Chord-distribution functions and Rice formulae. Application to random media.

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Abstract

We consider an isotropic bi-phasic (pore and solid) medium, draw many lines through it, and see each line as a one-dimensional level-cut process with values 0 or 1 according to whether a regular stationary process X is less or greater than a given level. The lengths of time intervals when X is in a given phase are named chord functions. We are interested into obtaining information on chord functions in order to get most characterizations of the medium such as the volume fractions, the two-point correlation function or the specific surface area. Working with the Palm probability measure and using level crossings techniques, Rice methods in particular, allow to obtain the exact analytical formula of the chord-distribution function as well as the joint distribution function of two successive chords and the two-point correlation function. Also the "Independent Interval Assumption" is discussed.

Introduction

The modelling of random porous media has been, and still is, subject to investigation in particular by physicists. Here we consider an isotropic bi-phasic medium, for which various models are proposed. Some authors chose a multidimensional approach based

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on the notion of random level surfaces ([1], [2], [22], [29]). Instead we adopt a one dimensional approach, inspired by stereology techniques, and motivated by previous works mainly in physics literature. It allows to obtain easily most characterizations of random composite and porous materials, such as the volume fractions, the two-point correlation function, the specific surface area, among others ([11] or [28] for instance for an overview of the microstructural descriptors).
This approach consists in drawing many lines through the medium, and to see each one as a one-dimensional level-cut process. More specifically, if $X = (X_t, t \in I\!R)$ denotes a regular stationary process, the level-cut process takes values 0 or 1 according to whether X is less or greater than a given level, thus defining two phases, pore or solid. The time intervals when X is in a given phase are named chords. Our main goal is to obtain information on chord lengths distribution to characterize the media. To this aim, we choose to work both with Palm probability measures and crossings.
The introduction of Palm probability measures provides a theoretical framework to the study, formalizing known results and getting new ones in view of applications. We also borrow other tools to the queueing theory ([21], [6] for more recent reference) as well as to the literature on level crossings ([1], [9], [15]), in particular by using Rice type methods ([5], [20], [26], [29]).
The study of chords is clearly related to work on crossings. Crossings theory started to be developed mainly in the 60's by mathematicians such as Cramér and Leadbetter ([9] and references therein) who obtained in particular the distribution function of the sum of successive chords, De Maré ([19] and references therein) who worked on the chords independence in the Gaussian case, and engineers such as Longuet-Higgins ([18]). It got some revival of interest in the 90's (for instance [7], [16], [24], [25]). In most papers related to this subject, the chord functions are estimated through simulations or on real data, under the assumption of stochastically independence of the successive chords. Alternatively, Rychlik [27] rejects the independence assumption and provides an estimation of the joint distribution of two successive chords by using a regression method in the Gaussian case. In the present study we adopt an analytical point of view, consider a general frame and discuss the independence hypothesis.

General results on chord distribution functions under Palm measures are presented in the first section. In section 2, the Rice method is applied to express not only the distribution function of the sum of successive chords but also the joint distribution function of any two successive chords, as well as the marginals, under the suitable Palm probability. The assumption of chords independence is not required for those results. Applications to bi-phasic media are developed in the last section, as well as some discussions on the chord dependence, crucial point in practice. This section may have interest on its own for non mathematician readers and the technical details of previous sections are not necessary for its understanding.

1 Chord lengths distributions

Let $X = (X_t, t \in \mathbb{R})$ be a real, centered, strictly stationary and ergodic process, with one dimensional continuous distribution, a.s. continuous sample functions and covariance function ρ such that $\rho(0) = 1$. Let λ_2 denote the second spectral moment of X; we have $\lambda_2 = -\rho''(0)$ that we suppose to be finite.

Let us introduce $C(s,t)$, $U(s,t)$ and $D(s,t)$, the number of crossings, upcrossings and downcrossings respectively, of a given level γ by X in the interval $[s,t]$; the level γ does not appear in the chosen notation and will not be anymore mentioned.
Let μ denote the mean upcrossings number on $[0,1]$, assumed to be finite. In this case

$$\mu := \mathbb{E}[U(0,1)] = \mathbb{E}[D(0,1)] = \frac{1}{2}\mathbb{E}[C(0,1)] \tag{1}$$

and the stationary stream of events $U(0,t)$ is regular ([9]), *i.e.*

$$\mathbb{P}[U(0,t) \geq 1] = \mu t + o(t), \text{ as } t \to 0. \tag{2}$$

If X is a Gaussian process, the Rice formula ([23], or [12], [30]) given by

$$\mu = e^{-\gamma^2/2}\sqrt{\lambda_2}/(2\pi), \tag{3}$$

holds whenever $\lambda_2 < \infty$.
As noticed in the physics literature, a chord is the interval between two crossings. It may be necessary to distinguish between upward or downward chords depending on whether the first crossing is an up or a down crossing. More precisely, let us define the random variables L_k by

$$L_1 = \inf\{s > 0 \ : \ X_s = \gamma\} \quad \text{and} \quad L_k = \inf\{s > 0 \ : \ X_{s+L_{k-1}} = \gamma\}, \text{ for } k > 1. \tag{4}$$

The distributions of these variables depend on the location in time of the first crossing (by stationarity we can suppose that it occurs in $t = 0$). So, we need to introduce probability measures P_{0^+} or P_{0^-} depending on whether the first crossing is an up- or a down-crossing, whereas P_0 stands for any crossing. These probability measures are conditional relative to a zero probability event and called Palm probability measures in the theory of stationary point process. Many equivalent definitions are possible, we choose one based on the horizontal conditioning and the regularity assumption (2) ([21], or [6] Th 1.5.1).

PROPOSITION 1.1 *For all event A, the limit of $\mathbb{P}[A \mid U(-\tau,0) \geq 1]$ as τ goes to 0^+ does exist. It defines a probability measure denoted by P_{0^+} that satisfies*

$$P_{0^+}(A) = \mu^{-1} \lim_{\tau \to 0^+} \frac{1}{\tau} \mathbb{P}\left(A \cap [U(-\tau,0) \geq 1]\right) .$$

The same result holds for P_{0^-} defined as P_{0^+} when replacing $U(-\tau,0)$ by $D(-\tau,0)$. We can also define P_0 considering all the crossings $C(-\tau,0)$ and replacing μ by 2μ. This measure satisfies $P_0 = \frac{1}{2}(P_{0^+} + P_{0^-})$.

In the literature on queueing theory, the correspondence between the stationary distribution of a point process and its associated Palm distribution has been well studied ([21] or [6] for more recent references). It allows, in particular, the study of the durations between two upcrossings with respect to the Palm distribution associated to the point process of upcrossings.
Our concern is different since it involves the simultaneous study of both, upward and downward chords, with respect to the Palm distribution associated with the point process of upcrossings only.

Since $(C(0,t) \geq k) = (L_1 + \cdots + L_k \leq t)$, thus the function $F_k^{\pm}$ defined for $k \geq 0$ by

$$F_k^{\pm}(t) \ := \ P_{0^{\pm}}\left[C(0,t) \geq k\right] = \ P_{0^{\pm}}\left[L_1 + \cdots + L_k \leq t\right], \tag{5}$$

is the conditional distribution function of the random variable $\sum_{i=1}^{k} L_i$ given that there is an up/down crossing at 0.
The distribution functions $F_k^{\pm}$ have extensively been studied by Cramér and Leadbetter ([9], Chap. 10 & 11). Whereas those authors were concerned with the conditional distribution of the sum of successive chords, we are also interested in the joint distribution of successive chords under $P_{0^{\pm}}$. The following theorem confirms intuitive ideas of invariance or stationarity of the successions of chords.

Let $\mathcal{L}_{0^{\pm}}(Z)$ denote the law under $P_{0^{\pm}}$ of any random variable Z.

Theorem 1.1 *For all finite set of indices $I \subset I\!N \backslash \{0\}$,*

$$\mathcal{L}_{0^+}(L_i, i \in I) \ = \ \mathcal{L}_{0^-}(L_{i+1}, i \in I) \ ;$$

consequently, for all $k \in I\!N$, $\mathcal{L}_{0^+}(L_i, i \in I) \ = \ \mathcal{L}_{0^+}(L_{i+2k}, i \in I)$, and so

$$\mathcal{L}_{0^+}(L_{2k+1}) = \mathcal{L}_{0^+}(L_1) \quad \text{and} \quad \mathcal{L}_{0^+}(L_{2k}) = \mathcal{L}_{0^+}(L_2) \ .$$

Similar identities hold when exchanging $\mathcal{L}_{0^+}$ and $\mathcal{L}_{0^-}$.

Remark 1.1

i) The last two relations state that the upward chord lengths are identically distributed under $P_{0^{\pm}}$, as well as the downward ones. Furthermore, if X is symmetric with respect to γ, all chord lengths are identically distributed.

ii) Theorem 1.1 gives us back the known result that the chord lengths are identically distributed under the Palm probability P_0. The same result does hold when considering, for instance, the lengths between two upcrossings and the Palm probability P_{0^+}, since $\mathcal{L}_{0^+}(L_1 + L_2) = \mathcal{L}_{0^+}(L_{1+2k} + L_{2+2k})$, $\forall k \in I\!N$.

Proof of Theorem 1.1
We choose here to proceed with the crossings approach since it is constructive and more intuitive. The result may also be obtained when applying the Neveu exchange

formula ([21] or [6]).
Let us prove that for all $n \geq 1$ and all $t_1, t_2, \ldots, t_n \in I\!\!R^+$

$$P_{0^+}[L_{i+1} \leq t_i; 1 \leq i \leq n] = P_{0^-}[L_i \leq t_i; 1 \leq i \leq n] \ .$$

Clearly the same will hold when P_{0^+} and P_{0^-} are exchanged. Actually we will prove the relation only in the case $n = 2$. The generalization to any n is tedious but straightforward. So, let us compute $P_{0^+}[L_2 > s, L_3 > t]$ for $s, t \in I\!\!R^+$.
By Proposition 1.1 we have that

$$P_{0^+}[L_2 > s, L_3 > t] = \mu^{-1} \lim_{\tau \to 0^+} \frac{1}{\tau} g(\tau, s, t),$$

where for $\tau > 0$

$$g(\tau, s, t) := I\!\!P[U(-\tau, 0) \geq 1, L_2 > s, L_3 > t] \ .$$

Since the event of two or more crossings appearing in the interval $(-\tau, 0)$ is $o(\tau)$,

$$g(\tau, s, t) = I\!\!P[C(-\tau, 0) = U(-\tau, 0) = 1, L_2 > s, L_3 > t] + o(\tau) \ .$$

We will make use of this last argument several times in the following computations without more precision unless writing a $o(\tau)$-term.
For $k \in I\!\!N$, we consider the event

$$A_{0,k}(\tau) = \{C(0, (k-1)\tau) = 0, C((k-1)\tau, k\tau) = D((k-1)\tau, k\tau) = 1\},$$

and we denote by $A_{-k,0}$ its translation by $-k\tau$.
Choosing τ s.t. $0 < \tau < \frac{s}{2} \wedge t$, we can write

$$\begin{aligned} g(\tau, s, t) &= I\!\!P[U(-\tau, 0) = C(-\tau, 0) = 1, \cup_{k \geq 1} A_{0,k}(\tau), L_2 > s, L_3 > t] + o(\tau) \\ &= I\!\!P[U(-\tau, 0) \geq 1, \cup_{k \geq 1} A_{0,k}(\tau), L_2 > s, L_3 > t] + o(\tau) \\ &= \sum_{k \geq 1} I\!\!P[U(-\tau, 0) \geq 1, A_{0,k}(\tau), L_2 > s, L_3 > t] + o(\tau). \end{aligned}$$

The event $A_{0,k}$ involves a down-crossing in the interval $[(k-1)\tau, k\tau]$. The next crossing, which is an upcrossing, lies in some interval $[(k-1+j)\tau, (k+j)\tau]$ with $j > 1$. Then necessarily $(j-1)\tau \leq L_2 \leq (j+1)\tau$. This fact allows us to produce an upper and a lower bound for $g(\tau, s, t)$ up to a $o(\tau)$-term:

$$\begin{gathered} \sum_{k \geq 1} I\!\!P[U(-\tau, 0) \geq 1, A_{0,k}(\tau), \cup_{j \geq 1, (j-1)\tau \geq s} B_{k,k+j}(\tau, t)] + o(\tau) \\ \leq \ g(\tau, s, t) \ \leq \qquad (6) \\ \sum_{k \geq 1} I\!\!P[U(-\tau, 0) \geq 1, A_{0,k}(\tau), \cup_{j \geq 1, (j+1)\tau \geq s} B_{k,k+j}(\tau, t-\tau)] + o(\tau) \end{gathered}$$

where for $0 < \tau < u$,

$$\begin{aligned} B_{k,k+j}(\tau, u) \ := \ &\{C(k\tau, (k+j-1)\tau) = 0, \\ &C((k+j-1)\tau, (k+j)\tau) = U((k+j-1)\tau, (k+j)\tau) = 1, \\ &C((k+j)\tau, (k+j)\tau + u) = 0\}. \end{aligned}$$

But the stream stationarity and the fact that $I\!\!P[U(-\infty,-\tau)\geq 1]=1$ allow to write the upper bound in (6) as

$$\begin{aligned}
&\sum_{k\geq 1} I\!\!P[U(-\tau,0)\geq 1, A_{0,k}(\tau), \cup_{j\geq 1,(j+1)\tau\geq s} B_{k,k+j}(\tau,t-\tau)]\\
=\;&\sum_{k\geq 1} I\!\!P[U(-(k+1)\tau,-k\tau)\geq 1, A_{-k,0}(\tau), \cup_{j\geq 1,(j+1)\tau\geq s} B_{0,j}(\tau,t-\tau)]\\
=\;& I\!\!P[D(-\tau,0)=C(-\tau,0)=1, \cup_{j\geq 1,(j+1)\tau\geq s} B_{0,j}(\tau,t-\tau)]\\
\leq\;& I\!\!P[D(-\tau,0)=C(-\tau,0)=1, L_1>s-2\tau, L_2>t-\tau]\ .
\end{aligned}$$

A similar computation yields the following lower bound in (6)

$$\begin{aligned}
&\sum_{k\geq 1} I\!\!P[U(-\tau,0)\geq 1, A_{0,k}(\tau), \cup_{j\geq 1,(j-1)\tau\geq s} B_{k,k+j}(\tau,t)]\\
&\geq\; I\!\!P[C(-\tau,0)=D(-\tau,0)=1, L_1>s+2\tau, L_2>t+\tau]\ .
\end{aligned}$$

Hence, (6) becomes

$$\begin{aligned}
& I\!\!P[D(-\tau,0)\geq 1, L_1>s-2\tau, L_2>t-\tau]+o(\tau)\\
&\leq g(\tau,s,t)\leq \qquad\qquad (7)\\
& I\!\!P[D(-\tau,0)\geq 1, L_1>s+2\tau, L_2>t+\tau]+o(\tau)
\end{aligned}$$

which yields, when taking the limit as τ goes to 0,

$$\mu^{-1}\lim_{\tau\to 0^+}\frac{1}{\tau}\, g(\tau,s,t)=P_{0^-}[L_1>s, L_2>t]$$

and concludes the proof. □

Formulae linking the probability under $I\!\!P$ and under $P_{0^\pm}$ of some events are stated in the following lemma. They provide useful tools for the next section.
Let us denote by $A(t,s)$, for all $t,s\in I\!\!R^+$, the event "the first crossing after t is followed by an interval of length s free of crossing", and for $k\in I\!\!N$,

$$\begin{aligned}
u_k^+(t) &:= I\!\!P[U(0,t)=k] \quad\text{and}\quad u_k^-(t):=I\!\!P[D(0,t)=k]\ ,\\
v_k^+(t) &:= I\!\!P[X_0>\gamma, C(0,t)=k] \quad\text{and}\quad v_k^-(t):=I\!\!P[X_0<\gamma, C(0,t)=k]\ ,\\
w_k^+(t,s) &:= I\!\!P[X_0>\gamma, C(0,t)=k, A(t,s)] \quad\text{and}\quad w_k^-(t,s):=I\!\!P[X_0<\gamma, C(0,t)=k, A(t,s)]\ ,
\end{aligned}$$

LEMMA 1.1 *The functions $u_k^\pm$, $v_k^\pm$ and $w_k^\pm(.,s)$ admit a right derivative ∂_t at any point $t\in I\!\!R^+$ which satisfy*

$$\partial_t v_0^\pm(t) = -\mu\ P_{0^\pm}[L_1>t] \qquad (8)$$
$$\partial_t w_0^\pm(t,s) = -\mu\ P_{0^\pm}[L_1>t, L_2>s] \qquad (9)$$
$$\partial_t u_0^\pm(t) = -\mu\ P_{0^\pm}[L_1+L_2>t] \qquad (10)$$

and for $k \geq 1$, the distribution functions $F_k^{\pm}$ are recursively linked by

$$\begin{aligned}
F_1^{\pm}(t) &= 1 + \mu^{-1}\ \partial_t v_0^{\pm}(t) \\
F_2^{\pm}(t) &= 1 + \mu^{-1}\partial_t u_0^{\pm}(t) \\
F_{2k+1}^{\pm}(t) &= F_{2k}^{\pm}(t) + F_{2k}^{\mp}(t) - F_{2k-1}^{\mp}(t) + \mu^{-1}\ \partial_t v_{2k}^{\pm}(t) \qquad (11)\\
F_{2k+2}^{\pm}(t) &= F_{2k}^{\pm}(t) + \mu^{-1} \sum_{0\leq i\leq k} \partial_t u_i^{\pm}(t)\ . \qquad (12)
\end{aligned}$$

Proof: Let us start with relation (9). From the definition of w_0^+, using stationarity and regularity, we write for $0 < \tau < t$,

$$\begin{aligned}
w_0^+(t,s) - w_0^+(t+\tau,s) &= I\!P[X(0) > \gamma, C(0,t) = 0, A(t,s)] \\
&- I\!P[X(-\tau) > \gamma, C(-\tau,t) = 0, A(t,s)] \\
&= I\!P[X(-\tau) \leq \gamma, X(0) > \gamma, C(0,t) = 0, A(t,s)] \\
&= I\!P[U(-\tau,0) \geq 1, L_1 > t, L_2 > s] + o(\tau)\ .
\end{aligned}$$

Hence, by definition of the Palm probability measure P_{0^+}, the right hand side term divided by τ converges to $\mu P_{0^+}[L_1 > t, L_2 > s]$ as τ goes to 0^+. Therefore w_0^+ admits a right derivative with respect to t and so (9) holds.
The other relations are well known ([9], pp 225-232 and [6], pp 18), nevertheless can be derived from (9). Once noticing that $w_0^+(t,0) = v_0^+(t)$, relation (8) follows and similar arguments applied on u_k^+ ($k = 0$ or $k \geq 1$) instead of w_0^+ yield (10) and (12). For (11), we write

$$\begin{aligned}
v_k^+(t+\tau) - v_k^+(t) &= I\!P[D(-\tau,0) \geq 1, C(0,t) = k-1] \\
&\quad - I\!P[U(-\tau,0) \geq 1, C(0,t) = k] + o(\tau)
\end{aligned}$$

so that for $k \geq 1$,

$$\partial_t v_k(t) = \mu\left((F_{k-1}^-(t) - F_k^-(t)) - (F_k^+(t) - F_{k+1}^+(t))\right),$$

from which the recursive identity follows. □

2 Rice type formulae

We are interested in computing the chord lengths distribution functions under $P_{0^\pm}$. We turn to the Rice method to express these distributions, either explicitely as a Rice series in terms of factorial moments of the crossings number, or implicitely with a conditional expectation.
We first recall formulae concerning the factorial moments of the crossings number.
Let us introduce the following notation:

$$\begin{aligned}
\alpha_m^+(t) &:= I\!E[C(0,t)^{[m]}\mathbb{1}_{(X_0>\gamma)}], & \alpha_m^-(t) &:= I\!E[C(0,t)^{[m]}\mathbb{1}_{(X_0<\gamma)}] \\
\beta_m^+(t) &:= I\!E[U(0,t)^{[m]}], & \beta_m^-(t) &:= I\!E[D(0,t)^{[m]}]
\end{aligned}$$

where $I\!E[Z^{[m]}]$ denotes the mth factorial moment of any integer-valued random variable Z. The following holds ([9], §10.6, and [29] for instance):

$$\alpha^+_m(t)=\int_{[0,t]^m}\left(\int_{(\gamma,+\infty)} G_m(\tau,x)dx\right) d\tau, \quad \alpha^-_m(t)=\int_{[0,t]^m}\left(\int_{(-\infty,\gamma)} G_m(\tau,x)dx\right) d\tau \tag{13}$$

$$\beta^\pm_m(t) = \int_{[0,t]^m} H^\pm_m(\tau)\ d\tau, \tag{14}$$

where

$$G_m(t_1,\ldots,t_m,x) := I\!E\left[\left|\dot{X}_{t_1}\ldots\dot{X}_{t_m}\right| |X_0 = x, X_{t_1} = \ldots = X_{t_m} = \gamma\right] p_{X_0,X_{t_1},\ldots,X_{t_m}}(x,\gamma,\ldots,\gamma)$$

and

$$H^\pm_m(t_1,\ldots,t_m) \quad := \quad I\!E\left[(\dot{X}_{t_1})^\pm\ldots(\dot{X}_{t_m})^\pm|\ X_{t_1} = \ldots = X_{t_m} = \gamma\right] p_{X_{t_1},\ldots,X_{t_m}}(\gamma,\ldots,\gamma)$$

p_Z denoting the probability density function of any random vector Z and $\dot{X}$ the derivative process.

We have

$$I\!P[X_0 > \gamma, C(0,t) \geq 1] \quad = \quad \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m!}\alpha^+_m(t) \tag{15}$$

$$I\!P[U(0,t) \geq 1] \quad = \quad \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m!}\beta^+_m(t) \tag{16}$$

and analogous versions for the crossings number starting below γ and for the down-crossings number, whenever the series do converge.
General conditions (applying not only to Gaussian processes) to ensure these convergences have been proposed ([9], [29] or [4] for references). For instance, in the Gaussian case, the Rice series converge whenever the covariance function of X does have a Taylor expansion at 0 absolutely convergent at $2t$.

In the same spirit as the Rice series formulae (15) and (16), we provide explicit formulae in terms of the factorial moments of the crossings numbers for the $P_{0^\pm}$-distributions of the chord length L_1 and of the length between two successive up/down-crossings. We also give lower and upper bounds which should be of interest for numerical applications.

Theorem 2.1

(i) For all positive integer n,

$$\sum_{m=1}^{2n} \frac{(-1)^{m+1}}{m!}\ \dot{\alpha}^\pm_m(t) \leq \mu P_{0^\pm}[L_1 > t] \leq \sum_{m=1}^{2n-1} \frac{(-1)^{m+1}}{m!}\ \dot{\alpha}^\pm_m(t)$$

$$\sum_{m=1}^{2n} \frac{(-1)^{m+1}}{m!}\ \dot{\beta}_m^{\pm}(t) \le \mu P_{0^\pm}[L_1 + L_2 > t] \le \sum_{m=1}^{2n-1} \frac{(-1)^{m+1}}{m!}\ \dot{\beta}_m^{\pm}(t)$$

where, for $m \ge 1$, $\dot{\alpha}_m^{\pm}(t)$ *and* $\dot{\beta}_m^{\pm}(t)$ *are the derivatives w.r.t.* t *of (13) and (14) respectively, i.e.*

$$\dot{\alpha}_m^{+}(t) = m \int_{[0,t]^{m-1}} d\sigma \int_{(\gamma,+\infty)} dx\ G_m(\sigma,t,x),\ \ \dot{\alpha}_m^{-}(t) = m \int_{[0,t]^{m-1}} d\sigma \int_{(-\infty,\gamma)} dx\ G_m(\sigma,t,x) \tag{17}$$

and

$$\dot{\beta}_m^{\pm}(t) = m \int_{[0,t]^{m-1}} H_m^{\pm}(\sigma,t)\ d\sigma\ . \tag{18}$$

(ii) If the series with general term $(-1)^m \dot{\alpha}_m^{\pm}(t)/m!$, *and* $(-1)^m \dot{\beta}_m^{\pm}(t)/m!$ *respectively, do converge, then*

$$F_1^{\pm}(t) = P_{0^\pm}[L_1 \le t] = 1 + \mu^{-1} \sum_{m=1}^{\infty} \frac{(-1)^m}{m!}\ \dot{\alpha}_m^{\pm}(t)$$

$$F_2^{\pm}(t) = P_{0^\pm}[L_1 + L_2 \le t] = 1 + \mu^{-1} \sum_{m=1}^{\infty} \frac{(-1)^m}{m!}\ \dot{\beta}_m^{\pm}(t)$$

and for $k \ge 1$,

$$\begin{aligned} F_{2k+1}^{\pm}(t) &= F_{2k}^{\pm}(t) + F_{2k}^{\mp}(t) - F_{2k-1}^{\mp}(t) + \mu^{-1} \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(2k)!}\ \dot{\alpha}_{m+2k}^{\pm}(t) \\ F_{2k+2}^{\pm}(t) &= F_{2k}^{\pm}(t) + \mu^{-1} \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \left(\sum_{i=1}^{k} \frac{1}{i!} \dot{\beta}_{m+i}^{\pm}(t) \right)\ . \end{aligned}$$

Proof of Theorem 2.1.
Let us recall the following lemma used in the Rice method, proved in Lindgren ([17]), which is the major step of the proof.

Lemma 2.1 *If* ξ *and* ξ' *are two integer valued random variables such that* $0 \le \xi \le \xi'$ *, a.s., then for all positive integer* n,

$$\sum_{m=1}^{2n} \frac{(-1)^{m+1}}{m!}\ \left(\mathbb{E}(\xi^{[m]}) - \mathbb{E}(\xi'^{[m]})\right) \le \mathbb{P}[\xi = 0, \xi' \ge 1] \le \sum_{m=1}^{2n-1} \frac{(-1)^{m+1}}{m!}\ \left(\mathbb{E}(\xi^{[m]}) - \mathbb{E}(\xi'^{[m]})\right)\ .$$

(i) Let $t \in \mathbb{R}^+$ and $\tau > 0$. We apply Lemma 2.1 for

$$\xi = C(0,t)\mathbb{1}_{(X_0>\gamma)} \qquad \text{and} \qquad \xi' = C(0,t+\tau)\mathbb{1}_{(X_0>\gamma)} \qquad \text{to obtain}$$

$\mathbb{P}[\xi = 0, \xi' \ge 1] = \mathbb{P}[X_0 > \gamma, C(0,t+\tau) \ge 1] - \mathbb{P}[X_0 > \gamma, C(0,t) \ge 1] = v_0^+(t) - v_0^+(t+\tau)$.

then

$$\sum_{m=1}^{2n} \frac{(-1)^{m+1}}{m!}\big(\alpha_m^+(t+\tau)-\alpha_m^+(t)\big) \le v_0^+(t)-v_0^+(t+\tau) \le \sum_{m=1}^{2n-1} \frac{(-1)^{m+1}}{m!}\big(\alpha_m^+(t+\tau)-\alpha_m^+(t)\big)\,. \tag{19}$$

It is easy to see that $t \mapsto \alpha_m^+(t)$ is differentiable and since G_m defined in (17) is symmetric in $(t_1, ..., t_m)$, then its derivative $\dot\alpha_m^+(t)$ satisfies (17) for all $t \in I\!R^+$.
Hence letting τ going to 0 in (19) and using (8) in Lemma 1.1 yield the required inequality for $\mu P_{0^+}(L_1 > t)$.
Similar arguments lead to obtain the required inequality for $P_{0^+}[L_1 + L_2 > t]$.
(ii) The first two identities are straightforward from (i) as soon as the series converge. For the last two identities we can write $v_k^+(t)$ and $u_k^+(t)$ for $k \ge 1$, as

$$v_k^+(t) = I\!P[X_0 > \gamma, C(0,t) = k] = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\ k!}\ \alpha_{m+k}^+(t)$$

and

$$u_k^+(t) = I\!P[U(0,t) = k] = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\ k!}\ \beta_{m+k}^+(t)\ .$$

The result follows using Lemma 1.1 and the derivative of the v_k's and u_i's. □

Now, for numerical consideration, we provide implicit formulae for those distributions by using a variation of Rice method ([29]). Note that Rychlik ([26]) proceeded in this way to express the density function of the first upcrossing at the level γ between 0 and T, that is the distribution of L_1 under $I\!P$, according to

$$I\!P\,[X_0 < \gamma,\ L_1 \le T] = \int_0^T I\!E\,[(\dot X_t)^+ 1\!\!1_{(X_s<\gamma,\forall s<t)} \mid X_t = \gamma]\ p_{X_t}(\gamma)\ dt\ .$$

This implicit formula has been used and extended to two-parameter Gaussian processes by Mercadier ([20]) to propose numerical bounds that turn to be very accurate in the Gaussian case for levels not too large and improve those of Azaïs and Wschebor ([4], [5]) when the interval length becomes large. We also extend this implicit technique to get the joint distribution of two successive chords lengths. It is given as an analytical formula for any process X, whereas an estimation in the case X is Gaussian can be found in [27].

Theorem 2.2 *The distribution of L_1 and the joint distribution of (L_1, L_2) under the Palm probability $P_{0^\pm}$ are given by*

$$P_{0^+}[L_1 > t] \ = \ \mu^{-1} I\!E\,[|\dot X_t| 1\!\!1_{(X_s>\gamma:\ 0<s<t)}|\ X_t = \gamma]\ p_{X_0}(\gamma) \tag{20}$$

and

$$P_{0^+}[L_1 > t_1, L_2 > t_2] \ = \ \mu^{-1} I\!E\,[|\dot X_{t_1}| 1\!\!1_{(X_s>\gamma>X_u\ :\ 0<s<t_1<u<t_1+t_2)}|\ X_{t_1} = \gamma]\ p_{X_0}(\gamma). \tag{21}$$

It may also be expressed as

$$F_1^+(t) = P_{0^+}[L_1 \le t] = \mu^{-1} \int_{[0,t]} I\!E\left[|\dot X_0 \dot X_u| \mathbb{1}_{(X_s>\gamma:\ 0<s<u)}|\ X_0 = X_u = \gamma\right] p_{X_0,X_u}(\gamma,\gamma)\, du \tag{22}$$

and

$$P_{0^+}[L_1 \le t_1, L_2 \le t_2] = \mu^{-1} \int_{[-t_1,0]\times[0,t_2]} I\!E[|\dot X_u \dot X_0 \dot X_v| \mathbb{1}_{(X_{u_1}>\gamma>X_{v_1}:\ u<u_1<0<v_1<v)}|X_u = X_0 = X_v = \gamma]$$
$$p_{X_u,X_0,X_v}(\gamma,\gamma,\gamma) du dv. \tag{23}$$

The corresponding relations under P_{0^-} can be written straightaway.

Proof of Theorem 2.2.
The proof relies mainly on the relations given in Lemma 1.1, Rice-type formulae for marked-crossings in (24) and in (25) below, and the continuity in time-parameters of the integrands appearing in those formulae ([29], [4] and [2]).
We use that for any process $Y = (Y_t, t \in I\!R)$ with values in $I\!R^p$ and any open set $B \subset I\!R^p$ satisfying regularity assumption, the following identity holds

$$I\!E[C([0,t], Y, B)] = \int_0^t I\!E\left[|\dot X_s| \mathbb{1}_{(Y_s \in B)}|X_s = \gamma\right] p_{X_s}(\gamma) ds, \tag{24}$$

where $C([0,t], Y, B)$ denotes the number of crossings of the level γ by X in the interval $[0,t]$ occurring at time s such that $Y_s \in B$.
A similar formula holds for disjoint intervals (s_1, s_2) and (t_1, t_2) with $s_2 < t_1$ and two open sets $B, B' \subset I\!R^p$,

$$\begin{aligned} & I\!E[C([s_1,s_2], Y, B) C([t_1,t_2], Y, B')] \\ = & \int_{s_1}^{s_2}\int_{t_1}^{t_2} I\!E\left[|\dot X_s \dot X_t| \mathbb{1}_{(Y_s\in B)} \mathbb{1}_{(Y_t \in B')}|X_s = X_t = \gamma\right] p_{X_s,X_t}(\gamma,\gamma) ds dt\ . \end{aligned} \tag{25}$$

• Combining (8) and (24) for $Y_s = \inf_{(0,s)} X_t$ and $B = (\gamma, +\infty)$ provides (20), whereas (9) and (24) for $Y_s = \left(\inf_{(0,s)} X_t,\ \sup_{(s,s+t_2)} X_t\right)$ and $B = (\gamma, +\infty) \times (-\infty, \gamma)$ lead to (21).
• Let us prove (23). Note that by definition,

$$\mu P_{0^+}[L_1 \le t_1, L_2 \le t_2] \ = \ \lim_{\tau \to 0^+} \frac{1}{\tau}\left(P_{t_1,t_2}(\tau) + o(\tau)\right),\ \text{with}$$

$$\begin{aligned} P_{t_1,t_2}(\tau) \ &:= \ I\!P[U(-\tau,0) = 1, L_1 \le t_1, L_2 \le t_2] \\ &= \sum_{k\ge 1: k\tau \le t_1} I\!P[U(-\tau,0) = 1, D((k-1)\tau, k\tau) = 1, C(0,(k-1)\tau) = 0, L_2 \le t_2] \\ &= \sum_{k\ge 1: k\tau \le t_1} I\!P[U(-(k+1)\tau, -k\tau) = 1, D(-\tau,0) = 1, C(-k\tau,-\tau) = 0, \\ &\qquad \exists j : 0 \le (j-1)\tau \le t_2 \le j\tau, C(0,(j-1)\tau) = 0, U((j-1)\tau, j\tau) = 1], \end{aligned}$$

by using stationarity in the last equation.
We proceed in the same way as for g in the proof of Theorem 1.1 to provide lower and upper bounds of $P_{0^+}[L_1 \leq t_1, L_2 \leq t_2]$. These bounds can be expressed via the equivalent of (25) for three crossings and the suitable marks as

$$\int_{-t_1 \pm a\tau}^{-\tau} \int_{-\tau}^{0} \int_{0}^{t_2 \pm b\tau} I\!E[|\dot{X}_s \dot{X}_u \dot{X}_t| 1\!\!1_{(X_{u_1} > \gamma > X_{u_2} : -\tau < u_2 < u < u_1 < 0)} \\ 1\!\!1_{(X_{t_1} < \gamma < X_{s_1} : s < s_1 < -\tau < 0 < t_1 < t)} | X_s = X_u = X_t = \gamma] \; p_{X_s, X_u, X_t}(\gamma, \gamma, \gamma) ds \; du \; dt \; ,$$

with $a = \pm 1$ and $b = \pm 1$.
We conclude by continuity when taking the limit as $\tau \to 0^+$.
• Equation (22) may be obtained in a similar way. $\square$

3 Application to bi-phasic media

3.1 Parameters

In this section, we focus on the modelling of a bi-phasic medium through the level cut of a regular stationary process. We restrict our study to lines extracted from the 3D material. The two phases, described for instance by solid and pore, are associated with the intervals where the process is above or under a given level γ. The microstructure is characterized by well known stereological descriptors such as the volume fractions, the specific surface area, the two point correlation functions and others ([11], Chap. 2, or [28]). By using results of previous sections we precise some of those parameters and provide a way to estimate them.
It has been proved in Section 1 that the chord lengths of a given phase are identically distributed under $P_{0^\pm}$. It allows the interpretation of $F_1^\pm$ as the conditional distribution function of an "arbitrary" chord length in each phase. Using (8) in Lemma 1.1, we can compute the mean chord length in each phase.

PROPOSITION 3.1

$$E_{0^+}(L_1) = \mu^{-1} I\!P[X_0 > \gamma] \quad \textit{and} \quad E_{0^-}(L_1) = \mu^{-1} I\!P[X_0 < \gamma]$$

This proposition focuses on main descriptors of an heterogeneous random medium: the volume fractions ϕ_+ and ϕ_- of each phase, defined by

$$\phi_+ := I\!P[X_0 > \gamma] \text{ and } \phi_- := I\!P[X_0 < \gamma] \; , \tag{26}$$

and the mean number of phase changes per unit length, i.e. $2\mu = I\!E[C(0,1)]$. In consequence these parameters satisfy the following intuitive relations:

$$\phi_\pm = \frac{E_{0^\pm}(L_1)}{E_{0^+}(L_1) + E_{0^-}(L_1)} \quad , \quad \mu = \frac{\phi_+}{E_{0^+}(L_1)} = \frac{\phi_-}{E_{0^-}(L_1)},$$

and then can be estimated along any extracted line.
The parameter ϕ_+ corresponds also to the constant mean of the level-cut process Y defined for each t by $Y_t = 1\!\!1_{(X_t>\gamma)}$, that is

$$\phi_+ = I\!\!E[Y_t], \quad \forall t \in I\!\!R.$$

Note that the parameter μ is traditionally linked to the specific surface S_v by $S_v = 4\mu$. As a consequence of the previous section, we can also derive some information about the two-point correlation function defined by

$$S_{ij}(t) = I\!\!P[Y_0 = i, Y_t = j] \ , \ i, j = 0 \text{ or } 1.$$

¿From a practical point of view, we are particularly interested in formulae that link the two-point correlation function with the chord lengths distributions rather than with the distribution of X. Indeed, the chord lengths are observable quantities, whereas the field X is not. Without any condition on the chords, it is straightforward to write

$$S_{11}(t) = I\!\!P[X_0 > \gamma, X_t > \gamma] = \sum_{k=0}^{\infty} v_{2k}^+(t) \ , \tag{27}$$

$v_j^+(t)$ being defined in Lemma 1.1. Using the recursive identity (11), we obtain the following.

Proposition 3.2

$$S_{11}(t) = \phi_+ - \mu t \ - \ \mu \sum_{k=1}^{\infty} (-1)^k \int_0^t \left(F_k^+(s) + F_k^-(s) \right) ds. \tag{28}$$

Taking the derivative at the origin yields $\dot{S}_{11}(0) = -\mu = -S_v/4$. Hence the specific surface area S_v can be evaluated through the estimation of the correlation function. Relation (28) shows that explicit formulae for the distribution functions $F_k^\pm$ defined in (5) are essential. The series expansions of Theorem 2.1 allow numerical computation and statistical estimation of the $F_k^\pm$ using the factorial moments.

3.2 Discussion on chord lengths dependence

The direct computation of the two-point correlation function S_{11} as well as the cumulative chord length distribution functions $F_k^\pm$ requires the knowledge of the joint distributions of $(L_1, \ldots, L_k)$ for any k. It motivates the following discussion about chord lengths dependence. First would it be reasonable to assume independence or at least to consider it as a good approximation? If not, which type of dependence could be handled with? The non-independence hypothesis is certainly more realistic but requires tractable tools.

3.2.1 Independent chords

The chord lengths independence is generally assumed in the physics literature ([10], [25], where one speaks of *Independent Interval Approximation*), but with no precision on the concerned probability measure. Let us point out the importance of the choice of the probability measure under which the assumption is stated.

First let us explore the links between the independence assumptions according to which probability measure is under consideration.
We know that on one hand, independence of the $(L_k)_k$ under $I\!P$ is equivalent to the independence under P_0 defined in Proposition 1.1 ([21], Proposition II.19). In this case, $(C(0,t), P_{0^+})$ is a renewal process ([6], §1.4.1).
On the other hand, by Theorem 1.1, the $(L_k)_k$ independence under P_{0^+} is equivalent to the independence under P_{0^-}.
But the independence under $P_{0^\pm}$ does not imply the independence under P_0, unless the chord lengths in both phases are identically distributed. It is the case for instance if the process X is symmetric with respect to the level γ.
Indeed, if we assume the independence of L_i and L_{i+1} under P_{0^+} and P_{0^-}, then by using Theorem 1.1, we have

$$P_0[L_i \leq t, L_{i+1} \leq t] = P_{0^+}[L_i \leq t] P_{0^-}[L_i \leq t]$$

which is different from

$$P_0[L_i \leq t] P_0[L_{i+1} \leq t] = 1/4 \left(P_{0^+}[L_i \leq t] + P_{0^-}[L_i \leq t]\right)^2$$

unless $P_{0^+}[L_i \leq t] = P_{0^-}[L_i \leq t]$.
At last, whether the independence under $I\!P$ does imply the independence under $P_{0^\pm}$ is to our knowledge still an open problem.

Let us focus now on the validity of those assumptions.
Let us assume for a while that the Independent Interval Approximation is valid under the Palm probability measures $P_{0^\pm}$. It allows to complete a result given in [10] and [25] in the following way.

LEMMA 3.1
Suppose that $\mathcal{L}_{0^\pm}(L_1)$ have probability density functions $p_\pm$: $F_1^\pm(t) = \int_0^t p_\pm(s)ds$, $t \in I\!R^+$. If the random variables $(L_k)_{k\geq 1}$ are independent with respect to P_{0^+} and P_{0^-}, then

$$\widehat{S_{11}}(\lambda) = \frac{\phi_+}{\lambda} - \frac{\mu}{\lambda^2} \frac{(1-\widehat{p_-}(\lambda))(1-\widehat{p_+}(\lambda))}{(1-\widehat{p_-}(\lambda)\widehat{p_+}(\lambda))} \ , \ \lambda \in I\!R^+ \tag{29}$$

where $\widehat{f}$ denotes the Laplace transform of f.

Proof. The lemma is a consequence of Proposition 3.2 and the independence assumption.
The probability density functions associated to the distribution functions $F_k^\pm$ can be

expressed as iterated convolutions of the probability density functions p_- and p_+ by Theorem 1.1. Therefore, by taking the Laplace transform, relation (28) becomes

$$\begin{aligned}\widehat{S_{11}}(\lambda) &= \frac{\phi_+}{\lambda} - \frac{\mu}{\lambda^2}\left[1 + 2\sum_{k=1}^{\infty}\widehat{p_-}^k(\lambda)\widehat{p_+}^k(\lambda) + (\widehat{p_-}(\lambda) + \widehat{p_+}(\lambda))\sum_{k=1}^{\infty}\widehat{p_-}^{k-1}(\lambda)\widehat{p_+}^{k-1}(\lambda)\right] \\ &= \frac{\phi_+}{\lambda} + \frac{\mu}{\lambda^2}\left[1 - \frac{(2 - (\widehat{p_-}(\lambda) + \widehat{p_+}(\lambda))}{1 - \widehat{p_-}(\lambda)\widehat{p_+}(\lambda)}\right] . \qquad \square\end{aligned}$$

The result (29) is of an easy use since it only involves the probability density functions p_- and p_+, which may be estimated from the observations.
The formula (29) could be used as a first approximation. Indeed, it has been done in [10] and [25]; the authors showed that the observed two-point correlation function based on simulations of well chosen Gaussian fields, fits perfectly with the one given in formula (29). Hence those simulations do surprisingly agree with this independence assumption, although it is not justified theoretically.
Actually, De Maré [19] or Rychlik [27] proved that in the stationary Gaussian case, the independence assumption fails unless the correlation function of the process X is given by $\rho(t) = \cos(at)$ for some positive constant a.
Furthermore, studies on real data can exhibit dependence between successive chords ([13] for instance, where a dependence between two chords is suggested).
This motivates the next section.

3.2.2 Markovian chords

Looking for a type of dependence which is both theoretically and practically adequate leads us to consider the Markovian assumption.
Indeed, dealing with Markovian chords only requires the two-dimensional distributions $\mathcal{L}_{0^\pm}(L_1, L_2)$. Let us emphasize that they can be computed numerically using the previous section (see Theorem 2.2) or estimated directly on observations.
Suppose that $\mathcal{L}_{0^\pm}(L_1, L_2)$ have probability density functions given by $p_{+-}(x, y) := f^+_{L_1,L_2}(x, y)$ and $p_{-+}(x, y) := f^-_{L_1,L_2}(x, y)$.
By Theorem 1.1, we have for all integer $i \geq 1$,

$$\begin{aligned} & f^+_{L_{2i-1},L_{2i}}(x, y) = p_{+-}(x, y) \ , \quad f^-_{L_{2i-1},L_{2i}}(x, y) = p_{-+}(x, y) \\ \text{and} \quad & f^\pm_{L_{2i-1},L_{2i}}(x, y) = f^\mp_{L_{2i},L_{2i+1}}(x, y) \ . \end{aligned} \tag{30}$$

Assume that the random variables $(L_k)_{k\geq 1}$ satisfy the Markov property under $P_{0^\pm}$, namely

$$P_{0^\pm}\left[h(L_i, k+1 \leq i \leq n) \mid L_1, \cdots, L_k\right] = P_{0^\pm}\left[h(L_i, k+1 \leq i \leq n) \mid L_k\right], \ n > k,$$

for all bounded Borel function h.
Then it is straightforward to obtain:

Lemma 3.2 *Under the Markovian assumption, the probability density function $f^{\pm}_{L_{i_1},\cdots,L_{i_k}}$ of $(L_{i_1},\cdots,L_{i_k})$, $1<i_1<\cdots<i_k$, satisfies*

$$\begin{aligned} f^{\pm}_{L_1,\ldots,L_{2k}}(x_1,\cdots,x_k) &= f^{\pm}_{L_1}(x_1)\prod_{j=2}^{2k} f^{\pm}_{L_j|L_{j-1}=x_{j-1}}(x_j) \\ &= p_+(x_1)\frac{p_{+-}(x_{2k-1},x_{2k})}{p_+(x_{2k-1})}\prod_{j=1}^{k-1}\frac{p_{+-}(x_{2j-1},x_{2j})p_{-+}(x_{2j},x_{2j+1})}{p_+(x_{2j-1})p_-(x_{2j})} \end{aligned}$$

and

$$f^{+}_{L_1,\ldots,L_{2k+1}}(x_1,\cdots,x_{2k+1}) = p_+(x_1)\prod_{j=1}^{k}\frac{p_{+-}(x_{2j-1},x_{2j})p_{-+}(x_{2j},x_{2j+1})}{p_+(x_{2j-1})p_-(x_{2j})}.$$

Another way of relaxing the Markov assumption may be to assume the m-dependence of the $(L_k)_{k\geq 1}$, *i.e.* to assume that L_i is independent of L_{i+k}, for all integer $i\geq 1$ and $k>m\geq 2$. It would then require the knowledge of all joint probability density functions of order $m+1$.

Perspectives

A mathematical framework has been proposed in this paper, as well as explicit formulae to compute characteristic parameters and distribution functions appearing in the modelling of porous media.
In order to be complete and totally effective, our theoretical relations need to be numerically tested. A first step in this direction has been done in the Gaussian case ([8]). More will be studied in a forthcoming paper providing for instance the rate of convergence of the series appearing in Theorem 2.1 and numerical comparisons between the formulae in Theorems 2.1 and 2.2.
Another direction has to be explored: the independence assumption. We are working on the construction of a statistical test which could be applied both to simulations and real data. We also might estimate, at least numerically, the error made when supposing the chords independence. How to relate (in)dependence conditions on the chords and assumptions on the process X itself still deserves further study.

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