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Syllepsis in Homotopy Type Theory

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Abstract

The Eckmann-Hilton argument shows that any two monoid structures on the same set satisfying the interchange law are in fact the same operation, which is moreover commutative. When the monoids correspond to the vertical and horizontal composition of a sufficiently higher-dimensional category, the Eckmann-Hilton argument itself appears as a higher cell. This cell is often required to satisfy an additional piece of coherence, which is known as the syllepsis. We show that the syllepsis can be constructed from the elimination rule of intensional identity types in Martin-Löf type theory.

CCS Concepts: • Theory of computation → Type theory.

Keywords: homotopy type theory, syllepsis, higher coherences, higher category theory, Eckmann-Hilton

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1 Introduction

The Eckmann and Hilton [1962] (EH) argument proves that if we have two binary operations \circ and \star on the same set S which share a common, two-sided unit element $1 \in S$, and satisfy the *interchange law*

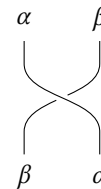
$$(a \circ b) \star (c \circ d) = (a \star c) \circ (b \star d) \quad (1)$$

then the operations coincide ($\circ = \star$), and moreover they are commutative [Leinster 2004, Lemma 1.2.4].

The intuitive idea behind the EH argument is perhaps more easily understood in the graphical form shown in fig. 1, which is due to Baez and Dolan [1995]. Elements of the set S are drawn as boxes. Horizontal juxtaposition corresponds to the operation \star , while vertical juxtaposition corresponds to \circ . The fact that we can stack boxes both vertically and horizontally without parentheses corresponds to the interchange law (1). Starting with two elements $\alpha, \beta \in S$ we make

the unit appear vertically, then absorb it horizontally. We then repeat this process backwards to show commutativity.

Picturing the proof of fig. 1 as a ‘film of frames’ allows to draw lines tracing the paths of α and β , as in fig. 2. We thus obtain a higher-dimensional visualisation of the proof as a braid [Baez and Dolan 1995, §6]:



This elementary argument has numerous applications. Most famously it forms the core of the proof that higher homotopy groups are abelian [Arkowitz 2011, Prop. 2.2.12]. In the context of categorical quantum mechanics it shows that the multiplication of abstract scalars, i.e. endomorphisms $m : I \rightarrow I$ of the monoidal unit object I , is commutative [Heunen and Vicary 2019, §§2.1.1–2.1.3]. Finally, the EH argument is the reason that in concurrency theory there cannot be a simple algebra of sequential and parallel composition that satisfies (1). For example, the axioms of *concurrent Kleene algebra* are forced to weaken (1) to an inequality in order to escape the hold of EH. However, practitioners do not shy away from using inequational EH-type arguments to reason about the interaction of sequential and parallel composition [Hoare et al. 2011, §6].

Eckmann-Hilton in type theory. In the late 2000s a connection was discovered between the *intensional identity types* of Martin-Löf Type Theory (MLTT) and homotopy theory, which swiftly led to the Homotopy Type Theory (HoTT) project [Univalent Foundations Program 2013]. The premise is that type theory can be given a homotopical interpretation [Awodey and Warren 2009]. For example, an element $p : a = b$ of the identity type can be thought of as a *path* from point $a : X$ to point $b : X$ of space X .

Consequently, it is natural to wonder whether the EH argument applies to identity types—and indeed it does. Write $p \cdot q : a = c$ for the concatenation of $p : a = b$ and $q : b = c$. Let there be a type A , and an arbitrary point $*$: A . Furthermore, write $1_* : * = *$ for the reflexivity element of the identity type. Then, given any 2-loops $p, q : 1_* = 1_*$ we have $p \cdot q = q \cdot p$. A proof of this fact is recorded as Theorem 2.1.6 in *loc. cit.*

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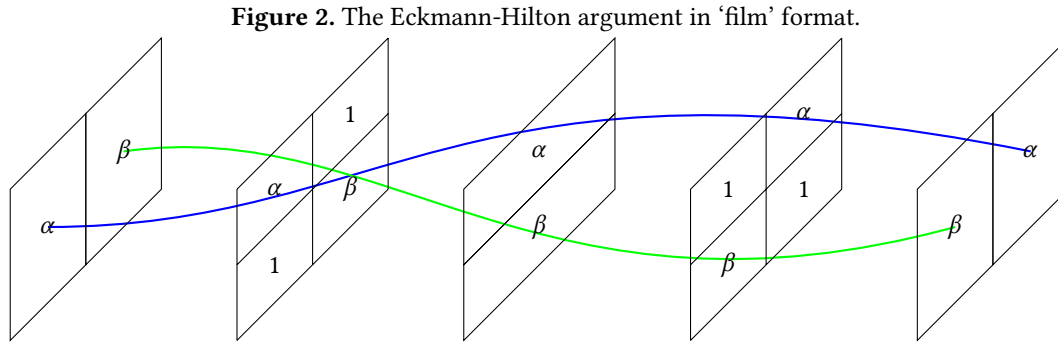
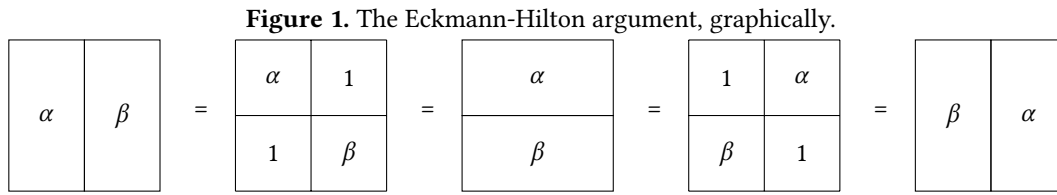


Figure 3. The periodic table of n -categories, as updated by Baez and Shulman [2010] and seen on the nLab.

$k \backslash n$	-2	-1	0	1	2	...
0	trivial	truth value	set	category	2-category	...
1	“	trivial	monoid	monoidal category	monoidal 2-category	...
2	“	“	commutative monoid	braided monoidal cat.	braided monoidal 2-cat.	...
3	“	“	“	symmetric monoidal cat.	syllaptic monoidal 2-cat.	...
4	“	“	“	“	symmetric monoidal 2-cat.	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Eckmann-Hilton and the periodic table. The Eckmann-Hilton argument also plays a important rôle in the *periodic table of n -categories*, which is due to Baez and Dolan [1995]. The most recent version of the table can be seen in fig. 3.

The periodic table generalises the well-known observations that (a) a one-object category is a monoid, and (b) a one-object bicategory is a monoidal category. More generally, the periodic table charts the object that we expect a $(n + k)$ -category with a unique j -cell for $j < k$ to be. This object is also sometimes called a *k -tuply monoidal n -category*. For example, fact (b) above can be seen in the cell $k = n = 1$, as a weak $k + n = 2$ -category with a unique trivial 0-cell (i.e. a unique object) is a monoidal category.

This correspondence is not exact. For example, the cell $k = 2, n = 0$ states that a bicategory with a unique object and unique 1-cell is a commutative monoid. This is not the complete story, as unfolding the definition of the bicategory we also obtain a distinguished invertible element of the monoid. Hence, constructing a categorical equivalence between the k -tuply monoidal n -categories and the category of objects

recorded in the cell can be tricky; some low-dimensional cases are discussed in detail by Cheng and Gurski [2007].

An interesting phenomenon arises whenever $k \geq 2$: when there are at least 2 levels of degenerate cells we obtain some sort of *commutativity*: first a commutative monoid, then braided and symmetric monoidal categories, and so on. This is because of the Eckmann-Hilton argument. For example, when $k = 2$ and $n = 0$ the table claims that a 2-category with a unique object and a unique 1-cell is a commutative monoid. A 2-category comes with both a horizontal and a vertical composition that satisfy (1). Hence, the EH argument can be used to show that these compositions are the same, and are moreover commutative.

The column for $n = 1$ shows that the progression from monoidal to braided to symmetric monoidal category is simply an increase in dimensionality.¹ However, a curious phenomenon arises for $n = 2$: a gap has appeared between braided monoidal and symmetric monoidal bicategories! Originally called weakly involutory, these categories were

¹A textbook account of these types of categories and their string diagram calculi can be found in Heunen and Vicary [2019, §1].

named *syllaptic monoidal 2-categories*, after their distinguishing gadget was named *syllapsis* by Day and Street [1997]. An explicit definition is given by McCrudden [2000].

Given that syllaptic monoidal 2-categories are 5-categories trivial below dimension 3, this is not surprising. As explained by Baez and Dolan [1995, §5] the case of $k = 1$ corresponds to the ability to multiply any two objects X, Y of the bicategory to obtain $X \otimes Y$. Then, $k = 2$ corresponds to having a braiding $\sigma_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$. For $k = 3$ we also obtain an invertible 2-cell $v_{X,Y} : \sigma_{x,y} \xrightarrow{\cong} \sigma_{y,x}^{-1}$, viz. the syllapsis. Finally, the case $k = 4$ requires a strict equality of horizontally-composed 2-cells. Thus, what was previously a strict equality for symmetric monoidal categories now becomes structure, and new coherence laws arise at higher dimensions.

Syllepsis in type theory. Recall that given $* : A$, and 2-loops $p, q : 1_* = 1_*$ we may construct a proof

$$\text{EH}(p, q) : p \cdot q = q \cdot p$$

We shift one dimension higher: starting from $1_* : * = *$, and given 3-loops $p, q : 1_* = 1_*$, the *type-theoretic syllapsis* is the statement that

$$\text{EH}(p, q) \cdot \text{EH}(q, p) = 1_{p \cdot q}$$

The dimensional shift upwards is expected given our previous discussion: the syllapsis arises in a 5-category which is trivial below dimension 3, so the least dimension in which the statement is provable should be 3. This observation is supported by a key step in the proof, which requires the application of the EH argument one dimension lower.

It is interesting to ponder why the syllapsis should morally be definable in intensional type theory. Path induction endows every type with the structure of an ∞ -groupoid. Thus, restricting our view to 2-loops on an arbitrary point we expect to see some kind of monoidal structure arise. But, as we have an ∞ -groupoid, we further expect “all possible coherence” to be definable: starting with 3-loops we expect to see at least a syllaptic monoidal 2-category; and if we start with 4-loops, we expect a symmetric monoidal 2-category. Nevertheless, our work shows that this is far from trivial.

Contributions. We construct both a proof of Eckmann-Hilton as well as the associated syllapsis from first principles, using only intensional identity types in MLTT. That this is possible has been believed for more than a decade. For example, Shulman stated that HoTT may be used to study braidings and syllapses in a comment on the *n-Category Café* back in 2011. The explicit question of constructing a syllapsis from identity types was posed again by Vicary [2020]. Thus, our work resolves a long-standing problem in generating higher coherences in HoTT.

Note that, while we rely heavily on the homotopical interpretation of intensional identity types in Martin-Löf Type Theory on an intuitive and informal level, we do not use any

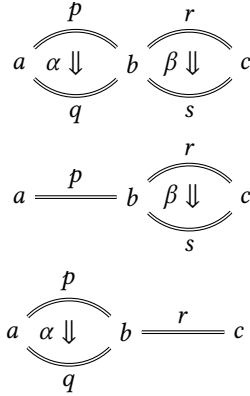
of the usual axioms of the formal system commonly referred to as “book-HoTT.” In particular, we use neither function extensionality, nor the univalence axiom.

To the best of our knowledge, this is the first time a higher coherence of this kind has been proven in a fully-fledged formal system, and in particular in Martin-Löf type theory. Work by Benjamin [2018] and Finster et al. [2022] has respectively attempted and succeeded to construct a cell with the type of syllapsis in specialised systems for weak ∞ -categories. In contrast, we work in a well-understood system with an established homotopical interpretation [Kapulkin and Lumsdaine 2021]. Moreover, our proof implies that the syllapsis can be constructed in any $(\infty, 1)$ -topos, through the interpretation of Shulman [2019].

Finally, we hope that our construction can be used in further results. For example, Shulman [2011] has conjectured that the Eckmann-Hilton argument can be used to construct the Hopf element of $\pi_3(S^2) \cong \mathbb{Z}$, and Vicary [2020] that the syllapsis may be implicated in a proof that $\pi_4(S^3) \cong \mathbb{Z}_2$ [Brunerie 2019]. Thus, we hope that our work can be used to construct novel, machine-checkable proofs of statements in synthetic homotopy theory.

Notation. Our notation will closely follow the informal style of the HoTT book [Univalent Foundations Program 2013, §1–2]. We write $a, b : X$ to mean that a and b are points of the type X . We write $a = b$ for the intensional identity type, so that $p : a = b$ means that p is a proof that a is equal to b . We write \cdot for the concatenation of paths; for example, if $p : a = b$ and $q : b = c$, then $p \cdot q : a = c$. Given any point $a : X$ we denote the reflexivity path on that point by $1_a : a = a$. We use the symbol \equiv for definitional equality. Finally, we make repeated use of the elimination rule for intensional identity types, in the informal style of *path induction* described in §§1.12, 2 of *loc. cit.*

Style. Throughout the paper we try to demonstrate the central rôle that path induction plays when generating higher coherences. Broadly speaking, there are always two ways to prove a higher-dimensional statement. The first is to use a number of well-known laws of path algebra, as described e.g. in §2 of the HoTT book. The second is to generalise the desired result; this is usually achieved by replacing specific paths with general ones, liberating the endpoints of a path, and so on. Used deftly, this technique reduces most higher-dimensional statements to a series of judicious path inductions. We strive to follow the second technique wherever possible. Sometimes this comes at a price: the resultant lemmas are often neither general nor intuitive. However, we believe that, more often than not, it vastly reduces the complexity of our results, and the effort required to prove them. In addition, this technique improves the computational behaviour of proofs. In short, using path induction almost exclusively scales well to higher-dimensional results.

Figure 4. Whiskering and horizontal composition

2 Monoidal structure

We first define some operations that witness the monoidal structure of concatenation. First, given any type A , points $a, b : A$, and $p : a = b$ we define paths

$$\mathsf{l}_p : 1_a \cdot p = p \quad \mathsf{r}_p : p \cdot 1_b = p$$

by path induction on p . The functoriality of concatenation is witnessed through whiskering operations. For any type A and points $x, y, z : A$, paths $p, q : a = b$ and $r, s : b = c$, and 2-paths $\alpha : p = q$ and $\beta : r = s$, we define the operations

$$\begin{aligned} \alpha \star \beta &: p \cdot r = q \cdot s \\ p \star_\ell \beta &: p \cdot r = p \cdot s \\ \alpha \star_r r &: p \cdot r = q \cdot r \end{aligned}$$

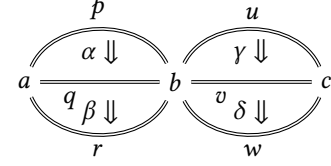
The ‘shape’ of these operations is depicted in fig. 4. Clearly the latter two—namely left and right whiskering—are special cases of the first, viz. horizontal composition. However, we find all three definitions convenient.

Left and right whiskering are natural with respect to units. This corresponds to the naturality of the unit isomorphisms in monoidal categories.

Lemma 2.1. *Given points $a, b : A$, paths $p, q : a = b$ and a 2-path $\alpha : p = q$, we have 3-paths that fill the following squares.*

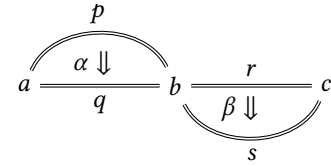
$$\begin{array}{ccc} 1_a \cdot p & \xrightarrow{\mathsf{l}_p} & p \\ 1_a \star_\ell \alpha \parallel & \text{ulnat}(\alpha) & \parallel \alpha \\ 1_a \cdot q & \xrightarrow{\mathsf{l}_q} & q \end{array} \quad \begin{array}{ccc} p \cdot 1_b & \xrightarrow{\mathsf{r}_p} & p \\ \alpha \star_r 1_b \parallel & \text{urnat}(\alpha) & \parallel \alpha \\ q \cdot 1_b & \xrightarrow{\mathsf{r}_q} & q \end{array}$$

As expected, the parallel and horizontal compositions satisfy the *interchange law*: given paths



we have $(\alpha \star \beta) \star (\gamma \star \delta) = (\alpha \star \gamma) \star (\beta \star \delta)$. Thus, the following lemma, which states that whiskering on the left commutes with whiskering on the right, is not surprising. Both of these facts correspond to the bifunctionality of the tensor in a monoidal category.

Lemma 2.2. *For paths as in the picture*



we have a filler for the square

$$\begin{array}{ccc} p \cdot r & \xrightarrow{\alpha \star_r r} & q \cdot r \\ p \star_\ell \beta \parallel & \text{wlrat}(\alpha, \beta) & \parallel q \star_\ell \beta \\ p \cdot s & \xrightarrow{\alpha \star_r s} & q \cdot s \end{array}$$

Proof. The filler is defined by induction on both α and β , whereupon everything reduces to reflexivity. \square

3 Eckmann-Hilton

Proofs of Eckmann-Hilton (EH) in HoTT have been given by various authors, including Licata [2011], Favonia [2012], and Christensen [2021]. We will be computing with the EH term, so we give a careful proof of our own. Our proof is propositionally equal to one of the two ‘evident’ proofs.

Let $\ast : A$, and $p, q : 1_\ast = 1_\ast$ be 2-loops. Instantiating lemma 2.1 with $\alpha \equiv p$ and $\beta \equiv q$ we obtain 3-paths

$$\begin{aligned} \text{ulnat}(p) &: (1_\ast \star_\ell p) \cdot 1_{1_\ast} = 1_{1_\ast} \cdot p \\ \text{urnat}(q) &: (q \star_r 1_\ast) \cdot 1_{1_\ast} = 1_{1_\ast} \cdot q \end{aligned}$$

Noting that $1_{1_\ast} \equiv \mathsf{r}_{1_\ast} \equiv 1_{1_\ast}$ this reduces to

$$\begin{aligned} \text{ulnat}(p) &: (1_\ast \star_\ell p) \cdot 1_{1_\ast} = 1_{1_\ast} \cdot p \\ \text{urnat}(q) &: (q \star_r 1_\ast) \cdot 1_{1_\ast} = 1_{1_\ast} \cdot q \end{aligned}$$

Cancelling the reflexivities on each side immediately gives $1_\ast \star_\ell p = p$ and $q \star_r 1_\ast = q$, which will yield EH as a corollary of lemma 2.2. This step of ‘squashing’ a commutative square whose opposing sides are reflexivities is a leitmotif in the proof, so we define the following shorthands.

Lemma 3.1. For any $a, b : A$ and paths $p, q : a = b$ we have equivalences

$$\begin{aligned} \Rightarrow & : (p \cdot 1_b = 1_a \cdot q) \simeq (p = q) \\ \Downarrow & : (1_a \cdot p = q \cdot 1_b) \simeq (p = q) \end{aligned}$$

Theorem 3.2 (Eckmann-Hilton). Given $*$: A and 2-loops $p, q : 1_* = 1_*$ we have a 3-path

$$EH(p, q) : p \cdot q = q \cdot p$$

Proof. The path is given by

$$\begin{array}{c} p \cdot q \\ \parallel \\ (\Rightarrow \text{ulnat}(p) \star \Rightarrow \text{ulnat}(q))^{-1} \\ (1_* \star_\ell p) \cdot (q \star_r 1_*) \\ \parallel \\ \text{wlrnat}(q, p) \\ (q \star_r 1_*) \cdot (1_* \star_\ell p) \\ \parallel \\ \Rightarrow \text{ulnat}(q) \star \Rightarrow \text{ulnat}(p) \\ q \cdot p \end{array}$$

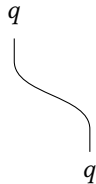
□

4 Eckmann-Hilton and reflexivity

In order to construct the syllepsis we will need to compute the effect of EH on reflexivity. Unfolding the definitions of its various parts we plainly see that

$$EH(1_{1_*}, 1_{1_*}) \equiv 1_{1_{1_*}}$$

However, the situation is not as immediate in the case of $EH(1_{1_*}, q) : 1_{1_*} \cdot q = q \cdot 1_{1_*}$. To understand what it must be, it is useful to picture $EH(1_{1_*}, q)$ as a braid:



If $p \equiv 1_{1_*}$, then the wire representing p is rendered invisible, leaving behind essentially the identity. Unfortunately, the amenities of string diagrams are not available within HoTT, and we must use the appropriate isomorphisms to deal with the lurking reflexivities. Hence, we will show that

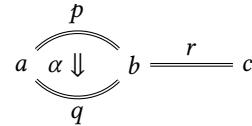
$$EH(1_{1_*}, q) = 1_{1_*} \cdot q \stackrel{l_q}{=} q \stackrel{r_q^{-1}}{=} q \cdot 1_{1_*} \quad (2)$$

Unfolding $EH(1_{1_*}, q)$ and normalizing we obtain

$$\begin{array}{c} 1_{1_*} \cdot q \\ \parallel \\ ((1_{1_*} \star \Rightarrow(\text{urnat}(q)))^{-1} \\ 1_{1_{1_*}} \cdot (q \star_r 1_*) \\ \parallel \\ \text{wlrnat}(1_{1_*}, q) \\ (q \star_r 1_*) \cdot 1_{1_{1_*}} \\ \parallel \\ \Rightarrow(\text{urnat}(q)) \star 1_{1_{1_*}} \\ q \cdot 1_{1_*} \end{array}$$

This path consists entirely of monoidal-type coherences, so it should fall under the spell of the coherence theorem: as “all diagrams commute,” it cannot be anything but $l_q \cdot r_q^{-1}$. However, we cannot prove that immediately: the endpoints of q are fixed, so we cannot induct on it. Thus, we must make an appropriate generalisation to which we will be able to apply path induction. The thing to do here is to abstract the endpoints, as well as the cell $\Rightarrow(\text{urnat}(q))$.

In more detail, given any type X , points $a, b, c : X$, paths



as well as a 2-cell $s : p \cdot r = q \cdot r$, and a 3-path

$$\theta : \alpha \star_r r = s$$

we define a filler for the pentagon

$$\begin{array}{ccc} 1_{p \cdot r} \cdot (\alpha \star_r 1_r) & \xrightarrow{\text{wlrnat}(1_r, \alpha)} & (\alpha \star_r 1_r) \cdot 1_{q \cdot r} \\ 1_{1_{p \cdot r}} \star \theta \parallel & & \parallel \theta \star 1_{1_{q \cdot r}} \\ 1_{p \cdot r} \cdot s & & s \cdot 1_{q \cdot r} \\ & \searrow l_s \quad \swarrow r_s & \\ & s & \end{array}$$

Notice that this is a 4-dimensional lemma! What is more, it is immediately provable by induction: first on θ , and then on α . Specialising it to $\theta \equiv \Rightarrow \text{urnat}(p)$ we see that the top, left and right sides are the aforementioned proof of $EH(1_{1_*}, q)$. Thus, we obtain a proof

$$EH_{1l}(q) : EH(1_{1_*}, q) \cdot r_q = l_q$$

A similar construction yields for any $p : 1_* = 1_*$ a path

$$EH_{1r}(p) : EH(p, 1_{1_*}) \cdot l_q = r_q$$

5 Squares

Many constructions in the rest of the paper will depend on the manipulation of squares of the form

$$\begin{array}{ccc} x & \xrightarrow{q} & z \\ p \parallel & \phi & \parallel s \\ y & \xrightarrow{r} & w \end{array} \quad (3)$$

Such squares are meant to be read first from top to bottom, and then from left to right. For example, the above depiction implies that $p : x = y$, $q : x = z$, and $\theta : p \cdot r = q \cdot s$.

Two squares

$$\begin{array}{ccccc} a_0 & \xrightarrow{ab_0} & b_0 & \xrightarrow{bc_0} & c_0 \\ a_{01} \parallel & \phi & \parallel b_{01} & \theta & \parallel c_{01} \\ a_1 & \xrightarrow{ab_1} & b_1 & \xrightarrow{bc_1} & c_1 \end{array}$$

that can be pasted side-by-side can be composed in the evident way to form a square

$$\phi \boxplus \theta : a_{01} \cdot (ab_1 \cdot bc_1) = (ab_0 \cdot bc_0) \cdot c_{01}$$

Similarly, two squares

$$\begin{array}{ccc} a_0 & \xrightarrow{a_{01}} & a_1 \\ ab_0 \parallel & \phi & \parallel ab_1 \\ b_0 & \xrightarrow{b_{01}} & b_1 \\ bc_0 \parallel & \theta & \parallel bc_1 \\ c_0 & \xrightarrow{c_{01}} & c_1 \end{array}$$

that can be placed vertically on top of each other can be composed to form

$$\phi \boxminus \theta : (ab_0 \cdot bc_0) \cdot c_{01} = a_{01} \cdot (ab_1 \cdot bc_1)$$

Degenerate squares. Squares of the form

$$\begin{array}{ccc} a & \xrightarrow{1_a} & a \\ p \parallel & \phi & \parallel q \\ b & \xrightarrow{1_b} & b \end{array}$$

which have reflexivities on their top and bottom boundaries will play a special rôle in our proof. Indeed, such squares can be ‘squeezed’ into paths between their sides by the equivalence $\Rightarrow : (p \cdot 1_b = 1_a \cdot q) \simeq (p = q)$. This equivalence interacts in a rich manner with other operations.

Lemma 5.1. *Given two squares*

$$\begin{array}{ccc} a & \xrightarrow{1} & a \\ p \parallel & \phi & \parallel q \\ b & \xrightarrow{1} & b \end{array} \quad \begin{array}{ccc} a & \xrightarrow{1} & a \\ q & \theta & r \\ b & \xrightarrow{1} & b \end{array}$$

we have that ‘squeezing’ the squares and then composing the resultant paths vertically is equal to horizontally composing the squares and then ‘squeezing’ the result. In other words,

$$\Rightarrow(\phi) \cdot \Rightarrow(\theta) = \Rightarrow(\phi \boxplus \theta)$$

Lemma 5.2. *Given two squares*

$$\begin{array}{ccc} a & \xrightarrow{1} & a \\ p \parallel & \phi & \parallel q \\ b & \xrightarrow{1} & b \\ r \parallel & \theta & \parallel s \\ c & \xrightarrow{1} & c \end{array}$$

we have that ‘squeezing’ both squares into paths and then composing them horizontally is the same as composing the squares vertically and then ‘squeezing’ them. In other words,

$$\Rightarrow(\phi) \star \Rightarrow(\theta) = \Rightarrow(\phi \boxminus \theta)$$

Both of these lemmas are readily proven by noticing that the squares θ and ϕ are equivalent to paths, and then performing path induction on these paths.

Cubical interchange. Finally, the vertical and horizontal composition of squares satisfy an interchange law.

Lemma 5.3 (Cubical interchange). *Four squares*

$$\begin{array}{ccccc} a_0 & \xrightarrow{i_0} & b_0 & \xrightarrow{j_0} & c_0 \\ p \parallel & \alpha & q \parallel & \beta & \parallel r \\ a_1 & \xrightarrow{i_1} & b_1 & \xrightarrow{j_1} & c_1 \\ u \parallel & \gamma & v \parallel & \delta & \parallel w \\ a_2 & \xrightarrow{i_2} & b_2 & \xrightarrow{j_2} & c_2 \end{array}$$

satisfy the interchange law

$$(\alpha \boxminus \gamma) \boxplus (\beta \boxminus \delta) = (\alpha \boxplus \beta) \boxminus (\gamma \boxplus \delta)$$

Proof. First induct on all the horizontal paths. This leaves a number of degenerate squares like the ones mentioned previously. We may replace these with ordinary paths under the equivalence \Rightarrow , and then perform path induction on them. Finally, inducting on the remaining paths p and u yields the result by reflexivity. \square

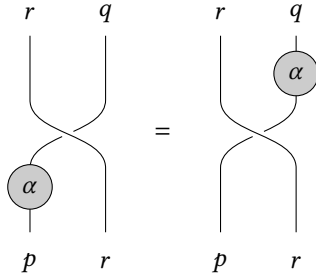
6 Naturality of Eckmann-Hilton

We can show that the proof of EH itself is natural with respect to whiskering in both arguments.

Lemma 6.1. *For 2-loops $p, q, r : 1_* = 1_*$ and any 3-path $\alpha : p = q$ we have fillers for the following squares.*

$$\begin{array}{ccc} p \cdot r \xrightarrow{EH(q, r)} r \cdot p & & r \cdot p \xrightarrow{EH(r, p)} p \cdot r \\ \alpha \star_r r \parallel EH_r^{nat}(\alpha, r) \parallel r \star_\ell \alpha & & r \star_\ell \alpha \parallel EH_r^{nat}(\alpha, r) \parallel \alpha \star_r r \\ q \cdot r \xrightarrow{EH(p, r)} r \cdot q & & r \cdot q \xrightarrow{EH(r, q)} q \cdot r \end{array}$$

Both of these fillers are defined by induction on α , and can be understood graphically. For example, the first one corresponds to the following string diagram equation:



Furthermore, these naturality squares satisfy their own coherence laws! For example, given $p : 1_{1_*} = 1_{1_*}$ we aim to show the following equality of squares.

$$\begin{array}{ccc} 1 \cdot 1 \xrightarrow{EH(1, 1)} 1 \cdot 1 \xrightarrow{I_1} 1 & & 1 \cdot 1 \xrightarrow{r_1} 1 \\ p \star_r 1 \parallel EH_r^{nat}(p, 1) \parallel ulnat(p) \parallel p & = & p \star_r 1 \parallel urnat(p) \parallel p \\ 1 \cdot 1 \xrightarrow{EH(1, 1)} 1 \cdot 1 \xrightarrow{I_1} 1 & & 1 \cdot 1 \xrightarrow{r_1} 1 \end{array}$$

The top boundaries match because $EH(1, 1) \equiv 1$, and $r_1 \equiv I_1 \equiv 1$. The middle vertical boundary, which is omitted due to space, is $1 \star_\ell p$.

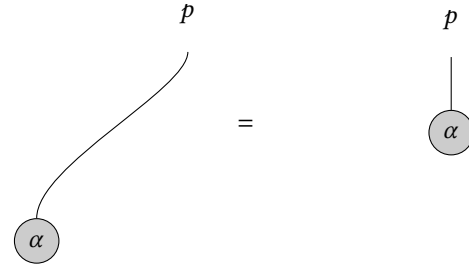
As before, p is now a 2-loop so we cannot induct on it. However, we can liberate one of the endpoints, and thereby prove the following more general lemma.

Lemma 6.2. *Given any 2-loop $p : 1_* = 1_*$ and any 3-path $\alpha : 1_* = p$ we have the following equality.*

$$\begin{array}{ccc} 1 \cdot 1 \xrightarrow{EH(1, 1)} 1 \cdot 1 \xrightarrow{I_1} 1 & & 1 \cdot 1 \xrightarrow{r_1} 1 \\ \alpha \star_r 1 \parallel EH_r^{nat}(\alpha, 1) \parallel ulnat(\alpha) \parallel \alpha & = & \alpha \star_r 1 \parallel urnat(\alpha) \parallel \alpha \\ p \cdot 1 \xrightarrow{EH(p, 1)} 1 \cdot p \xrightarrow{I_p} p & & p \cdot 1 \xrightarrow{r_p} p \\ & & EH(p, 1) \parallel I_p \end{array}$$

where the triangle is filled with $EH_{1r}(p) : EH(p, 1) \cdot I_p = r_p$.

This generalised lemma can be visualised slightly more easily. Both the LHS and RHS of this equation can be visualised as the string diagram equation



where there is an ‘invisible EH braid’ just above α , one of whose strands is the identity. Of course, string diagrams do not differentiate between these two constructions.

This lemma is immediate by induction on α . Then, letting α be a 3-loop on 1_{1_*} proves the aforementioned equality.

7 Syllepsis

We will now prove the main

Theorem 7.1 (Syllepsis). *Given a type A , a point $* : A$, and 2-loops $p, q : 1_{1_*} = 1_{1_*}$ we have*

$$Syl(p, q) : EH(p, q) \cdot EH(q, p) = 1_{1_{1_*}}$$

The paths implicated in the statement are depicted graphically as a pentagon in fig. 5. In order to avoid confusion regarding the direction of various paths we have added arrowheads.

One obvious strategy for proving this theorem is to split the pentagon into a square (b) and two triangles (a) and (c) by adding paths at the dotted lines. We immediately see that (b) appears to be some higher-dimensional naturality property. Moreover, it is easy to notice that (a) and (c) are very strongly related. We will first tackle (b), and then (a) and (c) together.

7.1 Square (b)

We may attempt to prove square (b) by ‘liberating’ the endpoints of all loops, so that we can perform induction. This does not work immediately.

To see this, suppose we start with the general situation

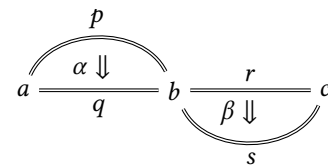
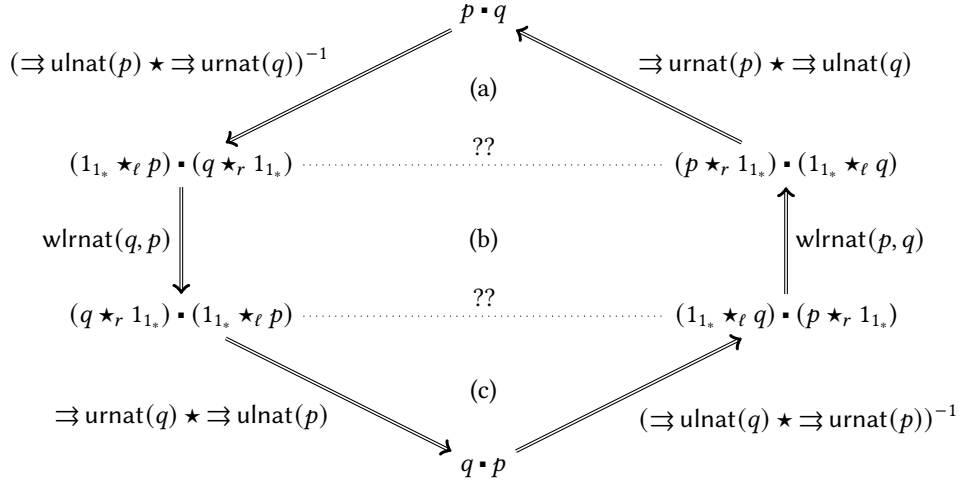
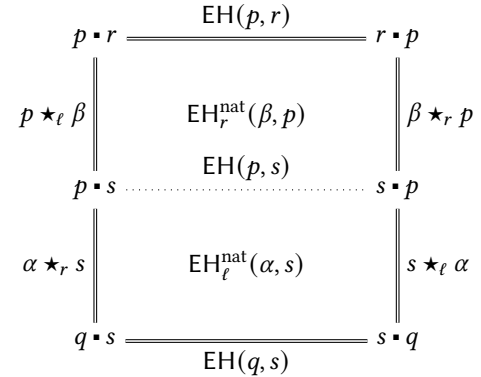
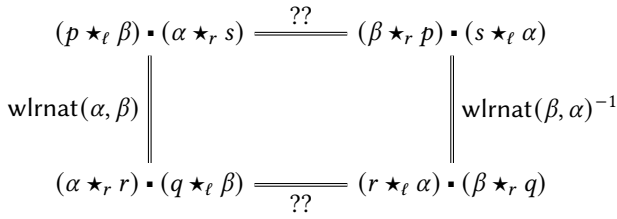


Figure 5. The statement of syllepsis



First, we see that $\text{wlrnat}(\beta, \alpha)$ does not make sense unless $a \equiv b \equiv c$. But even then, if we try to ‘complete the square’

squares



we notice that the path on the top left has boundary

$$(p \star_{\ell} \beta) \star (\alpha \star_r s) : p \star r = q \star s$$

while the path on the top right has boundary

$$(\beta \star_r p) \star (s \star_{\ell} \alpha) : r \star p = s \star q$$

Similarly, the bottom one is taken to be $\text{EH}_r^{\text{nat}}(\alpha, r) \boxminus \text{EH}_r^{\text{nat}}(\beta, q)$. The square itself is filled by induction on α and β , followed by taking \Downarrow of reflexivity.

As this is another 5-dimensional lemma, the square of fig. 6 cannot immediately be depicted in string diagrams. However, it is worth noticing that the top side is essentially

Hence, there cannot be a path between them. To mend that we must ‘correct’ both endpoints by an EH. That can only happen if p, q, r, s are 2-loops themselves, which forces $a \equiv b \equiv c \equiv 1_*$ for some $*$: A . Hence, we are proving a lemma that cannot even be stated below 5 dimensions: p, q, \dots are 2-paths, α, β, \dots are 3-paths, and $\text{wlrnat}(\alpha, \beta)$ is then a 4-path.

The completed version of the square—with the endpoints ‘corrected’ by EH—can be seen fig. 6. Both the top and bottom sides are easy to construct by pasting naturality squares for EH. The top one is given by the vertical composition of

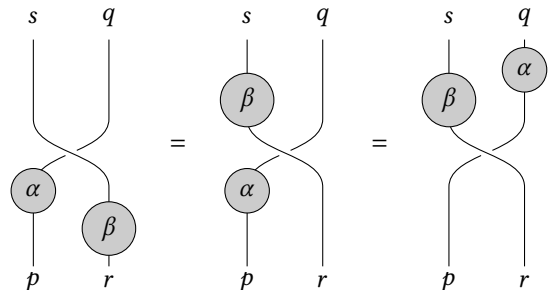


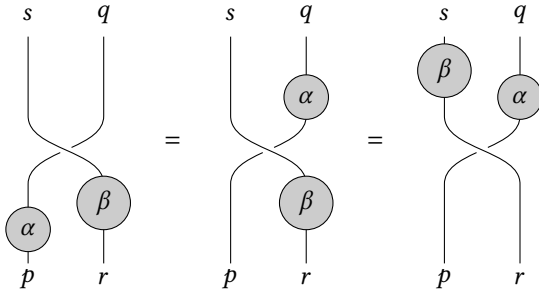
Figure 6. The adapted version of square (b).

$$\begin{array}{ccc}
 ((p \star_\ell \beta) \star (\alpha \star_r s)) \star \text{EH}(q, s) & \xrightarrow{\text{EH}_r^{\text{nat}}(\beta, p) \boxtimes \text{EH}_\ell^{\text{nat}}(\alpha, s)} & \text{EH}(p, r) \star ((\beta \star_r q) \star (s \star_\ell \alpha)) \\
 \text{wlrnat}(\alpha, \beta) \star 1_{\text{EH}(q, s)} \Big\| & \text{square}(\alpha, \beta) & \Big\| 1_{\text{EH}(p, r)} \star \text{wlrnat}(\beta, \alpha)^{-1} \\
 ((\alpha \star_r r) \star (q \star_\ell \beta)) \star \text{EH}(q, s) & \xrightarrow{\text{EH}_\ell^{\text{nat}}(\alpha, r) \boxtimes \text{EH}_r^{\text{nat}}(\beta, q)} & \text{EH}(p, r) \star ((r \star_\ell \alpha) \star (\beta \star_r q))
 \end{array}$$

Figure 7. The property square (q, p) for 2-loops $p, q : 1_{1_*} = 1_{1_*}$.

$$\begin{array}{ccc}
 ((1_{1_*} \star_\ell p) \star (q \star_r 1_{1_*})) \star 1_{1_*} & \xrightarrow{\text{EH}_r^{\text{nat}}(p, 1) \boxtimes \text{EH}_\ell^{\text{nat}}(q, 1)} & 1_{1_*} \star ((p \star_r 1_{1_*}) \star (1_{1_*} \star_\ell q)) \\
 \text{wlrnat}(q, p) \star 1_{1_{1_*}} \Big\| & \text{square}(q, p) & \Big\| 1_{1_{1_*}} \star \text{wlrnat}(p, q)^{-1} \\
 ((q \star_r 1_{1_*}) \star (1_{1_*} \star_\ell p)) \star 1_{1_*} & \xrightarrow{\text{EH}_\ell^{\text{nat}}(q, 1) \boxtimes \text{EH}_r^{\text{nat}}(p, 1)} & 1_{1_*} \star ((1_{1_*} \star_\ell q) \star (p \star_r 1_{1_*}))
 \end{array}$$

and the bottom side is



The sides of the square adjust for the different heights of α and β , a piece of naturality that string diagrams automatically quotient away. The lemma then states that these two witnesses for these equations are themselves equal. In a way, we are proving commutativity of the naturality of EH: it is immaterial which 2-cell slides through the braid first.

Going back to the setting of syllepsis and the 2-loops p, q on 1_{1_*} , we see that $\text{square}(q, p)$, which is shown in fig. 7, is very close to being a filler to square (b). The only difference is the appearance of various reflexivities, incl. the concatenation by $\text{EH}(1_{1_*}, 1_{1_*}) \equiv 1_{1_{1_*}}$ everywhere, as well as the horizontal composition with $1_{1_{1_*}}$ in places. We could deal with this systematically. For example, we could apply \rightrightarrows to a filled square and its sides, prove that $\rightrightarrows(1 \star \alpha) = \alpha$, and then eliminate the identities on the sides. Instead, we will be content with the fact that

$$\begin{aligned}
 &\rightrightarrows(\text{EH}_r^{\text{nat}}(p, 1) \boxtimes \text{EH}_\ell^{\text{nat}}(q, 1)) \\
 &\rightrightarrows(\text{EH}_\ell^{\text{nat}}(q, 1) \boxtimes \text{EH}_r^{\text{nat}}(p, 1))
 \end{aligned}$$

have the right boundary to fit in the dotted lines of fig. 5, and then mend the difference in section 7.3.

7.2 Triangles (a) and (c)

Consider now the triangles (a) and (c). Their completed form is depicted in fig. 8. These are clearly another piece of coherence. In fact, they are equations that are familiar to us, but are obscured by the way they are written.

We will prove these two triangles in two different ways. The first proof will be more conceptual: it will consist of path algebra involving squares, using the various lemmas we have developed in section 5. This proof will reveal what the triangles really mean. Then, we will show another, shorter proof, which proceeds by path induction.

First proof. The first thing we notice about triangle (a) is the ubiquity of the equivalence \rightrightarrows . We can simplify two of its sides by using lemma 5.2, which yields that

$$\begin{aligned}
 &\rightrightarrows \text{ulnat}(p) \star \rightrightarrows \text{urnat}(q) = \rightrightarrows(\text{ulnat}(p) \boxtimes \text{urnat}(q)) \\
 &\rightrightarrows \text{urnat}(p) \star \rightrightarrows \text{ulnat}(q) = \rightrightarrows(\text{urnat}(p) \boxtimes \text{ulnat}(q))
 \end{aligned}$$

We are then left with a triangle all of whose sides are of the form $\rightrightarrows \alpha$ for some α . Luckily, we can use lemma 5.1 to reduce the composition of paths, for it provides an equality

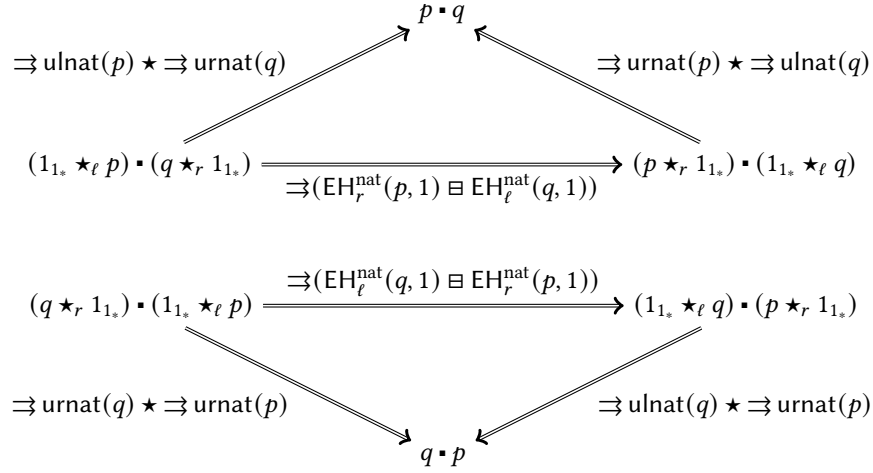
$$\begin{aligned}
 &\rightrightarrows(\text{EH}_r^{\text{nat}}(p, 1) \boxtimes \text{EH}_\ell^{\text{nat}}(q, 1)) \star \rightrightarrows(\text{urnat}(p) \boxtimes \text{ulnat}(q)) \\
 &= \rightrightarrows((\text{EH}_r^{\text{nat}}(p, 1) \boxtimes \text{EH}_\ell^{\text{nat}}(q, 1)) \boxtimes (\text{urnat}(p) \boxtimes \text{ulnat}(q)))
 \end{aligned}$$

This reduces the equation to one of the form $\rightrightarrows \alpha = \rightrightarrows \beta$, so it suffices to prove $\alpha = \beta$. This happens to be the statement

$$\begin{aligned}
 &(\text{EH}_r^{\text{nat}}(p, 1) \boxtimes \text{EH}_\ell^{\text{nat}}(q, 1)) \boxtimes (\text{urnat}(p) \boxtimes \text{ulnat}(q)) \\
 &= \text{ulnat}(p) \boxtimes \text{urnat}(q)
 \end{aligned}$$

By the cubical interchange law (lemma 5.3) we can replace the LHS with

$$(\text{EH}_r^{\text{nat}}(p, 1) \boxtimes \text{urnat}(p)) \boxtimes (\text{EH}_\ell^{\text{nat}}(q, 1) \boxtimes \text{ulnat}(q))$$

Figure 8. The completed triangles (a) and (c) of fig. 5.

Thus, it suffices to prove that

$$EH_r^{\text{nat}}(p, 1) \sqcap \text{urnat}(p) = \text{ulnat}(p)$$

$$EH_{\ell}^{\text{nat}}(q, 1) \sqcap \text{urnat}(q) = \text{urnat}(q)$$

Yet, we proved both of these equations in section 6! Thus, we have a filler for triangle (a). Likewise, a similar calculation provides a filler for triangle (c).

Second proof. While the above proof is intuitive and algebraic, it relies a lot on explicit compositions of paths. Moreover, it relies on lemmas 5.1 to 5.3. Here we may have proved these lemmas in a computationally sensible way, which might not have been the case had we imported them from a library. Moreover, modifying the proof to triangle (c) contains some tricky symmetry (left vs. right, p vs. q , etc.).

A much more scalable way of filling triangle (a) is abstracting all of its constituent parts, and generating exactly the right amount of coherence with path induction. What is more, the same proof will also apply to triangle (c). Unlike the first one there will be no fiddly path algebra.

Lemma 7.2. *Suppose we have six squares as follows:*

$$\begin{array}{ccc}
 \begin{array}{c} a \\ p \\ b \\ u \\ c \end{array} & \begin{array}{c} \xrightarrow{1} \\ \alpha \\ \xrightarrow{1} \\ \gamma \\ \xrightarrow{1} \end{array} & \begin{array}{c} a \\ q \\ b \\ v \\ c \end{array} \\
 \begin{array}{c} a \\ p \\ b \\ u \\ c \end{array} & \begin{array}{c} \xrightarrow{1} \\ \beta \\ \xrightarrow{1} \\ \delta \\ \xrightarrow{1} \end{array} & \begin{array}{c} a \\ r \\ b \\ w \\ c \end{array}
 \end{array}
 \quad
 \begin{array}{ccc}
 \begin{array}{c} a \\ p \\ b \\ u \\ c \end{array} & \begin{array}{c} \xrightarrow{1} \\ \phi \\ \xrightarrow{1} \\ \theta \\ \xrightarrow{1} \end{array} & \begin{array}{c} a \\ r \\ b \\ w \\ c \end{array}
 \end{array}$$

Suppose moreover that we have equalities

$$\alpha \sqcap \beta = \phi$$

$$\gamma \sqcap \delta = \theta$$

Then we can show that

$$\Rightarrow(\alpha \sqcap \gamma) \cdot (\Rightarrow \beta \star \Rightarrow \delta) = (\Rightarrow \phi \star \Rightarrow \theta)$$

Proof. First we induct on the given assumptions, which replaces ϕ with $\alpha \sqcap \beta$ and θ with $\gamma \sqcap \delta$ in the goal. Then, we notice that each of α , β , γ , δ is a square with trivial top and bottom. Replacing them under the equivalence \Rightarrow we obtain equalities $\Rightarrow \alpha : p = q$, $\Rightarrow \beta : q = r$, $\Rightarrow \gamma : u = v$ and $\Rightarrow \delta$. We then perform induction on these four equalities, which leaves only the paths $p : a = b$ and $u : b = c$. Performing induction on these two reduces the goal to reflexivity. \square

This proof combines cubical interchange with the various other lemmas used above to construct the first proof: it simply puts all the premises together, and annihilates them using path induction. If we apply it to the equations

$$EH_r^{\text{nat}}(p, 1) \sqcap \text{urnat}(p) = \text{ulnat}(p)$$

$$EH_{\ell}^{\text{nat}}(q, 1) \sqcap \text{urnat}(q) = \text{urnat}(q)$$

in two different formations, then we immediately obtain proofs of triangles (a) and (c).

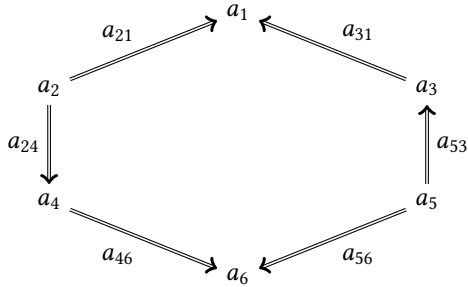
We believe that this style of proof is advantageous when proving higher-dimensional results.

7.3 Completing the proof

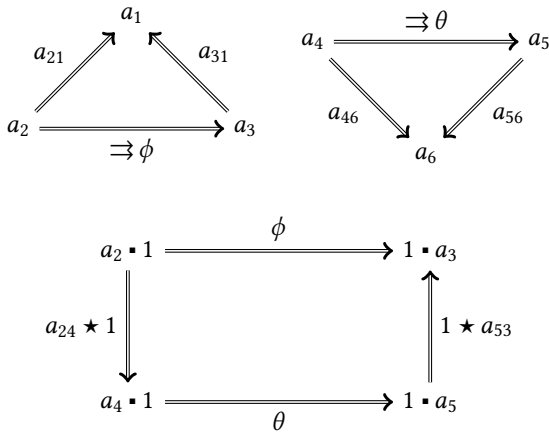
The final task in completing the proof of syllepsis is pasting together (a), (b) and (c). However, this is not immediately possible, as there is a mismatch: both the top and bottom of (b) are ‘off by a \Rightarrow ’ compared to the bottom of (a) and the top of (c). Similarly, both the sides of (b) are ‘off by a $\star 1$ ’ compared to the syllepsis construction of fig. 5.

We will use the same high-powered method of path induction to deal with this problem as well. In particular, we will construct a ‘syllepsis generator’ lemma, which will take in the individual proofs we have constructed, and fill the hexagon of fig. 5.

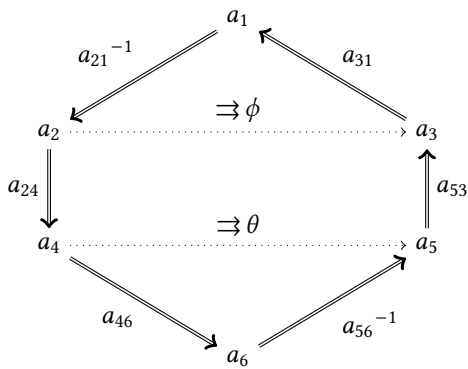
Lemma 7.3 (Syllepsis generator). *Suppose we have points $a, b : X$, paths $a_i : a = b$, and 2-paths as in the diagram*



Then, given squares $\phi : a_2 \cdot 1 = 1 \cdot a_3$ and $\theta : a_4 \cdot 1 = 1 \cdot a_5$ and fillers for



then we have a filler for the hexagon²



Proof. We first induct on the top left side a_{12} , the bottom left side a_{65} , and the two vertical sides a_{24} and a_{53} . This trivialises the assumed square into a path $\phi = \theta$ (modulo \Downarrow), on which we can then induct. Inducting further on the lower triangular assumption forces $a_{46} \equiv \Rightarrow \theta \equiv \Rightarrow \phi$. Cancelling some reflexivities, the remaining goal is $\Rightarrow \phi \cdot a_{31} = 1$, which is what is left from the upper triangular assumption. \square

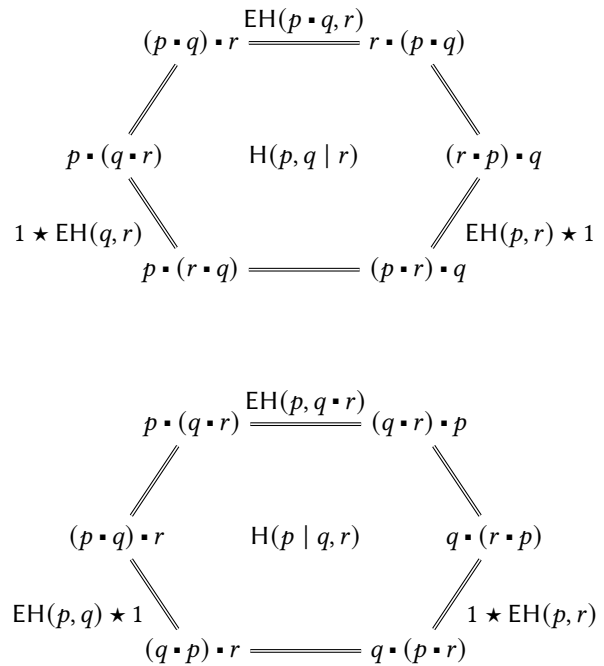
²The dotted lines are for illustration only; they are not part of the hexagon.

This lemma, which was once again shown by a series of path inductions, is simply the ‘shape’ of syllepsis. Instantiating with the proofs of (a), (b) and (c) that we have established before we obtain a proof of theorem 7.1.

8 Further coherence laws

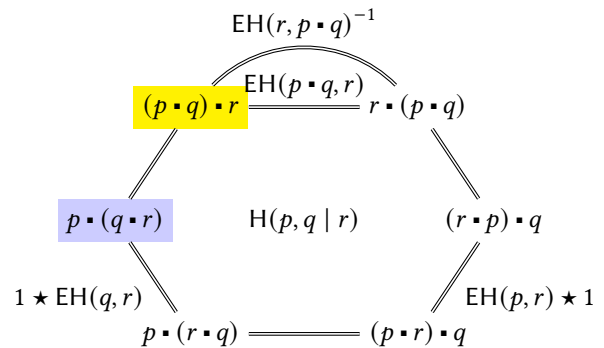
The syllepsis gadget as introduced by Day and Street [1997] needs to satisfy further coherence laws. However, even stating those takes some work.

First, recalling that EH is a kind of braid, it should satisfy a hexagon equation [Heunen and Vicary 2019, §1.2.1]. That is, given 2-loops $p, q : 1_* = 1_*$ we must have fillers for the diagrams

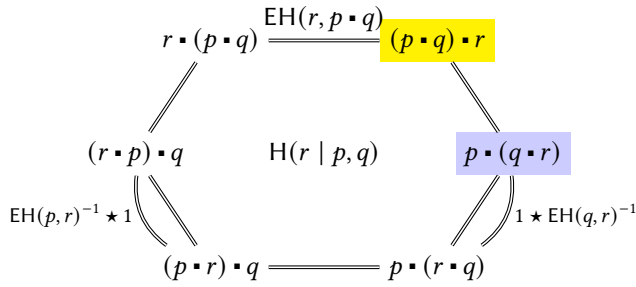


where the unlabelled edges are associativities.

These hexagons must satisfy additional coherences when adjusted by syllepses. For example, the hexagon



should be equal to the horizontally-flipped hexagon³



where in both diagrams the globular cells are filled by an instance of syllepsis. There is also a symmetric version of this diagram, beginning with $H(p | q, r)$.

Finally, to make the jump from a sylleptic monoidal 2-category to a *symmetric* monoidal 2-category, we need to show that if we start with 4-loops $p, q : 1_{1_{1_*}} = 1_{1_{1_*}}$ we have that

$$\begin{array}{ccc}
 q \bullet p & \xrightarrow{EH(q, p)} & p \bullet q \\
 \parallel \text{EH}(p, q) & \text{Syl}(p, q) & \parallel \text{EH}(p, q) \\
 & \text{1} & \\
 p \bullet q & \xrightarrow{EH(p, q)} & q \bullet p
 \end{array} \quad (4)$$

is equal to

$$\begin{array}{ccc}
 q \bullet p & \xrightarrow{EH(q, p)} & p \bullet q \\
 \parallel \text{EH}(p, q) & \text{Syl}(q, p) & \parallel \text{EH}(p, q) \\
 & \text{1} & \\
 p \bullet q & \xrightarrow{EH(p, q)} & q \bullet p
 \end{array} \quad (5)$$

where the unmarked triangles are filled with reflexivity.

We have been able to give pen-and-paper proofs of these facts. Together they amount to a proof that 2-loops in MLTT form a symmetric monoidal 2-category under concatenation. These are fairly involved results; for example, this last equation between (4) and (5) is in a way akin to ‘syllepsis one dimension higher’—much in the same way syllepsis is like ‘EH one dimension higher’ itself. We are thus forced to omit the proofs due to want of space. However, we have formalised the first two coherence laws on syllepsis itself.

9 Formalisation

The results in this paper have been formalised using the [Coq library for Homotopy Type Theory](#) [Bauer et al. 2017]. In the form it has at the time of writing, the Coq HoTT library is meant to be installed through the OCaml package manager,

³We must not forget to adjust the associativities appropriately.

opam. Our formalisation is thus a self-contained repository which imports the library; it may be found at

<https://github.com/lambdabetaeta/Syllepsis-in-Coq>

This repository is in the form of a Coq project, and comes with a Makefile that compiles it.

Our results are contained in `src/Syllepsis.v`. Finally, the hexagon equations are shown in the file `src/EH_Braid.v`.

10 Conclusion

In this paper we showed in detail how to construct a syllepsis for 2-loops in intensional Martin-Löf type theory. This result paves the way towards a fully-formalised proof that 2-loops in MLTT are a symmetric monoidal 2-category under concatenation. This exhausts the known elements of the periodic table of n -categories, as shown in fig. 3. In the future we hope to use these results to simplify proofs in synthetic homotopy theory.

Acknowledgments

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