AN EXTENSION OF BERWALD'S INEQUALITY AND ITS RELATION TO ZHANG'S INEQUALITY

DAVID ALONSO-GUTIÉRREZ, JULIO BERNUÉS, AND BERNARDO GONZÁLEZ MERINO

ABSTRACT. In this note prove the following Berwald-type inequality, showing that for any integrable log-concave function $f:\mathbb{R}^n\to[0,\infty)$ and any concave function $h:L\to[0,\infty)$, where $L=\{(x,t)\in\mathbb{R}^n\times[0,\infty)\,:\,f(x)\geq e^{-t}\|f\|_\infty\}$, then

$$p\to\left(\frac{1}{\Gamma(1+p)\int_L e^{-t}dtdx}\int_L h^p(x,t)e^{-t}dtdx\right)^{\frac{1}{p}}$$

is decreasing in $p\in(-1,\infty)$, extending the range of p where the monotonicity is known to hold true.

As an application of this extension, we will provide a new proof of a functional form of Zhang's reverse Petty projection inequality, recently obtained in [ABG].

1. Introduction and notation

Let $K\subseteq\mathbb{R}^n$ be a *convex body*, i.e., a compact, convex set with non-empty interior, and let us denote by $\mathcal{K}^n$ the set of all convex bodies in $\mathbb{R}^n$ and by $|K|$ the *Lebesgue measure* of K. We will also denote by $\mathcal{K}^n_0$ the set of convex bodies containing the origin. It is well known that, as a consequence of Hölder's inequality, for any integrable function $f:K\to[0,\infty)$ the function

$$p\to\left(\frac{1}{|K|}\int_K f(x)^p dx\right)^{\frac{1}{p}}$$

is increasing in $p\in(0,\infty)$.

A famous inequality proved by Berwald [Ber, Satz 7] (see also [AAGJV, Theorem 7.2] for a translation into English) provides a reverse Hölder's inequality for L_p-norms ($p>0$) of concave functions defined on convex bodies. It states that for any $K\in\mathcal{K}^n$ and any concave function $f:K\to[0,\infty)$, one has

$$p\to\left(\frac{\binom{p+n}{n}}{|K|}\int_K f(x)^p\,dx\right)^{\frac{1}{p}}\tag{1}$$

is decreasing in $p\in(0,\infty)$.

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A function $f:\mathbb{R}^n\to[0,\infty)$ is called log-concave if, for every $x,y\in\mathbb{R}^n, 0<\lambda<1$, $f(\lambda x+(1-\lambda)y)\geq (f(x))^\lambda (f(y))^{1-\lambda}$. Throughout the paper, we will denote by $\mathcal{F}(\mathbb{R}^n)$ the set of all integrable log-concave functions in $\mathbb{R}^n$.

In the context of log-concave functions, the following version of Berwald's inequality (1) on epigraphs of convex functions was proved in [AAGJV, Lemma 3.3]:

"Let $f\in\mathcal{F}(\mathbb{R}^n)$ and let $h:L\to[0,\infty)$ be a continuous concave non-identically null function, where $L=\{(x,t)\in\mathbb{R}^{n+1} : f(x)\geq e^{-t}\|f\|_\infty\}$ is the the epigraph of $-\log\frac{f}{\|f\|_\infty}$. Then, the function

$$p\to\left(\frac{1}{\Gamma(1+p)\int_L e^{-t}dtdx}\int_L h^p(x,t)e^{-t}dtdx\right)^{\frac{1}{p}} \tag{2}$$

is decreasing in $p\in(0,\infty)$."

When providing a new proof of Zhang's reverse Petty projection inequality, Gardner and Zhang [GZ] extended (1) to the larger range of values $p>-1$ (see [GZ, Theorem 5.1]). The first goal in this paper is to also extend (2) to the larger range of values $p>-1$.

Theorem 1.1. *Let $f\in\mathcal{F}(\mathbb{R}^n)$ and let $h:L\to[0,\infty)$ be a concave function, where $L=\{(x,t)\in\mathbb{R}^{n+1} : f(x)\geq e^{-t}\|f\|_\infty\}$. Then, the function*

$$p\to\left(\frac{1}{\Gamma(1+p)\int_L e^{-t}dtdx}\int_L h^p(x,t)e^{-t}dtdx\right)^{\frac{1}{p}}$$

is decreasing in $p\in(-1,\infty)$.

For any $K\in\mathcal{K}^n$, its *polar projection body* $\Pi^*(K)$ is the unit ball of the norm given by

$$\|x\|_{\Pi^*(K)} := |x||P_{x^\perp}K|, \qquad x\in\mathbb{R}^n,$$

where $P_{x^\perp}K$ is the *orthogonal projection* of K onto the hyperplane orthogonal to x, $|\cdot|$ denotes (besides the Lebesgue measure in the suitable space) the Euclidean norm and $\|\cdot\|_K$ denotes the *Minkowski functional* of K, defined for every $x\in\mathbb{R}^n$, as $\|x\|_K:=\inf\{\lambda>0 \mid x\in\lambda K\}\in[0,\infty]$. It is a norm if and only if K is centrally symmetric.

The expression $|K|^{n-1}|\Pi^*(K)|$ is affine invariant and its extremal convex bodies are well known: *Petty's projection inequality* [P] states that the (affine class of the) n-dimensional Euclidean ball, B_2^n, is the only maximizer and *Zhang's inequality* [Z1] proves that the (affine class of the) n-dimensional simplex Δ_n, is the only minimizer. That is, for any convex body $K\subseteq\mathbb{R}^n$,

$$\frac{\binom{2n}{n}}{n^n}=|\Delta_n|^{n-1}|\Pi^*(\Delta_n)|\leq|K|^{n-1}|\Pi^*(K)|\leq|B_2^n|^{n-1}|\Pi^*(B_2^n)|=\frac{|B_2^n|^n}{|B_2^{n-1}|^n}. \tag{3}$$

In recent years, many relevant geometric inequalities have been extended to the general context of log-concave functions (see for instance [AKM], [KM], [C], or [HJM] and the references therein). Let us recall that $\mathcal{K}^n$ and $\mathcal{K}_0^n$ naturally embed into $\mathcal{F}(\mathbb{R}^n)$, via the natural injections

$$K\to\chi_K \quad\text{ and }\quad K\to e^{-\|\cdot\|_K},$$

where χ_K is the characteristic function of K. These and other basic facts on convex bodies and log-concave functions used in the paper can be found in [BGVV] and [AGM].

For any $f \in \mathcal{F}(\mathbb{R}^n)$, the *polar projection body of* f, denoted as $\Pi^*(f)$, is the unit ball of the norm given by

$$\|x\|_{\Pi^*(f)} := 2|x|\|f\|_\infty \int_0^\infty |P_{x^\perp} K_t(f)|\ e^{-t} dt = 2\|f\|_\infty \int_0^\infty \|x\|_{\Pi^*(K_t(f))} e^{-t} dt,$$

where $K_t(f) := \{x \in \mathbb{R}^n \ : \ f(x) \geq e^{-t}\|f\|_\infty\}$, $t > 0$ (see [AGJV]).

In [ABG], an extension of Zhang's inequality (i.e., the left hand side inequality in (3)) was proved in the settings of log-concave functions.

Theorem 1.2. *Let* $f \in \mathcal{F}(\mathbb{R}^n)$. *Then,*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \min\{f(y), f(x)\}\, dy dx \leq 2^n n! \|f\|_1^{n+1} \left|\Pi^*(f)\right|. \tag{4}$$

Moreover, if $\|f\|_\infty = f(0)$ *then equality holds if and only if* $\frac{f(x)}{\|f\|_\infty} = e^{-\|x\|_{\Delta_n}}$ *for some* n*-dimensional simplex* Δ_n *containing the origin.*

Observe that when $f = e^{-\|\cdot\|_K}$ for some convex body $K \in \mathcal{K}_0^n$, then (4) recovers Zhang's inequality.

Our second goal here is to provide a new proof of the functional version of Zhang's inequality (4) by using the extension of Berwald's inequality given by Theorem 1.1, in a similar way as Gardner and Zhang [GZ] proved the geometrical version of Zhang's inequality via their extension of Berwald's inequality (1) to $p > -1$.

A common feature in both proofs, the one given in [ABG] and the one in this paper, is the crucial role played by the functional form of the covariogram function g_f associated to the function $f \in \mathcal{F}(\mathbb{R}^n)$. See [ABG] and its definition below. Recall that in the geometric setting the covariogram function of a convex body K is given by $g_K(x) = |K \cap (x + K)|$. Apart from this fact, the two proofs completely differ.

We introduce further notation: S^{n-1} denotes the Euclidean unit sphere in $\mathbb{R}^n$. If the origin is in the interior of a convex body K, the function $\rho_K\colon S^{n-1} \to [0, +\infty)$ given by $\rho_K(u) = \sup\{\lambda \geq 0 \mid \lambda u \in K\}$ is the *radial function* of K. It extends to $\mathbb{R}^n \setminus \{0\}$ via $t\rho_K(tu) = \rho_K(u)$, for any $t > 0, u \in S^{n-1}$.

Finally, for any function $f \in \mathcal{F}(\mathbb{R}^n)$ let g_f be the covariogram functional of f, defined by

$$g_f(x) := \int_0^\infty e^{-t} |K_t(f) \cap (x + K_t(f))| dt$$

(cf. [ABG]).

The paper is organized as follows: Section 2 contains the aforementioned extension, Theorem 1.1, of the functional Berwald inequality to the larger range of values of $p > -1$. In Section 3 we recall the celebrated family (with parameter $p > 0$) of convex bodies associated to any log-concave function introduced by Ball in [B, pg. 74]. We also recall the properties of the covariogram functional of a log-concave function, proven in [ABG]. Another main ingredient in the proof in [GZ] is an expression that connects the covariogram function of a convex body K and Ball's convex bodies. Such a connection can be extended to the functional form of the covariogram g_f of a log-concave function and moreover, the polar projection body of f will appear as a limiting case of this new expression when the value of the parameter p tends to -1.

2. An extension of Berwald's inequality

In this section we will prove the aforementioned extension of Berwald's inequality, see Theorem 1.1 above. We first state a 1-dimensional lemma that can be seen as a degenerate version of Theorem 1.1.

Lemma 2.1. *Let $\gamma : [0,\infty) \to [0,\infty)$ be a non-decreasing concave function and define*

$$\Phi_\gamma(p) = \left(\frac{1}{\Gamma(1+p)}\int_0^\infty \gamma(r)^p e^{-r}dr\right)^{\frac{1}{p}}, \; p > -1.$$

Then $\Phi_\gamma(p)$ is decreasing in p in $(-1,\infty)$. Furthermore, if there exist $-1 < p_1 < p_2$ such that $\Phi_\gamma(p_1) = \Phi_\gamma(p_2)$, then γ is a linear function and Φ_γ is constant on $(-1,\infty)$.

Remark 1. As usual, we define $\Phi_\gamma(0) = \lim_{p\to 0}\Phi_\gamma(p)$ which by straightforward computations (using L'Hôpital's rule, interchanging the integral and the derivative operations, and taking into account that $\frac{\partial\Gamma(1+x)}{\partial x}|_{x=0} = -A$, where $A \approx 0.577$ is the Euler-Mascheroni constant) yields $\Phi_\gamma(0) = e^A \exp\left(\int_0^\infty \log\gamma(r)e^{-r}dr\right)$.

Proof of Lemma 2.1. Fix $0 \neq p_1 > -1$ and write $\overline{\gamma}(r) = \Phi_\gamma(p_1)\cdot r,\; r \geq 0$. For any $p > -1$,

$$\Phi_{\overline{\gamma}}(p) = \left(\frac{1}{\Gamma(1+p)}\int_0^\infty \Phi_\gamma(p_1)^p r^p e^{-r}dr\right)^{\frac{1}{p}} = \Phi_\gamma(p_1).$$

Therefore

$$0 = \Phi_\gamma^{p_1}(p_1) - \Phi_{\overline{\gamma}}^{p_1}(p_1) = \frac{1}{\Gamma(1+p_1)}\int_0^\infty (\gamma(r)^{p_1} - \overline{\gamma}(r)^{p_1})e^{-r}dr, \tag{5}$$

or equivalently,

$$\int_0^1 (\gamma(-\log t)^{p_1} - \overline{\gamma}(-\log t)^{p_1})dt = 0.$$

We first consider the case $-1 < p_1 < p_2 < 0$.

Since the function γ is non-negative and concave and (5) holds, if γ is not identically equal to $\overline{\gamma}$, i.e., γ is not linear, there exists a unique $r_0 \in (0,\infty)$ such that $\gamma(r) > \overline{\gamma}(r)$ if $r \in (0,r_0)$ and $\gamma(r) < \overline{\gamma}(r)$ if $r \in (r_0,\infty)$. Denoting $t_0 = e^{-r_0}$, we have that $\gamma(-\log t) < \overline{\gamma}(-\log t)$ if $t \in (0,t_0)$ and $\gamma(-\log t) > \overline{\gamma}(-\log t)$ if $t \in (t_0,1)$. Now,

$$\begin{aligned}\Gamma(1+p_2)(\Phi_\gamma^{p_2}(p_2) - \Phi_{\overline{\gamma}}^{p_2}(p_2)) &= \int_0^\infty (\gamma(r)^{p_2} - \overline{\gamma}(r)^{p_2})e^{-r}dr \\ &= \int_0^1 (\gamma(-\log t)^{p_2} - \overline{\gamma}(-\log t)^{p_2})dt \\ &= \int_0^1 (\gamma(-\log t)^{p_1} - \overline{\gamma}(-\log t)^{p_1})\psi(t)dt,\end{aligned}$$

where

$$\psi(t) = \frac{\gamma(-\log t)^{p_2} - \overline{\gamma}(-\log t)^{p_2}}{\gamma(-\log t)^{p_1} - \overline{\gamma}(-\log t)^{p_1}}.$$

Since $w(x) = x^{\frac{p_2}{p_1}}$ is strictly concave in $(0,\infty)$, $\dfrac{w(x)-w(y)}{x-y}$ is strictly decreasing in x and y and, since $\gamma(-\log t)^{p_1}$ is non-decreasing and $\overline{\gamma}(-\log t)^{p_1}$ is strictly increasing in t, $\psi(t)$ is strictly decreasing. Now, by the mean value theorem, there exist $c_1 \in (0,t_0)$ and $c_2 \in (t_0,1)$ such that

$$\begin{aligned}
&\int_0^1 (\gamma(-\log t)^{p_1} - \overline{\gamma}(-\log t)^{p_1})\psi(t)dt \\
&= \int_0^{t_0} (\gamma(-\log t)^{p_1} - \overline{\gamma}(-\log t)^{p_1})\psi(t)dt + \int_{t_0}^1 (\gamma(-\log t)^{p_1} - \overline{\gamma}(-\log t)^{p_1})\psi(t)dt \\
&= \psi(c_1)\int_0^{t_0} (\gamma(-\log t)^{p_1} - \overline{\gamma}(-\log t)^{p_1})dt + \psi(c_2)\int_{t_0}^1 (\gamma(-\log t)^{p_1} - \overline{\gamma}(-\log t)^{p_1})dt \\
&= (\psi(c_1) - \psi(c_2))\int_0^{t_0} (\gamma(-\log t)^{p_1} - \overline{\gamma}(-\log t)^{p_1})dt > 0,
\end{aligned}$$

since ψ is strictly decreasing, $\gamma(-\log t) < \overline{\gamma}(-\log t)$ for $t \in (0,t_0)$ and $p_1 < 0$. Therefore, if γ is not linear, $\Phi_\gamma(p_2) < \Phi_{\overline{\gamma}}(p_2) = \Phi_{\overline{\gamma}}(p_1) = \Phi_\gamma(p_1)$.

The case $0 < p_1 < p_2$ follows analogously with straightforward changes (in this case, if γ is not linear w is strictly convex and ψ is strictly decreasing). The continuity of Φ_γ at 0 then implies that $\Phi_\gamma(p)$ is decreasing in $p > -1$.

If $\Phi_\gamma(p_1) = \Phi_\gamma(p_2)$ for some $-1 < p_1 < p_2$, since $\Phi_\gamma(p)$ would not be strictly decreasing in $[p_1,p_2]$, then γ would be linear, thus concluding the case of equality. □

Our next result is the aforementioned extension of [AAGJV, Lemma 3.3] to $p \in (-1,\infty)$.

Proof of Theorem 1.1. Consider the probability measure on $\mathbb{R}^{n+1}$ given by

$$d\mu(x,t) := \frac{e^{-t}\chi_L(x,t)}{\int_L e^{-t}dtdx}dtdx.$$

Denote $C_s(h) = \{(x,t) \in L : h(x,t) \geq s\}$ and define the function $I_h : [0,\infty) \to [0,\infty)$ as

$$I_h(s) := \frac{1}{\int_L e^{-t}dtdx}\int_{C_s(h)} e^{-t}dtdx = \mu(C_s(h)).$$

I_h is non-increasing, $I_h(0) = \mu(L) = 1$ and since h is concave, I_h is log-concave (see [AAGJV, Lemma 3.2]).

Observe that $(x,t) \in L$ if and only if $x \in K_t(f)$, which happens if and only if $\rho_{K_t(f)}(x) \geq 1$, and that, by Fubini's theorem, $\displaystyle\int_L e^{-t}dtdx = \int_0^\infty e^{-t}|K_t(f)|dt$. Now define $h_1 : L \to [0,\infty)$ as

$$h_1(x,t) := \sup\left\{ s \in [0,\infty) \,:\, I_h(s) > \frac{1}{\rho^n_{K_t(f)}(x)} \right\}.$$

h_1 has two important properties:

- h and h_1 are equally distributed with respect to μ, that is $I_{h_1} \equiv I_h$. In order to prove this, notice that for every $s \geq 0$, and every $(x,t) \in L$, we have that $h_1(x,t) > s$ if and only if $\rho^n_{K_t(f)}(x) > \frac{1}{I_h(s)}$ and so by Fubini's theorem,

$$I_{h_1}(s) = \int_{C_s(h_1)} d\mu(x,t) = \int_0^\infty e^{-t}|K_t(f)|I_h(s)\frac{dt}{\int_L e^{-t}dtdx} = I_h(s).$$

- $h_1(r\rho_{K_t(f)}(u)u,t)$ does not depend on t and u since for any $r,t>0$, $u\in S^{n-1}$,

$$h_1(r\rho_{K_t(f)}(u)u,t)=\sup\{s\in[0,\infty)\,:\,I_h(s)>r^n\}:=\gamma(r).$$

Therefore, for any $p>0$,

$$\begin{aligned}\int_L h^p(x,t)d\mu(x,t) &= \int_{\mathbb{R}^n}\int_0^\infty \chi_{\{h^p(x,t)\geq r\}}drd\mu(x,t)=\int_0^\infty I_h(r^{\frac{1}{p}})dr\\ &= \int_0^\infty I_{h_1}(r^{\frac{1}{p}})dr=\int_L h_1^p(x,t)d\mu(x,t).\end{aligned}$$

By Fubini's theorem and integrating in polar coordinates,

$$\begin{aligned}\int_L h_1^p(x,t)e^{-t}dxdt &= \int_0^\infty e^{-t}\int_{K_t(f)} h_1^p(x,t)dxdt\\ &= n|B_2^n|\int_0^\infty e^{-t}\int_{S^{n-1}}\int_0^{\rho_{K_t(f)}(u)} h_1^p(ru,t)r^{n-1}drd\sigma(u)dt\\ &= n|B_2^n|\int_0^\infty e^{-t}\int_{S^{n-1}}\int_0^1 \gamma^p(r)\rho^n_{K_t(f)}(u)r^{n-1}drd\sigma(u)dt\\ &= n\int_0^\infty e^{-t}|K_t(f)|\int_0^1\gamma^p(r)r^{n-1}drdt\end{aligned}$$

and so, since $\int_0^\infty e^{-t}|K_t(f)|dt=\int_L e^{-t}dtdx$,

$$\int_L h^p(x,t)d\mu(x,t)=n\int_0^1\gamma^p(r)r^{n-1}dr.$$

If $p<0$ the same equality holds. Indeed, we have

$$\begin{aligned}\int_L h^p(x,t)d\mu(x,t) &= \int_0^\infty\int_{\mathbb{R}^n}\chi_{\{h(x,t)\leq r^{\frac{1}{p}}\}}d\mu(x,t)dr=\int_0^\infty(1-I_h(r^{\frac{1}{p}}))dr\\ &= \int_0^\infty(1-I_{h_1}(r^{\frac{1}{p}}))dr=\int_L h_1^p(x,t)d\mu(x,t)\end{aligned}$$

and we proceed as before. If $p=0$ the equality is obviously true.

Notice that since I_h is log-concave the function γ is non-increasing and for every $r_1,r_2\in[0,1]$,

$$\gamma(r_1^{1-\lambda}r_2^\lambda)\geq(1-\lambda)\gamma(r_1)+\lambda\gamma(r_2).$$

If we denote $\gamma_1(r)=\gamma(e^{-r/n})$ the previous statement means that γ_1 is non-decreasing and concave in $[0,\infty)$ and we have

$$\int_L h^p(x,t)d\mu(x,t)=n\int_0^1\gamma^p(r)r^{n-1}dr=\int_0^\infty\gamma_1^p(r)e^{-r}dr.$$

We can apply now Lemma 2.1 to the function γ_1 and conclude that

$$\left(\frac{1}{\Gamma(1+p)\int_L e^{-t}dtdx}\int_L h^p(x,t)e^{-t}dtdx\right)^{\frac{1}{p}}$$

is non-decreasing in $(-1,\infty)$. □

3. Proof of functional Zhang's inequality

In this section we will give the proof of the functional version of Zhang's inequality (4). For any $g \in \mathcal{F}(\mathbb{R}^n)$ such that $g(0) > 0$ and $p > 0$, we will consider the following important family of convex bodies, which was introduced by K. Ball in [B, pg. 74]. We denote

$$\widetilde{K}_p(g) := \left\{ x \in \mathbb{R}^n : \int_0^\infty g(rx) r^{p-1} dr \geq \frac{g(0)}{p} \right\}.$$

It follows from the definition that the radial function of $\widetilde{K}_p(g)$ is given by

$$\rho^p_{\widetilde{K}_p(g)}(u) = \frac{1}{g(0)} \int_0^\infty p r^{p-1} g(ru) dr.$$

Remark 2. It is well known (cf. [BGVV, Proposition 2.5.7]) that, for any $g \in \mathcal{F}(\mathbb{R}^n)$ such that $\|g\|_\infty = g(0)$ and $0 < p \leq q$,

$$\frac{\Gamma(1+p)^{\frac{1}{p}}}{\Gamma(1+q)^{\frac{1}{q}}} \widetilde{K}_q(g) \subseteq \widetilde{K}_p(g) \subseteq \widetilde{K}_q(g).$$

We will make use of the following well known relation (cf. [B]) between the Lebesgue measure of $\widetilde{K}_n(g)$ and the integral of g.

Lemma 3.1 ([B])**.** *Let $g \in \mathcal{F}(\mathbb{R}^n)$ be such that $g(0) > 0$. Then*

$$|\widetilde{K}_n(g)| = \frac{1}{g(0)} \int_{\mathbb{R}^n} g(x) dx.$$

For any $f \in \mathcal{F}(\mathbb{R}^n)$, we collect below the properties of its covariogram functional g_f, whose proof can be found in [ABG, Lemma 2.1].

Lemma 3.2. *Let $f \in \mathcal{F}(\mathbb{R}^n)$. Then the function $g_f : \mathbb{R}^n \to \mathbb{R}$ defined by*

$$g_f(x) = \int_0^\infty e^{-t} |K_t(f) \cap (x + K_t(f))| dt$$

verifies that

$$g_f(x) = \int_{\mathbb{R}^n} \min \left\{ \frac{f(y)}{\|f\|_\infty}, \frac{f(y-x)}{\|f\|_\infty} \right\} dy$$

is even, log-concave, $0 \in \operatorname{int}(\operatorname{supp} g_f)$ with $\|g_f\|_\infty = g_f(0) = \int_0^\infty e^{-t} |K_t(f)| dt = \int_{\mathbb{R}^n} \frac{f(x)}{\|f\|_\infty} dx > 0$, and $\int_{\mathbb{R}^n} g_f(x) dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \min \left\{ \frac{f(y)}{\|f\|_\infty}, \frac{f(x)}{\|f\|_\infty} \right\} dy dx$.

In the particular case of g_f as in Lemma 3.2, we can provide an alternative definition for $\widetilde{K}_p(g_f)$ in terms of its radial function that will allow us to obtain the polar projection body of f as a limiting case of this expression when p tends to -1.

Lemma 3.3. *Let $f \in \mathcal{F}(\mathbb{R}^n)$ and let $g_f : \mathbb{R}^n \to \mathbb{R}$ be the function*

$$g_f(x) = \int_0^\infty e^{-t} |K_t(f) \cap (x + K_t(f))| dt.$$

Then, for any $u \in S^{n-1}$ and $p > 0$,

$$\rho^p_{\widetilde{K}_p(g_f)}(u) = \frac{1}{(p+1) \int_{\mathbb{R}^n} \frac{f(x)}{\|f\|_\infty} dx} \int_0^\infty e^{-t} \int_{P_{u^\perp} K_t(f)} |K_t(f) \cap (y + \langle u \rangle)|^{p+1} dy dt,$$

where $\langle u\rangle$ denotes the 1-dimensional subspace spanned by u.

Remark 3. Notice that the right hand side in the equality above is defined for $p>-1$ and that, since $(p+1)\Gamma(1+p)=\Gamma(2+p)$, if $p\to -1^+$ then

$$\frac{1}{(p+1)\Gamma(1+p)\int_{\mathbb{R}^n}\frac{f(x)}{\|f\|_\infty}dx}\int_0^\infty e^{-t}\int_{P_{u^\perp}K_t(f)}|K_t(f)\cap(y+\langle u\rangle)|^{p+1}dt\to\frac{\|u\|_{\Pi^*(f)}}{2\|f\|_1}.$$

Proof of Lemma 3.3. By Lemma 3.2, $g_f(0)>0$ and

$$\begin{aligned}
&\rho^p_{\widetilde{K}_p(g_f)}(u)=\\
&= \frac{p}{g_f(0)}\int_0^\infty r^{p-1}g_f(ru)dr\\
&= \frac{1}{g_f(0)}\int_0^\infty pr^{p-1}\int_0^\infty e^{-t}|K_t(f)\cap(ru+K_t(f))|dtdr\\
&= \frac{1}{g_f(0)}\int_0^\infty e^{-t}\int_0^{\rho_{K_t(f)-K_t(f)}(u)}pr^{p-1}|K_t(f)\cap(ru+K_t(f))|drdt\\
&= \frac{1}{g_f(0)}\int_0^\infty e^{-t}\int_0^{\rho_{K_t(f)-K_t(f)}(u)}pr^{p-1}\int_{P_{u^\perp}K_t(f)}\max\{|K_t(f)\cap(y+\langle u\rangle)|-r,0\}dydrdt\\
&= \frac{1}{g_f(0)}\int_0^\infty e^{-t}\int_{P_{u^\perp}K_t(f)}\int_0^{|K_t(f)\cap(y+\langle u\rangle)|}pr^{p-1}\left(|K_t(f)\cap(y+\langle u\rangle)|-r\right)\,drdydt\\
&= \frac{1}{(p+1)g_f(0)}\int_0^\infty e^{-t}\int_{P_{u^\perp}K_t(f)}|K_t(f)\cap(y+\langle u\rangle)|^{p+1}dydt.
\end{aligned}$$

□

Proof of inequality (4). Let $u\in S^{n-1}$ and consider a function $h:L\to[0,\infty)$ given by

$$h(x,t)=|K_t(f)\cap\{(x,t)+\lambda u\,:\,\lambda\geq 0\}|,$$

where L is the epigraph of $-\log\frac{f}{\|f\|_\infty}$. Since L is convex, h is concave. For any $p>-1$ we have,

$$\frac{1}{(p+1)}\int_0^\infty e^{-t}\int_{P_{u^\perp}K_t(f)}|K_t(f)\cap(y+\langle u\rangle)|^{p+1}dxdt=\int_0^\infty\int_{K_t(f)}e^{-t}h(x,t)^p dxdt.$$

Therefore, by Theorem 1.1, for every $-1<p<0$,

$$\frac{1}{(p+1)\Gamma(1+p)\int_{\mathbb{R}^n}\frac{f}{\|f\|_\infty}}\int_0^\infty e^{-t}\int_{P_{u^\perp}K_t(f)}|K_t(f)\cap(y+\langle u\rangle)|^{p+1}dtdx\leq$$

$$\leq\left(\frac{1}{(n+1)n!\int_{\mathbb{R}^n}\frac{f}{\|f\|_\infty}}\int_0^\infty e^{-t}\int_{P_{u^\perp}K_t(f)}|K_t(f)\cap(y+\langle u\rangle)|^{n+1}dtdx\right)^{\frac{p}{n}}=\frac{\rho_{\widetilde{K}_n(g_f)}(u)^p}{n!^{\frac{p}{n}}}.$$

Taking limit as $p\to -1$ and by Lemma 3.3 we obtain

$$\rho_{\widetilde{K}_n(g_f)}(u)\leq 2(n!)^{\frac{1}{n}}\|f\|_1\rho_{\Pi^*(f)}(u),$$

that is,

$$\widetilde{K}_n(g_f)\subseteq 2(n!)^{\frac{1}{n}}\|f\|_1\Pi^*(f).$$

Taking Lebesgue measure and using Lemmas 3.2 and 3.1 we obtain inequality (4).

□

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Área de Análisis Matemático, Departamento de Matemáticas, Facultad de Ciencias, Universidad de Zaragoza, Pedro Cerbuna 12, 50009 Zaragoza (Spain), IUMA
E-mail address, (David Alonso-Gutiérrez): alonsod@unizar.es

Área de análisis matemático, Departamento de matemáticas, Facultad de Ciencias, Universidad de Zaragoza, Pedro cerbuna 12, 50009 Zaragoza (Spain), IUMA
E-mail address, (Julio Bernués): bernues@unizar.es

Departamento de Didáctica de las Ciencias Matemáticas y Sociales, Facultad de Educación, Universidad de Murcia, Murcia, Spain
E-mail address, Bernardo González Merino: bgmerino@um.es