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# Best approximation of functions by log-polynomials ☆

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A B S T R A C T

Lasserre [32] proved that for every compact set $K \subset \mathbb{R}^n$ and every even number $d$ there exists a unique homogeneous polynomial $g_0$ of degree $d$ with $K \subset G_1(g_0) = \{x \in \mathbb{R}^n : g_0(x) \leq 1\}$ minimizing $|G_1(g)|$ among all such polynomials $g$ fulfilling the condition $K \subset G_1(g)$. This result extends the notion of the Löwner ellipsoid, not only from convex bodies to arbitrary compact sets (which was immediate if $d = 2$ by taking convex hulls), but also from ellipsoids to level sets of homogeneous polynomial of an arbitrary even degree.

In this paper we extend this result for the class of non-negative log-concave functions in two different ways. One of them is the straightforward extension of the known results, and the other one is a suitable extension with uniqueness of the solution in

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the corresponding problem and a characterization in terms of some 'contact points'.


## 1. Introduction

In [3] the authors proved that for any non-negative integrable log-concave function $f : \mathbb{R}^n \to [0, +\infty)$ with $\|f\|_\infty = f(0) = 1$, there exists a unique pair $(t_0, \mathcal{E}_0)$, with $t_0 \in (0, 1]$ and $\mathcal{E}_0 \subset \mathbb{R}^n$ an ellipsoid such that

$$t_0 \chi_{\mathcal{E}_0} \leq f \tag{1}$$

maximizing

$$\int_{\mathbb{R}^n} t\chi_{\mathcal{E}}(x)\, dx = t|\mathcal{E}|$$

among all the pairs $(t, \mathcal{E})$ verifying (1), where for any measurable set $K \subseteq \mathbb{R}^n$, $\chi_K$ denotes its characteristic function and $|K|$ denotes its $n$-dimensional Lebesgue measure.

This is a functional version of John's celebrated theorem [30], which provides the existence of a unique maximal volume ellipsoid contained in any convex body $K \subset \mathbb{R}^n$. This maximal volume ellipsoid is called the John ellipsoid of $K$. Whenever one takes, for any convex body $K \subseteq \mathbb{R}^n$ containing the origin, $f = \chi_K$ in the aforementioned maximization the solution is $t_0 = 1$ and $\mathcal{E}_0$ is the John ellipsoid of $K$. For other functional version of the problem, see [28].

Recall that if $f(x) = e^{-u(x)}$ with $u : \mathbb{R}^n \to [0, +\infty]$ a convex function, its polar function $f^\circ$ is defined as $f^\circ(x) = e^{-u^*(x)}$, where $u^* : \mathbb{R}^n \to \mathbb{R}$ is the Legendre transform of $u$ given by $u^*(x) = \sup_{y\in\mathbb{R}^n}(\langle x, y\rangle - u(y))$. If $K \subseteq \mathbb{R}^n$ is a convex body containing the origin in its interior, then $(\chi_K)^\circ = e^{-\|\cdot\|_{K^\circ}}$. Here, for any convex body $K$ containing the origin, $\|\cdot\|_K$ denotes the Minkowski gauge

$$\|x\|_K = \inf\{\lambda > 0 \,:\, x \in \lambda K\},$$

and $K^\circ$ denotes the polar body of $K$, defined by

$$K^\circ = \{x \in \mathbb{R}^n \,:\, \langle x, y\rangle \leq 1, \forall y \in K\}.$$

Besides, we will denote by $|x| = \|x\|_{B_2^n}$ the Euclidean norm, for every $x \in \mathbb{R}^n$, where $B_2^n$ denotes the Euclidean unit ball and the epigraph of a convex function $u : \mathbb{R}^n \to [0, +\infty)$ by

$$\mathrm{epi}(u) := \{(x, t) \in \mathbb{R}^n \times [0, +\infty) \,:\, u(x) \leq t\}.$$

Using this functional notion of polarity [31], taking into account that $f^{\circ\circ} = f$ for any log-concave function $f$, and the equality $|\mathcal{E}||\mathcal{E}^\circ| = |B_2^n|^2$ for any origin centered ellipsoid $\mathcal{E}$, whenever $f$ is even, the previous result can be stated as a minimizing volume problem, i.e., for any even integrable log-concave function $f : \mathbb{R}^n \to [0, +\infty)$ with $\|f\|_\infty = f(0) = 1$ there exists a unique pair $(t_1, \mathcal{E}_1)$ with $t_1 \geq 1$ and $\mathcal{E}_1 \subset \mathbb{R}^n$ a origin centered ellipsoid such that

$$f \leq t_1 \exp\{-\|\cdot\|_{\mathcal{E}_1}\} \tag{2}$$

minimizing

$$\int t_1 \exp\{-\|x\|_{\mathcal{E}_1}\}\, dx = t_1 n! |\mathcal{E}_1|$$

among all the pairs $(t, \mathcal{E})$ verifying (2).

In [35] the authors provided a definition of a functional Löwner ellipsoid also whenever $f$ is not necessarily even. They considered the corresponding integral minimization problem related to such functional ellipsoid. In that case the solution does not necessarily coincide with the polar of the functional John ellipsoid of the polar function. This result generalizes the dual version of John's Theorem, which states that for any convex body $K \subseteq \mathbb{R}^n$ there exists a unique ellipsoid, known as the Löwner ellipsoid of $K$, of minimal volume containing $K$. Whenever one takes, for any convex body $K$ containing the origin, $f = \exp\{-\|\cdot\|_K\}$, the solution of the minimization problem appearing in [35] also recovers the Löwner's ellipsoid of $K$. Interpreting and proving functional versions for log-concave functions of well-known geometric results has become increasingly popular in the last years, see for instance [2], [3], [4], [6], [7], [8], [15], [16], [17], [18], [20], [19], [31], [34], [41].

John and Löwner ellipsoids of convex bodies have been widely investigated in the literature (see, for example, [23], [24], [22], [25]). Furthermore, the John or the Löwner ellipsoid of a convex body $K \subseteq \mathbb{R}^n$ is characterized by the existence of some contact points between the boundary of $K$ and the Euclidean sphere, $S^{n-1}$ (see [9], [13]).

Other connections between convex bodies and ellipsoids can be found in the literature. For instance, the Legendre and Binet ellipsoids are well-known concepts from classical mechanics. For some references, see [33], [36], [38], [39], and [37] for recent developments.

On the other hand, Lasserre [32] generalized the definition of the Löwner ellipsoid for any compact (non-necessarily convex) set by means of replacing the bilinear form given by an ellipsoid by a homogeneous polynomial of even degree $d \geq 2$.

More precisely, if we denote by $\mathbb{H}_d(\mathbb{R}^n)$ the vector space of homogeneous polynomials of degree $d$ in $\mathbb{R}^n$, of dimension $h_d(n) = \binom{n+d-1}{d}$, it was proved that, given any compact set $K \subset \mathbb{R}^n$ with non-empty interior and an even integer $d \in \mathbb{N}$, there exists a unique homogeneous polynomial $g_0 \in \mathbb{H}_d(\mathbb{R}^n)$ of degree $d$, the *d-Lasserre-Löwner polynomial*, such that

$$K \subseteq G_1(g_0) = \{x \in \mathbb{R}^n : g_0(x) \leq 1\} \tag{3}$$

with minimum volume $|G_1(g_0)|$ among all $d$-homogeneous polynomial verifying (3).

Let $\mathbb{F}_d(\mathbb{R}^n)$ be the set in $\mathbb{H}_d(\mathbb{R}^n)$ of all $d$-homogeneous polynomials in $\mathbb{R}^n$ such that $|G_1(g)| < +\infty$. Note that $|G_1(g)| < +\infty$ implies $g \geq 0$. In particular, the previous minimization problem cannot be stated for odd $d$.

Moreover, the solution is also characterized in terms of some common contact points in the boundaries of $K$ and $G_1(g_0)$ (cf. [32]). More precisely, $|G_1(g_0)|$ is minimum among all $g \in \mathbb{H}_d(\mathbb{R}^n)$ verifying (3) if and only if there exist $y_1, \ldots, y_s \in K$, $\lambda_1, \ldots, \lambda_s > 0$, with $s \leq h_d(n)$, such that $g_0(y_i) = 1$ for $i = 1, \ldots, s$, and

$$\int_{\mathbb{R}^n} x^\alpha e^{-g_0(x)} dx = \sum_{i=1}^{s} \lambda_i y_i^\alpha \tag{4}$$

for every $\alpha \in \mathbb{N}^n$ such that $|\alpha| = \sum_{i=1}^n \alpha_i = d$, where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Note that the identity above implies a trace identity (see Lemma 2.1)

$$\frac{n}{d} \int_{\mathbb{R}^n} e^{-g_0(x)} dx = \int_{\mathbb{R}^n} g_0(x) e^{-g_0(x)} dx = \sum_{i=1}^{s} \lambda_i.$$

In this paper we will extend the result of [32] to the functional setting. Let us pose the following problem:

**Problem 1.** Given $f : \mathbb{R}^n \to [0, +\infty)$ with $\|f\|_\infty = f(0) = 1$, and $d \in \mathbb{N}$ even, minimize

$$\int_{\mathbb{R}^n} t e^{-g(x)^{\frac{1}{d}}} dx = tn!|G_1(g)|$$

among all $g \in \mathbb{H}_d(\mathbb{R}^n)$ and $t \geq 1$ such that

$$f(x) \leq t e^{-g(x)^{\frac{1}{d}}}. \tag{5}$$

Note that the functional $(t, g) \mapsto t|G_1(g)|$ to be minimized verifies a strong global convexity property on the space of pairs $(r, g)$ verifying (5), once the natural reparametrization $t = e^r$, together with an appropriate change in the integral to consider, is taken (see Lemma 2.2 and Lemma 4.6). Despite this global property, the set of pairs $(r, g)$ verifying (5), with $t = e^r$, does not verify a suitable convexity or compactness property, so the existence and uniqueness of a minimizing pair is not straightforwardly obtained. Considering, for instance, $f(x) = \chi_{B_2^n}$ and taking $r_0, r_1 > 0$ and polynomials of the form $g_i(x) = r_i^d |x|^d$, $i = 0, 1$, we have

$$\chi_{B_2^n} \leq \exp\{r_i - g_i^{1/d}\}$$

for $i = 0, 1$. However,

$$\chi_{B_2^n} \not\leq \exp\{r_\theta - g_\theta^{1/d}\}$$

for any $\theta \in (0,1)$, where $r_\theta = (1-\theta)r_0 + \theta r_1$ and $g_\theta = (1-\theta)g_0 + \theta g_1$.

In view of Lasserre's result, one might think of integrable functions $f : \mathbb{R}^n \to [0, +\infty)$ as the typical extension of compact sets to spaces of functions. Unfortunately, *Problem 1* does not make sense in such a general case (see Example 3.1). Motivated by this fact, we solve the problem in the setting of log-concave integrable functions. Let $\mathcal{F}(\mathbb{R}^n)$ be the set of all log-concave integrable functions on $\mathbb{R}^n$.

**Theorem 1.1.** *Let $f \in \mathcal{F}(\mathbb{R}^n)$ with $\|f\|_\infty = f(0) = 1$ and $d \in \mathbb{N}$ even. Then there exists $(t_1, g_1) \in [1, +\infty) \times \mathbb{F}_d(\mathbb{R}^n)$ a solution of Problem 1.*

Notice that the problem considered in Theorem 1.1 was solved with uniqueness for $d = 2$ in [3], in the even case, and in [35], in the general case, since for $g \in \mathbb{H}_2(\mathbb{R}^n)$ the set $G_1(g)$ is an ellipsoid provided that $|G_1(g)|$ is finite (see Lemma 2.3 (2.3)).

However, even this case is not solved with uniqueness in the proof of Theorem 1.1 with this point of view, since polarity does not work clearly between polynomials. More precisely, if we try to construct a proof by taking duals in the proof in [3], we would need to take the polar of the ellipsoid $G_1(g)$, but the expression of the polynomial defining the polar ellipsoid in terms of $g$ is not clear.

In the general case, the uniqueness is not straightforwardly obtained (although we do not know any example for which the minimization point is not unique). It seems to us that the proof would require some more convexity properties than the ones we have obtained.

The following similar problem is also posed. Unlike the case of Problem 1 above, we are able to show existence and uniqueness of the solution.

**Problem 2.** Given $f : \mathbb{R}^n \to [0, +\infty)$ with $\|f\|_\infty = f(0) = 1$, and $d \in \mathbb{N}$ even, minimize

$$\int_{\mathbb{R}^n} te^{-g(x)}\, dx = t\Gamma\left(\tfrac{n}{d} + 1\right)|G_1(g)|$$

among all $g \in \mathbb{H}_d(\mathbb{R}^n)$ and $t \geq 1$ such that

$$f(x) \leq te^{-g(x)}. \tag{6}$$

Again, the existence of a global minimum is not guaranteed using the convexity of the functional to be optimized, since the argument would need the feasible set of solutions to be a convex compact set. Lemma 4.5 proves that the set is convex, but compactness can not be assured.

This problem is solved with uniqueness, when imposing an extra condition, in the following result.

**Theorem 1.2.** *Let $f : \mathbb{R}^n \to [0, +\infty)$ be a log-concave function with $\|f\|_\infty = f(0) = 1$ and $d \in \mathbb{N}$ even, such that*

$$\widehat{H}_1(f) = \bigcup_{\lambda \in (0,1)} \log(1/\lambda)^{-1/d}\{x \in \mathbb{R}^n : f(x) \geq \lambda\}$$

*is bounded. Then there exists $(t_2, g_2) \in [1, +\infty) \times \mathbb{F}_d(\mathbb{R}^n)$ a unique solution of Problem 2.*

For an interior minimization point $(t_2, g_2) \in (1, +\infty) \times \mathrm{int}\mathbb{F}_d(\mathbb{R}^n)$, being the unique solution of *Problem 2* can be characterized by some *touching conditions*, via the Karush-Kuhn-Tucker conditions (see [5], [27]). For these conditions to hold, no hypothesis on the log-concavity of $f$ is needed.

**Theorem 1.3.** *Let $f : \mathbb{R}^n \to [0, +\infty)$ be a bounded function with $\|f\|_\infty = f(0) = 1$. Moreover, let $(t_2, g_2) \in (1, +\infty) \times \mathrm{int}(\mathbb{F}_d(\mathbb{R}^n))$ be such that $f(x) \leq t_2 \exp(-g_2(x))$ for every $x \in \mathbb{R}^n$. Then the following are equivalent:*

(i) *$(t_2, g_2)$ is the only solution of Problem 2.*
(ii) *There exist $x_1, \dots, x_m \in \mathbb{R}^n$, $m \leq \binom{n+d-1}{d} + 1$, with $f(x_i) = t_2 \exp(-g_2(x_i))$, and $\lambda_i > 0$, $1 \leq i \leq m$, such that*

$$t_2 \int_{\mathbb{R}^n} \exp(-g_2(x))dx = \sum_{i=1}^m \lambda_i \text{ and}$$

$$t_2 \int_{\mathbb{R}^n} x^\alpha \exp(-g_2(x))dx = \sum_{i=1}^m \lambda_i x_i^\alpha \quad \text{for all } \alpha \in \mathbb{N}_d^n.$$

The paper is organized as follows. In Section 2 we provide all the definitions and properties related to homogeneous polynomials which are needed for the study of both problems. Section 3 is devoted to give the existence of a minimization point in *Problem 1*. In Section 4 we study *Problem 2*, giving similar results as the ones given in Section 3, and new facts that allow to prove the existence and uniqueness of the minimization problem, under the additional assumption given in Theorem 1.2. Further, we give the characterization of the minimization point in terms of the contact points. Finally in Section 5 we introduce the $d$-outer volume and integral ratio of a convex body, and show an application of the $d$-Löwner-Lasserre polynomial to approximation of convex bodies.

## 2. Homogeneous polynomials

Let $\mathbb{H}_d(\mathbb{R}^n)$ be the vector space of homogeneous polynomials of degree $d$ in $\mathbb{R}^n$, with dimension $h_d(n) = \binom{n+d-1}{d}$. Any $g \in \mathbb{H}_d(\mathbb{R}^n)$ can be uniquely written as

$$g(x) = \sum_{\alpha \in \mathbb{N}_d^n} g_\alpha x^\alpha$$

where $\mathbb{N}_d^n = \{\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n : |\alpha| = \alpha_1 + \cdots + \alpha_n = d\}$ and for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_d^n$, $x^\alpha$ denotes the monomial $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$.

For any $g \in \mathbb{H}_d(\mathbb{R}^n)$, let us denote, for any $t > 0$, $G_t(g) = \{x \in \mathbb{R}^n : g(x) \leq t\}$. Notice that by the homogeneity of $g$, $G_t(g) = t^{\frac{1}{d}} G_1(g)$, and that if $|G_1(g)| < +\infty$ then necessarily $g$ must be non-negative on $\mathbb{R}^n$ and therefore $d$ must be even. Moreover, if $d = 2$, $G_1(g)$ is an ellipsoid. However, for $d > 2$, $G_1(g)$ can be non-convex, and even unbounded, as the example $g(x, y) = x^2 y^2 (x^2 + y^2)$ shows (see [32] and Lemma 2.3 below).

Let $\mathbb{F}_d(\mathbb{R}^n)$ be the set in $\mathbb{H}_d(\mathbb{R}^n)$ of all $d$-homogeneous polynomials in $\mathbb{R}^n$ such that $|G_1(g)| < +\infty$.

We first show the identities involving the integrals in *Problem 1* and *Problem 2* and the volume $|G_1(g)|$. They are particular cases of the following technical result (a particular case is given in [32, Thm. 2.2]).

**Lemma 2.1.** *Let $n \geq 1$, $k \geq 0$, $d \geq 2$ even, $r \in \mathbb{R}$ and $m > 0$ be such that $\frac{n+k}{d} + r > 0$. For $\alpha \in \mathbb{N}_k^n$, let $g \in \mathbb{H}_d(\mathbb{R}^n)$ be such that $x^\alpha$ is integrable in $G_1(g)$. Then*

$$\int_{\mathbb{R}^n} x^\alpha g(x)^r \exp(-g(x)^{1/m})\, dx = \tfrac{n+k}{d} m \Gamma\left(m\left(\tfrac{n+k}{d} + r\right)\right) \int_{G_1(g)} x^\alpha\, dx.$$

*In particular,*

$$\int_{\mathbb{R}^n} g(x) \exp(-g(x))\, dx = \tfrac{n}{d} \Gamma\left(\tfrac{n}{d} + 1\right) |G_1(g)|.$$

**Proof.** Let us define, for any $y > 0$, $w_\alpha(y) = \int_{\{x : g(x) \leq y\}} x^\alpha dx$. By the homogeneity of $g$ we have that $w_\alpha(y) = y^{\frac{n+k}{d}} w_\alpha(1)$. Therefore

$$\int_{\mathbb{R}^n} x^\alpha g(x)^r \exp(-g(x)^{1/m})\, dx = \int_{\mathbb{R}^n} x^\alpha \int_{g(x)^{1/m}}^{+\infty} (y - mr) y^{mr-1} e^{-y}\, dy\, dx$$

$$= \int_0^{+\infty} (y-mr)y^{mr-1}e^{-y} \int_{\{x:g(x)\leq y^m\}} x^\alpha \, dx \, dy$$

$$= w_\alpha(1) \int_0^{+\infty} (y-mr) y^{m(\frac{n+k}{d}+r)-1} e^{-y} \, dy$$

$$= \frac{n+k}{d} m\Gamma\left(m\left(\tfrac{n+k}{d}+r\right)\right) w_\alpha(1). \quad \square$$

**Lemma 2.2.** *Let $g \in \mathbb{F}_d(\mathbb{R}^n)$. For every $t \geq 0$, and $m > 0$,*

$$|G_t(g)| = \frac{t^{n/d}}{\Gamma\left(\frac{nm}{d}+1\right)} \int_{\mathbb{R}^n} \exp(-g(x)^{1/m}) \, dx.$$

*In particular,*

$$|G_1(g)| = \frac{1}{\Gamma\left(\frac{n}{d}+1\right)} \int_{\mathbb{R}^n} \exp(-g(x)) \, dx = \frac{1}{n!} \int_{\mathbb{R}^n} \exp(-g(x)^{1/d}) \, dx.$$

**Proof.** By the homogeneity of $g$, $G_t(g) = t^{\frac{1}{d}} G_1(G)$ and then $|G_t(g)| = t^{n/d}|G_1(g)| < +\infty$ for any $t > 0$. Besides, by Lemma 2.1 with $k = 0$, $\alpha = (0\ldots,0)$, and $r = 0$, we have that for any $m > 0$

$$\int_{\mathbb{R}^n} \exp(-g(x)^{1/m}) \, dx = \Gamma\left(\frac{nm}{d}+1\right) |G_1(g)|.$$

In particular, taking $m = 1$ or $m = d$ we obtain

$$|G_1(g)| = \frac{1}{\Gamma\left(\frac{n}{d}+1\right)} \int_{\mathbb{R}^n} \exp(-g(x)) \, dx = \frac{1}{n!} \int_{\mathbb{R}^n} \exp(-g(x)^{1/d}) \, dx. \quad \square$$

The following result states some topological properties of $\mathbb{F}_d(\mathbb{R}^n)$.

**Lemma 2.3.** *Let $d \in \mathbb{N}$ be an even integer.*

(1) *$\mathbb{F}_d(\mathbb{R}^n)$ is a convex cone in $\mathbb{H}_d(\mathbb{R}^n)$, which is not closed and has non-empty interior.*
(2) *For $d = 2$, $g \in \mathbb{F}_2(\mathbb{R}^n)$ if and only if $G_1(g)$ is bounded (an ellipsoid). Moreover, $\mathbb{F}_2(\mathbb{R}^n)$ is open.*
(3) *For $d = 4$, $n = 2$, $g \in \mathbb{F}_4(\mathbb{R}^2)$ if and only if $G_1(g)$ is bounded. Moreover $\mathbb{F}_4(\mathbb{R}^2)$ is open.*
(4) *For $d = 4$, $n \geq 3$, there exists $g \in \mathbb{F}_4(\mathbb{R}^n)$ so that $G_1(g)$ is not bounded. Moreover $\mathbb{F}_4(\mathbb{R}^n)$ is not open.*

(5) *For $d \geq 6$, $n \geq 2$, there exists $g \in \mathbb{F}_d(\mathbb{R}^n)$ so that $G_1(g)$ is not bounded. Moreover, $g \in \mathbb{F}_d(\mathbb{R}^n)$ is not open.*

**Proof.** (1) $\mathbb{F}_d(\mathbb{R}^n)$ is a convex cone, as proved in [32, Lemma 2.1].
The polynomial $g_0(x) = \sum_{i=1}^n x_i^d$ is an interior point in $\mathbb{F}_d(\mathbb{R}^n)$. In fact, if $g(x) = \sum_{\alpha \in \mathbb{N}_d^n} g_\alpha x^\alpha$ is such that $|g_\alpha - g_{0,\alpha}| < \varepsilon$, for some $\varepsilon < \frac{1}{2\left(1+\left(\binom{n+d-1}{d}-n\right)\right)}$ for every $\alpha \in \mathbb{N}_d^n$, then

$$\begin{aligned} g(x) &\geq (1-\varepsilon)\sum_{i=1}^n x_i^d - \varepsilon\left(\binom{n+d-1}{d} - n\right)\|x\|_\infty^d \\ &\geq \left(1 - \varepsilon\left(1 + \left(\binom{n+d-1}{d} - n\right)\right)\right)\|x\|_\infty^d \\ &\geq \frac{1}{2}\|x\|_\infty^d, \end{aligned}$$

where $\|x\|_\infty = \max\{|x_i| : 1 \leq i \leq n\}$. Then, $G_1(g)$ is bounded, so $g \in \mathbb{F}_d(\mathbb{R}^n)$.
The polynomial $tg_0$ belongs to $\mathbb{F}_d(\mathbb{R}^n)$ for any $t > 0$ but the zero polynomial does not belong to $\mathbb{F}_d(\mathbb{R}^n)$. Therefore $\mathbb{F}_d(\mathbb{R}^n)$ is not closed.

(2) Applying Sylvester's law of inertia [43] for quadratic forms, any $g \in \mathbb{H}_2(\mathbb{R}^n)$ can be written in the canonical form $g(x) = \sum_{i=1}^n \alpha_i x_i^2$ with an appropriate change of coordinates. Then $g \in \mathbb{F}_2(\mathbb{R}^n)$ if and only if $\alpha_i > 0$ for all $i = 1, \dots n$ (if and only if $G_1(g)$ is an ellipsoid). That clearly implies that $\mathbb{F}_2(\mathbb{R}^n)$ is open.

(3) Similarly, any $g \in \mathbb{H}_4(\mathbb{R}^2)$ can be written, with an appropriate change of coordinates, in the canonical form $g(x, y) = ax^4 + 2bx^2y^2 + cy^4$ (see [42, Les. XV]).
Notice that written in this canonical form $G_1(g)$ is bounded if and only if $a, c > 0$ and $b > -\sqrt{ac}$. Indeed, if $G_1(g)$ is bounded then necessarily $a > 0$ and $c > 0$. In such case, if $b \leq -\sqrt{ac}$ then $b^2 \geq ac$ and there exists some $\lambda = -\frac{b}{2a} > 0$ such that

$$a + 2b\lambda + c\lambda^2 \leq 0$$

and then all the points $(x, y) \in \mathbb{R}^2$ with $y = \sqrt{\lambda}x$ belong to $G_1(g)$. Conversely, if $a, c > 0$ and $h = b + \sqrt{ac} > 0$, writing

$$g(x, y) = (\sqrt{a}x^2 - \sqrt{c}y^2)^2 + 2hx^2y^2$$

we have that if $(x, y) \in G_1(g)$, then $|\sqrt{a}x^2 - \sqrt{c}y^2| \leq 1$, and $2hx^2y^2 \leq 1$. The two inequalities imply $|x|, |y|$ are bounded and then $G_1(g)$ is bounded.
Furthermore, $g \in \mathbb{F}_4(\mathbb{R}^2)$ if and only if $a, c > 0$ and $b > -\sqrt{ac}$. Indeed, if $|G_1(g)| < +\infty$, then $a, c > 0$, otherwise for every $y_0 \in \mathbb{R}$, $g(x, y_0) \leq 0$ for every $x$ large enough, or for every $x_0 \in \mathbb{R}$ $g(x_0, y) \leq 0$ for every $y$ large enough; in any case $\iint e^{-g(x,y)}\, dxdy = +\infty$. If $a, c > 0$ and $b \leq -\sqrt{ac}$, then $g(x, y) \leq (\sqrt{a}x^2 - \sqrt{c}y^2)^2 = (\sqrt{a}x + \sqrt{c}y)^2(\sqrt{a}x - \sqrt{c}y)^2$. The change of variables $u = \sqrt{a}x + \sqrt{c}y$, $v = \sqrt{a}x - \sqrt{c}y$

and the fact $\iint e^{-u^2v^2}\,dudv = +\infty$ show that $|G_1(g)| = +\infty$. Therefore, $a, c > 0$ and $b > -\sqrt{ac}$. Conversely, if $a, c > 0$ and $b > -\sqrt{ac}$, then $G_1(g)$ is bounded, and therefore $|G_1(g)| < +\infty$.
Consequently, $g \in \mathbb{F}_4(\mathbb{R}^2)$ if and only if written in its canonical form $a, c > 0$ and $b > -\sqrt{ac}$ and then $\mathbb{F}_4(\mathbb{R}^2)$ is open.

(4) Let $g(x, y, z) = x^4 + y^4 + z^4 - 2\sqrt{2}x^2yz$. Then $G_1(g)$ is unbounded (see Figure 1), since it contains the lines $y = z$, $x = \pm\sqrt[4]{2}y$. But $|G_1(g)| < +\infty$. In fact, using Lemma 2.2, and since $g$ is even with respect to $x$ and for $y, z > 0$, we have that $g(x, y, z) = g(x, -y, -z) \le g(x, -y, z) = g(x, y, -z)$, it is enough to prove that

$$\iiint\limits_{[0,+\infty)^3} e^{-g(x,y,z)}\,dxdydz < +\infty.$$

The change of variables $x = u, y = uv, z = uw$, with Jacobian $J(u, v, w) = u^2$, rewrites the previous integral as

$$\iiint\limits_{[0,+\infty)^3} u^2 e^{-u^4 h(v,w)}\,dudvdw$$

where $h(v, w) = 1 + v^4 + w^4 - 2\sqrt{2}vw$. The change of variables $\overline{u} = uh(v, w)^{\frac{1}{4}}$ shows that the previous integral equals

$$\iint\limits_{[0,+\infty)^2} \frac{dv\,dw}{h(v, w)^{3/4}} \int\limits_0^{+\infty} \overline{u}^2 e^{-\overline{u}^4}\,d\overline{u}.$$

Therefore, it suffices to see that $h(v, w)^{-3/4}$ is integrable in $[0, +\infty)^2$. Note that $h$ can be written as

$$h(v, w) = \sqrt{2}(v - w)^2 + (v^2 - \gamma^2)^2 + (w^2 - \gamma^2)^2$$

with $\gamma = 2^{-1/4}$. Notice that $h(\gamma, \gamma) = 0$ and that for every $(v, w) \in [0, +\infty)^2$ such that $(v, w) \neq (\gamma, \gamma)$ we have that $h(v, w) > 0$.
First, for $(v, w) \in [0, 2\gamma] \times [0, 2\gamma]$, the bound

$$h(v, w) \ge \gamma^2((v - \gamma)^2 + (w - \gamma)^2) \ge \sqrt{2}|v - \gamma||w - \gamma|$$

shows the integrability of $h(v, w)^{-3/4}$ in $[0, 2\gamma] \times [0, 2\gamma]$.
Second, for $(v, w) \in [0, 2\gamma] \times [2\gamma, +\infty)$, the bound

$$h(v, w) \ge \gamma^2(v - \gamma)^2 + \tfrac{1}{4}w^4 \ge \tfrac{1}{2^{1/4}}|v - \gamma|w^2$$

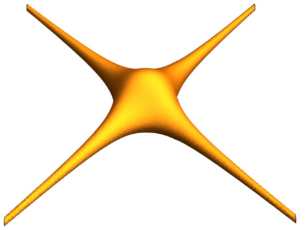

**Fig. 1.** $G_1(g)$ for $g(x, y, z) = x^4 + y^4 + z^4 - 2\sqrt{2}x^2yz$.

shows the integrability of $h(v, w)^{-3/4}$ in $[0, 2\gamma] \times [2\gamma, +\infty)$. A similar bound shows the integrability in $[2\gamma, +\infty) \times [0, 2\gamma]$.

Finally, for $(v, w) \in [2\gamma, +\infty) \times [2\gamma, +\infty)$, the bound

$$h(v, w) \geq \tfrac{1}{4}(v^4 + w^4) \geq \tfrac{1}{2}v^2w^2$$

shows the integrability of $h(v, w)^{-3/4}$ in $[2\gamma, +\infty) \times [2\gamma, +\infty)$.

Adding $x_i^4$ for the rest of variables, we can construct an example in $\mathbb{R}^n$ for $n \geq 3$ of a polynomial $g$ such that $G_1(g)$ is unbounded but $|G_1(g)| < +\infty$.

The polynomial $g_t(x, y, z) = x^4 + y^4 + z^4 - tx^2yz$ for any $t > 2\sqrt{2}$ does not belong to $\mathbb{F}_4(\mathbb{R}^n)$ (since $g_t(\sqrt[4]{y}, y, y) = (4 - t\sqrt{2})y^4 < 0$). A similar example can be constructed in $\mathbb{R}^n$ for $n \geq 3$ as well, so $\mathbb{F}_4(\mathbb{R}^n)$ is not open for $n \geq 3$.

(5) Consider $g(x, y) = (x^2 - y^2)^2(x^{d-4} + y^{d-4})$ in $\mathbb{R}^2$. Then, $g \in \mathbb{F}_d(\mathbb{R}^2)$, since $\{(x, y) \in \mathbb{R}^2 : x^{d-4} + y^{d-4} \leq x^2 + y^2\}$ is compact and $(x^2 - y^2)^2(x^2 + y^2) \in \mathbb{F}_6(\mathbb{R}^2)$, by Lemma 2.2 with $m = 3$, and integrating in polar coordinates:

$$\begin{aligned}
&\int_{\mathbb{R}^2} \exp\{-((x^2 - y^2)^2(x^2 + y^2))^{1/3}\}\, dxdy \\
&= \int_0^{2\pi} \int_0^{+\infty} r \exp\{-r^2(\cos^2\theta - \sin^2\theta)^{2/3}\} dr d\theta \\
&= \int_0^{2\pi} \frac{d\theta}{2(\cos^2\theta - \sin^2\theta)^{2/3}} < +\infty.
\end{aligned}$$

However, $G_1(g)$ is unbounded (see Figure 2), since it contains the lines $y = \pm x$. Moreover, $g(x, y) - (1 - t)x^d$ is not in $\mathbb{F}_d(\mathbb{R}^2)$ for any $t < 1$, since it takes negative values for $x = y$. Thus $\mathbb{F}_d(\mathbb{R}^2)$ is not open.

Adding $x_i^d$ for the rest of variables, we can construct an example in $\mathbb{R}^n$ for $n \geq 3$. □

**Remark.** It is worth mentioning here the connection of homogeneous positive multivariate polynomials with Hilbert's seventeenth problem, one of the 23 Hilbert problems set

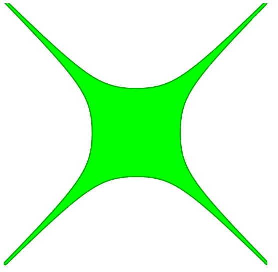

**Fig. 2.** $G_1(g)$ for $g(x,y) = (x^2-y^2)^2(x^2+y^2)$.

out in a celebrated list compiled in 1900 by David Hilbert. It concerns the expression of positive definite rational functions as sums of quotients of squares.

In 1888, Hilbert himself [26] showed that every non-negative homogeneous polynomial in $n$ variables and degree $d$ can be represented as sum of squares of other polynomials if and only if either (1) $n=2$, (2) $d=2$ or (3) $n=3$ and $d=4$. This result, however, cannot be used in the previous lemma, in the study of the structure of polynomials, since that canonical way of writing homogeneous polynomials is not specific enough to suggest a suitable change of variables, as it was done in (2) or (3) in Lemma 2.3.

The following result, of independent interest, will be needed for the study of the convergence of coefficients of polynomials.

**Proposition 2.4.** *Consider the map* $\Phi : \mathbb{F}_d(\mathbb{R}^n) \to \mathbb{R}^{h_d(n)}$ *given by*

$$\Phi(g) = \left( \frac{1}{\Gamma\left(\frac{n}{d}+1\right)} \int_{\mathbb{R}^n} x^\alpha \exp(-g(x))\, dx \right)_{\alpha \in \mathbb{N}_d^n}.$$

*The map* $\Phi$ *is one-to-one, continuous and differentiable, and its inverse (defined on the image set) is also continuous and differentiable.*

**Proof.** The integrability is guaranteed by Lemma 2.1, so $\Phi$ is well defined on $\mathbb{F}_d(\mathbb{R}^n)$. As it was shown in [32], the function

$$\omega(g) = |G_1(g)| = \frac{1}{\Gamma\left(\frac{n}{d}+1\right)} \int_{\mathbb{R}^n} \exp(-g(x))\, dx, \qquad g \in \mathbb{F}_d(\mathbb{R}^n)$$

is a strictly convex function. Moreover, its gradient is $\nabla\omega = -\Phi$. Consequently, its Hessian, a positive semi-definite matrix, is the Jacobian matrix of $-\Phi$. More precisely, as it was shown in [32],

$$\frac{\partial \Phi_\alpha}{\partial g_\beta} = -\frac{1}{\Gamma\left(\frac{n}{d}+1\right)} \int_{\mathbb{R}^n} x^{\alpha+\beta} \exp(-g(x))\, dx, \tag{7}$$

where $\beta \in \mathbb{N}_d^n$, and thus for every $(h_\alpha) \in \mathbb{R}^{h_d(n)}$,

$$\sum_{\alpha,\beta\in\mathbb{N}_d^n} h_\alpha h_\beta \frac{\partial \Phi_\alpha}{\partial g_\beta} = -\frac{1}{\Gamma\left(\frac{n}{d}+1\right)} \int_{\mathbb{R}^n} h(x)^2 \exp(-g(x))\, dx,$$

where $h = \sum_{\alpha\in\mathbb{N}_d^n} h_\alpha x^\alpha$. This shows that the matrix $(\frac{\partial \Phi_\alpha}{\partial g_\beta})$ is negative semi-definite. By [21, Theorem 6] we get that $\Phi$ is globally one-to-one. The continuity and differentiability of the inverse function follow from the Inverse Function Theorem. □

**Example 2.5.** For $n = d = 2$, the function $\Phi : \mathbb{F}_2(\mathbb{R}^2) \to \mathbb{R}^3$ is defined as

$$\Phi(g) = (4ac - b^2)^{-3/2}\, (4\pi c, -2\pi b, 4\pi a)$$

is bijective from $\{g \in \mathbb{F}_2(\mathbb{R}^2) : g(x, y) = ax^2 + bxy + cy^2\, (a, c > 0,\, 4ac > b^2)\}$ onto $\{(a', b', c') \in \mathbb{R}^3 : a', c' > 0,\, a'c' > (b')^2\}$.

## 3. Approximation of log-concave functions by polynomials

Before showing the existence of a solution for *Problem 1* whenever $f \in \mathcal{F}(\mathbb{R}^n)$ we will start by considering the following example, which shows that without any convexity assumption on $f$ *Problem 1* can be ill-posed (see also Example 3.5). Nevertheless, if $f : \mathbb{R}^n \to [0, +\infty)$ is a function with $\|f\|_\infty = f(0) = 1$ and compact support, considering $K$ the convex hull of supp$f$ we have that $\chi_K \in \mathcal{F}(\mathbb{R}^n)$ and $f \le \chi_K$. Therefore, a solution of *Problem 1* for $\chi_K$ will provide a function $g \in \mathbb{H}_d[x]$ and a $t \ge 1$ for which $f(x) \le t \exp(-g(x)^{1/d})$ and $|G_1(g)|$ is finite.

**Example 3.1.** Let $f = \chi_A$, where $A$ is the union of concentric spherical shells:

$$A = \bigcup_{k=1}^{\infty} \{x \in \mathbb{R}^n : k \le |x| \le k + \tfrac{1}{2^k}\}.$$

The function $f$ is integrable, but $f(x) \le t \exp(-g(x)^{1/d})$ for some $g \in \mathbb{H}_d[x]$ would imply $g(x) \le (\log t)^d$ for all $x \in A$. The only $d$-homogeneous bounded polynomial is $g = 0$, for which $|G_1(g)| = +\infty$.

Having this example in mind, we will solve *Problem 1* in the setting of log-concave integrable functions.

For $f \in \mathcal{F}(\mathbb{R}^n)$ with $\|f\|_\infty = f(0) = 1$, let $K_\lambda(f) = \{x \in \mathbb{R}^n : f(x) \ge \lambda\}$. These super-level sets are convex by the log-concavity of $f$.

Given any quasi-convex function $f$ (i.e., a function whose super-level sets are convex) and $t \ge 1$, let

$$H_t(f) = \bigcup_{\lambda\in(0,1)} \log(t/\lambda)^{-1} K_\lambda(f).$$

Since $K_\lambda(f)$ are convex sets that contain the origin, $H_t(f)$ is decreasing in $t \geq 1$.

The following lemma shows the relation between $H_t(f)$ and *Problem 1*.

**Lemma 3.2.** *Let $f : \mathbb{R}^n \to [0, +\infty)$ be a log-concave function with $\|f\|_\infty = f(0) = 1$, $d \in \mathbb{N}$ even $g \in \mathbb{H}_d(\mathbb{R}^n)$ and $t \geq 1$. The following conditions are equivalent:*

(i) $f(x) \leq t \exp(-g(x)^{1/d})$ *for all* $x \in \mathbb{R}^n$.
(ii) $H_t(f) \subset G_1(g)$.

**Proof.** Assume that condition *(i)* holds. Then, for any $x \in H_t(f)$, there exists $\lambda \in (0, 1)$ such that $(\log(t/\lambda))x \in K_\lambda(f)$. In other words, $f((\log(t/\lambda)x) \geq \lambda$. But then, using *(i)* and the homogeneity of $g$,

$$\lambda \leq f((\log(t/\lambda)x)) \leq t \exp(-g(\log(t/\lambda)x)^{1/d}) = t \exp(-\log(t/\lambda)g(x)^{1/d})$$

and hence $g(x) \leq 1$. So *(ii)* is proved.

Conversely, assume that *(ii)* holds and take any $x \in \mathbb{R}^n$.

If $f(x) < 1$, let $\lambda = f(x)$. Clearly, $(\log(t/\lambda))^{-1}x \in H_t(f)$, so using condition *(ii)*, $(\log(t/\lambda))^{-1}x \in G_1(g)$, and therefore, $f(x) \leq t \exp(-g(x)^{1/d})$.

Now assume that $f(x) = 1$. If $t > 1$, then $1 \leq t \exp(-g(x)^{1/d})$ is equivalent to $g(x) \leq (\log t)^d$. Take any $\lambda \in (0, 1)$. Then $x \in K_\lambda(f)$ and therefore

$$((\log(t/\lambda))^{-1}x \in ((\log(t/\lambda))^{-1}K_\lambda(f) \subset H_t(f) \subset G_1(g)$$

by condition *(ii)*. Consequently, $g(x) \leq (\log(t/\lambda))^d$ for any $\lambda \in (0, 1)$. But then $g(x) \leq (\log t)^d$.

Finally, assume that $f(x) = t = 1$. It is left to show that $g(x) = 0$. Since $H_t(f)$ is decreasing in $t \geq 1$,

$$H_t(f) \subset H_1(f) \subset G_1(g)$$

by condition *(ii)*. Using the case $t > 1$ proved above, $g(x) \leq (\log t)^d$. This inequality is true for any $t > 1$, and then $g(x) = 0$. □

The following result gives a monotonicity behaviour, crucial in the study of the minimization problem. Moreover, it will imply some consequences about the boundedness of $H_t(f)$.

**Lemma 3.3.** *Let $f : \mathbb{R}^n \to [0, +\infty)$ be a log-concave function with $\|f\|_\infty = f(0) = 1$. Then, for every $1 < t_0 < t_1$, we have that*

$$(\log t_0)H_{t_0}(f) \subset (\log t_1)H_{t_1}(f).$$

**Proof.** Since log is an increasing function and for every $\lambda \in (0,1)$ $K_\lambda(f)$ is star-shaped with respect to the origin, we have that

$$\begin{aligned} H_{t_0}(f) &= \frac{\log t_1}{\log t_0} \bigcup_{\lambda\in(0,1)} \frac{1}{\log t_1 + \frac{\log t_1}{\log t_0}\log(1/\lambda)} K_\lambda(f) \\ &\subset \frac{\log t_1}{\log t_0} \bigcup_{\lambda\in(0,1)} \frac{1}{\log t_1 + \log(1/\lambda)} K_\lambda(f) \\ &= \frac{\log t_1}{\log t_0} H_{t_1}(f). \quad \square \end{aligned}$$

**Remark.** In the previous two lemmas we have not assumed the integrability of $f$. Moreover, in Lemma 3.2 we have only used the log-concavity of $f$ in the case $t = 1$; indeed, in both lemmas, the only fact needed is that $\alpha_1 K_\lambda(f) \subset \alpha_2 K_\lambda(f)$ for any $0 < \alpha_1 \leq \alpha_2$, which is equivalent to the fact that $K_\lambda(f)$ is star-shaped with respect to the origin. Finally, the inclusions in Lemma 3.3 above are sharp, see for instance Example 3.9 (3.9).

Now we can state the boundedness of $H_t(f)$.

**Lemma 3.4.** *Let $f \in \mathcal{F}(\mathbb{R}^n)$ with $\|f\|_\infty = f(0) = 1$ and an even $d \in \mathbb{N}$. Then $H_t(f)$ is bounded for every $t > 1$. Moreover, if $H_1(f)$ is unbounded, then $|H_1(f)| = +\infty$.*

**Proof.** Let $t_0 \exp(-\|\cdot\|_{\mathcal{E}})$ be the unique minimization ellipsoid verifying (2) for the even log-concave function $f_s = \exp(-u_s)$, being $u_s$ the convex function whose epigraph is the convex hull of the functions $u(x)$ and $u_-(x) := u(-x)$. That is,

$$\operatorname{epi}(u_s) = \operatorname{conv}(\operatorname{epi}(u), \operatorname{epi}(u_-)).$$

Notice that $f \leq f_s$ and, since $f \in \mathcal{F}(\mathbb{R}^n)$, then also $f_s \in \mathcal{F}(\mathbb{R}^n)$ (see [1]).

Let us observe that $g = \|\cdot\|_{\mathcal{E}}^d \in \mathbb{H}_d(\mathbb{R}^n)$. Then $f \leq t_0 \exp(-g^{1/d})$, which by Lemma 3.2 means that $H_{t_0}(f) \subset G_1(g)$. In this case $G_1(g)$ is an ellipsoid, hence bounded, so $H_{t_0}(f)$ is bounded too. Since $H_t(f)$ is decreasing on $t \geq 1$ and by Lemma 3.3, $(\log t)H_t(f)$ is increasing on $t > 1$, we have that $H_t(f)$ is bounded for every $t > 1$.

Finally, let us assume that $H_1(f)$ is unbounded. Let us observe that

$$\frac{1}{\log(1/\lambda_1)} K_{\lambda_1}(f) \subset \frac{1}{\log(1/\lambda_2)} K_{\lambda_2}(f)$$

for any $0 < \lambda_1 < \lambda_2 \leq 1$. Indeed, take $x \in K_{\lambda_1}(f)$. Since $f(0) = 1$ and $\log(1/\lambda_2)/\log(1/\lambda_1) \in [0,1]$,

$$f\left(\frac{\log(1/\lambda_2)}{\log(1/\lambda_1)} x\right) \geq f(x)^{\frac{\log(1/\lambda_2)}{\log(1/\lambda_1)}} \geq \lambda_1^{\frac{\log(1/\lambda_2)}{\log(1/\lambda_1)}} = \lambda_2.$$

Therefore $H_1(f)=\bigcup_{\lambda\in(0,1)}(\log(1/\lambda))^{-1}K_\lambda(f)$ is an increasing union of convex sets, i.e., convex. We can assume that $\int_{\mathbb{R}^n}f(x)dx>0$, and hence $|K_\lambda(f)|>0$ for some $\lambda\in(0,1)$, so it has non empty interior, and so has $H_1(f)$. Since $H_1(f)$ is unbounded (and convex), then $|H_1(f)|=+\infty$, as desired. □

**Example 3.5.** The previous result is not true if the log-concavity assumption on $f$ is dropped. For instance, given $K\subseteq\mathbb{R}^n$ a convex body with $0\in K$ and $\alpha>n$ consider

$$f(x)=\begin{cases}1 & x\in K,\\ \|x\|_K^{-\alpha} & \text{otherwise.}\end{cases}$$

Then $f$ is quasi-concave and

$$\begin{aligned}\int_{\mathbb{R}^n}f(x)dx&=|K|+\int_{\mathbb{R}^n\setminus K}\|x\|_K^{-\alpha}dx\\&=|K|+\int_0^1|\{x\in\mathbb{R}^n\setminus K:\|x\|_K^{-\alpha}\geq t\}|dt\\&=|K|+\int_0^1|t^{-1/\alpha}K\setminus K|dt=\frac{\alpha}{\alpha-n}|K|<+\infty.\end{aligned}$$

Its super-level sets are

$$K_\lambda(f)=\begin{cases}\lambda^{-1/\alpha}K & 0<\lambda<1\\ K & \lambda=1,\end{cases}$$

(note that these sets are convex), and thus for every $t\geq 1$

$$H_t(f)=\bigcup_{\lambda\in(0,1)}\frac{1}{\lambda^{1/\alpha}\log(t/\lambda)}K=\mathbb{R}^n.\quad\square$$

The following result shows the relation between $H_t(f)$ and the super-level sets of $f^\circ$. Recall that the polar body of a convex body $K$ containing the origin is $K^\circ=\{x\in\mathbb{R}^n:\langle x,y\rangle\leq 1\text{ for every }y\in K\}$.

**Lemma 3.6.** *Let $f\in\mathcal{F}(\mathbb{R}^n)$ with $\|f\|_\infty=f(0)=1$. Then for any $t>1$ $H_t(f)$ is convex and*

$$\overline{H_t(f)}=\{x\in\mathbb{R}^n:f^\circ(x)\geq\tfrac{1}{t}\}^\circ=(K_{\frac{1}{t}}(f^\circ))^\circ.$$

**Proof.** Calling $t = e^r$ for some $r > 0$ and $\lambda = e^{-s}$ for $s \geq 0$, we have that

$$H_{e^r}(f) = \bigcup_{s\geq 0} \frac{K_{e^{-s}}(f)}{r+s}.$$

We start showing that $H_{e^r}(f)$ is convex. Let $x_1, x_2 \in H_t(f)$ and $0 \leq \theta \leq 1$. Then, there exist $s_1, s_2 \geq 0$ such that $y_1 = (r+s_1)x_1 \in K_{e^{-s_1}}(f)$ and $y_2 = (r+s_2)x_2 \in K_{e^{-s_2}}(f)$. Letting $0 \leq \lambda = \frac{\theta(r+s_1)}{(1-\theta)s_2+\theta s_1+r} \leq 1$ and $s_\theta = (1-\lambda)s_1 + \lambda s_2$ we have that

$$(r+s_\theta)[(1-\theta)x_1 + \theta x_2] = (1-\lambda)y_1 + \lambda y_2$$

and then, since

$$f((1-\lambda)y_1 + \lambda y_2) \geq e^{-[(1-\lambda)s_1+\lambda s_2]} = e^{-s_\theta},$$

we have that $(r+s_\theta)[(1-\theta)x_1+\theta x_2] \in K_{e^{-s_\theta}}$ or, equivalently, $(1-\theta)x_1+\theta x_2 \in \frac{K_{e^{-s_\theta}}}{r+s_\theta} \subseteq H_{e^r}(f)$.

Let $u : \mathbb{R}^n \to [0, +\infty]$ be the convex function such that $f = \exp(-u)$. Note that

$$f^\circ(x) \geq \frac{1}{t} \Leftrightarrow u^*(x) \leq r,$$

where $u^*(x)$ is the Legendre transform of $u$

$$u^*(x) = \sup_{y\in\mathbb{R}^n} (\langle x, y\rangle - u(y)).$$

Therefore $u^*(x) \leq r$ if and only if for every $y \in \mathbb{R}^n$

$$\langle x, y\rangle \leq u(y) + r,$$

which happens if and only if for every $s \geq 0$ and every $y \in \{y \in \mathbb{R}^n : u(y) \leq s\} = K_{e^{-s}}(f)$ we have

$$\langle x, y\rangle \leq s + r,$$

which is equivalent to the fact that for every $s \geq 0$, $x \in (s+r)K^\circ_{e^{-s}}$. Thus,

$$K_{\frac{1}{t}}(f^\circ) = \{x \in \mathbb{R}^n : f^\circ(x) \geq \frac{1}{t}\} = \bigcap_{s\geq 0}(s+r)K^\circ_{e^{-s}} = \bigcap_{s\geq 0}\left(\frac{K_{e^{-s}}}{s+r}\right)^\circ$$

and then

$$(K_{\frac{1}{t}}(f^\circ))^\circ = \left(\bigcap_{s\geq 0}\left(\frac{K_{e^{-s}}}{s+r}\right)^\circ\right)^\circ = \overline{\mathrm{conv}\left(\bigcup_{s\geq 0}\frac{K_{e^{-s}}}{s+r}\right)} = \overline{H_t(f)}. \quad \square$$

Using Lemmas 3.2 and 2.2 we can reformulate *Problem 1* as follows: Given $f \in \mathcal{F}(\mathbb{R}^n)$ with $\|f\|_\infty = f(0) = 1$, find $t_0 \geq 1$ and $g_0 \in \mathbb{H}_d[x]$ such that $H_{t_0}(f) \subset G_1(g_0)$ and

$$t_0|G_1(g_0)| = \inf t|G_1(g)| = \inf_{t \geq 1} \left( t \inf_{g \in \mathbb{H}_d(\mathbb{R}^n)} |G_1(g)| \right)$$

where the infimum is taken among all $(t, g)$ verifying $t \geq 1$, $g \in \mathbb{H}_d[x]$, and $H_t(f) \subset G_1(g)$.

Let us observe that for every $t > 1$, the infimum above over $g \in \mathbb{H}_d(\mathbb{R}^n)$ such that $H_t(f) \subset G_1(g)$ is a minimum. Indeed, we may apply the minimization problem solved by Lasserre to $\overline{H_t(f)}$, the closure of $H_t(f)$, (which by Lemma 3.4 is compact), and get $g_t \in \mathbb{H}_d(\mathbb{R}^n)$ the only polynomial verifying

$$H_t(f) \subset G_1(g_t)$$

with minimum volume $|G_1(g_t)|$ among all $g \in \mathbb{H}_d(\mathbb{R}^n)$ such that $H_t(f) \subset G_1(g)$.

We can also apply the argument to $H_1(f)$ (or its closure) if $H_1(f)$ is bounded (if it is unbounded, then Lemma 3.4 implies that $|H_1(f)| = +\infty$, and hence it does not play any role in the minimization problem). Let $g_1$ be the corresponding polynomial to $H_1(f)$ (if it is bounded). Then, the infimum in *Problem 1* can be rewritten as

$$\inf_{t \geq 1} t|G_1(g_t)|.$$

For any $t \geq 1$, let $v(t) = |G_1(g_t)|$ (consider $v(1) = +\infty$ if $H_1(f)$ is unbounded) and $\phi(t) = tv(t)$ be the function to be minimized. Some properties for these functions are needed to solve the problem. Let us start with a technical lemma.

**Lemma 3.7.** *Let $t_0, t_1, d \geq 1$, $\theta \in [0, 1]$, and $a \in (0, 1]$. Then*

$$(1 - \theta) \left(\log \tfrac{t_0}{a}\right)^d + \theta \left(\log \tfrac{t_1}{a}\right)^d \leq \left(\log \tfrac{t_\theta}{a}\right)^d,$$

*where $t_\theta$ is defined by the identity $(\log t_\theta)^d = (1 - \theta)(\log t_0)^d + \theta(\log t_1)^d$.*

**Proof.** If $d = 1$ the inequality in the statement is trivially an equality. Assume $d > 1$. The inequality above can be reformulated as

$$F(a) = a \exp \left[ (1 - \theta) \left(\log \tfrac{t_0}{a}\right)^d + \theta \left(\log \tfrac{t_1}{a}\right)^d \right]^{\frac{1}{d}} \leq t_\theta$$

for any $a \in (0, 1]$. Since $F(1) = t_\theta$, it is enough to prove that $F$ is increasing on $(0, 1]$. Indeed, considering the change of variables $a = e^{-b}$, $t_i = e^{s_i}$, $i = 0, 1$, $F$ is increasing if and only if $G$ is decreasing on $[0, +\infty)$, where

$$G(b) = \log F(e^{-b}) = \left( (1 - \theta)(s_0 + b)^d + \theta(s_1 + b)^d \right)^{\frac{1}{d}} - b, \qquad b \in [0, +\infty)$$

for any $s_0, s_1 \geq 0$. Its derivative equals

$$G'(b) = \left[ \frac{\left((1-\theta)(s_0+b)^{d-1} + \theta(s_1+b)^{d-1}\right)^{\frac{1}{d-1}}}{((1-\theta)(s_0+b)^d + \theta(s_1+b)^d)^{\frac{1}{d}}} \right]^{d-1} - 1.$$

Letting $u_i = (s_i + b)^{d-1}$, $i = 0, 1$, then $G'(b) \leq 0$ rewrites as

$$((1-\theta)u_0 + \theta u_1)^{\frac{d}{d-1}} \leq (1-\theta)u_0^{\frac{d}{d-1}} + \theta u_1^{\frac{d}{d-1}},$$

which is a consequence of the convexity of $u \mapsto u^{\frac{d}{d-1}}$. □

**Lemma 3.8.** *Let $f \in \mathcal{F}(\mathbb{R}^n)$ with $\|f\|_\infty = f(0) = 1$ and $d \in \mathbb{N}$ even. Then $v(t)$ is a decreasing function and $(\log t)^n v(t)$ is increasing in $t$.*

*Moreover, if $t_0, t_1 > 1$ and $\theta \in [0, 1]$, then*

$$v(t_\theta) \leq v(t_0)^{1-\theta} v(t_1)^\theta,$$

*where $(\log t_\theta)^d = (1-\theta)(\log t_0)^d + \theta(\log t_1)^d$.*

**Proof.** $v$ is decreasing by definition. On the other hand, taking volumes in the inclusion given in Lemma 3.3, for any $1 < t_0 < t_1$

$$(\log t_0)^n v(t_0) \leq (\log t_1)^n v(t_1).$$

Now, for any $i = 0, 1$, and for every $x \in \mathbb{R}^n$,

$$f(x) \leq t_i \exp(-g_{t_i}(x)^{1/d}).$$

Then, if $f(x) \neq 0$,

$$g_{t_i}(x) \leq \left(\log \frac{t_i}{f(x)}\right)^d$$

for $i = 0, 1$. Since $f(x) \in (0, 1]$ and $t_i \geq 1$, then by Lemma 3.7 we get that

$$(1-\theta)g_{t_0}(x) + \theta g_{t_1}(x) \leq (1-\theta)\left(\log \frac{t_0}{f(x)}\right)^d + \theta \left(\log \frac{t_1}{f(x)}\right)^d \leq \left(\log \frac{t_\theta}{f(x)}\right)^d,$$

and by Lemma 3.2, $H_{t_\theta}(f) \subset G_1((1-\theta)g_{t_0} + \theta g_{t_1})$. Since $(1-\theta)g_{t_0} + \theta g_{t_1} \in \mathbb{H}_d(\mathbb{R}^n)$, the minimization property of $g_{t_\theta}$ implies that $|G_1(g_{t_\theta})| \leq |G_1((1-\theta)g_{t_0} + \theta g_{t_1})|$, and using Lemma 2.2 and Hölder's inequality

$$
\begin{aligned}
|G_1(g_{t_\theta})| &\le |G_1((1-\theta)g_{t_0} + \theta g_{t_1})| \\
&= \Gamma(\tfrac{n}{d}+1)^{-1} \int\limits_{\mathbb{R}^n} \exp(-((1-\theta)g_{t_0}(x) + \theta g_{t_1}(x)))\, dx \\
&\le \left( \Gamma(\tfrac{n}{d}+1)^{-1} \int\limits_{\mathbb{R}^n} \exp(-g_{t_0}(x))\, dx \right)^{1-\theta} \left( \Gamma(\tfrac{n}{d}+1)^{-1} \int\limits_{\mathbb{R}^n} \exp(-g_{t_1}(x))\, dx \right)^{\theta} \\
&= |G_1(g_{t_0})|^{1-\theta} |G_1(g_{t_1})|^{\theta}. \quad \square
\end{aligned}
\tag{8}
$$

Now we can prove the first main result.

**Proof of Theorem 1.1.** By Lemma 3.8, the function $s \in (0,+\infty) \mapsto \log v\left(e^{s^{1/d}}\right)$ is a convex function. This implies that $v$ is a continuous function on $(1,+\infty)$. We will also prove that

$$\lim_{t\to 1^+} v(t) = v(1). \tag{9}$$

Recall that $H_t(f)$ is decreasing in $t \ge 1$. That implies that there exists

$$\lim_{t\to 1^+} |H_t(f)| \le |H_1(f)|.$$

We have that $\mu H_1(f) \subset \bigcup_{t>1} H_t(f)$ for any $\mu < 1$. Indeed, $x \in H_1(f)$ if and only if $f(\log(1/\lambda)x) \ge \lambda$ for some $\lambda \in (0,1)$. Take $t > 1$ so that $\mu \log(t/\lambda) = \log(1/\lambda)$. Then $\mu x \in \log(t/\lambda)^{-1} K_\lambda(f) \subset H_t(f)$.

The Monotone Convergence Theorem ensures that $\mu^n |H_1(f)| \le \lim_{t\to 1^+} |H_t(f)|$ (even if $|H_1(f)| = \infty$) for any $\mu < 1$. Then

$$\lim_{t\to 1^+} |H_t(f)| = |H_1(f)|.$$

First, assume that $H_1(f)$ is bounded. Using the minimization property for $|G_1(g_t)|$, we have that $(|G_1(g_t)|)_{t>1}$ is decreasing, so there exists $\lim_{t\to 1^+} |G_1(g_t)| \le |G_1(g_1)|$.

Using (4) (and repeating some of the contact points and the coefficients if necessary), for any $t \ge 1$, there are $y_1^{(t)}, \dots, y_{h_d(n)}^{(t)} \in H_t(f)$, $\lambda_1^{(t)}, \dots, \lambda_{h_d(n)}^{(t)} \ge 0$, such that $g_t(y_i^{(t)}) = 1$ for $i = 1, \dots, h_d(n)$, and

$$\int\limits_{\mathbb{R}^n} x^\alpha e^{-g_t(x)} dx = \sum_{i=1}^{h_d(n)} \lambda_i^{(t)} (y_i^{(t)})^\alpha \tag{10}$$

for every $\alpha \in \mathbb{N}_d^n$. Moreover, using the trace identity and Lemma 2.1,

$$\frac{n}{d}\Gamma(\tfrac{n}{d}+1)|G_1(g_t)| = \int_{\mathbb{R}^n} g_t(x)e^{-g_t(x)}dx = \sum_{i=1}^{h_d(n)} \lambda_i^{(t)}.$$

Using that $H_t(f) \subset H_1(f)$ and subsequently, $|G_1(g_t)| \leq |G_1(g_1)|$, all coefficients $\lambda_i^{(t)}$ are uniformly bounded by

$$0 \leq \lambda_i^{(t)} \leq \frac{n}{d}\Gamma(\tfrac{n}{d}+1)|G_1(g_1)|$$

and all the vectors $y_i^{(t)}$ lie in the same bounded set $H_1(f)$. These uniformly bounding conditions together with the set of equalities in (10), for every $\alpha \in \mathbb{N}_d^n$, imply that there exists a compact set $\Omega \subset \mathbb{R}^{h_d(n)}$ such that for every $t \geq 1$, $\Phi(g_t) \in \Omega$, where $\Phi$ is the map defined in Proposition 2.4. Using Proposition 2.4, we have that the coefficients of all $g_t$ are uniformly bounded. Thus, taking a sequence $(t_k)$ converging to 1, considering the sequence $g_{t_k}$, and passing to a convergent subsequence, we can construct a polynomial $g_0 \in \mathbb{H}_d(\mathbb{R}^n)$ whose coefficients are the limit of the coefficients of such a subsequence of $(g_{t_k})$. Using that $H_t(f) \subset G_1(g_t)$ for any $t > 1$ and Lemma 3.2, and taking limit, we get that $H_1(f) \subset G_1(g_0)$. Using that the miminizing property defining $g_1$, we have $|G_1(g_1)| \leq |G_1(g_0)|$, and using Lemma 2.2 and Fatou's lemma, $|G_1(g_0)| \leq \lim_{t\to 1^+} |G_1(g_t)|$ (since this limit exists). Then $|G_1(g_1)| \leq \lim_{t\to 1^+} |G_1(g_t)|$ and

$$\lim_{t\to 1^+} |G_1(g_t)| = |G_1(g_1)|,$$

as desired. Note that, using the equalities (10), and taking again a subsequence, we get the same equalities for $g_0$, for some coefficients and contact points in $H_1(f)$. Since these equalities characterize $g_1$, we have $g_0 = g_1$.

If $H_1(f)$ is unbounded, then $|H_1(f)| = +\infty$ by Lemma 3.4. The Monotone Convergence Theorem ensures again that $\lim_{t\to 1+} |H_t(f)| = +\infty$. Since $H_t(f) \subset G_1(g_t)$,

$$\lim_{t\to 1+} |G_1(g_t)| = +\infty$$

and the proof of (9) is completed.

Using Lemma 3.8, the function $\phi(t) = \frac{t}{(\log t)^n}(\log t)^n v(t)$ is the product of two positive increasing functions in $[e^n, +\infty)$. If $H_1(f)$ is bounded, $\phi$ attains its minimum in $[1, e^n]$ by continuity. If $H_1(f)$ is unbounded, then $\lim_{t\to 1^+} \phi(t) = +\infty$ and so, $\phi$ attains its minimum in $(1, e^n]$ by continuity. In both cases, this is the minimum of $\phi$ in $[1, +\infty)$. □

The end of this section is devoted to showing several examples where *Problem 1* can be explicitly solved.

**Example 3.9.** In the following examples, $K \subseteq \mathbb{R}^n$ is a convex body with $0 \in K$ and $g \in \mathbb{H}_d(\mathbb{R}^n)$ is the optimal polynomial verifying (3) for the given convex body $K$ given in [32].

(1) Let $f(x) = e^{-\|x\|_K}$. Then $K_\lambda(f) = (\log(1/\lambda))K$, $H_t(f) = \mathrm{int}(K)$ for $t > 1$, and $H_1(f) = K$. Therefore $g_t = g$ for every $t \geq 1$. Thus $\phi(t) = t|G_1(g)|$ and then $\min_{t\geq 1}\phi(t) = \phi(1)$.

(2) More generally, for $\alpha > 1$ let $f(x) = e^{-\|x\|_K^\alpha}$. Then $K_\lambda(f) = (\log(1/\lambda))^{1/\alpha}K$, $H_t(f) = \alpha^{-1/\alpha}(\alpha' \log t)^{-1/\alpha'}K$ for $t > 1$ ($\alpha^{-1} + (\alpha')^{-1} = 1$) and $H_1(f) = \mathbb{R}^n$. Then $g_t = \alpha^{d/\alpha}(\alpha' \log t)^{d/\alpha'}g$ for every $t > 1$ ($g_1$ is undefined since $H_1(f)$ is unbounded). Thus $\phi(t) = t\alpha^{-n/\alpha}(\alpha' \log t)^{-n/\alpha'}|G_1(g)|$ and then $\min_{t>1}\phi(t) = \phi(e^{n/\alpha'}) = (e/n)^{n/\alpha'}\alpha^{-n/\alpha}|G_1(g)|$.

(3) Let $f = \chi_K$. Then $K_\lambda(f) = K$, $H_t(f) = (\log t)^{-1}K$ for $t > 1$, and $H_1(f) = \mathbb{R}^n$. Therefore $g_t = (\log t)^d g$ for $t > 1$ ($g_1$ is undefined since $H_1(f)$ is unbounded). Thus $\phi(t) = t(\log t)^{-n}|G_1(g)|$ and $\min_{t>1}\phi(t) = \phi(e^n) = (e/n)^n|G_1(g)|$.

(4) Let

$$f(x) = \begin{cases} 1 & x \in K, \\ e^{1-\|x\|_K} & \text{otherwise.} \end{cases}$$

Then $f \in \mathcal{F}(\mathbb{R}^n)$. Moreover,

$$K_\lambda(f) = \begin{cases} \left(1 + \log \frac{1}{\lambda}\right)K & 0 < \lambda < 1 \\ K & \lambda = 1, \end{cases}$$

and thus

$$H_t(f) = \begin{cases} \frac{1}{\log t}K & 1 \leq t \leq e \\ \mathrm{int}(K) & t > e. \end{cases}$$

Hence,

$$g_t = \begin{cases} (\log t)^d g & 1 \leq t \leq e \\ g & t > e \end{cases}$$

and thus

$$\phi(t) = \begin{cases} \frac{t}{(\log t)^n}|G_1(g)| & 1 \leq t \leq e \\ t|G_1(g)| & t > e \end{cases}$$

is not differentiable at the point $t = e$, precisely where it attains the minimum $\min_{t\geq 1}\phi(t) = \phi(e) = e|G_1(g)|$.

## 4. A new approach to approximate log-concave functions by polynomials

With the purpose of getting uniqueness for the optimal polynomial, we pose *Problem 2* as a similar minimization problem, where the polynomial exponent $\frac{1}{d}$ is dropped in (5), turning into (6).

In order to solve *Problem 2*, for $t \geq 1$, we introduce

$$\widehat{H}_t(f) = \bigcup_{\lambda \in (0,1)} (\log(t/\lambda))^{-1/d} K_\lambda(f).$$

This case is not a generalization of Lasserre's problem, and moreover we can not assure $\widehat{H}_t(f)$ to be bounded, as in *Problem 1* (as one can see by taking $f(x) = e^{-\|x\|_2}$). In fact, the following example shows the existence of a log-concave function $f$ for which $\widehat{H}_1(f)$ is an unbounded set with finite volume.

**Example 4.1.** Let us consider $f(x) = (1 - \|(x_1, \dots, x_{n-1}, 0)\|_\infty)\chi_{[0,1]^n}(x)$. Notice that $f$ is integrable and concave in its support, and thus, also log-concave. Moreover, $K_\lambda(f) = [0, 1-\lambda]^{n-1} \times [0,1]$ for every $\lambda \in (0,1)$. Thus

$$\widehat{H}_1(f) = \bigcup_{0<\lambda<1} \left[0, (1-\lambda)(\log 1/\lambda)^{-1/d}\right]^{n-1} \times \left[0, (\log 1/\lambda)^{-1/d}\right].$$

Notice that the terms of the union when $\lambda \to 1^-$ contain points with arbitrarily large norm, thus $\widehat{H}_1(f)$ is unbounded.

The function $h(\lambda) = (1-\lambda)(\log(1/\lambda))^{-1/d}$ fulfills $h'(\lambda) = -\frac{d\lambda \log(1/\lambda) + \lambda - 1}{d\lambda (\log 1/\lambda)^{1+1/d}}$.

Let $\lambda_d \in (0,1)$ be the unique root of the equation $h'(\lambda) = 0$ in $(0,1)$. Then, $h$ is increasing in $(0, \lambda_d)$ and decreasing in $(\lambda_d, 1)$. Therefore

$$\begin{aligned} \widehat{H}_1(f) = &\left[0, (1-\lambda_d)(\log 1/\lambda_d)^{-1/d}\right]^{n-1} \times \left[0, (\log 1/\lambda_d)^{-1/d}\right] \cup \\ &\bigcup_{\lambda_d \leq \lambda \leq 1} \left\{(x_1, \dots, x_{n-1}, (\log 1/\lambda)^{-1/d}) : 0 \leq x_i \leq (1-\lambda)(\log 1/\lambda)^{-1/d},\ 1 \leq i \leq n-1\right\}. \end{aligned}$$

Since the first term in the union above is bounded, $|\widehat{H}_1(f)| < +\infty$ if and only if the second term in the union has finite volume. Letting $\mu = (\log(1/\lambda))^{-1/d}$, that term becomes

$$\{(x_1, \dots, x_{n-1}, \mu) : 0 \leq x_i \leq \mu(1 - e^{-\mu^{-d}}),\ \mu \geq \mu_d\}$$

where $\mu_d = (\log(1/\lambda_d))^{-1/d}$. Using Fubini's formula, its volume is

$$\int_{\mu_d}^{+\infty} \mu^{n-1} \left(1 - e^{-\mu^{-d}}\right)^{n-1} d\mu = \int_0^{1/\mu_d} \frac{\left(1 - e^{-\theta^d}\right)^{n-1}}{\theta^{n+1}} d\theta.$$

This last integral converges if and only if $(n-1)(d-1) > 1$, which turns out to be always true except in the case $n = d = 2$. In this last case, $\widehat{H}_1(f)$ is unbounded with infinite volume. Otherwise $|\widehat{H}_1(f)| < +\infty$, as desired.

Note also that $\widehat{H}_1(f)$ is not convex, in contrast to $H_1(f)$ (see Lemma 3.6), while $f$ is concave on its compact support. Regarding *Problem 2*, it would be interesting to solve Lasserre's problem for these type of sets $\widehat{H}_1(f)$ (bounded or not).

For that reason, we will restrict the study of *Problem 2* to $\mathcal{B}(\mathbb{R}^n)$, the set of all log-concave functions for which $\widehat{H}_1(f)$ is bounded. Note that $\mathcal{B}(\mathbb{R}^n) \subset \mathcal{F}(\mathbb{R}^n)$. Since $K_\lambda(f)$ are convex sets that contain the origin, $\widehat{H}_t(f)$ is decreasing in $t \geq 1$, so the boundedness of $\widehat{H}_t(f)$ for any $t \geq 1$ is guaranteed by the condition $f \in \mathcal{B}(\mathbb{R}^n)$.

Similar lemmas to those given in Section 3 are now provided. The proofs follow the same ideas as in the previous section.

**Lemma 4.2.** *Let $f : \mathbb{R}^n \to [0, +\infty)$ be a log-concave function with $\|f\|_\infty = f(0) = 1$, $d \in \mathbb{N}$ even, $g \in \mathbb{H}_d(\mathbb{R}^n)$ and $t \geq 1$. The following are equivalent:*

(1) *$f(x) \leq t \exp(-g(x))$ for all $x \in \mathbb{R}^n$.*
(2) *$\widehat{H}_t(f) \subset G_1(g)$.*

**Lemma 4.3.** *Let $f : \mathbb{R}^n \to [0, +\infty)$ be a log-concave function with $\|f\|_\infty = f(0) = 1$ and $d \in \mathbb{N}$ even. Then for every $1 < t_0 < t_1$ we have that*

$$(\log t_0)^{\frac{1}{d}} \widehat{H}_{t_0}(f) \subset (\log t_1)^{\frac{1}{d}} \widehat{H}_{t_1}(f).$$

**Remark.** In these two lemmas we have not assumed $f \in \mathcal{B}(\mathbb{R}^n)$ or the integrability of $f$. Moreover, in Lemma 4.2 the log-concavity of $f$ is only used in the case $t = 1$; indeed, in both lemmas the only fact needed is that $\alpha_1 K_\lambda(f) \subset \alpha_2 K_\lambda(f)$ for any $0 < \alpha_1 \leq \alpha_2$, which is equivalent to the fact that $K_\lambda(f)$ is star-shaped with respect to the origin. Finally, the inclusions in Lemma 4.3 above are sharp (take $f = \chi_K$ for a convex body $K \subset \mathbb{R}^n$).

For every $t \geq 1$, the minimum in *Problem 2* over $g \in \mathbb{H}_d(\mathbb{R}^n)$ such that $\widehat{H}_t(f) \subset G_1(g)$ is attained. Indeed, we may apply the minimization problem solved by Lasserre to $\widehat{H}_t(f)$ (actually its closure) and get $\widehat{g}_t \in \mathbb{H}_d(\mathbb{R}^n)$ the only polynomial verifying

$$\widehat{H}_t(f) \subset G_1(\widehat{g}_t)$$

with minimum volume $|G_1(\widehat{g}_t)|$ among all $g \in \mathbb{H}_d(\mathbb{R}^n)$ such that $\widehat{H}_t(f) \subset G_1(g)$.

Then, the infimum in *Problem 2* can be rewritten as

$$\inf_{t \geq 1} t |G_1(\widehat{g}_t)|.$$

For any $t \geq 1$, let $\widehat{v}(t) = |G_1(\widehat{g}_t)|$ and $\widehat{\phi}(t) = t\widehat{v}(t)$ be the function to be minimized.

**Lemma 4.4.** *Let $f \in \mathcal{B}(\mathbb{R}^n)$ with $\|f\|_\infty = f(0) = 1$ and $d \in \mathbb{N}$ even. Then $\widehat{v}$ is a decreasing function and $(\log t)^{\frac{n}{d}}\widehat{v}(t)$ is increasing on $t$. As a consequence, $\widehat{\varphi}$ is increasing on $[e^{\frac{n}{d}}, +\infty)$.*

**Lemma 4.5.** *Let $f \in \mathcal{B}(\mathbb{R}^n)$ with $\|f\|_\infty = f(0) = 1$ and let $d \in \mathbb{N}$ even. Then $r \in [0, +\infty) \mapsto \widehat{\phi}(e^r)$ is log-convex. Moreover, if $\widehat{\phi}(t_0) = \widehat{\phi}(t_1) = \min_{t\geq 1} \widehat{\phi}(t)$, then $t_0 = t_1$.*

**Proof.** We have that for any $t_0, t_1 \in [1, +\infty)$, $f(x) \leq \exp(r_i - \widehat{g}_{t_i}(x))$ for $i = 0, 1$, where $t_i = e^{r_i}$. Then

$$\begin{aligned} f(x) &\leq (\exp(r_0 - \widehat{g}_{t_0}(x)))^{1-\theta} (\exp(r_1 - \widehat{g}_{t_1}(x)))^{\theta} \\ &= \exp((1-\theta)r_0 + \theta r_1 - ((1-\theta)\widehat{g}_{t_0}(x) + \theta\widehat{g}_{t_1}(x))). \end{aligned}$$

Since $(1-\theta)\widehat{g}_{t_0} + \theta\widehat{g}_{t_1} \in \mathbb{H}_d(\mathbb{R}^n)$, the minimality of $\widehat{g}_{\exp((1-\theta)r_0+\theta r_1)}$ implies that $|G_1(\widehat{g}_{\exp((1-\theta)r_0+\theta r_1)})| \leq |G_1((1-\theta)\widehat{g}_{t_0} + \theta\widehat{g}_{t_1})|$. Therefore, using Hölder's inequality and Lemma 2.2,

$$\begin{aligned} |G_1(\widehat{g}_{\exp((1-\theta)r_0+\theta r_1)})| &\leq |G_1((1-\theta)\widehat{g}_{t_0} + \theta\widehat{g}_{t_1})| \\ &= \Gamma(\tfrac{n}{d}+1)^{-1} \int_{\mathbb{R}^n} e^{-((1-\theta)\widehat{g}_{t_0}(x)+\theta\widehat{g}_{t_1}(x))}\, dx \\ &\leq \left(\Gamma(\tfrac{n}{d}+1)^{-1} \int_{\mathbb{R}^n} e^{-\widehat{g}_{t_0}(x)}\, dx\right)^{1-\theta} \left(\Gamma(\tfrac{n}{d}+1)^{-1} \int_{\mathbb{R}^n} e^{-\widehat{g}_{t_1}(x)}\, dx\right)^{\theta} \\ &= |G_1(\widehat{g}_{t_0})|^{1-\theta} |G_1(\widehat{g}_{t_1})|^{\theta}, \end{aligned}$$

and thus $\widehat{\phi}(\exp((1-\theta)r_0 + \theta r_1)) \leq \widehat{\phi}(e^{r_0})^{1-\theta}\widehat{\phi}(e^{r_1})^{\theta}$, i.e., $\widehat{\phi}(e^r)$ is log-convex for $r \in [0, +\infty)$.

Let us now assume that $0 \leq r_0 \leq r_1$, $t_i = e^{r_i}$ are such that $\widehat{\phi}(e^{r_0}) = \widehat{\phi}(e^{r_1}) = \min_{r\geq 0} \widehat{\phi}(e^r)$. This implies that for every $\theta \in [0, 1]$,

$$|G_1(\widehat{g}_{\exp((1-\theta)r_0+\theta r_1)})| = |G_1(\widehat{g}_{t_0})|^{1-\theta} |G_1(\widehat{g}_{t_1})|^{\theta},$$

which by Hölder's equality cases means that $e^{-\widehat{g}_{t_0}} = ce^{-\widehat{g}_{t_1}}$, for some $c > 0$. Since $\widehat{g}_{t_i}(0) = 0$ for $i = 0, 1$, then $c = 1$, thus $\widehat{g}_{t_0} = \widehat{g}_{t_1}$, and hence $|G_1(\widehat{g}_{t_0})| = |G_1(\widehat{g}_{t_1})|$ from which we get that $t_0 = e^{r_0} = e^{r_1} = t_1$, concluding the proof. □

Now we prove the existence and uniqueness of a global minimum for *Problem 2*.

**Proof of Theorem 1.2.** By Lemma 4.5, $\widehat{\phi}(e^r) = e^r|G_1(\widehat{g}_t)|$ is a log-convex function, and thus convex in $[0, +\infty)$. This shows, in particular, that $\widehat{\phi}$ is continuous in $(1, +\infty)$.

The same ideas used in the proof of Theorem 1.1 can be used to see that

$$\lim_{t\to 1^+} \widehat{v}(t) = \widehat{v}(1).$$

By Lemma 4.4, $\widehat{\phi}$ is increasing on $[e^{\frac{n}{d}}, +\infty)$, and since $\widehat{\phi}$ is continuous, it attains its minimums in $t_0 \in [1, e^{\frac{n}{d}}]$.

Finally, Lemma 4.5 shows that if $\widehat{\phi}$ attains the minimum, it must be at a single point $t = t_0$, concluding the proof. □

Now we can characterize the minimization point using the Karush-Kuhn-Tucker conditions (see [5], [27]). In order to do so, we first show a global convexity property of the function to be minimized.

**Lemma 4.6.** *The feasible set $\mathbb{F}_d(\mathbb{R}^n)$ is convex and the objective function $W : [0, +\infty) \times \mathbb{F}_d(\mathbb{R}^n)$ given by*

$$W(r, g) = e^r \int_{\mathbb{R}^n} \exp(-g(x))dx$$

*is log-convex and strictly convex.*

**Proof.** Let $(r_i, g_i) \in [0, +\infty) \times \mathbb{F}_d(\mathbb{R}^n)$, $i = 0, 1$. Then, Hölder's inequality implies that

$$\int_{\mathbb{R}^n} e^{-((1-\theta)g_0(x)+\theta g_1(x))}\,dx \leq \left(\int_{\mathbb{R}^n} e^{-g_0(x)}\,dx\right)^{1-\theta} \left(\int_{\mathbb{R}^n} e^{-g_1(x)}\,dx\right)^{\theta}, \tag{11}$$

thus showing that $\mathbb{F}_d(\mathbb{R}^n)$ is convex and

$$W((1-\theta)(r_0, g_0) + \theta(r_1, g_1)) \leq W(r_0, g_0)^{1-\theta} W(r_1, g_1)^{\theta},$$

and hence the log-convexity of $W$.

Notice that the Arithmetic-Geometric mean inequality implies that $W$ is convex.

Furthermore, we show now that it is strictly convex. First of all, let us suppose $(r_0, g_0), (r_1, g_1)$ and $\theta \in [0, 1]$ with

$$W((1-\theta)(r_0, g_0) + \theta(r_1, g_1)) = (1-\theta)W(r_0, g_0) + \theta W(r_1, g_1).$$

The equality case of the AG-mean inequality directly implies that $W(r_0, g_0) = W(r_1, g_1)$. Moreover, it also means that there is equality in (11). Hence, the equality case of Hölder's inequality implies the existence of $c > 0$ such that $e^{-g_0} = ce^{-g_1}$. Since $g_0(0) = g_1(0) = 0$, $c = 1$. Therefore $g_0 = g_1$. Thus $|G_1(g_0)| = |G_1(g_1)|$, and then $r_0 = r_1$, hence showing the strict convexity of $W$. □

**Proof of Theorem 1.3.** Let $W$ be defined as in Lemma 4.6. *Problem 2* then rewrites as the following minimization problem:

$$\min_{(r,g)\in C} W(r,g),$$

where

$$C=\{(r,g)\in[0,+\infty)\times\mathbb{F}_d(\mathbb{R}^n): r-\log f(x)-g(x)\geq 0 \text{ for all } x\in S_f\}$$

with $S_f=\{x\in\mathbb{R}^n : f(x)\neq 0\}$.

Any $g\in\mathbb{H}_d(\mathbb{R}^n)$ can be uniquely written as $g=\sum_{\alpha\in\mathbb{N}_d^n} g_\alpha x^\alpha$, so we can identify each $g$ with its coordinate vector $(g_\alpha)_\alpha\in\mathbb{R}^{h_d(n)}$. Notice that $r-g(x)=r-\sum_{\alpha\in\mathbb{N}_d^n} g_\alpha x^\alpha=\langle(r,(g_\alpha)_\alpha),(1,-(x^\alpha)_\alpha)\rangle$. Thus the feasible set can be rewritten as

$$\begin{aligned}C=\{&(r,(g_\alpha)_\alpha)\in[0,+\infty)\times\mathbb{R}^{h_d(n)}:\\ &g\in\mathbb{F}_d(\mathbb{R}^n),\ \langle(r,(g_\alpha)_\alpha),(1,-(x^\alpha)_\alpha)\rangle\geq\log f(x) \text{ for all } x\in S_f\},\end{aligned}\tag{12}$$

so it is convex, as it is the intersection of half-spaces.

Assume condition (i) holds. Notice that, taking $t_2=e^{r_2}$, $(r_2,g_2)\in\partial C$. Otherwise, we can take $(r,g_2)\in C$ with $r<r_2$, and $W(r,g_2)<W(r_2,g_2)$ contradicting that $W$ attains its minimum on $C$ at $(r_2,g_2)$.

Since $C$ is described in (12) as intersection of halfspaces, then the supporting cone $S_C(r_2,g_2)$ of $C$ at $(r_2,g_2)$ is given by the set of all such halfspaces whose boundaries contain $(r_2,g_2)$, i.e.,

$$\begin{aligned}S_C(r_2,g_2)=\{&(r,(g_\alpha)_\alpha)\in[0,+\infty)\times\mathbb{R}^{h_d(n)}:\\ &g\in\mathbb{F}_d(\mathbb{R}^n),\ \langle(r,(g_\alpha)_\alpha),(1,-(x^\alpha)_\alpha)\rangle\geq\log f(x) \text{ for all } x\in S_f^*\},\end{aligned}$$

where $S_f^*=\{x\in S_f : r_2-g_2(x)=\log f(x)\}$. Thus we have that

$$N_C(r_2,g_2)=\operatorname{pos}(\{(-1,(x^\alpha)_\alpha): x\in S_f^*\}),$$

where $N_C(z)$ is the outer normal cone of $C$ at $z$, for every convex set $C$ and every $z\in\partial C$, and $\operatorname{pos}(R)$ is the positive hull of $R$, the smallest convex cone containing $R$.

Since $W$ is a differentiable strictly convex function, and $C$ is a convex set, under the assumption $(r_2,g_2)\in(0,+\infty)\times\operatorname{int}(\mathbb{F}_d(\mathbb{R}^n))$ the Karush-Kuhn-Tucker conditions (see [5]) characterize $(r_2,g_2)$ by

$$-\nabla W(r_2,g_2)\in N_C(r_2,g_2).\tag{13}$$

Besides, by (7)

$$\nabla W(r,g) = \left( W(r,g), \left( -e^r \int_{\mathbb{R}^n} x^\alpha \exp(-g(x))dx \right)_\alpha \right).$$

Moreover, since $N_C(r_2, g_2) \subset \mathbb{R}^{h_d(n)+1}$ is a convex cone, using Carathéodory's theorem for cones, the previous condition (13) is equivalent to the existence of $x_1, \dots, x_m \in \mathbb{R}^n$, $m \le h_d(n) + 1$, with $r_2 - \log f(x_i) - g_2(x_i) = 0$, and $\lambda_i > 0$, $1 \le i \le m$, such that

$$\left( -W(r_2, g_2), \left( e^r \int_{\mathbb{R}^n} x^\alpha \exp(-g(x))dx \right)_\alpha \right) = \sum_{i=1}^m \lambda_i(-1, (x_i^\alpha)_\alpha)$$

which proves (ii).

Conversely, suppose condition (ii) holds. The Karush-Kuhn-Tucker conditions imply that $(r_2, g_2)$ is an extreme point, and thus by the convexity of $W$, a local minimization point of $W$ on $C$. Since $W$ is strictly convex and $C$ is a convex set, this local minimization point must be the only global minimization point, and (i) is proved. □

**Remark.** Note that our arguments work for the case $|\widehat{H}_1(f)| = +\infty$ (as long as $\widehat{H}_t(f)$ is bounded for some $t > 1$), since then the minimum is attained at some $t_2 \in (1, +\infty)$. This remark allows us to apply our results to a more general set of functions outside $\mathcal{B}(\mathbb{R}^n)$, as shown in the following example. The only case we can not use our arguments is when $\widehat{H}_1(f)$ is unbounded but $|\widehat{H}_1(f)| < +\infty$ (see Example 4.1). For this reason, it would be very interesting to get an extension of Lasserre's theorem for sets of the form $\widehat{H}_1(f)$.

**Example 4.7.** Let $f(x) = \exp(-\|x\|_2^d)$ for some $d \in \mathbb{N}$ even. *Problem 2* then makes sense for $f$ for every even $d' \in \{2, \dots, d\}$. Since

$$\begin{aligned}\widehat{H}_t(f) &= \bigcup_{0<\lambda<1} \frac{1}{(\log t - \log \lambda)^{\frac{1}{d'}}} K_\lambda(f) \\ &= \bigcup_{0<\lambda<1} \frac{(-\log \lambda)^{\frac{1}{d}}}{(\log t - \log \lambda)^{\frac{1}{d'}}} B_2^n.\end{aligned}$$

Note that $\widehat{H}_1(f) = \mathbb{R}^n$. Since the maximum of $(-\log \lambda)^{\frac{1}{d}} / (\log t - \log \lambda)^{\frac{1}{d'}}$ is attained at $\lambda_M = t^{-\frac{d'}{d-d'}}$, then

$$\widehat{H}_t(f) = \frac{d'^{\frac{1}{d}}}{d^{\frac{1}{d'}}} \left( \frac{d - d'}{\log t} \right)^{\frac{1}{d'} - \frac{1}{d}} B_2^n = G_1(g_t),$$

where

$$g_t(x) = \frac{d}{d'^{\frac{d'}{d}} (d - d')^{1 - \frac{d'}{d}}} (\log t)^{1 - \frac{d'}{d}} \|x\|_2^{d'}.$$

Then

$$\min_{t\geq 1} t|G_1(g_t)| = \frac{d'^{\frac{n}{d}}}{d^{\frac{n}{d'}}}(d-d')^{n(\frac{1}{d'}-\frac{1}{d})}\omega_n \min_{t\geq 1} \frac{t}{(\log t)^{n(\frac{1}{d'}-\frac{1}{d})}},$$

where $\omega_n = |B_2^n|$. The minimum above is attained at $t_2 = e^{n(\frac{1}{d'}-\frac{1}{d})}$, and then

$$f(x) \leq t_2 e^{-g_{t_0}(x)} = e^{n(\frac{1}{d'}-\frac{1}{d})} \exp\left(-\frac{d^{\frac{d'}{d}}}{d'} n^{1-\frac{d'}{d}} \|x\|_2^{d'}\right)$$

is the unique solution to *Problem 2* for $f$ with

$$t_2 \int_{\mathbb{R}^n} \exp(-g_{t_0}(x))dx = \omega_n \Gamma(\tfrac{n}{d'}+1) \left(\frac{n}{ed}\right)^{n/d} \Big/ \left(\frac{n}{ed'}\right)^{n/d'} .$$

## 5. Application: *d*-outer volume and integral ratio

Given a compact set $K \subset \mathbb{R}^n$, it is a natural question to consider how well does the volume of the level set of the $d$-Lasserre Löwner polynomial approximate the volume of $K$. In the context of convex bodies, $K \in \mathcal{K}^n$ (resp. centrally symmetric convex bodies $K \in \mathcal{K}_0^n$), it was already Ball in [10, Thms. 1 & 2] who showed, by means of the Brascamp-Lieb inequality, that the largest ratio between the volumes of a compact convex set $K$ and its John ellipsoid is attained when $K$ is a simplex (resp. a cube when $K \in \mathcal{K}_0^n$). Later on, Barthe (see [11, Thms. 2 & 3]) showed, by means of a reverse Brascamp-Lieb inequality [12], that, in the case of the Löwner ellipsoid, the analogous largest ratio between the volume of $G_1(g_2)$ and the volume of $K$ (assuming $G_1(g_2)$ is the Löwner ellipsoid of $K$) is attained when $K$ is a centered simplex (resp. a crosspolytope when $K \in \mathcal{K}_0^n$).

The existence of the $d$-Lasserre-Löwner polynomial $g_d$ naturally leads to define the *$d$-outer volume ratio* $\mathrm{o.v.r}_d(K)$ for any given $K \in \mathcal{K}^n$ as

$$\mathrm{o.v.r}_d(K) = \left(\frac{|G_1(g_d)|}{|K|}\right)^{1/n},$$

for every even $d \in \mathbb{N}$.

Since $g_d$ is homogeneous of degree $d$, $g_d$ is an even function, and thus $G_1(g_d)$ is a centrally symmetric star-shaped with respect to the origin set. The first non-trivial examples on how well we can approximate $K \in \mathcal{K}_0^n$ by $G_1(g_d)$ were computed by Lasserre (see [32, Thm. 3.4]), for the 2-dimensional cube in the cases $d = 4$ and $d = 6$.

Benko and Kroó showed (see Theorem 2 and Lemma 5 in [14]) that if $K \in \mathcal{K}_0^n$ has $C^{1+\varepsilon}$ boundary, for some $\varepsilon \in (0,1]$, then for any $\tau \in (0,1)$ and any even degree $d$ there exists a sequence of polynomials $g_d \in \mathbb{H}_d(\mathbb{R}^n)$ such that $|g_d(x) - 1| \leq cd^{-\tau\varepsilon}$, for every $x \in \partial K$ and some constant $c > 0$, only depending on $K$. By the homogeneity of $g_d$ and of the Minkowski gauge $\|\cdot\|_K$, the inequality above can be rewritten as

$$\forall x \in \mathbb{R}^n \quad (1 - cd^{-\tau\varepsilon})\|x\|_K^d \le g_d(x) \le (1 + cd^{-\tau\varepsilon})\|x\|_K^d.$$

This inequality leads to the following theorem:

**Theorem 5.1.** *Let* $K \in \mathcal{K}_0^n$*. Then* $\lim_{d\to+\infty} \mathrm{o.v.r}_d(K) = 1$.

**Proof.** Fix $\delta > 1$. A standard approximation argument gives us some $Q \in \mathcal{K}_0^n$ of $C^2$ boundary, with $K \subset Q$ and $(|Q|/|K|)^{1/n} \le \sqrt{\delta}$. Let us apply Benko and Kroó result above to $Q$ (with $\varepsilon = 1$ and any fixed $\tau \in (0, 1)$) to get, for any even $d \ge 2$ a sequence of homogeneous polynomials $g_{d,\delta} \in \mathbb{H}_d(\mathbb{R}^n)$ and a constant $c_\delta > 0$ such that

$$(1 - c_\delta d^{-\tau})\|x\|_Q^d. \le g_d(x) \le (1 + c_\delta d^{-\tau})\|x\|_Q^d \quad \forall x \in \mathbb{R}^n.$$

Define $\overline{g}_{d,\delta} = (1 + c_\delta d^{-\tau})^{-1} g_{d,\delta} \in \mathbb{H}_d(\mathbb{R}^n)$. We have $\overline{g}_{d,\delta}(x) \le 1$ for every $x \in Q$, which means $Q \subset G_1(\overline{g}_{d,\delta})$.

On the other hand, if $x \in G_1(g_{d,\delta})$, then $\|x\|_Q \le (1 - c_\delta d^{-\tau})^{-1/d}$, which implies $G_1(g_{d,\delta}) \subset (1 - c_\delta d^{-\tau})^{-1/d} Q$.

Using the fact that $|G_1(\overline{g}_{d,\delta})| = (1 + c_\delta d^{-\tau})^{n/d} |G_1(g_{d,\delta})|$,

$$\begin{aligned}\left(\frac{|G_1(\overline{g}_{d,\delta})|}{|K|}\right)^{\frac{1}{n}} &= (1 + c_\delta d^{-\tau})^{\frac{1}{d}} \left(\frac{|Q|}{|K|}\frac{|G_1(g_{d,\delta})|}{|Q|}\right)^{\frac{1}{n}} \\ &\le \sqrt{\delta}\left(\frac{1 + c_\delta d^{-\tau}}{1 - c_\delta d^{-\tau}}\right)^{\frac{1}{d}} \le \delta,\end{aligned}$$

for any $d \ge d_\delta$ and large enough even $d_\delta$. Since $K \subset Q \subset G_1(\overline{g}_{d,\delta})$, choosing the sequence $\delta_k = \frac{k+1}{k}$ and taking $d_{\delta_{k+1}} > d_{\delta_k}$, the sequence of polynomials

$$g_{d_2,2}, g_{d_2+2,2}, \dots, g_{d_{3/2},3/2}, g_{d_{3/2}+2,3/2}, \dots$$

immediately proves the result. □

**Remark.** Rogers and Shephard showed (see [40]) that if $K \in \mathcal{K}^n$ with $0 \in K$, then $|\mathrm{conv}(K \cup (-K))| \le 2^n |K|$. Considering $g_d^{(1)}$ and $g_d^{(2)}$ the $d$-Lasserre-Löwner polynomials of $K$ and $\mathrm{conv}(K \cup (-K))$, respectively, we have that $K \subset \mathrm{conv}(K \cup (-K)) \subset G_1(g_d^2)$, and

$$\mathrm{o.v.r}_d(K) = \left(\frac{|G_1(g_d^{(1)})|}{|K|}\right)^{\frac{1}{n}} \le 2\left(\frac{|G_1(g_d^{(2)})|}{|\mathrm{conv}(K \cup (-K))|}\right)^{\frac{1}{n}}.$$

Therefore, if $K \in \mathcal{K}^n$ with $0 \in K$, we have that

$$\limsup_{d\to+\infty} \mathrm{o.v.r}_d(K) \le 2.$$

A natural functional extension of the $d$-outer volume ratio for any $f \in \mathcal{F}(\mathbb{R}^n)$ with $\|f\|_\infty = f(0) = 1$ is the *d-outer integral ratio*

$$\text{o.i.r}_d(f) = \left( t \int\limits_{\mathbb{R}^n} \exp(-g(x)^{1/d})\, dx \Big/ \int\limits_{\mathbb{R}^n} f(x)\, dx \right)^{1/n}$$

where $(t, g)$ minimizes *Problem 1*. For $d = 2$, a similar definition is considered in [29].

Theorem 5.1 can also be extended to some examples whenever we approximate log-concave functions. For instance, we can show that if $f \in \mathcal{F}(\mathbb{R}^n)$, then

$$\lim_{d \to +\infty} \text{o.i.r}_d(f) = 1$$

whenever $f(x) = e^{-\|x\|_K}$ with $K \in \mathcal{K}_0^n$. Indeed, it was shown in Example 3.9(1) that $H_t(f) = \text{int}(K)$ if $t > 1$ and $H_1(f) = K$. Since $K \in \mathcal{K}_0^n$, we can take $g_d$ a sequence of homogeneous polynomials given by Theorem 5.1 such that $K \subset G_1(g_d)$ with $|G_1(g_d)|/|K| \to 1$ when $d \to +\infty$. Since $\min_{t \geq 1} t|G_1(g_d)| = |G_1(g_d)|$, and $H_1(f) = K \subset G_1(g_d)$ for every even $d \geq 2$, we have that $f(x) \leq \exp(-g_d(x)^{1/d})$ (see Lemma 3.2) and

$$\text{o.i.r}_d(f) \leq \left( \int\limits_{\mathbb{R}^n} \exp(-g_d(x)^{1/d})\, dx \Big/ \int\limits_{\mathbb{R}^n} f(x)\, dx \right)^{\frac{1}{n}} = \left( \frac{|G_1(g_d)|}{|K|} \right)^{\frac{1}{n}} \to 1$$

as $d \to +\infty$ (see also Lemma 2.2).

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