An asymptotic study to explain the role of active transport in models with countercurrent exchangers

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AN ASYMPTOTIC STUDY TO EXPLAIN THE ROLE OF ACTIVE TRANSPORT IN MODELS WITH COUNTERCURRENT EXCHANGERS

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Abstract

We study a solute concentrating mechanism that can be represented by coupled transport equations with specific boundary conditions. Our motivation for considering this system is urine concentrating mechanism in nephrons. The model consists in 3 tubes arranged in a countercurrent manner. Our equations describe a countercurrent exchanger, with a parameter V which quantifies the active transport. In order to understand the role of active transport in the mechanism, we consider the limit $V \longrightarrow \infty$. We prove that when V goes to infinity, the system converges to a profile which stays uniformly bounded in V and which presents a boundary layer at the border of the domain. The effect is that the solute is concentrated at a specific point in the tubes. When considering urine concentration, this is physilogically optimal because the composition of final urine is determined at this point.

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1 Introduction

Many problems occurring in biology or physiology [14, 9] can be described by transport equations with different propagation velocities that combine together to produce specific effects. For instance, typical cases are propagation of particules waves in neurones [5], or diphasic propagation arising in chemical engineering, which can describe chromatography or distillation [7]. Usually, the stationary states are of interest and their study constitute a classical field of analysis (see for instance [13] and the references therein).

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Here, we study a model of a countercurrent exchanger combined with an active transport pump [6]. Countercurrent exchanges accross parallel tubes can be used for building up concentration or heat gradient. Our equations come from the modelisation of kidney nephrons, in which a concentration gradient is amplified by an active transport pump [16], which plays a fundamental role for urine concentration [11]. In this particular study, we investigate the effects of active transport using a limiting case.

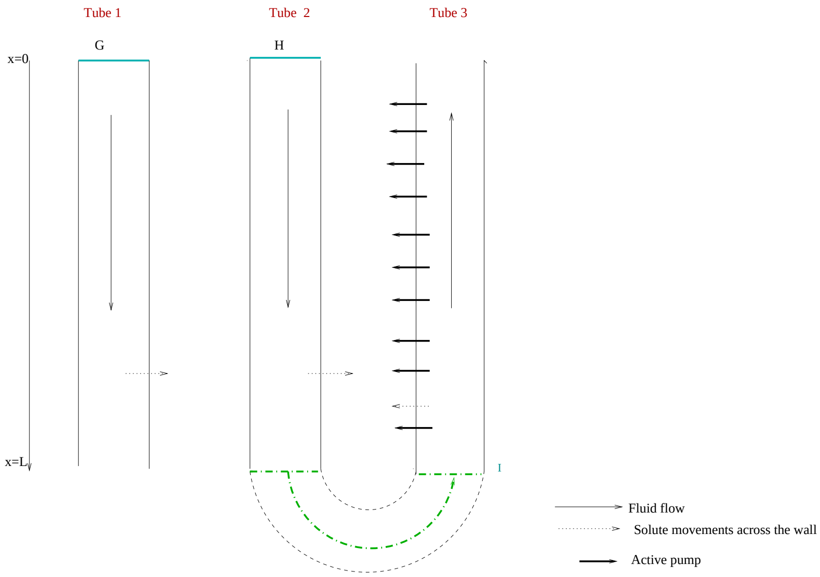

Figure 1: Representation of the 3-tube architecture in which the fluid circulates. Tubes are water-impermeable but can exchange solutes with the bath. C_0^1, C_0^2 and $C^3(L) = C^2(L)$

The model consists in a fluid circulating at a constant velocity in 3 tubes arranged in a countercurrent architecture. The 3 tubes are bathing in a common bath in which no solute can accumulate. Each tube can exchange solute with the bath and solute transport accross tubes wall is driven by diffusion in all tubes and by an active pump in tube 3. This active pump extracts solute from tube 3 and carries it into the bath and is assumed to follow Michaelis-Menten kinetics. We call V the maximum rate achieved by the pump at saturating concentrations. We call $C^i(x)$ the solute concentration in tube i at depth x. The nonlinearity $V\dfrac{C_V^3}{1+C_V^3}$ represents the effect of active pumps along tube 3. The fluid enters tube 1 with a concentration value C_0^1 and tube 2 with a concentration value C_0^2. The outlet of tubes 1 and 3 are open at $x = L$ and we have $C^2(L) = C^3(L)$. See Figure 1 for a drawing of the system. The stationary state is of particular interest in renal physiology, given that the kidney acts to preserve homeostasis.

The differential system satisfied by C^1, C^2, C^3 is written as

$$
\begin{cases}
\dfrac{dC^1(x)}{dx} = \dfrac{1}{3}\Big[C^1(x) + C^2(x) + C^3(x) + V\dfrac{C^3(x)}{1+C^3(x)}\Big] - C^1(x), \\
\dfrac{dC^2(x)}{dx} = \dfrac{1}{3}\Big[C^1(x) + C^2(x) + C^3(x) + V\dfrac{C^3(x)}{1+C^3(x)}\Big] - C^2(x), \\
-\dfrac{dC^3(x)}{dx} = \dfrac{1}{3}\Big[C^1(x) + C^2(x) + C^3(x) + V\dfrac{C^3(x)}{1+C^3(x)}\Big] - C^3(x) \\
\qquad\qquad - V\dfrac{C^3(x)}{1+C^3(x)}, \\
C^1(0) = C^1_0, \qquad C^2(0) = C^2_0, \qquad C^3(L) = C^2(L).
\end{cases}
\tag{1}
$$

The specific boundary conditions relate the solutions at different points and make this system not a mere ordinary differential equation. We call $C_V = (C^1_V, C^2_V, C^3_V)$ the solution of (1). We wish to explain through the analysis of the system for large values of V, why this combination of active pump and boundary conditions (tube arrangement) is performing well the task of concentrating the solute at $x = L$, where the composition of final urine is determined. We already know [17] that each C^i_V is continuous and nonnegative on $[0, L]$. The question we want to answer is : How do the solutions of (1) behave when V tends to ∞?

Other asymptotic studies have been done for similar systems in the context of hyperbolic relaxation where a parameter is assumed to be small in comparison to the typical size of the problem [12, 8, 7]. This approach comes from the concept of mean free path in Boltzmann equation [3]. For example, in [5], the length of the domain is large, and they establish the asymtotic behavior of the solution in the limiting case of an infinite domain.

In our case, for answering our question, we prove that C_V converges toward a limit $C = (C^1, C^2, C^3)$ that we calculate. Our analysis uses only direct a priori estimates and weak limits obtained by compact injections which do not use the specific smooth form of the non-linearity and makes it very general. We identify completely the limit as $V \longrightarrow \infty$ including boundary layers. Compact injections give us the convergence of some particular subsequences, but as we point out that the limit only depends on the problem data, we are able to prove that the whole sequence converges. The boundary layers are coming from the particular boundary conditions in the model, which can be seen as reflection conditions and make the problem specific and interesting.

We state our main results in next section. Section 3 and 4 are devoted to the proofs of the asymptotic results. Numerical illustrations are given in section 5.

2 Main results

It is possible to identify completely the profiles of the limiting values for the solutions C^1, C^2, C^3 almost everywhere as $V \longrightarrow \infty$. This is stated in the

Theorem 1 (Asymptotics) *Solutions to* (1) *satisfy*

$$C_V^1 \underset{V\longrightarrow+\infty}{\longrightarrow} C^1, \quad C_V^2 \underset{V\longrightarrow+\infty}{\longrightarrow} C^2, \quad C_V^3 \underset{V\longrightarrow+\infty}{\longrightarrow} C^3, \quad L^p(1\le p<\infty),\ a.e., \tag{2}$$

with

$$\begin{aligned} &C^1(x) = \frac{C_0^1+C_0^2}{2} + \frac{C_0^1-C_0^2}{2}e^{-x}, \quad C^2(x) = \frac{C_0^1+C_0^2}{2} + \frac{C_0^2-C_0^1}{2}e^{-x}, \\ &C^3(x) = 0 \quad a.e. \end{aligned} \tag{3}$$

This result is somewhat sharp since we will see that a boundary layer occurs and thus the convergence does not hold in L^∞. To state our next result, we need to define the quantity

$$M = \operatorname{ess\,inf}\{\frac{1}{C_V^3(x)}; x\in[0,L], V\in\mathbb{R}^+\}. \tag{4}$$

We prove in the next section that $M>0$. The second result is more accurate and states that C^3 decreases exponentially fast to zero. We describe also the boundary layer that appears at $x=L$.

Theorem 2 (The boundary layer) *The limits of the boundary values are*

$$\begin{aligned} C_V^1(L) &\underset{V\longrightarrow+\infty}{\longrightarrow} C_0^1 + C_0^2, \\ C_V^2(L) = C_V^3(L) &\underset{V\longrightarrow+\infty}{\longrightarrow} C_0^1 + C_0^2 + (C_0^2 - C_0^1)e^{-L}, \end{aligned} \tag{5}$$

The behavior of C_V^3 for $x\simeq L$ is given by the inequalities

$$C_V^3(x) \le C_V^3(L)\exp\Big(-\frac{2}{3}VM(L-x)\Big) + \frac{K}{V}\Big[1-\exp\Big(-\frac{2}{3}VM(L-x)\Big)\Big], \tag{6}$$

$$C_V^3(x) \ge C_V^3(L)\exp\Big(-\frac{2}{3}V(L-x)\Big) + \frac{\overline{K}}{V}\Big[1-\exp\Big(-\frac{2}{3}V(L-x)\Big)\Big], \tag{7}$$

where K and $\overline{K}$ are two constants which do not depend on V.

The next section is dedicated to the proof of these results.

3 Proof of the asymptotic results(Theorem 1)

First step: Uniform bounds on the solution.

Lemma 3 *There is a constant K depending only on C_0^1, C_0^2 but not on V such that*

$$C_V^1(L) \le K, \qquad C_V^3(0) \le C_0^1 + C_0^2, \tag{8}$$

$$\int_0^L C_V^i(x)dx \le K; V\int_0^L C_V^3(x)dx \le K; \int_0^L |\frac{dC_V^3}{dx}(x)|dx \le K; 0 \le C_V^i \le K. \tag{9}$$

Proof. To prove (8), we sum the three lines of (1), and we obtain a quantity which does not depend on x,

$$C_V^1(x) + C_V^2(x) - C_V^3(x) =: K(V). \tag{10}$$

Using the boundary values, we find uniform bounds on $K(V)$

$$\begin{aligned} K(V) &= C_0^1 + C_0^2 - C_V^3(0) \leq C_0^1 + C_0^2, \\ K(V) &= C_V^1(L) + C_V^2(L) - C_V^3(L) = C_V^1(L) \geq 0, \end{aligned} \tag{11}$$

and thus

$$0 \leq K(V) \leq C_0^1 + C_0^2. \tag{12}$$

The combination of (11) and (12) proves (8).

Then, we prove the first two bounds in (9). The first equation can be written

$$\frac{dC_V^1(x)}{dx} + C_V^1(x) = Q_V(x) \geq 0, \tag{13}$$

with

$$Q_V(x) = \frac{1}{3}\Big[C^1(x) + C^2(x) + C^3(x) + V\frac{C^3(x)}{1+C^3(x)}\Big].$$

Therefore we also have

$$\frac{d}{dx}\Big(C_V^1(x)e^x\Big) = Q_V(x)e^x.$$

By integration over $[0, L]$, we obtain

$$\int_0^L Q_V(x)dx \leq \int_0^L Q_V(x)e^x dx = C_V^1(L)e^L - C_0^1 \leq (C_0^1 + C_0^2)e^L. \tag{14}$$

We conclude that

$$\int_0^L V\frac{C_V^3(x)}{1+C_V^3(x)}dx,\ \int_0^L C_V^i(x)dx,\ i = 1, 2, 3, \tag{15}$$

are uniformly bounded by $(C_0^1 + C_0^2)e^L$. Then, by injecting equation (14) in (13) and because the C_V^i are positive, we have

$$\int_0^L |\frac{dC_V^i}{dx}(s)|ds \leq (C_0^1 + C_0^2)e^L \qquad i = 1, 2, 3. \tag{16}$$

We finally prove that the functions $\Big(C_V^i(x)\Big)_V$ are uniformly bounded in V. We write

$$|C_V^i(x)| = |C_V^i(0) + \int_0^x \frac{dC_V^i}{dx}(s)ds| \leq |C_V^i(0)| + \int_0^L |\frac{dC_V^i}{dx}(s)|ds.$$

Thanks to (16) and (8), we conclude

$$\|C_V^i\|_\infty \leq (C_0^1 + C_0^2)(e^L + 1). \qquad (17)$$

□

The upper bound (17) on C_V^3 gives us that $M > 0$.

Second step: The behaviour of C_V^i when $V \longrightarrow \infty$

Lemma 4 *After extraction of a subsequence,*

$$C_V^1 \underset{V\longrightarrow+\infty}{\longrightarrow} C^1, \quad C_V^2 \underset{V\longrightarrow+\infty}{\longrightarrow} C^2, \quad C_V^3 \underset{V\longrightarrow+\infty}{\longrightarrow} 0, \quad L^p(1 \leq p < \infty),\ a.e., \qquad (18)$$

and $C^1 + C^2 = K^0$ for some constant K^0.

Proof. We know from Lemma 3 that $\left(C_V^i\right)_V$ is bounded in BV, then, using the Rellich-Kondrachov compact injection [4]

$$C_V^i \underset{V\longrightarrow+\infty}{\longrightarrow} C^i, \qquad \text{in } L^p(1 \leq p < \infty) \text{ and } a.e. \qquad (19)$$

On the other hand, we have thanks to (15),

$$\int_0^L \frac{C_V^3(x)}{1 + C_V^3(x)} dx \underset{V\longrightarrow+\infty}{\longrightarrow} 0,$$

and thus

$$C^3 \equiv 0 \qquad a.e. \qquad (20)$$

Combining (10) with (20), we have $C^1 + C^2 = K^0$ for some constant K^0. □

Third step : The behavior of $\frac{dC_V^3}{dx}$. We define $\mathcal{M}^1[0, L]$ the set of Radon measures on $[0, L]$, taken with the weak convergence of measures.

Lemma 5 *There exists a nonnegative constant B such that, after extraction,*

$$C_V^3(L) \underset{V\longrightarrow+\infty}{\longrightarrow} B, \qquad \frac{dC_V^3}{dx} \underset{V\longrightarrow+\infty}{\longrightarrow} B\delta_{x=L} \qquad \textit{in } \mathcal{M}^1[0, L],$$

Proof. The information (16) implies that $\left(\frac{dC_V^3}{dx}\right)_V$ is bounded in $L^1[0, L]$, then [2] there exists $\mu \in \mathcal{M}^1[0, L]$ a Radon measure so that, after extraction,

$$\frac{dC_V^3}{dx} \underset{V\longrightarrow+\infty}{\longrightarrow} \mu \qquad \text{in the sense of measures.} \qquad (21)$$

For all functions $\phi \in C^1[0, L]$ such as $\phi(0) = \phi(L) = 0$, we have using (18)

$$\int_0^L \phi(x) \frac{dC_V^3}{dx}(x) dx = \int_0^L C_V^3(x) \frac{d\phi}{dx}(x) dx \underset{V\longrightarrow+\infty}{\longrightarrow} 0, \qquad (22)$$

which means,

$$\mu = 0 \qquad \text{on }]0, L[. \tag{23}$$

Therefore, we can write in the sense of measures

$$\frac{dC_V^3}{dx} \underset{V \longrightarrow +\infty}{\longrightarrow} \beta\delta_{x=L} + \alpha\delta_{x=0}. \tag{24}$$

It remains to compute α and β. To do so, we notice that $\Big(C_V^3(L)\Big)_V$ and $\Big(C_V^3(0)\Big)_V$ are both real value bounded sequences, so, there are two nonnegative real numbers A, B such that, after extraction,

$$\lim_{V\to\infty} C_V^3(L) = B \geq 0, \qquad \lim_{V\to\infty} C_V^3(0) = A \geq 0. \tag{25}$$

For $\phi \in C^1\Big([0, L]\Big)$, we compute

$$\int_0^L \phi(x)\frac{dC_V^3}{dx}(x)dx = C_V^3(L)\phi(L) - C_V^3(0)\phi(0) - \int_0^L C_V^3(x)\frac{d\phi}{dx}(x)dx \underset{V \longrightarrow +\infty}{\longrightarrow} B\phi(L) - A\phi(0),$$

which means

$$\frac{dC_V^3}{dx} \underset{V \longrightarrow +\infty}{\longrightarrow} B\delta_{x=L} - A\delta_{x=0} \qquad \text{in the sense of measures.} \tag{26}$$

We still have to prove $A = 0$. To do so we use the system of equations (1) which gives us

$$\frac{dC_V^3}{dx} \geq -\frac{1}{3}(C_V^1 + C_V^2).$$

As we know from Lemma 3 that C_V^i is uniformly bounded from above by K, we also have,

$$\frac{dC_V^3}{dx} \geq -\frac{2}{3}K,$$

which implies $A = 0$. □

Fourth step: The limiting equation.

Lemma 6 *In the limit $V \longrightarrow \infty$, we have*

$$C^1(x) + C^2(x) = C_0^1 + C_0^2, \qquad V\frac{C_V^3}{1 + C_V^3} - \frac{C_V^1 + C_V^2}{2} \longrightarrow \frac{3}{2}B\delta_L$$

in the sense of measures.

Proof. We deduce from (19) and (26), by injecting in the third line of (1) that

$$V\frac{C_V^3}{1+C_V^3} - \frac{C_V^1 + C_V^2}{2} - \frac{3}{2}B\delta_L \longrightarrow 0 \qquad \text{in the sense of measures.} \qquad (27)$$

Reinjecting in the first lines of (1), we find the limit equations on C^1 et C^2

$$\begin{cases} \dfrac{dC^1}{dx} = -\dfrac{1}{2}C^1 + \dfrac{1}{2}C^2 + \dfrac{1}{2}B\delta_L, \\ \dfrac{dC^2}{dx} = -\dfrac{1}{2}C^2 + \dfrac{1}{2}C^1 + \dfrac{1}{2}B\delta_L, \\ C^1(0) = C_0^1, \qquad C^2(0) = C_0^2. \end{cases} \qquad (28)$$

Then, summing the two lines,

$$\begin{cases} \dfrac{d(C^2 + C^1)}{dx} = B\delta_L, \\ (C^1 + C^2)(0) = C_0^1 + C_0^2. \end{cases} \qquad (29)$$

By integrating this differential equation, we deduce [15] that

$$C^1(x) + C^2(x) = C_0^1 + C_0^2 \qquad \text{a.e.}$$

Indeed, the weak formulation of (29) is

$$\forall \phi \in \mathcal{C}^1[0,L], \qquad \int_0^L \frac{d\phi}{dx}(x)[C^1 + C^2](x)dx + \phi(0)[C_0^1 + C_0^2] = 0. \qquad (30)$$

By choosing ϕ such as $\phi(0) = 0$, we obtain $C^1 + C^2 \equiv \alpha$ a.e., for some constant α. and then, by choosing any $\phi \in \mathcal{C}^1[0,L]$, we have that $\alpha = C_0^1 + C_0^2$. □

The limit equation on C^i then becomes

$$\begin{cases} \dfrac{dC^i}{dx}(x) = -C^i(x) + \dfrac{C_0^1 + C_0^2}{2} + \dfrac{1}{2}B\delta_L(x), \qquad i = 1,2, \\ C^i(0) = C_0^i. \end{cases}$$

Fifth step: Explicit solution for the limit Using the variation of parameters, we compute easily C^1 and C^2. We find

$$C^1(x) = \frac{C_0^1 + C_0^2}{2} + \frac{C_0^1 - C_0^2}{2}e^{-x}, \qquad C^2(x) = \frac{C_0^1 + C_0^2}{2} + \frac{C_0^2 - C_0^1}{2}e^{-x}. \quad (31)$$

In particular, (C^1, C^2, C^3) are $\mathcal{C}^\infty$ functions.

4 Proof of theorem 2

The limiting profiles are $\mathcal{C}^\infty$ in $[0, L]$, nevertheless, the Dirac mass at $x = L$ indicates a boundary layer. The derivatives of the profiles for $V = \infty$ are given by

$$\begin{cases} \dfrac{dC^1}{dx} = \dfrac{1}{2}\Big[(C_0^2 - C_0^1)e^{-x} + B\delta_L\Big], \\ \dfrac{dC^2}{dx} = \dfrac{1}{2}\Big[(C_0^1 - C_0^2)e^{-x} + B\delta_L\Big], \\ \dfrac{dC^3}{dx} = B\delta_L. \end{cases} \qquad \text{in the sense of measures,} \tag{32}$$

First Step: The limiting values of C at $x = L$

Lemma 7

$$\begin{aligned} C_V^1(L) &\underset{V \longrightarrow +\infty}{\longrightarrow} C_0^1 + C_0^2, \\ C_V^2(L) = C_V^3(L) &\underset{V \longrightarrow +\infty}{\longrightarrow} C_0^1 + C_0^2 + (C_0^2 - C_0^1)e^{-L}. \end{aligned} \tag{33}$$

Proof. We already have defined in Lemma 5

$$B = \lim_{V\to\infty} C_V^2(L) = \lim_{V\to\infty} C_V^3(L).$$

We know that the $C_V^i(L)$ are bounded real numbers, then we define

$$B' = \lim_{V\to\infty} C_V^1(L).$$

Our first task is to determine B. We compute for all $\phi \in \mathcal{C}^1[0, L]$,

$$\int_0^L \phi(x)\frac{dC_V^2(x)}{dx}(x)dx = \phi(L)C_V^2(L) - \phi(0)C_V^2(0) - \int_0^L \frac{\phi}{dx}(x)C_V^2(x)dx$$

which converges when $V \longrightarrow +\infty$ toward

$$\begin{aligned} &B\phi(L) - C_0^2\phi(0) - \int_0^L \frac{\phi}{dx}(x)[\frac{C_0^1 + C_0^2}{2} + \frac{C_0^2 - C_0^1}{2}e^{-x}]dx \\ &= B\phi(L) - C_0^2\phi(0) - \frac{C_0^1 + C_0^2}{2}\phi(L) + \frac{C_0^1 + C_0^2}{2}\phi(0) \\ &- \frac{C_0^2 - C_0^1}{2}\Big[e^{-L}\phi(L) - \phi(0) + \int_0^L e^{-x}\phi(x)\Big] \\ &= B\phi(L) - \frac{C_0^1 + C_0^2}{2}\phi(L) + \frac{C_0^2 - C_0^1}{2}e^{-L}\phi(L) + \frac{C_0^2 - C_0^1}{2}\int_0^L e^{-x}\phi(x)dx. \end{aligned} \tag{34}$$

On the other hand, thanks to (32)

$$\int_0^L \phi(x)\frac{dC_V^2(x)}{dx}(x)dx \underset{V\longrightarrow+\infty}{\longrightarrow} \frac{C_0^2-C_0^1}{2}\int_0^L e^{-x}\phi(x)dx+\frac{B}{2}\phi(L). \tag{35}$$

By equalizing (34) and (35), we find

$$B = C_0^1 + C_0^2 + (C_0^2 - C_0^1)e^{-L}, \tag{36}$$

which is the unique limit of $C_V^2(L)$ and $C_V^3(L)$. In particular, $B > 0$. Our second task is to obtain B'. We perform the same computation for C_V^1. On the one hand, for all $\phi \in \mathcal{C}^1[0,L]$,

$$\int_0^L \phi(x)\frac{dC_V^1(x)}{dx}(x)dx = \phi(L)C_V^1(L) - \phi(0)C_V^1(0) - \int_0^L \frac{\phi}{dx}(x)C_V^1(x)dx$$

which converges when $V \longrightarrow +\infty$ toward

$$B'\phi(L) - \frac{C_0^1+C_0^2}{2}\phi(L) + \frac{C_0^1-C_0^2}{2}e^{-L}\phi(L) + \frac{C_0^1-C_0^2}{2}\int_0^L e^{-x}\phi(x)dx, \tag{37}$$

and on the other hand,

$$\int_0^L \phi(x)\frac{dC_V^1(x)}{dx}(x)dx$$

converges toward

$$\frac{C_0^2-C_0^1}{2}\int_0^L e^{-x}\phi(x)dx + \frac{B}{2}\phi(L). \tag{38}$$

This gives us

$$B' = C_0^1 + C_0^2, \tag{39}$$

and ends the proof of Lemma 7. □

We proved in passing that the limits of the subsequences we deal with are only determined by the problem data and do not depend on the subsequence we choose. Thus, the whole sequences converge.

Second step: The boundary layer.

Lemma 8 *For all $x \in [0,L]$ the inequalities hold*

$$C_V^3(x) \le C_V^3(L)exp\Big(-\frac{2}{3}VM(L-x)\Big) + \frac{K}{V}\Big[1-exp\Big(-\frac{2}{3}VM(L-x)\Big)\Big], \tag{40}$$

$$C_V^3(x) \ge C_V^3(L)exp\Big(-\frac{2}{3}V(L-x)\Big) + \frac{\overline{K}}{V}\Big[1-exp\Big(-\frac{2}{3}V(L-x)\Big)\Big], \tag{41}$$

which means that $C_V^3(x)$ relaxes exponentially fast with V to zero, away from the boundary layer at $X = L$.

Proof. We can write the third line of (1) as

$$-\frac{dC_V^3(x)}{dx}+\frac{2}{3}V\frac{C_V^3(x)}{1+C_V^3(x)}=\frac{1}{3}\Big[C^1(x)+C^2(x)-2C^3(x)\Big]. \tag{42}$$

We multiply this equation by the exponential factor

$$F(x)=\exp\Big(-\frac{2}{3}V\int_L^x\frac{1}{1+C_V^3(s)}ds\Big), \tag{43}$$

$$-\frac{dC_V^3(x)}{dx}F(x)+\frac{2}{3}V\frac{C_V^3(x)}{1+C_V^3(x)}F(x)=\frac{1}{3}\Big[C_V^1(x)+C_V^2(x)-2C_V^3(x)\Big]F(x), \tag{44}$$

and we obtain

$$\frac{d}{dx}\Big[C_V^3(x)F(x)\Big]=-\frac{1}{3}\Big[C_V^1(x)+C_V^2(x)-2C_V^3(x)\Big]F(x).$$

Integrating this equation between L and x, we find,

$$\begin{aligned}&C_V^3(x)F(x)-C_V^3(L)=\\&\qquad-\int_L^x\frac{1}{3}\Big[C_V^1(x)+C_V^2(x)-2C_V^3(x)\Big]\exp\Big(-\frac{2}{3}V\int_L^u\frac{1}{1+C_V^3(s)}ds\Big)du.\end{aligned}$$

By Lemma 3, we have,

$$\begin{aligned}&C_V^3(x)\exp\Big(-\frac{2}{3}V\int_L^x\frac{1}{1+C_V^3(s)}ds\Big)\le\\&\qquad C_V^3(L)+K'\int_x^L\exp\Big(-\frac{2}{3}V\int_L^u\frac{1}{1+C_V^3(s)}ds\Big)du.\end{aligned}$$

We can now complete our calculation. We estimate

$$\begin{aligned}C_V^3(x)&\le C_V^3(L)\exp\Big(-\frac{2}{3}V\int_x^L\frac{1}{1+C_V^3(s)}ds\Big)\\&+K'\Big[\int_x^L\exp\Big(-\frac{2}{3}V\int_L^u\frac{1}{1+C_V^3(s)}ds\Big)\exp\Big(-\frac{2}{3}V\int_x^L\frac{1}{1+C_V^3(s)}ds\Big)\Big]du\\&=C_V^3(L)\exp\Big(-\frac{2}{3}V\int_x^L\frac{1}{1+C_V^3(s)}ds\Big)\\&+K'\int_x^L\exp\Big(-\frac{2}{3}V\int_x^u\frac{1}{1+C_V^3(s)}ds\Big)du\\&\le C_V^3(L)\exp\Big(-\frac{2}{3}V\int_x^L\frac{1}{1+C_V^3(s)}ds\Big)+K'\int_x^L\exp\Big(-\frac{2}{3}VM(u-x)\Big)du\\&=C_V^3(L)\exp\Big(-\frac{2}{3}V\int_x^L\frac{1}{1+C_V^3(s)}ds\Big)+\frac{K}{V}\Big[1-\exp\Big(-\frac{2}{3}VM(L-x)\Big)\Big].\end{aligned}$$

where in the last equality we have used that $0<M<\dfrac{1}{1+C_V^3}$ and (40) is proved.

We can prove in the same way the second part of Lemma 8. □

5 Numerical results

5.1 The numerical algorithm

Numerical simulations illustrate the solute concentration mechanism at $x = L$ that is proved in the theoretical result. The system (1) can be seen as the stationary state associated with the following dynamical problem

$$\begin{cases} \dfrac{\partial C^1}{\partial t}(x,t) + \dfrac{\partial C^1}{\partial x}(x,t) = \\ \qquad \dfrac{1}{3}\Big[C^1(x,t) + C^2(x,t) + C_3(x,t) + V\dfrac{C^3(x,t)}{1+C^3(x,t)}\Big] - C^1(x,t), \\ \dfrac{\partial C^2}{\partial t}(x,t) + \dfrac{\partial C^2}{\partial x}(x,t) = \\ \qquad \dfrac{1}{3}\Big[C^1(x,t) + C^2(x,t) + C^3(x,t) + V\dfrac{C^3(x,t)}{1+C^3(x,t)}\Big] - C^2(x,t), \\ \dfrac{\partial C^3}{\partial t}(x,t) - \dfrac{\partial C^3}{\partial x}(x,t) = \\ \qquad \dfrac{1}{3}\Big[C^1(x,t) + C^2(x,t) + C^3(x,t) + V\dfrac{C^3(x,t)}{1+C^3(x,t)}\Big] - C^3(x,t) \\ \qquad - V\dfrac{C^3(x,t)}{1+C^3(x,t)}, \\ \qquad C^1(0,t) = C_0^1, \qquad C^2(0,t) = C_0^2, \qquad C^3(L,t) = C^2(L,t), \end{cases} \tag{45}$$

which we complete with nonnegative initial concentrations $C^1(x,0)$, $C^2(x,0)$, $C^3(x,0)$ in $BV[0,L]$. We proved in [17] that for every nonnegative initial condition, the system relaxes in L^1 toward the unique solution to the stationary system (1), which is written, by denoting here $C = (C^1, C^2, C^3)$ the solution to (45) and $\overline{C} = (\overline{C^1}, \overline{C^2}, \overline{C^3})$ the solution to (1),

$$\|C(x,t) - \overline{C}(x)\|_{L^1} \underset{t\to\infty}{\searrow} 0. \tag{46}$$

To obtain this result, we prove that this is the case when the initial condition of (45) is a sub- or super-solution to (1), then we remark that every initial condition in $\Big(BV[0,L]\Big)^3$ can be stuck between a sub and a super-solution and we conclude with a comparison principle and with an argument of monotony. In [17], we developp an algorithm to solve (45). This algorithm is based on a finite volume type method [1, 10]. We use a time step Δt and a mesh of size $\Delta x = L/N$ with N the number of cells $Q_k = (x_{k-1/2}, x_{k+1/2})$ (that means $x_{1/2} = 0$ and $x_{N+1/2} = L$). The discrete times are denoted by $t^n = n\Delta t$. We discretize the initial states as

$$C_k^{j,0} = \frac{1}{\Delta x}\int_{Q_k} C^j(x,0)dx, \qquad i = 1,\ 2,\ 3, \quad k = 1, ..., N. \tag{47}$$

We call $C_k^{j,n}$ the discrete solution at time t^n in tube i that approximates equation (45), for $k \in [0, N]$. We use the scheme

$$\begin{cases} C_k^{1,n+1} = C_k^{1,n} - \dfrac{\Delta t}{\Delta x}(C_k^{1,n} - C_{k-1}^{1,n}) \\ \qquad + \Delta t\Big(\dfrac{1}{3}\Big[C_k^{1,n} + C_k^{2,n} + C_k^{3,n} + V\dfrac{C_k^{3,n}}{1 + C_k^{3,n}}\Big] - C_k^{1,n}\Big), \\ C_k^{2,n+1} = C_k^{2,n} - \dfrac{\Delta t}{\Delta x}(C_k^{2,n} - C_{k-1}^{2,n}) \\ \qquad + \Delta t\Big(\dfrac{1}{3}\Big[C_k^{1,n} + C_k^{2,n} + C_k^{3,n} + V\dfrac{C_k^{3,n}}{1 + C_k^{3,n}}\Big] - C_k^{2,n}\Big), \\ C_k^{3,n+1} = C_k^{3,n} + \dfrac{\Delta t}{\Delta x}(C_{k+1}^{3,n} - C_k^{3,n}) \\ \qquad + \Delta t\Big(\dfrac{1}{3}\Big[C_k^{1,n} + C_k^{2,n} + C_k^{3,n} + V\dfrac{C_k^{3,n}}{1 + C_k^{3,n}}\Big] - C_k^{3,n} - V\dfrac{C_k^{3,n}}{1 + C_k^{3,n}}\Big). \end{cases} \tag{48}$$

For boundary conditions, at each time we choose: $C_0^{1,n} = C_0^1$, $C_0^{2,n} = C_0^2$, $C_{N+1}^{3,n} = C_N^{2,n}$.

Because this is an explicit scheme, departing from (47), we obtain directly the solution $C_k^{1,n+1}$ at time t^{n+1} from that at time t^n.

Thus, we can reach the solution to (1) by iterating this scheme for n large enough. The stability condition which ensures the positivity of the scheme is given by

$$\Delta t \leq \frac{3\Delta x}{3 + 2\Delta x(1 + V)}, \tag{49}$$

and detailed in [17]. This CFL condition becomes a tough constraint on Δt when we choose V large. The constraint on Δx also depends on V because it is function of the size of the boundary layer. For each V, to be accurate enough around the boundary layer point L, we discretize the space in N_V cells and we validate a posteriori that this number of cells is high enough since we know from the analytical solution the behaviour of the solution for large values of V.

5.2 Concentration profiles for different V

We present in figure 2 concentration profiles for $V = 1, V = 10, V = 50$ and $V = 100$.

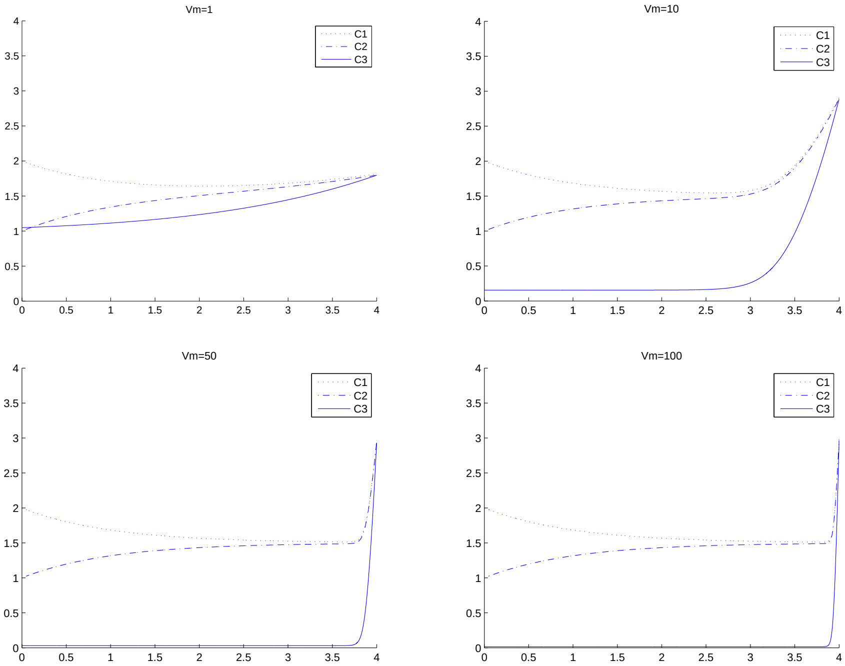

Figure 2: Concentration profiles for $V = 1, 10, 50, 100$, on a domain of length $L = 4$.

When the value of the rate V has the same order of magnitude as the parameters of (1), concentration is hardly increasing in tubes 2 and 3, but decreasing in tube 1. It comes from the fact that we chose $C_0^1 > C_0^2$, but it would have been the contrary in the opposite case. With low values of V, the difusive part of the system (1) is paramount and concentrations tend to homogenize along the tubes. If we increase the pump rate by a factor 10, the concentration tends to approach zero in tube 3 and is abruptly increasing from $L = 3$ and it achieves at $L = 4$ a value greater than $\max(C_0^1, C_0^2)$. We clearly observe the limit profiles and the boundary layer appear for $V \geq 50$.

Illustration of Theorem 2. We want to illustrate that the bounds from above and from below found in Theorem 2 give an accurate descrition of the qualitative behavior of C^3.

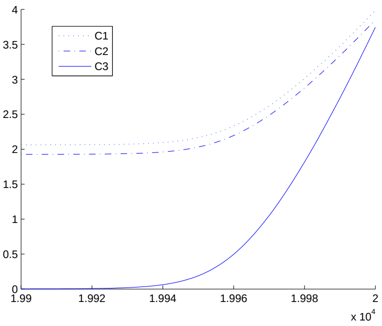

Figure 3: Zoom on the interval $[0.99L, L]$ of figure 2 for $V = 1000$.

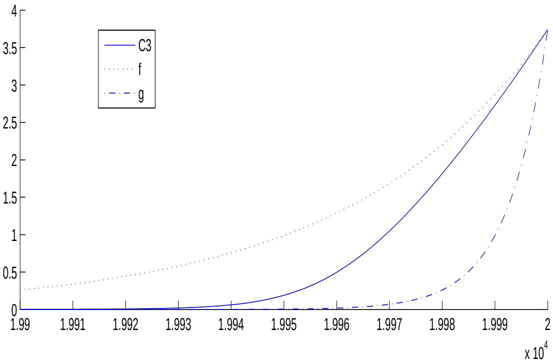

Figure 4: The curve in the middle represents C^3_{1000} on $[0.99L, L]$. The upper curve represents the upperbound f for C^3 found in Lemma 8 and the the lower curve represents g, the bound from below.

Figure 3 displays a zoom of the numerical aproximation of $C^3_{1000}(L)$ that we will denote $C^3_{num,1000}(L)$. This approximative value enables us to define $M_{num} = \dfrac{1}{C^3_{num,1000}(L)}$. To illustrate Lemma 8, we define

$$f(x) = C^3_{num,1000}(L) \exp\Big(- \frac{2 \times 1000}{3} M_{num}(L-x)\Big)$$

and

$$g(x) = C^3_{num,1000}(L) \exp\Big(- \frac{2}{3} \times 1000(L-x)\Big).$$

In Figure 4, we depict C^3_{1000} on the interval $[\frac{99L}{100}, L]$ and the profiles of the two functions f and g which control C^3_{1000}. We observe that $g \leq C^3_{num,1000} \leq f$, as expected in theorem 2, and then that the component C^3_V decreases exponentially to zero.

6 Conclusion

Motivated by renal flows, we have studied a concentration mechanism with an active pump characterized by a parameter V. As expected, for V large enough, a large axial solute concentration gradient appears in all tubes. The result of our analysis is that the concentrations are uniformly bounded in V for all $x \in [0, L]$, and so are their derivatives, except at $x = L$. In the limit $V = \infty$, the concentration gradient converges to a Dirac profile at $x = L$. We obtain a limit concentration profile in all tubes which presents a boundary layer at $x = L$. In the urine concentrating model, we are mostly interested on the behaviour at $x = L$, because it is at this depth that the composition of final urine is determined. Therefore, our analysis explains why active transport plays a very specific role, which is to increase solute concentration at $x = L$ and only at $x = L$.

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