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# Moment bounds for dependent sequences in smooth Banach spaces

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## Abstract

We prove a Marcinkiewicz-Zygmund type inequality for random variables taking values in a smooth Banach space. Next, we obtain some sharp concentration inequalities for the empirical measure of  $\{T, T^2, \dots, T^n\}$ , on a class of smooth functions, when  $T$  belongs to a class of nonuniformly expanding maps of the unit interval.

## 1 Introduction and notations

Let  $(\mathbb{B}, |\cdot|_{\mathbb{B}})$  be a separable Banach space. The notion of  $p$ -smooth Banach spaces ( $1 < p \leq 2$ ) was introduced in a famous paper by Pisier ([17], Section 3). These spaces play the same role with respect to martingales as spaces of type  $p$  do with respect to the sums of independent random variables.

We shall follow the approach of Pinelis [16], who showed that 2-smoothness is in some sense equivalent to a control of the second directional derivative of the map  $\psi_2$  defined by  $\psi_2(x) = |x|_{\mathbb{B}}^2$ . In particular, if there exists  $C > 0$  such that, for any  $x, u$  in  $\mathbb{B}$ ,

$$D^2\psi_2(x)(u, u) \leq C|u|_{\mathbb{B}}^2, \quad (1.1)$$

then the space  $\mathbb{B}$  is 2-smooth (here  $D^2g(x)(u, v)$  denotes the second derivative of  $g$  at point  $x$ , in the directions  $u, v$ ). In his 1994 paper, Pinelis [16] used the property (1.1) to derive Burkholder and Rosenthal moment inequalities as well as exponential bounds for  $\mathbb{B}$ -valued martingales.

We shall consider two different classes of 2-smooth Banach spaces, whose smoothness properties are described as follows. Let  $p$  be a real number in  $[2, \infty[$  and let  $\psi_p$  be the function from  $\mathbb{B}$  to  $\mathbb{R}$  defined by

$$\psi_p(x) = |x|_{\mathbb{B}}^p. \quad (1.2)$$

We say that the separable Banach space  $(\mathbb{B}, |\cdot|_{\mathbb{B}})$  belongs to the class  $\mathcal{C}_2(p, c_p)$  if the function  $\psi_p$  is two times differentiable and satisfies for all  $x$  and  $u$  in  $\mathbb{B}$ ,

$$|D^2\psi_p(x)(u, u)| \leq c_p|x|_{\mathbb{B}}^{p-2}|u|_{\mathbb{B}}^2. \quad (1.3)$$

We say that  $(\mathbb{B}, |\cdot|_{\mathbb{B}})$  belongs to the class  $\tilde{\mathcal{C}}_2(p, \tilde{c}_p)$  if the more restrictive inequality holds: for all  $x, u, v$  in  $\mathbb{B}$ ,

$$|D^2\psi_p(x)(u, v)| \leq \tilde{c}_p|x|_{\mathbb{B}}^{p-2}|u|_{\mathbb{B}}|v|_{\mathbb{B}}. \quad (1.4)$$

Before describing our results, let us quote that the class  $\tilde{\mathcal{C}}_2(p, \tilde{c}_p)$  contains the  $\mathbb{L}^q$ -spaces for  $q \geq 2$ , for which one can compute the constant  $\tilde{c}_p$ . The following lemma will be proved in Appendix.

**Lemma 1.1.**

1. For any  $q \in [2, \infty[$  and any measure space  $(\mathcal{X}, \mathcal{A}, \mu)$ , the space  $\mathbb{L}^q = \mathbb{L}^q(\mathcal{X}, \mathcal{A}, \mu)$  belongs to the class  $\mathcal{C}_2(p, c_p)$  with  $c_p = p(\max(p, q) - 1)$ , and to the class  $\tilde{\mathcal{C}}_2(p, \tilde{c}_p)$  with  $\tilde{c}_p = p(\max(p, 2q - p) - 1)$ .
2. If  $\mathbb{B}$  is a separable Hilbert space then it belongs to the class  $\tilde{\mathcal{C}}_2(p, \tilde{c}_p)$  with  $\tilde{c}_p = p(p - 1)$ .

The main result of this paper is a Marcinkiewicz-Zygmund type inequality for the moment of order  $p$  of partial sums  $S_n$  of  $\mathbb{B}$ -valued random variables, when  $\mathbb{B}$  belongs to the class  $\tilde{\mathcal{C}}_2(p, \tilde{c}_p)$ . The upper bound is expressed in terms of conditional expectations of the random variables with respect to a past  $\sigma$ -field, and extends the corresponding upper bound by Dedecker and Doukhan [3] for real-valued random variables. As in [18] and [3], the proof is done by writing  $\psi_p(S_n)$  as a telescoping sum. The property (1.3) enables to use the Taylor integral formula at order 2 to control the terms of the telescoping sums.

This Marcinkiewicz-Zygmund type bound together with the Rosenthal type bound given in [6] and the deviation inequality given in [5] provide a complete picture of the moment bounds for sums of  $\mathbb{B}$ -valued random variables, when  $\mathbb{B}$  belongs to the class  $\tilde{\mathcal{C}}_2(p, \tilde{c}_p)$ . As we shall see, these bounds apply to a large class of dependent sequences, in the whole range from short to long dependence.

As an application, we shall focus on the  $\mathbb{L}^q$ -norm of the centered empirical distribution function  $G_n$  of the iterates of a nonuniformly expanding map  $T$  of the unit interval (modelled by a Young tower with polynomial tails). On the probability space  $[0, 1]$  equipped with the  $T$ -invariant probability  $\nu$ , the covariance between two Hölder observables of  $T$  and  $T^n$  is of order  $n^{-(1-\gamma)/\gamma}$  for some  $\gamma \in (0, 1)$ . Hence the sequence of the iterates  $(T^i)_{i \geq 1}$  is short-range dependent if  $\gamma < 1/2$  and long-range dependent if  $\gamma \in [1/2, 1)$ . Moment and deviation bounds for the  $\mathbb{L}^q$ -norm of  $G_n$  are given in Theorem 4.1 in the short range dependent case, and in Theorems 4.2 and 4.3 in the long range dependent case. In Remark 4.1, we give some arguments, based on a limit theorem for the  $\mathbb{L}^2$ -norm of  $G_n$ , showing that the deviations bounds of Theorem 4.3 are in some sense optimal.

As a consequence of these results, we obtain in Corollary 4.1 a complete picture of the behavior of  $\|W_1(\nu_n, \nu)\|_p$  for  $p \geq 1$ , where  $W_1(\nu_n, \nu)$  is the Wasserstein distance between the empirical measure  $\nu_n$  of  $\{T, T^2, \dots, T^n\}$  and the invariant distribution  $\nu$ . These results are different but complementary to the moment bounds on  $W_1(\nu_n, \nu) - \mathbb{E}(W_1(\nu_n, \nu))$  obtained by Chazottes and Gouëzel [1] and Gouëzel and Melbourne [10] as a consequence of a concentration inequality for separately Lipschitz functionals of  $(T, T^2, \dots, T^n)$ . See Section 4.3 for a deeper discussion.

All along the paper, the notation  $a_n \ll b_n$  means that there exists a numerical constant  $C$  not depending on  $n$  such that  $a_n \leq Cb_n$ , for all positive integers  $n$ .

## 2 A Marcinkiewicz-Zygmund type inequality

Our first result extends Proposition 4 of Dedecker and Doukhan [3] to smooth Banach spaces belonging to  $\tilde{\mathcal{C}}_2(p, \tilde{c}_p)$ .

**Theorem 2.1.** *Let  $p$  be a real number in  $[2, \infty[$  and let  $(\mathbb{B}, |\cdot|_{\mathbb{B}})$  be a Banach space belonging to the class  $\tilde{\mathcal{C}}_2(p, \tilde{c}_p)$ . Let  $(X_i)_{i \in \mathbb{N}}$  be a sequence of centered random variables in  $\mathbb{L}^p(\mathbb{B})$ . Let  $(\mathcal{F}_i)_{i \geq 0}$  be an increasing sequence of  $\sigma$ -algebras such that  $X_i$  is  $\mathcal{F}_i$ -measurable, and denote by  $\mathbb{E}_i(\cdot) = \mathbb{E}(\cdot | \mathcal{F}_i)$  the conditional expectation with respect to  $\mathcal{F}_i$ . Define then*

$$b_{i,n} = \max_{i \leq \ell \leq n} \left( \mathbb{E}_0 \left( |X_i|_{\mathbb{B}}^{p/2} \left| \sum_{k=i}^{\ell} \mathbb{E}_i(X_k) \Big|_{\mathbb{B}} \right|^{p/2} \right) \right)^{2/p}.$$

For any integer  $n \geq 0$ , the following inequality holds:

$$\mathbb{E}_0(|S_n|_{\mathbb{B}}^p) \leq K^p \left( \sum_{i=1}^n b_{i,n} \right)^{p/2} \text{ almost surely, where } K = \sqrt{2p^{-1}} \sqrt{\max(\tilde{c}_p, p/2)}. \quad (2.1)$$

**Remark 2.1.** Taking  $\mathcal{F}_0 = \{\Omega, \emptyset\}$ , it follows that, for any integer  $n \geq 0$ ,

$$\mathbb{E}(|S_n|_{\mathbb{B}}^p) \leq K^p \left( \sum_{i=1}^n \max_{i \leq \ell \leq n} \left\| |X_i|_{\mathbb{B}} \left\| \sum_{k=i}^{\ell} \mathbb{E}(X_k | \mathcal{F}_i) \right\|_{\mathbb{B}} \right\|_{p/2} \right)^{p/2} \text{ where } K = \sqrt{2p^{-1}} \sqrt{\max(\tilde{c}_p, p/2)}. \quad (2.2)$$

In addition, if we assume that  $\mathbb{P}(|X_k|_{\mathbb{B}} \leq M) = 1$  for any  $k \in \{1, \dots, n\}$ , Inequality (2.2) combined with Proposition 5.2 of the appendix leads to the bound

$$\mathbb{E} \left( \max_{1 \leq k \leq n} |S_k|_{\mathbb{B}}^p \right) \leq C_p M^{p-1} n^{p/2} \left( \sum_{k=0}^{n-1} \theta^{2/p}(k) \right)^{p/2}, \quad (2.3)$$

where

$$C_p = \frac{1}{2} \left( \frac{2pK}{p-1} \right)^p + 2^{3p-4} 3^p p \quad \text{and} \quad \theta(k) = \max \left\{ \mathbb{E}(|\mathbb{E}(X_i | \mathcal{F}_{i-k})|_{\mathbb{B}}), i \in \{k+1, \dots, n\} \right\}.$$

A complete proof of Inequality 2.3 will be given in Section 5.4.

When  $\mathbb{B} = \mathbb{L}^q$  for  $q \geq 2$ , the constant  $K$  of Inequality (2.2) is equal to  $\sqrt{\max(4q - 2p, 2p) - 2}$ . However we notice that we can obtain a better constant when the underlying sequence is a martingale differences sequence. More precisely, the following extension of the Marcinkiewicz-Zygmund type inequality obtained by Rio (2009) when the random variables are real-valued holds:

**Theorem 2.2.** Let  $p$  be a real number in  $[2, \infty[$  and let  $(\mathbb{B}, |\cdot|_{\mathbb{B}})$  be a Banach space belonging to the class  $\mathcal{C}_2(p, c_p)$ . Let  $(d_i)_{i \in \mathbb{N}}$  be a sequence of martingale differences with values in  $\mathbb{B}$  with respect to an increasing filtration  $(\mathcal{F}_i)_{i \in \mathbb{N}}$  and such that for all  $i \in \mathbb{N}$ ,  $\|d_i\|_p < \infty$ . Then, setting  $M_n = \sum_{i=1}^n d_i$ , the following inequality holds:

$$\mathbb{E}(|M_n|_{\mathbb{B}}^p) \leq (p^{-1} c_p)^{p/2} \left( \sum_{i=1}^n \|d_i\|_p^2 \right)^{p/2}. \quad (2.4)$$

In particular if  $\mathbb{B} = \mathbb{L}^q(\mathcal{X}, \mathcal{A}, \mu)$  with  $q \in [2, \infty[$  and  $(T, \mathcal{A}, \nu)$  a measure space, Inequality (2.4) combined with Lemma 1.1 leads to

$$\mathbb{E}(|M_n|_q^p) \leq (\max(p, q) - 1)^{p/2} \left( \sum_{i=1}^n \|d_i\|_q^2 \right)^{p/2}, \quad (2.5)$$

$|\cdot|_q$  being the norm on  $\mathbb{L}^q(\mathcal{X}, \mathcal{A}, \mu)$ .

**Proof of Theorem 2.1.** As in [18] and [3], we shall prove the result by induction. For any  $t \in [0, 1]$  let

$$h_n(t) = \mathbb{E}_0(|S_{n-1} + tX_n|_{\mathbb{B}}^p). \quad (2.6)$$

Our induction hypothesis at step  $n-1$  is the following: for any  $k \leq n-1$ ,

$$h_k(t) \leq K^p \left( \sum_{i=1}^{k-1} b_{i,k} + t b_{k,k} \right)^{p/2}. \quad (2.7)$$

Since  $K \geq 1$ , the above inequality is clearly true for  $k = 1$ . Assuming that it is true for  $n - 1$ , let us prove it at step  $n$ .

Assume that one can prove that

$$h_n(t) \leq \max(\tilde{c}_p, p/2) \left( \sum_{k=1}^{n-1} b_{k,n} \int_0^1 (h_k(s))^{1-2/p} ds + b_{n,n} \int_0^t (h_n(s))^{1-2/p} ds \right), \quad (2.8)$$

then, using our induction hypothesis, it follows that

$$\begin{aligned} h_n(t) &\leq \max(\tilde{c}_p, p/2) \left( \sum_{k=1}^{n-1} b_{k,n} \int_0^1 K^{p-2} \left( \sum_{i=1}^{k-1} b_{i,k} + sb_{k,k} \right)^{(p-2)/2} ds + b_{n,n} \int_0^t (h_n(s))^{1-2/p} ds \right) \\ &\leq \max(\tilde{c}_p, p/2) \left( K^{p-2} \sum_{k=1}^{n-1} b_{k,n} \int_0^1 \left( \sum_{i=1}^{k-1} b_{i,n} + sb_{k,n} \right)^{(p-2)/2} ds + b_{n,n} \int_0^t (h_n(s))^{1-2/p} ds \right). \end{aligned}$$

Integrating with respect to  $s$ , we get

$$b_{k,n} \int_0^1 \left( \sum_{i=1}^{k-1} b_{i,n} + sb_{k,n} \right)^{(p-2)/2} ds = \frac{2}{p} \left( \sum_{i=1}^k b_{i,n} \right)^{p/2} - \frac{2}{p} \left( \sum_{i=1}^{k-1} b_{i,n} \right)^{p/2},$$

implying that

$$\sum_{k=1}^{n-1} b_{k,n} \int_0^1 \left( \sum_{i=1}^{k-1} b_{i,n} + sb_{k,n} \right)^{(p-2)/2} ds = 2p^{-1} \left( \sum_{i=1}^{n-1} b_{i,n} \right)^{p/2}.$$

Therefore, since  $K^2 = 2p^{-1} \max(\tilde{c}_p, p/2)$ ,

$$h_n(t) \leq K^p \left( \sum_{i=1}^{n-1} b_{i,n} \right)^{p/2} + \max(\tilde{c}_p, p/2) b_{n,n} \int_0^t (h_n(s))^{1-2/p} ds. \quad (2.9)$$

Let  $H_n(t) = \int_0^t (h_n(s))^{1-2/p} ds$ . The differential integral inequation (2.9) writes

$$H'_n(s) \left( K^p \left( \sum_{i=1}^{n-1} b_{i,n} \right)^{p/2} + \max(\tilde{c}_p, p/2) b_{n,n} H(s) \right)^{-1+2/p} \leq 1.$$

Setting

$$R_n(s) = \left( K^p \left( \sum_{i=1}^{n-1} b_{i,n} \right)^{p/2} + \max(\tilde{c}_p, p/2) b_{n,n} H(s) \right)^{2/p},$$

the previous inequality can be rewritten as

$$R'_n(s) \leq 2p^{-1} \max(\tilde{c}_p, p/2) b_{n,n}.$$

Integrating between 0 and  $t$ , we derive

$$(h_n(t))^{2/p} - K^2 \sum_{i=1}^{n-1} b_{i,n} \leq R_n(t) - R_n(0) \leq 2tp^{-1} \max(\tilde{c}_p, p/2) b_{n,n}.$$

Taking into account that  $K^2 = 2p^{-1} \max(\tilde{c}_p, p/2)$ , it follows that

$$(h_n(t))^{2/p} \leq K^2 \left( \sum_{i=1}^{n-1} b_{i,n} + tb_{n,n} \right),$$

showing that our induction hypothesis holds true at step  $n$ . To end the proof it suffices to prove (2.8). We shall proceed as in the proof of Theorem 2.3 in [18]. With this aim, let

$$S_n(t) = \sum_{i=1}^n Y_i(t), \quad \text{where } Y_i(t) = X_i \text{ for } 1 \leq i \leq n-1 \text{ and } Y_n(t) = tX_n.$$

Notice that for any integer  $k$  in  $[1, n-1]$ ,  $S_k(t) = S_k$ . Let now  $\psi_p$  be defined by (1.2). Applying Taylor integral formula at order 2, we get

$$\begin{aligned} \psi_p(S_n(t)) &= \sum_{i=1}^n (\psi_p(S_i(t)) - \psi_p(S_{i-1}(t))) \\ &= \sum_{k=1}^n D\psi_p(S_{k-1})(Y_k(t)) + \sum_{i=1}^n \int_0^1 (1-s) D^2\psi_p(S_{i-1} + sY_i(t))(Y_i(t), Y_i(t)) ds. \end{aligned}$$

But, for any integer  $k$  in  $[1, n]$ ,

$$\begin{aligned} D\psi_p(S_{k-1})(Y_k(t)) &= \sum_{i=1}^{k-1} (D\psi_p(S_i)(Y_k(t)) - D\psi_p(S_{i-1})(Y_k(t))) \\ &= \sum_{i=1}^{k-1} \int_0^1 D^2\psi_p(S_{i-1} + sX_i)(Y_k(t), X_i) ds. \end{aligned}$$

Notice now that for any  $x$  and  $u$  in  $\mathbb{B}$ ,  $D^2\psi_p(x)(u, u) \geq 0$ . Indeed, the function  $x \mapsto \psi_p(x) = |x|_{\mathbb{B}}^{p/2}$  is convex for any  $p \geq 2$  and is by assumption 2-times differentiable, implying that the second differentiable derivative at  $x$  in the direction  $u$  is non-negative. So, overall, using the fact that  $D^2\psi_p(x)(u, u) \geq 0$ ,

$$\begin{aligned} \psi_p(S_n(t)) &\leq \sum_{i=1}^{n-1} \int_0^1 D^2\psi_p(S_{i-1} + sX_i) \left( \sum_{k=i+1}^n Y_k(t), X_i \right) ds \\ &\quad + \sum_{i=1}^n \int_0^1 D^2\psi_p(S_{i-1} + sY_i(t))(Y_i(t), Y_i(t)) ds. \end{aligned}$$

Taking the conditional expectation w.r.t.  $\mathcal{F}_0$  and recalling the definition (2.6) of  $h_n(t)$ , it follows that, for any  $t \in [0, 1]$ ,

$$\begin{aligned} h_n(t) &\leq \sum_{i=1}^{n-1} \int_0^1 \mathbb{E}_0 \left( D^2\psi_p(S_{i-1} + sX_i) \left( \sum_{k=i}^{n-1} X_k + tX_n, X_i \right) ds \right) \\ &\quad + t^2 \int_0^1 \mathbb{E}_0 \left( D^2\psi_p(S_{n-1} + stX_n)(X_n, X_n) ds \right). \end{aligned}$$

Using again the fact that  $D^2\psi_p(v)(u, u) \geq 0$ , we have

$$t^2 \int_0^1 \mathbb{E}_0 \left( D^2\psi_p(S_{n-1} + stX_n)(X_n, X_n) ds \right) \leq \int_0^t \mathbb{E}_0 \left( D^2\psi_p(S_{n-1} + uX_n)(X_n, X_n) du \right).$$

Hence setting

$$a_{i,n}(t) = X_i + \sum_{k=i+1}^{n-1} \mathbb{E}(X_k | \mathcal{F}_i) + t\mathbb{E}(X_n | \mathcal{F}_i),$$

and using the fact that  $(\mathcal{F}_i)$  is an increasing sequence of  $\sigma$ -algebras, we derive

$$h_n(t) \leq \sum_{i=1}^{n-1} \int_0^1 \mathbb{E}_0 \left( D^2 \psi_p(S_{i-1} + sX_i)(a_{i,n}(t), X_i) ds \right) + \int_0^t \mathbb{E}_0 \left( D^2 \psi_p(S_{n-1} + sX_n)(X_n, X_n) ds \right).$$

Using (1.4), we then get

$$h_n(t) \leq \tilde{c}_p \sum_{i=1}^{n-1} \int_0^1 \mathbb{E}_0 \left( |S_{i-1} + sX_i|_{\mathbb{B}}^{p-2} |a_{i,n}(t)|_{\mathbb{B}} |X_i|_{\mathbb{B}} \right) ds + \tilde{c}_p \int_0^t \mathbb{E}_0 \left( |S_{n-1} + sX_n|_{\mathbb{B}}^{p-2} |X_n|_{\mathbb{B}}^2 \right) ds.$$

Hölder's inequality implies that

$$\begin{aligned} h_n(t) &\leq \tilde{c}_p \sum_{i=1}^{n-1} \int_0^1 (h_i(s))^{(p-2)/p} \left( \mathbb{E}_0(|a_{i,n}(t)|_{\mathbb{B}}^{p/2} |X_i|_{\mathbb{B}}^{p/2}) \right)^{2/p} ds \\ &\quad + \tilde{c}_p \int_0^t (h_n(s))^{(p-2)/p} \left( \mathbb{E}_0(|X_n|_{\mathbb{B}}^p) \right)^{2/p} ds. \end{aligned} \quad (2.10)$$

Let  $G_{i,n}(t) = \mathbb{E}_0(|a_{i,n}(t)|_{\mathbb{B}}^{p/2} |X_i|_{\mathbb{B}}^{p/2})$ . Since it is a convex function, for any  $t \in [0, 1]$ ,

$$G_{i,n}(t) \leq \max(G_{i,n}(0), G_{i,n}(1)) \leq b_{i,n}^{p/2}. \quad (2.11)$$

Starting from (2.10), using (2.11) and the fact that  $(\mathbb{E}_0(|X_n|_{\mathbb{B}}^p))^{2/p} \leq b_{n,n}$ , the inequality (2.8) follows.  $\diamond$

**Proof of Theorem 2.2.** The proof follows the lines of the proof of Proposition 2.1 in [19]. The only difference is that Inequality (1.3) is used to get his bound (2.1). For the reader's convenience, let us give the main steps of the proof. For any  $t \in [0, \infty[$ , let  $\varphi_n(t) = \|M_{n-1} + td_n\|_{\mathbb{B}}^p$ . Using Taylor's integral formula at order two together with Inequality (1.3), we infer that

$$\varphi_n(t) \leq \varphi_n(0) + c_p \|d_n\|_{\mathbb{B}}^2 \int_0^t (t-s) (\varphi(s))^{1-2/p} := \phi_n(t).$$

Proceeding as at the top of page 150 in [19], it follows that for any non-negative real  $x$ ,

$$\phi'_n(x) \leq \|d_n\|_{\mathbb{B}}^2 \sqrt{\frac{pc_p}{(p-1)}} \sqrt{(\phi_n(x))^{2-2/p} - (\varphi_n(0))^{2-2/p}}.$$

Next, using lemma 2.1 in [19] and the arguments following it, we derive

$$\phi'_n(x) \leq \|d_n\|_{\mathbb{B}}^2 \sqrt{pc_p} (\phi_n(x))^{1-2/p} \sqrt{(\phi_n(x))^{2/p} - (\varphi_n(0))^{2/p}},$$

and then

$$(\phi_n(x))^{2/p} \leq (\varphi_n(0))^{2/p} + p^{-1} c_p x^2 \|d_n\|_{\mathbb{B}}^2.$$

Since  $\varphi_n(x) \leq \phi_n(x)$ , it follows that

$$\|M_n\|_{\mathbb{B}}^2 = (\varphi_n(1))^{2/p} \leq \|M_{n-1}\|_{\mathbb{B}}^2 + p^{-1} c_p \|d_n\|_{\mathbb{B}}^2,$$

proving the theorem.  $\diamond$

### 3 Hoeffding type inequalities for martingales

In the following corollary, we give an exponential inequality for the deviation of the  $\mathbb{L}^q$ -norm of martingales.

**Corollary 3.1.** *Let  $q \in [2, \infty[$  and  $(\mathcal{X}, \mathcal{A}, \mu)$  a measure space. Let  $(d_i)_{i \in \mathbb{N}}$  be a sequence of martingale differences with values in  $\mathbb{L}^q = \mathbb{L}^q(\mathcal{X}, \mathcal{A}, \mu)$  (equipped with the norm  $|\cdot|_q$ ) with respect to an increasing filtration  $(\mathcal{F}_i)_{i \in \mathbb{N}}$ . Assume that for all  $i \in \mathbb{N}$ , there exists a positive real  $b$  such that  $\| |d_i|_q \|_\infty \leq b$ . Let  $M_n = \sum_{i=1}^n d_i$ . For any positive integer  $n$  and any positive real  $x$ , the following inequality holds*

$$\mathbb{P}\left(\max_{1 \leq k \leq n} |M_k|_q \geq x\right) \leq \begin{cases} 1 & \text{if } x < b\sqrt{(q-1)n} \\ \frac{(b^2(q-1)n)^{q/2}}{x^q} & \text{if } b\sqrt{(q-1)n} < x < b\sqrt{e(q-1)n} \\ \frac{1}{\sqrt{e}} \exp\left(-\frac{x^2}{2eb^2n}\right) & \text{if } x \geq b\sqrt{e(q-1)n}. \end{cases} \quad (3.1)$$

Under the assumptions of Corollary 3.1, Theorem 3.5 in [16] gives the following upper bound: for any positive integer  $n$  and any positive real  $x$ ,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} |M_k|_q \geq x\right) \leq 2 \exp\left(-\frac{x^2}{2(q-1)b^2n}\right). \quad (3.2)$$

It is noteworthy to indicate that for any  $q \geq e + 1$ , the bound in (3.1) is always better than the one given in (3.2).

**Proof of Corollary 3.1.** Let  $p$  be a real number in  $[2, \infty[$ . By the Doob-Kolmogorov maximal inequality,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} |M_k|_q \geq x\right) \leq x^{-p} \mathbb{E}(|M_n|_q^p).$$

Therefore, using Inequality (2.5), we derive that for any  $p \geq q$ ,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} |M_k|_q \geq x\right) \leq \left(\frac{\sqrt{a_p b^2 n}}{x}\right)^p, \text{ where } a_p = \max(p, q) - 1.$$

Taking  $p = q$  if  $x < ((q-1)eb^2n)^{1/2}$  (so in this case  $a_p = q - 1$ ) and  $p = 1 + \frac{x^2}{eb^2n}$  if  $x \geq ((q-1)eb^2n)^{1/2}$  (so in this case  $a_p = p - 1$ ), the inequality (3.1) follows.  $\diamond$

In the following corollary, we give an exponential inequality for the deviation of the  $\mathbb{L}^q$ -norm of partial sums. The proof is omitted since it is exactly the same as that of Corollary 3.1, by using Inequality (2.2) instead of Inequality (2.5).

**Corollary 3.2.** *Let  $q \in [2, \infty[$  and  $(\mathcal{X}, \mathcal{A}, \mu)$  a measure space. Let  $(X_i)_{i \in \mathbb{N}}$  be a sequence of random variables with values in  $\mathbb{L}^q = \mathbb{L}^q(\mathcal{X}, \mathcal{A}, \mu)$  (equipped with the norm  $|\cdot|_q$ ). Let  $(\mathcal{F}_i)_{i \geq 0}$  be an increasing sequence of  $\sigma$ -algebras such that  $X_i$  is  $\mathcal{F}_i$ -measurable, and denote by  $\mathbb{E}_i(\cdot) = \mathbb{E}(\cdot | \mathcal{F}_i)$  the conditional expectation with respect to  $\mathcal{F}_i$ . For any positive integer  $n$ , let  $S_n = \sum_{i=1}^n X_i$ . Assume that for any integer  $i \in [1, n]$ ,*

$$\left\| |X_i|_q \left| \sum_{k=i}^n \mathbb{E}_i(X_k) \right|_q \right\|_\infty \leq b_n^2.$$

Then, for any positive real  $x$ , the following inequality holds

$$\mathbb{P}\left(|S_n|_q \geq x\right) \leq \begin{cases} 1 & \text{if } x < b_n \sqrt{2(q-1)n} \\ \frac{(2b_n^2(q-1)n)^{q/2}}{x^q} & \text{if } b_n \sqrt{2(q-1)n} < x < b_n \sqrt{2e(q-1)n} \\ \frac{1}{\sqrt{e}} \exp\left(-\frac{x^2}{4eb_n^2n}\right) & \text{if } x \geq b_n \sqrt{2e(q-1)n}. \end{cases}$$



## 4 Moment and deviation inequalities for the empirical process of nonuniformly expanding maps

In this section, we shall apply Theorem 2.1 and the inequalities recalled in Appendix to obtain moment and deviation inequalities for the  $\mathbb{L}^q$  norm of the centered empirical distribution function of nonuniformly expanding maps of the interval. More precisely, our results apply to the iterates of a map  $T$  from  $[0, 1]$  to  $[0, 1]$  that can be modelled by a Young tower with polynomial tails of the return time.

In Section 4.1, we recall the formalism of Young towers, which has been described in many papers (see for instance [20] and [13]) with sometimes slight differences. Here we borrow the formalism described in Chapter 1 of Gouëzel's PhD thesis [8].

The moment inequalities are stated in Section 4.2, and an application to the Wassertein metric between the empirical measure of  $\{T, T^2, \dots, T^n\}$  and the  $T$ -invariant distribution is presented in Section 4.3. To be complete, we give in Section 4.4 some upper bounds for the maximum of the partial sums of Hölder observables, which can be proved as in Section 4.2.

### 4.1 One dimensional maps modelled by Young towers

Let  $T$  be a map from  $[0, 1]$  to  $[0, 1]$ , and  $\lambda$  be a probability measure on  $[0, 1]$ . Let  $Y$  be a Borel set of  $[0, 1]$ , with  $\lambda(Y) > 0$ . Assume that there exist a partition (up to a negligible set)  $\{Y_k\}_{k \in \{1, \dots, K\}}$  of  $Y$  (note that  $K$  can be infinite) and a sequence  $(\varphi_k)_{k \in \{1, \dots, K\}}$  of increasing numbers such that  $T^{\varphi_k}(Y_k) = Y$ . Let then  $\varphi_Y$  be the function from  $Y$  to  $\{\varphi_k\}_{k \in \{1, \dots, K\}}$  such that  $\varphi_Y(y) = \varphi_k$  if  $y \in Y_k$ .

We then define a space

$$X = \{(y, i) : y \in Y, i < \varphi_Y(y)\}$$

and a map  $\bar{T}$  on  $X$ :

$$\bar{T}(y, i) = \begin{cases} (y, i + 1) & \text{if } i < \varphi_Y(y) - 1 \\ (T^{\varphi_Y(y)}(y), 0) & \text{if } i = \varphi_Y(y) - 1. \end{cases}$$

The space  $X$  is the Young tower. One can define the floors  $\Delta_{k,i}$  for  $k \in \{1, \dots, K\}$  and  $i \in \{0, \dots, \varphi_k - 1\}$ :  $\Delta_{k,i} = \{(y, i) : y \in Y_k\}$ . These floors define a partition of the tower:

$$X = \bigcup_{k \in \{1, \dots, K\}, i \in \{0, \dots, \varphi_k - 1\}} \Delta_{k,i}.$$

On  $X$ , the measure  $m$  is defined as follows: if  $\bar{B}$  is a set included in  $\Delta_{k,i}$ , that can be written as  $\bar{B} = B \times \{i\}$  with  $B \subset Y_k$ , then  $m(\bar{B}) = \lambda(B)$ . Consequently, for a set  $\bar{A} \subset \bigcup_{\{k: \varphi_k > i\}} \Delta_{k,i}$ , which can be written as  $\bar{A} = A \times \{i\} = (\bigcup_{\{k: \varphi_k > i\}} B_k) \times \{i\}$  with  $B_k \subset Y_k$ , one has

$$m(\bar{A}) = \lambda(A) = \sum_{\{k: \varphi_k > i\}} \lambda(B_k).$$

Let  $\pi$  be the ‘‘projection’’ from  $X$  to  $[0, 1]$  defined by  $\pi(y, i) = T^i(y)$ . Then, one has

$$\pi \circ \bar{T} = T \circ \pi.$$

Indeed, if  $i < \varphi_Y(y) - 1$ , then  $\bar{T}(y, i) = (y, i + 1)$  so that

$$\pi \circ \bar{T}(y, i) = \pi(y, i + 1) = T^{i+1}(y) = T \circ \pi(y, i).$$

If  $i = \varphi_Y(y) - 1$ , then  $\bar{T}(y, i) = (T^{\varphi_Y(y)}(y), 0)$  so that

$$\pi \circ \bar{T}(y, \varphi_Y(y) - 1) = T^{\varphi_Y(y)}(y) = T(T^{\varphi_Y(y)-1}(y)) = T \circ \pi(y, \varphi_Y(y) - 1).$$

Assume now that  $\bar{T}$  preserves the probability  $\bar{\nu}$  on  $X$ , and let  $\nu$  be the image measure of  $\bar{\nu}$  by  $\pi$ . Then, for any measurable and bounded function  $f$ ,

$$\nu(f(T)) = \bar{\nu}(f(T \circ \pi)) = \bar{\nu}((f \circ \pi)(\bar{T})) = \bar{\nu}(f \circ \pi) = \nu(f),$$

and consequently  $\nu$  is invariant by  $T$ .

The map  $T$  can be modelled by a Young tower if:

1. For any  $k \in \{1, \dots, K\}$ ,  $T^{\varphi_k}$  is a measurable isomorphism between  $Y_k$  and  $Y$ . Moreover there exists  $C > 0$  such that, for any  $k \in \{1, \dots, K\}$  and almost every  $x, y$  in  $Y_k$ ,

$$\left| 1 - \frac{(T^{\varphi_k})'(x)}{(T^{\varphi_k})'(y)} \right| \leq C |T^{\varphi_k}(x) - T^{\varphi_k}(y)|.$$

2. There exists  $C > 0$  such that, for any  $k \in \{1, \dots, K\}$  and almost every  $x, y$  in  $Y_k$ , for any  $i < \varphi_k$ ,

$$|T^i(x) - T^i(y)| \leq C |T^{\varphi_k}(x) - T^{\varphi_k}(y)|.$$

3. There exists  $\tau > 1$  such that, for any  $k \in \{1, \dots, K\}$  and almost every  $x, y$  in  $Y_k$ :

$$|T^{\varphi_k}(x) - T^{\varphi_k}(y)| \geq \tau |x - y|.$$

4.  $\sum_{k=1}^K \varphi_k \lambda(Y_k) < \infty$ .

If  $T$  can be modelled by a Young tower, then, on the tower, there exists a unique  $\bar{T}$ -invariant probability measure  $\bar{\nu}$  which is absolutely continuous with respect to  $m$ . Hence, there exists a unique  $T$ -invariant measure  $\nu$  which is absolutely continuous with respect to the measure  $\lambda$  (see [8], Proposition 1.3.18). This measure is the image measure of  $\bar{\nu}$  by the projection  $\pi$  and is supported by

$$\Lambda = \bigcup_{n \geq 0} T^n(Y).$$

Let  $\bar{Y}$  be the basis of the tower, that is  $\bar{Y} = \{(y, 0), y \in Y\}$ . Let  $\varphi_{\bar{Y}}$  be the function from  $\bar{Y}$  to  $\{\varphi_k\}_{k \in \{1, \dots, K\}}$  such that  $\varphi_{\bar{Y}}((y, 0)) = \varphi_Y(y)$ . By definition of  $\bar{T}$  one gets  $\bar{T}^{\varphi_k}(\Delta_{k,0}) = \bar{Y}$ . In addition, the quantity  $\bar{\nu}(\{(y, 0) \in \bar{Y} : \varphi_{\bar{Y}}((y, 0)) > k\})$  is exactly of the same order as  $\lambda(\{y \in Y : \varphi_Y(y) > k\})$  (see [8], Proposition 1.1.24).

On the tower, one defines the distance  $s$  as follows:  $s(x, y) = 0$  if  $x$  and  $y$  do not belong to the same partition element  $\Delta_{k,i}$ . If  $x = (a, i)$  and  $y = (b, i)$  belong to the same  $\Delta_{k,i}$  (meaning that  $a$  and  $b$  belong to  $Y_k$ ), then  $\delta(x, y) = \beta^{s(x,y)}$  for  $\beta = 1/\tau$ , where  $s(x, y)$  is the smallest integer  $n$  such that  $S^n(a)$  and  $S^n(b)$  are not in the same  $Y_j$ .

Because of Item 3, we know that  $|S'| \geq \tau > 1$ , so that  $S$  is uniformly expanding. For  $x = (a, i)$  and  $y = (b, i)$  in  $\Delta_{k,i}$ , one has

$$|\pi(x) - \pi(y)| = |T^i(a) - T^i(b)| \leq C |T^{\varphi_k}(a) - T^{\varphi_k}(b)|$$

by Item 2. Since  $T^{\varphi_k} = S$  on  $Y_k$ , and since  $|S'| \geq \tau$ , it follows that

$$|\pi(x) - \pi(y)| \leq C \beta^{s(x,y)-1} \leq \frac{C}{\beta} \beta^{s(x,y)}.$$

Now, if  $x$  and  $y$  do not belong to the same partition element  $\Delta_{k,i}$ , then  $|\pi(x) - \pi(y)| \leq \beta^{s(x,y)} = 1$ . It follows that there exists a positive constant  $K$  such that

$$|\pi(x) - \pi(y)| \leq K \beta^{s(x,y)},$$

meaning that  $\pi$  is Lipschitz with respect to the distance  $\delta$ .

Among the maps that can be modelled by a Young tower, we shall consider the maps defined as follows.

**Definition 4.1.** One says that the map  $T$  can be modelled by a Young tower with polynomial tails of the return times of order  $1/\gamma$  with  $\gamma \in (0, 1)$  if  $\lambda(\{y \in Y : \varphi_Y(y) > k\}) \leq Ck^{-1/\gamma}$ .

Let us briefly describe some properties of such maps. For  $\alpha \in (0, 1]$ , let  $\delta_\alpha = \delta^\alpha$ , let  $L_\alpha$  be the space of Lipschitz functions with respect to  $\delta_\alpha$ , and let

$$L_\alpha(f) = \sup_{x, y \in X} \frac{|f(x) - f(y)|}{\delta_\alpha(x, y)}. \quad (4.1)$$

For any positive real  $a$ , let  $L_{\alpha, a}$  be the set of functions such that  $L_\alpha(f) \leq a$ .

Denote by  $P$  the Perron-Frobenius operator of  $\bar{T}$  with respect to  $\bar{\nu}$ : for any bounded measurable functions  $\varphi, \psi$ ,

$$\bar{\nu}(\varphi \cdot \psi \circ \bar{T}) = \bar{\nu}(P(\varphi)\psi).$$

Let  $T$  be a map that can be modelled by a Young tower with polynomial tails of the return times of order  $1/\gamma$ . Then one can prove that (see [13] and Lemma 2.2 in [7]): for any  $m \geq 1$  and any  $\alpha \in (0, 1]$ , there exists  $C_\alpha > 0$  such that, for any  $\psi \in L_\alpha$ ,

$$|P^m(\psi)(x) - P^m(\psi)(y)| \leq C_\alpha \delta_\alpha(x, y) L_\alpha(\psi). \quad (4.2)$$

Moreover, starting from the results by Gouëzel [8], we shall prove in Proposition 5.3 of the appendix that, for any  $\alpha \in (0, 1]$  there exists  $K_\alpha > 0$  such that

$$\bar{\nu}\left(\sup_{f \in L_{\alpha, 1}} |P^n(f) - \bar{\nu}(f)|\right) \leq \frac{K_\alpha}{n^{(1-\gamma)/\gamma}}. \quad (4.3)$$

A well known example of map which can be modelled by a Young tower with polynomial tails of the return times is the intermittent map  $T_\gamma$  introduced by Liverani *et al.* [12]: for  $\gamma \in (0, 1)$ ,

$$T_\gamma(x) = \begin{cases} x(1 + 2^\gamma x^\gamma) & \text{if } x \in [0, 1/2[ \\ 2x - 1 & \text{if } x \in [1/2, 1]; \end{cases} \quad (4.4)$$

For this map,  $\lambda$  is the Lebesgue measure on  $[0, 1]$  and one can take  $Y = ]1/2, 1]$ . Let  $x_0 = 1$ , and define recursively  $x_{n+1} = T_\gamma^{-1}(x_n) \cap [0, 1/2]$ . One can prove that  $x_n = \frac{1}{2}(\gamma n)^{-1/\gamma}$ . Let then  $y_n = T_\gamma^{-1}(x_{n-1}) \cap ]1/2, 1]$ . The  $y_k$ 's are built in such a way that  $Y_k = ]y_{k+1}, y_k]$  is the set of points  $y$  in  $Y$  for which  $T_\gamma^k(Y_k) = Y$ . One can verify, by controlling explicitly the distortion, that the items 1, 2 and 3 are satisfied with  $\varphi_k = k$ . Item 4 follows from the fact that  $\sum_{k=1}^{\infty} k \lambda(Y_k) \leq C \sum_{k=1}^{\infty} k k^{-(\gamma+1)/\gamma} < \infty$ , since  $\gamma \in (0, 1)$ . Moreover, one has

$$\lambda(\{y \in Y : \varphi_Y(y) > k\}) = \sum_{i=k+1}^{\infty} \lambda(Y_i) \leq Ck^{-1/\gamma},$$

so that the tail of the return times is of order  $1/\gamma$ .

## 4.2 Moment and deviation inequalities for the empirical process

For any  $q \in [2, \infty[$ , let

$$D_{n, q} = \left( \int_0^1 |G_n(t)|^q dt \right)^{1/q}, \quad (4.5)$$

where  $G_n$  is defined by

$$G_n(t) = \sum_{k=1}^n (\mathbf{1}_{T^k \leq t} - \nu([0, t])), \quad t \in [0, 1]. \quad (4.6)$$

Applying Lemma 1 in [4], we see that

$$\frac{1}{n}D_{n,q} = \sup_{f \in W_{q',1}} \left| \frac{1}{n} \sum_{k=1}^n (f(T^k) - \nu(f)) \right|,$$

where  $q' = q/(q-1)$  and  $W_{q',1}$  is the Sobolev ball

$$W_{q',1} = \left\{ f : \int_0^1 |f'(x)|^{q'} dx \leq 1 \right\}. \quad (4.7)$$

Consequently, a moment inequality on  $D_{n,q}$  provides a concentration inequality of the empirical measure of  $\{T, T^2, \dots, T^n\}$  around  $\nu$ , on a class of smooth functions. Note that, the class  $W_{q',1}$  is larger as  $q$  increases, and always contains the class of Lipschitz functions with Lipschitz constant 1.

In what follows, we shall denote by  $\|\cdot\|_{p,\nu}$  the  $\mathbb{L}^p$ -norm on  $([0,1], \nu)$

**Theorem 4.1.** *Let  $T$  be a map that can be modelled by a Young tower with polynomial tails of the return times of order  $1/\gamma$  with  $\gamma \in (0, 1/2)$ , and let  $p_\gamma = 2(1-\gamma)/\gamma$ . For  $q \in [2, \infty[$  let  $D_{n,q}$  be defined by (4.5). Then, there exists a positive constant  $C$  such that for any  $n \geq 1$ ,*

$$\left\| \max_{1 \leq k \leq n} D_{k,q} \right\|_{p_\gamma, \nu} \leq C\sqrt{n}.$$

As a consequence of Theorem 4.1, for any  $\gamma \in (0, 1/2)$  and any positive real  $x$ ,

$$\nu \left( \max_{1 \leq k \leq n} D_{k,q} \geq x\sqrt{n} \right) \leq \frac{C}{x^{2(1-\gamma)/\gamma}}.$$

In addition, proceeding as at the beginning of page 872 of the paper [1], we infer that, under the assumptions of Theorem 4.1, for any real  $p > 2(1-\gamma)/\gamma$ , there exists a positive constant  $C$  such that, for any  $n \geq 1$ ,

$$\left\| \max_{1 \leq k \leq n} D_{k,q} \right\|_{p,\nu} \leq Cn^{(\gamma p + \gamma - 1)/(\gamma p)}.$$

Let us examine now the case where  $\gamma \geq 1/2$ .

**Theorem 4.2.** *Let  $T$  be map that can be modelled by a Young tower with polynomial tails of the return times of order  $1/\gamma$  with  $\gamma \in [1/2, 1)$ . For  $q \in [2, \infty[$ , let  $D_{n,q}$  be defined by (4.5).*

1. *There exists a positive constant  $C$  such that for any  $n \geq 1$ ,*

$$\left\| \max_{1 \leq k \leq n} D_{n,q} \right\|_{1/\gamma, \nu} \leq C(n \log n)^\gamma.$$

2. *If  $p > 1/\gamma$ , then there exists a positive constant  $C$  such that for any  $n \geq 1$ ,*

$$\left\| \max_{1 \leq k \leq n} D_{k,q} \right\|_{p,\nu} \leq Cn^{(\gamma p + \gamma - 1)/(\gamma p)}.$$

For the optimality of the moment bounds of Theorems 4.1 and 4.2, we refer the paper by Melbourne and Nicol [14] and to the recent paper by Gouëzel and Melbourne [10]. Since, for  $q \geq 2$ , the class  $W_{q',1}$  contains the class of Lipschitz functions with Lipschitz constant 1, one can apply Proposition 1.1 and 1.2 in [10], showing that these bounds are optimal. See also Remark 4.1 below for more comments about the optimality.

**Theorem 4.3.** *Let  $T$  be map that can be modelled by a Young tower with polynomial tails of the return times of order  $1/\gamma$  with  $\gamma \in (1/2, 1)$ . For  $q \in [2, \infty[$ , let  $D_{n,q}$  be defined by (4.5). Then, there exists a positive constant  $C$  such that for any  $n \geq 1$  and any positive real  $x$ ,*

$$\nu \left( \max_{1 \leq k \leq n} D_{k,q} \geq xn^\gamma \right) \leq Cx^{-1/\gamma}. \quad (4.8)$$

Applying Theorem 4.3, one gets for  $p \in [1, 1/\gamma[$ ,

$$\left\| \max_{1 \leq k \leq n} D_{k,q} \right\|_{p,\nu}^p = p \int_0^\infty x^{p-1} \nu \left( \max_{1 \leq k \leq n} D_{k,q} \geq x \right) dx \leq p \int_0^{n^\gamma} x^{p-1} dx + Cnp \int_{n^\gamma}^\infty \frac{1}{x^{1+\gamma^{-1}-p}} dx.$$

Consequently, for  $p \in [1, 1/\gamma[$ , there exists a positive constant  $C$  such that

$$\left\| \max_{1 \leq k \leq n} D_{k,q} \right\|_{p,\nu} \leq Cn^\gamma.$$

**Remark 4.1.** *Inequality (4.8) cannot hold for  $\gamma = 1/2$ . Indeed, for the map  $T_\gamma$  defined in (4.4), Item 1 of Theorem 1.1 in [2] implies that, for any positive real  $x$ ,*

$$\lim_{n \rightarrow \infty} \nu \left( \frac{1}{\sqrt{n \log n}} D_{n,2} > x \right) = \mathbb{P}(|N| > x) > 0,$$

where  $N$  is a real-valued centered Gaussian random variable with positive variance. In addition, for  $\gamma \in (1/2, 1)$ , Item 2 of the same paper implies that

$$\lim_{n \rightarrow \infty} \nu \left( \frac{1}{n^\gamma} D_{n,2} > t \right) = \mathbb{P}(|Z_\gamma| > t) > 0,$$

where  $Z_\gamma$  is an  $1/\gamma$ -stable random variable such that  $\lim_{x \rightarrow \infty} x^{1/\gamma} \mathbb{P}(|Z_\gamma| > x) = c > 0$ .

### 4.3 Application to the Wasserstein metric between the empirical measure and the invariant measure

Let us give an application of the results of Section 4.2 to the Wasserstein distance between the empirical measure of  $\{T, T^2, \dots, T^n\}$  and the invariant distribution  $\nu$ . Recall that Wasserstein distance  $W_1$  between two probability measures  $\nu_1$  and  $\nu_2$  on  $[0, 1]$  is defined as

$$W_1(\nu_1, \nu_2) = \inf \left\{ \int |x - y| \mu(dx, dy), \mu \in \mathcal{M}(\nu_1, \nu_2) \right\}.$$

where  $\mathcal{M}(\nu_1, \nu_2)$  is the set of probability measures on  $[0, 1] \times [0, 1]$  with margins  $\nu_1$  and  $\nu_2$ . Recall also that, in this one dimensional setting,

$$W_1(\nu_1, \nu_2) = \int_0^1 |F_{\nu_1}(t) - F_{\nu_2}(t)| dt,$$

where  $F_{\nu_1}$  and  $F_{\nu_2}$  are the distribution functions of  $\nu_1$  and  $\nu_2$  respectively. Therefore, setting

$$\nu_n = \frac{1}{n} \sum_{i=1}^n \delta_{T^i}$$

we get that for any  $q \geq 2$ ,

$$W_1(\nu_n, \nu) \leq \frac{1}{n} D_{n,q}.$$

The following corollary is a direct consequence of the results of Section 4.2.

**Corollary 4.1.** *Let  $T$  be map that can be modelled by a Young tower with polynomial tails of the return times of order  $1/\gamma$  with  $\gamma \in (0, 1)$ .*

1. *If  $\gamma \in (0, 1/2)$ , then  $\|W_1(\nu_n, \nu)\|_{p,\nu}^p \ll n^{-(1-\gamma)/\gamma}$  for any  $p \geq 2(1-\gamma)/\gamma$ .*
2. *If  $\gamma \in [1/2, 1)$ , then*

$$\|W_1(\nu_n, \nu)\|_{p,\nu}^p \ll \begin{cases} n^{-(1-\gamma)/\gamma} \log n & \text{if } p = 1/\gamma \\ n^{-(1-\gamma)/\gamma} & \text{if } p > 1/\gamma. \end{cases}$$

3. If  $\gamma \in (1/2, 1)$ , then, for any  $n \geq 1$  and any positive real  $x$ ,

$$\nu(W_1(\nu_n, \nu) \geq x n^{\gamma-1}) \ll x^{-1/\gamma}.$$

In their Theorem 1.4, Gouëzel and Melbourne [10] obtain general bounds for the moment of separately Lipschitz functionals of  $(T, T^2, \dots, T^n)$ , where  $T$  is a (non necessarily one-dimensional) map that can be modelled by a Young tower with polynomial tails of the return times.

As a consequence of their results, one gets the same inequalities as in Corollary 4.1 but for the quantity  $W_1(\nu_n, \nu) - \mathbb{E}(W_1(\mu_n, \nu))$  instead of  $W_1(\nu_n, \nu)$ . Note that the upper bounds for  $W_1(\nu_n, \nu) - \mathbb{E}(W_1(\mu_n, \nu))$  are valid if  $T$  is nonuniformly expanding from  $\mathcal{X}$  to  $\mathcal{X}$ , where  $\mathcal{X}$  can be any bounded metric space.

The two results are not of the same nature. However, in our one dimensional setting, the moments bounds of Corollary 4.1 imply the same moment bounds for  $W_1(\nu_n, \nu) - \mathbb{E}(W_1(\mu_n, \nu))$ , because  $(\mathbb{E}(W_1(\nu_n, \nu)))^p \leq \|W_1(\mu_n, \nu)\|_p^p$ . The same remark does not hold for the deviation bounds, which are not directly comparable.

To conclude this section, let us mention that there is no hope to extend Corollary 4.1 to higher dimension with the same bounds. To see this, let us consider the case of  $\mathbb{R}^d$ -valued random variables  $(X_1, X_2, \dots, X_n)$  that are bounded, independent, and identically distributed. Let  $\nu_n$  be the empirical measure of  $\{X_1, X_2, \dots, X_n\}$  and  $\nu$  be the common distribution of the  $X_i$ 's. It is well known that, when  $d \geq 3$  and  $\nu$  has a component which is absolutely continuous with respect to the Lebesgue measure,  $\mathbb{E}(W_1(\nu_n, \nu))$  is exactly of order  $n^{-1/d}$ , which is much slower than  $n^{-1/2}$ .

#### 4.4 Moment and deviation inequalities for partial sums

In this section, we assume that  $T$  is a nonuniformly expanding map on  $(\mathcal{X}, \lambda)$  with  $\lambda$  a probability measure on  $\mathcal{X}$ , and that  $T$  can be modelled by a Young tower. Contrary to the previous sections,  $\mathcal{X}$  can be any bounded metric space and not necessarily the unit interval. Let  $f$  be a Hölder continuous function from  $\mathcal{X}$  to  $\mathbb{R}$  and  $S_n(f) = \sum_{i=1}^n (f \circ T^i - \nu(f))$ .

**Theorem 4.4.** *Let  $T$  be map that can be modelled by a Young tower with polynomial tails of the return times of order  $1/\gamma$  with  $\gamma \in (0, 1)$ .*

1. If  $\gamma \in (0, 1/2)$  then  $\left\| \max_{1 \leq k \leq n} |S_k(f)| \right\|_{p, \nu}^p \ll n^{p-(1-\gamma)/\gamma}$  for any  $p \geq 2(1-\gamma)/\gamma$ .

2. If  $\gamma \in [1/2, 1)$ , then

$$\left\| \max_{1 \leq k \leq n} |S_k(f)| \right\|_{p, \nu}^p \ll \begin{cases} n \log n & \text{if } p = 1/\gamma \\ n^{p-(1-\gamma)/\gamma} & \text{if } p > 1/\gamma. \end{cases}$$

3. If  $\gamma \in (1/2, 1)$ , for any  $n \geq 1$  and any positive real  $x$ ,

$$\nu\left(\max_{1 \leq k \leq n} |S_k(f)| \geq x n^\gamma\right) \ll x^{-1/\gamma}.$$

The proof is omitted since it is a simpler version of the proofs of Theorems 4.1, 4.2 and 4.3. Indeed the norm  $|\cdot|_q$  is replaced by the absolute values and we do not need to deal with the supremum over a subset of the class of Hölder functions of order  $1/q$ .

After this paper was written, we became aware that, using different methods based on martingale approximations, Gouëzel and Melbourne [10] had independently obtained the upper bounds given in Theorem 4.4 (but for  $|S_n(f)|$  instead of  $\max_{1 \leq k \leq n} |S_k(f)|$ ).

As in Section 4.2, applying Propositions 1.1 and 1.2 in [10], we see that the moments bounds of Theorem 4.4 cannot be improved.

Note also that, for the map  $T_\gamma$  defined in (4.4), we can make a similar remark as Remark 4.1: Firstly, Inequality (4.8) cannot hold for  $\gamma = 1/2$ . Indeed by Item 3 page 88 [9], if  $f(0) \neq \nu(f)$ , for any positive real  $x$ ,

$$\lim_{n \rightarrow \infty} \nu \left( \frac{1}{\sqrt{n \log n}} |S_n(f)| > x \right) = \mathbb{P}(|N| > x) > 0,$$

where  $N$  is a real-valued centered Gaussian random variable with positive variance. In addition, for  $\gamma \in (1/2, 1)$ , Theorem 1.3 of the same paper implies that

$$\lim_{n \rightarrow \infty} \nu(|S_n(f)| > xn^\gamma) = \mathbb{P}(|Z_\gamma| > x) > 0,$$

where  $Z_\gamma$  is an  $1/\gamma$ -stable random variable such that  $\lim_{x \rightarrow \infty} x^{1/\gamma} \mathbb{P}(|Z_\gamma| > x) = c > 0$ .

#### 4.5 Proofs of Theorems 4.1, 4.2 and 4.3.

**Proof of Theorem 4.1.** For any  $t$ , let  $f_t$  be the function defined by  $f_t(x) = \mathbf{1}_{x \leq t}$ . Notice first that, for any  $p \geq 1$ ,

$$\begin{aligned} \left\| \max_{1 \leq k \leq n} D_{k,q} \right\|_{p,\nu}^p &= \nu \left( \max_{1 \leq k \leq n} \left| \int_0^1 \left| \sum_{i=1}^k (\mathbf{1}_{T^i \leq t}) - \nu([0, t]) \right|^q dt \right|^{p/q} \right) \\ &= \bar{\nu} \left( \max_{1 \leq k \leq n} \left| \int_0^1 \left| \sum_{i=1}^k (f_t \circ T^i \circ \pi - \bar{\nu}(f_t \circ \pi)) \right|^q dt \right|^{p/q} \right) \\ &= \bar{\nu} \left( \max_{1 \leq k \leq n} \left| \int_0^1 \left| \sum_{i=1}^k (f_t \circ \pi \circ \bar{T}^i - \bar{\nu}(f_t \circ \pi)) \right|^q dt \right|^{p/q} \right). \end{aligned}$$

Let  $g_t := f_t \circ \pi$  and  $G(x) = \{g_t(x), t \in [0, 1]\}$ . Denote by  $|\cdot|_q$  the norm associated to the Banach space  $\mathbb{B} = \mathbb{L}^q([0, 1], dt)$ . With these notations, we then have

$$\left\| \max_{1 \leq k \leq n} D_{k,q} \right\|_{p,\nu}^p = \bar{\nu} \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^k (G(\bar{T}^i) - \bar{\nu}(G(\bar{T}^i))) \right|_q^p \right). \quad (4.9)$$

Let now  $(X_i)_{i \in \mathbb{N}}$  be a stationary Markov chain defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , with state space  $X$ , transition probability  $P$  and invariant distribution  $\bar{\nu}$ . Recall then (see for instance Lemma XI.3 [11]) that for every  $n \geq 1$ , we have the following equalities in law (where in the left-hand side the law is meant under  $\bar{\nu}$  and in the right-hand side the law is meant under  $\mathbb{P}$ )

$$\begin{aligned} (\bar{T}^n, \dots, \bar{T}) &\stackrel{d}{=} (X_1, \dots, X_n) \\ \max_{1 \leq k \leq n} \left| \sum_{i=1}^k (G(\bar{T}^i) - \bar{\nu}(G(\bar{T}^i))) \right|_q &\stackrel{d}{=} \max_{1 \leq k \leq n} \left| \sum_{i=k}^n (G(X_i) - \mathbb{E}(G(X_i))) \right|_q^p. \end{aligned} \quad (4.10)$$

Therefore, starting from (4.9) and using (4.10), we infer that for any real  $p \in [1, \infty[$ ,

$$\begin{aligned} \left\| \max_{1 \leq k \leq n} D_{k,q} \right\|_{p,\nu}^p &= \mathbb{E} \left( \max_{1 \leq k \leq n} \left| \sum_{i=k}^n (G(X_i) - \mathbb{E}(G(X_i))) \right|_q^p \right) \\ &\leq 2^p \mathbb{E} \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^k (G(X_i) - \mathbb{E}(G(X_i))) \right|_q^p \right). \end{aligned} \quad (4.11)$$

Whence, Theorem 4.1 will follow if one can prove that there exists a positive constant  $C$  such that for any  $n \geq 1$ ,

$$\mathbb{E} \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^k (G(X_i) - \mathbb{E}(G(X_i))) \right|_q^{\frac{2(1-\gamma)}{\gamma}} \right) \leq C n^{\frac{1-\gamma}{\gamma}}. \quad (4.12)$$

With this aim, we shall apply the Rosenthal type inequality (5.2) given in Appendix, with  $p = 2(1-\gamma)/\gamma$  (note that  $p > 2$  since  $\gamma \in (0, 1/2)$ ). Letting  $\mathcal{F}_k = \sigma(X_i, i \leq k)$  and  $G^{(0)} = G - \mathbb{E}(G(X_1))$ , this leads to

$$\begin{aligned} \mathbb{E} \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^k (G(X_i) - \mathbb{E}(G(X_i))) \right|_q^{\frac{2(1-\gamma)}{\gamma}} \right) &\ll n \mathbb{E} \left( |G(X_1)|_q^{\frac{2(1-\gamma)}{\gamma}} \right) \\ &+ n \left( \sum_{k=1}^n \frac{1}{k^{1+\delta\gamma/(1-\gamma)}} \left\| \mathbb{E}_0 \left( \left| \sum_{i=1}^k G^{(0)}(X_i) \right|_q^2 \right) \right\|_{(1-\gamma)/\gamma}^\delta \right)^{\frac{(1-\gamma)}{\delta\gamma}}. \end{aligned} \quad (4.13)$$

where  $\delta = \min(1/2, \gamma/(2-4\gamma))$ . To handle the terms  $\left\| \mathbb{E}_0 \left( \left| \sum_{i=1}^k G^{(0)}(X_i) \right|_q^2 \right) \right\|_{(1-\gamma)/\gamma}$  in Inequality (4.13), we shall use Inequality (2.2) which together with Item 1 of Lemma 1.1 leads to

$$\begin{aligned} \mathbb{E}_0 \left( \left| \sum_{i=1}^k G^{(0)}(X_i) \right|_q^2 \right) &\leq 2(2q-3) \sum_{i=1}^k \sum_{\ell=i}^k \mathbb{E}_0(|G^{(0)}(X_i)|_q |\mathbb{E}_i(G^{(0)}(X_\ell))|_q) \\ &\leq 2(2q-3) \sum_{i=1}^k \sum_{\ell=i}^k \mathbb{E}_0(|\mathbb{E}_i(G^{(0)}(X_\ell))|_q), \end{aligned}$$

where for the last inequality, we have used the fact that for any  $i$ ,  $|G^{(0)}(X_i)|_q \leq 1$  almost surely. Hence

$$\left\| \mathbb{E}_0 \left( \left| \sum_{i=1}^k G^{(0)}(X_i) \right|_q^2 \right) \right\|_{(1-\gamma)/\gamma} \leq 2(2q-3) \sum_{i=1}^k \sum_{\ell=i}^k \left\| \mathbb{E}_0(|\mathbb{E}_i(G^{(0)}(X_\ell))|_q) \right\|_{(1-\gamma)/\gamma}. \quad (4.14)$$

Let us now handle the term  $\left\| \mathbb{E}_0(|\mathbb{E}_i(G^{(0)}(X_\ell))|_q) \right\|_{(1-\gamma)/\gamma}$  in Inequality (4.14). With this aim, we first notice that

$$|\mathbb{E}_i(G^{(0)}(X_\ell))|_q^q = \int_0^1 |\mathbb{E}(\mathbf{1}_{\pi(X_\ell) \leq t} | X_i) - \mathbb{E}(\mathbf{1}_{\pi(X_\ell) \leq t})|^q dt$$

Using Lemma 1 in [4], we have

$$\int_0^1 |\mathbb{E}(\mathbf{1}_{\pi(X_\ell) \leq t} | X_i) - \mathbb{E}(\mathbf{1}_{\pi(X_\ell) \leq t})|^q dt = \sup_{h \in W_{q',1}} |P_{\pi(X_\ell)|X_i}(h) - P_{\pi(X_\ell)}(h)|^q,$$

where the Sobolev ball  $W_{q',1}$  is defined in (4.7),  $P_{\pi(X_\ell)|X_i}$  is the conditional distribution of  $\pi(X_\ell)$  given  $X_i$ , and  $P_{\pi(X_\ell)}$  is the distribution of  $\pi(X_\ell)$ . Therefore

$$|\mathbb{E}_i(G^{(0)}(X_\ell))|_q = \sup_{h \in W_{q',1}} |P_{\pi(X_\ell)|X_i}(h) - P_{\pi(X_\ell)}(h)| = \sup_{h \in W_{q',1}} |P_{X_\ell|X_i}(h \circ \pi) - P_{X_\ell}(h \circ \pi)|,$$

where  $P_{X_\ell|X_i}$  is the conditional distribution of  $X_\ell$  given  $X_i$ , and  $P_{X_\ell}$  is the distribution of  $X_\ell$ . Notice now that if  $f \in W_{q',1}$  then for any  $x$  and  $y$  in  $[0, 1]$ ,

$$|f(x) - f(y)| = \left| \int_x^y f'(t) dt \right| \leq |x - y|^{1/q} \left( \int_0^1 |f'(x)|^{q'} dx \right)^{1/q'}.$$



Therefore,

$$W_{q',1} \subset H_{1/q,1},$$

where  $H_{1/q,1}$  is the set of functions that are  $1/q$ -Hölder with Hölder constant 1. It follows that, for any  $h \in W_{q',1}$ , there exists a positive constant  $C$  such that

$$|h \circ \pi(x) - h \circ \pi(y)| \leq |\pi(x) - \pi(y)|^{1/q} \leq C\delta_{1/q}(x, y),$$

proving that  $h \circ \pi$  belongs to the set  $L_{1/q,C}$  defined right after (4.1). Let now

$$f_{\ell-i,h}(x) := |P_{X_\ell|X_i=x}(h \circ \pi) - P_{X_\ell}(h \circ \pi)| = |P^{\ell-i}(h \circ \pi)(x) - \bar{\nu}(h \circ \pi)|.$$

Using the triangle inequality, we have

$$|f_{\ell-i,h}(x) - f_{\ell-i,h}(y)| \leq |P^{\ell-i}(h \circ \pi)(x) - P^{\ell-i}(h \circ \pi)(y)|.$$

Since  $h \circ \pi$  belongs to  $L_{1/q,C}$ , the contraction property (4.2) entails that

$$|f_{\ell-i,h}(x) - f_{\ell-i,h}(y)| \leq CC_{1/q}\delta_{1/q}(x, y).$$

Let  $\tilde{C} = CC_{1/q}$ . We have shown that, for any  $h \in W_{q',1}$ ,  $f_{\ell-i,h} \in \mathcal{F}_{\ell-i} \subset L_{1/q,\tilde{C}}$ . Then, setting

$$m_{\ell-i}(x) = \sup_{h \in W_{q',1}} f_{\ell-i,h}(x)$$

we have  $m_{\ell-i}(x) = \sup_{g \in \mathcal{F}_{\ell-i}} g(x)$ . Therefore, if  $m_{\ell-i}(x) \geq m_{\ell-i}(y)$ ,

$$m_{\ell-i}(x) - m_{\ell-i}(y) = g_x(x) - g_y(y) \leq g_x(x) - g_x(y) \leq \tilde{C}\delta_{1/q}(x, y),$$

since  $\mathcal{F}_{\ell-i} \subset L_{1/q,\tilde{C}}$ . So overall,

$$|\mathbb{E}_i(G^{(0)}(X_\ell))|_q - \mathbb{E}|\mathbb{E}_i(G^{(0)}(X_\ell))|_q = m_{\ell-i}(X_i) - \mathbb{E}(m_{\ell-i}(X_i)),$$

with  $m_{\ell-i} \in L_{1/q,\tilde{C}}$ . Next, using (4.3), it follows that there exists a positive constant  $C$  such that, for any  $i \geq 1$ ,

$$\|\mathbb{E}_0(|\mathbb{E}_i(G^{(0)}(X_\ell))|_q) - \mathbb{E}|\mathbb{E}_i(G^{(0)}(X_\ell))|_q\|_1 = \|P^i(m_{\ell-i}) - \bar{\nu}(m_{\ell-i})\|_1 \leq Ci^{-(1-\gamma)/\gamma}. \quad (4.15)$$

Using similar arguments we infer that there exists a positive constant  $C$  such that, for any  $\ell \geq i + 1$ ,

$$\|\mathbb{E}_i(G^{(0)}(X_\ell))|_q\|_1 = \|\mathbb{E}_0(G^{(0)}(X_{\ell-i}))|_q\|_1 \leq \bar{\nu} \left( \sup_{g \in L_{1/q,\tilde{C}}} |P^{\ell-i}(g) - \bar{\nu}(g)| \right) \leq C(\ell - i)^{-(1-\gamma)/\gamma}. \quad (4.16)$$

We control now the quantity  $\sum_{i=1}^k \sum_{\ell=i}^k \|\mathbb{E}_0(|\mathbb{E}_i(G^{(0)}(X_\ell))|_q)\|_{(1-\gamma)/\gamma}$  with the help of (4.15) and (4.16). With this aim, we first write the following decomposition:

$$\begin{aligned} \sum_{i=1}^k \sum_{\ell=i}^k \|\mathbb{E}_0(|\mathbb{E}_i(G^{(0)}(X_\ell))|_q)\|_{(1-\gamma)/\gamma} &\leq \sum_{i=1}^k \sum_{\ell=2i+1}^k \|\mathbb{E}_i(G^{(0)}(X_\ell))|_q\|_{(1-\gamma)/\gamma} \\ &+ \sum_{i=1}^k \sum_{\ell=i}^{2i} \|\mathbb{E}_0(|\mathbb{E}_i(G^{(0)}(X_\ell))|_q) - \mathbb{E}|\mathbb{E}_i(G^{(0)}(X_\ell))|_q\|_{(1-\gamma)/\gamma} + \sum_{i=1}^k \sum_{\ell=i}^{2i} \|\mathbb{E}_i(G^{(0)}(X_\ell))|_q\|_1 \end{aligned}$$

Next, since  $(1 - \gamma)/\gamma > 1$  and for any  $i$ ,  $|G^{(0)}(X_i)|_q \leq 1$  almost surely, we get

$$\begin{aligned} & \sum_{i=1}^k \sum_{\ell=i}^k \|\mathbb{E}_0(|\mathbb{E}_i(G^{(0)}(X_\ell))|_q)\|_{(1-\gamma)/\gamma} \leq \sum_{i=1}^k \sum_{\ell=2i+1}^k \|\mathbb{E}_i(G^{(0)}(X_\ell))|_q\|_1^{\gamma/(1-\gamma)} \\ & + 2^{\frac{1-2\gamma}{1-\gamma}} \sum_{i=1}^k \sum_{\ell=i}^{2i} \|\mathbb{E}_0(|\mathbb{E}_i(G^{(0)}(X_\ell))|_q) - \mathbb{E}|\mathbb{E}_i(G^{(0)}(X_\ell))|_q\|_1^{\gamma/(1-\gamma)} + \sum_{i=1}^k \sum_{\ell=i}^{2i} \|\mathbb{E}_i(G^{(0)}(X_\ell))|_q\|_1. \end{aligned}$$

Therefore, using (4.15) and (4.16), we derive that

$$\begin{aligned} & \sum_{i=1}^k \sum_{\ell=i}^k \|\mathbb{E}_0(|\mathbb{E}_i(G^{(0)}(X_\ell))|_q)\|_{(1-\gamma)/\gamma} \\ & \ll \sum_{i=1}^k \sum_{\ell=2i+1}^k \frac{1}{\ell-i} + \sum_{i=1}^k \sum_{\ell=i}^{2i} \frac{1}{i} + k + \sum_{i=1}^k \sum_{\ell=i+1}^{2i} \frac{1}{(\ell-i)^{\frac{1-\gamma}{\gamma}}} \ll k. \end{aligned} \quad (4.17)$$

So starting from (4.13) and taking into account (4.14), (4.17) and the fact that  $\gamma/(1-\gamma) < 1$ , we get

$$\mathbb{E}\left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k (G(X_i) - \mathbb{E}(G(X_i))) \right|_q^{\frac{2(1-\gamma)}{\gamma}}\right) \ll n + n \left( \sum_{k=1}^n \frac{k^\delta}{k^{1+\delta\gamma/(1-\gamma)}} \right)^{(1-\gamma)/(\delta\gamma)} \ll n^{(1-\gamma)/\gamma},$$

which completes the proof of (4.12) and then of the theorem.  $\diamond$

**Proof of Theorem 4.2.** We keep the same notations as in the proof of Theorem 4.1.

We start by proving Item 1. By (4.11), it suffices to prove that there exists a positive constant  $C$  such that for any  $n \geq 1$ ,

$$\mathbb{E}\left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k (G(X_i) - \mathbb{E}(G(X_i))) \right|_q^{1/\gamma}\right) \leq C n \log n. \quad (4.18)$$

Assume first that  $\gamma = 1/2$ . Applying Inequality (2.3), taking into account the stationarity and the fact that  $|G(X_1) - \mathbb{E}(G(X_1))|_q \leq 1$  almost surely, we derive

$$\mathbb{E}\left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k (G(X_i) - \mathbb{E}(G(X_i))) \right|_q^{1/\gamma}\right) \ll n + n \sum_{k=1}^n \|\mathbb{E}_0(G^{(0)}(X_k))|_q\|_1.$$

Therefore, using (4.16), it follows that

$$\mathbb{E}\left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k (G(X_i) - \mathbb{E}(G(X_i))) \right|_q^{1/\gamma}\right) \ll n + n \sum_{k=1}^n k^{-1}.$$

proving (4.18) in the case  $\gamma = 1/2$ . We turn now to the proof of (4.18) when  $\gamma \in (1/2, 1)$ . With this aim, we apply the moment inequality (with  $p = 1/\gamma$ ) stated in Proposition 5.1. This leads to

$$\mathbb{E}\left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k (G(X_i) - \mathbb{E}(G(X_i))) \right|_q^{1/\gamma}\right) \leq C_\gamma n \sum_{k=0}^{n-1} (k+1)^{(1-2\gamma)/\gamma} \|\mathbb{E}_0(G^{(0)}(X_k))|_q\|_1,$$

where  $C_\gamma$  is a positive constant depending only on  $\gamma$ . Therefore, for any  $\gamma \in (1/2, 1)$  using (4.16), we get

$$\mathbb{E}\left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k (G(X_i) - \mathbb{E}(G(X_i))) \right|_q^{1/\gamma}\right) \leq \tilde{C}_\gamma n \left(1 + \sum_{k=1}^{n-1} k^{-1}\right),$$

proving (4.18) in case  $\gamma \in (1/2, 1)$ . This ends the proof of Item 1.

We turn now to the proof of Item 2. By (4.11), it suffices to prove that, for  $\gamma \in [1/2, 1)$  and  $p > 1/\gamma$ , there exists a positive constant  $C$  such that for any  $n \geq 1$ ,

$$\mathbb{E} \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^k (G(X_i) - \mathbb{E}(G(X_i))) \right|_q^p \right) \leq C n^{p+(\gamma-1)/\gamma}. \quad (4.19)$$

We shall distinguish two cases: ( $p \geq 2$  and  $p > 1/\gamma$ ) or  $p \in ]1/\gamma, 2[$ . We first consider the case where  $p \geq 2$  and  $p > 1/\gamma$ . To prove (4.19), we shall apply Inequality (2.3). Taking into account the stationarity and the fact that  $|G(X_1) - \mathbb{E}(G(X_1))|_q \leq 1$  almost surely, we derive

$$\mathbb{E} \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^k (G(X_i) - \mathbb{E}(G(X_i))) \right|_q^p \right) \ll n^{p/2} \left( \sum_{k=0}^n \|\mathbb{E}_0(G^{(0)}(X_k))\|_q^{2/p} \right)^{p/2}.$$

Next, using (4.16) and the fact that  $2(1-\gamma)/(\gamma p) < 1$ , Inequality (4.19) follows.

We consider now the case where  $p \in ]1/\gamma, 2[$ . Using, once again, the moment inequality stated in Proposition 5.1, we get

$$\mathbb{E} \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^k (G(X_i) - \mathbb{E}(G(X_i))) \right|_q^p \right) \leq C_p n \sum_{k=0}^{n-1} (k+1)^{p-2} \|\mathbb{E}_0(G^{(0)}(X_k))\|_q,$$

where  $C_p$  is a positive constant depending only on  $p$ . Using then (4.16) and the fact that  $p > 1/\gamma$ , (4.19) follows. This ends the proof of the theorem.  $\diamond$

**Proof of Theorem 4.3.** We keep the same notations as in the proof of Theorem 4.1. Notice first that, for any non-negative  $x$ ,

$$\begin{aligned} \nu \left( \max_{1 \leq k \leq n} D_{k,q} \geq x \right) &= \bar{\nu} \left( \max_{1 \leq k \leq n} \left| \int_0^1 \left| \sum_{i=1}^k (f_t \circ T^i \circ \pi - \bar{\nu}(f_t \circ \pi)) \right|^q dt \right|^{1/q} \geq x \right) \\ &= \bar{\nu} \left( \max_{1 \leq k \leq n} \left| \int_0^1 \left| \sum_{i=1}^k (f_t \circ \pi \circ \bar{T}^i - \bar{\nu}(f_t \circ \pi)) \right|^q dt \right|^{1/q} \geq x \right) \\ &= \bar{\nu} \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^k (G(\bar{T}^i) - \bar{\nu}(G(\bar{T}^i))) \right|_q \geq x \right). \end{aligned}$$

According to (4.10),

$$\begin{aligned} \nu \left( \max_{1 \leq k \leq n} D_{k,q} \geq x \right) &= \mathbb{P} \left( \max_{1 \leq k \leq n} \left| \sum_{i=k}^n (G(X_i) - \mathbb{E}(G(X_i))) \right|_q \geq x \right) \\ &\leq \mathbb{P} \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^k (G(X_i) - \mathbb{E}(G(X_i))) \right|_q \geq x/2 \right). \end{aligned}$$

The theorem will then follow if we can prove that, for any positive real  $x$ ,

$$\mathbb{P} \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^k (G(X_i) - \mathbb{E}(G(X_i))) \right|_q \geq 4x \right) \ll n x^{-1/\gamma}. \quad (4.20)$$

To prove this inequality, we shall apply Proposition 5.1 with lag  $[x]$ . Using (4.16), this leads to the following inequality: for any positive real  $x$ ,

$$\mathbb{P} \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^k (G(X_i) - \mathbb{E}(G(X_i))) \right|_q \geq 4x \right) \ll \frac{n}{x^{1/\gamma}} + \frac{n}{x^2} \sum_{k=0}^{[x]} \frac{1}{(k+1)^{(1-\gamma)/\gamma}},$$

and (4.20) follows.  $\diamond$

## 5 Appendix

### 5.1 A Rosenthal-type inequality for stationary sequences

In this section, for the reader convenience, we recall the Rosenthal-type inequality stated in [6] (see Inequality (3.11) therein). This inequality is the extension to Banach-valued random variables of the Rosenthal type inequality given by Merlevède and Peligrad [15].

Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space, and  $\theta : \Omega \mapsto \Omega$  be a bijective bimeasurable transformation preserving the probability  $\mathbb{P}$ . For a  $\sigma$ -algebra  $\mathcal{F}_0$  satisfying  $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$ , we define the nondecreasing filtration  $(\mathcal{F}_i)_{i \in \mathbb{Z}}$  by  $\mathcal{F}_i = \theta^{-i}(\mathcal{F}_0)$ . We shall use the notations  $\mathbb{E}_k(\cdot) = \mathbb{E}(\cdot | \mathcal{F}_k)$ .

Let  $X_0$  be a random variable with values in  $\mathbb{B}$ . Define the stationary sequence  $(X_i)_{i \in \mathbb{Z}}$  by  $X_i = X_0 \circ T^i$ , and the partial sum  $S_n$  by  $S_n = X_1 + X_2 + \dots + X_n$ .

**Theorem 5.1.** *Assume that  $X_0$  belongs to  $\mathbb{L}^p(\mathbb{B})$  where  $(\mathbb{B}, |\cdot|_{\mathbb{B}})$  is a separable Banach space and  $p$  is a real number in  $]2, \infty[$ . Assume that  $X_0$  is  $\mathcal{F}_0$ -measurable. Then, for any  $r \geq 0$ ,*

$$\mathbb{E}\left(\max_{1 \leq j \leq 2^r} |S_j|_{\mathbb{B}}^p\right) \ll 2^r \mathbb{E}(|X_0|_{\mathbb{B}}^p) + 2^r \left(\sum_{k=0}^{r-1} \frac{\|\mathbb{E}_0(|S_{2^k}|_{\mathbb{B}}^2)\|_{p/2}^{\delta}}{2^{2\delta k/p}}\right)^{p/(2\delta)}, \quad (5.1)$$

where  $\delta = \min(1/2, 1/(p-2))$ .

**Remark 5.1.** *The inequality in the above theorem implies that for any positive integer  $n$ ,*

$$\mathbb{E}\left(\max_{1 \leq j \leq n} |S_j|_{\mathbb{B}}^p\right) \ll n \mathbb{E}(|X_0|_{\mathbb{B}}^p) + n \left(\sum_{k=1}^n \frac{1}{k^{1+2\delta/p}} \|\mathbb{E}_0(|S_k|_{\mathbb{B}}^2)\|_{p/2}^{\delta}\right)^{p/(2\delta)}. \quad (5.2)$$

### 5.2 A deviation inequality

The following proposition is adapted from Proposition 4 in [5]. It also extends Proposition 6.1 in [2] to random variables taking values in a separable Banach space belonging to the class  $\tilde{\mathcal{C}}_2(2, \tilde{c}_2)$ .

**Proposition 5.1.** *Let  $Y_1, Y_2, \dots, Y_n$  be  $n$  random variables with values in a separable Banach space  $(\mathbb{B}, |\cdot|_{\mathbb{B}})$  belonging to the class  $\tilde{\mathcal{C}}_2(2, \tilde{c}_2)$ . Assume that  $\mathbb{P}(|Y_k|_{\mathbb{B}} \leq M) = 1$  for any  $k \in \{1, \dots, n\}$ . Let  $\mathcal{F}_1, \dots, \mathcal{F}_n$  be an increasing filtration such that  $Y_k$  is  $\mathcal{F}_k$ -measurable for any  $k \in \{1, \dots, n\}$ . Let  $S_n = \sum_{k=1}^n Y_k$ , and for  $k \in \{0, \dots, n-1\}$ , let*

$$\theta(k) = \max\left\{\mathbb{E}(|\mathbb{E}(Y_i | \mathcal{F}_{i-k})|_{\mathbb{B}}), i \in \{k+1, \dots, n\}\right\}. \quad (5.3)$$

*Then, for any  $q \in \{1, \dots, n\}$ , and any  $x \geq qM$ , the following inequality holds*

$$\mathbb{P}\left(\max_{1 \leq k \leq n} |S_k|_{\mathbb{B}} \geq 4x\right) \leq \frac{n\theta(q)}{x} \mathbf{1}_{q < n} + \frac{4\tilde{c}_2 K^2 n M}{x^2} \sum_{k=0}^{q-1} \theta(k), \quad (5.4)$$

where  $K = \sqrt{\max(\tilde{c}_2, 1)}$ . In addition, for any  $p \in [1, 2]$ ,

$$\mathbb{E}\left(\max_{1 \leq k \leq n} |S_k|_{\mathbb{B}}^p\right) \leq \left(4^p p + \frac{4^{p+1} p \tilde{c}_2 K^2}{2-p}\right) M^{p-1} n \sum_{k=0}^{n-1} (k+1) \theta(k). \quad (5.5)$$

**Proof of Proposition 5.1.** Let  $S_0 = 0$  and define the random variables  $U_i$  by:  $U_i = S_{iq} - S_{(i-1)q}$  for  $i \in \{1, \dots, [n/q]\}$  and  $U_{[n/q]+1} = S_n - S_{q[n/q]}$ . By Proposition 4 in [5], for any  $x \geq Mq$ ,

$$\begin{aligned} \mathbb{P}\left(\max_{1 \leq k \leq n} |S_k|_{\mathbb{B}} \geq 4x\right) &\leq \frac{1}{x} \sum_{i=3}^{[n/q]+1} \mathbb{E}(|\mathbb{E}(U_i | \mathcal{F}_{(i-2)q})|_{\mathbb{B}}) + \frac{\tilde{c}_2}{x^2} \sum_{i=1}^{[n/q]+1} \mathbb{E}(|U_i - \mathbb{E}(U_i | \mathcal{F}_{(i-2)q})|_{\mathbb{B}}^2) \\ &\leq \frac{1}{x} \sum_{i=3}^{[n/q]+1} \mathbb{E}(|\mathbb{E}(U_i | \mathcal{F}_{(i-2)q})|_{\mathbb{B}}) + \frac{4\tilde{c}_2}{x^2} \sum_{i=1}^{[n/q]+1} \mathbb{E}(|U_i|_{\mathbb{B}}^2). \end{aligned} \quad (5.6)$$

Since  $(\theta(k))_{k \geq 0}$  is a non-increasing sequence, it is not hard to see that

$$\sum_{i=3}^{[n/q]+1} \mathbb{E}(|\mathbb{E}(U_i | \mathcal{F}_{(i-2)q})|_{\mathbb{B}}) \leq n\theta(q)\mathbf{1}_{q < n}. \quad (5.7)$$

To handle the second term in (5.6), we use Inequality (2.2) with  $p = 2$ . This leads to the following upper bounds: for any  $i \in \{1, \dots, [n/q]\}$ ,

$$\mathbb{E}(|U_i|_{\mathbb{B}}^2) \leq K^2 \sum_{k=(i-1)q+1}^{iq} \sum_{j=k}^{iq} \mathbb{E}(|Y_k|_{\mathbb{B}} |\mathbb{E}(Y_j | \mathcal{F}_k)|_{\mathbb{B}}),$$

and

$$\mathbb{E}(|U_{[n/q]+1}|_{\mathbb{B}}^2) \leq K^2 \sum_{k=q[n/q]+1}^n \sum_{j=k}^n \mathbb{E}(|Y_k|_{\mathbb{B}} |\mathbb{E}(Y_j | \mathcal{F}_k)|_{\mathbb{B}}),$$

where  $K = \sqrt{\max(\tilde{c}_2, 1)}$ . Using the fact that  $\mathbb{P}(|Y_k|_{\mathbb{B}} \leq M) = 1$  for any  $k \in \{1, \dots, n\}$  and that  $(\theta(k))_{k \geq 0}$  is a non-increasing sequence, we then derive that, for any  $i \in \{1, \dots, [n/q]\}$ ,

$$\mathbb{E}(|U_i|_{\mathbb{B}}^2) \leq K^2 M \sum_{k=(i-1)q+1}^{iq} \sum_{j=k}^{iq} \theta(j-k) \leq K^2 M q \sum_{k=0}^{q-1} \theta(k),$$

and

$$\mathbb{E}(|U_{[n/q]+1}|_{\mathbb{B}}^2) \leq K^2 M \sum_{k=q[n/q]+1}^n \sum_{j=k}^n \theta(j-k) \leq K^2 M (n - q[n/q]) \sum_{k=0}^{q-1} \theta(k).$$

Whence

$$\sum_{i=1}^{[n/q]+1} \mathbb{E}(|U_i|_{\mathbb{B}}^2) \leq K^2 M n \sum_{k=0}^{q-1} \theta(k). \quad (5.8)$$

Starting from (5.6) and using the upper bounds (5.7) and (5.8), Proposition 5.1 follows.  $\diamond$

### 5.3 A maximal inequality

**Proposition 5.2.** Let  $n \geq 2$  be an integer and  $Y_1, Y_2, \dots, Y_n$  be  $n$  random variables with values in a separable Banach space  $(\mathbb{B}, |\cdot|_{\mathbb{B}})$ . Assume that  $\mathbb{P}(|Y_k|_{\mathbb{B}} \leq M) = 1$  for any  $k \in \{1, \dots, n\}$ . Let  $\mathcal{F}_1, \dots, \mathcal{F}_n$  be an increasing filtration such that  $Y_k$  is  $\mathcal{F}_k$ -measurable for any  $k \in \{1, \dots, n\}$ . Let  $S_n = \sum_{k=1}^n Y_k$  and  $\theta(k)$  be defined by (5.3). Then, for any real  $p > 1$ , the following inequality holds:

$$\mathbb{E}\left(\max_{1 \leq k \leq n} |S_k|_{\mathbb{B}}^p\right) \leq \frac{1}{2} \left(\frac{2p}{p-1}\right)^p \mathbb{E}(|S_n|_{\mathbb{B}}^p) + 2^{p-1} 3^p p M^{p-1} n \sum_{k=0}^{n-2} (k+1)^{p-2} \theta(k).$$

**Proof of Proposition 5.2.** All along the proof,  $\mathbb{E}_k(\cdot) = \mathbb{E}(\cdot | \mathcal{F}_k)$ . We start by noticing that

$$S_k = \mathbb{E}_k(S_n) + \mathbb{E}_k(S_k - S_n).$$

Therefore

$$\mathbb{E}\left(\max_{1 \leq k \leq n} |S_k|_{\mathbb{B}}^p\right) \leq 2^{p-1} \mathbb{E}\left(\max_{1 \leq k \leq n} |\mathbb{E}_k(S_n)|_{\mathbb{B}}^p\right) + 2^{p-1} \mathbb{E}\left(\max_{1 \leq k \leq n} |\mathbb{E}_k(S_n - S_k)|_{\mathbb{B}}^p\right).$$

Notice now that  $(|\mathbb{E}_k(S_n)|, \mathcal{F}_k)_{1 \leq k \leq n}$  is a submartingale. Therefore by the Doob's maximal inequality,

$$\mathbb{E}\left(\max_{1 \leq k \leq n} |\mathbb{E}_k(S_n)|_{\mathbb{B}}^p\right) \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}(|S_n|_{\mathbb{B}}^p).$$

So, overall,

$$\mathbb{E}\left(\max_{1 \leq k \leq n} |S_k|_{\mathbb{B}}^p\right) \leq 2^{-1} \left(\frac{2p}{p-1}\right)^p \mathbb{E}(|S_n|_{\mathbb{B}}^p) + 2^{p-1} \mathbb{E}\left(\max_{1 \leq k \leq n} |\mathbb{E}_k(S_n - S_k)|_{\mathbb{B}}^p\right).$$

To end the proposition, it remains to prove that

$$\mathbb{E}\left(\max_{1 \leq k \leq n} |\mathbb{E}_k(S_n - S_k)|_{\mathbb{B}}^p\right) \leq 3^p p M^{p-1} n \sum_{k=0}^{n-2} (k+1)^{p-2} \theta(k). \quad (5.9)$$

With this aim, we write

$$\mathbb{E}\left(\max_{1 \leq k \leq n} |\mathbb{E}_k(S_n - S_k)|_{\mathbb{B}}^p\right) = p \int_0^{nM} x^{p-1} \mathbb{P}\left(\max_{1 \leq k \leq n} |\mathbb{E}_k(S_n - S_k)|_{\mathbb{B}} > x\right) dx.$$

Let  $q$  be a non-negative integer such that  $q \leq n$ . Notice that

$$|\mathbb{E}_k(S_n - S_k)|_{\mathbb{B}} = \left| \sum_{i=k+1}^n \mathbb{E}_k(X_i) \right|_{\mathbb{B}} \leq \left| \sum_{i=k+1}^n \mathbb{E}_k(X_i - \mathbb{E}_{i-q}(X_i)) \right|_{\mathbb{B}} + \left| \sum_{i=k+1}^n \mathbb{E}_k(\mathbb{E}_{i-q}(X_i)) \right|_{\mathbb{B}}.$$

But

$$\left| \sum_{i=k+1}^n \mathbb{E}_k(X_i - \mathbb{E}_{i-q}(X_i)) \right|_{\mathbb{B}} = \left| \sum_{i=k+1}^{q+k} (\mathbb{E}_k(X_i) - \mathbb{E}_{i-q}(X_i)) \right|_{\mathbb{B}} \leq 2qM.$$

Therefore, for any real  $x$  such that  $x \in [0, n]$ , choosing  $q = [x]$ , we get

$$\begin{aligned} \mathbb{P}\left(\max_{1 \leq k \leq n} |\mathbb{E}_k(S_n - S_k)|_{\mathbb{B}} > 3Mx\right) &\leq \mathbb{P}\left(\max_{1 \leq k \leq n} \left| \sum_{i=k+1}^n \mathbb{E}_k(\mathbb{E}_{i-[x]}(X_i)) \right|_{\mathbb{B}} > Mx\right) \\ &\leq \mathbb{P}\left(\max_{1 \leq k \leq n} \mathbb{E}_k\left(\sum_{i=2}^n |\mathbb{E}_{i-[x]}(X_i)|_{\mathbb{B}}\right) > Mx\right). \end{aligned}$$

But  $(\mathbb{E}_k(\sum_{i=2}^n |\mathbb{E}_{i-[x]}(X_i)|), \mathcal{F}_k)_{1 \leq k \leq n}$  is a martingale, so the Doob-Kolmogorov's inequality implies

$$\mathbb{P}\left(\max_{1 \leq k \leq n} \mathbb{E}_k\left(\sum_{i=2}^n |\mathbb{E}_{i-[x]}(X_i)|_{\mathbb{B}}\right) > Mx\right) \leq \frac{1}{Mx} \sum_{i=2}^n \mathbb{E}(|\mathbb{E}_{i-[x]}(X_i)|_{\mathbb{B}}) \leq \frac{n\theta([x])}{Mx}.$$

So, overall,

$$\begin{aligned} \mathbb{E}\left(\max_{1 \leq k \leq n} |\mathbb{E}_k(S_n - S_k)|_{\mathbb{B}}^p\right) &= p(3M)^p \int_0^{n/3} x^{p-1} \mathbb{P}\left(\max_{1 \leq k \leq n} |\mathbb{E}_k(S_n - S_k)|_{\mathbb{B}} > 3Mx\right) dx \\ &\leq 3^p p M^{p-1} n \int_0^{n/3} x^{p-2} \theta([x]) dx, \end{aligned}$$

proving (5.9) by using the fact that  $(\theta(k))_k$  is a non-increasing sequence. The proof of the proposition is therefore complete.  $\diamond$

## 5.4 Proof of Inequality (2.3)

Proposition 5.2 together with Inequality (2.2) leads to

$$\begin{aligned} \mathbb{E}\left(\max_{1 \leq k \leq n} |S_k|_{\mathbb{B}}^p\right) &\leq 2^{-1} \left(\frac{2p}{p-1}\right)^p K^p \left(\sum_{i=1}^n \max_{i \leq \ell \leq n} \left\| |X_i|_{\mathbb{B}} \left| \sum_{k=i}^{\ell} \mathbb{E}(X_k | \mathcal{F}_i) \right|_{\mathbb{B}} \right\|_{p/2}\right)^{p/2} \\ &\quad + 2^{p-1} 3^p p M^{p-1} n \sum_{k=0}^{n-2} (k+1)^{p-2} \theta(k). \end{aligned} \quad (5.10)$$

Since  $\mathbb{P}(|X_k|_{\mathbb{B}} \leq M) = 1$  for any  $k \in \{1, \dots, n\}$ , it follows that

$$\sum_{i=1}^n \max_{i \leq \ell \leq n} \left\| |X_i|_{\mathbb{B}} \left| \sum_{k=i}^{\ell} \mathbb{E}(X_k | \mathcal{F}_i) \right|_{\mathbb{B}} \right\|_{p/2} \leq n M^{2-2/p} \sum_{k=0}^{n-1} \theta^{2/p}(k). \quad (5.11)$$

On the other hand, since  $(\theta(k))_{k \geq 1}$  is non-increasing,

$$\sum_{k=1}^{n-2} (k+1)^{p-2} \theta(k) = \sum_{\ell=0}^{\log_2(n-1)-1} \sum_{k=2^\ell}^{2^{\ell+1}-1} (k+1)^{p-2} \theta(k) \leq 2^{p-2} \sum_{\ell=0}^{\log_2(n-1)} 2^{\ell(p-1)} \theta(2^\ell).$$

Hence, using the fact that  $p \geq 2$  and again that  $(\theta(k))_{k \geq 1}$  is non-increasing, we successively derive

$$\begin{aligned} \sum_{k=1}^{n-2} (k+1)^{p-2} \theta(k) &\leq 2^{p-2} \left( \sum_{\ell=0}^{\log_2(n-1)} 2^{\ell(2-2/p)} \theta^{2/p}(2^\ell) \right)^{p/2} \\ &\leq 2^{p-2} \left( \theta^{2/p}(1) + 2 \sum_{\ell=1}^{\log_2(n-1)} \sum_{k=2^{\ell-1}+1}^{2^\ell} 2^{\ell(1-2/p)} \theta^{2/p}(2^\ell) \right)^{p/2} \leq 2^{2p-3} \left( \sum_{k=1}^{n-1} k^{1-2/p} \theta^{2/p}(k) \right)^{p/2}. \end{aligned}$$

Since  $p \geq 2$ , it follows that

$$\sum_{k=1}^{n-2} (k+1)^{p-2} \theta(k) \leq 2^{2p-3} n^{p/2-1} \left( \sum_{k=1}^{n-1} k^{1-2/p} \theta^{2/p}(k) \right)^{p/2}. \quad (5.12)$$

Starting from (5.10) and considering the upper bounds (5.11) and (5.12), the inequality (2.3) follows.  $\diamond$

## 5.5 Dependence properties of Young towers

In this section, we assume that  $T$  is a nonuniformly expanding map on  $(\mathcal{X}, \lambda)$  with  $\lambda$  a probability measure on  $\mathcal{X}$ , and that  $T$  can be modelled by a Young tower. As in Section 4.4,  $\mathcal{X}$  can be any bounded metric space and not necessarily the unit interval.

**Proposition 5.3.** *Let  $T$  be map that can be modelled by a Young tower with polynomial tails of the return times of order  $1/\gamma$  with  $\gamma \in (0, 1)$ . Then the inequality (4.3) holds, that is: for any  $\alpha \in (0, 1]$  there exists  $K_\alpha > 0$  such that*

$$\bar{\nu} \left( \sup_{f \in L_{\alpha,1}} |P^n(f) - \bar{\nu}(f)| \right) \leq \frac{K_\alpha}{n^{(1-\gamma)/\gamma}}.$$

**Proof of Proposition 5.3.** The proof is a slight modification of the proof of Theorem 2.3.6 in [8] and is included here for the sake of completeness. In this proof,  $C$  is a positive constant, and  $C_\alpha$  is a positive constant depending only on  $\alpha$ . Both constants may vary from line to line.

We keep the same notations as in Subsection 4.1. For  $f \in L_\alpha$ , let

$$\|f\|_{L_\alpha} = L_\alpha(f) + \|f\|_\infty.$$

Let  $f^{(0)} = f - \bar{\nu}(f)$ . Since  $\|f^{(0)}\|_\infty \leq L_\alpha(f)$ , it follows that

$$\|f - \bar{\nu}(f)\|_{L_\alpha} \leq 2L_\alpha(f). \quad (5.13)$$

Recall that one has the decomposition

$$P^n f = \sum_{a+k+b=n} \lambda_b(f) A_a(\mathbf{1}_{\bar{Y}}) + \sum_{a+k+b=n} A_a E_k B_b f + C_n f, \quad (5.14)$$

where the operators  $A_n$ ,  $B_n$ ,  $C_n$  and  $E_n$  and are defined in Chapter 2 of Gouëzel's PhD thesis [8] and  $\lambda_b(f) = \bar{\nu}(B_b(f))$ . In particular, Gouëzel has proved that

$$\|E_k f\|_{L_\alpha} \leq \frac{C_\alpha \|f\|_{L_\alpha}}{(k+1)^{(1-\gamma)/\gamma}} \quad \text{and} \quad \|B_k f\|_{L_\alpha} \leq \frac{C_\alpha \|f\|_{L_\alpha}}{(k+1)^{1/\gamma}}. \quad (5.15)$$

Following the proof of Lemma 2.3.5 in [8], there exists a set  $Z_n$  such that, for any bounded measurable function  $g$ ,

$$|C_n(g)| \leq C \|g\|_\infty \mathbf{1}_{Z_n}, \quad (5.16)$$

and

$$\bar{\nu}(Z_n) \leq \frac{C}{(n+1)^{(1-\gamma)/\gamma}}. \quad (5.17)$$

We now turn to the term  $\sum_{a+k+b=n} A_a E_k B_b f$  in (5.14). Following the proof of Lemma 2.3.3 in [8], there exist a set  $U_n$  such that, for any bounded measurable function  $g$ ,

$$|A_n(g)| \leq C \|g\|_\infty \mathbf{1}_{U_n}, \quad (5.18)$$

and

$$\bar{\nu}(U_n) \leq \frac{C}{(n+1)^{1/\gamma}}. \quad (5.19)$$

Using successively (5.18) and (5.15), we obtain that

$$\begin{aligned} \left| \sum_{a+k+b=n} A_a E_k B_b f \right| &\leq C \sum_{a+k+b=n} \|E_k B_b f\|_\infty \mathbf{1}_{U_a} \\ &\leq C_\alpha \sum_{a+k+b=n} \|B_b f\|_{L_\alpha} \frac{\mathbf{1}_{U_a}}{(k+1)^{(1-\gamma)/\gamma}} \\ &\leq C_\alpha \|f\|_{L_\alpha} \sum_{a+k+b=n} \frac{\mathbf{1}_{U_a}}{(k+1)^{(1-\gamma)/\gamma} (b+1)^{1/\gamma}}. \end{aligned} \quad (5.20)$$

We now turn to the term  $\sum_{a+k+b=n} A_a(\mathbf{1}_{\bar{Y}}) \cdot \bar{\nu}(B_b f)$  in (5.14). From the last equality of (2.21) in [8], if  $\bar{\nu}(f) = 0$ ,

$$\begin{aligned} \left| \sum_{b=0}^{n-a} \bar{\nu}(B_b f) \right| &= \left| \sum_{b>n-a} \bar{\nu}(B_b f) \right| \leq \sum_{b>n-a} \|B_b f\|_{L_\alpha} \leq \sum_{b>n-a} \frac{C_\alpha \|f\|_{L_\alpha}}{(b+1)^{1/\gamma}} \\ &\leq \frac{C_\alpha \|f\|_{L_\alpha}}{(n+1-a)^{(1-\gamma)/\gamma}}. \end{aligned} \quad (5.21)$$



From (5.21) and (5.18), if  $\bar{\nu}(f) = 0$ ,

$$\left| \sum_{a=0}^n A_a(\mathbf{1}_{\bar{Y}}) \cdot \left( \sum_{b=0}^{n-a} \bar{\nu}(B_b f) \right) \right| \leq C_\alpha \|f\|_{L_\alpha} \sum_{a=0}^n \frac{\mathbf{1}_{U_a}}{(n+1-a)^{(1-\gamma)/\gamma}}. \quad (5.22)$$

From (5.13),  $\|f - \nu(f)\|_{L_\alpha} \leq 2L_\alpha(f)$ . Hence, it follows from (5.14), (5.16), (5.20) and (5.22) that

$$|P^n(f - \nu(f))| \leq C_\alpha L_\alpha(f) \left( \mathbf{1}_{Z_n} + \sum_{a=0}^n \frac{\mathbf{1}_{U_a}}{(n+1-a)^{(1-\gamma)/\gamma}} + \sum_{a+k+b=n} \frac{\mathbf{1}_{U_a}}{(k+1)^{(1-\gamma)/\gamma} (b+1)^{1/\gamma}} \right). \quad (5.23)$$

From (5.23), (5.17) and (5.19), it follows that

$$\begin{aligned} \bar{\nu} \left( \sup_{f \in L_{\alpha,1}} |P^n(f) - \bar{\nu}(f)| \right) &\leq C_\alpha \left( \frac{1}{(n+1)^{(1-\gamma)/\gamma}} + \sum_{a=0}^n \frac{1}{(a+1)^{1/\gamma} (n+1-a)^{(1-\gamma)/\gamma}} \right. \\ &\quad \left. + \sum_{a+k+b=n} \frac{1}{(a+1)^{1/\gamma} (k+1)^{(1-\gamma)/\gamma} (b+1)^{1/\gamma}} \right). \end{aligned} \quad (5.24)$$

All the sums on right hand being of the same order (see the end of the proof of Proposition 6.2 in [2]), it follows that there exists  $K_\alpha > 0$  such that

$$\bar{\nu} \left( \sup_{f \in L_{\alpha,1}} |P^n(f) - \bar{\nu}(f)| \right) \leq \frac{K_\alpha}{n^{(1-\gamma)/\gamma}},$$

and the proof is complete.  $\diamond$

## 5.6 Proof of Lemma 1.1

We shall prove here that Lemma 1.1 also holds for the derivative in the sense of Fréchet. Hence in the proof  $D$  and  $D^2$  are the first and second derivatives in the sense of Fréchet.

Set  $|x|_q = \left( \int_{\mathcal{X}} |x(t)|^q d\nu(t) \right)^{1/q}$  and observe that for, any  $x$  and  $h$  in  $\mathbb{L}^q$ , by the Taylor integral formula at order 2,

$$\begin{aligned} |x + h|_q^q - |x|_q^q &= q \int_{\mathcal{X}} h(t) |x(t)|^{q-1} \text{sign}(x(t)) \mu(dt) \\ &\quad + q(q-1) \int_{\mathcal{X}} h^2(t) \int_0^1 (1-s) |x(t) + sh(t)|^{q-2} ds \mu(dt). \end{aligned}$$

implying that

$$|x + h|_q^q - |x|_q^q = q \int_{\mathcal{X}} h(t) |x(t)|^{q-2} x(t) \mu(dt) + O(|h|_q^2). \quad (5.25)$$

Define now the function  $\ell$  from  $\mathbb{L}^q$  to  $\mathbb{R}$  by

$$\ell(x) = |x|_q^2.$$

Using (5.25), we derive that, for any  $x$  and  $h$  in  $\mathbb{L}^q$ ,

$$\begin{aligned} \ell(x+h) - \ell(x) &= 2q^{-1} (\ell(x))^{1-q/2} \left( \int_{\mathcal{X}} (|x(t) + h(t)|^q - |x(t)|^q) d\nu(t) \right) + o(|h|_q) \\ &= 2(\ell(x))^{1-q/2} \int_{\mathcal{X}} h(t) |x(t)|^{q-2} x(t) \mu(dt) + o(|h|_q). \end{aligned} \quad (5.26)$$

Therefore  $\ell$  is Fréchet differentiable and

$$D\ell(x)(h) = 2(\ell(x))^{1-q/2} \int_{\mathcal{X}} h(t)|x(t)|^{q-2}x(t)\mu(dt). \quad (5.27)$$

Let us prove that  $\ell$  is two times Fréchet differentiable. Starting from (5.27), we first write that, for any  $x, h, v$  in  $\mathbb{L}^q$ ,

$$\begin{aligned} D\ell(x+v)(h) - D\ell(x)(h) &= 2(\ell(x+v))^{1-q/2} \int_{\mathcal{X}} h(t)|x(t) + v(t)|^{q-2}(x(t) + v(t))\mu(dt) \\ &\quad - 2(\ell(x))^{1-q/2} \int_{\mathcal{X}} h(t)|x(t)|^{q-2}x(t)d\nu(t) \end{aligned}$$

Notice that

$$\begin{aligned} \int_{\mathcal{X}} h(t)|x(t) + v(t)|^{q-2}(x(t) + v(t))\mu(dt) &- \int_{\mathcal{X}} h(t)|x(t)|^{q-2}x(t)\mu(dt) \\ &= (q-1) \int_{\mathcal{X}} h(t)v(t)|x(t)|^{q-2}\mu(dt) + o(|h|_q|v|_q). \end{aligned}$$

Hence

$$\begin{aligned} D\ell(x+v)(h) - D\ell(x)(h) &= 2(q-1)(\ell(x))^{1-q/2} \int_{\mathcal{X}} h(t)v(t)|x(t)|^{q-2}\mu(dt) \\ &\quad + 2(q-1)((\ell(x+v))^{1-q/2} - (\ell(x))^{1-q/2}) \int_{\mathcal{X}} h(t)v(t)|x(t)|^{q-2}\mu(dt) \\ &\quad + 2((\ell(x+v))^{1-q/2} - (\ell(x))^{1-q/2}) \int_{\mathcal{X}} h(t)|x(t)|^{q-2}x(t)\mu(dt) + o(|h|_q|v|_q). \end{aligned}$$

Using (5.26), we infer that

$$((\ell(x+v))^{1-q/2} - (\ell(x))^{1-q/2}) = (2-q)(\ell(x))^{1-q} \int_{\mathcal{X}} v(t)|x(t)|^{q-2}x(t)\mu(dt) + o(|v|_q).$$

So, overall,

$$\begin{aligned} D\ell(x+v)(h) - D\ell(x)(h) &= 2(q-1)(\ell(x))^{1-q/2} \int_{\mathcal{X}} h(t)v(t)|x(t)|^{q-2}\mu(dt) \\ &\quad + 2(2-q)(\ell(x))^{1-q} \int_{\mathcal{X}} v(t)x(t)|x(t)|^{q-2}\mu(dt) \int_{\mathcal{X}} h(t)x(t)|x(t)|^{q-2}\mu(dt) + o(|h|_q|v|_q). \end{aligned}$$

Therefore  $\ell$  is two-times Fréchet differentiable and

$$\begin{aligned} D^2\ell(x)(h, v) &= 2(q-1)(\ell(x))^{1-q/2} \int_{\mathcal{X}} h(t)v(t)|x(t)|^{q-2}\mu(dt) \\ &\quad + 2(2-q)(\ell(x))^{1-q} \int_{\mathcal{X}} v(t)x(t)|x(t)|^{q-2}\mu(dt) \int_{\mathcal{X}} h(t)x(t)|x(t)|^{q-2}\mu(dt). \quad (5.28) \end{aligned}$$

Since  $\psi_p(x) = (\ell(x))^{p/2}$ ,  $\psi_p$  is also two-times Fréchet differentiable. Moreover

$$D\psi_p(x)(h) = 2(\ell(x))^{(p-q)/2} \int_{\mathcal{X}} h(t)|x(t)|^{q-2}x(t)\mu(dt),$$

and

$$\begin{aligned} D^2\psi_p(x)(h, v) &= p(q-1)(\ell(x))^{(p-q)/2} \int_{\mathcal{X}} h(t)v(t)|x(t)|^{q-2}\mu(dt) \\ &\quad + p(p-q)(\ell(x))^{-q+p/2} \int_{\mathcal{X}} v(t)x(t)|x(t)|^{q-2}\mu(dt) \int_{\mathcal{X}} h(t)x(t)|x(t)|^{q-2}\mu(dt). \quad (5.29) \end{aligned}$$

Starting from (5.29) and using the fact that  $\ell(x) = |x|_q^2$ , we get

$$D^2\psi_p(x)(h, v) = p(q-1)|x|_q^{p-q} \int_{\mathcal{X}} h(t)v(t)|x(t)|^{q-2} \mu(dt) \\ + p(p-q)|x|_q^{p-2q} \int_{\mathcal{X}} v(t)x(t)|x(t)|^{q-2} \mu(dt) \int_{\mathcal{X}} h(t)x(t)|x(t)|^{q-2} \mu(dt), \quad (5.30)$$

and an application of Hölder's inequality shows that  $\mathbb{L}^q$  belongs to the class  $\tilde{\mathcal{C}}_2(p, \tilde{c}_p)$  with  $\tilde{c}_p = p(\max(p, 2q-p)-1)$ . To prove that  $\mathbb{L}^q$  belongs to the class  $\mathcal{C}_2(p, c_p)$  with  $c_p = p(\max(p, q)-1)$ , it suffices to write (5.30) with  $h = v$ , and to use the fact that  $(\int_{\mathcal{X}} v(t)|x(t)|^{q-2}x(t)\mu(dt))^2$  is non-negative. This ends the proof of Item 1.

The proof of Item 2 is omitted since it uses the same arguments as for  $\mathbb{L}^2$ .  $\diamond$

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