On a high-dimensional nonlinear stochastic partial differential equation

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ON A HIGH-DIMENSIONAL NONLINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATION

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Abstract. In this paper we investigate a nonlinear stochastic partial differential equation (spde in short) perturbed by a space-correlated Gaussian noise in arbitrary dimension $d \geq 1$, with a non-Lipschitz coefficient noisy term. The equation studied coincides in one dimension with the stochastic Burgers equation. Existence of a weak solution is established through an approximation procedure.

1. Introduction

Let $d \geq 1$ be an integer and $D := [0,1]^d$. In this paper, we study the following nonlinear stochastic partial differential equation on $\mathbb{R}^+ \times D$

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_i}u(t,x) + \frac{1}{2}u^2(t,x)\right) \\ &\quad + \sigma(u(t,x))\dot{F}(t,x),\ t > 0,\ x \in D,\ i = 1,...,d \end{aligned} \tag{1.1}$$

with Dirichlet boundary conditions

$$u(t,x) = 0, \quad t > 0, \quad x \in \partial D, \tag{1.2}$$

and the initial condition

$$u(0,x) = u_0(x), \quad x \in D, \tag{1.3}$$

where u_0 is a continuous function on D with values in the interval $[0,1]$ and $\dot{F}$ is a noise on $\mathbb{R}^+ \times D$ that is white in time and colored in space. The noisy coefficient term σ is a real-valued function which will be described precisely later. We will refer to the problem $(1.1) - (1.3)$ by $Eq(d, u_0, \sigma)$.

It is worth noting that the equation $Eq(d, u_0, 0)$ gives the classic Burgers equation, which arises in physics and have extensively been studied in the literature, see e.g. [21] and [23].

A class of equations of type $Eq(d, u_0, \sigma)$, with noise depending only on time, was studied by Gyöngy and Rovira [17]. The authors took the space variable in a bounded convex domain of $\mathbb{R}^d$, and the class investigated contains a version of the one-dimensional Burgers equation [3, 5, 8, 9, 14, 16, 18] as a special case. They

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proved existence, uniqueness and a comparison theorem under assumptions on coefficients which include global Lipschitz condition on the diffusion coefficient σ.

However, the present work is concerned with a non-Lipschitz diffusion coefficient. Since the F is a space-time noise and σ is non-Lipschitz the results of [17] cannot be applied here.

The equation $Eq(1, u_0, \sigma)$ was studied by Kolkovska [20]. The author proved the existence of a weak solution by using an argument of tightness and solving a martingal problem. The case considered corresponds to the stepping stone model which describes migrating populations consisting of two types when the total population size does not change over time. For more details on the genealogy of a variation of this model, one can see Chapter 7 in [29]. Our study extends this of [20] to a multidimensional case, and with a more general class of diffusion coefficients.
It is also worth mentioning that a one dimensional version of (1.1), perturbed with a space-time white noise, was investigated by Adler and Bonnet [1, 4] on the whole real line and with the non-Lipschitz noisy coefficient $\sigma(r) = \sqrt{r}$. The authors established the existence of a weak solution and its Hölder continuity. Moreover, they discussed (but did not prove) the uniqueness of the solution.

In the present paper, we deal with a d-dimensional spdes. As it is well known, we can no longer use the space-time white noise for the perturbation; indeed, in dimension $d \geq 2$, the fundamental solution associated with the operator $\frac{\partial}{\partial t} - \Delta$ is not square integrable when $d \geq 2$, while the one associated with the operator $\frac{\partial}{\partial t^2} - \Delta$ becomes less smooth as the dimension increases. Hence, the martingale measure approach, introduced by Walsh [30], to investigate multidimensional SPDEs driven by a space-time white noise gives solutions only in the space of random distributions (see e.g. [2, 11, 26]).

In order to circumvent this difficulty, Dalang and Frangos [7] suggested to replace the space time white noise by some Gaussian noise which is white in time and which has some space correlation in order to obtain solutions in the space of real-valued stochastic processes. See [6] and [27] and the references therein for more literature in the subject.

The aim of this paper is to study the existence of solution of a non-linear equation $Eq(d, u_0, \sigma)$ in arbitrary dimension $d \geq 1$, with a non-Lipschitz coefficient noisy term. The idea to prove our main result is to adopt an approach going back to Funaki [13].

The paper is organized as follows. In the second section we give a formulation of the problem. Section 3 is devoted to define a spatial discretization scheme of (1.1) and obtain a system of stochastic differential equations. The existence and uniqueness of a unique strong solution for this system. Section 4 is devoted to the tightness of these approximating solutions. The existence of a weak solution of $Eq(d, u_0, \sigma)$ is established in Section 5 by solving a martingale problem. We conclude with an Appendix containing some technical results needed along the paper.

Note that all real positive constants are denoted by c regardless of their values and some of the standing parameters are not mentioned.

2. Framework

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Let $F = \{F(\varphi), \varphi \in \mathcal{D}(\mathbb{R}^{d+1}\}$ be a mean-zero $\mathrm{L}^2(\Omega, \mathcal{F}, \mathbb{P})$-valued Gaussian process with the covariance functional

$$\mathbb{E}(F(\varphi)F(\psi)) = \int_0^{+\infty} ds \int_D dx \int_D dy \varphi(s,x) f(x,y) \psi(s,y), \tag{2.1}$$

where $f : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$ denotes the correlation kernel of the noise F.

Several authors discussed the weaker assumptions in such a way that (2.1) defines a covariance functional, see e.g. the references [7, 6, 29].

In this paper, we will take f symmetric and belonging to $\mathcal{C}_b(D \times D)$, the space of continuous bounded functions. Following the same approach in [6], the Gaussian process F can be extended to a worthy martingale measure $M = \{M(t,A) := F([0,t] \times A) : t \geq 0, A \in \mathcal{B}_b(D)\}$ which shall acts as integrator, in the sense of Walsh [30], where $\mathcal{B}_b(D)$ denotes the bounded Borel subsets of D. Let $\mathcal{G}_t$ be the completion of the σ–field generated by the random variables $\{M(s,A),\ 0 \leq s \leq t,\ A \in \mathcal{B}_b(D)\}$. The properties of F ensure that the process $M = \{M(t,A),\ t \geq 0,\ A \in \mathcal{B}_b(D)\}$, is a martingale with respect to the filtration $\{\mathcal{G}_t : t \geq 0\}$. Then one can give a rigorous meaning to solution of the formal equation (1.1). A stochastic process $u : \Omega \times \mathbb{R}^+ \times D \to \mathbb{R}$, which is jointly measurable and $\mathcal{G}_t$-adapted, is said to be a weak solution to the equation $Eq(d, u_0, \sigma)$ if there exists a noise F of the form (2.1) such that for each $\varphi \in \mathcal{C}^2(D)$ such that $\varphi = 0$ on ∂D, and a.s. for almost all $t \geq 0$ and $x \in D$,

$$\begin{aligned}\int_D u(t,x)\varphi(x)dx &= \int_D u_0(x)\varphi(x)dx + \int_0^t \int_D u(s,x)\Delta\varphi(x)dxds \\ &\quad -\frac{1}{2}\int_0^t \int_D u^2(s,x) \sum_{i=1}^d \frac{\partial \varphi}{\partial x_i}(x)dxds + \int_0^t \int_D \sigma(u(s,x))\varphi(x)F(ds,dx).\end{aligned}$$

In order to formulate our main result, we assume the following hypothesis :

(H) The function σ is Hölder continuous of order $\frac{1}{2} \leq \alpha < 1$ i.e. there exists a positive constant c such that for all $x, y \in [0,1]$

$$|\sigma(x) - \sigma(y)| \leq c|x-y|^\alpha.$$

Moreover, we assume that σ is not identidically zero and satisfies $\sigma(0) = \sigma(1) = 0$.

Theorem 2.1. *Assume that u_0 is a continuous function on D with values in the interval $[0,1]$, and that σ satisfies* (**H**). *Then there exists a weak solution to the equation $Eq(d, u_0, \sigma)$.*

Remark 2.1. *The hypothesis* (**H**) *implies the existence of a positive constant c such that for all $x \in [0,1]$*

$$\sigma(x) \leq c \min(x^\alpha, (1-x)^\alpha).$$

Examples

1.: The function $\sigma_1 : x \longmapsto \sqrt{x(1-x)}$, which corresponds to the stepping stone model, satisfies (**H**) with $\alpha = \frac{1}{2}$.

2.: Let $\gamma \in]\frac{1}{2}, 1]$. The function σ_2 defined on the interval $[0,1]$ by $\sigma_2(0) = 0$ and $\sigma_2(x) = -x^\gamma \log(x)$ for $x \in]0,1]$ satisfies (**H**). It is a concave function with values in $[0,1]$. Indeed, σ_2 is twice continuously differentiable on $]0,1[$ and $\sigma_2''(x) = x^{\gamma-2}(1-2\gamma+\gamma(1-\gamma)\log(x))$, for all $x \in]0,1[$. Moreover, taking into account the continuity of σ_2 on $[0,1]$ we deduce its concavity on $[0,1]$. It follows that $|\sigma_2(x)-\sigma_2(y)| \le \sup\{|\sigma_2(x-y)-\sigma_2(0)|, |\sigma_2(1)-\sigma_2(1-x+y)|\}$, for all $x, y \in [0,1]$ such that $x \ge y$. Finally, we point out that $|\sigma_2(x-y)-\sigma_2(0)| = |\sigma_2(x-y)| \le c|x-y|^{\gamma-\epsilon}$, where $\epsilon \in]0, \gamma - \frac{1}{2}[$, and $|\sigma_2(1)-\sigma_2(1-x+y)| = |\sigma_2(1-x+y)| \le c|x-y|^\eta$ for any $\eta \in]0,1[$. The result follows.

3. The approximating processes

Let $n \ge 2$ be an integer. For $\mathbb{k} := (k_1, ..., k_d) \in \{1, ..., n-1\}^d$, set $x^n_{\mathbb{k}} = \left(\frac{k_1}{n}, \frac{k_2}{n}, ..., \frac{k_d}{n}\right)$ and define the interval

$$\mathbb{I}^n_{\mathbb{k}} = \Pi^d_{j=1}[\frac{k_j}{n}, \frac{k_j+1}{n}).$$

Consider the discretized version of $Eq(d, u_0, \sigma)$:

$$\begin{cases} du^n(t,x^n_{\mathbb{k}}) = \Delta^n u^n(t,x^n_{\mathbb{k}})dt + \frac{1}{2}\sum_{i=1}^d \nabla^n_i (u^n(t,x^n_{\mathbb{k}})^2)dt + n^d\sigma(u^n(t,x^n_{\mathbb{k}}))F(dt,\mathbb{I}^n_{\mathbb{k}}), \\ u^n(0,x^n_{\mathbb{k}}) \quad = u_0(x^n_{\mathbb{k}}), \end{cases} \tag{3.1}$$

where, the operator in the first term is given by

$$\Delta^n u^n(t,x^n_{\mathbb{k}}) = n^2 \sum_{i=1}^d \left(u^n(t,x^n_{\mathbb{k}}+\frac{e_i}{n}) + u^n(t,x^n_{\mathbb{k}}-\frac{e_i}{n}) - 2u^n(t,x^n_{\mathbb{k}})\right),$$

and in second term by

$$\nabla^n_i(u^n(t,x^n_{\mathbb{k}})^2) = n\left(u^n(t,x^n_{\mathbb{k}}+\frac{e_i}{n})^2 - u^n(t,x^n_{\mathbb{k}})^2\right),$$

with $\{e_i, 1 \le i \le d\}$ denotes the canonical basis of $\mathbb{R}^d$, and

$$F(t,\mathbb{I}^n_{\mathbb{k}}) = \int_0^t \int_{\mathbb{I}^n_{\mathbb{k}}} F(ds,dx).$$

The noises $\{F(t,\mathbb{I}^n_{\mathbb{k}}), t \ge 0\}$, derived from colored noise F on $\mathbb{R}^d$ with covariance given by (2.1), are one dimensional correlated Brownian motions. Moreover, the boundedness of f implies that $\langle F(\cdot,\mathbb{I}^n_{\mathbb{k}})\rangle_t \le ctn^{-2d}$, where c is a constant depending on f. Indeed

$$\begin{aligned} \langle F(\cdot,\mathbb{I}^n_{\mathbb{k}})\rangle_t &= \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}_{\mathbb{I}^n_{\mathbb{k}}}(x) f(x,y) \mathbf{1}_{\mathbb{I}^n_{\mathbb{k}}}(y) ds dx dy \\ &= t\int_{\mathbb{I}^n_{\mathbb{k}}}\int_{\mathbb{I}^n_{\mathbb{k}}} f(x,y)dxdy \\ &\le c\,t\,n^{-2d}, \end{aligned}$$

where $c = \max\{f(x,y);\ (x,y) \in D \times D\}$.

Now, we proceed as in [24] and we consider the bijection between the grid $D_n^d := \{x_{\mathbb{k}}^n = \left(\frac{k_1}{n}, \frac{k_2}{n}, ..., \frac{k_d}{n}\right),\ (k_1, ..., k_d) \in \{1, ..., n-1\}^d\}$, which assigns to each $\left(\frac{k_1}{n}, \frac{k_2}{n}, ... \frac{k_d}{n}\right)$ the integer $k_1 + (k_2 - 1)(n-1) + ... + (k_d - 1)(n-1)^{d-1}$.

Set $N := (n-1)^d$, and for each $i \in \{1, .., N\}$, let $\underline{i}^n$ be the unique element $\left(\frac{k_1}{n}, \frac{k_2}{n}, ..., \frac{k_d}{n}\right)$ such that $i = k_1 + (k_2 - 1)(n-1) + ... + (k_d - 1)(n-1)^{d-1}$ and $1 \leq k_j \leq n-1$ for each $j \in \{1, .., d\}$.

Finally, let $u_i^n(t)$ denotes $u^n(t, \underline{i}^n)$. Hence, the system (3.1) can be written, for $i = 1, ..., d$,

$$\begin{cases} du_i^n(t) = \sum_{j=1}^{N} [a_{i,j} u_j^n(t) + \frac{1}{2} b_{i,j} u_j^n(t)^2(t)]dt + n^d \sigma(u_i^n(t)) F(dt, \mathbb{I}_{\underline{i}^n}^n) \\ u_i^n(0) = u_0(\underline{i}^n). \end{cases} \tag{3.2}$$

Let $A_n^{(d)}$ and $B_n^{(d)}$be the $N \times N$ matrices such that $A_n^{(d)} : = (a_{i,j})_{1 \leq i,j \leq N}$ and $B_n^{(d)} : = (b_{i,j})_{1 \leq i,j \leq N}$.

For $d = 1$, it is well known that we have

$$a_{i,j} = \begin{cases} n^2 & \text{if} \quad |i-j| = 1, \\ -2n^2 & \text{if} \quad j = i, \\ 0 & \quad otherwise, \end{cases} \tag{3.3}$$

and,

$$b_{i,j} = \begin{cases} n & \text{if} \quad j = i+1, \\ -n & \text{if} \quad j = i, \\ 0 & \quad otherwise. \end{cases} \tag{3.4}$$

For $d \geq 2$, it is known from [24] that $A_n^{(d)}$ can be obtained by induction using a relation of recurrence and the fact that $A_n^{(1)}$ is known. For the coefficients $b_{i,j}$, let us first introduce some notations. For an integer $\ell \geq 1$, let I_ℓ be the $\ell \times \ell$ identity matrix and set $M := (n-1)^{d-1}$. One can easily show that

$$B_n^{(d)} = \begin{pmatrix} B_n^{(d-1)} & nI_M & 0 & \cdots & \cdots & 0 \\ 0 & B_n^{(d-1)} & nI_M & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & \cdots & \ddots & \ddots & nI_M \\ 0 & \cdots & \cdots & \cdots & 0 & B_n^{(d-1)} \end{pmatrix} - nI_N.$$

Note that $B_n^{(1)}$ is the $(n-1) \times (n-1)$-matrix defined by (3.4).
The main result of this section is the following

Proposition 3.1. *For any integer $n \geq 1$ and for any initial random condition $u^n(0) = (\xi_1^n, ..., \xi_N^n) \in [0,1]^N$, the system*

$$\begin{cases} du_i^n(t) = \sum_{j=1}^{N} [a_{i,j} u_j^n(t) + \frac{1}{2} b_{i,j} u_j^n(t)^2] dt + n^d \sigma(u_i^n(t)) F(dt, \mathrm{I\!I}_{\underline{i}^n}^n), \\ u_i^n(0) \ = \xi_i^n, \end{cases} \tag{3.5}$$

admits a unique strong solution $u^n(.) \in \mathcal{C}([0, +\infty[, [0,1]^N)$.

Proof.
We consider first the following system

$$\begin{cases} du_i^n(t) = \sum_{j=1}^{N} \left[a_{i,j} u_j^n(t) + \frac{1}{2} b_{i,j} g(u_j^n(t)) \right] dt + h(u_i^n(t)) F(dt, I_i^n), \\ u_i^n(0) = \xi_i^n, \end{cases} \tag{3.6}$$

where $h : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $h(x) = n^d \sigma(x) \mathbf{1}_{\{0 \leq x \leq 1\}}$, and $g : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $g(x) = x^2 \mathbf{1}_{\{-1 \leq x \leq 1\}}$. In order to show that the continuous coefficients of the finite dimensional system (3.6) satisfy the growth linear condition, we write it under the following form

$$\begin{cases} dU^n(t) = A_n^{(d)} U^n(t) dt + \frac{1}{2} B_n^{(d)} \Sigma(U^n(t)) dt + K(U^n(t)) dF_t \\ U^n(0) = u^n(0), \end{cases} \tag{3.7}$$

with $U^n(t) := (u_i^n(t), 1 \leq i \leq N)$, $\Sigma(U^n(t)) := (g(u_i^n(t)), 1 \leq i \leq N)$, $K(U^n(t)) := diag(h(u_i^n(t)), 1 \leq i \leq N)$, *and* $F_t := \Big(F(t, \mathrm{I\!I}_{\underline{i}^n}^n), \ 1 \leq i \leq N \Big)$.

Since the first term in the right hand side of (3.7) is linear, then it satisfies the linear growth condition. For the second one, using an elementary property of matrix norms and the equivalence of norms on $\mathbb{R}^N$, we can write

$$\begin{aligned} \|B_n^{(d)} \Sigma(U^n(t))\| &\leq \|B_n^{(d)}\| \|L(U^n(t))\| \\ &\leq c \sum_{i=1}^{N} |u_i^n(t)| \leq c \|U^n(t)\|, \end{aligned}$$

where the constant c depends on n and d. Concerning the noise coefficient, owing to Remark 2.1 we obtain a.s. for $t \geq 0$,

$$\begin{aligned} \sum_{i=1}^{N} h(u_i^n(t))^2 &\leq cn^{2d} \sum_{i=1}^{N} u_i^n(t)^{2\alpha} \mathbf{1}_{\{0 \leq u_i^n(t) \leq 1\}} \\ &\leq cn^{2d} \sum_{i=1}^{N} (1 + (u_i^n(t))^2) \\ &\leq cn^{2d} N \left(1 + \|U^n(t)\|^2\right). \end{aligned}$$

Thus, the existence of a weak solution with continuous sample paths for (3.6) is given by Theorem 6.1 in the Appendix.

Now, we will show that $0 \leq u_i^n(t) \leq 1$ for all $i = 1, ..., n$ and $t \geq 0$. Taking $\rho(x) = x^\beta$ with $1 \leq \beta \leq 2\alpha$, and using the hypothesis (**H**) and the continuity of the trajectories of u_i^n, we get fot $0 \leq t \leq T$

$$\begin{aligned}
\int_0^t \frac{\mathbf{1}_{\{u_i^n(s)>0\}}}{\rho(u_i^n(s))} d\langle u_i^n\rangle_s &= n^{2d} \int_0^t \frac{\mathbf{1}_{\{u_i^n(s)>0\}} h^2(u_i^n(s))}{\rho(u_i^n(s))} ds \\
&\leq n^{2d} T \\
&< +\infty.
\end{aligned}$$

Then, by Lemma 6.1 we obtain that the local time, $L_t^0(u_i^n)$, of the semi-martingale u_i^n satisfies $L_t^0(x_i^n) = 0$ for all $t > 0$, a.s, and $i = 1, \cdots, N$.

Moreover, set $x^- := \max(0, -x)$ and by applying the Tanaka formula and summing over all indices $i = 1, .., N$, we get for $t \geq 0$,

$$\begin{aligned}
\sum_{i=1}^N u_i^n(t)^- &= -\int_0^t \sum_{i=1}^N \mathbf{1}_{\{u_i^n(s)\leq 0\}} \sum_{j=1}^N a_{i,j} u_j^n(s) ds - \frac{1}{2}\int_0^t \sum_{i=1}^N \mathbf{1}_{\{u_i^n(s)\leq 0\}} \sum_{j=1}^N b_{i,j} g(u_j^n(s)) ds \\
&:= I_1 + I_2.
\end{aligned}$$

Since $-x \leq x^-$ and taking into account the fact that $a_{i,j} \geq 0$ for all $j \neq i$ and $\sum_{i=1, i\neq j}^N a_{i,j} \leq 2dn^2$

$$\begin{aligned}
I_1 &= -\int_0^t \sum_{j=1}^N u_j^n(s) \sum_{i=1}^N a_{i,j} \mathbf{1}_{\{u_i^n(s)\leq 0\}} ds \\
&= -\int_0^t \sum_{j=1}^N \big[u_j^n(s)\big(-2n^2 \mathbf{1}_{\{u_j^n(s)\leq 0\}} + \sum_{i=1, i\neq j}^N a_{i,j} \mathbf{1}_{\{u_i^n(s)\leq 0\}}\big)\big] \\
&= -\int_0^t \sum_{j=1}^N \big[u_j^n(s)^- \, 2n^2 \mathbf{1}_{\{u_j^n(s)\leq 0\}} + u_j^n(s) \sum_{i=1, i\neq j}^N a_{i,j} \mathbf{1}_{\{u_i^n(s)\leq 0\}}\big] \\
&= \int_0^t \sum_{j=1}^N \big[u_j^n(s)^- \, (-2n^2) \mathbf{1}_{\{u_j^n(s)\leq 0\}} - u_j^n(s) \sum_{i=1, i\neq j}^N a_{i,j} \mathbf{1}_{\{u_i^n(s)\leq 0\}}\big] \\
&\leq \int_0^t \sum_{j=1}^N u_j^n(s)^- \left(\sum_{i=1}^N a_{i,j} \mathbf{1}_{\{u_i^n(s)\leq 0\}}\right) ds \leq 2dn^2 \int_0^t \sum_{i=1}^N u_i^n(s)^- ds. \quad (3.8)
\end{aligned}$$

For I_2, notice first that by using the definition of g,

$$I_2 = \frac{1}{2}\int_0^t \sum_{i=1}^N \mathbf{1}_{\{-1\leq u_i^n(s)\leq 0\}} \left(nd u_i^n(s)^2 - \sum_{j=1, j\neq i}^N b_{i,j} u_j^n(s)^2 \right) ds.$$

Taking into account the fact that $b_{i,j} \geq 0$ for all $j \neq i$, we infer that

$$I_2 \leq \frac{1}{2} nd \int_0^t \sum_{i=1}^N \mathbf{1}_{\{-1\leq u_i^n(s)\leq 0\}} u_i^n(s)^2 ds.$$

Moreover, for $0 \leq s \leq t$ such that $-1 \leq u_i^n(s) \leq 0$, we have

$$u_i^n(s)^2 = (-u_i^n(s))^2 \leq -u_i^n(s) = u_i^n(s)^-.$$

Hence,

$$I_2 \leq \frac{1}{2} nd \int_0^t \sum_{i=1}^N u_i^n(s)^- ds. \tag{3.9}$$

From the estimates (3.8) and (3.9), we deduce

$$\sum_{i=1}^N u_i^n(t)^- \leq c \int_0^t \sum_{i=1}^N u_i^n(s)^- ds.$$

Then, by Gronwall's Lemma, we obtain that $\sum_{i=1}^N u_i^n(t)^- = 0$. Consequently, the solution is non-negative for each $t \geq 0$.
Concerning the fact that $u_i^n(t) \leq 1$, it can be easily checked by using the same arguments as above for $(1 - u_i^n(t))$.

Therefore, the system (3.5) admits a weak solution with trajectories in $\mathcal{C}([0, +\infty[, [0,1]^N)$.

To show the pathwise uniqueness for (3.5), we assume that $u^{1,n}(t) := (u_i^{1,n}(t), 1 \leq i \leq N)$ and $u^{2,n}(t) := (u_i^{2,n}(t), 1 \leq i \leq N)$ are two weak solutions to (3.5), with the same noise F and the same initial data.

Set $y_i^n := u_i^{1,n} - u_i^{2,n}$, then for $t > 0$,

$$\begin{aligned} y_i^n(t) &= \sum_{j=1}^N a_{i,j} \int_0^t y_j^n(s) ds + \frac{1}{2} \sum_{j=1}^N b_{i,j} \int_0^t \left(u_j^{1,n}(s)^2 - u_j^{2,n}(s)^2 \right) ds \\ &\quad + n^d \int_0^t \left[\sigma(u_i^{1,n}(s)) - \sigma(u_i^{2,n}(s)) \right] F(ds, \mathbb{I}^n_{\underline{i}^n}). \end{aligned}$$

Consequently, by the boundedness of the correlation kernel f,

$$\langle y_i^n(\cdot) \rangle_t \leq c \int_0^t \left[\sigma(u_i^{1,n}(s)) - \sigma(u_i^{2,n}(s)) \right]^2 ds.$$

Owing to Remark 2.1 and Lemma 6.1, we obtain that a.s. $L_t^0(y_i^n) = 0$, for all $i \in \{1, ..., N\}$. Applying the Tanaka's formula for the continuous semimartingale y_i^n, it follows that

$$\begin{aligned} |y_i^n(t)| &= \sum_{j=1}^N a_{i,j} \int_0^t sgn(y_j^n(s)) y_j^n(s) ds \\ &\quad + \frac{1}{2} \sum_{j=1}^N b_{i,j} \int_0^t sgn(y_j^n(s)) \left(u_j^{1,n}(s)^2 - u_j^{2,n}(s)^2 \right) ds \\ &\quad + n^d \int_0^t sgn(y_i^n(s)) \left[\sigma(u_i^{1,n}(s)) - \sigma(u_i^{2,n}(s)) \right] F(ds, \mathbb{I}^n_{\underline{i}^n}). \end{aligned}$$

Noting that the sign function is defined by

$$sgn(x) = \begin{cases} +1, & x \geq 0; \\ -1, & x < 0. \end{cases}$$

By summing over all $i \in \{1, ..., N\}$ and taking the expectation, we get

$$\begin{aligned}
\mathbb{E}\left(\sum_{i=1}^{N} |y_i^n(t)|\right) &= \mathbb{E}\left(\sum_{i,j=1}^{N} a_{i,j} \int_0^t sgn(y_j^n(s)) y_j^n(s) ds\right) \\
&\quad + \frac{1}{2}\mathbb{E}\left(\sum_{i,j=1}^{N} b_{i,j} \int_0^t sgn(y_j^n(s)) \left(u_j^{1,n}(s)^2 - u_j^{2,n}(s)^2\right) ds\right) \\
&:= J_1 + J_2. \qquad (3.10)
\end{aligned}$$

For J_1, the boundedness of the coefficients $a_{i,j}$ yields

$$\begin{aligned}
J_1 &= \mathbb{E}\int_0^t \sum_{j=1}^{N} sgn(y_j^n(s)) y_j^n(s) \left(\sum_{i=1}^{N} a_{i,j}\right) ds \\
&\leq c\,\mathbb{E}\int_0^t \sum_{j=1}^{N} |y_j^n(s)| ds, \qquad (3.11)
\end{aligned}$$

where c is a constant depending on n. Concerning J_2, using the boundedness of the coefficients $b_{i,j}$ and the fact that the solutions are with values in the interval $[0, 1]$, we can write

$$\begin{aligned}
J_2 &= \frac{1}{2}\mathbb{E}\int_0^t \sum_{j=1}^{N} sgn(y_j^n(s)) y_j^n(s)(u_i^{1,n}(s) + u_i^{2,n}(s)) \left(\sum_{i=1}^{N} b_{i,j}\right) ds \\
&\leq c\mathbb{E}\int_0^t \sum_{j=1}^{N} |y_j^n(s)| ds. \qquad (3.12)
\end{aligned}$$

Combining the estimates (3.10)–(3.12) and applying Gronwall's Lemma, we obtain for all $t \geq 0$

$$\mathbb{E}\sum_{i=1}^{N} |y_i^n(t)| = 0.$$

Finally, the well-known theorem of Yamada and Watanabe [31] (see also Corollary 3.23 in [19]) implies that (3.5) has a unique strong solution. □

4. The tightness of the approximating processes

As in [28], let Y^n be a simple random walk whose generator is the discrete Laplacian Δ^n on the lattice D_n^d. To get the system (3.2) in its mild form we define the fundamental solution p_d^n associated with Δ^n.

For $\underline{i}^n, \underline{j}^n \in D_n^d$, set

$$p_d^n(t, \underline{i}^n, \underline{j}^n) = n^d \mathbb{P}(Y_t^n = \underline{j}^n \mid Y_0^n = \underline{i}^n).$$

The system (3.2) can be now written in its variation of constant form

$$\begin{aligned} u_i^n(t) &= \frac{1}{n^d}\sum_{j=1}^{N} p_d^n(t,\underline{i}^n,\underline{j}^n)u_0(\underline{j}^n) + \frac{1}{2n^d}\int_0^t \sum_{j=1}^{N} p_d^n(t-s,\underline{i}^n,\underline{j}^n)b_{i,j}u_j^n(s)^2 ds \\ &\quad + \int_0^t \sum_{j=1}^{N} p_d^n(t-s,\underline{i}^n,\underline{j}^n)\sigma(u_j^n(s))F(ds,\mathrm{I\!I}_{\underline{i}^n}^n). \end{aligned}$$

In the remainder of this paper, we denote $u_i^n(t)$ by $u^n(t,\underline{i}^n)$, where the relationship between i and $\underline{i}^n$ is described in the previous section. Moreover, in order to define $u^n(t,\cdot)$ for all $x \in D$, we set $\kappa_n(x) = \underline{i}^n$ for $x \in I_i^n$, and $\bar{p}_d^n(t,x,y) = p_d^n(t,\kappa_n(x),\kappa_n(y))$ for $x,y \in D$. Using a polygonal interpolation like as in [28], we can write for all $x \in D$

$$\begin{aligned} u^n(t,x) &= \int_D p_d^n(t,x,\kappa_n(y))u_0(\kappa_n(y))dy + \frac{1}{2}\int_0^t\int_D p_d^n(t-s,x,\kappa_n(y))u^n(s,y)^2 dyds \\ &\quad + \int_0^t\int_D p_d^n(t-s,x,\kappa_n(y))\sigma(u^n(s,y))F(ds,dy) \\ &\equiv u_1^n(t,x) + u_2^n(t,x) + u_3^n(t,x). \end{aligned} \tag{4.1}$$

The main result of this section is as follows.

Proposition 4.1. *For each $T > 0$, the sequence $\{u^n, n \in \mathbb{N}\}$ is tight in the space $\mathcal{C}([0,T],\mathcal{H})$, where $\mathcal{H} := \mathcal{C}(D,[0,1])$.*

Proof. Note firstly that $u^n(t,x)$ are in $[0,1]$. Consequently, there exists a constant $c(p,T) > 0$ such that for any $p \geq 1$,

$$\sup_n \mathbb{E}\left[\sup_{0\leq t\leq T}\sup_{x\in D}|u^n(t,x)|^p\right] \leq c(p,T). \tag{4.2}$$

To handle the stochastic integral, we will use the factorization method introduced by Da Prato el al. in [10]. Indeed, it is well known that one can write for $\beta \in (0,1)$,

$$\begin{aligned} &\int_0^t\int_D \bar{p}_d^n(t-s,x,y)\sigma(u^n(s,y))F(ds,dy) \\ &= \frac{\sin(\beta\pi)}{\pi}\int_0^t\int_D (t-s)^{\beta-1}\bar{p}_d^n(t-s,x,y)Y(s,y)dyds, \end{aligned} \tag{4.3}$$

where

$$Y(s,y) := \int_0^s\int_D (s-r)^{-\beta}\bar{p}_d^n(s-r,y,z)\sigma(u^n(r,z))F(dr,dz).$$

In the sequel, we will take $0 < \beta < \frac{1}{2}$. Hence, by Burkholder's inequality, the boundedness of σ and f and the estimate $i)$ of Lemma 6.2, we obtain for any $T > 0$ and $p > 1$, that there exists a constant $c(T,p) > 0$ such that

$$\sup_{s\in[0,T]}\sup_{y\in D}\mathbb{E}|Y(s,y)|^p \leq c(p,T). \tag{4.4}$$

Let us now prove an estimate on the increments in both the space and time variables. Firstly, we are concerned with the third term in (4.1). Clearly, for all $x,y \in D$ and $0 \leq s \leq t \leq T$

$$\mathbb{E}|u_3^n(t,x) - u_3^n(s,y)|^p \leq c\left(\mathbb{E}|u_3^n(t,x) - u_3^n(t,y)|^p + \mathbb{E}|u_3^n(t,y) - u_3^n(s,y)|^p\right).$$

By Burkholder's inequality and the fact that σ and the kernel noise f are bounded, we can write

$$\begin{aligned}
\mathbb{E}|u_3^n(t,x)-u_3^n(t,y)|^p &= \mathbb{E}\left|\int_0^t\int_D \left[\bar{p}_d^n(t-s,x,z)-\bar{p}_d^n(t-s,y,z)\right]\sigma(u^n(s,z))F(dz,ds)\right|^p \\
&\le c\left(\int_0^t\left(\int_D |\bar{p}_d^n(t-s,x,z)-\bar{p}_d^n(t-s,y,z)|\,dz\right)^2 ds\right)^{p/2} \\
&\le c\|x-y\|^{\left(\frac{1}{q}-\frac{1}{2}\right)\frac{p}{2}}, \qquad (4.5)
\end{aligned}$$

where $1<q<2$. Note that in the last inequality, we have used the estimate $ii)$ of Lemma 6.2. On the other hand, we have

$$\begin{aligned}
&\mathbb{E}|u_3^n(t,y)-u_3^n(s,y)|^p \\
&= \mathbb{E}\left|\int_0^t\int_D \left[(t-r)^{\beta-1}\bar{p}_d^n(t-r,y,z)-(s-r)^{\beta-1}\bar{p}_d^n(s-r,y,z)\mathbf{1}_{[0,s]}(r)\right]Y(r,z)dzdr\right|^p \\
&\le c\left(\mathbb{E}\left|\int_0^s\int_D (s-r)^{\beta-1}(\bar{p}_d^n(t-r,y,z)-\bar{p}_d^n(s-r,y,z))Y(r,z)dzdr\right|^p\right. \\
&\quad + \mathbb{E}\left|\int_0^s\int_D ((t-r)^{\beta-1}-(s-r)^{\beta-1})\bar{p}_d^n(t-r,y,z)Y(r,z)dzdr\right|^p \\
&\quad + \left.\mathbb{E}\left|\int_s^t\int_D (t-r)^{\beta-1}\bar{p}_d^n(t-r,y,z)Y(r,z)dzdr\right|^p\right) \\
&\equiv A_1+A_2+A_3.
\end{aligned}$$

By Hölder's inequality, (4.4) and $i)$ of the Lemma 6.2, we get

$$A_3 \le c|t-s|^{(\beta-\lambda d)p}, \qquad (4.6)$$

where $0<\beta<\frac{1}{2}$ and $0<\lambda<\frac{\beta}{d}$. Concerning A_2, the same arguments as above yield

$$\begin{aligned}
A_2 &\le c\left(\int_0^s [(s-r)^{\beta-1}-(t-r)^{\beta-1}](t-r)^{-\lambda d}dr\right)^p \\
&\le c|t-s|^{-2\lambda dp}\left(\int_0^s [(s-r)^{\beta-1}-(t-r)^{\beta-1}]dr\right)^p \\
&\le c|t-s|^{p(\beta-\lambda d)}. \qquad (4.7)
\end{aligned}$$

For A_1, we use firstly Burkholder's inequality

$$\begin{aligned}
A_1 &= \mathbb{E}\left|\int_0^s\int_D (\bar{p}_d^n(t-r,y,z)-\bar{p}_d^n(s-r,y,z))\sigma(u^n(r,z))F(dr,dz)\right|^p \\
&\le c\mathbb{E}\left(\int_0^s\int_D\int_D (\bar{p}_d^n(t-r,y,z)-\bar{p}_d^n(s-r,y,z))\sigma(u^n(r,z))f(z,x)\right. \\
&\qquad \left.\times(\bar{p}_d^n(t-r,y,x)-\bar{p}_d^n(s-r,y,x))\sigma(u^n(r,x))drdzdx\right)^{p/2},
\end{aligned}$$

and since σ and f are bounded, and by $iv)$ of the Lemma 6.2

$$\begin{aligned} A_1 &\leq c\left(\int_0^s\int_D\int_D(\bar{p}_d^n(t-r,y,z)-\bar{p}_d^n(s-r,y,z))|z-x|^{-\alpha}\right. \qquad (4.8)\\ &\quad \times(\bar{p}_d^n(t-r,y,x)-\bar{p}_d^n(s-r,y,x)))^{p/2} \\ &\leq c\left(\int_0^s \|\bar{p}_d^n(t-r,y,\cdot)-\bar{p}_d^n(s-r,y,\cdot)\|_\alpha^2 dr\right)^{p/2} \\ &\leq c|t-s|^{(1-\frac{\alpha}{2})\frac{p}{2}}. \qquad (4.9)\end{aligned}$$

Therefore, taking into account (4.5)-(4.8), there exists a positive constant c and $0<\delta_1<\frac{1}{2}$ and $0<\delta_2<\frac{1}{4}$ such that

$$\mathbb{E}|u_3^n(t,y)-u_3^n(s,y)|^{2p}\leq c\left(|t-s|^{\delta_1 p}+\|x-y\|^{\delta_2 p}\right). \qquad (4.10)$$

From the Proposition 3.2 in [24], we deduce that (4.10) holds for u_1^n. Concerning u_2^n, using Hölder's inequality, (4.2) and the estimates on $\bar{p}_d^n$ given by the Lemma 6.2, we deduce that (4.10) holds for u_2^n. Finally, we deduce the tightness of the sequence $\{u^n, n\geq 1\}$ by the Totoki-kolmogorov criterion, see Theorem 4.10 and Problem 4.11 in [19]. □

5. The weak solution

Since the sequence $\{u^n,\ n\geq 1\}$, is tight in $\mathcal{C}([0,T],\mathcal{H})$, it is relatively compact in this space. Thus, there exists a subsequence, that we still denote by u^n, which converges weakly in $\mathcal{C}([0,T],\mathcal{H})$ to a process u. By Skorohod's Representation Theorem, there exists a probability space $\tilde{\Omega}$, and on it a sequence $\tilde{u}^n$, as well as a noise $\tilde{F}$ and a process $\tilde{u}$ such that $(u^n,u,F)\overset{\mathcal{D}}{\equiv}(\tilde{u}^n,\tilde{u},\tilde{F})$ and $\tilde{u}^n$ converges almost surely to $\tilde{u}$ in $\mathcal{C}([0,T],\mathcal{H})$. We will show, by solving the corresponding martingale problem, that $\tilde{u}$ is a weak solution of the equation $Eq(d,u_0,\sigma)$.

Proposition 5.1. *For any $\varphi\in\mathcal{C}^2(D)$, such that $\varphi\equiv 0$ on ∂D, we have*

$$\begin{aligned}\mathcal{M}_\varphi(t): &= \int_D\tilde{u}(t,x)\varphi(x)dx-\int_D u_0(x)\varphi(x)dx-\int_0^t\int_D\tilde{u}(s,x)\Delta\varphi(x)dxds \\ &\quad +\frac{1}{2}\sum_{i=1}^d\int_0^t\int_D\tilde{u}^2(s,x)\frac{\partial\varphi}{\partial x_i}(x)dxds \qquad (5.1)\end{aligned}$$

is a martingale with the quadratic variation

$$\langle\mathcal{M}_\varphi\rangle_t=\int_0^t\int_D\int_D\sigma(\tilde{u}(s,x))\varphi(x)f(x,y)\sigma(\tilde{u}(s,y))\varphi(y)dxdyds. \qquad (5.2)$$

Proof. We consider the scheme (3.1) and by multiplying both its sides by $\frac{1}{n^d}\varphi(x^n_{\mathbb{k}})$ and summing over all the elements of the grid D^d_n, we obtain

$$\begin{aligned}\mathcal{M}^n_\varphi(t): &= \frac{1}{n^d}\sum_{x^n_{\mathbb{k}}} u^n(t,x^n_{\mathbb{k}})\varphi(x^n_{\mathbb{k}}) - \frac{1}{n^d}\sum_{x^n_{\mathbb{k}}} u(0,x^n_{\mathbb{k}})\varphi(x^n_{\mathbb{k}}) - \int_0^t \frac{1}{n^d}\sum_{x^n_{\mathbb{k}}}\Delta^n u^n(s,x^n_{\mathbb{k}})\varphi(x^n_{\mathbb{k}})ds \\ &\quad -\frac{1}{2}\int_0^t \frac{1}{n^d}\sum_{x^n_{\mathbb{k}}}\sum_{i=1}^d \nabla^n_i (u^n(s,x^n_{\mathbb{k}}))^2\varphi(x^n_{\mathbb{k}})ds \\ &= \frac{1}{n^d}\sum_{x^n_{\mathbb{k}}} \tilde{u}^n(t,x^n_{\mathbb{k}})\varphi(x^n_{\mathbb{k}}) - \frac{1}{n^d}\sum_{x^n_{\mathbb{k}}} u(0,x^n_{\mathbb{k}})\varphi(x^n_{\mathbb{k}}) - \int_0^t \frac{1}{n^d}\sum_{x^n_{\mathbb{k}}}\tilde{u}^n(s,x^n_{\mathbb{k}})\Delta^n\varphi(x^n_{\mathbb{k}})ds \\ &\quad +\frac{1}{2}\int_0^t \frac{1}{n^d}\sum_{x^n_{\mathbb{k}}}\sum_{i=1}^d (\tilde{u}^n(s,x^n_{\mathbb{k}}))^2\nabla^n_i\varphi(x^n_{\mathbb{k}})ds \\ &= \int_0^t \sum_{x^n_{\mathbb{k}}}\varphi(x^n_{\mathbb{k}})\sigma(\tilde{u}^n(s,x^n_{\mathbb{k}}))F(ds,\mathbb{I}^n_{\mathbb{k}}).\end{aligned}$$

Note that $\{\mathcal{M}^n_\varphi(t), t\geq 0\}$ is a martingale as it is a finite sum of martingales. Moreover, by (2.1) and the boundedness of σ, we can write

$$\begin{aligned}\mathbb{E}(\mathcal{M}^n_\varphi(t))^2 &= \sum_{x^n_{\mathbb{k}},x^n_{\mathbb{k}'}}\int_0^t ds\int_{\mathbb{I}^n_{\mathbb{k}}}dx\int_{\mathbb{I}^n_{\mathbb{k}'}}dy\varphi(x)\sigma(\tilde{u}(s,x))\varphi(y)\sigma(\tilde{u}(s,y))f(x,y) \\ &\leq t\sum_{x^n_{\mathbb{k}},x^n_{\mathbb{k}'}}\int_{\mathbb{I}^n_{\mathbb{k}}}dx\int_{\mathbb{I}^n_{\mathbb{k}'}}dy\varphi(x)f(x,y)\varphi(y) \\ &\leq t\int_D dx\int_D dy\varphi(x)f(x,y)\varphi(y).\end{aligned}$$

Hence, there exists a constant c depending only on φ, k and T such that

$$\sup_{0\leq t\leq T}\sup_n \mathbb{E}(\mathcal{M}^n_\varphi(t))^2 \leq c < \infty.$$

Consequently, $\{\mathcal{M}^n_\varphi, n\geq 1\}$ converges a.s., as $n\longrightarrow\infty$, to the martingale $\mathcal{M}_\varphi$ given by (5.1). Also, we have

$$\langle\mathcal{M}_\varphi\rangle_t = \lim_{n\longrightarrow\infty}\langle\mathcal{M}^n_\varphi\rangle_t,$$

where

$$\begin{aligned}\langle\mathcal{M}^n_\varphi\rangle_t &= \left\langle\sum_{x^n_{\mathbb{k}}}\int_0^{\cdot}\int_{\mathbb{I}^n_{\mathbb{k}}}\varphi(x)\sigma(\tilde{u}^n(s,x))F(ds,dx)\right\rangle_t \\ &= \int_0^t\sum_{x^n_{\mathbb{k}},x^n_{\mathbb{k}'}}\int_{\mathbb{I}^n_{\mathbb{k}}}\int_{\mathbb{I}^n_{\mathbb{k}'}}\varphi(\kappa_n(x))\sigma(\tilde{u}^n(s,\kappa_n(x)))f(x,y)\varphi(\kappa_n(y))\sigma(\tilde{u}^n(s,\kappa_n(y)))dsdxdy,\end{aligned}$$

which converges a.s. to (5.2) as $n\longrightarrow\infty$. □

Proof of the Theorem 2.1 Following Walsh [30], there exists a martingale measure M with quadratic variation

$$\nu(dxdyds) = \sigma(\tilde{u}(s,x))f(x,y)\sigma(\tilde{u}(s,y))dxdyds,$$

which corresponds to the quadratic variation $\langle \mathcal{M}_\varphi \rangle_t$. Furthermore, we consider a measure martingale N independent of M and characterized by (2.1), and we set

$$F(t,\varphi) = \int_0^t \int_D \frac{1}{\sigma(\tilde{u}(s,x))} \mathbf{1}_{\{\tilde{u}(s,x)\notin\{0,1\}\}} \varphi(x) M(dxds) + \int_0^t \int_D \mathbf{1}_{\{\tilde{u}(s,x)\in\{0,1\}\}} \varphi(x) N(dxds).$$

Therefore, $F(t,\varphi)$ satisfies

$$\langle F(\cdot,\varphi)\rangle_t = \int_0^t \int_D \int_D \varphi(x) f(x,y) \varphi(y) dxdyds.$$

Hence, because of the form of its quadratic variation, it has the same distribution as N, and by Proposition 2.5.7 in [29], we can write

$$\mathcal{M}_\varphi(t) = \int_0^t \int_D \sigma(\tilde{u}(s,x)) \varphi(x) F(dx,ds),$$

therefore, $\tilde{u}$ is a weak solution of the equation $Eq(d,u_0,\sigma)$. □

6. Appendix

Firstly, we recall the Lemma 1.0 in [22],

Lemma 6.1. *Let $Z \equiv \{Z(t), t \geq 0\}$ be a real-valued semi-martingale. Suppose that there exists a function $\rho : [0,+\infty) \longrightarrow [0,+\infty)$ such that*

$$\int_0^\varepsilon \frac{dx}{\rho(x)} = +\infty,$$

for all $\varepsilon > 0$, and

$$\int_0^t \frac{\mathbf{1}_{\{Z_s>0\}}}{\rho(Z_s)} d\langle Z\rangle_s < +\infty,$$

for all $t > 0$ a.s. Then, the local time of Z at zero, $L_t^0(Z)$ is identically zero for all $t > 0$ a.s.

The following lemma gives some useful estimates satisfied by $\bar{p}_d^n$.

Lemma 6.2. *i) There exists a constant $c > 0$ such that for any $t > 0$ and $\lambda > 0$,*

$$\sup_{n\geq 1} \sup_{x\in D} \|\bar{p}_1^n(t,x,\cdot)\|_1 \leq ct^{-\lambda} \exp(-ct)$$

ii) There exist constants $c > 0$, $1 < q < 2$ such that for any $t > 0$ and $x, y \in D$,

$$\sup_{n\geq 1} \int_0^t \left(\int_D |\bar{p}_d^n(t-s,x,z) - \bar{p}_d^n(t-s,y,z)|\, dz \right)^2 ds \leq c\|x-y\|^{\frac{1}{q}-\frac{1}{2}};$$

this estimate holds with $q = 1$ when $d = 1$.
iii) For any $T > 0$, there exists a constant $c > 0$ such that for $h > 0$

$$\sup_{n\geq 1} \sup_{x\in D} \sup_{t\in[0,T]} \int_0^t \|\bar{p}_d^n(t-s,x,\cdot) - \bar{p}_d^n(t+h-s,x,\cdot)\|_1 ds \leq ch^{\frac{1}{2}}$$

iv) For any $T > 0$, there exists a constant $c > 0$ such that for $h > 0$

$$\sup_{n\geq 1} \sup_{x\in D} \sup_{t\in[0,T]} \int_0^t \|\bar{p}_d^n(t-s,x,\cdot) - \bar{p}_d^n(t-s+h,x,\cdot)\|_\alpha^2 ds \leq ch^{1-\frac{\alpha}{2}}$$

where $0 < \alpha < 2 \wedge d$, *and* $\|\cdot\|_\alpha$ *is defined on an appropriate class of functions* φ *by*

$$\|\varphi\|_{(\alpha)}^2 = \int_D \int_D |\varphi(x)||x-y|^{-\alpha}|\varphi(y)|dxdy$$

Proof. The estimates $i)$, $iii)$ and $iv)$ were proved in [24], that are respectively, (A.16) of the Lemma A.4, (A.36) and (A.38) of the Lemma A.6. Concerning $ii)$, using the fact that $\bar{p}_d^n(t,x,z) = \prod_{i=1}^d \bar{p}_1^n(t-s,x_i,z_i)$, where $x = (x_i, 1 \leq i \leq d)$ and $z = (z_i, 1 \leq i \leq d)$, the inequality $\left|\prod_{i=1}^d a_i - \prod_{i=1}^d b_i\right| \leq \sum_{i=1}^d |a_i - b_i| \prod_{j=i+1}^d |a_j| \prod_{j=1}^{i-1} |b_j|$ for all real numbers $a_1, ..., a_d$, $b_1..., b_d$, the estimate $i)$ and the Cauchy-Schwarz inequality, we can write

$$\begin{aligned}
&\int_0^t \left(\int_D |\bar{p}_d^n(t-s,x,z) - \bar{p}_d^n(t-s,y,z)|\,dz\right)^2 ds \\
&= \int_0^t \left(\int_D \left|\prod_{j=1}^d \bar{p}_1^n(t-s,x_j,z_j) - \prod_{j=1}^d \bar{p}_1^n(t-s,y_j,z_j)\right| dz\right)^2 ds \\
&\leq \int_0^t \Bigg(\sum_{i=1}^d \prod_{j=1}^{i-1} \|\bar{p}_1^n(t-s,x_j,\cdot)\|_1 \times \int_0^1 |\bar{p}_1^n(t-s,x_i,z_i) - \bar{p}_1^n(t-s,y_i,z_i)|dz_i \\
&\qquad \times \prod_{j=i+1}^d \|\bar{p}_1^n(t-s,y_j,\cdot)\|_1\Bigg)^2 ds \\
&\leq c\int_0^t \left(\sum_{i=1}^d (t-s)^{-\lambda(d-1)} \int_0^1 |\bar{p}_1^n(t-s,x_i,z_i) - \bar{p}_1^n(t-s,y_i,z_i)|dz_i\right)^2 ds \\
&\leq c\sum_{i=1}^d \int_0^t (t-s)^{-2\lambda(d-1)} \int_0^1 |\bar{p}_1^n(t-s,x_i,z_i) - \bar{p}_1^n(t-s,y_i,z_i)|^2 dz_i ds.
\end{aligned}$$

On the other hand, using notations of Gyöngy [15] and Hölder's inequality, we get

$$\begin{aligned}
&\int_0^t (t-s)^{-2\lambda(d-1)} \int_0^1 |\bar{p}_1^n(t-s,x_i,z_i) - \bar{p}_1^n(t-s,y_i,z_i)|^2 dz_i ds \\
&\leq \sum_{j=1}^{n-1} \int_0^t (t-s)^{-2\lambda(d-1)} \exp(-8j^2(t-s))ds|\varphi_j^n(x_i) - \varphi_j^n(y_i)|^2 \\
&\leq c\sum_{j=1}^{n-1} \left(\int_0^t \exp(-8q(t-s)j^2)\right)^{1/q} (j^2|x_i - yi|^2 \wedge 1) \\
&\leq c\sum_{j=1}^{n-1} \frac{1}{j^{2/q}}(j^2|x_i - yi|^2 \wedge 1) \\
&\leq c|x_i - y_i|^{\frac{1}{q}-\frac{1}{2}}.
\end{aligned}$$

Note that we choose λ and q such that $1 < q < 2$ and $0 < 2\lambda(d-1)\xi < 1$, where $\frac{1}{q} + \frac{1}{\xi} = 1$. $\square$

We have also used the following result in Section 3, for the proof see Theorem 3.10 of Chapter 5 in [12].

Theorem 6.1. *Consider the SDE*

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dM_s, \tag{6.1}$$

where $t \geq 0$, $b : \mathbb{R}^+ \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ *and* $\sigma : \mathbb{R}^+ \times \mathbb{R}^d \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$, *are Borel continuous functions,* M *is a* d*-dimensional martingale in* $\mathcal{M}_c^2$ *and* X *is a* d*-dimensional process. Assume further that the* $\mathbb{R}^d$*–valued random variable* X_0 *is* $\mathcal{F}_0$*–measurable such that* $\mathbb{P}X_0^{-1} = \mu$*. If there exists a constant* c *such that*

$$\|b(t,x)\| \leq c(1 + \|x\|) \qquad \text{and} \qquad \left(\sum_{i,j=1}^d \sigma_{i,j}^2(t,x)\right)^{\frac{1}{2}} \leq c(1 + \|x\|),$$

for all $t \geq 0$ *and* $x \in \mathbb{R}^d$*, then for all initial condition probability measure* μ *on* $\mathbb{R}^d$*, there exists a weak solution of the SDE (6.1).*

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