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C_0 -semigroups and resolvent operators approximated by Laguerre expansions

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Abstract

In this paper we introduce Laguerre expansions to approximate vector-valued functions. We apply this result to approximate C_0 -semigroups and resolvent operators in abstract Banach spaces. We study certain Laguerre functions in order to estimate the rate of convergence of these expansions. Finally, we illustrate the main results of this paper with some examples: shift, convolution and holomorphic semigroups, where the rate of convergence is improved.

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1. Introduction

Representations of functions through series of orthogonal polynomials such as Legendre, Hermite or Laguerre are well known in mathematical analysis and applied mathematics. They

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allow us to approximate functions by series of orthogonal polynomials using different types of convergence: a pointwise way, uniformly, or in Lebesgue norm. Two classical monographs where we can find this kind of results are [21, Chapter 4] and [28, Chapter IX]. In this paper we are concentrated on Laguerre expansions.

Rodrigues’ formula gives a representation of generalized Laguerre polynomials,

$$L_n^{(\alpha)}(t) = e^t \frac{t^{-\alpha}}{n!} \frac{d^n}{dt^n} (e^{-t} t^{n+\alpha}), \quad t \in \mathbb{R},$$

for $\alpha \in \mathbb{R}$ and $n \in \mathbb{N} \cup \{0\}$. The following theorem, which appears in [21, Sec. 4.23] and whose statement was originally proved by J.V. Uspensky in [30], gives a pointwise approximation for scalar functions in terms of Laguerre polynomials. As we prove in [Theorem 3.3](#), this result also holds for vector-valued functions in abstract Banach spaces.

Theorem 1.1. *Let $\alpha > -1$ and $f : (0, \infty) \rightarrow \mathbb{C}$ be a differentiable function such that the integral $\int_0^\infty e^{-t} t^\alpha |f(t)|^2 dt$ is finite. Then the series $\sum_{n=0}^\infty c_n(f) L_n^{(\alpha)}(t)$ converges pointwise to $f(t)$ for $t > 0$, where*

$$c_n(f) := \frac{n!}{\Gamma(n + \alpha + 1)} \int_0^\infty e^{-t} t^\alpha f(t) L_n^{(\alpha)}(t) dt.$$

There exists a large amount of results about Laguerre expansions: for example, Laguerre expansions of analytic functions are considered in [26]; the decay of coefficients is also studied in [31] in connection with the other properties of the function given by their Laguerre expansions; and the algebraic structure related to the Laguerre expansions is covered in detail in [19].

In particular, we apply [Theorem 1.1](#) to the function e_a (where $e_a(t) := e^{-at}$) in order to show that

$$e^{-at} = \sum_{n=0}^\infty \frac{a^n}{(a + 1)^{n+\alpha+1}} L_n^{(\alpha)}(t), \quad a > 0, \tag{1.1}$$

and the Laguerre expansion converges pointwise for $t > 0$.

Through Laguerre polynomials, Laguerre functions are defined by

$$\mathcal{L}_n^{(\alpha)}(t) := \sqrt{\frac{n!}{\Gamma(n + \alpha + 1)}} t^{\frac{\alpha}{2}} e^{-\frac{t}{2}} L_n^{(\alpha)}(t), \quad t > 0,$$

for $\alpha > -1$. These functions form an orthonormal basis in the Hilbert space $L^2(\mathbb{R}_+)$. Furthermore, let f be in $L^p(\mathbb{R}_+)$, $\frac{4}{3} < p < 4$, and $a_k(f) := \int_0^\infty \mathcal{L}_k^{(\alpha)}(t) f(t) dt$ for $k \in \mathbb{N} \cup \{0\}$. Then $\|S_n(f) - f\|_p \rightarrow 0$ as $n \rightarrow \infty$, with

$$S_n(f)(t) := \sum_{k=0}^n a_k(f) \mathcal{L}_k^{(\alpha)}(t), \quad t > 0,$$

see [3, Theorem 1].

On the other hand, a C_0 -semigroup $(T(t))_{t \geq 0}$ is a one parameter family of linear and bounded operators on a Banach space X which may be interpreted, approximately, as $(e^{-tA})_{t \geq 0}$. The (densely defined) operator $-A$, defined by

$$-Ax := \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t}, \quad \text{when the limit exists, } x \in D(A),$$

is called the infinitesimal generator of $(T(t))_{t \geq 0}$. See more details in Section 4 and in monographs [2,14]. It seems natural to consider the formula (1.1) in the context of C_0 -semigroups; we prove that

$$T(t)x = \sum_{n=0}^{\infty} A^n (A + 1)^{-n-\alpha-1} x L_n^{(\alpha)}(t), \quad x \in D(A),$$

in Theorem 4.1(iii). The rate of convergence of $\sum_{n=0}^m A^n (A + 1)^{-n-\alpha-1} x L_n^{(\alpha)}(t)$ to $T(t)x$ when $m \rightarrow \infty$ is estimated in Theorem 4.4.

In semigroup theory, there are different types of approximations for C_0 -semigroups such as Trotter–Kato theorem, Yosida or Euler approximations. In a recent paper [17], a new functional calculus is introduced to improve some rate of these approximations. Also, Padé approximations and the rate of convergence to the semigroup have been covered in [7,12,25]. Note that Padé approximants are closer to generalized Laguerre polynomials [7, p. 688] but both approaches are essentially different. In [7, Section 2], the authors give rational approximations for C_0 -semigroups whenever the rational function is A -acceptable. We denote by r_m the partial sum of Laguerre expansion for the exponential function e_{-1} , i.e.,

$$r_m(z) := \sum_{n=0}^m (1/2)^{n+\alpha+1} L_n^{(\alpha)}(-z).$$

Note that the function r_m is a rational function and $e^z = \lim_{m \rightarrow \infty} r_m(z)$ for $\Re z < 0$. However, the function r_m is not bounded on the half plane $\{\Re z < 0\}$, therefore r_m is not A -acceptable for $m \geq 1$ and the approximation results shown in [7] are not applied in this context.

The approximation of C_0 -semigroups by resolvent series has been considered in [18] using some analytic functional calculus given by integration of exponential functions. In this paper, authors use the Laguerre polynomials $(L_n^{(-1)})_{n \geq 0}$ and they provide the following representation,

$$T(t) = x + \sum_{n=1}^{\infty} A^n (A + 1)^{-n} x L_n^{(-1)}(t), \quad x \in D(A),$$

which has not been proved in this paper. However, both representations are linked as Remark 4.2 shows. Note that the rate of convergence of partial sums of Laguerre expansions to the C_0 -semigroup for $\alpha = -1$ is slightly sharper than ours, see Remark 4.5. To obtain it, the authors of [18] use advanced techniques of harmonic analysis such as Carlson’s inequality and Fourier multipliers. Our approach is based on a different point of view: the extension of Uspensky’s theorem to the Banach space setting, i.e., Theorem 3.3. The proof of this theorem illustrates the importance of geometry of the underlying Banach space, see in particular Remark 3.4. Moreover, in Theorem 5.2(ii), we considerably improve the rate of convergence of the expansions for holomorphic semigroups, by making it dependent on the index α .

The paper is organized as follows. In Section 2, we consider the functions $t \mapsto \frac{n!}{\Gamma(n+\alpha+1)} e^{-t} t^\alpha L_n^{(\alpha)}(t)$ (for $\alpha > -1$ and $n \in \mathbb{N} \cup \{0\}$) which play a key role in Theorem 1.1. They satisfy interesting properties (similar to Laguerre polynomials, see Propositions 2.2 and 2.3). We estimate their L^p -norms in Theorem 2.4 as well as the L^p -norms of their Laplace transforms in Theorem 2.7.

In Section 3, we consider certain Laguerre expansions in Lebesgue spaces $L^p(\mathbb{R}_+)$ and abstract Banach spaces. In particular, we obtain the Laguerre expansions for the fractional

semigroup and for the exponential function e_a in $L^1(\mathbb{R}_+)$ (Theorem 3.2). To finish this section, we prove a vector-valued version of Theorem 1.1 on an abstract Banach space X .

Main applications of Theorem 3.3 appear in Sections 4 and 5. In Theorem 4.1, we express C_0 -semigroup and resolvent operators through Laguerre expansions. We apply Theorem 4.6 to express the fractional powers of the resolvent operator of the semigroup generators as a binomial type series. In Theorem 4.4, we give the rate of convergence of the Laguerre expansion to the C_0 -semigroup, which depends on the regularity of the initial data.

In the last section, we present some examples of C_0 -semigroups and its Laguerre expansion: shift semigroup, convolution semigroups and in particular, Gauss and Poisson semigroups in Lebesgue spaces. For some differentiable and analytic semigroups, previous results are improved.

Throughout the paper, we work with functions and operators defined in \mathbb{R}_+ , but there are also other results on expansions of functions defined in \mathbb{R} using Hermite polynomials. Hermite polynomials are defined by Rodrigues' formula

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} (e^{-t^2}) \quad \text{for } t \in \mathbb{R} \text{ and } n \geq 0.$$

A similar result to Theorem 1.1 holds for functions defined on \mathbb{R} and involves Hermite polynomials, see [21, Sec. 4.15], in particular for $\lambda \in \mathbb{C}$,

$$e^{\lambda t} = e^{\frac{\lambda^2}{4}} \sum_{n=0}^{\infty} \frac{\lambda^n}{2^n n!} H_n(t), \quad t \in \mathbb{R}.$$

In article [1], we studied vector-valued approximations defined by Hermite polynomials, and their applications to C_0 -groups and cosine operator families.

Notation. We write $\mathbb{R}_+ := [0, \infty)$. We call $C_0(\mathbb{R}_+)$ the set of continuous functions defined in $[0, \infty)$ such that $\lim_{t \rightarrow \infty} f(t) = 0$, with the norm $\| \cdot \|_{\infty}$; and $\mathcal{H}_0(\mathbb{C}_+)$ the Banach of bounded holomorphic functions defined in $\mathbb{C}_+ = \{z \in \mathbb{C} : \Re z > 0\}$ such that $\lim_{z \rightarrow \infty} f(z) = 0$, with the norm $\|f\|_{\infty} := \sup_{z \in \mathbb{C}_+} |f(z)|$. Furthermore, note the following equivalence which appears several times throughout the paper: for $\alpha > -1$ there exist $c_1 < c_2$ such that

$$\frac{c_1}{n^\alpha} \leq \frac{n!}{\Gamma(n + \alpha + 1)} \leq \frac{c_2}{n^\alpha}, \quad n \in \mathbb{N}. \quad (1.2)$$

2. Laguerre functions and their Laplace transform

Generalized Laguerre polynomials $\{L_n^{(\alpha)}\}_{n \geq 0}$ ($\alpha \in \mathbb{R} \setminus \{-1, -2, \dots\}$) are given by

$$L_n^{(\alpha)}(t) = \sum_{k=0}^n (-1)^k \binom{n + \alpha}{n - k} \frac{t^k}{k!}, \quad t \geq 0.$$

The generalized Laguerre polynomials are solutions of the second order differential equation

$$t y'' + (\alpha + 1 - t) y' + n y = 0.$$

For $\alpha > -1$, they satisfy the following condition of orthogonality:

$$\frac{n!}{\Gamma(n + \alpha + 1)} \int_0^\infty e^{-t} t^\alpha L_n^{(\alpha)}(t) L_m^{(\alpha)}(t) dt = \delta_{n,m},$$

for $n, m = 0, 1, 2, \dots$, with $\delta_{n,m}$ is the Kronecker delta. The following identities will be crucial for the sequel:

$$\begin{aligned} nL_n^{(\alpha)}(t) &= (n + \alpha)L_{n-1}^{(\alpha)}(t) - tL_{n-1}^{(\alpha+1)}(t), \\ tL_n^{(\alpha+1)}(t) &= (n + \alpha)L_{n-1}^{(\alpha)}(t) - (n - t)L_n^{(\alpha)}(t), \\ nL_n^{(\alpha)}(t) &= (\alpha + 1 - t)L_{n-1}^{(\alpha+1)}(t) - tL_{n-2}^{(\alpha+2)}(t), \\ 0 &= (n + 1)L_{n+1}^{(\alpha)}(t) + (t - \alpha - 2n - 2)L_n^{(\alpha)}(t) + (n + \alpha)L_{n-1}^{(\alpha)}(t), \end{aligned}$$

see for example [21,28].

As consequence of Muckenhoupt’s estimates, see [24] and [29, Lemma 1.5.3], we have

$$\|\mathcal{L}_n^{(0)}\|_1 \sim \sqrt{n}, \quad n \geq 1, \tag{2.1}$$

$$|L_n^{(\alpha)}(\lambda)| \leq C_\lambda n^{\frac{\alpha}{2}}, \quad n \geq n_0, \tag{2.2}$$

for $\lambda > 0$, see [23, Lemma 1] and [29, Lemma 1.5.4.(i)]. Similar results can be found in [13, Lemma 5.1] and [17, Theorem A.1].

In this paper, we are mainly interested in the following functions.

Definition 2.1. For $\alpha > -1$ and $n \in \mathbb{N} \cup \{0\}$, we denote by $\ell_n^{(\alpha)}$ the function defined in $(0, \infty)$ by

$$\ell_n^{(\alpha)}(t) := \frac{n!}{\Gamma(n + \alpha + 1)} t^\alpha e^{-t} L_n^{(\alpha)}(t).$$

Note that $\ell_n^{(\alpha)}(t) = \frac{1}{\Gamma(n+\alpha+1)} \frac{d^n}{dt^n} (e^{-t} t^{n+\alpha})$, for $t > 0$, with $n = 0, 1, 2, \dots$ and $\ell_0^{(\alpha)} = I_{\alpha+1}$, where $(I_s)_{s \in \mathbb{C}_+}$ is the fractional semigroup given by

$$I_s(t) := \frac{1}{\Gamma(s)} e^{-t} t^{s-1}, \quad t > 0, s \in \mathbb{C}_+.$$

In [27, Theorem 2.6] the fractional semigroup $(I_s)_{s \in \mathbb{C}_+}$ is studied in detail; in particular $I_s * I_t = I_{t+s}$ for $s, t \in \mathbb{C}_+$ (i.e. algebraic semigroup) where the convolution product of $f * g$ is defined by

$$(f * g)(t) := \int_0^t f(t - s)g(s)ds, \quad f, g \in L^1(\mathbb{R}_+), t \geq 0.$$

If $M(\mathbb{R}_+)$ denotes the non-negative Borel measures on \mathbb{R}_+ of finite total variation and $\mu \in M(\mathbb{R}_+)$, then the convolution product $f * \mu$ is given by

$$(f * \mu)(t) := \int_0^t f(t - s)d\mu(s), \quad f \in L^1(\mathbb{R}_+), t \geq 0.$$

The Dirac delta δ_0 verifies that $f * \delta_0 = f$ for any $f \in L^1(\mathbb{R}_+)$. We write $f^{*n} = f * f^{*(n-1)}$ for $n \geq 2$ and $f^{*1} = f$. The functions $\{\ell_n^{(\alpha)}\}_{n \geq 0}$ satisfy the following algebraic property (for convolution product $*$).

Proposition 2.2. For $\alpha > -1$, $n \in \mathbb{N}$, and $e_1(t) := e^{-t}$ for $t > 0$, we have that

$$\ell_n^{(\alpha)} = (\delta_0 - e_1)^{*n} * \ell_0^{(\alpha)}. \tag{2.3}$$

Then the equality $\ell_n^{(\alpha)} * \ell_m^{(\beta)} = \ell_{n+m}^{(\alpha+\beta+1)}$ holds for all $\alpha, \beta > -1$ and $n, m \geq 0$.

Proof. Note that $e_1^{*n} = I_n$. Denoting $I_0 := \delta_0$, we have

$$\begin{aligned} (\delta_0 - e_1)^{*n} * \ell_n^{(\alpha)} &= \sum_{k=0}^n (-1)^k \frac{n!}{k!(n-k)!} I_k * I_{\alpha+1} = \sum_{k=0}^n (-1)^k \frac{n!}{k!(n-k)!} I_{k+\alpha+1} \\ &= \frac{n!}{\Gamma(n+\alpha+1)} t^\alpha e^{-t} \sum_{k=0}^n (-1)^k \frac{\Gamma(n+\alpha+1)}{(n-k)! \Gamma(k+\alpha+1)} \frac{t^k}{k!} = \ell_n^{(\alpha)}, \end{aligned}$$

for $\alpha > -1, n \in \mathbb{N}$. The algebraic equality $\ell_n^{(\alpha)} * \ell_m^{(\beta)} = \ell_{n+m}^{(\alpha+\beta+1)}$ for $\alpha, \beta > -1$ and $n, m \geq 0$ is a straightforward consequence of the equality (2.3) and $\ell_0^{(\alpha)} = I_{\alpha+1}$. \square

The functions $\{\ell_n^{(\alpha)}\}_{n \geq 0}$ are rich in algebraic structure since they present recurrence relations, differential equations and some additional identities as the next proposition shows. The proof is left to the reader.

Proposition 2.3. *For $\alpha > -1$ the family of functions $\{\ell_n^{(\alpha)}\}_{n \geq 0}$ satisfies:*

- (i)
$$\ell_n^{(\alpha)}(t) = \ell_{n-1}^{(\alpha)}(t) - \ell_{n-1}^{(\alpha+1)}(t).$$
- (ii)
$$(n + \alpha + 1)\ell_n^{(\alpha+1)}(t) = n\ell_{n-1}^{(\alpha)}(t) - (n - t)\ell_n^{(\alpha)}(t).$$
- (iii)
$$t\ell_n^{(\alpha)}(t) = (\alpha + 1 - t)\ell_{n-1}^{(\alpha+1)}(t) - (n - 1)\ell_{n-2}^{(\alpha+2)}(t).$$
- (iv)
$$(n + \alpha + 1)\ell_{n+1}^{(\alpha)}(t) + (t - \alpha - 2n - 1)\ell_n^{(\alpha)}(t) + n\ell_{n-1}^{(\alpha)}(t) = 0.$$
- (v)
$$t \frac{d^2}{dt^2} \ell_n^{(\alpha)}(t) + (1 - \alpha + t) \frac{d}{dt} \ell_n^{(\alpha)}(t) + (n + 1)\ell_n^{(\alpha)}(t) = 0.$$
- (vi)
$$\frac{d^k}{dt^k} \ell_n^{(\alpha)}(t) = \ell_{n+k}^{(\alpha-k)}(t), \quad \alpha - k > -1.$$

In the next theorem, we present some estimates of L^p -norms for the functions $\{\ell_n^{(\alpha)}\}_{n \geq 0}$.

Theorem 2.4. *Let $\mathcal{Z}(L_n^{(\alpha)})$ be the set of zeros of $L_n^{(\alpha)}$.*

- (i) *For $\alpha > 0$, and $n \geq 0$, the inequality $\|\ell_n^{(\alpha)}\|_\infty \leq \|\ell_{n+1}^{(\alpha-1)}\|_1$ holds.*
- (ii) *The set of functions $\{\ell_n^{(\alpha)}\}_{n \geq 0} \subset L^1(\mathbb{R}_+)$ for $\alpha > -1$, and*

$$\max_{t \in \mathcal{Z}(L_n^{(\alpha)})} |\ell_{n-1}^{(\alpha)}(t)| \leq \|\ell_n^{(\alpha)}\|_1 \leq \frac{C_\alpha}{n^{\frac{\alpha}{2}}}, \quad n \geq 1.$$

- (iii) *The set of functions $\{\ell_n^{(\alpha)}\}_{n \geq 0} \subset L^p(\mathbb{R}_+)$ for $\alpha > -\frac{1}{p}$ with $p \geq 1$, and $\|\ell_n^{(\alpha)}\|_p \leq C_{p,\alpha} n^{\frac{1}{2}}$, for $n \geq 1$.*

Proof. To show (i), it is sufficient to use that

$$|\ell_n^{(\alpha)}(t)| \leq \int_t^\infty \left| \frac{d}{ds} \ell_n^{(\alpha)}(s) \right| ds \leq \|\ell_{n+1}^{(\alpha-1)}\|_1, \quad t > 0,$$

where we have applied Proposition 2.3(vi). The second inequality in (ii) is in [28, (5.7.16)]. Taking into account (i), and due to $\lim_{t \rightarrow 0^+} \ell_{n-1}^{(\alpha+1)}(t) = 0 = \lim_{t \rightarrow \infty} \ell_{n-1}^{(\alpha+1)}(t)$, we obtain that

$$\begin{aligned} \|\ell_n^{(\alpha)}\|_1 &\geq \|\ell_{n-1}^{(\alpha+1)}\|_\infty = \max_{\{t \in \mathbb{R}_+ | (\ell_{n-1}^{(\alpha+1)})'(t) = 0\}} |\ell_{n-1}^{(\alpha+1)}(t)| \\ &= \max_{\{t \in \mathbb{R}_+ | \ell_n^{(\alpha)}(t) = 0\}} |\ell_{n-1}^{(\alpha+1)}(t)| = \max_{\{t \in \mathbb{R}_+ | \ell_n^{(\alpha)}(t) = 0\}} |\ell_{n-1}^{(\alpha)}(t)|, \end{aligned}$$

where we have used Proposition 2.3(vi) and (i) to get the result. The statement (iii) is a direct consequence of Muckenhoupt’s estimates ([24] and [29, Lemma 1.5.3]), see also [23, Lemma 1]. \square

Remark 2.5. By [9, Corollary 2.2], there is a positive constant C_α such that

$$\|\ell_n^{(\alpha)}\|_p \leq C_\alpha n^{2+[\alpha]-\alpha}, \quad \alpha > -1, \quad p \geq 1,$$

where $[\alpha]$ is the integer part of the real number α . For $\alpha = 0$, the bound $|\ell_n^{(0)}(t)| \leq e^{\frac{t}{2}}$ for $t \geq 0$ and $n \in \mathbb{N} \cup \{0\}$ appears in [28, (7.21.3)], and then

$$\|\ell_n^{(0)}\|_p \leq \left(\frac{p}{2}\right)^{\frac{1}{p}}, \quad p > 1.$$

In [28, Theorems 7.6.2 and 7.6.4], [10, Theorem 3] and [22, Theorem 2], other pointwise bounds for Laguerre polynomials are obtained. From them, bounds for $\|\ell_n^{(\alpha)}\|_p$ can be deduced.

Let \mathcal{L} be the Laplace transform, $\mathcal{L} : L^1(\mathbb{R}_+) \rightarrow \mathcal{H}_0(\mathbb{C}_+)$ given by

$$\mathcal{L}(f)(z) := \int_0^\infty e^{-zt} f(t) dt, \quad f \in L^1(\mathbb{R}_+), \quad z \in \mathbb{C}_+.$$

Recall that the Laplace transform is a contraction, $\|\mathcal{L}(f)\|_\infty \leq \|f\|_1$, and $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$ for $f, g \in L^1(\mathbb{R}_+)$. Observe that

$$\mathcal{L}(\ell_n^{(\alpha)})(z) = \frac{z^n}{(z+1)^{n+\alpha+1}}, \quad z \in \mathbb{C}_+, \tag{2.4}$$

see [4, p. 110]. For convenience, we set

$$\varphi_{n,\alpha}(z) := \frac{z^n}{(z+1)^{n+\alpha+1}}, \quad z \in \mathbb{C}_+. \tag{2.5}$$

Lemma 2.6. Let $\varphi_{n,\alpha}$ be defined by (2.5) for $n \in \mathbb{N} \cup \{0\}$ and $\alpha > -1$.

- (i) The equality $\varphi'_{n,\alpha} = n\varphi_{n-1,\alpha+2} - (\alpha+1)\varphi_{n,\alpha+1}$, holds for $n \geq 1$.
- (ii) For $j \geq 1$, we have that

$$\varphi_{n,\alpha}^{(j)} = \sum_{l=0}^{\min(j,n)} \frac{n!}{(n-l)!} b_{l,\alpha} \varphi_{n-l,\alpha+j+l},$$

where $b_{l,\alpha}$ is a real number dependent on l and α .

Proof. To prove (i) it suffices to note that

$$\varphi'_{n,\alpha}(z) = \frac{nz^{n-1}}{(z+1)^{n+\alpha+1}} - \frac{(n+\alpha+1)z^n}{(z+1)^{n+\alpha+2}} = n\varphi_{n-1,\alpha+2}(z) - (\alpha+1)\varphi_{n,\alpha+1}(z),$$

for $z \in \mathbb{C}_+$ and $n \geq 1$. To show (ii), we firstly consider $1 \leq j < n$. We prove the equality by induction in j . If $j = 1$, the equality coincides with the one proved in (i). Taking $j \leq n - 1$ and using (i), we get the following:

$$\begin{aligned} \varphi_{n,\alpha}^{(j+1)} &= \left(\sum_{l=0}^j \frac{n!}{(n-l)!} b_{l,\alpha} \varphi_{n-l,\alpha+j+l} \right)' = \sum_{l=0}^j \frac{n!}{(n-l)!} b_{l,\alpha} \varphi'_{n-l,\alpha+j+l} \\ &= \sum_{l=1}^{j+1} \frac{n!}{(n-l)!} b_{l-1,\alpha} \varphi_{n-l,\alpha+j+l+1} - \sum_{l=0}^j \frac{n!}{(n-l)!} b_{l,\alpha} (\alpha+j+l+1) \varphi_{n-l,\alpha+j+l+1} \\ &= \sum_{l=0}^{j+1} \frac{n!}{(n-l)!} \widetilde{b}_{l,\alpha} \varphi_{n-l,\alpha+j+l+1}, \end{aligned}$$

and the identity follows in this case. If $j \geq n$, then $\varphi'_{0,\alpha} = -(\alpha+1)\varphi_{0,\alpha+1}$, hence

$$\varphi_{n,\alpha}^{(j)} = \sum_{l=0}^n \frac{n!}{(n-l)!} b_{l,\alpha} \varphi_{n-l,\alpha+j+l},$$

and (ii) holds, where we have applied induction in j again. \square

For $j \in \mathbb{N}$, denote by $AC^{(j)}$ the Sobolev Banach space obtained as the completion of the Schwartz class $\mathcal{S}(\mathbb{R}_+)$ (the set of restrictions to $[0, \infty)$ of the Schwartz class $\mathcal{S}(\mathbb{R})$), in the norm

$$\|f\|_{(j)} := \frac{1}{(j-1)!} \int_0^\infty |f^{(j)}(x)|x^{j-1} dx, \quad f \in \mathcal{S}(\mathbb{R}_+),$$

for more details see [9, Definition 1.1] and [16]. Note that the following continuous embeddings hold:

$$(AC^{(j+1)}, \|\cdot\|_{(j+1)}) \hookrightarrow (AC^{(j)}, \|\cdot\|_{(j)}) \hookrightarrow (C_0(\mathbb{R}_+), \|\cdot\|_\infty),$$

see [16, Proposition 3.1.(i)]. These function spaces were considered by several authors, see [16] and references therein.

Theorem 2.7. *Let $\alpha > -1$.*

(i) *For $1 \leq p < \infty$, $\alpha > 1/p - 1$ one has $\varphi_{n,\alpha} \in L^p(\mathbb{R}_+)$ for each $n \geq 0$, and*

$$\|\varphi_{n,\alpha}\|_p^p = \frac{\Gamma(np+1)\Gamma(p\alpha+p-1)}{\Gamma(p(n+\alpha+1))}.$$

(ii) *For each $n \geq 1$, there exists $N_\alpha > M_\alpha > 0$ such that $\frac{M_\alpha}{n^{\frac{\alpha+1}{2}}} \leq \|\varphi_{n,\alpha}\|_\infty \leq \frac{N_\alpha}{n^{\frac{\alpha+1}{2}}}$.*

(iii) *For $j \in \mathbb{N}$, $\varphi_{n,\alpha} \in AC^{(j)}$ for each $n \geq 0$, and $\|\varphi_{n,\alpha}\|_{(j)} \leq \frac{C_{j,\alpha}}{n^{\alpha+1}}$ for $n \geq 1$.*

Proof. (i) Note that $\int_0^\infty |\varphi_{n,\alpha}(t)|^p dt = \int_0^\infty \frac{t^{np}}{(t+1)^{p(n+\alpha+1)}} dt = \beta(np+1, p(\alpha+1)-1)$, where β is the Euler's beta function. (ii) By (2.4), $\varphi_{n,\alpha} \in \mathcal{H}_0(\mathbb{C}_+)$, and

$$\|\varphi_{n,\alpha}\|_\infty = \sup_{z \in \mathbb{C}_+} \frac{|z|^n}{|z+1|^{n+\alpha+1}}.$$

Furthermore, $|\varphi_{n,\alpha}(z)| \leq 1$ for all $z \in \mathbb{C}_+$. We apply the Maximum Modulus Theorem to get that

$$\|\varphi_{n,\alpha}\|_\infty = \max_{x \in \mathbb{R}} \frac{|ix|^n}{|ix + 1|^{n+\alpha+1}} = \max_{x \in \mathbb{R}} \frac{|x|^n}{(x^2 + 1)^{\frac{n+\alpha+1}{2}}}.$$

Define $g(t) := \frac{t^n}{(t^2+1)^{\frac{n+\alpha+1}{2}}}$ for $t > 0$ and note that

$$\max_{t \geq 0} g(t) = (\alpha + 1)^{\frac{\alpha+1}{2}} \frac{n^{\frac{n}{2}}}{(n + \alpha + 1)^{\frac{n+\alpha+1}{2}}}.$$

Therefore the statement (ii) holds. (iii) It is enough to consider $n \geq j$. We apply Lemma 2.6(ii) to get that

$$\begin{aligned} \|\varphi_{n,\alpha}\|_{(j)} &\leq \frac{1}{(j-1)!} \sum_{l=0}^j \frac{n!}{(n-l)!} |b_{l,\alpha}| \int_0^\infty \frac{t^{n+j-l-1}}{(1+t)^{n+\alpha+j+1}} dt \\ &\leq C_{j,\alpha} \frac{\Gamma(\alpha + j + 1)}{(j-1)! \Gamma(n + \alpha + j + 1)} \sum_{l=0}^j \frac{n!(n+j-l-1)!}{(n-l)!} \\ &\leq (j+1) \tilde{C}_{j,\alpha} \frac{(n+j-1)!}{\Gamma(n + \alpha + j + 1)} \leq \frac{C_{j,\alpha}}{n^{\alpha+1}}, \end{aligned}$$

where we use the inequality (1.2). \square

Remark 2.8. Observe that if $\alpha > 0$, $\varphi_{n,\alpha} \in L^1(\mathbb{R}_+)$ for each $n \geq 0$, and

$$\mathcal{L}(\varphi_{n,\alpha})(\lambda) = n! \lambda^{\frac{\alpha-1}{2}} e^{\frac{\lambda}{2}} W_{-n-\frac{(\alpha+1)}{2}, -\frac{\alpha}{2}}(\lambda), \quad \lambda \in \mathbb{C}_+$$

where $W_{k,\mu}$ is the Whittaker function, see [4, p. 24].

3. Laguerre expansions in Banach spaces

In this section, we study Laguerre expansions on Lebesgue spaces $L^p(\mathbb{R}_+)$ and on abstract Banach spaces X . First of all, we show in Theorem 3.1(ii) that the span generated by the Laguerre functions $\{\ell_n^{(\alpha)}\}_{n \geq 0}$ is dense in $L^p(\mathbb{R}_+)$ for $\alpha > -1/p$. The fractional semigroup $\{I_\alpha\}_{\alpha > 0}$ and exponential functions $\{e_a\}_{a > 0}$ may be expressed via Laguerre expansions, see Theorem 3.2. These expansions will be applied to obtain various series representations of operator families arising in semigroup theory and dealt with in the next section. Following the original proof of J.V. Uspensky, we prove a vector-valued theorem of Laguerre expansions for continuous functions (Theorem 3.3).

Theorem 3.1. Let $1 \leq p < \infty$.

(i) For $\alpha > -\frac{1}{p}$, $\lambda > 0$, and $\xi_{\alpha,\lambda}(t) := t^\alpha e^{-\lambda t}$ for $t > 0$, one has

$$\xi_{\alpha,\lambda+1} = \sum_{n=0}^\infty \frac{\lambda^n}{(\lambda + 1)^{n+\alpha+1}} \frac{\Gamma(n + \alpha + 1)}{n!} \ell_n^{(\alpha)} \quad \text{in } L^p(\mathbb{R}_+).$$

(ii) The linear span of $\{\ell_n^{(\alpha)} \mid n \geq 0\}$ is dense in $L^p(\mathbb{R}_+)$ if $\alpha > -\frac{1}{p}$.

Proof. (i) Note that for all $\lambda > 0$, $e^{-\lambda t} = \sum_{n=0}^{\infty} \frac{\lambda^n}{(\lambda+1)^{n+\alpha+1}} L_n^{(\alpha)}(t)$ pointwise, see (1.1). Then

$$t^\alpha e^{-(\lambda+1)t} = \sum_{n=0}^{\infty} \frac{\lambda^n}{(\lambda+1)^{n+\alpha+1}} \frac{\Gamma(n+\alpha+1)}{n!} \ell_n^{(\alpha)}(t), \quad t > 0.$$

Furthermore this convergence is in $L^p(\mathbb{R}_+)$ since

$$\left\| \sum_{n=0}^{\infty} \frac{\lambda^n}{(\lambda+1)^{n+\alpha+1}} \frac{\Gamma(n+\alpha+1)}{n!} \ell_n^{(\alpha)} \right\|_p \leq C_{p,\alpha} \sum_{n=0}^{\infty} \frac{\lambda^n}{(\lambda+1)^{n+\alpha+1}} n^{\alpha+\frac{1}{2}} < \infty, \quad \lambda > 0,$$

where we have applied Theorem 2.4(iii). (ii) By (i), it is enough to show that $\text{span}\{t^\alpha e^{-(\lambda+1)t} \mid \lambda > 0\}$ is dense in $L^p(\mathbb{R}_+)$ to get the result. Let $f \in L^q(\mathbb{R}_+)$ with $\frac{1}{p} + \frac{1}{q} = 1$ be such that

$$\int_0^\infty f(t) t^\alpha e^{-(\lambda+1)t} dt = 0, \quad \lambda > 0.$$

By Hölder’s inequality, the function $f \xi_{\alpha,1} \in L^1(\mathbb{R}_+)$ and then

$$0 = \int_0^\infty f(t) t^\alpha e^{-(\lambda+1)t} dt = \mathcal{L}(f \xi_{\alpha,1})(\lambda), \quad \lambda > 0.$$

Since the Laplace transform is injective in $L^1(\mathbb{R}_+)$, we conclude that $f = 0$. \square

Theorem 3.2. *The following statements hold.*

- (i) For $2\beta > \alpha > -1$, the equality $I_{\beta+1} = \sum_{n=0}^{\infty} \binom{\alpha-\beta+n-1}{n} \ell_n^{(\alpha)}$ holds in $L^1(\mathbb{R}_+)$. In particular, for $\alpha > 2$, we have that $I_\alpha = \sum_{n=0}^{\infty} \ell_n^{(\alpha)}$ in $L^1(\mathbb{R}_+)$.
- (ii) For all $a > 0$, the equality $e_a = \sum_{n=0}^{\infty} \frac{(a-\frac{1}{2})^n}{(a+\frac{1}{2})^{n+1}} \mathcal{L}_n^{(0)}$ holds in $L^1(\mathbb{R}_+)$.

Proof. To prove (i) we note that $\frac{1}{t^{\alpha-\beta}} = \sum_{n=0}^{\infty} \binom{\alpha-\beta+n-1}{n} \frac{n! \Gamma(\beta+1)}{\Gamma(n+\alpha+1)} L_n^{(\alpha)}(t)$ for all $t > 0$ with $2\beta > \alpha - 1$ (see [21, p. 89]), and then

$$\frac{t^\beta e^{-t}}{\Gamma(\beta+1)} = \sum_{n=0}^{\infty} \binom{\alpha-\beta+n-1}{n} \ell_n^{(\alpha)}(t), \quad t > 0.$$

This convergence is in $L^1(\mathbb{R}_+)$ when $2\beta > \alpha > -1$: we apply Theorem 2.4(ii) to get that

$$\left\| \sum_{n=0}^{\infty} \binom{\alpha-\beta+n-1}{n} \ell_n^{(\alpha)} \right\|_1 \leq \sum_{n=0}^{\infty} C_\alpha \frac{n^{\alpha-\beta-1}}{n^{\frac{\alpha}{2}}}.$$

In order to prove (ii), observe that $\{\mathcal{L}_n^{(0)}\}$ is an orthonormal basis in $L^2(\mathbb{R}_+)$ and $e_a \in L^2(\mathbb{R}_+)$ for $a > 0$. Then, the series $\sum_{n=0}^{\infty} \frac{(a-\frac{1}{2})^n}{(a+\frac{1}{2})^{n+1}} \mathcal{L}_n^{(0)}$ converges to e_a in $L^2(\mathbb{R}_+)$ (in fact in $L^p(\mathbb{R}_+)$ for $\frac{4}{3} < p < 4$, see Introduction and [3]) for $a > 0$:

$$\langle e_a, \mathcal{L}_n^{(0)} \rangle = \int_0^\infty e^{-at} e^{\frac{-t}{2}} L_n^{(0)}(t) dt = \int_0^\infty e^{-(a-\frac{1}{2})t} \ell_n^{(0)}(t) dt = \frac{\left(a - \frac{1}{2}\right)^n}{\left(a + \frac{1}{2}\right)^{n+1}},$$

where we have applied (2.4). This convergence also holds in $L^1(\mathbb{R}_+)$:

$$\left\| \sum_{n=0}^{\infty} \frac{\left(a - \frac{1}{2}\right)^n}{\left(a + \frac{1}{2}\right)^{n+1}} \mathcal{L}_n^{(0)} \right\|_1 \leq \sum_{n=0}^{\infty} \frac{b^n}{\left|a + \frac{1}{2}\right|} \|\mathcal{L}_n^{(0)}\|_1 \leq C \sum_{n=0}^{\infty} \frac{b^n}{\left|a + \frac{1}{2}\right|} \sqrt{n},$$

with $b = \left|\frac{a - \frac{1}{2}}{a + \frac{1}{2}}\right| < 1$ for $a > 0$, where we have applied $\|\mathcal{L}_n^{(0)}\|_1 \sim \sqrt{n}$, see (2.1).

As $e^{-at} = \sum_{n=0}^{\infty} \frac{(a - \frac{1}{2})^n}{(a + \frac{1}{2})^{n+1}} \mathcal{L}_n^{(0)}(t)$ pointwise for $t > 0$ by Theorem 1.1, the statement (ii) follows. \square

Let X be a Banach space, and $f : (0, \infty) \rightarrow X$ be a continuous function. We say that this function f is differentiable at t if the limit

$$\lim_{h \rightarrow 0^+} \frac{1}{h} (f(t+h) - f(t))$$

exists in X . Now, we provide a vector-valued version of Theorem 1.1. The proof is similar to the scalar case shown in [30], with some natural changes in the vector-valued setting. We include a sketch of the proof in order to keep our exposition self-contained.

Theorem 3.3. *Let X be a Banach space and let $f : (0, \infty) \rightarrow X$ be a differentiable function such that the integral $\int_0^\infty e^{-t^\alpha} \|f(t)\|^2 dt$ is finite. Then for every $t > 0$ the series $\sum_{n=0}^\infty c_n(f) L_n^{(\alpha)}(t)$ converges pointwise to $f(t)$, where*

$$c_n(f) = \int_0^\infty \ell_n^{(\alpha)}(t) f(t) dt.$$

Proof. Since $\{\mathcal{L}_n^{(\alpha)}(t)\}_{n \geq 0}$ is an orthonormal basis in $L^2(\mathbb{R}_+)$ and f satisfies the hypothesis, we get that $c_n(f) \in X$. Let $S_m(f)$ be the sum of the first $m + 1$ terms of the series,

$$S_m(f)(t) := \sum_{n=0}^m c_n(f) L_n^{(\alpha)}(t) = \int_0^\infty e^{-y} y^\alpha \varphi_m(t, y) f(y) dy, \quad t > 0,$$

where $\varphi_m(t, y) = \sum_{n=0}^m \frac{n!}{\Gamma(n + \alpha + 1)} L_n^{(\alpha)}(t) L_n^{(\alpha)}(y)$, for $t, y > 0$. Note that $\varphi_m(t, y) = \varphi_m(y, t)$, for $t, y > 0$, and

$$\varphi_m(t, y) = \frac{(m + 1)!}{\Gamma(m + \alpha + 1)} \left(\frac{L_{m+1}^{(\alpha)}(t) L_m^{(\alpha)}(y) - L_{m+1}^{(\alpha)}(y) L_m^{(\alpha)}(t)}{y - t} \right), \quad t \neq y > 0,$$

for $m \geq 0$ [30, p. 611]. Taking into account that

$$S_m(f)(t) - f(t) = \int_0^\infty e^{-y} y^\alpha \varphi_m(t, y) (f(y) - f(t)) dy, \quad t > 0,$$

we prove in the following that $S_m f(t) - f(t) \rightarrow 0$ when $m \rightarrow \infty$ for $t > 0$. Take $a < t < b$ and observe that there exist H, G such that $0 < H < a < b < G$ and

$$\int_0^H y^\alpha e^{-y} \varphi_m^2(y, t) dy < C, \quad \int_G^\infty y^\alpha e^{-y} \varphi_m^2(y, t) dy < C,$$

for a constant $C > 0$, see [30, p. 614, formula (27)].

Take $\varepsilon > 0$. There exist $H, G > 0$ as above such that

$$\left\| \int_0^H e^{-y} y^\alpha \varphi_m(t, y) (f(y) - f(t)) dy \right\| \leq \frac{\varepsilon}{3},$$

$$\left\| \int_G^\infty e^{-y} y^\alpha \varphi_m(t, y) (f(y) - f(t)) dy \right\| \leq \frac{\varepsilon}{3}.$$

Note that

$$\varphi_m(t, y) = \frac{\sqrt{(m+1)(m+\alpha+1)} (ty)^{\frac{-\alpha}{2}-\frac{1}{4}} e^{\frac{t+y}{2}}}{\pi m (t-x)} \left(T_m(t, y) + \frac{U_m(t, y)}{\sqrt{m}} \right),$$

where T_m and U_m are defined and studied in [30, pp. 612–613]. Now, we introduce

$$F(t, y) := \frac{f(y) - f(t)}{y - t}, \quad t \neq y > 0.$$

Observe that as a function of y , $F(t, \cdot)$ is continuous in $(0, \infty)$ for any $t > 0$ and

$$\int_H^G e^{-y} y^\alpha \varphi_m(t, y) (f(y) - f(t)) dy$$

$$= C_m t^{\frac{-\alpha}{2}-\frac{1}{4}} e^{\frac{t}{2}} \int_H^G e^{-\frac{y}{2}} y^{\frac{\alpha}{2}-\frac{1}{4}} \left(T_m(t, y) + \frac{U_m(t, y)}{\sqrt{m}} \right) F(t, y) dy,$$

where $\sup_{m \geq 1} C_m < \infty$. Note that if $m \rightarrow \infty$, then

$$\int_H^G e^{-\frac{y}{2}} y^{\frac{\alpha}{2}-\frac{1}{4}} \frac{U_m(t, y)}{\sqrt{m}} F(t, y) dy \rightarrow 0, \quad \int_H^G e^{-\frac{y}{2}} y^{\frac{\alpha}{2}-\frac{1}{4}} T_m(t, y) F(t, y) dy \rightarrow 0,$$

where we have applied the Riemann–Lebesgue lemma in the second limit, see for example [2, Theorem 1.8.1(c)]. We conclude that

$$\left\| \int_H^G e^{-y} y^\alpha \varphi_m(t, y) (f(y) - f(t)) dy \right\| \leq \frac{\varepsilon}{3},$$

and $\lim_{m \rightarrow \infty} \|S_m(f)(t) - f(t)\| = 0$, for $t > 0$. \square

Remark 3.4. In fact, Theorem 3.3 is proved in [30] under less restrictive conditions on the function f . In UMD spaces, the convergence of the Laguerre expansion for continuous functions could be proved following the original proof given in [30].

On the other hand, a straightforward application of Theorem 1.1 allows us to obtain the weak convergence of the partial series $S_m(f)(t)$ to the function $f(t)$ for $t > 0$.

4. C_0 -semigroups and resolvent operators given by Laguerre expansions

In this section, we are interested in representing C_0 -semigroups and resolvent operators by the series of Laguerre polynomials. To develop this idea, we will apply several results obtained in the previous sections. However, first of all, we recall several basic results from semigroup theory, for further details see monographies [2, 14].

Let $-A$ be a closed operator in a Banach space X with domain $D(A)$. Its resolvent operator $\lambda \rightarrow (\lambda + A)^{-1}$ is analytic on the resolvent set $\rho(-A)$, and

$$\frac{d^n}{d\lambda^n} (\lambda + A)^{-1} = (-1)^n n! (\lambda + A)^{-n-1} \quad \text{for all } n \in \mathbb{N},$$

see [14, p. 240]. We say that a C_0 -semigroup $(T(t))_{t \geq 0}$ is uniformly bounded if $\|T(t)\| \leq M$ for all $t \geq 0$, with M a positive constant. Let $-A$ be the infinitesimal generator of a uniformly bounded C_0 -semigroup $(T(t))_{t \geq 0}$. For $\alpha > 0$ and $\lambda > 0$ we define the fractional powers of the resolvent operator as below

$$(\lambda + A)^{-\alpha} x := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\lambda t} T(t)x \, dt, \quad x \in X, \tag{4.1}$$

see [20, Proposition 11.1]. The Cayley transform of $-A$, i.e., $V := (A - 1)(A + 1)^{-1}$, defines a bounded operator called the cogenerator of the C_0 -semigroup, see for example [13,17]. It is easy to prove that

$$A^n (A + 1)^{-n-\alpha-1} x = \left(\frac{V + 1}{2}\right)^n \left(\frac{1 - V}{2}\right)^{\alpha+1} x, \quad x \in X. \tag{4.2}$$

For a number of identities relating the powers of the cogenerator and generalized Laguerre polynomials we refer to [6, Lemma 4.4] and [5, Lemma 6.7].

Theorem 4.1. *Let $(T(t))_{t \geq 0}$ be a uniformly bounded C_0 -semigroup in a Banach space X with infinitesimal generator $-A$.*

(i) *For $n \in \mathbb{N} \cup \{0\}$ and $\alpha > -1$,*

$$\int_0^\infty \ell_n^{(\alpha)}(t) T(t)x \, dt = A^n (A + 1)^{-n-\alpha-1} x, \quad x \in X.$$

(ii) *For $n \in \mathbb{N} \cup \{0\}$ and $\alpha > 0$,*

$$\int_0^\infty \ell_n^{(\alpha)}(t) (t + A)^{-1} x \, dt = \int_0^\infty \varphi_{n,\alpha}(t) T(t)x \, dt, \quad x \in X.$$

(iii) *For $x \in D(A)$ and $\alpha > -1$,*

$$T(t)x = \sum_{n=0}^\infty A^n (A + 1)^{-n-\alpha-1} x L_n^{(\alpha)}(t), \quad t > 0. \tag{4.3}$$

(iv) *For $x \in D(A)$ and $\alpha > -1$,*

$$T(t)x = \left(\frac{1 - V}{2}\right)^{\alpha+1} \sum_{n=0}^\infty \left(\frac{V + 1}{2}\right)^n x L_n^{(\alpha)}(t), \quad t > 0.$$

(v) *For $x \in X$ and $\alpha > 1$,*

$$(\lambda + A)^{-1} x = \sum_{n=0}^\infty \left(\int_0^\infty \varphi_{n,\alpha}(t) T(t)x \, dt\right) L_n^{(\alpha)}(\lambda), \quad \lambda > 0.$$

Proof. (i) For $\alpha > -1$ and $n \in \mathbb{N} \cup \{0\}$, we have that

$$\int_0^\infty \ell_n^{(\alpha)}(t) T(t)x \, dt = \frac{1}{\Gamma(n + \alpha + 1)} \int_0^\infty \frac{d^n}{dt^n} (e^{-t} t^{n+\alpha}) T(t)x \, dt, \quad x \in X.$$

We integrate by parts to obtain that

$$\int_0^\infty \ell_n^{(\alpha)}(t) T(t)x \, dt = A^n \frac{1}{\Gamma(n + \alpha + 1)} \int_0^\infty e^{-t} t^{n+\alpha} T(t)x \, dt = A^n (A + 1)^{-n-\alpha-1} x,$$

for $x \in D(A^n)$, where we use the formula (4.1). The identity is valid for $x \in X$ using the density of $D(A^n)$ in X .

(ii) Take $x \in X$ and $\alpha > 0$. The integral $\int_0^\infty \ell_n^{(\alpha)}(t)(t + A)^{-1}x dt$ converges due to $\|(t + A)^{-1}\| \leq \frac{M}{t}$ for $t > 0$. Then

$$\begin{aligned} \int_0^\infty \ell_n^{(\alpha)}(t)(t + A)^{-1}x dt &= \frac{1}{\Gamma(n + \alpha + 1)} \int_0^\infty \frac{d^n}{dt^n} (e^{-t}t^{n+\alpha})(t + A)^{-1}x dt \\ &= \frac{1}{\Gamma(n + \alpha + 1)} \int_0^\infty e^{-t}t^{n+\alpha} \left(\int_0^\infty s^n e^{-ts} T(s)x ds \right) dt \\ &= \int_0^\infty \frac{s^n}{(s + 1)^{n+\alpha+1}} T(s)x ds = \int_0^\infty \varphi_{n,\alpha}(s) T(s)x ds. \end{aligned}$$

(iii) For each $x \in D(A)$, the function $T(\cdot)x : \mathbb{R}_+ \rightarrow X$ is differentiable at every point and $\frac{d}{dt} T(t)x = -T(t)Ax$, see [14, Definition 1.2, Chapter II]. In addition note that

$$\int_0^\infty e^{-t}t^\alpha \|T(t)x\|^2 dt \leq M^2 \int_0^\infty e^{-t}t^\alpha \|x\|^2 dt < \infty.$$

Then from Theorem 3.3 it follows that

$$\left\| T(t)x - \sum_{n=0}^m c_n(T(\cdot)x) L_n^{(\alpha)}(t) \right\| \rightarrow 0, \quad m \rightarrow \infty,$$

for all $t > 0$, with

$$c_n(T(\cdot)x) = \int_0^\infty \ell_n^{(\alpha)}(t) T(t)x dt = A^n(A + 1)^{-n-\alpha-1}x,$$

where we have applied (i).

(iv) The proof of (iv) is a straightforward consequence of (iii) and (4.2).

(v) Note that

$$\int_0^\infty e^{-t}t^\alpha \|(t + A)^{-1}\|^2 dt \leq M^2 \int_0^\infty \frac{e^{-t}}{t^{2-\alpha}} dt < \infty$$

if only if $\alpha > 1$. In this case

$$(\lambda + A)^{-1}x = \sum_{n=0}^\infty c_n((\cdot + A)^{-1}x) L_n^{(\alpha)}(\lambda), \quad x \in X,$$

with

$$c_n((\cdot + A)^{-1}x) = \int_0^\infty \ell_n^{(\alpha)}(t)(t + A)^{-1}x dt = \int_0^\infty \varphi_{n,\alpha}(t) T(t)x dt,$$

where we have applied (ii). \square

Remark 4.2. Generalized Laguerre polynomials $\{L_n^{(-1)}\}_{n \geq 0}$ are defined by

$$L_0^{(-1)}(t) = 1, \quad L_n^{(-1)}(t) = -\frac{t}{n} L_{n-1}^{(1)}(t), \quad n \geq 1,$$

see for example [28, Section 5.2]. In [18, Theorem 5.1], another approximation of C_0 -semigroups is presented:

$$T(t) = x + \sum_{n=1}^{\infty} A^n (A + 1)^{-n} x L_n^{(-1)}(t), \quad t > 0, x \in D(A). \tag{4.4}$$

For $t > 0$ and $x \in D(A)$, both approximations, (4.3) for $\alpha = 0$ and (4.4), are equivalent:

$$\begin{aligned} T(t)x &= \sum_{n=0}^{\infty} A^n (A + 1)^{-n-1} x L_n^{(0)}(t) = \sum_{n=0}^{\infty} A^n (A + 1)^{-n} x L_n^{(0)}(t) \\ &\quad - \sum_{n=0}^{\infty} A^{n+1} (A + 1)^{-n-1} x L_n^{(0)}(t) \\ &= x + \sum_{n=1}^{\infty} A^n (A + 1)^{-n} x (L_n^{(0)}(t) - L_{n-1}^{(0)}(t)) = x + \sum_{n=1}^{\infty} A^n (A + 1)^{-n} x L_n^{(-1)}(t), \end{aligned}$$

where the recurrence relation $L_n^{(-1)} = L_n^{(0)} - L_{n-1}^{(0)}$ is used for $n \geq 1$.

For C_0 -semigroups which are not uniformly bounded, we may perturb the infinitesimal generator to obtain uniformly bounded C_0 -semigroups, and in this way, $-A - w$ generates a uniformly bounded C_0 -semigroup, see for example [14, p. 60].

Corollary 4.3. *Let $(T(t))_{t \geq 0}$ be a C_0 -semigroup in a Banach space X with infinitesimal generator $(-A, D(A))$ and exponential bound w , that is, $\|T(t)\| \leq Me^{wt}$. Then for $x \in D(A)$ and $\alpha > -1$,*

$$T(t)x = e^{wt} \sum_{n=0}^{\infty} (A + w)^n (A + w + 1)^{-n-\alpha-1} x L_n^{(\alpha)}(t), \quad t > 0.$$

The approximation theory of C_0 -semigroups has a great importance in many mathematical areas, as PDE, mathematical physics and probability theory, and there is a large number of results on different types of approximation of C_0 -semigroups. A good summary of these different approximations can be found in the recent paper [17]. Subdiagonal Padé approximations to C_0 -semigroups are given in [7,12,25].

In the following, we focus in the rate of the approximation for the C_0 -semigroup by the m th partial sum of the Laguerre series, where,

$$T_{m,\alpha}(t)x := \sum_{n=0}^m A^n (A + 1)^{-n-\alpha-1} x L_n^{(\alpha)}(t), \quad x \in X, t > 0.$$

This approximation is computationally better than other known rational approximations in the following sense. For example, if we consider the Euler approximation,

$$T(t)x = \lim_{m \rightarrow \infty} \left(1 + \frac{t}{m} A \right)^{-m} x, \quad x \in X,$$

to increase the approximation of order m , one has to start to calculate over again $(1 + \frac{t}{m+1} A)^{-1} x$ and then $(1 + \frac{t}{m+1} A)^{-(m+1)} x$. In other hand, one has to calculate $A^{m+1} (A + 1)^{-(m+1)-\alpha-1} x = A^m (A + 1)^{-m-\alpha-1} (A(A + 1)^{-1} x)$ in Laguerre expansions. Similar troubles take place with A -acceptable rational approximations [7] and Padé approximations [12].

Theorem 4.4. Let $(T(t))_{t \geq 0}$ be a uniformly bounded C_0 -semigroup in a Banach space X , $\|T(t)\| \leq M$, with infinitesimal generator $-A$, $\alpha > -1$.

(i) For $x \in D(A^p)$ and $n \geq p$,

$$\|A^n(A + 1)^{-n-\alpha-1}x\| \leq \frac{CM^p}{n^{\frac{\alpha+p}{2}}} \|A^p x\|,$$

with C a positive constant depending on α , p and M .

(ii) For each $t > 0$ there is a $m_0 \in \mathbb{N}$ such that for all integer $2 < p \leq m + 1$ with $m \geq m_0$,

$$\|T(t)x - T_{m,\alpha}(t)x\| \leq \frac{C_{t,p} \|A^p x\|}{m^{\frac{p}{2}-1}} \quad x \in D(A^p),$$

where $C_{t,p}$ is a constant which depends only on $t > 0$ and p .

Proof. (i) We write $B(x) := A^n(A + 1)^{-n-\alpha-1}x = \int_0^\infty \ell_n^{(\alpha)}(t)T(t)x dt$ for $x \in X$, see Theorem 4.1(i). We apply Proposition 2.3(vi) and integrate by parts to obtain that

$$B(x) = \int_0^\infty \frac{d^p}{dt^p}(\ell_{n-p}^{(\alpha+p)}(t))T(t)x dt = \int_0^\infty \ell_{n-p}^{(\alpha+p)}(t)A^p T(t)x dt \quad x \in D(A^p),$$

where we have used that $\frac{d^{p-j}}{dt^{p-j}}(\ell_{n-p}^{(\alpha+p)}(t))_{t=0} = 0$, and $\frac{d^{p-j}}{dt^{p-j}}(\ell_{n-p}^{(\alpha+p)}(t))_{t=\infty} = 0$, for $j = 1, 2, \dots, p$. By Theorem 2.4(ii),

$$\|B(x)\| \leq \frac{C}{n^{\frac{\alpha+p}{2}}} \sup_{t \geq 0} \|A^p T(t)x\| < \frac{CM^p}{n^{\frac{\alpha+p}{2}}} \|A^p x\|,$$

and we obtain the desired result.

(ii) By (2.2) there is a $m_0 \in \mathbb{N}$ such that $|L_m^{(\alpha)}(t)| \leq C_t m^{\frac{\alpha}{2}}$, for all $m \geq m_0$. So, for $m \geq m_0$ and $x \in D(A^p)$ with $p \leq m + 1$, we apply Theorem 4.1(iii) and (i) to get that

$$\begin{aligned} \|T(t)x - T_{m,\alpha}(t)x\| &\leq \sum_{n=m+1}^\infty \|A^n(A + 1)^{-n-\alpha-1}x\| |L_n^{(\alpha)}(t)| \\ &\leq \sum_{n=m+1}^\infty \frac{C_t}{n^{\frac{p}{2}}} \|A^p x\| \leq \frac{C_{t,p} \|A^p x\|}{m^{\frac{p}{2}-1}}, \end{aligned}$$

where we have used the bound of (i). \square

Remark 4.5. By [18, Theorem 5.1], the following rate of approximation holds

$$\left\| T(t)x - x - \sum_{n=1}^m A^n(A + 1)^{-n}x L_n^{(-1)}(t) \right\| \leq C \frac{t^p}{m^{\frac{p}{2}-\frac{1}{4}}} \|A^p x\|, \quad x \in D(A^p).$$

This rate of convergence is greater than the ones in Theorem 4.4(ii). To get it, authors use a holomorphic functional calculus, Carlson’s inequality and the Fourier multiplier norms for exponential integrators, see more details in [18, Section 5]. However, the rate of convergence improves for certain holomorphic semigroups considered in Theorem 5.2(ii).

Note that the rate of approximation for smooth data is also estimated for A -acceptable rational approximations in [7, Theorem 3,4] and for Padé approximations in [12, Theorem 4.1].

The following result gives some representation formulae (via series of operators) for fractional powers of $(\lambda + A)^{-1}$. The Hille–Phillips functional calculus is a linear and bounded map $f \mapsto f(A)$, $L^1(\mathbb{R}_+) \rightarrow \mathcal{B}(X)$, where

$$f(A)x := \int_0^\infty f(t)T(t)x dt, \quad x \in X, \quad f \in L^1(\mathbb{R}_+),$$

and $(T(t))_{t \geq 0}$ is a uniformly bounded semigroup generated by $-A$ (see for example [27, Chapter 3]). Note that if $f = \sum_{n=1}^\infty f_n$ in $L^1(\mathbb{R}_+)$ then $f(A) = \sum_{n=1}^\infty f_n(A)$ in $\mathcal{B}(X)$.

Theorem 4.6. *Let $(-A, D(A))$ be the infinitesimal generator of a uniformly bounded C_0 -semigroup $(T(t))_{t \geq 0}$ in a Banach space X .*

(i) For $2\beta > \alpha > -1$, and $x \in X$,

$$(A + 1)^{-\beta-1} = \sum_{n=0}^\infty \binom{\alpha - \beta + m - 1}{m} A^n (A + 1)^{-n-\alpha-1} \quad \text{in } \mathcal{B}(X).$$

In particular for $\alpha > 2$, we have that $(A + 1)^{-\alpha} = \sum_{n=0}^\infty A^n (A + 1)^{-n-\alpha-1}$ in $\mathcal{B}(X)$.

(ii) For all $\lambda > 0$, $(\lambda + A)^{-1} = \sum_{n=0}^\infty \frac{(\lambda - \frac{1}{2})^n}{(\lambda + \frac{1}{2})^{n+1}} (A - \frac{1}{2})^n (A + \frac{1}{2})^{-n-1}$ holds in $\mathcal{B}(X)$.

Proof. (i) By Theorems 3.2(i) and 4.1(i), we have that

$$\begin{aligned} (A + 1)^{-\beta-1}x &= \frac{1}{\Gamma(\beta + 1)} \int_0^\infty t^\beta e^{-t} T(t)x dt \\ &= \sum_{n=0}^\infty \binom{\alpha - \beta + n - 1}{n} \int_0^\infty \ell_n^{(\alpha)}(t) T(t)x dt \\ &= \sum_{n=0}^\infty \binom{\alpha - \beta + n - 1}{n} A^n (A + 1)^{-n-\alpha-1}x. \end{aligned}$$

(ii) For $\lambda > 0$ and $x \in X$, we have by Theorem 3.2(ii),

$$\begin{aligned} (\lambda + A)^{-1}x &= \int_0^\infty e^{-\lambda t} T(t)x dt = \sum_{n=0}^\infty \frac{(\lambda - \frac{1}{2})^n}{(\lambda + \frac{1}{2})^{n+1}} \int_0^\infty e^{-\frac{1}{2}t} L_n^{(0)}(t) T(t)x dt \\ &= \sum_{n=0}^\infty \frac{(\lambda - \frac{1}{2})^n}{(\lambda + \frac{1}{2})^{n+1}} \int_0^\infty \ell_n^{(0)}(t) S(t)x dt, \end{aligned}$$

with $S(t) := e^{\frac{1}{2}t} T(t)$ is the C_0 -semigroup generated by $\frac{1}{2} - A$. By Theorem 3.2(i), we conclude that $(\lambda + A)^{-1}x = \sum_{n=0}^\infty \frac{(\lambda - \frac{1}{2})^n}{(\lambda + \frac{1}{2})^{n+1}} (A - \frac{1}{2})^n (A + \frac{1}{2})^{-n-1}x$, for $x \in X$. \square

5. Examples, applications and final comments

In this section, we apply our results to concrete examples: translation, convolution and multiplication semigroups in Lebesgue space. Holomorphic semigroups allow us to improve some previous results, compare Theorems 5.2 and 4.1.

5.1. Translation semigroup

Let $1 \leq p < \infty$. Let $(T(t))_{t \geq 0}$ be the right shift semigroup on $L^p(\mathbb{R}_+)$ defined by

$$T(t)f(x) := (\delta_t * f)(x) = \begin{cases} f(x - t), & x > t, \\ 0, & x \leq t. \end{cases}$$

The infinitesimal generator $-A$ is the usual derivation operator, $-A = -\frac{d}{dx}$ [14, Section I.4(c)]. Furthermore, $(A + 1)^{-n-\alpha-1} f = I_{n+\alpha+1} * f$, and $A^n(A + 1)^{-n-\alpha-1} f = \ell_n^{(\alpha)} * f$, for $n \in \mathbb{N} \cup \{0\}$ and $\alpha > -1$. By Theorem 4.1(iii), we obtain the formula

$$\delta_t * f = \sum_{n=0}^{\infty} (\ell_n^{(\alpha)} * f) L_n^{(\alpha)}(t), \quad f \in W^{1,p}(\mathbb{R}_+),$$

where $W^{1,p}(\mathbb{R}_+)$ is a standard Sobolev space. In particular for $\alpha = 0$ this formula has been considered in [9, Section 3, Examples 3.1(4)] where Laguerre expansions of tempered distributions are studied. This nice formula seems to be new for $\alpha \neq 0$.

5.2. Convolution and multiplication semigroups

Let $1 \leq p < \infty$. For $t > 0$, let $k_t : \mathbb{R}^m \rightarrow \mathbb{R}$ be a convolution kernel on $L^p(\mathbb{R}^m)$ such that $\sup_{t>0} \|k_t\|_1 < \infty$ and $(T(t))_{t \geq 0}$, defined by

$$T(t)f(s) := (k_t * f)(s) = \int_{\mathbb{R}^m} k_t(s - u) f(u) du, \quad f \in L^p(\mathbb{R}^m),$$

is a uniformly bounded C_0 -semigroup, whose generator is denoted by $-A$.

Fixed $s \in \mathbb{R}^m$, we suppose that for some $\alpha > -1$ the map $t \mapsto k_t(s)$ is differentiable with respect to t and

$$\int_0^{\infty} e^{-t^\alpha} |k_t(s)|^2 dt < \infty. \tag{5.1}$$

By Theorem 1.1, we have that $k_t(s) = \sum_{n=0}^{\infty} a_n(s) L_n^{(\alpha)}(t)$ for $s \in \mathbb{R}^m$, and $t > 0$, where

$$a_n(s) = \int_0^{\infty} \ell_n^{(\alpha)}(t) k_t(s) dt, \quad s \in \mathbb{R}^m.$$

Furthermore, $a_n \in L^1(\mathbb{R}^m)$ and $\|a_n\|_1 \leq \sup_{t>0} \|k_t\|_1 \|\ell_n^{(\alpha)}\|_1$. Applying Fubini’s Theorem and Theorem 4.1(i), we obtain that

$$A^n(A + 1)^{-n-\alpha-1} f = a_n * f, \quad f \in L^p(\mathbb{R}^m),$$

and therefore by Theorem 4.1(iii), we have $k_t * f = \sum_{n=0}^{\infty} (a_n * f) L_n^{(\alpha)}(t)$ for $f \in D(A)$.

The standard examples of convolution C_0 -semigroups are provided by Gaussian and Poisson kernels with

$$g_t(s) := \frac{1}{(4\pi t)^{\frac{m}{2}}} e^{-\frac{\|s\|^2}{4t}}, \quad p_t(s) := \frac{\Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m+1}{2}}} \frac{t}{(t^2 + \|s\|^2)^{\frac{m+1}{2}}},$$

respectively, see [27, p. 25]. Both kernels satisfy (5.1) if $s \neq 0$ for $\alpha > -1$. If $s = 0$, then one needs to consider $\alpha > m - 1$ for the Gaussian kernel and $\alpha > 2m - 1$ for Poisson the kernel.

Now, let $q : \mathbb{R}^m \rightarrow \mathbb{R}_-$ be a continuous function, where $\mathbb{R}_- := (-\infty, 0]$. The family of operators $(S_q(t))_{t \geq 0}$ defined by $S_q(t)f := e^{tq}f$ is a contraction C_0 -semigroup in $C_0(\mathbb{R}^m)$ whose infinitesimal generator $-B$ is given by $-Bf := qf$ and $D(B) = \{f \in C_0(\mathbb{R}^m) \mid qf \in C_0(\mathbb{R}^m)\}$ [14, Section I.4(b)]. Note that

$$B^n(B + 1)^{-n-\alpha-1} f(s) = \frac{(-q)^n(s)}{(1 - q(s))^{n+\alpha+1}} f(s), \quad f \in C_0(\mathbb{R}^m), s \in \mathbb{R}^m,$$

and

$$S_q(t)f = \sum_{n=0}^{\infty} \frac{(-q)^n}{(1 - q)^{n+\alpha+1}} f L_n^{(\alpha)}(t), \quad f \in D(B), t > 0.$$

Examples for this setting are provided by the Fourier transform of the Gaussian and Poisson semigroups, $q(s) = -\|s\|^2$ and $q(s) = -\|s\|$ [14, p. 69], or by semigroups subordinated to them with $q(s) = -\log(1 + \|s\|^2)$ and $q(s) = -\log(1 + \|s\|)$, studied in details in [8].

To finish this subsection, we are interested in identifying the functions a_n in the cases of Poisson and Gaussian semigroup. We denote by $\mathcal{F} : L^1(\mathbb{R}^m) \rightarrow C_0(\mathbb{R}^m)$ the usual Fourier transform, defined by $\mathcal{F}(f)(s) := \int_{\mathbb{R}^m} e^{-is \cdot u} f(u) du$, with $s \cdot u$ the inner product in \mathbb{R}^m . For the Poisson semigroup, we have $\mathcal{F}(a_n)(s) = \varphi_{n,\alpha}(\|s\|)$ for $s \in \mathbb{R}^m$ and $\mathcal{F}(a_n) \in L^1(\mathbb{R}^m)$ for $\alpha > m - 1$ and it belongs to $L^2(\mathbb{R}^m)$ for $\alpha > \frac{m-2}{2}$, see functions $\varphi_{n,\alpha}$ defined in (2.5). For the Gaussian semigroup, $\mathcal{F}(a_n)(s) = \varphi_{n,\alpha}(\|s\|^2)$ for $s \in \mathbb{R}^m$ and $\mathcal{F}(a_n) \in L^1(\mathbb{R}^m)$ for $\alpha > \frac{m-2}{2}$ and belongs to $L^2(\mathbb{R}^m)$ for $\alpha > \frac{m-4}{4}$. To show both results, we use spherical coordinates and apply Theorem 2.7(i). Note that in these cases a_n is a radial function. Then $\mathcal{F}(a_n) = 2\pi \mathcal{F}^{-1}(a_n)$ and

$$a_n(s) = \frac{C_m}{(2\pi)^m} \int_0^\infty \mathcal{F}(a_n)(r) j_m(\|s\|r) r^{m-1} dr, \quad s \in \mathbb{R}^m,$$

where C_m is the area of the unit $(m - 1)$ -dimensional sphere and j_m is the spherical Bessel function for $m \geq 2$, see more details in [11, p. 133]. If $m = 1$, then we have

$$a_n(s) = \frac{1}{2\pi} \int_0^\infty e^{isr} \mathcal{F}(a_n)(r) dr + \frac{1}{2\pi} \int_0^\infty e^{-isr} \mathcal{F}(a_n)(r) dr,$$

since $\mathcal{F}(a_n)$ is an even function. For $\alpha > 0$ and the Poisson semigroup, we have that

$$\begin{aligned} a_n(s) &= \frac{1}{2\pi} \left(\lim_{\lambda \rightarrow is} \mathcal{L}(\varphi_{n,\alpha})(\lambda) + \lim_{\lambda \rightarrow -is} \mathcal{L}(\varphi_{n,\alpha})(\lambda) \right) \\ &= \frac{n!}{2\pi} \left((-is)^{\frac{\alpha-1}{2}} e^{\frac{-is}{2}} W_{-n-\frac{(1+\alpha)}{2}, -\frac{\alpha}{2}}(-is) + (is)^{\frac{\alpha-1}{2}} e^{\frac{is}{2}} W_{-n-\frac{(1+\alpha)}{2}, -\frac{\alpha}{2}}(is) \right), \end{aligned}$$

for $s \in \mathbb{R}$ and $W_{k,\mu}$ is the Whittaker function; for the Gaussian semigroup,

$$a_n(s) = \frac{1}{2\pi} \left(\lim_{\lambda \rightarrow is} \mathcal{L}(\varphi_{n,\alpha}(t^2))(\lambda) + \lim_{\lambda \rightarrow -is} \mathcal{L}(\varphi_{n,\alpha}(t^2))(\lambda) \right), \quad s \in \mathbb{R}.$$

To calculate $\mathcal{L}(\varphi_{n,\alpha}(t^2))(\lambda)$ with $\lambda \in \mathbb{C}_+$, note that

$$\begin{aligned} \mathcal{L}(\varphi_{n,\alpha}(t^2))(\lambda) &= \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{\lambda^2}{4u^2}} \mathcal{L}(\varphi_{n,\alpha})(u^2) du \\ &= \frac{n!}{\sqrt{\pi}} \int_0^\infty e^{-\frac{\lambda^2}{4u^2}} u^{\alpha-1} e^{\frac{u^2}{2}} W_{-n-\frac{(1+\alpha)}{2},-\frac{\alpha}{2}}(u^2) du, \end{aligned}$$

where we have used the Laplace transform properties and Remark 2.8. These computations to identity function $(a_n)_{n \geq 0}$ in the cases of Gaussian and Poisson semigroups seem to be original.

5.3. Differentiable and holomorphic semigroups

So far, we have considered C_0 -semigroups defined in $[0, \infty)$. If we consider differentiable or holomorphic semigroups with certain growth assumptions, some previous results will be improved.

A C_0 -semigroup $(T(t))_{t \geq 0}$ is called immediately differentiable if the orbit $t \mapsto T(t)x$ is differentiable for $t > 0$, see [14, Definition II.4.1]. A straightforward consequence of Theorem 4.1(iii) is the following corollary. A similar result seems that it may hold in UMD spaces for every uniformly bounded C_0 -semigroup, see Remark 3.4.

Corollary 5.1. *Let $-A$ be the infinitesimal generator of an immediately differentiable uniformly bounded C_0 -semigroup $(T(t))_{t \geq 0}$ in a Banach space X . Then*

$$T(t)x = \sum_{n=0}^\infty A^n (A + 1)^{-n-\alpha-1} x L_n^{(\alpha)}(t), \quad t > 0, x \in X.$$

A holomorphic semigroup $(T(z))_{z \in \mathbb{C}_+}$ is said to be of type HG_β if $\|T(z)\| \leq C_\nu \left(\frac{|z|}{\Re z}\right)^\nu$, for $\Re z > 0$ and every $\nu > \beta$ [16]. Classical holomorphic semigroups such as Gaussian and Poisson semigroups satisfy the HG_β condition for some $\beta > 0$. Among semigroups satisfying the HG_β condition one may mention:

- (a) semigroups generated by $-\sqrt{\mathcal{L}}$, where \mathcal{L} is the sub-Laplacian in $L^p(\mathbb{G})$ and \mathbb{G} a Lie group.
- (b) semigroups generated by $-(-\log(\lambda) + H)$, where H is the strongly elliptic operator affiliated with a strongly continuous representation of a Lie group into a Banach space.

A through discussion of these classes of semigroups can be found in [15, Section 5].

Theorem 5.2. *Let $-A$ be the infinitesimal generator of a uniformly bounded holomorphic C_0 -semigroup $(T(z))_{z \in \mathbb{C}_+}$ of type HG_β for some $\beta \geq 0$ in a Banach space X .*

(i) *For $\alpha > -1$, there exists $C_{\alpha,\beta} > 0$ such that*

$$\|A^n (A + 1)^{-n-\alpha-1} x\| \leq \frac{C_{\alpha,\beta} \|x\|}{n^{\alpha+1}}, \quad x \in X.$$

(ii) *For $\alpha > 0$ and $t > 0$,*

$$\|T(t)x - \sum_{n=0}^m A^n (A + 1)^{-n-\alpha-1} x L_n^{(\alpha)}(t)\| \leq \frac{C_{\alpha,\beta,t} \|x\|}{m^{\frac{\alpha}{2}}}, \quad x \in X.$$

Proof. (i) By Theorem 2.7(iii), $\varphi_{n,\alpha} \in AC^{(j)}$ for $j, n \in \mathbb{N}$, and $\alpha > -1$. We apply holomorphic functional calculus $f \mapsto f(A)$ defined in [16, Theorem 6.2.] to get that

$$\|A^n(A+1)^{-n-\alpha-1}x\| = \|\varphi_{n,\alpha}(A)x\| \leq C\|x\| \|\varphi_{n,\alpha}\|_{(j)} \leq \frac{C_{\alpha,\beta}\|x\|}{n^{\alpha+1}}, \quad x \in X,$$

for all $j > \beta + 1$ and $x \in X$. (ii) For $t > 0$ there is a $n_0 \in \mathbb{N}$ such that the inequality $|L_n^{(\alpha)}(t)| \leq C_t n^{\frac{\alpha}{2}}$ holds for $n \geq n_0$, see formula (2.2). We use part (i) to get that

$$\left\| T(t)x - \sum_{n=0}^m A^n(A+1)^{-n-\alpha-1}x L_n^{(\alpha)}(t) \right\| \leq \sum_{n=m+1}^{\infty} \frac{C_{\alpha,\beta,t}}{n^{1+\frac{\alpha}{2}}} \|x\| \leq \frac{C_{\alpha,\beta,t}}{m^{\frac{\alpha}{2}}} \|x\|, \quad x \in X,$$

and we conclude the result. \square

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