-1.8905	-1.8930

Table 1.2: Numerical values for different estimation times: $SNR = 10dB$

1.6.4 A time-varying amplitude polynomial phase signal

In this example, we address the parameter estimation problem for a class of non-stationary signals which generalizes the one of the preceding example. This class is that of polynomial phase signals with time-varying amplitude. Typical representatives of this class are AM-FM signals which are encountered in many applications such as numerical communications, voiced speech modeling, radar and sonar systems (see, *e.g.*, [33] and the references therein). Recall that a complex (analytic signal) polynomial phase signal with time-varying amplitude, $x(t)$, is modeled as

$$x(t) = a(t)e^{i\varphi(t)} \tag{1.13}$$

where the phase is a polynomial in t, $\varphi(t) = \sum_{\text{finite}} \varphi_k t^k$ and where the amplitude $a(t)$ is a given continuous function of time. Since the signal is to be analyzed in a finite interval of time, one usually invoke the Weierstrass approximation theorem to represent the amplitude also as a polynomial in t, unless a model is available for $a(t)$ [27]. So from now on, we set $a(t) = \sum_{\text{finite}} a_k t^k$.

Now, one can readily check that the signal x in equation (1.13) is a solution of the following linear differential equation with time-varying coefficients

$$a(t)\dot{x}(t) - \{\dot{a}(t) + ia(t)\dot{\varphi}(t)\}x(t) = 0.$$

The parameters $\mathbf{\Theta} = (\{a_k\}_{k\geqslant 0}, \{\varphi_k\}_{k\geqslant 0})$ can then be identified, using the same developments as in the chirp example. However the two polynomials $a(t)$ and $\varphi(t)$ can also be identified seperately, in a more simple and straightforward way from the polar decomposition of the signal. Indeed, if $z(t)$ is a polynomial signal of degree $\kappa - 1$, then it satisfies the differential equation $\frac{d^\kappa}{dt^\kappa}z(t) = 0$, which corresponds in the operational domain to:

$$s^\kappa z - s^{\kappa-1}z(0) - \ldots - z^{(\kappa-1)}(0) = 0.$$

The initial conditions $z^{(i)}(0) \triangleq \frac{d^i}{dt^i} z(t)\Big|_{t=0}$, $i = 0, \ldots, \kappa - 1$ and hence the coefficients of the polynomial, are then obtained from the linear triangular system of equations

$$s^{-\nu} \frac{d^m}{ds^m} \left\{ z^{(\kappa-1)}(0) + z^{(\kappa-2)}(0)s + \ldots + z(0)s^{\kappa-1} \right\} = s^{-\nu} \frac{d^m}{ds^m} \left\{ s^\kappa z \right\},$$

for $m = 0, \ldots, \kappa - 1$ and where $\nu \geqslant \kappa$.

In the following simulations we have considered second order polynomials for both $a(t)$ and $\varphi(t)$. The two polynomials were estimated separatly, with $\nu = 6$, from a noisy version of equation (1.13): $y(t) = x(t) + n(t)$. The signal to noise ratio where set to $10dB$. The curves in figure 1.9 represent the graphs of the estimated amplitude (in solid line) together with that of the true amplitude (in dotted line). The results of the phase estimation are displayed in figure 1.10.

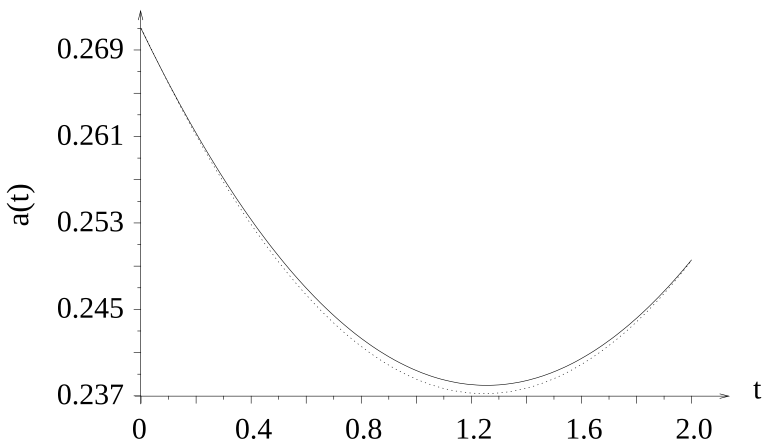

Figure 1.9: Noisy polynomial phase signal with time-varying amplitude–estimation of $a(t)$: $SNR = 10dB$

In these simulations, the plotted curves correspond to results obtained by averaging 10 runs. This then shows that the mean value of our estimates coincide with the true values of the parameters.

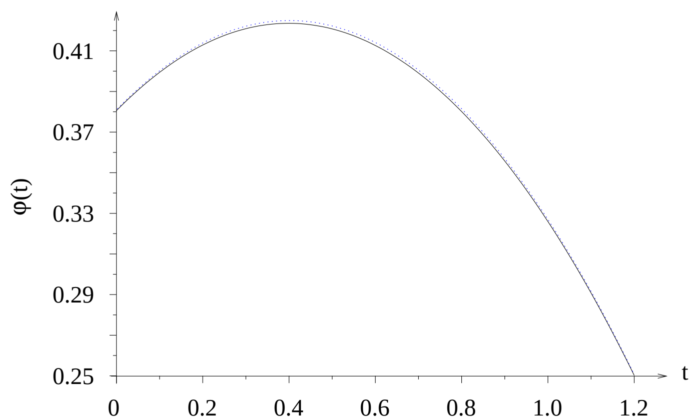

Figure 1.10: Noisy polynomial phase signal with time-varying amplitude–estimation of $\varphi(t)$: $SNR = 10dB$

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